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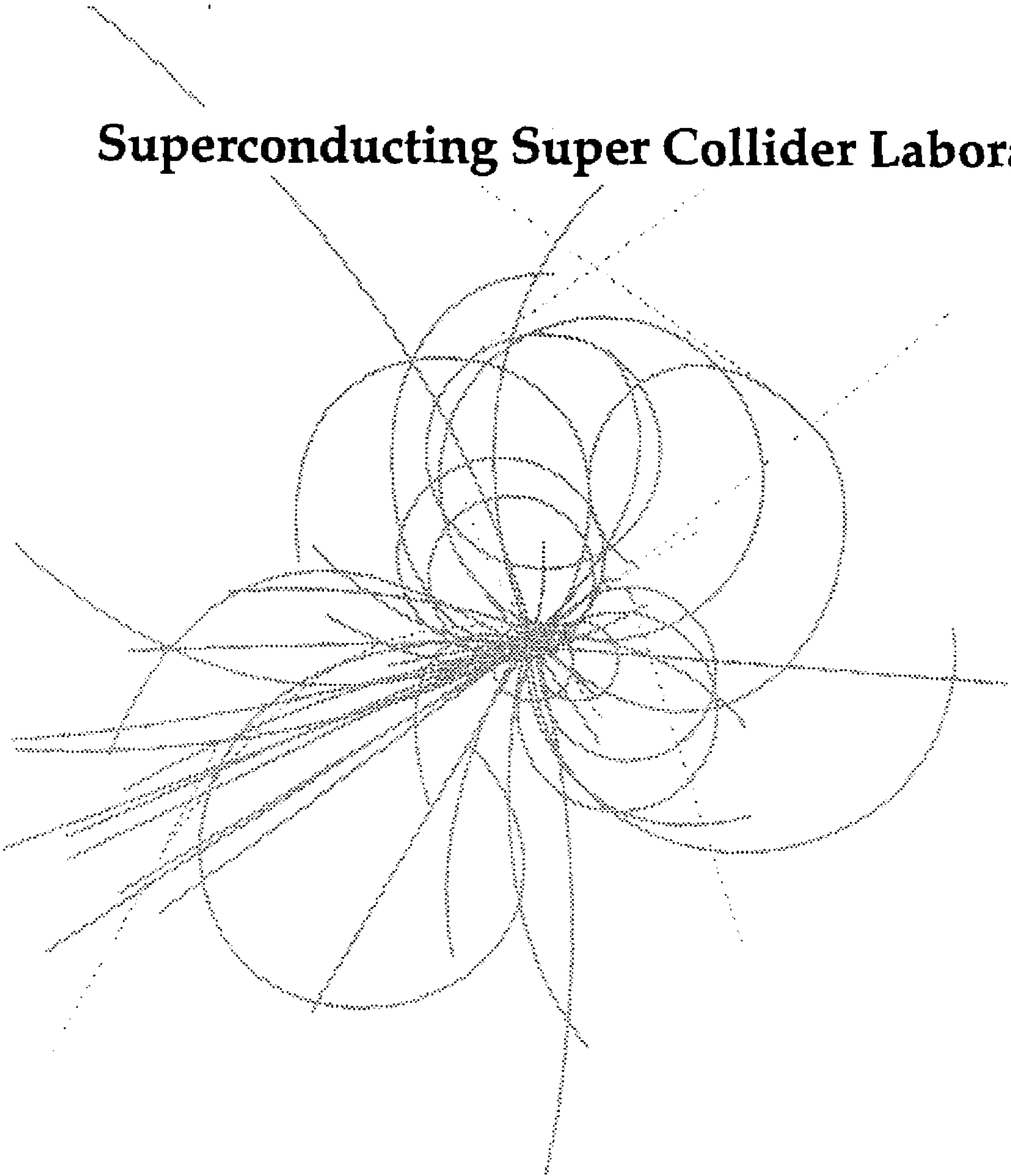

Normal Form Algorithm on Non-Linear Symplectic Maps

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NORMAL FORM ALGORITHM ON NON-LINEAR SYMPLECTIC MAPS

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I. Introduction

In this paper we will look at normal form procedures applied on non-linear symplectic maps. Our attention is focused on the non-linear part of the map only. A knowledge of Lie operators is assumed.

The map we will examine is relevant to the study of storage rings. We will start by assuming that we have computed the map representing one turn (or one superperiod) of the ring. We denote this symplectic map by M.

$$\vec{\xi}^{1} = M\vec{\xi} \tag{1.1a}$$

$$\vec{\xi} = (x,p_x,y,p_y,t,p_t) \quad . \tag{1.1b}$$

The vector ξ is made of a space part (x,p_x,y,p_y) and longitudinal or time part (t,p_t). In canonical variables p_t is an energy deviation variable. First we have to make a statement about our machine. We assume that the particle is not accelerated, hence p_t is a constant. Mathematically, this is the same as saying that the Hamiltonian of our system is time independent. More relevant to us, is the fact that the Lie algebraic polynomials representing M will not contain the time t.

Now, let us make some remarks about M_2, the linear part of M. The map M_2 has a linear matrix representation $\overline{M}$. If our machine is stable in the space variables it must have 4 unit modulus eigenvalues associated to the transverse coordinates $(x\text{-}p_x\text{-}y\text{-}p_y)$. We will denote them as follows:

$$\lambda_{\pm}^{x} = \exp(\pm i\ w_x) \tag{1.2a}$$

$$\lambda_{\pm}^{y} = \exp(\pm i\ w_y) \quad . \tag{1.2b}$$

On the other hand, the motion of t is "drift"-like in nature. That is, if a particle is launched with only p_t different from zero, time will linearly accumulate proportionally to p_t. For this particular set of conditions, the linear map M_2 can be transformed into a new map M_2^+ by a linear symplectic similarity transformation:

$$M_2^+ = \tilde{A}\ M_2\ \tilde{A}^{-1} \quad , \tag{1.3}$$

where M_2^+ as a very simple Lie algebraic representation:

$$M_2^+ = \exp(:f_2:) \tag{1.4a}$$

$$f_2 = f_2^x + f_2^y + f_2^t \tag{1.4b}$$

$$f_2^x = -\frac{w_x}{2}(x^2 + p_x^2) \tag{1.4c}$$

$$f_2^y = -\frac{w_y}{2}(y^2 + p_y^2) \tag{1.4d}$$

$$f_2^t = -\frac{\alpha}{2}\, p_t^2 \tag{1.4e}$$

The details of this transformation will not concern us here. They are outlined in the appendix.

Assuming that we now know the map $\tilde{A}$, we can compute its effect on M. We assume that M has a Lie algebraic representation:

$$M = M_2 \exp(:f_3:) \exp(:f_4:) \ldots \tag{1.5}$$

Applying $\tilde{A}$ on M leads to the result:

$$\begin{aligned} M^+ &= \tilde{A}\ M\tilde{A}^{-1} = (\tilde{A}M_2\tilde{A}^{-1})(\tilde{A}\ \exp(:f_3:)\tilde{A}^{-1})(\tilde{A}\ \exp(:f_4:)\tilde{A}^{-1}) \\ &= M_2^+\ \exp(:f_3^+:)\ \exp(:f_4^+:) \ldots \end{aligned} \tag{1.6}$$

The factors in (1.6) were transformed by the rule:

$$\tilde{A}\,\exp(:g(\vec{\xi}):)\tilde{A}^{-1} = \exp(\tilde{A}:g(\vec{\xi}):\tilde{A}^{-1})$$

$$= \exp(:\tilde{A}g(\vec{\xi}):)$$

$$= \exp(:g(\vec{\tilde{A}\xi}):)$$

$$= \exp(:g^{+}:) \quad . \tag{1.7}$$

Our task is to find a non-linear similarity transformation which will simplify the expression (1.6) for M^{+}. One extremely useful concept in the linear theory is the eigenvector basis. We will see in the next section how one decomposes the f_m^{+} into eigenvectors of f_2. These eigenvectors will represent the resonances and the amplitude dependent tune shifts.

II. The Eigenbasis for the Polynomials

Consider $J_x(= x^2 + p_x^{\,2})$, it is clear that $x + ip_x$ and $x - ip_x$ are eigenfunctions of $:J_x:$. In fact we must have:

$$:f_2^x:(x\pm ip_x) = -\frac{w_x}{2}\,:J_x:(x\pm ip_x) = \mp\, iw_x(x\pm ip_x) \tag{2.1a}$$

and similarly for $:f_2^y:$,

$$:f_2^y:(y\pm ip_y) = -\frac{w_y}{2}\,:J_y:\,(y\pm ip_y) = \mp iw_y(y\pm ip_y) \tag{2.1b}$$

The $(t\text{-}p_t)$ space is made of the eigenfunction p_t and the generalized eigenfunction t:

$$:f_2^t:\; t = -\alpha p_t \tag{2.2a}$$

$$:f_2^t:\; p_t = 0 \tag{2.2b}$$

Since the polynomials $f_m^{\,+}$ do not contain t we need only to consider the 5 remaining eigenfunctions. Let us denote them as follows:

$$h_{\mp} = x \pm i p_x \tag{2.3a}$$

$$V_{\mp} = y \pm i p_y \tag{2.3b}$$

In general, f_m^+ can be expressed in terms of $(x,\ p_x,\ y,\ p_y,\ p_t)$ or $(h_{\mp},\ V_{\mp},\ p_t)$. In the latter representation we get:

$$f_m^+ = \sum_{a=o}^{m} p_t^a \sum_{b+c+d+e=m-a} A_{abcde} h_+^b\ h_-^c\ V_+^d\ V_-^e \tag{2.4}$$

Let us compute the effect of $:f_2:$ on one monomial of f_m^+.

$$\begin{aligned} :f_2: |abcde> &= :f_2: p_t^a\ h_+^b\ h_-^c\ V_+^d\ V_-^e \\ &= i\left\{(b-c)w_x + (d-e)w_y\right\} |abcde> \end{aligned} \tag{2.5}$$

Clearly, each monomial in (2.4) is an eigenvector of $:f_2:$. Because of this, any map which is a function of $:f_2:$ is diagonal in the basis that we selected. In particular, M_2^+, which is just $\exp(:f_2:)$, is very simply expressed.

In the next section we examine the structure of $M_2^{+-1} - I$. This strange map, which we denote T, is essential to the normal problem.

III. The Maps M_2^+, T and T^{-1}

Suppose we are only interested in understanding the second order effects. (Transport-like formalism.) In this case, the knowledge of f_3^+ is sufficient.

$$M^+ = M_2^+ \exp(:f_3^+:) \tag{3.1}$$

Our hope is to "clean up" f_3^+ as much as we can. To achieve this purpose, we applied a non-linear transformation A. The map A does not contain any linear terms since they would affect M_2^+. However, it must have a third order Lie polynomial to cancel f_3^+. It follows that A must be of the form:

$$A = \ldots \exp(:F_4:)\exp(:F_3:) \tag{3.2}$$

To the order of f_3^+, we apply A on M^+.

$$= A\ M^+ A^{-1} = \exp(:F_3:)M_2^+\exp(:f_3^+:)\exp(-:F_3:)$$

$$= M_2^+(M_2^{+^{-1}} \exp(:F_3:)M_2^+)\exp(:f_3^+:)\exp(:-F_3:)$$

$$= M_2^+\exp(:M_2^{+^{-1}} F_3 - F_3 + f_3^+:)\ldots \tag{3.3}$$

To order f_3^+, all the exponents add. The new map, which we denote Λ, will depend on the exponent in (3.3), that is:

$$\Lambda_3 = (M_2^{+^{-1}} - I)\ F_3 + f_3^+ = T\ F_3 + f_3^+ \quad . \tag{3.4}$$

Can we select F_3, such that Λ_3 is zero. And should we do it?

To study these questions, we restrict ourselves to the subspace of homogeneous polynomials of degree m in (x, p_x, y, p_y). Every such polynomial g_m can be expanded in the eigenbasis of Section 2.

Using the result of line (2.5) we get:

$$:f_2:\ h_+^b\ h_-^c\ V_+^d\ V_-^e = :f_2:|0bcde>$$

$$= i\left\{(b-c)w_x + (d-e)w_y\right\}|0bcde> \quad . \tag{3.5}$$

Hence T acting on $|0bcde\rangle$ is just given by:

$$T|0bcde\rangle = (\exp(-i\{(b-c)w_x + (d-e)w_y\} -1)|0bcde\rangle \tag{3.6}$$

Replacing w_x by $2\pi\ \nu_x$ and w_y by $2\pi\ \nu_y$, we see that $|0bcde\rangle$ is in the null space KerT if:

$$(b-c)\nu_x + (d-e)\nu_y = \text{integer}. \tag{3.7}$$

Independently of ν_x and ν_y, this will always be satisfied if b=c and d=e. These vectors have the following form:

$$\begin{aligned} |0bbdd\rangle &= (h_+h_-)^b\ (V_+V_-)^d \\ &= (x^2+p_x^2)^b\ (y_+^2+p_y^2)^d \\ &= J_x^b\ J_y^d \end{aligned} \tag{3.8}$$

One sees that these "permanent" members of KerT are just power of J_x and J_y. They are the tune shifts. Occasionally, equation (3.7) will be exactly or nearly satisfied. These cases correspond to the so-called non-linear resonances.

In the next section, we study equation (3.4) in greater detail.

IV. Full Decomposition of the Polynomial Spaces

Let us come back to the eigenbasis. Our eigenbasis is complex. However, we are studying real polynomials. With the help of the complex basis, we can construct a very useful basis for the m^{th} degree real polynomials in transverse variables.

Consider the following real basis:

$$r_1(bcde) = \frac{1}{2} (|0bcde> + |0cbed>) \tag{4.1a}$$

$$r_2(bcde) = \frac{i}{2} (-|0bcde> + |0cbed>) \tag{4.1b}$$

It is worth noticing that $|0cbed>$ is the complex conjugate of $|0bcde>$. This makes r_1 and r_2 real.

One also notices that r_1 and r_2 belong to the same resonance. That is if Tr_1 is zero so is Tr_2. Now, let us apply the map $\exp(\alpha:f_2:)$ on r_1 and r_2.

$$\phi(b,c,d,e) = (b-c)w_x + (d-e)w_y \tag{4.2a}$$

$$\exp(\alpha:f_2:)r_1 = \frac{1}{2} \{\exp(i\alpha\phi)\ 0bcde> + \exp(-i\alpha\phi)|0cbed>\}$$

$$= \cos(\alpha\phi)r_1 - \sin(\alpha\phi)r_2 \tag{4.2b}$$

$$\exp(\alpha:f_2:)r_2 = \frac{i}{2}\{-\exp(i\alpha\phi)\,|0bcde> + \exp(-i\alpha\phi)\,|0cbed>\}$$

$$= \cos(\alpha\phi)r_2 + \sin(\alpha\phi)r_1 \tag{4.2c}$$

Upon application of the map $\exp(\alpha:f_2:)$, the real polynomials r_1 and r_2 rotate into one another by an angle $\alpha\phi$. We can now decompose entirely a m^{th} degree polynomial in (x, p_x, y, p_y) as follows:

$$f_m = \sum_{\substack{b\neq c\\ d\neq e\\ b+c+d+e=m}} \left\{A_\phi\ r_1(\phi) + B_\phi\ r_2(\phi)\right\} + \sum_{i+j=\frac{m}{2}} C_{ij}\ J_i^x J_j^y \tag{4.3}$$

Equation (4.3) indicates that f_m is made of pairs of polynomials and of individual null vectors.

The pairs are globally invariant under the action of $\exp(\alpha:f_2:)$ or T. One should notice that for odd degrees, f_m does not contain any "permanent" null space members.

V. Computation of A for the Special Cases of f_3^+ and f_4^+ of a Ring

Using the decomposition of the last section we can write f_3^+ as follows:

$$f_3^+ = a\, p_t^3 + p_t^2 h_1 + p_t h_2 + h_3 \tag{5.1}$$

The polynomials h_i are of degree i in the transverse variables. Similarly, we can write F_3 of equation (3.4) using the same ordering in degrees of p_t:

$$F_3 = b\, p_t^3 + p_t^2 g_1 + p_t g_2 + g_3 \tag{5.2}$$

Let us substitute F_3 and f_3^+ in equation (3.4). The resulting Λ_3 is decomposed in the Λ_3^i, where the superscript "i" denotes the power in transverse variables. Obviously, since T does not change the power of p_t, we can proceed in every subspace separately. Away from resonant conditions, we can find g_3 and g_1.

$$i = 1 \text{ or } 3: \Lambda_3^i = 0 \rightarrow g_i = -(T^{-1} h_i) \quad . \tag{5.3}$$

Obviously, p_t^3 is invariant, hence we set b_1=0.

$$\Lambda_3^0 = A p_t^3 \tag{5.4}$$

The case of Λ_3^2 is more interesting. From this term will emerge the tune shift. Indeed in general, h_2 will contain powers of J_x and J_y. We can rewrite h_2, in terms of the basis (4.3):

$$h_2 = \sum_{\phi} \left\{ A(\phi)\ r_1(\phi) + B(\phi) r_2(\phi) \right\} + \frac{w_x'}{w_x}\ f_2^x + \frac{w_y'}{w_y}\ f_2^y \tag{5.5a}$$

$$\phi = \left\{ 2w_x,\ 2w_y,\ w_x + w_y,\ w_x - w_y \right\} \tag{5.5b}$$

We chose Λ_3^2 to be just the tune shift terms in (5.5a).

$$\Lambda_3^2 = \left\{ \frac{w_x'}{w_x}\ f_2^x + \frac{w_y'}{w_y}\ f_2^y \right\} p_t \tag{5.6}$$

This implies that g_2 must remove the first term (5.5):

$$g_2 = -T^{-1} \sum_{\phi} \left\{ A(\phi)\ r_1(\phi) + B(\phi)\ r_2(\phi) \right\} \tag{5.7}$$

In the $(r_1(\phi),\ r_2(\phi))$ subspace this problem is reduced to inverting a two by two matrix.

The case of f_4^+ is similar; we will outline the expected results. To remove f_4^+, we will require that f contain one more map:

$$A = \exp(:F_4:)\exp(:F_3) \quad . \tag{5.8}$$

Applying A on M^+ gives the result:

$$\begin{aligned} A\ M^+ A^{-1} &= A\ M_2^+\ \exp(:f_3^+:)\exp(:f_4^+:)\ A^{-1} \\ &= M_2^+\ \exp(:M_2^{+^{-1}}\ F_4:)\exp(:M_2^{+^{-1}}\ F_3:)\exp(:f_3^+:)\exp(:f_4^+:) \\ &\quad \times \exp(:-F_3:)\exp(:-F_4:) \quad . \end{aligned} \tag{5.9}$$

We now combine the exponents to degree 4:

$$\Lambda_3 = TF_3 + f_3^+ \tag{5.10a}$$

$$\Lambda_4 = TF_4 + f_4^+ + \frac{1}{2}\,[f_3^+, -F_3] + \frac{1}{2}\,[\Lambda_3, f_3^+ - F_3] \tag{5.10b}$$

Again, the right-hand side is decomposed in powers of p_t. Let us write the final result:

$$\Lambda_4 = \beta_{hh}\, f_2^{x^2} + \beta_{vv}\, f_2^{y^2} + \beta_{hv}\, f_2^x f_2^y + \frac{1}{2}\frac{w_x''}{w_x} p_t^2\, f_2^x$$

$$+ \frac{1}{2}\frac{w_y''}{w_y} p_t^2\, f_2^y + b\, p_t^4 \tag{5.11}$$

Now, we will examine the interpretation of Λ_3 and Λ_4. This is the purpose of the next section.

VI. The Tune Shifts

We first notice that f_2, Λ_3 and Λ_4 all mutually commute with one another. In fact all the terms in Λ_3 and Λ_4 are made of products of the invariants f_2^x, f_2^y and p_t. Because of this feature we can join the exponents:

$$A\, M^+ A^{-1} = \exp(:f_2 + \Lambda_3 + \Lambda_4:) \quad . \tag{6.1}$$

The motion of a ray, under the action of M^+ can be computed exactly from the pseudo-Hamiltonian $H = (f_2 + \Lambda_3 + \Lambda_4)$. In fact, H will produce tune shifts with amplitude and energy. To see what they are let us compute $\frac{\partial H}{\partial x}$:

$$\frac{\partial H}{\partial x} = \frac{\partial f_2^x}{\partial x} + p_t \frac{w_x'}{w_x} \frac{\partial f_2^x}{\partial x} + \frac{1}{2} p_t^2 \frac{w_x''}{w_x} \frac{\partial f_2^x}{\partial x} + 2\beta_{hh}\, f_2^x\, \frac{\partial f_2^x}{\partial x}$$

$$+ \beta_{hv}\, f_2^y \frac{\partial f_2^x}{\partial x}$$

$$= \left\{1 + p_t \frac{w_x'}{w_x} + \frac{p_t^2}{2} \frac{w_x''}{w_x} + 2\beta_{hh}\, f_2^x + \beta_{hv}\, f_2^y\right\} \frac{\partial f_2^x}{\partial x} \quad . \tag{6.2}$$

Hence the tune shift $\delta w_x/w_x$ is given by the expression in (6.2):

$$\frac{\delta w_x}{w_x} = p_t \frac{w_x'}{w_x} + \frac{p_t^2}{2} \frac{w_x''}{w_x} + 2\beta_{hh} f_2^x + \beta_{hv} f_2^x \tag{6.3a}$$

$$\frac{\delta w_y}{w_y} = p_t \frac{w_y'}{w_y} + \frac{p_t^2}{2} \frac{w_y''}{w_y} + 2\beta_{vv} f_2^y + \beta_{hv} f_2^x \quad . \tag{6.3b}$$

The effect on the time t is similarly computed by evaluating $\partial H/\partial p_t$:

$$\frac{\partial H}{\partial t} = \frac{w_x'}{w_x} f_2^x + \frac{w_y'}{w_y} f_2^y + 3a\ p_t^2 + \frac{w_x''}{w_x} f_2^x\ p_t + \frac{w_y''}{w_y} f_2^y\ p_t + 4bp_t^3 - \alpha p_t \tag{6.3c}$$

This last expression (6.3c), contains the various shifts to the coefficient α of f_2^t.

We conclude that the computation of A gives all the tune shifts one ordinarily desires. In the next section, we look at the invariants and the new closed orbit for a given p_t.

VII. Generalized Map Invariants and the Fixed Point

Consider the two function F_x and F_y generated as follows:

$$F_x = A^{-1}\ f_2^x \tag{7.1c}$$

$$F_y = A^{-1}\ f_2^y \tag{7.1b}$$

If we express the map M^+ in term of Λ_3 and Λ_4 we can easily check the invariance of F_x and F_y:

$$M^+ = A^{-1}\ e^{:f_2+\Lambda_3+\Lambda_4:}\ A = A^{-1}\ e^{:H:}\ A \quad . \tag{7.2}$$

Applying M^+ on F_x, we obtain

$$M^+ F_x = A^{-1} e^{:H:} A A^{-1} f_2^x \tag{7.3}$$

$$= A^{-1} e^{:H:} f_2^x$$

$$= A^{-1} f_2^x, \text{ since } [H, f_2^x] = 0$$

$$= F_x$$

A similar result applies for F_y.

F_x and F_y are p_t dependent non-linear invariants. The trajectory of a particle will be on the surface of constant F_x and F_y.

In addition, we can construct a pseudo-Hamiltonian h, to describe the motion of the particle.

$$M^+ = A^{-1} \exp(:H:) A$$

$$= \exp(:A^{-1}H:) = \exp(:h:) \tag{7.4}$$

Clearly, h is a function of F_x and F_y alone (as well as p_t).

Our next topic concerns the fixed point. Consider the following ray:

$$\vec{\zeta}^{inv} = (0,0,0,0,0,p_t) \tag{7.5}$$

It is clear that $\vec{\zeta}^{inv}$ is an invariant of H.

$$\vec{\zeta}^{inv} \xrightarrow{\text{under H}} \vec{\zeta}^{inv} \tag{7.6a}$$

or

$$\exp(:H(\vec{\zeta}):)\vec{\zeta}\,\Big|_{\vec{\zeta} = \vec{\zeta}^{inv}} = \vec{\zeta}^{inv} \tag{7.6b}$$

Now, let us go back to the map M^+, we must have:

$$M^+ = A^{-1} \exp(:H:) A \qquad (7.7)$$

This map is expressed in term of the physical variables, $\vec{\xi}$. In this representation A^{-1} operate first. Therefore $\vec{\zeta}$, the variables undergoing pure oscillations are given by

$$\vec{\zeta} = A^{-1} \vec{\xi} \quad . \qquad (7.8)$$

As a result, the fixed point in physical variables must be:

$$\vec{\xi}^{\,inv} = A\vec{\zeta} \Big|_{\vec{\zeta} = \vec{\zeta}^{\,inv} = (0,0,0,0,0,p_t)} \qquad (7.9)$$

Derivative of this fixed point can be computed exactly from the expression for A.

VIII. Conclusion

In this paper, we outlined the techniques used for normalizing a symplectic map. These techniques assume the existence of such a map. In our case, we use the map provided by the computer code Marylie. We have implemented two normal form algorithms. The first one we just described. It assumes only that p_t (or δ_p) is a constant. The second option assumes that the longitudinal phase space undergoes oscillations with a tune w_t. What we have implemented does not assume mid-plane symmetry or any other kind of symmetry.

A last word about other techniques should be said. Before the advent of the Lie algebraic representation of a map, the process of normalizing was done on the Hamiltonian. This implies that the normal form algorithm must be path length dependent. It can be shown that these approaches, simultaneously compute the map M as well as the transformation A. Our technique is more modular.

Appendix

Computing the linear part A_2 involves two steps. First we find the new fixed point off-energy $\eta(=(x, p_x, y, p_y))$. Indeed, if p_t is zero, the closed orbit is just the origin. What if p_t is not zero? To compute η we write the matrix representation of M, $\bar{M}$:

$$\bar{M} = \begin{array}{r} 4\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ \\ \\ \end{array} \left[\begin{array}{c|c} \overbrace{\qquad\quad N \qquad\quad}^{4} & \begin{matrix} 0 & m_1 \\ 0 & m_2 \\ 0 & m_3 \\ 0 & m_4 \end{matrix} \\ \hline \begin{matrix} n_1 & n_2 & n_3 & n_4 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} 1 & \alpha \\ 0 & \end{matrix} \end{array}\right] , \qquad \text{(A1)}$$

We act with $\bar{M}$ on $(\eta, 0, p_t)$. If η is a fixed point, the following equation is satisfied:

$$\eta = N\eta + m\, p_t \quad . \qquad \text{(A2)}$$

Writing η as δp_t, we get for δ:

$$\delta = (I-N)^{-1}\, m \quad . \qquad \text{(A3)}$$

Can we construct a canonical transformation B_2 which shift the origin to the location η? The answer is yes and this transformation is just:

$$x^{New} = B_2^{-1}\ x = x - \delta_1\ p_t$$

$$p_x^{New} = B_2^{-1}\ p_x = p_x - \delta_2\ p_t$$

$$y^{New} = B_2^{-1}\ y = y - \delta_3\ p_t$$

$$p_y^{New} = B_2^{-1}\ p_y = p_y - \delta_4\ p_t$$

$$t^{New} = B_2^{-1}\ t = t - \delta_1\ p_x + \delta_2\ x - \delta_3\ p_y + \delta_4\ y$$

$$p^{New} = B_2^{-1}\ p_t = p_t \tag{A4}$$

Notice that B_2 preserves the Poisson bracket and therefore is canonical. Applying B_2 to M, brings its matrix representation to the form:

$$\tilde{M} = \left(\begin{array}{c|c} N & 0 \\ \hline 0 & \begin{matrix} 1 & \alpha \\ 0 & 1 \end{matrix} \end{array}\right) \tag{A5}$$

The final step consists in reducing N in two blocks, each being a rotation.

For more generality, let us assume that N is a n by n symplectic matrix acting on a vector $Z = (x_1, p_1, \ldots, x_n, p_n)$. We first compute the eigenfunctions associated to the Lie map N. In general, a linear function of Z can be represented as follows:

$$f = \sum_{i=1}^{2N} \beta_i\ Z_i \quad . \tag{A6}$$

We can think of the 2n monomials Z_i as a basis. What is the action of N on f?

$$Nf = \sum_{i=1}^{2N} \beta_i \, NZ_i = \sum_{j=1}^{2N} \sum_{i=1}^{2N} \beta_i \, N_{ij} \, Z_j$$

$$= \sum_{j=1} \beta_j^{New} \, Z_j \qquad (A7)$$

The action of N on the coordinates $\vec{\beta}$ is given by the transpose of N.

$$\vec{\beta}^{New} = N^{tr} \vec{\beta} \qquad (A8)$$

We assume that N^{tr} as n distinct pairs of complex conjugate, unimodular eigenvalues. From them and their associated eigenvectors we construct the eigenfunctions f_i.

$$f_i^{\pm} = f_i^R \pm i \, f_i^I \qquad i=1,n$$

$$Nf_i^{\pm} = e^{\pm i w_i}(f_i^R \pm i \, f_i^I) \qquad (A9)$$

Now, consider the following transformation:

$$\bar{X}_i = \frac{f_i^R}{\chi} \qquad \bar{p}_i = \frac{f_i^I}{\varepsilon\chi} \qquad (A10)$$

$$\chi = [f_i^R, f_i^I]^{1/2}$$

$$\varepsilon = \mathrm{sign}[f_i^R, f_i^I] \quad .$$

It is remarkable that (A10) is canonical and turns N into blocks of rotations.

We will leave the proof to the reader. It involves only one trick. One computes the action of N on the Poisson bracket $[f_i^{\pm}, f_j^{\pm}]$ two different ways.

$$\text{First way:}\quad N[f_i^{\pm}, f_j^{\pm}] = [f_i^{\pm}, f_j^{\pm}] \quad , \tag{A11}$$

since $[f_i^{\pm}, f_j^{\pm}]$ is a C-number.

$$\text{Second way:}\quad N[f_i^{\pm}, f_j^{\pm}] = [Nf_i^{\pm}, Nf_i^{\pm}] \tag{A12}$$

$$= \exp(\pm iw_i \pm iw_j)[f_i^{\pm}, f_j^{\pm}].$$

Assuming that $w_i \neq w_j$, one can deduce from (A11) and (A12) that (A10) is canonical or Poisson bracket preserving.

We also find that $\bar{J}_i = (\bar{X}_i^2 + \bar{p}_i^2)$ are invariants of N. From this we deduce that in the new and the old basis N must be given by:

$$N = \exp(:\sum_{i=1} - \frac{w_i}{2} \bar{J}_i:) \quad . \tag{A13}$$

This is just the desired form for the normalization algorithm.

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