Spacetime Locality of the Antifield Formalism: General Theorems Illustrated by Means of Examples

Marc Henneaux

Faculté des Sciences, Université Libre de Bruxelles, Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium.
and
Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile

Abstract. Some general techniques and theorems on the spacetime locality of the antifield formalism are illustrated in the familiar cases of the free scalar field, electromagnetism and Yang-Mills theory. Common misconceptions in the field are corrected.

1 Introduction

The antifield-BRST formalism [1] provides a powerful approach to the quantization of gauge systems. Its geometric and algebraic features have been clarified in [2, 3, 4], where it was shown how the general BRST construction implements gauge invariance in cohomology. The crucial equation of the theory, namely the TTmaster equation", was in particular justified and derived from this point of view. A general exposition of these ideas with pedagogical emphasis may be found in [5].

A major feature of the theory is that the solution of the master equation is determined perturbatively as a power series in the antifields. As it has been shown in [2, 3], the rationale for introducing the antifields is that these provide a resolution of the algebra of functionals of on-shell field configurations. Namely, the antifields are there to implement the equations of motion when one passes to the BRST cohomology. The resolution associated with the antifields is called "Koszul-Tate" resolution, because it is patterned after a construction due to Koszul [6], supplemented, when the equations of motion are not independent, by the introduction of further variables killing unwanted homology along lines due to Tate [7]. The acyclicity of the Koszul-Tate differential in stricly positive resolution degree is crucial for the existence of the higher order terms in the perturbative expansion of the solution of the master equation. [We assume some familiarity with the general ideas of the antifield formalism; we refer to [5] for a detailed exposition].

The analysis presented in [3] did not address the question of the spacetime locality of the construction. More precisely, it did not address the question as to whether the acyclicity of the Koszul-Tate differential in stricly positive resolution degree still holds in the space of local functionals. A few years ago, that question has been investigated and completely solved [8] (see also [5], chapters 12 and 17). The purpose of this paper is to make it clear how the approach developed in [8] works and does indeed solve the issue of locality by illustrating it in the familiar cases of the Klein-Gordon field, the electromagnetic field and the Yang-Mills field.

We shall analyse only the specific question of locality of the Koszul-Tate complex. The reference [5] contains a discussion as to why this complex is so useful in the quantization of gauge systems.

2 Definitions

Consider a field theory with field variables ϕ^i. We shall deal with both local functionals and local functions of ϕ^i. Local functions are functions of ϕ^i and a finite number of their derivatives, which may also involve the spacetime coordinates explicitly. So, a local function is given by

$$f(x^\mu, \phi^i, \partial_\mu \phi^i, ..., \partial_{\mu_1 \dots \mu_k} \phi^i). \tag{1}$$

Local functionals are integrals of local functions. Hence,

$$F[\phi^i] = \int f(x^\mu, \phi^i, \partial_\mu \phi^i, ..., \partial_{\mu_1 \dots \mu_k} \phi^i) d^n x \tag{2}$$

is a local functional.

The appropriate way to deal with local functions is well known and has been used quite a lot in the algebraic study of anomalies. The corresponding mathematical framework is the one of jet bundle theory (see e.g. [9, 10]). However, in order to keep the discussion simple, we shall not adopt here the jet bundle terminology. This is permissible because we shall assume that spacetime is R^n, so that there are no global subtleties.

Let V^0 be the space with coordinates (x, ϕ^i). More generally, let V^k be the space with coordinates $(x, \phi^i, \partial_\mu \phi^i, ..., \partial_{\mu_1 \dots \mu_k} \phi^i)$. If f is a smooth local function, then there exists k such that $f \in C^\infty(V^k)$. For this reason, the V^k's are the natural spaces in which to analyze locality. These spaces arose first in the geometric study of differential equations, which can naturally be regarded as representing surfaces in the V^k's. In that context, the spaces V^k are called k-th jet bundles and are denoted by $J^k(E)$.

We stress that the jet bundle spaces are quite familiar not only in mathematics but also in physics since these are the spaces in which the Lagrangians of local field theories live. These spaces are finite dimensional for each k. For this reason, all the standard algebraic tools of the antifield formalism (contracting homotopy, counting operators, recursive introduction of the antifields of antifields by successive killing of unwanted cohomology, model for the exterior derivative along the gauge orbits, antibracket cohomology, role of zeroth order terms - see [5]) are available in the jet bundle spaces without functional complications.

In order to discuss local functionals, it is useful to consider the algebra $A_k \equiv C^\infty(V^k) \otimes \bigwedge[dx^\mu]$ of exterior forms on R^n with coefficients that are functions on V^k,

$$\omega \in A_k \Leftrightarrow \omega = \Sigma\, \omega_{\nu_1 \dots \nu_j}(x, \phi^i, \partial_\mu \phi^i, \dots, \partial_{\mu_1 \dots \mu_k} \phi^i)\, dx^{\nu_1} \wedge \dots \wedge dx^{\nu_j} \tag{3}$$

One can define a differential $d : A_k \to A_{k+1}$ as follows,

$$d\omega = \Sigma\, d\omega_{\mu_1 \dots \mu_j} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_j} \tag{4}$$

where d acting on a function $f \in A_k$ is defined by

$$df = \frac{\partial^T f}{\partial x^\mu}\, dx^\mu, \tag{5}$$

$$\frac{\partial^T f}{\partial x^\mu} \equiv \frac{\partial f}{\partial x^\mu} + \frac{\partial f}{\partial \phi^i} \partial_\mu \phi^i + \dots + \frac{\partial f}{\partial(\partial_{\mu_1 \dots \mu_k})\phi^i}\, \partial_{\mu_1 \dots \mu_k \mu} \phi^i. \tag{6}$$

One crucial property of d is that

$$\int d\omega = 0 \tag{7}$$

(we assume here and throughout that the boundary conditions are such that the surface terms appearing in the equations vanish. If not, one must carefully keep track of the relevant surface integrals).

Conversely let ρ be a n-form such that $\int \rho = 0$ for all field configurations. Then $\rho = d\omega$ (see e.g. [5]). Accordingly, two local functions determine the same local functional if and only if they differ by a d-exact term. For that reason, one can, following Gel'fand and Dorfman [11], identify local functionals with the quotient space $H^n(d)$ of local n-forms (which are automatically closed) modulo exact ones.

The Lagrangian $\mathcal{L}(\phi^i, \partial_\mu \phi^i, \dots, \partial_{\mu_1 \dots \mu_s} \phi^i)$ of the theory is a smooth function on V^s. The equations of motion[1]

$$\frac{\delta \mathcal{L}}{\delta \phi^i} \equiv \frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} + \dots + (-1)^s \partial_{\mu_1 \dots \mu_s} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1 \dots \mu_s} \phi^i)}, \tag{8}$$

together with their derivatives $\partial_\mu(\delta\mathcal{L}/\delta\phi^i) = 0$, $\partial_{\mu_1\mu_2}(\delta\mathcal{L}/\delta\phi^i) = 0$ ... determine surfaces Σ_k in V^k. For a fixed k, only a finite number of equations are relevant. The surfaces Σ_k are called "stationary surfaces".

[1] From now on, we shall drop the suffix T on ∂^T_μ: ∂_μ always stands for ∂^T_μ.

In the antifield formalism, the algebra $C^\infty(\Sigma_k)$ of smooth functions on Σ_k plays an important role because it is related to the observables [5]. The Koszul-Tate construction provides a resolution of $C^\infty(\Sigma_k)$ for each k. The idea is to view $C^\infty(\Sigma_k)$ as the quotient algebra $C^\infty(V^k)/\mathcal{N}_k$, where $\mathcal{N}_k$ is the ideal of functions of $C^\infty(V^k)$ that vanish on Σ_k. The Koszul-Tate differential is such that the elements of $\mathcal{N}_k$ are exact, i.e., are pure boundaries.

3 The Koszul-Tate differential for the massless scalar field

To illustrate the construction, we consider first the massless Klein-Gordon theory. One has a single scalar field ϕ with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \tag{9}$$

The equations of motion are

$$\Delta\phi \equiv \partial_\mu\partial^\mu\phi = 0. \tag{10}$$

In V^0, the equations of motion imply no relation and Σ_0 is empty: two functions f and g in V^0 coincide "on-shell" (i.e., when the equations of motion hold) if and only if they are identical. Similarly, there is no relation in V^1. One has to go to V^2 to see the first effect of the equations of motion, which restrict the second derivatives of ϕ. The surface Σ_2 is defined by $\Delta\phi = 0$ in V^2. Then, in V^3, Σ_3 is the surface $\Delta\phi = 0$, $\partial_\mu\Delta\phi = 0$. More generally, the surface Σ_k in V^k is defined by the equations

$$\Sigma_k : \Delta\phi = 0, ..., \Delta\partial_{\mu_1}...\partial_{\mu_{k-2}}\phi = 0. \tag{11}$$

The equations of motion (11) are independent in V^k. This is most easily seen by introducing a new coordinate system in V^k, which has the left hand side of the equations (11) as independent coordinates. One such coordinate system is given by

$$\phi,\ \partial_\mu\phi,\ \partial_{m_1m_2}\phi,\ \partial_{m_10}\phi, \Delta\phi, ..., \partial_{m_1...m_{k-3}m_k}\phi,\ \partial_{m_1...m_{k-1}0}\phi,\ \partial_{\mu_1...\mu_{k-2}}\Delta\phi. \tag{12}$$

One can easily verify that any function f on V^k that vanishes on Σ_k ($f \approx 0$) takes the form,

$$f \approx 0 \Leftrightarrow f = h\Delta\phi + h^\mu\partial_\mu\Delta\phi + ... + h^{\mu_1\cdots\mu_{k-2}}\Delta\partial_{\mu_1}...\partial_{\mu_{k-2}}\phi \tag{13}$$

where the h's are functions on V^k (see for instance [5], chapter 1 with $\phi_m = 0$ replaced by (11)).

In order to construct a resolution of $C^\infty(\Sigma_k)$, one introduces one independent odd generator for each (independent) equation (11). That is, one considers the differential algebra $C^\infty(V^k) \otimes \bigwedge[\phi^*, \partial_\mu\phi^*, ..., \partial_{\mu_1}...\partial_{\mu_{k-2}}\phi^*]$ with differential

$$\delta\phi = 0, \delta\phi^* = \Delta\phi, \tag{14}$$

extended to the derivatives of the field and "antifield" ϕ^* so as to commute with ∂_μ,

$$\delta\partial_{\mu_1\ldots\mu_j}\phi = 0, \delta\partial_{\mu_1\ldots\mu_j}\phi^* = \partial_{\mu_1\ldots\mu_j}\Delta\phi. \tag{15}$$

One defines also the antighost number through

$$antigh(\phi) = 0,\ antigh(\phi^*) = 1. \tag{16}$$

By (14), (15), every equation of motion is δ-exact and so, is identified with zero when one passes to the δ-homology. More precisely, standard arguments from homological algebra show that

$$H_0(\delta) = C^\infty(\Sigma_k),\ \ H_j(\delta) = 0 \text{ for } j \neq 0. \tag{17}$$

This result may be derived by observing that the coordinates of $C^\infty(V^k) \otimes \bigwedge[\phi^*, \partial_\mu\phi^*, ..., \partial_{\mu_1}...\partial_{\mu_{k-2}}\phi^*]$ split into three groups $(x_i, z_\alpha, J\mathcal{P}_\alpha)$ such that δ takes the form

$$\delta x_i = 0,\ \delta\mathcal{P}_\alpha = z_\alpha,\ \delta z_\alpha = 0 \tag{18}$$

or equivalently

$$\delta = z_\alpha \frac{\partial}{\partial \mathcal{P}_\alpha}. \tag{19}$$

Explicitly, the coordinates x_i stand for the field ϕ and its derivatives with at most one ∂_0, the z_α stand for $\Delta\phi$ and its derivatives, while the $\mathcal{P}_\alpha$ stand for ϕ^* and its derivatives. A contracting homotopy may be defined through

$$\sigma x_i = 0, \sigma\mathcal{P}_\alpha = 0, \sigma z_\alpha = \mathcal{P}_\alpha \Leftrightarrow \sigma = \mathcal{P}_\alpha \frac{\partial}{\partial z_\alpha}, \tag{20}$$

i.e.,

$$\sigma = \phi^* \frac{\partial}{\partial(\Delta\phi)} + \partial_\mu\phi^* \frac{\partial}{\partial(\partial_\mu\Delta\phi)} + ... + \partial_{\mu_1}...\partial_{\mu_{k-2}}\phi^* \frac{\partial}{\partial(\partial_{\mu_1\ldots\mu_{k-2}}\Delta\phi)} \tag{21}$$

where the derivatives with respect to $\partial_{\mu_1\ldots\mu_j}\Delta\phi$ are computed in the coordinates (12) of V^k. One has

$$\sigma\delta + \delta\sigma = N \tag{22}$$

where N

$$N = \mathcal{P}_\alpha \frac{\partial}{\partial\mathcal{P}_\alpha} + z_\alpha \frac{\partial}{\partial z_\alpha} \tag{23}$$

is the operator counting the number of $\mathcal{P}_\alpha$ and z_α. The relation (22) crucially uses the derivation property of $\partial/\partial z_\alpha$. It follows from (22) and (23) that $\mathcal{P}_\alpha$ and z_α drop from the homology of δ ("they belong to the contractible part of the complex"), which is given by the functions of x_i ([5], sections 8.3.2 and 9.A.2. The G_a's there play the role of the equations of motion here). Since the functions of x_i are the functions on Σ_k and have antighost number equal to zero, formula (17) is established.

The argument is valid for any k, i.e. for any local function involving the derivatives of the field and antifield up to an arbitrarily high (but finite) order. One sometimes summarize (17) by saying that δ is acyclic in the space of local functions.

It should be noted that even though covariant-looking, the contracting homotopy (21) is not covariant. For instance, one finds

$$\sigma(\partial_\mu \partial_\nu \phi) = \delta_{\mu 0} \delta_{\nu 0} \phi^*. \tag{24}$$

Nevertherless, one can show that the homology of δ in the algebra of Lorentz invariant functions is trivial for positive k; that is, if $\delta f = 0$ and $antigh(f) = k \neq 0$, where f is Lorentz invariant, then $f = \delta g$ where g may also be taken to be Lorentz invariant. This can be proved either by redefining the homotopy, or equivalently, by following the methods of [12], theorem 2.

We close this section by a few remarks concerning incorrect statements that have been made in the literature.

1. First, it should be stressed that $f \approx 0$ does not imply $f = h\Delta\phi$ with h a local function. Rather, f may also involve the derivatives of $\Delta\phi$, i.e., one has the full expansion (13).

2. The homotopy σ given by (21) is well defined everywhere because the equations of motion are simple. For more general theories, however, a globally defined homotopy constructed along the above lines may just simply not exist. This is because obstructions for defining the derivation $\partial/\partial(\delta\mathcal{L}/\delta\phi^i)$ may be present (one needs to tell what is kept fixed when differentiating with respect to $\delta\mathcal{L}/\delta\phi^i$). Attempts for using a formula similar to (21) would then necessarily fail. This would show up in non convergence of power series, etc., which must be handled carefully. One way to handle correctly this problem is to introduce partitions of unity, as in [5], appendix 9A.

To make this point clear, consider the Lagrangian $L = L(q)$ where the function $h(q) \equiv dL/dq$ is such that (i) $h(q) = -1$ for $q \leq -1$; (ii) $h(q) = 1$ for $q \geq 1$; and (iii) $h(q)$ interpolates in a smooth way from -1 to $+1$ between -1 to $+1$ and vanishes only at the origin where $h'(0) = 1$. It is clear that it is impossible to define df/dh for all functions f's (with d/dh a derivation) since this would imply in particular that dq/dh is well-defined and such that $(dq/dh)(dh/dq) = 1$, in contradiction with $dh/dq = 0$ for $q \leq -1$ or $q \geq 1$. It turns out not to be necessary, however, to define df/dh in the open sets where $h \neq 0$. Indeed, in those sets ("of type V" according to [5]), any δ-closed function f is trivially δ-exact, $f = \delta(q^* f/h)$. The proof of acyclicity of δ proceeds by patching the V-sets with an open set covering the origin by means of a partition of unity.

One may also construct polynomial counterexamples. For instance, the Lagrangian

$$L(q) = \frac{1}{4}q^4 + \frac{5}{3}q^3 + \frac{1}{2}q^2 + 5q \tag{25}$$

for a real variable q leads to the equation of motion $h(q) \equiv dL/dq = (q + 5)(q^2 + 1) = 0$, whose sole solution is $q = -5$. The equation of motion is regular ($h'(q) \neq 0$ on-shell), but yet, one cannot define dq/dh everywhere since dh/dq

has two real roots. One may build other counteramples based on a non trivial topology of the stationary surface.

4 The Koszul-Tate differential for the electromagnetic field

We now turn to the electromagnetic case. The equations of motion are

$$\mathcal{L}^\rho \equiv \frac{\delta \mathcal{L}}{\delta A_\rho} = \partial_\mu F^{\mu\rho} = 0 \tag{26}$$

and define a surface in V^2. The new feature compared with the previous situation is that the derived equations

$$\partial_\mu \mathcal{L}^\rho = 0, \partial_{\mu_1\mu_2} \mathcal{L}^\rho = 0, ... \tag{27}$$

in V^3, V^4, ... are no longer independent. Because of the gauge invariance of the electromagnetic field Lagrangian, one has rather (identically)

$$\partial_\rho \mathcal{L}^\rho \equiv 0, \partial_{\mu_1}(\partial_\rho \mathcal{L}^\rho) \equiv 0... \tag{28}$$

(for any field configuration). For that reason, one needs "antifields of antifields" J[3, 5].

We start with V^2. There are clearly no relations among the equations $\mathcal{L}^\rho = 0$ in V^2 since one can solve these equations for n of the coordinates in V^2 (we work in n dimensions). Namely, one can solve $\mathcal{L}^k = 0$ for $\partial_{00}A_k$ and $\mathcal{L}^0 = 0$ for $\partial_{11}A_0$ (say). Hence, if one defines in $C^\infty(V^2) \otimes \bigwedge(A^{*\mu})$ the differential

$$\delta A_\mu = 0, \delta\partial_\rho A_\mu = 0, \delta\partial_{\rho\sigma} A_\mu = 0, \delta A^{*\mu} = \partial_\nu F^{\nu\mu} \tag{29}$$

one gets that $H_k(\delta) = 0$ for $k \neq 0$ and $H_0(\delta) = C^\infty(\Sigma_2)$. To verify this statement, one repeats the argument of the previous section and splits the variables of the complex in three groups. The coordinates A_μ, $\partial_\rho A_\mu$, $\partial_{\rho\sigma} A^k$ $((\rho,\sigma) \neq (0,0))$ and $\partial_{\rho\sigma} A^0$ $((\rho,\sigma) \neq (1,1))$ are of the x_i-type, the coordinates $\mathcal{L}^\rho$ are of the z_α-type, while the $A^{*\mu}$ are of the $\mathcal{P}_\alpha$-type. The appropriate contracting homotopy in $C^\infty(V^2) \otimes \bigwedge(A^{*\mu})$ reads

$$\sigma = A^{*\mu} \frac{\partial}{\partial \mathcal{L}^\mu}. \tag{30}$$

Thus, only the variables not constrained by the equations of motion, namely, A_μ, $\partial_\rho A_\mu$, $\partial_{\rho\sigma} A^k$ $((\rho,\sigma) \neq (0,0))$ and $\partial_{\rho\sigma} A^0$ $((\rho,\sigma) \neq (1,1))$ remain in homology. The other variables drop out.

Turn now to $C^\infty(V^3) \otimes \bigwedge(A^{*\mu}, \partial_\rho A^{*\mu})$, with differential δ (29) extended to the derivatives so that

$$\delta\partial_\mu = \partial_\mu \delta \tag{31}$$

i.e.,

$$\delta\partial_{\rho\sigma\alpha} A_\mu = 0, \;\; \delta\partial_\rho A^{*J\mu} = \partial_\rho(\partial_\nu F^{\nu\mu}) \tag{32}$$

The equations $\partial_\nu F^{\nu\mu} = 0$ and $\partial_\sigma \partial_\nu F^{\nu\mu} = 0$ are *not* independent in V^3 since they are subject to the (single) condition $\partial_\rho \mathcal{L}^\rho = 0$. There are no other identity in V^3 because one can solve n^2+n-1 of the n^2+n equations $\mathcal{L}^\rho = 0$, $\partial_\mu \mathcal{L}^\rho = 0$ for n^2+n-1 independent variables, namely $\partial_{00}A_k$ (from $\mathcal{L}^k = 0$), $\partial_{11}A_0$ (from $\mathcal{L}^0 = 0$), $\partial_{\rho 00}A_k$ (from $\partial_\rho \mathcal{L}^k = 0$) and $\partial_{s11}A_0$ (from $\partial_s \mathcal{L}^0 = 0$). The derivative $\partial_{011}A^0$ cannot be determined from $\partial_0 \mathcal{L}^0 = 0$, which is not an independent equation ($\partial_0 \mathcal{L}^0 = -\partial_k \mathcal{L}^k$). Hence, in V^3, there are n^2+n-1 independent equations and 1 dependent one.

Because the equations of motion in V^3 are not independent, there is one non trivial cycle at antighost number 1, namely $\partial_\rho A^{*\rho}$. Thus, $H_1(\delta) \neq 0$ in $C^\infty(V^3) \otimes \bigwedge(A^{*\mu}, \partial_\rho A^{*\mu})$. In order to achieve acyclicity of the Koszul-Tate differential, one needs to introduce one further even variable, denoted by C^* and called "antifield of antifield" [5], with grading

$$antigh C^* = 2. \tag{33}$$

This new variable must kill the non trivial cycle $\partial_\rho A^{*\rho}$ in homology, so that one defines

$$\delta C^* = \partial_\rho A^{*\rho}. \tag{34}$$

Once C^* is introduced, one can redefine the variables of the differential complex $C^\infty(V^3) \otimes C[A^{*\mu}, \partial_\rho A^{*\mu}, C^*]$ in such a way that δ takes again the characteristic form[2]

$$\delta x_i = 0,\ \delta \mathcal{P}_\alpha = z_\alpha,\ \delta z_\alpha = 0, \tag{35}$$

which makes manifest that $H_*(\delta) = C^\infty(x_i)$. The variables x_i have antighost number zero and parametrize Σ_3. They are explicitly given by A_μ, $\partial_\rho A_\mu$, $\partial_{\rho\sigma}A_k$ ($(\rho,\sigma) \neq (0,0)$), $\partial_{\rho\sigma}A_0$ ($(\rho,\sigma) \neq (1,1)$), $\partial_{\rho\sigma\nu}A_k$ (with at most one time derivative) and $\partial_{\rho\sigma\nu}A_0$ (with $(\rho,\sigma,\nu) \neq (k,1,1)$ even up to a permutation). The variables $\mathcal{P}_\alpha$ are $A^{*\mu}, \partial_\alpha A^{*k}, \partial_k A^{*0}$ *and* C^*. The variables z_α are the left hand sides of the equations of motion $\mathcal{L}^\rho, \partial_\alpha \mathcal{L}^k, \partial_k \mathcal{L}^0$ *and* $\partial_\rho A^{*\rho}$.

The same pattern goes on with the higher order derivatives. In $C^\infty(V^k) \otimes C[A^{*\mu}, \partial_\rho A^{*\mu}, ..., \partial_{\rho_1 \dots \rho_{k-2}} A^{*\mu}, C^*, ..., \partial_{\rho_1 \dots \rho_{k-3}} C^*]$, one may introduce new coordinates as follows:

(i) Coordinates of x_i-type : A_k and its derivatives with at most one ∂_0; A_0 and its derivatives except $\partial_{s_1 s_2 \dots s_m} A_0$ with at least two ∂_1. These variables parametrize Σ_k.

(ii) Coordinates of z_α-type: $\mathcal{L}^k$ and its derivatives; $\mathcal{L}^0$ and its spatial derivatives; $\partial_\rho A^{*\rho}$ and its derivatives.

(iii) Coordinates of $\mathcal{P}_\alpha$-type : A^{*k} and its derivatives; A^{*0} and its spatial derivatives; C^* and its derivatives.

Thus, again, $H_0(\delta) = C^\infty(V^k)$ and $H_m(\delta) = 0$, $m \neq 0$. The contracting homotopy has the standard form

$$\sigma = \mathcal{P}_\alpha \frac{\partial}{\partial z_\alpha}, \tag{36}$$

[2]From now on, we shall use the notation $C[A^{*\mu}, \partial_\rho A^{*\mu}, C^*]$ for the algebra $\bigwedge(A^{*\mu}, \partial_\rho A^{*\mu}) \otimes R[C^*]$. The symmetry properties are taken care of by the gradings of $A^{*\mu}$ (odd) and C^* (even).

where the sum runs over all the z_α's. At each stage, one can separate the equations $\mathcal{L}^\rho = 0$ and their derivatives into independent ones and dependent ones *without going out of the spaces V^k, i.e., in a manner compatible with spacetime locality.* Statements to the contrary are thus wrong.

It is true that the dependent equations at order $k+1$ are not just the derivatives of the dependent equations at order k. One cannot separate the n equations $\mathcal{L}^\rho = 0$ into two groups, so that the independent (respectively, dependent) equations would simply be all the derivatives of the equations of the first (respectively second) group. To achieve this property, one would have to make a non local split. But a split with this property is not necessary once one formulates the problem in terms of the standard spaces V^k of jet bundle theory, as appropriate for dealing with locality.

Similarly, although we have not done it, one could define a Lorentz-invariant homotopy by decomposing the derivatives of the fields along the irreducible representations of the Lorentz group. Hence, acyclicity of the Koszul-Tate differential also holds in the algebra of Lorentz-invariant local functions. This same result can equivalently be established along the lines of [12], by using the facts that δ commutes with the representation and that the Lorentz group is semi-simple.

5 The Koszul-Tate differential for the Yang-Mills field

The Yang-Mills case can be treated in the same manner. This is because the terms with the highest (second) order derivatives of the gauge potential in the Yang-Mills equations of motion are exactly the same as in the Abelian case. Hence, the change of variables such that the left hand sides of the equations of motion and their derivatives are new coordinates is still permissible, and one can proceed as above.

For instance, in V^3, one would take as new variables $A^a_\mu, \partial_\rho A^a_\mu, \partial_{\rho J\sigma} A^a_k$ $((\rho,\sigma) \neq (0,0))$, $\partial_{\rho J\sigma} A^a_0$, $((\rho,\sigma) \neq (1,1))$ and $\mathcal{L}^\mu_a$. The expression of $\partial_{00} A^a_k$ in terms of $\mathcal{L}^k_a$ is the same as in the abelian case up to terms containing lower order derivatives (which are independent coordinates in the previous space V^2). A similar analysis holds for higher order derivatives.

We leave it to the reader to check also that an analogous derivation can be performed for p-form gauge fields. The only difference is that one needs this time more antifields for antifields because the reducibility equations are not independent.

6 Acyclicity of Koszul-Tate differential and local functionals

The above sections establish the acyclicity of δ in the space of local functions. Does this property also hold in the space of local functionals? That is, if f is a n-form such that

$$\delta \int f = 0, \ antighf \geq 1 \tag{37}$$

does one have

$$\int f = \delta \int g \tag{38}$$

for some n-form g? [f and g are n-forms with coefficients that are local functions]. Equivalently, in terms of the integrands, does

$$\delta f = dj, \ antighf \geq 1 \tag{39}$$

imply

$$f = \delta g + dk \tag{40}$$

for some n-form g and $n-1$-form k? The presence of the d-exact terms in (39), (40) follows from (7) and *must be taken into account.* Failure to do so would be incorrect. The extra d-terms in (39) and (40) show that the relevant cohomology when dealing with local functionals is the cohomology of δ modulo d in the space of local n-forms. The corresponding cohomological spaces are denoted $H_k(\delta/d)$.

As pointed out in [8], the answer to this question is in general negative. Constants of the motion define non trivial solutions of $H_1(\delta/d)$. Indeed, the equation $\delta f + dj$ with $antighf = -1$ and $antighj = 0$ defines a conserved current j. If f is trivial (of the form (40)), then j is a trivial conserved current ($j = -\delta k + dm$). Since there exist in general non trivial conserved currents, $H_1(\delta/d)$ is not empty.

However, if f involves the ghosts[3] - which is the case encountered in homological perturbation theory -, then (39) does imply (40). To see this, consider first the case where f is linear in the ghosts. By making integrations by parts if necessary, one can assume that f does not involve the derivatives of the C^α,

$$f = \lambda_\alpha C^\alpha, \ antigh\lambda_\alpha = 0. \tag{41}$$

Then, $\delta f = \delta(\lambda_\alpha)C^\alpha$. If $\delta f = dj$, then δf *and* dj must separately vanish because dj would otherwise necessarily involve derivatives of the ghosts. Thus $\delta\lambda_\alpha = 0$, which implies $\lambda_\alpha = \delta\mu_\alpha$ since $H_k(\delta) = 0$ in the space of local functions. Consequently, $f = (\delta\mu_\alpha)C^\alpha = \delta(\mu_\alpha C^\alpha)$, which is the sought-for result. How to formalize the argument so that it applies also to forms f that are non linear in the ghosts is done in [8]. Thus, acyclicity of δ holds in the space of local functionals involving both the antifields and the ghosts.

[3]How the ghosts are introduced may be found for example in [5]. The ghosts will be denoted by C^α and are annihilated by the differential δ. Once the ghosts are introduced, the cohomology of δ is given by $C^\infty(\Sigma_k) \otimes \bigwedge(C^\alpha, \partial_\rho C^\alpha ...)$.

7 Conclusion

We have illustrated in this paper how to handle locality in the case of the antifield-antibracket formalism for gauge field theories. The tools involve both standard homological algebraic techniques applied to finitely generated algebras and ideas from jet bundle theory. We have shown in particular how the equations of motion for electromagnetism and Yang-Mills theory can split into independent and dependent ones in the "jet bundle" spaces V^k. The tools illustrated here have been used recently to prove a long-standing conjecture on the renormalization of Yang-Mills models [13].

We close this letter with two observations :
(i) The method of homological perturbation theory is quite general and does not depend on the precise form of the differential algebra on which the derivations act, provided these derivations fulfill the properties explained in [5] (chapter 8). Thus, one may modify the algebra of local functions by imposing restrictions if one wishes to do so. For instance, the well-known theorem that a BRST cohomological class is determined by its component of order zero in the antifields is quite standard and follows from the general principles of homological perturbation theory (see again [5], chapter 8, proof of main theorem and section 8.4.4).
(ii) Similarly, one may consider field theories for which the equations of motion are not "regular", in the sense that their gradients would vanish on the stationary surface. A theory with equation of motion $\delta\mathcal{L}'/\delta\phi = \phi^2 = 0$ (rather than the equivalent equation $\delta\mathcal{L}/\delta\phi = \phi = 0$) would provide such an example. This case does not arise in usual gauge theories, as we have just seen, but does occur in, say, Siegel formulation of chiral bosons [14]. Again, a lot of work already exists on this subject, especially in the Hamiltonian context. The algebraic framework is well developed. The real question is, however, what is the physical meaning of the BRST construction in those cases. The relation between the BRST cohomology and the cohomology of the geometrical longitudinal derivative on the stationary surface may no longer hold (this is why the BRST analysis performed in chapters 9 and 10 of [5] excludes these somewhat pathological cases). To the author, the question has not been fully resolved.

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