Scattering of Dirac fermions from spherically symmetric black holes: analytical phase shifts analysis

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The scattering of the Dirac fermions by the black-holes with spherical symmetry is studied by applying the method of partial wave analysis. The analytical expressions for the phase shifts and analytical formulas for the differential scattering cross section are written down for the fermion scattering from Schwarzschild, Reissner-Nordsrtöm and Bardeen black holes. A brief comment about the principal features of these cases is outlined.

Keywords: Black holes, fermions, scattering.

1. Introduction

In this paper we want to present a comparative study of the scattering of Dirac fermions by black-holes with spherical symmetry. For that we choose the Schwarzschild, the Reissner-Nordsrtöm and Bardeen black holes, which were each analysed separately in our previous works[1–4]. Other works that studied numerically the scattering of fermions by these three types of black holes can be found for example in Refs. 5–19.

The main steps for solving the Dirac equation in the geometries that describe black-holes with spherical symmetry are detailed. We obtain the scattering modes resulted solving the Dirac equation in the asymptotic zone of the black hole, which help us to obtain the phase shifts and the analytical expression of the differential scattering cross section. The differential scattering cross section will be analysed in terms of the relevant parameters such as the mass of the black hole and the charge of the black hole and scattering angle. From the analytical expression for the cross section the zero mass limit can be obtained.

We begin in the second section with the basic notions about the Dirac equation in geometries that describe black holes with spherical symmetry. The third section is dedicated to the phase shifts obtained using the partial wave analysis and in the fourth section we make a graphical analysis of the cross section in terms of scattering angle.

2. Dirac Equation and Black Holes

In this work we are discussing fermion scattering by black holes with spherical symmetry that have the following generic metric

$$ds^2 = h(r)dt^2 - \frac{dr^2}{h(r)} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

In the case of Schwarzschild, electrically charged Reissner-Nordsrtöm and (regular) Bardeen black holes the function $h(r)$ reads:

$$h(r)_S = 1 - \frac{2M}{r}, \qquad h(r)_{RN} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \qquad h(r)_B = 1 - \frac{2Mr^2}{\sqrt{(r^2+Q_m^2)^3}} \quad (2)$$

where M is the mass of each black hole; Q is the total electric charge and Q_m is the nonlinear magnetic monopole charge[20].

It can be shown (for details see our previous papers[1,2,4,23]) that the Dirac equation, $i\gamma^a D_a\psi - m\psi = 0$, in the black holes geometries given by eq. (1) can be reduced to only a radial equation, namely

$$\begin{pmatrix} m\sqrt{h(r)} + V(r) & -h(r)\frac{d}{dr} + \frac{\kappa}{r}\sqrt{h(r)} \\ h(r)\frac{d}{dr} + \frac{\kappa}{r}\sqrt{h(r)} & -m\sqrt{h(r)} + V(r) \end{pmatrix} \begin{pmatrix} f^+_{E,\kappa}(r) \\ f^-_{E,\kappa}(r) \end{pmatrix} = E \begin{pmatrix} f^+_{E,\kappa}(r) \\ f^-_{E,\kappa}(r) \end{pmatrix} \quad (3)$$

where $f^{\pm}(r)$ are two radial wave function and $V(r) = eQ/r$. The angular part of the Dirac equation is contained into the standard 4-component angular spinors $\Phi^{\pm}_{m,\kappa}(\theta,\phi)$[21]. Thus the particle-like solutions with energy E can be expressed as

$$\psi_{E,j,m,\kappa}(t,r,\theta,\phi) = \frac{e^{-iEt}}{r\,h(r)^{1/4}} \left\{ f^+_{E,\kappa}(r)\Phi^+_{m,\kappa}(\theta,\phi) + f^-_{E,\kappa}(r)\Phi^-_{m,\kappa}(\theta,\phi) \right\} \quad (4)$$

The separation of angular variables as in eq. (4) was possible by using the Cartesian gauge (see eqs. (1)–(4) in Ref. 2). In this gauge the Dirac equation is manifestly covariant under rotations.

The radial equation (3) can not be solved analytically for any of the three black holes (2) and up to now a combination of analytical and numerical methods was used to solve it and to compute numerical phase shifts as was done for ex. in Ref. 6 for Schwarzschild BH or in Ref. 19 for Bardeen BH.

2.1. *Scattering modes*

In the following we will briefly show how to obtain analytical phase shifts by finding (approximative) analytical scattering modes resulted from solving the Dirac equation in the asymptotic zone of the black hole as done in Refs. 1–4. One starts by changing the variable r to a Novikov-like one, namely

$$x = \sqrt{\frac{z}{r_+} - 1}, \quad (5)$$

with r_+ the radius of the outer black hole horizon. For Schwarzschild BH $z = r$ and $r_+ = r_0 = 2M$; for Reissner-Nordsrtöm BH $z = r$ and $r_+ = M + \sqrt{M^2 - Q^2}$; while for Bardeen BH $z = \sqrt{r^2 + Q_M^2}$ and r_+ has a more complicated analytical expression not given here.

By rewriting eq. (3) in terms of the new variable x and making a Taylor expansion with respect to $1/x$ one can show, after discarding the $O(1/x^2)$ terms and higher, that in the asymptotic zone of the black hole the new equation becomes

$$\begin{pmatrix} \frac{1}{2}\frac{d}{dx} + \frac{\kappa}{x} & -x(\mu+\varepsilon) - \frac{1}{x}(\zeta+\beta) \\ x(\varepsilon-\mu) - \frac{1}{x}(\zeta-\beta) & \frac{1}{2}\frac{d}{dx} - \frac{\kappa}{x} \end{pmatrix} \begin{pmatrix} f^{+}_{E,\kappa}(x) \\ f^{-}_{E,\kappa}(x) \end{pmatrix} = 0 \tag{6}$$

where $\mu = r_+ m$, $\varepsilon = r_+ E$ and

- for Schwarzschild black hole one has:

$$\zeta = \frac{\mu}{2}, \quad \beta = \varepsilon, \quad r_+ = 2M \tag{7}$$

- for Reissner-Nordström black hole:

$$\zeta = \frac{1}{2}\mu\left(1 - \frac{r_-}{r_+}\right), \quad \beta = \varepsilon - eQ, \quad r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{8}$$

- for regular Bardeen black hole:

$$\zeta = \mu\left(1 - \frac{M}{r_+}\right), \quad \beta = \varepsilon, \tag{9}$$

The scattering mode solutions (with $\varepsilon > \mu$) of eq. (6) can be found in terms of Whittaker functions[1,2,4,23]

$$\begin{pmatrix} f^{+}(x) \\ f^{-}(x) \end{pmatrix} = \begin{pmatrix} -i\sqrt{\mu+\varepsilon} & i\sqrt{\mu+\varepsilon} \\ \sqrt{\varepsilon-\mu} & \sqrt{\epsilon-\mu} \end{pmatrix} \begin{pmatrix} C_1^{+}\,\frac{1}{x}M_{\rho_+,s}(2i\nu x^2) + C_2^{+}\,\frac{1}{x}W_{\rho_+,s}(2i\nu x^2) \\ C_1^{-}\,\frac{1}{x}M_{\rho_-,s}(2i\nu x^2) + C_2^{-}\,\frac{1}{x}W_{\rho_-,s}(2i\nu x^2) \end{pmatrix} \tag{10}$$

were $\nu = \sqrt{\varepsilon^2 - \mu^2}$. The four integration constants $C_1^{\pm}$, $C_2^{\pm}$ are not independent, satisfying the following relations[1,23]:

$$\frac{C_1^-}{C_1^+} = \frac{s - iq}{\kappa - i\lambda}, \quad \frac{C_2^-}{C_2^+} = -\frac{1}{\kappa - i\lambda} \tag{11}$$

and were the following parameters are used:

$$s = \sqrt{\kappa^2 + \zeta^2 - \beta^2}, \quad \rho_\pm = \mp\frac{1}{2} - iq, \quad q = \frac{\beta\varepsilon - \zeta\mu}{\nu}, \quad \lambda = \frac{\beta\mu - \zeta\varepsilon}{\nu} \tag{12}$$

2.2. *Analytical Phase Shifts*

Applying the partial wave method on the scattering modes (10) as done in Refs. 1,2,4 we obtained for the first time in the literature analytical expressions for the phase shifts associated to the scattering of Dirac fermions by black holes, namely:

$$S_\kappa = e^{2i\delta_\kappa} = \left(\frac{\kappa - i\lambda}{s - iq}\right)\frac{\Gamma(1+s-iq)}{\Gamma(1+s+iq)}e^{i\pi(l-s)}\,. \tag{13}$$

Using these phase shifts one can now calculate analytical expressions for the differential scattering cross section

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |g(\theta)|^2 \tag{14}$$

were the scalar amplitudes $f(\theta)$ and $g(\theta)$ are defined in terms of Legendre polynomials[22] and the phase shifts $e^{2i\delta_l}$

$$
\begin{aligned}
f(\theta) &= \sum_{l=0}^{\infty} \frac{1}{2ip} \left[(l+1)(e^{2i\delta_{-l-1}} - 1) + l(e^{2i\delta_l} - 1)\right] P_l^0(\cos\theta) \\
g(\theta) &= \sum_{l=1}^{\infty} \frac{1}{2ip} \left[e^{2i\delta_{-l-1}} - e^{2i\delta_l}\right] P_l^1(\cos\theta)
\end{aligned}
\tag{15}
$$

The above infinite series have a singularity present at $\theta = 0$ that can not be removed. This makes the scattering section to be divergent in the forward direction. However, by using a method first proposed in Ref. 24 one can define new reduced series for $f(\theta)$ and $g(\theta)$ (see eqs. (72)–(75) in Ref. 1 for more details) that converge more quickly.

3. Main Results and Brief Comments

In this section we perform a graphical analysis of the cross section in terms of the scattering angle. The relevant parameters in our analysis are the black hole mass M, the fermion mass m, fermion speed v, the charge of the black hole Q and the scattering angle θ.

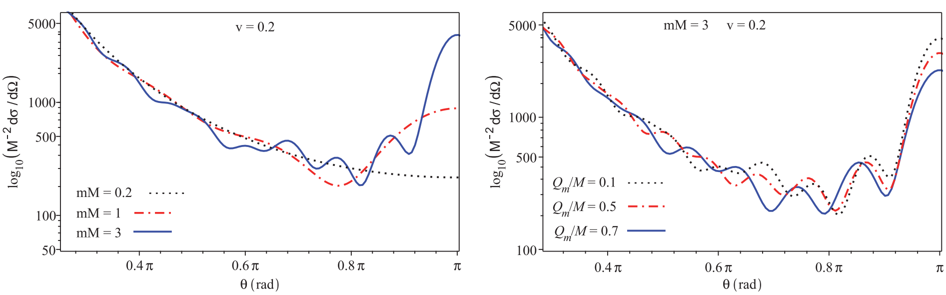

Fig. 1. Differential scattering cross section for fermion scattering by a Schwarzschild black hole (left panel), respectively for a Bardeen black hole (right panel).

In Fig. 1, we make a comparative study of the differential scattering cross section for fermion scattering by a Schwarzschild black hole and a Bardeen black hole. Our graphical results prove that the cross section has a oscillatory behaviour which become more pronounced as we increase the product of masses mM in the case of scattering on Schwarzschild black hole. When the scattering angle approaches the value $\theta = 0$ the cross section increase proving that the forward scattering is dominant. In the case of Bardeen black hole the differential cross section depend on the ratio between the charge of the magnetic monopole and the black hole mass and we observe a variation of the differential cross section when we modify the ratio Q_m/M. In both cases the differential cross section have a maximum for $\theta = \pi$, which prove

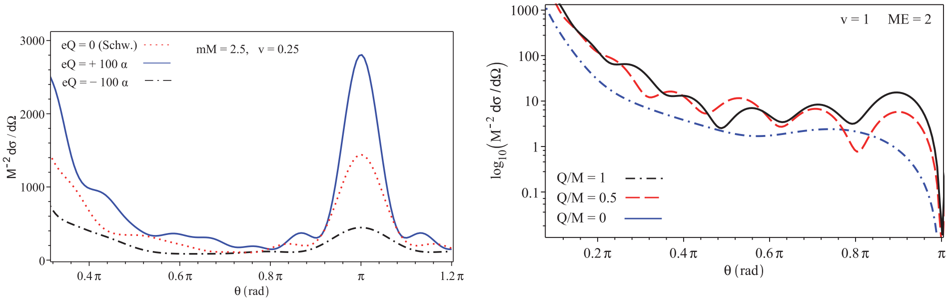

Fig. 2. Differential scattering cross section for fermion scattering by a Reisnner-Nordstrom black hole (left panel). Scattering of massless fermions (right panel).

that the backward scattering is also relevant. In the case of Schwarzschild black hole the backward scattering become important as the parameter mM increase.

The results presented in Fig. 2, shows the dependence of the differential scattering cross section for fermions scattered by a Reisnner-Nordsrtöm black hole on the scattering angle. Here we analyse the case of massive fermions as well as the case when the mass of the fermion is zero. In the case of scattering of massive fermions we observe the differential scattering cross section variation in terms of the charge of the black hole and the charge of the fermion in comparison with the scattering on Schwarzschild black hole (left panel). When the charge of the black hole and the fermion charge have the same sign, the backward scattering become important because of the electrostatic repulsion between charges. For scattering of fermions with zero mass we observe the variation of the differential scattering cross section in terms of the ratio Q/M and that in the backward direction the differential scattering cross section is vanishing.

4. Conclusions

The principal conclusion is that the scattering processes analyzed here can be studied by using exclusively analytical methods with suitable asymptotic conditions. In this manner we complete the previous studies which combine analytical and numerical methods obtaining new results and a synthetic overview over the considered processes, pointing out that these are described by similar formulas.

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