

4d-polytopes described by Coxeter diagrams and quaternions

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Abstract 4D-polytopes and their dual polytopes can be described as the orbits of the rank-4 Coxeter-Weyl groups. Their symmetries follow from the quaternionic descriptions of the rank-4 Coxeter-Dynkin diagrams. There exists a one to one correspondence between the finite subgroups of quaternions and the rank-4 Coxeter-Weyl groups.

1. Introduction

Discovery of the Platonic solids; tetrahedron, cube, octahedron, icosahedron and dodecahedron dates back to the people of Scotland lived 1000 years earlier than the ancient Greeks and the models carved on the stones are now kept in the Ashmolean Museum at Oxford [1]. Plato associated tetrahedron with fire, cube with earth, air with octahedron, and water with icosahedron. Archimedes discovered the semi-regular convex solids and several centuries later they were rediscovered by the renaissance mathematicians. By introducing prisms and anti-prisms as well as four regular non-convex polyhedra, Kepler completed the work in 1620. Nearly two centuries later, in 1865, Catalan constructed the dual solids of the Archimedean solids now known as Catalan solids [2]. Extensions of the platonic solids to 4D dimensions have been made in 1855 by L. Schläfli [3] and their generalizations to higher dimensions in 1900 by T. Gosset [4]. Further important contributions are made by W. A. Wythoff [5] among many others and in particular by the contemporary mathematicians H.S.M. Coxeter [6] and J.H. Conway [7].

The 3D and 4D convex polytopes single out as compared to the polytopes in higher dimensions. The number of Platonic solids is five in 3D and there exist six regular polytopes in 4D contrary to the higher dimensional cases where there exist only three platonic polytopes which are the generalizations of tetrahedron, octahedron and cube to higher dimensions. The Platonic and Archimedean solids [8] as well as the Catalan solids [9] can be described with the rank-3 Coxeter groups $W(A_3)$, $W(B_3)$ and $W(H_3)$. The 4D polytopes are described by the rank-4 Coxeter groups $W(A_4)$, $W(B_4)$, $W(H_4)$ and the group $W(F_4)$.

This paper studies the regular and semi regular 4D polytopes as the orbits of the rank-4 Coxeter groups expressed in terms of quaternions. There is a one-to-one correspondence between the finite subgroups of quaternions and the symmetries of the regular and semi regular 4D polytopes. In Section 2 we introduce the finite subgroups of quaternions. Constructions of the rank-4 Coxeter groups in terms of quaternions are given in Section 3. The analysis of the 4D polytopes requires a study of the 3D polyhedra which will be

introduced in Section 4. Examples of the 4D polytopes and their dual polytopes will be given in Section 5. Concluding remarks are made in Section 6.

2. Finite subgroups of quaternions

In this section we introduce the quaternions and its relevance to the orthogonal transformations in 4 dimensions and give the list of finite subgroups of quaternions.

2.1. Quaternions and $O(4)$ Transformations

Let $q = q_0 + q_i e_i$, ($i=1,2,3$) be a real unit quaternion with its conjugate defined by $\bar{q} = q_0 - q_i e_i$ and the norm $q\bar{q} = \bar{q}q = 1$. The imaginary units satisfy

$$e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k, \quad (i, j, k = 1, 2, 3). \quad (1)$$

Let p, q be unit quaternions and r represents an arbitrary quaternion. Then the transformations [10]

$$r \rightarrow prq : [p, q]; \quad r \rightarrow p\bar{r}q : [p, q]^* \quad (2)$$

define the orthogonal group $O(4)$ which preserves the norm $r\bar{r} = \bar{r}r$. The first term in (2) represents a proper rotation and the second includes also the reflection, generally called rotary reflection. In particular, the group element

$$r \rightarrow -p\bar{r}p : [p, -p]^* \quad (3)$$

represents the reflection with respect to the hyperplane orthogonal to the quaternion p .

2.2. Finite Subgroups of Quaternions

The finite subgroups of quaternions are well known and its classification can be found in the Coxeter's book on regular complex polytopes and in the du Val's book on quaternions [11]. They are given as follows.

(a) Cyclic group of order $2n$ is generated by $\langle p = \exp(e_1 \frac{\pi}{n}) \rangle$ and dicyclic group of order $4n$ is

generated by the generators $\langle p = \exp(e_1 \frac{\pi}{n}), e_2 \rangle$.

(b) The binary tetrahedral group can be represented by the set of 24 unit quaternions:

$$T = \{ \pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \}. \quad (4)$$

(c) The binary octahedral group consists of 48 unit quaternions. Let the set

$T' = (V_1 \oplus V_2 \oplus V_3)$ represents the 24 unit quaternions where

$$V_1 = \{ \frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3) \}, \quad V_2 = \{ \frac{1}{\sqrt{2}}(\pm 1 \pm e_2), \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1) \} \quad (5)$$

$$V_3 = \{ \frac{1}{\sqrt{2}}(\pm 1 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2) \}.$$

Then the union of the set $O = T \oplus T'$ represents the binary octahedral group.

(d) The binary icosahedral group $I = \langle b, c \rangle$ of order 120 can be generated by two unit quaternions

$b = \frac{1}{2}(\tau + \sigma e_1 + e_2) \in 12(1)_+$ and $c = \frac{1}{2}(\tau - \sigma e_1 + e_2) \in 12(1)_+$. We display its elements in table 1 as

Table 1. Conjugacy classes of the binary icosahedral group I represented by quaternions

Order of the elements	Conjugacy classes denoted by the number of elements and order of elements
1	1
2	-1
10	$12_+ : \frac{1}{2}(\tau \pm e_1 \pm \sigma e_3), \frac{1}{2}(\tau \pm e_2 \pm \sigma e_1), \frac{1}{2}(\tau \pm e_3 \pm \sigma e_2)$
5	$12_- : \frac{1}{2}(-\tau \pm e_1 \pm \sigma e_3), \frac{1}{2}(-\tau \pm e_2 \pm \sigma e_1), \frac{1}{2}(-\tau \pm e_3 \pm \sigma e_2)$
10	$12'_+ : \frac{1}{2}(\sigma \pm e_1 \pm \tau e_2), \frac{1}{2}(\sigma \pm e_2 \pm \tau e_3), \frac{1}{2}(\sigma \pm e_3 \pm \tau e_1)$
5	$12'_- : \frac{1}{2}(-\sigma \pm e_1 \pm \tau e_2), \frac{1}{2}(-\sigma \pm e_2 \pm \tau e_3), \frac{1}{2}(-\sigma \pm e_3 \pm \tau e_1)$
6	$20_+ : \frac{1}{2}(1 \pm e_1 \pm e_2 \pm e_3), \frac{1}{2}(1 \pm \tau e_1 \pm \sigma e_2), \frac{1}{2}(1 \pm \tau e_2 \pm \sigma e_3), \frac{1}{2}(1 \pm \tau e_3 \pm \sigma e_1)$
3	$20_- : \frac{1}{2}(-1 \pm e_1 \pm e_2 \pm e_3), \frac{1}{2}(-1 \pm \tau e_1 \pm \sigma e_2), \frac{1}{2}(-1 \pm \tau e_2 \pm \sigma e_3), \frac{1}{2}(-1 \pm \tau e_3 \pm \sigma e_1)$
4	$30 : \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm \sigma e_1 \pm \tau e_2 \pm e_3), \frac{1}{2}(\pm \sigma e_2 \pm \tau e_3 \pm e_1), \frac{1}{2}(\pm \sigma e_3 \pm \tau e_1 \pm e_2)$

In table 1 we used the golden ratio $\tau = \frac{1+\sqrt{5}}{2}$ and $\sigma = \frac{1-\sqrt{5}}{2}$ which satisfy

$$\tau + \sigma = 1, \quad \tau\sigma = -1, \quad \tau^2 = \tau + 1, \quad \sigma^2 = \sigma + 1. \tag{6}$$

3. Constructions of the rank-4 Coxeter-Weyl groups by finite subgroups of quaternions

Let $I_2(n)$ denotes the Coxeter diagram representing two vectors with the angle $\frac{2\pi}{n}$ between them.

Then the Coxeter diagram $I_2(n) \oplus I_2(n)$ of rank-4 is represented by the figure 1.



Figure 1. The Coxeter diagram $I_2(n) \oplus I_2(n)$.

Let $p = \exp(e_1 \frac{\pi}{n})$ and $q = \exp(e_1 \frac{\pi}{n})e_2$ be two orthogonal unit quaternions. Then the following set of quaternions describes the root system of the diagram of figure 1

$$I_2(n) \oplus I_2(n) = \{p^k, q^k\}; \quad k = 1, 2, \dots, 2n. \tag{7}$$

If $s, t \in (I_2(n) \oplus I_2(n))$ are arbitrary elements of the root system then the group $Aut(I_2(n) \oplus I_2(n)) = \{[s, t] \oplus [s, t]^*\}$ of order $4n \times 4n$ is represented by the elements of the dicyclic group.

The F_4 diagram shown in figure 2 with its quaternionic roots [12] leads to its automorphism group $Aut(F_4) = W(F_4) : \gamma = \{[O, O] \oplus [O, O]^*\}$ which is of the order $48 \times 48 = 2304$. Here γ is the Dynkin diagram symmetry generator.

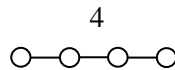


Figure 2. The Coxeter diagram of F_4 .

The automorphism group of H_4 with its quaternionic simple roots [13] is given as $W(H_4) = \{[I, I] \oplus [I, I]^*\}$ with the order of $120 \times 120 = 14,400$. The rank-4 Coxeter-Weyl groups $W(B_4)$ of order 384 and $W(D_4)$ of order 192 are the subgroups of the group $W(F_4)$. On the other hand the Coxeter-Weyl group $W(A_4) \approx S_5$ of order 120 is a subgroup of the group $W(H_4)$ [13].

4. Rank-3 Coxeter-Weyl groups and polyhedra

These groups are discussed extensively in the references [8-9]. Quaternionic representations of these groups can be classified as follows:

- (a) Icosahedral group: $W(H_3) = \{[I, \bar{I}] \oplus [I, \bar{I}]^*\} \approx A_5 \times C_2$
- (b) Octahedral group: $W(B_3) \approx Aut(A_3) = \{[T, \bar{T}] \oplus [T', \bar{T}'] \oplus [T, \bar{T}]^* \oplus [T', \bar{T}']^*\}$
- (c) Tetrahedral group: $W(A_3) = \{[T, \bar{T}] \oplus [T', \bar{T}']^*\}$
- (d) Symmetry of prisms: $W(I_2(n) \oplus A_1) \approx D_n \times C_2$.

Below we display some polyhedra as the orbits of these Coxeter groups. Let α_i and ω_i denote respectively the set of simple roots and basis vectors of the dual space satisfying the relations

$$(\alpha_i, \omega_j) = \delta_{ij}, (\alpha_i, \alpha_j) = C_{ij}, (\omega_i, \omega_j) = (C^{-1})_{ij}. \tag{8}$$

Define an arbitrary vector in the dual space by $\Lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3 \equiv (a_1 a_2 a_3)$ and the Λ -orbit by $O(\Lambda) = W(G)\Lambda$.

The Platonic and Archimedean solids as well as Catalan solids can be described as the orbits of the rank-3 Coxeter groups. Some examples are given below.

4.1. The tetrahedral group $W(A_3) \approx S_4$

The orbits $O(100) = \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3)$ with even number of (-) sign and $O(001) = \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3)$ with odd number of (-) sign represent two tetrahedra dual to each other.

The union of two tetrahedra constitutes a cube. Similarly the orbits $O(110)$ and $O(011)$ represent two truncated tetrahedra.

4.2. The octahedral group $W(B_3) \approx S_4 \times C_2$

The orbit $O(100) = (\pm e_1, \pm e_2, \pm e_3)$ represents an octahedron and the orbit $O(001) = \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3)$ represents a cube which is the dual of octahedron. All non-chiral Archimedean solids with octahedral symmetry can be constructed as the orbits of the octahedral group.

4.3. The icosahedral group $W(H_3) \approx A_5 \times C_2$

The platonic as well as Archimedean solids possessing the icosahedral symmetry can be constructed as the orbits of the Coxeter group $W(H_3)$. For example, two dual platonic solids, the icosahedron and dodecahedron, are represented by the orbits $O(100)$ and $O(001)$ respectively.

The Catalan solids, as we already stated, are the duals of the Archimedean solids [9]. Two examples which are well known in crystallography and viral structures are given below.

- (a) Dual of the cuboctahedron possessing the octahedral symmetry is the rhombic dodecahedron. Its 14 vertices is the union of the orbits $(100) \oplus \frac{1}{\sqrt{2}}O(001)$ which is represented in figure 3.

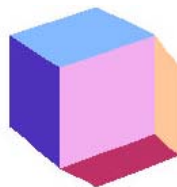


Figure 3. The rhombic dodecahedron.

- (b) Dual of the icosidodecahedron is the rhombic triacontahedron. The 32 vertices is the union of the orbits $O(100) \oplus \sqrt{\frac{\tau+2}{3}}O(001)$ which is displayed in figure 4.



Figure 4. The rhombic triacontahedron.

5. Rank-4 Coxeter groups and 4D Polytopes

5.1. The Coxeter group $W(B_4)$ of order 384

This group is an extension of the octahedral group $W(B_3)$ to four dimensions. As it is expected the generalizations of the octahedron and the cube in terms of quaternions are straightforward. The hyper octahedron, also known as (16-cell), is given by the orbit $O(1000) = \{\pm 1, \pm e_1, \pm e_2, \pm e_3\}$. It consists of 16 tetrahedra like $\{1, e_1, e_2, e_3\}$. Similarly the hyper cube, known as 8-cell, is represented by the orbit $O(1000) = \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$. It consists of 8 cubic cells. Hyper octahedron and hyper cube are dual to each other as expected.

5.2. *The Coxeter group $W(F_4)$ of order 1152*

It is a unique Coxeter group in the sense that it has no correspondence in any dimensions. It has a self dual polytope with 24 vertices with 24 cells made of octahedra. It is simply called 24-cell which is represented in terms of quaternions either by the orbit

$$O(1000) = T = \{\pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)\} \text{ or by } O(0001) = T' = (V_1 \oplus V_2 \oplus V_3).$$

A Dynkin diagram symmetry of F_4 transforms these two sets of vertices to each other. Thus the polytope 24-cell is said to be self dual.

5.3. *The Coxeter group $W(H_4) \subset W(E_8)$*

The Coxeter group $W(H_4)$ is a maximal subgroup of the group $W(E_8)$. Its dual platonic polytopes are called 120-cell and 600-cell represented by quaternions as follows:

$$120\text{-cell: } O(1000) = \sum_{i,j=0}^4 \oplus p^i T p^j \text{ which has 600 vertices. Here } p \in I, p^5 = \pm 1.$$

Each cell is a dodecahedron. At each vertex there are 4 dodecahedra. The 600-cell is represented by the set of quaternions I which also represents the binary icosahedral group:

$$600\text{-cell: } O(0001) = I = \sum_{i=0}^4 \oplus p^i T.$$

Each cell is a tetrahedron. At each vertex there are 20 tetrahedra. When the sphere S^3 representing the 600-cell is sliced by parallel hyperplanes one obtains 3D polyhedra represented by the conjugacy classes of the binary icosahedral group I .

They consist of two points ± 1 corresponding to the poles of the sphere S^3 , 4 icosahedra, 2 dodecahedra and one icosidodecahedron which can be readily seen from the table 1. The 120-cell and 600-cell are dual to each other. In addition to these two platonic solids we will study two more polytopes possessing the symmetry $W(H_4)$.

720-cell = 120 + 600: This is the orbit represented by $O(0100)$ consisting of 1200 vertices. Its cells are made of 120 icosidodecahedra and 600 tetrahedra. At any vertex there are 3 icosidodecahedra and 2 tetrahedra whose centers are represented respectively by the vectors $\omega_4, r_3 r_4 \omega_4, (r_3 r_4)^2 \omega_4$ up to a scale factor and by the vectors $\omega_1, r_1 \omega_1$.

The dual polytope of the 720-cell consists of 720 vertices which is the union of the orbits [14] $\frac{3}{2\sqrt{2}} O(0001) \oplus O(1000)$. Each cell of the dual polytope is a dipyrmaid with a triangular base. The dual

polytope is the projection of the Voronoi cell of E_8 [15] with 19,440 vertices into 4D space. Another semi regular polytope of the Coxeter group $W(H_4)$ has 720 vertices and 720 cells which we denote by *720'-cell = 120 + 600*. The *720'-cell* is represented by the orbit $O(0010)$. It consists of the cells of 120 icosahedra and 600 octahedra. At each vertex there are 2 icosahedra and 5 octahedra. Their centers can be represented respectively by $\omega_4, r_4 \omega_4$ up to a scale factor $\frac{\sqrt{2}}{2 + \sigma}$

and $\omega_1, r_1 r_2 \omega_1, (r_1 r_2)^2 \omega_1, (r_1 r_2)^3 \omega_1, (r_1 r_2)^4 \omega_1$. These 7 vertices form a dipyrmaid with a pentagonal

base. The 720 vertices of the dual polytope consist of the set $\frac{\sqrt{2}}{2 + \sigma} O(0001) \oplus O(1000)$.

It is also interesting to note that the symmetries of the semi regular 4D polytopes, the grand antiprism and the snub 24-cell, are represented by two maximal subgroups of the Coxeter group $W(H_4)$. We display the maximal subgroups of the Coxeter group $W(H_4)$ in table 2 [13].

Table 2. Maximal subgroups of the Coxeter group $W(H_4)$.

Maximal subgroup	Order	Polytope
$Aut(A_2 \oplus A_2)$	144	
$Aut(H_2 \oplus H_2)$	400	Grand antiprism with 100 vertices
$W(H_3 \oplus A_1)$	240	
$Aut(A_4)$	240	
$W(D(4))/C_2 : S_3$	576	Snub 24-cell with 96 vertices

One of these semi regular polytope is the Grand Antiprism (GA) which is first determined by a computer calculation by Conway and Guy in 1965 [16]. This semi regular polytope has 100 vertices which can be obtained as a subset of the set of quaternions I (600-cell) under the decomposition of its maximal subgroup $Aut(H_2 \oplus H_2)$ [17]. Let us take the quaternions

$$b = \frac{1}{2}(\tau + \sigma e_1 + e_2) \in 12(1)_+ \quad \text{and} \quad c = \frac{1}{2}(\tau - \sigma e_1 + e_2) \in 12(1)_+.$$

Then the vertices of the GA are given by the set of quaternions

$$GA = \{b^m c b^n, b^m e_3 c b^n\}, \quad m, n = 0, 1, \dots, 9 \tag{9}$$

which consist of 100 elements of the set of quaternions I from which the set of 20 quaternionic roots of the Coxeter diagram $H_2 \oplus H_2' = \{b^m, e_3 b^m\}; m = 0, 1, \dots, 9$ are removed. The GA has 100 vertices, 320 cells (20 pentagonal antiprisms+300 tetrahedra), 500 edges and 720 faces (700 triangles+20 pentagons). Dual Polytope of the GA consists of 320 vertices which are given as the orbits of the group $Aut(H_2 \oplus H_2)$ as follows:

200 vertices like $J_1 = \sum_{i,j=0}^4 \oplus b^i V_1 b^j$; 100 vertices like $J_3' = \sum_{i,j=0}^4 \oplus b^i \frac{1}{\sqrt{2}} (\pm 1 \pm e_3) b^j$ and

20 vertices like $\{\frac{\tau}{\sqrt{2}} b^m, \frac{\tau}{\sqrt{2}} e_3 b^m\}, m = 0, 1, \dots, 9.$

The typical cell of the dual polytope has 14 vertices with faces of 4 pentagons, 4 kites and 2 isosceles trapezoids as shown in figure 5.



Figure 5. A typical cell of the dual polytope of the grand antiprism.

Another interesting semi regular polytope is the snub 24-cell. The snub 24-cell [18] consists of 96 vertices $S = I - T = \sum_{i=1}^4 p^i T$ with the symmetry group $\left(\frac{W(D_4)}{C_2} \right): S_3 = \{[T, T] \oplus [T, T]^*\}$ of order 576. Its 144 cells consist of 24 icosahedra and 120 tetrahedra. The dual of the snub 24-cell has the vertices $T \oplus T' \oplus S'$ with $S' = \sum_{i=1}^4 \oplus p^i \bar{p}^i T'$ where p^\dagger is obtained from p by exchanging $\tau \leftrightarrow \sigma$. One cell of the dual polytope is a solid with 8 vertices as shown in figure 6.



Figure 6. A typical cell of the dual polytope of the snub 24-cell.

5.4. The Coxeter group $W(A_4)$ of order 120

The automorphism group $Aut(A_4) \approx W(A_4): \gamma$ is an extension of the Coxeter group $W(A_4)$ by the Dynkin diagram symmetry γ . It is a maximal subgroup of the group $W(H_4)$ and has its own Platonic and Archimedean polytopes. We discuss the following examples. The 5-cell is an extension of tetrahedron to 4D and can be represented either by the orbit $O(1000)$ or $O(0001)$. It has 5 vertices consisting of 5 tetrahedra 4 of which are meeting at one vertex. Similarly the orbit $O(0100)$ represents the rectified 5-cell consisting of 5 tetrahedral and 5 octahedral cells. The rectified cell has 10 vertices. At each vertex there are 2 tetrahedra and 3 octahedra whose centers respectively consist of the vertices $\omega_1, r_1 \omega_1$ and $\omega_4, r_3 r_4 \omega_4, (r_3 r_4)^2 \omega_4$ up to a scale factor. These vertices form a dipyrmaid which is a typical cell of the dual polytope [19]. The vertices of the dual polytope consist of the union of the orbits $O(0100) \oplus \frac{3}{2} O(0001)$. Another polytope with the symmetry group $Aut(A_4)$ is called runcinated 5-cell consisting of 20 vertices and 30 cells composed of 10 tetrahedra and 20 triangular prisms. It is represented by the orbit $O(1001)$. At each vertex there are 2 tetrahedra and 6 triangular prisms. The dual polytope has 30 vertices consisting of the union of the orbits $O(100) \oplus O(0100) \oplus O(0010) \oplus O(0001)$. The dual polytope has 20 cells each made of rhombohedra.

6. Concluding Remarks

We presented a few examples of the 4D polytopes along with their duals. Their cells are made of Platonic and Archimedean polyhedra. A systematic study of the 4D polytopes and their dual polytopes will follow [14, 19].

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