Cosmological perturbations of a perfect fluid and non-commutative geometry

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Abstract

I provide an action that describes the linear cosmological perturbations of a perfect fluid. This action is suited not only to perfect fluids with a barotropic equation of state, but also to those for which the pressure depends on two thermodynamical variables. By quantizing the system we find that 1) some perturbation fields exhibit a non-commutativity quite analogous to the one observed for a charged particle moving in a strong magnetic field, 2) curvature and pressure perturbations cannot be measured at the same time at the same point, 3) ghosts appear if the null energy condition is violated.

1 Action

In this proceeding I will report the work done in collboration with Jean-Marc Gérard and Teruaki Suyama [1].

The action considered here has been introduced by Schutz [3] and is defined as follows

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} + p(\mu, s)\right]. \tag{1}$$

Alternative functionals have been proposed, all being physically equivalent as shown in [4]. We chose the version (1) as it was the most convenient for our purpose. The four-velocity of the perfect fluid is defined via potentials:

$$u_\nu = \frac{1}{\mu}\left(\partial_\nu\ell + \theta\partial_\nu s + A\partial_\nu B\right), \tag{2}$$

where ℓ, θ, A and B are all scalar fields. The normalization for the four-velocity, $u^\nu u_\nu = -1$, gives μ in terms of the other fields. The fundamental fields over which the action (1) will be varied are $g_{\mu\nu}$, ℓ, θ, s, A, and B.

Having chosen the Lagrangian for gravity to be the one of GR, we recover $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ by varying with respect to the metric field. Besides the conservation of particle number and entropy already discussed, the other equations of motion derived from Eq. (1) are

$$u^\alpha\partial_\alpha\theta = T, \quad u^\alpha\partial_\alpha A = 0, \quad u^\alpha\partial_\alpha B = 0. \tag{3}$$

In a FLRW universe, $u_i = 0$ and $u_0 = -1$ such that the solutions to Eq. (3) are simply

$$A = A(\vec{x}), \quad B = B(\vec{x}), \quad \theta = \int^t T(t')\mathrm{d}t' + \tilde{\theta}(\vec{x}). \tag{4}$$

There is a complete freedom for the functions A, B, and $\tilde{\theta}$ [2], any choice leading to the same physical background. We will take advantage of this freedom to simplify our study of the scalar and vector perturbations.

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[2] Since $u_\nu = (-1, \vec{0})$, we also have that $\ell = -\int^t \mu(t')\mathrm{d}t' + \tilde{\ell}$, and $\vec{\nabla}\tilde{\ell} = -A\vec{\nabla}B$, which implies that $\vec{\nabla}A \times \vec{\nabla}B = 0$.

1.1 Scalar type perturbations

Let us simply consider the choice $A = B = \tilde{\theta} = 0$, to remove the vector perturbations arising from W_i. Regarding the metric, δg_{00} and δg_{0i} are auxiliary fields such that the only scalar component which will be dynamical is the curvature perturbation ϕ defined by $\delta g_{ij} = 2a^2\phi\,\delta_{ij}$.

We make a field redefinition, $v = \delta\ell + \theta(t)\delta s$, and introduce two gauge invariant fields, $\Phi = \phi + Hv/\mu$ and $\delta\bar{\theta} = \delta\theta + Tv/\mu$, to expand the action (1) at second order:

$$S_S = \int \mathrm{d}t\mathrm{d}^3\vec{x}\left\{\frac{a^3 Q_S}{2}\left[\dot{\Phi}^2 - \frac{c_s^2}{a^2}(\vec{\nabla}\Phi)^2\right] + C\delta s\dot{\Phi} - \frac{D}{2}\delta s^2 - E(\delta\bar{\theta}\dot{\delta s} - \delta s\dot{\delta\bar{\theta}} + \delta A\dot{\delta B} - \delta B\dot{\delta A})\right\}. \quad (5)$$

The coefficients for Φ are given by $Q_S = \frac{\rho+p}{c_s^2 H^2}$, $c_s^2 \equiv \frac{\dot{p}}{\dot{\rho}} = \left(\frac{\partial p}{\partial \rho}\right)_s$, whereas the remaining coefficients are

$$C = \frac{na^3}{H}\left[\mu\left(\frac{\partial T}{\partial \mu}\right)_s - T\right], \quad E = \frac{na^3}{2}, \quad D = na^3\left[T\left(\frac{\partial T}{\partial \mu}\right)_s + \left(\frac{\partial T}{\partial s}\right)_\mu\right]. \quad (6)$$

The general solution for δs, δA, and δB is their initial values since Eq. (5) forces them to be time-independent. As a consequence, the non trivial equations of motion are

$$\frac{1}{a^3 Q_S}\frac{\mathrm{d}}{\mathrm{d}t}(a^3 Q_S\dot{\Phi}) - \frac{c_s^2}{a^2}\nabla^2\Phi = -\frac{\dot{C}}{a^3 Q_S}\,\delta s\,, \quad (7)$$

$$na^3\dot{\delta\bar{\theta}} - D\delta s + C\dot{\Phi} = 0\,. \quad (8)$$

If $T = f(s)\,\mu$, which is equivalent to having a barotropic equation of state $p = p(\rho)$ [3], then $C = 0$. Both radiation and dust fulfill this condition. In these cases, the field Φ completely decouples from δs and propagates with a sound speed c_s, if $c_s^2 > 0$. Note that a cosmological constant has vanishing Q_S so that no contribution for perturbations arises, as is well known.

1.2 Vector type perturbations

To arrive at the desired action via the shortest path, let us first assume that all the perturbation variables propagate only in one direction, say the z-direction. This should be allowed, as we know that perturbations with different wavenumber vectors do not mix in a FLRW universe. Once we obtain the action for this particular mode, we can then easily infer the general action.

Taking again advantage of the freedom to select these background functions, we can make the simplest choice that contains all the information needed for the vector modes, namely $A = \tilde{\theta} = 0$, $B_{,i} = b_i$, where $\vec{b} = (b, 0, 0)$ is a constant vector orthogonal to the z-direction.

We find

$$S_V = \int \mathrm{d}^4x\left[\frac{a}{32\pi G}(\partial_j V_i)(\partial_j V_i) + a^3(\rho+p)\dot{C}_i\delta u_i + a^2(\rho+p)V_i\delta u_i - \tfrac{1}{2}\,a\,(\rho+p)\delta u_i\delta u_i\right], \quad (9)$$

where we substituted δu_i for $b_i\delta A/\mu$. Variations with respect to V_i and C_i yield the following equations,

$$\triangle V_i = 16\pi G a(\rho+p)\delta u_i, \qquad \frac{\mathrm{d}}{\mathrm{d}t}[(\rho+p)a^3\delta u_i] = 0, \quad (10)$$

respectively. Again, these equations exactly coincide with those derived by perturbing the Einstein equations and the energy-momentum conservation law.

2 Quantization

The most important advantage of the action approach proposed in this letter is that it allows us to quantize the system. Although the inhomogeneities of the present universe, such as the galaxy distribution, are

[3] In this case, we obtain $(\partial\mu/\partial s)_\rho = T$ such that $(\partial p/\partial s)_\rho = n[(\partial\mu/\partial s)_\rho - T] = 0$.

clearly described by the classical theory, the quantization of a perfect fluid may have something to do with the early universe if the seeds for structure formation are provided by quantum fluctuations of fields generated during inflation. Yet, besides its practical utility, our action approach also opens new theoretical prospects, as discussed below. In the following, we will again treat the quantization for the scalar and vector type perturbations separately.

2.1 Scalar type perturbations

To quantize the scalar perturbations, let us first introduce the canonical field $\psi \equiv \sqrt{a^3 Q_S}\Phi$. To avoid the appearance of a ghost, we assume that Q_S is positive. This means that $(\rho+p)/c_s^2 > 0$. Such a constraint, together with the stability of the perturbations, $c_s^2 > 0$, lead to the null energy condition $\rho+p>0$. Using the new variable ψ, the action (5) is rewritten as

$$S_S = \int \mathrm{d}^4x \left[\frac{\dot{\psi}^2}{2} - \frac{c_s^2}{2a^2}(\vec{\nabla}\psi)^2 + C_1\delta s\dot{\psi} + C_2\delta s\psi - \frac{N}{2}(\delta\bar{\theta}\dot{\delta s} - \delta s\dot{\delta\bar{\theta}}) - \frac{D}{2}\delta s^2\right], \tag{11}$$

where we have neglected δA and δB as they do not contribute to the Hamiltonian. The field ψ has a canonical kinetic term, whereas the quadratic terms for δs and $\delta\bar{\theta}$ are at most linear in their time derivatives. Yet, it is known [6] that a consistent quantization of such a singular Lagrangian can be done provided one introduces the following commutation conditions,

$$\left[\hat{\psi}(t,\vec{x}), \hat{\pi}(t,\vec{y})\right] = i\delta(\vec{x}-\vec{y})\,, \tag{12}$$

$$\left[\hat{\delta s}(t,\vec{x}), \hat{\delta\bar{\theta}}(t,\vec{y})\right] = -\frac{i}{N}\delta(\vec{x}-\vec{y})\,. \tag{13}$$

All the other commutators are zero and π is the canonical conjugate momentum of ψ. The corresponding Hamiltonian is given by

$$\hat{H} = \int \mathrm{d}^3\vec{x}\left[\frac{1}{2}\left(\hat{\pi} - C_1\hat{\delta s}\right)^2 + \frac{c_s^2}{2a^2}(\vec{\nabla}\hat{\psi})^2 - C_2\hat{\delta s}\hat{\psi} + \frac{D}{2}\hat{\delta s}^2\right]. \tag{14}$$

One can easily check that the Heisenberg equations, with the help of the commutation relations, yield the same equations of motion as the classical ones derived from the variation of Eq. (11).

The commutation relation (13) shows that $\hat{\delta s}$ and $\hat{\delta\bar{\theta}}$ are non-commuting variables. At this level, it is quite interesting to compare the action (11) with the one of the Landau problem, an archetype of non-commutative geometry. Regarding $\hat{\delta s}$ and $\hat{\delta\bar{\theta}}$, the action (11) is essentially the same as the one for a charged particle moving on a two-dimensional surface with a constant magnetic field background in the transverse direction:

$$S = \int \mathrm{d}t\left[\frac{m}{2}(\dot{x}^2+\dot{y}^2) - \frac{\mathcal{B}}{2}(\dot{x}y - \dot{y}x) - V(x,y)\right]. \tag{15}$$

Within this analogy, the perturbation fields $(\delta s, \delta\bar{\theta})$ correspond to the (x, y) space coordinates for the particle, and the number of particles $N = na^3$ plays the role of the constant magnetic field $\mathcal{B}$. Interestingly enough, while the non-commutative relation $[\hat{x}, \hat{y}] = -i/\mathcal{B}$ in the Landau problem [6] holds only in the absence of the kinetic term in Eq. (15), which is valid in the large magnetic field limit, the non-commutative relation (13) of a perfect fluid is exact for any finite number of particles. So, perfect fluids provide a nice example of non-commutativity.

The non-commutation relation (12) leads to another interesting physical consequence. By using once more the Einstein equations and the energy-momentum conservation law, we find that the pressure perturbation in the comoving gauge $(v = 0)$ is given by $\hat{\delta p} = -(\rho+p)\dot{\hat{\phi}}/H$. Then, the commutator between ϕ and δp becomes

$$\left[\hat{\phi}(t,\vec{x}), \hat{\delta p}(t,\vec{y})\right] = -ic_s^2 H\delta(\vec{x}-\vec{y})/a^3. \tag{16}$$

Consequently, curvature and pressure perturbations cannot be measured at the same time, at the same point.

2.2 Vector type perturbations

Time derivatives of V_i and δu_i do not appear in the action (9). Therefore, those are auxiliary fields which can be eliminated through their equations of motion. The action (9) becomes then a functional which depends only on C_i. To make this action canonical, we introduce a new variable $F_i(\vec{k},t) = \sqrt{a^3 Q_V(k,t)} C_i^{\parallel}(\vec{k},t)$, where $C_i^{\parallel}(\vec{k},t)$ is the Fourier transform of $C_i^{\parallel}(\vec{x},t)$ and Q_V is given by

$$Q_V(k,t) = \frac{a^2 k^2 (\rho+p)}{k^2 + 16\pi G a^2 (\rho+p)}. \tag{17}$$

To avoid the appearance of ghosts, Q_V must be positive. So, as for the scalar modes we require $\rho+p>0$, i.e. the null energy condition to hold. In terms of F_i, the canonical action in Fourier space is given by

$$S_V = \int \mathrm{d}t\mathrm{d}^3k \left(\tfrac{1}{2}\dot{F}_i^* \dot{F}_i - \tfrac{1}{2} m_k^2 F_i^* F_i \right), \tag{18}$$

with

$$m_k^2 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log(a^3 Q_V) - \frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}t}\log a^3 Q_V\right)^2. \tag{19}$$

Now the quantization is done by imposing the following canonical condition for F_i and its conjugate momentum

$$\left[\hat{F}_i(t,\vec{k}), \hat{\pi}_j{}^\dagger(t,\vec{k}')\right] = i\delta(\vec{k}-\vec{k}')\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right). \tag{20}$$

The corresponding Hamiltonian is given by

$$\hat{H} = \int \mathrm{d}^3k \left(\tfrac{1}{2}\hat{\pi}_i^\dagger \hat{\pi}_i + \tfrac{1}{2} m_k^2 \hat{F}_i^\dagger \hat{F}_i \right), \tag{21}$$

and the evolution of the operators is given by the Heisenberg equation with the help of the commutation relation (20). The quantum version of Eq. (10) implies $\left[\hat{V}_i(t,\vec{x}), \hat{\delta u}_j(t,\vec{y})\right] = 0$. Therefore, the gauge invariant metric perturbation and the vorticity of the perfect fluid can be measured at the same time, at the same point.

I have introduced a new frame to study the theory of cosmological perturbations for a perfect fluid. Starting from the action itself, first reproduced the known results derived from the equations of motion. Quantizing then the perturbation fields we found that some of them do not commute, leading thus to a non-commutative field-geometry. I also concluded that a simultaneous measurement of curvature perturbations and pressure inhomogeneities is not allowed. Finally I proved that both the null energy condition and a positive $c_s^2 = \dot{p}/\dot{\rho}$ have to hold at all times in order to avoid ghost degrees of freedom.

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