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Schrödinger equation in higher-dimensional curved space: a test for the existence of higher dimensions in the quantum realm

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Abstract By considering the possibility of higher dimensions for nonrelativistic quantum particles, we rederive the Schrödinger equation (SE) for such particles in a (d-1)-dimensional curved space embedded within a d-dimensional flat space. This approach generalizes de Costa's formalism, which describes a nonrelativistic quantum particle confined to a two-dimensional curved surface embedded in three-dimensional Euclidean space. The original d-dimensional SE is separated into two parts: a one-dimensional global SE, which includes a confining potential to ensure the particle's wavefunction does not propagate into the extra dimension, and a (d-1)-dimensional local SE. The local equation reveals an induced geometric potential, a distinctive feature arising from the presence of higher dimensions. This provides a hypothetical framework for probing the existence of higher-dimensional spaces. We apply this formalism to curved spaces generated by massive central objects, such as black holes or stars, and specifically revisit the behavior of a quantum particle near the Ellis wormhole.

1 Introduction

Higher dimensions are a well-established notion in physics. Tangherlini in [1] investigates the problem of spatial dimensionality, focusing on why space appears three-dimensional despite the formal extensibility of physical laws to higher dimensions. Furthermore, in [1] the bound state postulate is proposed, which asserts that stable bound orbits or states must exist for physical systems, such as planetary motion or atomic structures. By generalizing the Schwarzschild spacetime to n dimensions and analyzing the resulting geodesic equations, it is demonstrated that stable bound orbits are only possible in three spatial dimensions, both in Newtonian mechanics and general relativity. This conclusion is further supported by examining the Schrödinger equation for hydrogen atoms in n dimensions, where stable bound states are also found to be exclusive to $n = 3$. In [1] it is argued that the three-dimensionality of space is not an arbitrary assumption but a consequence of the requirement for stable interactions, aligning with Mach's principle and the need for measurable distances in physical theories. The findings suggest that the dimensionality of space can be derived from fundamental physical principles rather than being imposed a priori.

In another study, Caruso et al. in [2] revisit the question of whether hydrogen atoms can exist in spaces with more than three dimensions ($D > 3$). By solving the Schrödinger equation using Numerov's method, the authors determine the lowest quantum mechanical stable states and corresponding wave functions for various angular momentum quantum numbers and dimensionalities. They find that hydrogen atoms could indeed exist in higher dimensions, with stable states characterized by positive energy eigenvalues. The most probable distance between the electron and the nucleus increases with dimensionality, contrary to some earlier predictions. The results suggest that while stable hydrogen atoms can exist in higher-dimensional Euclidean spaces, the energy levels and spatial distributions differ significantly from those in three-dimensional space, indicating a preference for tridimensional space in nature.

Moreover [3] explores the implications of space dimensionality on the description of a hydrogen atom using a generalized wave equation for dimensions $D > 3$. The authors extend the Schrödinger equation by incorporating an iterated Laplacian and a Coulomb-like potential $r^{-\beta}$. They employ the $\frac{1}{N}$ expansion method to solve this generalized equation and analyze the ground state energy of hydrogen atoms in higher-dimensional spaces. The study reveals that the stability and energy levels of hydrogen atoms are intricately linked to the dimensionality of space, with the three-dimensional

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case ($D = 3$) uniquely favoring stable bound states. The findings provide new insights into Paul Ehrenfest's approach [4] to understanding space dimensionality and suggest that the measured ground state energy of hydrogen atoms is a direct consequence of the three-dimensional nature of space. The paper concludes that while higher-dimensional spaces can support bound states under specific conditions, the three-dimensional framework remains the most stable and physically significant.

In addition to these three particular papers there have been several other studies on the concept of quantum mechanics in higher dimensions such as [5–9] (also see [10] for a collection of related informations).

On the other hand, the traditional academic textbooks on quantum mechanics focus primarily on $D = 3$ and lower-dimensional systems. Technically, the flat space is regarded three-dimensional for lower-dimensional quantum systems, with one or two dimensions frozen depending on the problem being studied. This treatment is mathematically accurate if and only if the lower dimensional systems are flat. For example, the SE of a particle on a flat two- dimensional plane can be derived simply by freezing the normal axis to the plane and writing the SE in three dimensions, thereby making it two-dimensional. The same is not true for curved surfaces. For example, to determine the SE of a particle confined to a cylindrical surface, we cannot use the three-dimensional SE in cylindrical coordinates while freezing the radial coordinate. Several authors have proven this in the literature (see for instance [11–16]). In [13] de Costa developed the mathematical approach for the first time, obtaining the SE of a nonrelativistic quantum particle confined to a curved surface embedded in three-dimensional Euclidean space. In the SE of a nonrelativistic quantum particle on a curved surface, there is a geometrically generated potential that is described in terms of the surface curvature. Such an induced potential is not trivial, and its effects can be studied both theoretically and empirically.

While the standard treatment of quantum systems like the hydrogen atom typically occurs in a three-dimensional flat space, there are studies that explore quantum particles in three-dimensional curved spaces. For example, Dandoloff in [22] investigated the dynamics of a spinless quantum particle in a curved space defined by Eq. (38). Building on the insights gained from the aforementioned earlier works, we pose the following question: How does the energy spectrum of such a quantum particle change when the three-dimensional curved space is embedded in a four-dimensional flat space? This research aims to address this question and explore its implications. To do so, we consider a three-dimensional curved space embedded within a four-dimensional flat space. We derive the Schrödinger equation for a quantum particle confined to the three-dimensional curved space. Given that the bulk space is four-dimensional and flat, we adopt the approach outlined by de Costa in [13] to account for the effects of the extrinsic curvatures of the three-dimensional space. This method represents a direct extension of earlier works [17–21].

It is important to note that this study does not aim to solve the Schrö dinger equation in 4 dimensions, as it is well-established that certain potentials, such as the Coulomb potential, do not yield solutions for systems like the hydrogen atom in 4 dimensions. Instead, the quantum particle in our analysis is confined to a three-dimensional curved space that is embedded within a four-dimensional bulk space.

2 The theory

While the content in this section has been extensively discussed in [17–21], we find it beneficial to present the entire formalism in a way that aligns with the familiarity of contemporary readers. For this reason, we begin our discussion from the foundational principles. To ensure comprehensiveness, we develop the theory in a general higher-dimensional space rather than restricting it to the specific case of 4 dimensions. Following the approach in [13], we examine a nonrelativistic quantum particle in a d-dimensional flat Euclidean space M, which is constrained to a $d-1$-dimensional hypersurface Σ through the application of a normal force directed toward Σ. It is important to note that the concept of a hypersurface in higher dimensions is analogous to that of a two-dimensional surface embedded in a three-dimensional flat space.

The corresponding d-dimensional SE is given by

$$-\frac{\hbar^2}{2m_0}\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\right)\psi\left(x^\mu,t\right)+V_c\left(x^d\right)\psi\left(x^\mu,t\right)$$
$$=i\hbar\partial_t\psi\left(x^\mu,t\right), \tag{1}$$

in which m_0 is the mass of the particle, $g^{\mu\nu}$ is the inverse metric tensor of the d-dimensional bulk flat space $\mathcal{M}$, $g = \det g_{\mu\nu}$ is the determinant of the metric tensor, $\partial_\mu = \frac{\partial}{\partial x^\mu}$ is the partial derivative with respect to x^μ with $\mu = 1, 2, \ldots, d$, and $V_c\left(x^d\right)$ is the confining potential which forces the particle to remain in the $d-1$-dimensional hypersurface Σ. Moreover, the global line element describing the d-dimensional bulk space is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{2}$$

whereas the global metric tensor is defined to be

$$g_{\mu\nu} = \frac{\partial \mathbf{R}}{\partial x^\mu}\cdot\frac{\partial \mathbf{R}}{\partial x^\nu}, \tag{3}$$

in which $\mathbf{R}\left(x^\alpha\right)$ is the position vector at point x^α. In the hypersurface Σ the position vector is given by $\mathbf{r}\left(x^i\right)$ such that the local line element is described by

$$ds^2 = h_{ij}dx^i dx^j, \tag{4}$$

where

$$h_{ij}=\frac{\partial\mathbf{r}}{\partial x^{i}}\cdot\frac{\partial\mathbf{r}}{\partial x^{j}},\tag{5}$$

is the local metric tensor and x^i is the local coordinates of a point in Σ with $i=1,2,\ldots,d-1$. We note that the relation between the global and the local position vectors i.e., $\mathbf{R}$ and $\mathbf{r}$ maybe expressed as

$$\mathbf{R}=\mathbf{r}+w\mathbf{n},\tag{6}$$

in which $\mathbf{n}$ is the unit normal vector to Σ (outward) and w is the distance of the point in the balk – in the vicinity of Σ – with the position vector $\mathbf{R}\left(x^{\alpha}\right)$ from the hypersurface. By cognition of (6), (5) and (3), one writes

$$g_{ij}=\left(\frac{\partial\mathbf{r}}{\partial x^{i}}+w\frac{\partial\mathbf{n}}{\partial x^{i}}\right)\cdot\left(\frac{\partial\mathbf{r}}{\partial x^{j}}+w\frac{\partial\mathbf{n}}{\partial x^{j}}\right)\tag{7}$$

which is simplified to

$$\begin{aligned}g_{ij}&=h_{ij}+w\left(\frac{\partial\mathbf{r}}{\partial x^{i}}\cdot\frac{\partial\mathbf{n}}{\partial x^{j}}+\frac{\partial\mathbf{n}}{\partial x^{i}}\cdot\frac{\partial\mathbf{r}}{\partial x^{j}}\right)\\&\quad+w^{2}\frac{\partial\mathbf{n}}{\partial x^{i}}\cdot\frac{\partial\mathbf{n}}{\partial x^{j}}.\end{aligned}\tag{8}$$

From the equation of Weingarten i.e., $\frac{\partial\mathbf{n}}{\partial x^{a}}=k_{a}^{b}\frac{\partial\mathbf{r}}{\partial x^{b}}$ in differential geometry where k_a^b is the Weingarten tensor, the latter equation implies

$$g_{ij}=h_{ij}+2wh_{ib}k_{j}^{b}+w^{2}k_{i}^{a}k_{j}^{b}h_{ab},\tag{9}$$

where a Latin index runs over $1,2,\ldots,d-1$. Furthermore, the components of the bulk's metric tensor $g_{id}=g_{di}$, we find (for the sake of clearness, we use directly w for the last Greek index $\mu=d$)

$$g_{iw}=g_{wi}=\frac{\partial\mathbf{R}}{\partial w}\cdot\frac{\partial\mathbf{R}}{\partial x^{i}}=\mathbf{n}\cdot\frac{\partial\mathbf{r}}{\partial x^{i}}=0,\tag{10}$$

because $\frac{\partial\mathbf{r}}{\partial x^{i}}$ is tangent to the hypersurface and $\mathbf{n}$ is perpendicular to it. Finally, we obtain

$$g_{ww}=\frac{\partial\mathbf{R}}{\partial w}\cdot\frac{\partial\mathbf{R}}{\partial w}=\mathbf{n}\cdot\mathbf{n}=1.\tag{11}$$

In order to determine explicitly the SE (1), our next step is to obtain the determinant of $g_{\mu\nu}$ i.e., g. To do so we recall that for the hypersurface Σ, in principle there exists a local coordinates system such that the local metric tensor becomes diagonal. Such a system of coordinates is called the principal frame where the local metric tensor is written as

$$\bar{h}_{ij}=diag\left[\bar{h}_{11},\bar{h}_{22},\ldots,\bar{h}_{d-1,d-1}\right].\tag{12}$$

To distinguish the principal coordinates from the arbitrary ones, we have used a dash on the local metric tensor. From (9), the diagonal local principal metric tensor implies

$$g_{ii}=\bar{h}_{ii}\left(1+2w\bar{k}_{i}^{i}+\left(w\bar{k}_{i}^{i}\right)^{2}\right),\ \text{no summation}\tag{13}$$

such that

$$g=\prod_{i=1}^{d-1}\bar{h}_{ii}\left(1+w\bar{k}_{i}^{i}\right)^{2}=\bar{h}\left(\prod_{i=1}^{d-1}\left(1+w\bar{k}_{i}^{i}\right)\right)^{2},\tag{14}$$

in which $\bar{h}$ is the det $\left(\bar{h}_{ij}\right)$ and is invariant under the diagonalization of the local metric tensor, i.e., $\bar{h}=\det\left(\bar{h}_{ij}\right)=\det\left(h_{ij}\right)=h$. Let us also add that $\bar{k}_i^i$ are called the principal curvatures and the Weingarten tensor is diagonal i.e.,

$$\bar{k}_{i}^{j}=diag\left[\bar{k}_{1}^{1},\bar{k}_{2}^{2},\ldots,\bar{k}_{d-1}^{d-1}\right].\tag{15}$$

In (14), one may write

$$g=h\left(\prod_{i=1}^{d-1}\left(1+w\bar{k}_{i}^{i}\right)\right)^{2}=h\left(1+\sum_{s=1}^{d-1}\frac{w^{s}}{s!}\Gamma^{(s)}\right)^{2}\tag{16}$$

in which the general terms $\Gamma^{(s)}$ is the summation of all same-order combinations of the invariants of the hypersurface given by

$$K^{(1)}=k_{i}^{i},\tag{17}$$

$$K^{(2)}=k_{i}^{j}k_{j}^{i},\tag{18}$$

$$K^{(3)}=k_{i}^{j}k_{j}^{a}k_{a}^{i},\tag{19}$$

and in general

$$K^{(s)}=\overbrace{k_{a_{1}}^{a_{2}}k_{a_{2}}^{a_{3}}k_{a_{3}}^{a_{4}}\ldots k_{a_{s}}^{a_{1}}}^{s\text{-times}},\tag{20}$$

with appropriate coefficients. For instance, we explicitly calculate the first five orders which are given by

$$\Gamma^{(1)}=K^{(1)},\tag{21}$$

$$\Gamma^{(2)}=\left(K^{(1)}\right)^{2}-K^{(2)},\tag{22}$$

$$\Gamma^{(3)}=\left(K^{(1)}\right)^{3}-3K^{(1)}K^{(2)}+2K^{(3)},\tag{23}$$

$$\begin{aligned}\Gamma^{(4)}&=\left(K^{(1)}\right)^{4}-6\left(K^{(1)}\right)^{2}K^{(2)}+8K^{(1)}K^{(3)}\\&\quad+3\left(K^{(2)}\right)^{2}-6K^{(4)}\end{aligned}\tag{24}$$

and

$$\begin{aligned}\Gamma^{(5)}&=\left(K^{(1)}\right)^{5}-10\left(K^{(1)}\right)^{3}K^{(2)}+40K^{(2)}K^{(3)}\\&\quad-40\left(K^{(1)}\right)^{2}K^{(3)}+15\left(K^{(2)}\right)^{2}K^{(1)}+30K^{(4)}K^{(1)}\\&\quad-36K^{(5)}.\end{aligned}\tag{25}$$

The next step is to apply g given by (16) in the d-dimensional SE (1). Moreover, we restrict ourselves to the time-independent potential $V_c(w)$ and write $\psi\left(x^{\mu},t\right)=\frac{1}{\sqrt{\Lambda}}e^{-iEt/\hbar}\psi^{(N)}(w)\,\psi^{(T)}\left(x^{i}\right)$ to get

$$-\frac{\hbar^{2}}{2m_{0}}\frac{1}{\sqrt{h}\sqrt{\Lambda}\psi^{(T)}\left(x^{i}\right)}\partial_{i}\left(\sqrt{h}\Lambda g^{ij}\partial_{j}\right)\frac{1}{\sqrt{\Lambda}}\psi^{(T)}\left(x^{i}\right)$$

$$-\frac{\hbar^2}{2m_0}\frac{1}{\sqrt{\Lambda}\psi^{(N)}(w)}\partial_w\left(\Lambda\partial_w\frac{\psi^{(N)}(w)}{\sqrt{\Lambda}}\right)+V_c(w)=E \quad (26)$$

in which $\Lambda = 1+\sum_{s=1}^{d-1}\frac{w^s}{s!}\Gamma^{(s)}$. We note that $\psi^{(T)}\left(x^i\right)$ and $\psi^{(N)}(w)$ stand for the tangent or local and normal or global wavefunction. Since the hypersurface is located at $w=0$, and (26) eventually is considered at the limit $w\to 0$ where

$$\lim_{w\to 0}\Lambda = \lim_{w\to 0}\left(1+\sum_{s=1}^{d-1}\frac{w^s}{s!}\Gamma^{(s)}\right)=1, \quad (27)$$

$$\lim_{w\to 0}\Lambda_{,w} = \lim_{w\to 0}\sum_{s=1}^{d-1}\frac{w^{s-1}}{(s-1)!}\Gamma^{(s)}=\Gamma^{(1)}, \quad (28)$$

and

$$\lim_{w\to 0}\Lambda_{,ww} = \lim_{w\to 0}\sum_{s=2}^{d-1}\frac{w^{s-2}}{(s-2)!}\Gamma^{(s)}=\Gamma^{(2)}, \quad (29)$$

we obtain

$$\lim_{w\to 0}\frac{1}{\sqrt{\Lambda}\psi^{(N)}(w)}\partial_w\left(\Lambda\partial_w\frac{\psi^{(N)}(w)}{\sqrt{\Lambda}}\right) = -\frac{1}{2}\left(\Gamma^{(2)}-\frac{1}{2}\left(\Gamma^{(1)}\right)^2\right)+\frac{\psi^{(N)\prime\prime}(w)}{\psi^{(N)}(w)}. \quad (30)$$

Considering (30) in (26), the original d-dimensional SE is separated into two equations: (i) a local $d-1$-dimensional SE describing the dynamic of the particle within the hypersurface Σ and (ii) a one-dimensional global SE describing the dynamic of the particle across the extra dimension expressed by

$$-\frac{\hbar^2}{2m_0}\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}h^{ij}\partial_j\right)\psi^{(T)}\left(x^i\right) +\frac{\hbar^2}{2m_0}\left(\frac{1}{2}\Gamma^{(2)}-\left(\frac{1}{2}\Gamma^{(1)}\right)^2\right)\psi^{(T)}\left(x^i\right) = E^{(T)}\psi^{(T)}\left(x^i\right), \quad (31)$$

and

$$-\frac{\hbar^2}{2m_0}\psi^{(N)\prime\prime}(w)+V_c(w)\psi^{(N)}(w)=E^{(N)}\psi^{(N)}(w), \quad (32)$$

respectively. Herein we introduced the total energy of the particle to be addition of the local and global energies i.e.,

$$E=E^{(T)}+E^{(N)}. \quad (33)$$

This is very exciting to observe that similar to de Costa's formalism for $d=3$, the higher order invariants haven't appeared in the local SE (31), yet, $d=3$ and $d>3$ seem to differ due to the structure of $\Gamma^{(2)}$. According to the Eq. (22), $\frac{1}{2}\Gamma^{(2)}=\frac{1}{2}\left(\left(K^{(1)}\right)^2-K^{(2)}\right)$ which can be proved that in $d=3$ is simply the determinant of the Weingarten tensor i.e., $\frac{1}{2}\Gamma^{(2)}=k_1^1k_2^2-k_1^2k_2^1$. This, however, is not valid for $d>3$. Precisely speaking in any arbitrary dimensions d the determinant of the Weingarten tensor is $\frac{\Gamma^{(d-1)}}{(d-1)!}$. Therefore, although in $d=3$, the induced geometric potential i.e.,

$$V_G^{(d)}\left(x^i\right)=\frac{\hbar^2}{2m_0}\left(\frac{1}{2}\Gamma^{(2)}-\left(\frac{1}{2}\Gamma^{(1)}\right)^2\right), \quad (34)$$

reduces to de Costa's potential i.e.,

$$V_G^{(3)}\left(x^i\right) = \frac{\hbar^2}{2m_0}\left(\det\left(k_i^j\right)-\left(\frac{trace\left(k_i^j\right)}{2}\right)^2\right) = \frac{\hbar^2}{2m_0}\left(K_G-\left(\frac{K}{2}\right)^2\right), \quad (35)$$

in which for the curved surface the total and Gaussian curvatures are defined to be $K=trace\left(k_i^j\right)$ and $K_G=\det\left(k_i^j\right)$, respectively, in $d>3$ it doesn't. Therefore one has to consider the induced geometric potential as its universal form given in (34).

Apart from the local equation (31), the global equation (32) is considered as follows. Since the quantum particle is confined in the hypersurface Σ, one may propose an infinite walls potential across Σ. On the hand, we also assume that Σ extends in $\mathbf{n}$ direction with thickness ϵ centered at $w=0$ i.e., from $w=-\epsilon/2$ to $w=+\epsilon/2$ with infinitesimally small ϵ. In other words, Eq. (32) describes a one-dimensional infinite well of width ϵ whose energy eigenvalues and eigenvectors are simply given by

$$\psi_n^{(N)}(w)=\sqrt{\frac{\epsilon}{2}}\sin\left(\frac{n\pi}{\epsilon}w\right), \quad (36)$$

and

$$E_n^{(N)}=\frac{n^2\pi^2\hbar^2}{2m_0\epsilon^2}, \quad n=1,2,3,\ldots. \quad (37)$$

Therefore, the only equation left to be solved in different curved spaces is the local SE i.e., Eq. (31).

3 An example of the three-dimensional curved space

In this section, we provide an example in $d=4$ which has been studied in [22]. In Ref. [22], Dandoloff studied the non-relativistic quantum particle in the three-dimensional curved space with the line element given by

$$ds_3^2=dl^2+\left(b_0^2+l^2\right)\left(d\theta^2+\sin^2\theta d\varphi^2\right), \quad (38)$$

in which b_0 is a constant and $l\in(-\infty,\infty)$, $\theta\in[0,\pi)$, and $\varphi\in[0,2\pi)$ are the local coordinates. In [22] the usual three-dimensional SE is employed and the effect of curved space (38) is reflected through the expression of ∇^2 operator in the SE which is given by

$$\nabla^2 = \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}g^{ij}\partial_j\right), \tag{39}$$

as appeared in the local equation (31). Moreover, in [22], the SE is separated and while the angular part is still the spherical harmonic functions i.e., $Y_{\ell,m}(\theta,\varphi)$ the radial part is reduced to a one-dimensional SE with a repulsive effective potential given by (see Eqs. (8) and (9) in Ref. [22])

$$V_{eff}(l) = \frac{\hbar^2}{2m_0}\left(\frac{\ell(\ell+1)}{b_0^2+l^2} + \frac{b_0^2}{\left(b_0^2+l^2\right)^2}\right). \tag{40}$$

Going further and introducing $l = b_0 x$, the corresponding one-dimensional SE i.e.,

$$-\frac{\hbar^2}{2m_0}\nabla^2\psi(l) = E\psi(l) \tag{41}$$

reduces to

$$-\Phi''(x) + V_{eff}(x)\Phi(x) = \epsilon\Phi(x) \tag{42}$$

in which

$$V_{eff}(x) = \frac{\ell(\ell+1)}{1+x^2} + \frac{1}{\left(1+x^2\right)^2}, \tag{43}$$

$\epsilon = \frac{2m_0 b_0^2}{\hbar^2}E$ and the wavefunction $\psi(l) = \frac{\Phi(l)}{\sqrt{b_0^2+l^2}}$ or equivalently $\psi(x) \sim \frac{\Phi(x)}{\sqrt{1+x^2}}$ up to a constant coefficient which can be absorbed in the normalization constant. Herein $\ell = 0, 1, 2, \ldots$ is the angular quantum number and therefore the dimensionless effective potential is positive definite for all values of ℓ. This in turn means that the one-dimensional potential is repulsive as shown in Fig. 1 and consequently admits no bound state. Therefore, in summary, according to [22], there is no bound state with negative energy for a non-relativistic quantum particle confined to the curved space described by the line element (38).

Here we solve the same problem by assuming the existence of four-dimensional flat space where the curved space (or the hypersurface) with the line element (38) is embedded there. First of all, let us introduce $r^2 = b_0^2 + l^2$ and accordingly (38) transforms into

$$ds_3^2 = \frac{r^2}{r^2-b_0^2}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{44}$$

where $r \in [b_0, \infty)$ is the limited ordinary radial coordinate. Next, we define the four-dimensional flat bulk space to be described by the line element

$$ds_4^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + dz^2 \tag{45}$$

which in addition to r, θ and φ which are the same as ds_3^2 the z-axis is the fourth dimension. According to our formalism, (44) describes the three-dimensional curved hypersurface embedded in the four-dimensional flat bulk space (45). This amounts to

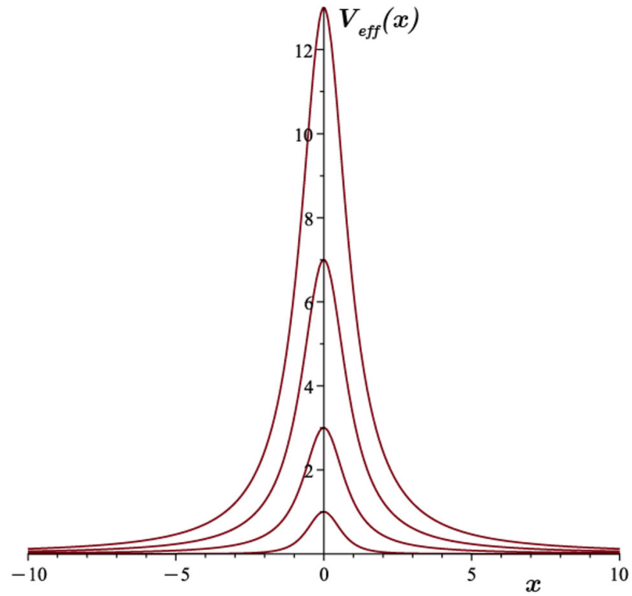

Fig. 1 The plot of $V_{eff}(x)$ versus x given in Eq. (43) for $\ell = 0, 1, 2, 3$ (from the bottom to the top curves). As is observed, the effective potential for any value for ℓ is a repulsive potential implying the absence of bound states

$$h_{ij} = diag\left[\frac{r^2}{r^2-b_0^2}, r^2, r^2\sin^2\theta\right] \tag{46}$$

with $\sqrt{h} = \frac{r^3\sin\theta}{\sqrt{r^2-b_0^2}}$ and

$$g_{\mu\nu} = diag\left[1, r^2, r^2\sin^2\theta, 1\right]. \tag{47}$$

To calculate the Weingarten tensor k_i^j, we introduce $z = z(r)$ in (45) and match it with (44) which yields the global definition of the hypersurface (44) given by

$$z(r) = b_0\ln\left(r + \sqrt{r^2-b_0^2}\right). \tag{48}$$

Having the definition of the hypersurface in the form of $\Sigma := z - b_0\ln\left(r + \sqrt{r^2-b_0^2}\right) = 0$ from (48), we can proceed to calculate the Weingarten tensor. First of all, we obtain the unit normal vector to the hypersurface which is defined as

$$n_\gamma = \frac{1}{\sqrt{\frac{\partial\Sigma}{\partial x^\alpha}\frac{\partial\Sigma}{\partial x^\beta}g^{\alpha\beta}}}\frac{\partial\Sigma}{\partial x^\gamma}, \tag{49}$$

and it satisfies $n_\gamma n^\gamma = 1$. Hence, the explicit calculation using the definition of Σ yields

$$n_\gamma = \left[-\frac{b_0}{r}, 0, 0, \frac{\sqrt{r^2 - b_0^2}}{r}\right]. \tag{50}$$

Moreover, the extrinsic curvature tensor is defined to be

$$k_{ij} = -n_\gamma \left(\frac{\partial^2 x^\gamma}{\partial x^i \partial x^j} + \Gamma^\gamma_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^i} \frac{\partial x^\beta}{\partial x^i}\right), \tag{51}$$

in which $\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\lambda} \left(g_{\alpha\lambda,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}\right)$ is the Christoffel symbol of the second kind of the bulk space. Our explicit calculation reveals that

$$k_{ij} = diag\left[\frac{b_0}{r^2 - b_0^2}, -b_0, -b_0 \sin^2\theta\right], \tag{52}$$

and consequently from (46), we get

$$k_i^j = h^{js} k_{is} = diag\left[\frac{b_0}{r^2}, -\frac{b_0}{r^2}, -\frac{b_0}{r^2}\right], \tag{53}$$

which is the Weingarten tensor. We note that k_i^j in (53) is diagonal that implies $k_i^j = \bar{k}_i^j$ and therefore its components are the principal curvatures. Furthermore, by cognition of (53), we write

$$\Gamma^{(1)} = K^{(1)} = k_i^i = -\frac{b_0}{r^2}, \tag{54}$$

and

$$\Gamma^{(2)} = -2\left(\frac{b_0}{r^2}\right)^2, \tag{55}$$

which amount to

$$V_G = -\frac{5\hbar^2 b_0^2}{8 m_0 r^4}. \tag{56}$$

Hence, the local SE becomes

$$-\frac{\hbar^2}{2m_0}\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}h^{ij}\partial_j\right)\psi^{(T)}(r,\theta,\varphi) - \frac{5\hbar^2 b_0^2}{8 m_0 r^4}\psi_t(r,\theta,\varphi) = E^{(T)}\psi^{(T)}(r,\theta,\varphi). \tag{57}$$

By introducing

$$\psi^{(T)}(r,\theta,\varphi) = \frac{1}{q}\Phi(x(q))\, Y_{\ell,m}(\theta,\varphi), \tag{58}$$

in which $Y_{\ell,m}(\theta,\varphi)$ are the standard spherical harmonics corresponding to the angular orbital number $\ell = 0, 1, 2, \ldots$ and magnetic quantum number $m = 0, \pm 1, \pm 2, \ldots, \pm \ell$, $x(q) = \sqrt{q^2 - 1}$ and $q = \frac{r}{b_0}$, (57) reduces to Eq. (42) where $\varepsilon = \frac{2 m_0 b_0^2}{\hbar^2} E^{(T)}$ and the effective potential given by

$$V_{eff}(x) = \frac{\ell(\ell+1)}{1+x^2} - \frac{1}{4\left(x^2+1\right)^2}. \tag{59}$$

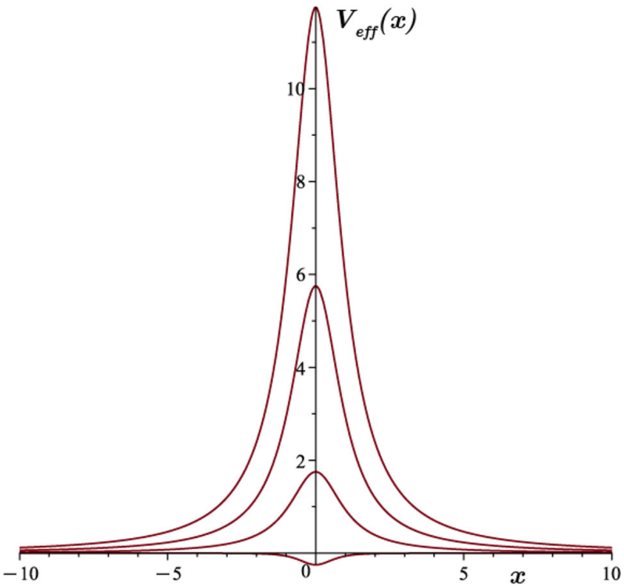

Fig. 2 The plot of $V_{eff}(x)$ versus x given in Eq. (59) for $\ell = 0, 1, 2, 3$ (from the bottom to the top curves). Unlike Fig. 1, the effective potential for $\ell = 0$ is attractive, implying the possible bound states. Furthermore, the effective potential for $\ell \geq 1$ is repulsive similar to Fig. 1

Equations (59) and (43) are corresponding to each other. In Fig. 2, we plot (59) in terms of x for different values of ℓ. The only potential well belongs to $\ell = 0$ in (59). We observe that the effective potential in (59) and the effective potential in [19] are different in the second term. In other words, in (42) the sign is positive. In Fig. 3, we plot both potentials i.e., (43) and (59) in one single frame to compare their strength. Clearly, the effective potential in the curved space embedded in a flat four-dimensional Euclidean space provides a weaker barrier toward the scattering of the quantum particle under consideration. As we already mentioned, the case $\ell = 0$ in (59) To obtain the bound state, we solve the following equation

$$-\Phi''(x) - \frac{1}{4\left(x^2+1\right)^2}\Phi(x) = \varepsilon\Phi(x). \tag{60}$$

which gives only one bound state with

$$\Phi(x) = N_0\left(x^2+1\right)^{\frac{2+\sqrt{5}}{4}} HeunC\left(0, -\frac{1}{2}, \frac{\sqrt{5}}{2}, -\frac{\varepsilon}{4}, \frac{9}{16} + \frac{\varepsilon}{4}, -x^2\right) \tag{61}$$

and the energy

$$\varepsilon \simeq -0.028924589997. \tag{62}$$

Herein, $HeunC$ is the confluent Heun function and N_0 is the normalization constant. In terms of dimensionless radial

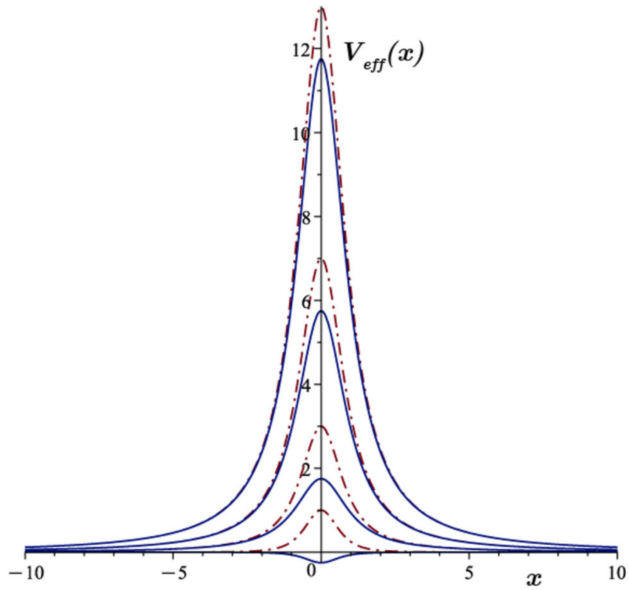

Fig. 3 The plots of $V_{eff}(x)$ versus x given in Eq. (59) (solid curves) and (43) (dashed curves) for $\ell = 0, 1, 2, 3$ (from the bottom to the top curves). The differences between the two effective potentials are highlighted in this figure. Not only does $\ell = 0$ imply a characteristic difference between the two approaches, but for $\ell \geq 1$ also the strength of the effective potential (43) is more than (59) for any given $\ell \geq 1$

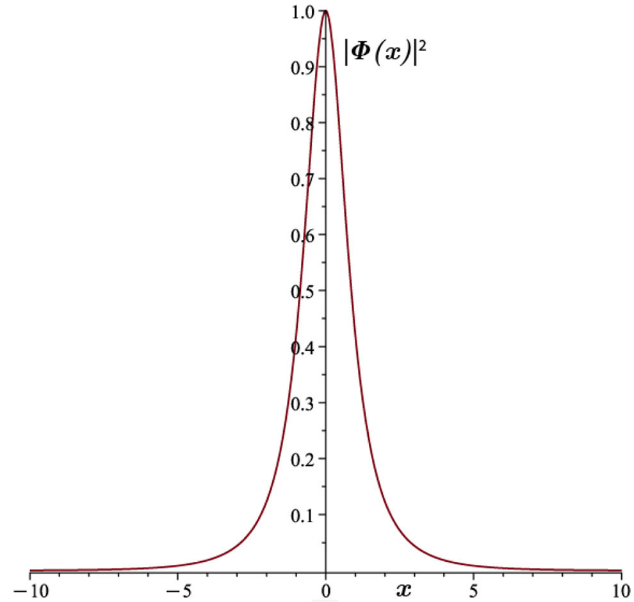

Fig. 4 $|\Phi(x)|^2$ in terms of x with $\Phi(x)$ the unnormalized wavefunction given in Eq. (61). This is the only bound state that exists with considering the higher dimensions only

variable i.e., x the full solution becomes

$$\psi^{(T)}(x,\theta,\varphi) = N_0\left(x^2+1\right)^{\frac{\sqrt{5}}{4}} HeunC \left(0,-\frac{1}{2},\frac{\sqrt{5}}{2},-\frac{\varepsilon}{4},\frac{9}{16}+\frac{\varepsilon}{4},-x^2\right) Y_{\ell,m}(\theta,\varphi) \quad (63)$$

and consequently in terms of r the only bounded wavefunction reads as

$$\psi^{(T)}(r,\theta,\varphi) = N_0\left(\frac{r}{b_0}\right)^{\frac{\sqrt{5}}{2}} HeunC\left(0,-\frac{1}{2},\frac{\sqrt{5}}{2},-\frac{\varepsilon}{4},\frac{9}{16}+\frac{\varepsilon}{4},1-\frac{r^2}{b_0^2}\right) Y_{\ell,m}(\theta,\varphi). \quad (64)$$

In Fig. 4, we plot $|\Phi(x)|^2$ from (61) for $N_0 = 1$ (unnormalized) in terms of x.

Having or not the bound state for a quantum particle confined to the curved space (38) is equivalent to the existence or nonexistence of the higher dimensions.

4 Space curved by a massive cosmological object

In this section, we apply the formalism to a quantum particle outside a massive static spherically symmetric cosmological object such as a static black hole or star with the line element given by

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (65)$$

in which the corresponding three-dimensional curved space is described by

$$ds_3^2 = \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (66)$$

Following the proposed formalism in this study, we embed (66) in a four -dimensional flat space with the line element given by (45). Hence, we obtain

$$\frac{dz(r)}{dr} = \sqrt{\frac{1}{f(r)} - 1}, \quad (67)$$

or equivalently

$$z(r) = \int^r \sqrt{\frac{1}{f(r')} - 1} dr'. \quad (68)$$

The equation of the hypersurface, therefore, is expressed by

$$\Sigma := z - \int^r \sqrt{\frac{1}{f(r')} - 1} dr' = 0. \quad (69)$$

Next, we calculate the unit normal which is defined in (49) such that

$$n_\gamma = \left(-\sqrt{1-f(r)}, 0, 0, \sqrt{f(r)}\right). \quad (70)$$

Applying (51), the second fundamental form or the extrinsic curvature is calculated to be

$$k_{ij} = diag\left[\frac{f'(r)}{2f(r)\sqrt{1-f(r)}}, -r\sqrt{1-f(r)}, -r\sin^2\theta\sqrt{1-f(r)}\right], \quad (71)$$

which in turn implies the Weingarten tensor expressed by

$$k_i^j = diag\left[\frac{f'(r)}{2\sqrt{1-f(r)}}, -\frac{1}{r}\sqrt{1-f(r)}, -\frac{1}{r}\sqrt{1-f(r)}\right]. \quad (72)$$

By virtue of (72), one finds

$$\Gamma^{(1)} = K^{(1)} = k_i^i = \frac{f'(r)}{2\sqrt{1-f(r)}} - \frac{2}{r}\sqrt{1-f(r)}, \quad (73)$$

and

$$\Gamma^{(2)} = \left(K^{(1)}\right)^2 - K^{(2)} = -\frac{2f'(r)}{r} + \frac{2(1-f(r))}{r^2}. \quad (74)$$

Finally, the geometric potential is obtained to be

$$V_G = -\frac{\hbar^2 f'}{32m_0}\left(\frac{8}{r} + \frac{f'}{1-f}\right), \quad (75)$$

which is followed by the local Schrödinger equation

$$-\frac{\hbar^2}{2m_0}\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}h^{ij}\partial_j\right)\psi^{(T)}(r,\theta,\phi) - \frac{\hbar^2 f'}{32m_0}\left(\frac{8}{r} + \frac{f'}{1-f}\right)\psi^{(T)}(r,\theta,\phi) = E^{(T)}\psi^{(T)}(r,\theta,\phi), \quad (76)$$

in which the induced metric tensor of the hypersurface is given by

$$h_{ij} = diag\left[\frac{1}{f}, r^2, r^2\sin^2\theta\right], \quad (77)$$

with $\sqrt{h} = \frac{r^2\sin\theta}{\sqrt{f}}$. Introducing $\psi^{(T)}(r,\theta,\phi) = R(r)Y_{\ell,m}(\theta,\phi)$, we separate the Schrödinger equation upon which the radial equation becomes

$$-\frac{\sqrt{f}}{r^2}\frac{d}{dr}\left(r^2\sqrt{f}\frac{dR}{dr}\right) + \left(\frac{\ell(\ell+1)}{r^2} + \tilde{V}_G\right)R = \epsilon R, \quad (78)$$

in which $V_G = \frac{\hbar^2}{2m_0}\tilde{V}_G$ and $E^{(T)} = \frac{\hbar^2}{2m_0}\epsilon$. By introducing

$$R(r) = \frac{U(r)}{r\sqrt[4]{f(r)}}, \quad (79)$$

the resultant radial Schrödinger equation reduces to

$$-fU'' + \left(\frac{\ell(\ell+1)}{r^2} + \frac{f''}{4} - \frac{3f'^2}{16f} + \frac{f'}{2r} + \tilde{V}_G\right)U = \epsilon U, \quad (80)$$

where

$$\tilde{V}_G = -\frac{f'}{16}\left(\frac{8}{r} + \frac{f'}{1-f}\right). \quad (81)$$

It can be seen that in a general case, the possible bound states as well as the continuous states – in terms of the solutions of the final equation i.e., (80) – differ with or without $\tilde{V}_G$. With a particular arrangement, one may find analytical solutions and figure it out in terms of the initial spacetime's physical properties.

As an example let's consider the dynamic of a nonrelativistic quantum particle in the curved space of anti-de Sitter spacetime where

$$f(r) = 1 + \frac{r^2}{\lambda^2}, \quad (82)$$

with $\lambda^2 = -\frac{3}{\Lambda}$ and $\Lambda < 0$ the cosmological constant. Considering (82) in (81) one finds

$$\tilde{V}_G = -\frac{3}{4\lambda^2} \quad (83)$$

and consequently after introducing $R(r) = \frac{1}{r}U(z(r))$ where $z(r) = \lambda\sinh^{-1}\left(\frac{r}{\lambda}\right)$, (78) simply reads

$$-U''(z) + \left(\frac{\ell(\ell+1)}{\lambda^2\sinh^2\left(\frac{z}{\lambda}\right)}\right)U(z) = \left(\epsilon - \frac{1}{4\lambda^2}\right)U(z) \quad (84)$$

The simplest case is the s-wave scattering solution where $\ell = 0$ and therefore

$$U_4(z) \sim \exp\left(\pm i\sqrt{\epsilon - \frac{1}{4\lambda^2}}z\right), \quad (85)$$

or in terms of r it reads as

$$R_4(r) \sim \frac{1}{r}\exp\left(\pm i\sqrt{\epsilon - \frac{1}{4\lambda^2}}\lambda\sinh^{-1}\left(\frac{r}{\lambda}\right)\right). \quad (86)$$

Herein, the subindex 4 stands for the solution with the higher dimension considered. The plane wave solution in the absence of the higher dimension is given also by

$$U_3(z) \sim \exp\left(\pm i\sqrt{\epsilon - \frac{1}{\lambda^2}}z\right), \quad (87)$$

or in terms of r one finds

$$R_3(r) \sim \frac{1}{r}\exp\left(\pm i\sqrt{\epsilon - \frac{1}{\lambda^2}}\lambda\sinh^{-1}\left(\frac{r}{\lambda}\right)\right) \quad (88)$$

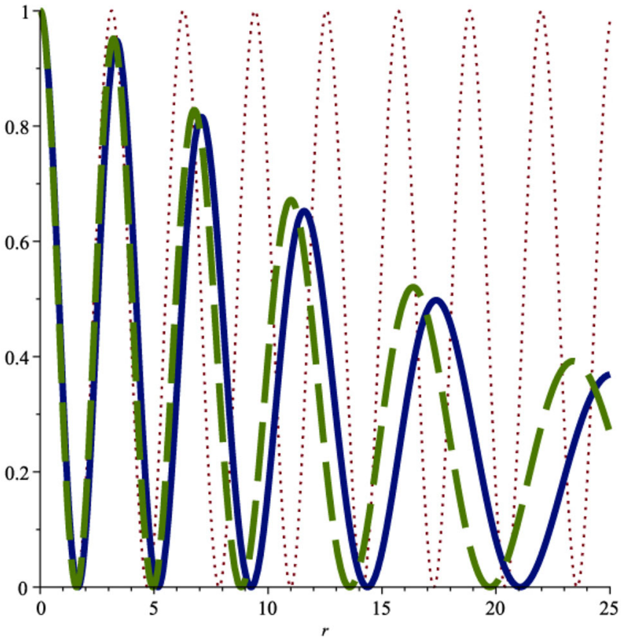

Fig. 5 The plots of the probability density i.e. $\mathrm{Re}\,(R(r))^2\,\frac{r^2}{\sqrt{f}}$ from Eq. (86), (88) and (89) in terms of r for three different scenarios. The dot-curve represents the plane wave of the quantum particle in three-dimensional flat space. The dashed and solid curves represent the same quantum particle in three-dimensional curved space with and without extra dimensions respectively. The wavelength of the wavefunctions in curved space is smaller than in the flat space. Furthermore, considering the extra dimensions reduces the change of the wavelength and in the curved space the probability density of the wavefunction is also reduced with larger r

In the limit of $\lambda \to \infty$ the space becomes flat and therefore

$$\lim_{\lambda\to\infty} R_4(r) = \lim_{\lambda\to\infty} R_3(r) = \frac{1}{r}\exp\left(\pm i\sqrt{\epsilon} r\right). \tag{89}$$

The differences between the three cases are shown in Fig. 5 where we plotted $\rho(r) = \mathrm{Re}\,(R(r))^2\,\frac{r^2}{\sqrt{f}}$ in terms of r for $\lambda = 10$ and $\epsilon = 1$. This figure reveals the effects of the curved space on the probability density $\rho(r)$ and the wavelength of the scattering s-wave as well as the influence of considering the higher dimension. We should add that, though our approach is theoretical in finding the effect of the higher dimensions, the different wavelengths may be observable in the phenomenon of electron interferometry. We leave the details of such thought experiment to be an open problem for further investigation.

5 Conclusion

In summary, by assuming that any $d-1$-dimensional curved space can be embedded in a d-dimensional flat space, we rederived the Schrödinger equation (SE) for a nonrelativistic quantum particle confined to a $d-1-$ dimensional curved space. It was demonstrated that, analogous to da Costa's formalism - which applies to two-dimensional curved surfaces embedded in three- dimensional flat space - there exists a geometric potential, given in (34), that depends on $\Gamma^{(1)} = K^{(1)}$ and $\Gamma^{(2)} = \left(K^{(1)}\right)^2 - K^{(2)}$. Here, $K^{(1)} = k_i^i$ and $K^{(2)} = k_i^j k_j^i$ are invariants of the hypersurface (the term used for the embedded curved space), and k_i^j is the Weingarten tensor. For $d = 3$, the hypersurface reduces to a two-dimensional curved surface, where $\Gamma^{(1)} = K$ represents the total curvature, and $\frac{1}{2}\Gamma^{(2)} = \det\left(k_i^j\right) = K_G$ corresponds to the Gaussian curvature of the surface. In this case, the geometric potential (34) reproduces da Costa's potential. For dimensions $d > 3$, while $\Gamma^{(1)} = k_i^i$ remains the trace of the Weingarten tensor, $\frac{1}{2}\Gamma^{(2)}$ no longer represents its determinant. Consequently, the local SE differs from that in $d = 3$. We applied this proposed theory to a quantum particle confined to a three-dimensional curved space, building on previous work in [22]. We have demonstrated that, while the absence of higher dimensions - as assumed in [22] - leads to no bound states for such particles, the introduction of higher dimensions results in a single bound state for this specific configuration. Additionally, we extended the example from [22] to a generic curved space near a static, spherically symmetric spacetime, such as a black hole or a star. We explicitly derived the effect of higher dimensions through a geometric potential, expressed in Eq. (81). For the curved space associated with an AdS spacetime, we formulated the Schrödinger equation (SE) and solved it for the s-wave case, obtaining the scattered wavefunction of the quantum particle. Unlike the flat space solution, which is a spherical plane wave with a constant probability density, the solution in the curved AdS spacetime is a spherical plane wave with a decreasing probability density. Furthermore, the wavelength of the spherical plane wave in the curved space is shorter than in flat space, and the presence of higher dimensions mitigates this wavelength reduction. These effects are illustrated in Fig. 5.

We conclude by posing a question for future research: If the Schrödinger equation for the electron in a hydrogen atom in three-dimensional flat space with a central Coulomb potential is replaced by the same electron in a curved space (without the central potential), how would embedding this curved space into a four-dimensional flat space affect the hydrogen spectrum? Addressing this question would require identifying a three-dimensional curved space where the Schrödinger equation for a free electron yields an energy spectrum identical to that of the standard hydrogen atom. Once such a curved space is defined, embedding it into a four-dimensional flat bulk space would allow us to explore the influence of higher dimensions. To our knowledge, such a curved space has not yet been defined or studied in the literature, and we leave this intriguing problem open for future investigation.

Data Availability Statement This manuscript has no associated data. [Author's comment: No datasets were generated or analyzed during this study, so data sharing is not applicable to this article.].

Code Availability Statement This manuscript has no associated code/software. [Author's comment: The sharing of code or software is not relevant to this article, as no code or software was developed or analyzed during the course of this study.].

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