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Special Issue

Advances in Applied Algebra and Related Topics

Edited by

Dr. Filipa Soares de Almeida and Dr. Fabio Calderola



<https://doi.org/10.3390/axioms15020110>

Article

Polycyclic Codes and Their Application in Constructing AEAQECCs

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Abstract

In this article, we study polycyclic codes over the ring $R = \mathbb{F}_q[v]/\langle v^2 - 1 \rangle$, where $q = p^m$ with p being an odd prime. First, we introduce polycyclic codes and sequential codes over R , and characterize the structural properties of these polycyclic codes. Next, we analyze the Euclidean dual codes, annihilator dual codes, annihilator self-orthogonal codes, and annihilator linear complementary dual (LCD) codes associated with this family of codes. Finally, some asymmetric entanglement-assisted quantum error-correcting codes (AEAQECCs) are constructed from polycyclic codes over R . Moreover, the parameters of our AEAQECCs are new in the existing literature.

Keywords: polycyclic codes; euclidean dual codes; annihilator dual codes; AEAQECCs

MSC: 94B05; 94B15

1. Introduction

A cyclic code of length n over a finite field \mathbb{F} can be regarded as an ideal of the quotient ring $\mathbb{F}[x]/\langle x^n - 1 \rangle$. It is natural to replace $x^n - 1$ with some other monic polynomial $f(x)$; the ideals of the quotient ring $\mathbb{F}[x]/\langle f(x) \rangle$ also form a code, called a polycyclic code. Therefore, polycyclic codes are a generalization of cyclic and constacyclic codes. In 1972, Peterson introduced pseudo-cyclic codes [1]. Later, López-Permouth et al. [2] redefined this kind of code from the point of view of linear algebra and called them polycyclic codes. There are many papers on polycyclic codes over finite fields and rings, such as [3–11].

We now briefly survey the recent advances in this area. Most recently, Karbaski and Abualrub et al. introduced polycyclic codes over the finite field \mathbb{F}_4 in [12]. Hassan et al. [13] analyzed the algebraic structure of quasi-polycyclic codes over \mathbb{F}_q . Due to the rich algebraic properties of codes over non-commutative rings and the greater flexibility in considering multiple possibilities, skew polycyclic codes have been used to produce new and better error-correcting codes and quantum codes. Patel et al. [14] and Yadav et al. [15] conducted a systematic investigation of skew polycyclic codes and their duals, while Bag et al. [16] performed an in-depth characterization of the algebraic properties and generator structures of skew quasi-cyclic codes over \mathbb{F}_q .

Quantum error-correcting codes, built upon stabilizer theory, enabling error detection and correction via the construction of code spaces with specific algebraic structures [17], are a critical tool for mitigating decoherence in quantum computing and communication. Among these, entanglement-assisted quantum error-correcting codes (EAQECCs)—a core technology for the reliable transmission and storage of quantum information—represent a major breakthrough in this field. Brun et al. [18] first proposed EAQECCs in 2006, which



Academic Editor: Cristina Flaut

Received: 24 December 2025

Revised: 25 January 2026

Accepted: 28 January 2026

Published: 2 February 2026

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incorporate pre-shared entangled resources and which can effectively suppress the effects of decoherence and quantum noise on quantum information. Using algebraic methods, researchers have constructed a large number of novel EAQECCs based on various classical codes [19–24], thereby driving the rapid advancement of this field. Previous work on the construction of EAQECCs has centered on symmetric quantum channels. Nevertheless, the inherent asymmetric nature of errors in practical quantum systems imposes constraints on the utility of such codes [25]. This practical limitation has motivated the transition of research focus from symmetric to physically relevant asymmetric channels, thus paving the way for the investigation of asymmetric entanglement-assisted quantum error-correcting codes (AEAQECCs). Galindo et al. [26] derived explicit parameters for AEAQECCs constructed from BCH codes, whereas Liu et al. [27,28] presented two distinct construction strategies: one based on matrix-produced codes derived from constacyclic codes, and another that yields three new families of AEAQECCs with improved parameters via Vandermonde matrices, extended generalized Reed–Solomon (GRS) codes, and cyclic codes.

Compared to finite fields or chain rings, non-chain rings provide a richer algebraic setting, which can offer additional design freedom for code constructions. Moreover, for many such rings there exist Gray maps that relate ring-linear codes to classical codes over finite fields while preserving useful structure. Focusing on the finite non-chain ring $R = \mathbb{F}_q[v]/\langle v^2 - 1 \rangle$, we are motivated by the notion of ℓ -linear complementary pairs (ℓ -LCP) over \mathbb{F}_q introduced in [29] for EAQECC constructions, and we extend the ℓ -LCP notion to codes over R . In contrast to [29], we employ ℓ -LCP codes over R to construct AEAQECCs, rather than EAQECCs. Motivated also by recent chain-ring-based entanglement-assisted quantum code constructions [30], we investigate polycyclic codes over R as a concrete class of ring-linear codes with useful ℓ -LCP properties. In particular, the parameter families reported in Section 5 (see the parameter summary and comparisons therein) compare favorably with representative chain-ring-based constructions, indicating practical advantages of working over R in these regimes and illustrating the flexibility of our generalized framework.

The main contributions of this paper are summarized as follows:

1. We determine the algebraic structure of polycyclic codes over R .
2. We provide a detailed description of the annihilator dual codes, annihilator self-orthogonal codes, and annihilator LCD (linear complementary dual) codes of polycyclic codes over R .
3. We construct some new AEAQECCs (asymmetric entanglement-assisted quantum error-correcting codes) from (ℓ, λ) -LCP (linear complementary pairs) of polycyclic codes over R .

This paper is organized as follows: In Section 2, we review basic notations and fundamental concepts. In Section 3, we introduce polycyclic codes over R and analyze their structural properties. In Section 4, we define and discuss the annihilator dual codes of polycyclic codes over R . In Section 5, AEAQECCs are derived from (ℓ, λ) -LCP of polycyclic codes over R .

2. Preliminaries

2.1. Notations

We fix some notations used in the sequel unless otherwise stated.

- $q = p^m$, where p is an odd prime and $m \geq 1$.
- \mathbb{F}_q : The finite field with q elements.
- \mathbb{F}_q^* : The multiplicative group of nonzero elements in \mathbb{F}_q .

- $\dim_R(C)$: The dimension of the linear code C over R .
- $\mathbf{0}$: The zero vector $(0, 0, \dots, 0) \in R^n$.
- \mathbf{e} : The vector $(e_0, e_1, \dots, e_{n-1}) \in R^n$.
- \mathbf{e}_i : The vector $(e_{0,i}, e_{1,i}, \dots, e_{n-1,i}) \in \mathbb{F}_q^n$ for $i = 1, 2$.
- \mathbf{v}^\top : The transpose of the vector \mathbf{v} .
- $R_n = R[x]/\langle x^n - e(x) \rangle$: The quotient ring used to represent \mathbf{e} -polycyclic codes of length n over R .
- R^n : The set of all vectors of length n over the ring R .
- C^\perp : The Euclidean dual of C .
- C° : The annihilator dual of C .
- ℓ -LCP: The ℓ -linear complementary pair.
- LCD: The linear complementary dual.
- $\langle \mathbf{w}, \mathbf{v} \rangle$: Euclidean inner product on R^n .
- $\langle \mathbf{w}, \mathbf{v} \rangle_{\mathbf{e}}$: Annihilator product of \mathbf{w} and \mathbf{v} .

2.2. Some Results on Linear Codes over R

Let $q = p^m$ with p being an odd prime, and let $R = \mathbb{F}_q[v]/\langle v^2 - 1 \rangle$, where v denotes the residue class of the indeterminate v in the quotient ring, so that $v^2 = 1$ in R . For convenience, we write $R = \mathbb{F}_q + v\mathbb{F}_q$. Then, every element of R can be written uniquely as $a + bv$ with $a, b \in \mathbb{F}_q$. Since $\text{char}(\mathbb{F}_q) \neq 2$, we have $v - 1 \neq v + 1$ and $(v - 1)(v + 1) = v^2 - 1 = 0$; hence, $v - 1$ and $v + 1$ are non-zero zero divisors. Therefore R is not an integral domain. Moreover, by the Chinese Remainder Theorem, we obtain

$$R \cong \mathbb{F}_q[v]/\langle v - 1 \rangle \oplus \mathbb{F}_q[v]/\langle v + 1 \rangle \cong \mathbb{F}_q \oplus \mathbb{F}_q.$$

Consequently, R has at least two distinct maximal ideals, and therefore R is a non-chain ring. Let $\eta_1 = \frac{1-v}{2}$ and $\eta_2 = \frac{1+v}{2}$. It is straightforward to verify that these elements satisfy $\eta_i^2 = \eta_i$ for $i \in \{1, 2\}$, $\eta_i\eta_j = 0$ for $i \neq j$, and $\eta_1 + \eta_2 = 1$. Therefore, we have the decomposition $R = \eta_1R \oplus \eta_2R$. In particular, every element $r \in R$ admits a unique representation as $r = \eta_1r_1 + \eta_2r_2$ with $r_1, r_2 \in \mathbb{F}_q$ (indeed, for $r = a + bv$, one may take $r_1 = a - b$ and $r_2 = a + b$).

Let M be a 2×2 square matrix over \mathbb{F}_q satisfying $MM^\top = kI_2$, where M^\top denotes the transpose of M and $k \in \mathbb{F}_q^*$. We first define a Gray map $\phi : R \rightarrow \mathbb{F}_q^2$ by $\phi(r) = (r_1, r_2)M$ for $r = \eta_1r_1 + \eta_2r_2 \in R$. This map is \mathbb{F}_q -linear and can be extended component-wise to a map on R^n as follows:

$$\begin{aligned} \Phi : R^n &\longrightarrow \mathbb{F}_q^{2n} \\ (c_0, c_1, \dots, c_{n-1}) &\mapsto (\phi(c_0), \phi(c_1), \dots, \phi(c_{n-1})), \end{aligned}$$

where for each $j \in \{0, 1, \dots, n - 1\}$, $c_j = \eta_1c_{j,1} + \eta_2c_{j,2}$ with $c_{j,1}, c_{j,2} \in \mathbb{F}_q$ and $\phi(c_j) = (c_{j,1}, c_{j,2})M$.

The Gray weight of an element $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in R^n$ is defined as the Hamming weight of its extended Gray image, i.e.,

$$\text{wt}_G(\mathbf{c}) = \sum_{j=0}^{n-1} \text{wt}_H(\phi(c_j)), \tag{1}$$

where $\text{wt}_H(\phi(c_j))$ denotes the Hamming weight of $\phi(c_j)$.

A non-empty subset of R^n is called a code of length n over R , denoted by C . If C is an R -submodule of R^n , then C is referred to as a linear code of length n over R . Throughout this paper, all codes C are assumed to be linear.

Let C be a linear code, and let $\mathbf{w}, \mathbf{v} \in C$ be two codewords. The Gray distance between \mathbf{w} and \mathbf{v} is defined as $d_G(\mathbf{w}, \mathbf{v}) = \text{wt}_G(\mathbf{w} - \mathbf{v})$, where wt_G denotes the Gray weight defined in Equation (1). The minimum Gray distance of C is given by

$$d_G(C) = \min\{\text{wt}_G(\mathbf{c}) \mid \mathbf{0} \neq \mathbf{c} \in C\}.$$

A linear $[n, k, d]$ code is Maximum Distance Separable (MDS) if its minimum distance attains the Singleton bound $d = n - k + 1$.

Lemma 1. *Let Φ be the Gray map defined above. Then,*

1. Φ is an \mathbb{F}_q -linear, distance-preserving map from R^n (Gray distance d_G) to \mathbb{F}_q^{2n} (Hamming distance d_H).
2. If C is an (n, M, d_G) linear code over R , where M denotes the number of codewords in C , then $\Phi(C)$ is an $[2n, \log_q(M), d_H]$ linear code over \mathbb{F}_q .

Proof. The proof follows a similar argument to that of Lemmas 1 and 2 in [31]. \square

For a linear code C over R , we define the following

$$C_1 = \left\{ \mathbf{a} \in \mathbb{F}_q^n \mid \exists \mathbf{b} \in \mathbb{F}_q^n \text{ such that } \eta_1 \mathbf{a} + \eta_2 \mathbf{b} \in C \right\},$$

$$C_2 = \left\{ \mathbf{b} \in \mathbb{F}_q^n \mid \exists \mathbf{a} \in \mathbb{F}_q^n \text{ such that } \eta_1 \mathbf{a} + \eta_2 \mathbf{b} \in C \right\}.$$

It is straightforward to verify that C_1 and C_2 are linear codes of length n over \mathbb{F}_q . Furthermore, the linear code C admits a unique decomposition as $C = \eta_1 C_1 \oplus \eta_2 C_2$. Let G_1 and G_2 denote the generator matrices of C_1 and C_2 over \mathbb{F}_q , respectively. Then the generator matrix of the Gray image $\Phi(C)$ is given by

$$\begin{pmatrix} \Phi(\eta_1 G_1) \\ \Phi(\eta_2 G_2) \end{pmatrix}.$$

Let $\mathbf{v} = (v_0, v_1, \dots, v_{n-1})$ and $\mathbf{w} = (w_0, w_1, \dots, w_{n-1})$ be elements of R^n . The Euclidean inner product on R^n is defined as

$$\langle \mathbf{w}, \mathbf{v} \rangle = \sum_{j=0}^{n-1} w_j v_j,$$

where the product $w_j v_j$ is computed in the ring R . The Euclidean dual code C^\perp of a linear code C over R is defined as

$$C^\perp = \{ \mathbf{v} \in R^n \mid \langle \mathbf{w}, \mathbf{v} \rangle = 0 \text{ for all } \mathbf{w} \in C \}.$$

In [29], Liu et al. introduced ℓ -linear complementary pairs (ℓ -LCP) of codes over the finite field \mathbb{F}_q . Next, we extend this definition to the ring R .

Definition 1. *Let C and C' be two linear codes of length n over R . If $\dim_R(C \cap C') = \ell$ and $C + C' = R^n$, then we call such (C, C') an ℓ -linear complementary pair (ℓ -LCP) of codes over R .*

The next lemma characterizes ℓ -LCPs in terms of code dimensions.

Lemma 2. *Let C and C' be two linear codes of length n over R . Then (C, C') is an ℓ -LCP of codes over R if and only if $\dim_R(C) + \dim_R(C') = n + \ell$ and $\dim_R(C + C') = n$.*

Proof. It is clear that

$$\dim_R(C + C') = \dim_R(C) + \dim_R(C') - \dim_R(C \cap C').$$

By Definition 1, (C, C') is an ℓ -LCP if and only if $\dim_R(C \cap C') = \ell$ and $C + C' = R^n$, i.e.,

$$\begin{aligned} \dim_R(C \cap C') &= \dim_R(C) + \dim_R(C') - \dim_R(C + C') = \ell, \\ \dim_R(C + C') &= n. \end{aligned}$$

Thus, $\dim_R(C) + \dim_R(C') = n + \ell$. Therefore, the conclusion holds. \square

3. Polycyclic Codes over R

In this section, we introduce the definitions of polycyclic codes and sequential codes over R . Furthermore, we provide some relevant results of polycyclic codes, including their structure and Euclidean dual codes.

Definition 2. Let C be a linear code of length n over R . If there exists a vector $\mathbf{e} = (e_0, e_1, \dots, e_{n-1}) \in R^n$ such that for any $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in C$,

$$T_{\mathbf{e}}(\mathbf{c}) = (0, c_0, c_1, \dots, c_{n-2}) + c_{n-1}(e_0, e_1, \dots, e_{n-1}) \in C,$$

then C is said to be an \mathbf{e} -polycyclic code of length n over R .

In this paper, we always assume that e_0 is a unit in R .

Definition 3. Let C be a linear code of length n over R . If there exists a vector $\mathbf{e} = (e_0, e_1, \dots, e_{n-1}) \in R^n$ such that for any $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in C$,

$$S_{\mathbf{e}}(\mathbf{c}) = (c_1, \dots, c_{n-1}, c_0e_0 + c_1e_1 + \dots + c_{n-1}e_{n-1}) \in C,$$

then C is said to be an \mathbf{e} -sequential code of length n over R .

We consider the quotient ring $R_n = R[x] / \langle x^n - e(x) \rangle$, where $e(x) = \sum_{i=0}^{n-1} e_i x^i \in R[x]$. There exists a natural R -module isomorphism between R^n and R_n , which maps a vector $\mathbf{c} = (c_0, \dots, c_{n-1})$ to the polynomial $c(x) = \sum_{i=0}^{n-1} c_i x^i \in R_n$. Under this correspondence, we state the following result.

Lemma 3. A linear code C of length n over R is an \mathbf{e} -polycyclic code if and only if C is an ideal of R_n .

In the sequel, we identify each \mathbf{e} -polycyclic code of length n over R with its corresponding ideal in R_n . Let $\mathbf{e} = (e_0, e_1, \dots, e_{n-1}) \in R^n$, where each component e_j admits a decomposition $e_j = \eta_1 e_{j,1} + \eta_2 e_{j,2}$ for $j = 0, 1, \dots, n - 1$. We may thus express \mathbf{e} as a linear combination: $\mathbf{e} = \eta_1 \mathbf{e}_1 + \eta_2 \mathbf{e}_2$, where $\mathbf{e}_i = (e_{0,i}, e_{1,i}, \dots, e_{n-1,i}) \in \mathbb{F}_q^n$ for $i = 1, 2$. For brevity, we retain the notations \mathbf{e} and \mathbf{e}_i to denote the vectors $(e_0, e_1, \dots, e_{n-1})$ and $(e_{0,i}, e_{1,i}, \dots, e_{n-1,i})$, respectively, throughout the remainder of this paper. Below, we establish several fundamental results concerning \mathbf{e} -polycyclic codes over R .

Theorem 1. Let $C = \eta_1 C_1 \oplus \eta_2 C_2$ be a linear code of length n over R , where $\eta_1, \eta_2 \in R$ satisfy $\eta_1 + \eta_2 = 1, \eta_1^2 = \eta_1, \eta_2^2 = \eta_2$, and $\eta_1 \eta_2 = 0$. Then,

1. C is an \mathbf{e} -polycyclic code over R if and only if each C_i is an \mathbf{e}_i -polycyclic code over \mathbb{F}_q ;
2. C is an \mathbf{e} -sequential code over R if and only if each C_i is an \mathbf{e}_i -sequential code over \mathbb{F}_q .

Proof. The proof of 1. is given below; that of 2. is analogous.

Assume C is an \mathbf{e} -polycyclic code over R . For any $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in C$, since $C = \eta_1 C_1 \oplus \eta_2 C_2$, there exist unique vectors $\mathbf{a} = (a_0, a_1, \dots, a_{n-1}) \in C_1$ and $\mathbf{b} = (b_0, b_1, \dots, b_{n-1}) \in C_2$, such that $c_j = \eta_1 a_j + \eta_2 b_j$ for all $j = 0, 1, \dots, n - 1$. Using the decomposition $e_j = \eta_1 e_{j,1} + \eta_2 e_{j,2}$ and the idempotent properties of η_1, η_2 , we obtain

$$c_{n-1}e_j = (\eta_1 a_{n-1} + \eta_2 b_{n-1})(\eta_1 e_{j,1} + \eta_2 e_{j,2}) = \eta_1 a_{n-1}e_{j,1} + \eta_2 b_{n-1}e_{j,2}.$$

By Definition 2, $T_{\mathbf{e}}(\mathbf{c}) \in C$. Combining the decompositions $\mathbf{c} = \eta_1 \mathbf{a} + \eta_2 \mathbf{b}$ and $\mathbf{e} = \eta_1 \mathbf{e}_1 + \eta_2 \mathbf{e}_2$ with the idempotent properties of η_1 and η_2 , we derive

$$T_{\mathbf{e}}(\mathbf{c}) = \eta_1 T_{\mathbf{e}_1}(\mathbf{a}) + \eta_2 T_{\mathbf{e}_2}(\mathbf{b}).$$

From the direct sum decomposition $C = \eta_1 C_1 \oplus \eta_2 C_2$, we conclude that $T_{\mathbf{e}_1}(\mathbf{a}) \in C_1$ and $T_{\mathbf{e}_2}(\mathbf{b}) \in C_2$. Since $\mathbf{a} \in C_1$ and $\mathbf{b} \in C_2$ are arbitrary, C_1 and C_2 are respectively \mathbf{e}_1 -polycyclic and \mathbf{e}_2 -polycyclic over \mathbb{F}_q .

Conversely, let C_i be \mathbf{e}_i -polycyclic. For any $\mathbf{c} = \eta_1 \mathbf{a} + \eta_2 \mathbf{b} \in C$ with $\mathbf{a} \in C_1, \mathbf{b} \in C_2$, we compute

$$T_{\mathbf{e}}(\mathbf{c}) = \eta_1 T_{\mathbf{e}_1}(\mathbf{a}) + \eta_2 T_{\mathbf{e}_2}(\mathbf{b}) \in \eta_1 C_1 + \eta_2 C_2 = C,$$

since $T_{\mathbf{e}_i}(\mathbf{a}) \in C_i$. Hence, C is \mathbf{e} -polycyclic. \square

Theorem 2. Let C be an \mathbf{e} -polycyclic code over R . Then, its Euclidean dual code C^\perp is an \mathbf{e} -sequential code over R .

Proof. Let $\mathbf{v} \in C^\perp$. Since C is \mathbf{e} -polycyclic, then $T_{\mathbf{e}}(\mathbf{w}) \in C$ for every $\mathbf{w} \in C$. As $\mathbf{v} \in C^\perp$, we obtain

$$0 = \langle T_{\mathbf{e}}(\mathbf{w}), \mathbf{v} \rangle = \sum_{j=0}^{n-2} w_j v_{j+1} + w_{n-1} \sum_{j=0}^{n-1} e_j v_j = \langle \mathbf{w}, S_{\mathbf{e}}(\mathbf{v}) \rangle,$$

which implies $S_{\mathbf{e}}(\mathbf{v}) \in C^\perp$. By Definition 3, C^\perp is an \mathbf{e} -sequential code over R . \square

In [32], Ge et al. established the structural properties of polycyclic codes over \mathbb{F}_q . These results can be naturally extended from finite fields to general finite rings. For completeness, we first recall the key result over \mathbb{F}_q : if $f(x) \in \mathbb{F}_q[x]$ is a monic polynomial of degree $n > 1$, then every non-zero ideal C of the principal ideal ring $\mathbb{F}_q[x]/\langle f(x) \rangle$ corresponds to an \mathbf{e} -polycyclic code of length n over \mathbb{F}_q (where $f(x) = x^n - e(x)$ for the associated vector \mathbf{e}). Moreover, there exists a monic polynomial $g(x) \in \mathbb{F}_q[x]$ such that $C = \langle g(x) \rangle$ and $g(x) \mid f(x)$.

Theorem 3. Let $C = \eta_1 C_1 \oplus \eta_2 C_2$ be an \mathbf{e} -polycyclic code of length n over R . For each $i = 1, 2$, let $C_i = \langle g_i(x) \rangle$ be the ideal of $\mathbb{F}_q[x]/\langle x^n - e_i(x) \rangle$ corresponding to the \mathbf{e}_i -polycyclic code C_i over \mathbb{F}_q , where $g_i(x) \in \mathbb{F}_q[x]$ is a monic polynomial with $g_i(x) \mid (x^n - e_i(x))$, and $e_i(x) \in \mathbb{F}_q[x]$ denotes the polynomial representation of \mathbf{e}_i . Then, there exists a unique monic generator polynomial $g(x) = \eta_1 g_1(x) + \eta_2 g_2(x) \in R[x]$ such that $C = \langle g(x) \rangle$ and $g(x) \mid (x^n - e(x))$, where $e(x) = \eta_1 e_1(x) + \eta_2 e_2(x) \in R[x]$ is the polynomial representation of \mathbf{e} .

Proof. Since $C = \eta_1 C_1 \oplus \eta_2 C_2$ and $C_i = \langle g_i(x) \rangle$, every codeword has the following form:

$$c(x) = \eta_1 p_1(x)g_1(x) + \eta_2 p_2(x)g_2(x) = (\eta_1 p_1(x) + \eta_2 p_2(x))(\eta_1 g_1(x) + \eta_2 g_2(x)) = p(x)g(x),$$

where $\eta_1^2 = \eta_1, \eta_2^2 = \eta_2$, and $\eta_1\eta_2 = 0$. Thus, $C = \langle g(x) \rangle$. Moreover, since $g_i(x) \mid (x^n - e_i(x))$, write $x^n - e_i(x) = h_i(x)g_i(x)$, for $i = 1, 2$. Define $h(x) = \eta_1h_1(x) + \eta_2h_2(x)$. Then,

$$h(x)g(x) = \eta_1(x^n - e_1(x)) + \eta_2(x^n - e_2(x)) = x^n(\eta_1 + \eta_2) - (\eta_1e_1(x) + \eta_2e_2(x)) = x^n - e(x),$$

as $\eta_1 + \eta_2 = 1$. Hence, $g(x) \mid (x^n - e(x))$. Uniqueness follows from the uniqueness of $g_1(x)$ and $g_2(x)$ and the orthogonal decomposition $R = \eta_1R \oplus \eta_2R$. \square

4. Annihilator Dual Codes of Polycyclic Codes

From Theorem 2, the Euclidean dual of an \mathbf{e} -polycyclic code over R is an \mathbf{e} -sequential code. As shown in [2], such codes are polycyclic only if they are constacyclic, implying that the Euclidean dual generally fails to preserve the \mathbf{e} -polycyclic structure. To maintain structural consistency, we adopt the annihilator duality framework of Alahmadi et al. [3], which preserves the \mathbf{e} -polycyclic property. This allows us to naturally define self-orthogonal and linear complementary dual (LCD) codes within the same algebraic setting.

In this section, we define and investigate the annihilator dual codes of \mathbf{e} -polycyclic codes over R .

Definition 4. Let $\mathbf{w}, \mathbf{v} \in R^n$, and let C be an \mathbf{e} -polycyclic code of length n over R . Denote by $w(x), v(x) \in R_n = R[x]/\langle x^n - e(x) \rangle$ the polynomial representations of \mathbf{w}, \mathbf{v} .

1. The annihilator product of \mathbf{w} and \mathbf{v} is defined as follows:

$$\langle \mathbf{w}, \mathbf{v} \rangle_{\mathbf{e}} = \{w(x)v(x)\}_{\mathbf{e}}(0),$$

where $\{w(x)v(x)\}_{\mathbf{e}}$ denotes the unique remainder polynomial of $w(x)v(x)$ modulo $(x^n - e(x))$, i.e., $\{w(x)v(x)\}_{\mathbf{e}} = r(x) \in R[x]$ such that $w(x)v(x) \equiv r(x) \pmod{x^n - e(x)}$ and $\deg(r(x)) \leq n - 1$.

2. The annihilator dual code C° of C is defined as follows:

$$C^\circ = \{\mathbf{v} \in R^n \mid \langle \mathbf{w}, \mathbf{v} \rangle_{\mathbf{e}} = 0 \text{ for all } \mathbf{w} \in C\}.$$

An \mathbf{e} -polycyclic code C is called annihilator self-orthogonal if $C \subseteq C^\circ$, and annihilator LCD if $C \cap C^\circ = \{0\}$.

Lemma 4. Let $\mathbf{w}, \mathbf{v} \in R^n$. Then the annihilator product $\langle \cdot, \cdot \rangle_{\mathbf{e}}$ is a non-degenerate symmetric R -bilinear form.

Proof. We verify the properties of $\langle \cdot, \cdot \rangle_{\mathbf{e}}$ in three steps: R -bilinearity, symmetry, and non-degeneracy.

R -bilinearity: Let $\mathbf{u}, \mathbf{w}, \mathbf{v} \in R^n$ and $k_1, k_2 \in R$. By Definition 4, it is straightforward that

$$\langle k_1\mathbf{u} + k_2\mathbf{w}, \mathbf{v} \rangle_{\mathbf{e}} = r(0) = k_1r_1(0) + k_2r_2(0) = k_1\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{e}} + k_2\langle \mathbf{w}, \mathbf{v} \rangle_{\mathbf{e}}.$$

A similar argument applies to the second argument. Thus, $\langle \cdot, \cdot \rangle_{\mathbf{e}}$ is R -bilinear.

Symmetry: Since R is commutative, then

$$\langle \mathbf{w}, \mathbf{v} \rangle_{\mathbf{e}} = \{w(x)v(x)\}_{\mathbf{e}}(0) = \{v(x)w(x)\}_{\mathbf{e}}(0) = \langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{e}}.$$

Hence, the annihilator product is symmetric.

Non-degeneracy: A bilinear form is non-degenerate if $\langle \mathbf{w}, \mathbf{v} \rangle_{\mathbf{e}} = 0$ for all $\mathbf{v} \in R^n$ implies $\mathbf{w} = \mathbf{0}$. Let $w(x) = \sum_{j=0}^{n-1} w_jx^j$ and assume $\langle \mathbf{w}, \mathbf{v} \rangle_{\mathbf{e}} = 0$ for all $\mathbf{v} \in R^n$. We show $w_j = 0$ for all j by induction. For $j = 0$, take $v(x) = 1$. Then, $\langle \mathbf{w}, \mathbf{v} \rangle_{\mathbf{e}} = w(0) = w_0 = 0$.

Assume $w_0 = \dots = w_{t-1} = 0$ for some $t \geq 1$, so $w(x) = x^t \tilde{w}(x)$ with $\tilde{w}(x) \in R[x]$. Since e_0 is a unit in R , define

$$v(x) = \left(-\frac{1}{e_0} \sum_{j=0}^{n-2} e_{j+1} x^j + \frac{1}{e_0} x^{n-1} \right)^t.$$

By construction, $x^t v(x) \equiv 1 \pmod{x^n - e(x)}$, so

$$w(x)v(x) = x^t \tilde{w}(x)v(x) \equiv \tilde{w}(x) \pmod{x^n - e(x)}.$$

Hence,

$$0 = \langle \mathbf{w}, \mathbf{v} \rangle_{\mathbf{e}} = \tilde{w}(0) = w_t,$$

which completes the induction. Thus, $\mathbf{w} = \mathbf{0}$, and the form is non-degenerate. \square

Lemma 5. Let C be an \mathbf{e} -polycyclic code of length n over R . Let $A = (\langle \epsilon_i, \epsilon_j \rangle_{\mathbf{e}})_{1 \leq i, j \leq n}$, where $\epsilon_i \in R^n$ is the standard basis vector with one in the i -th position and zero elsewhere. Define $C \cdot A = \{ \mathbf{c}A \mid \mathbf{c} \in C \}$. Then, $C^\circ = (C \cdot A)^\perp$, where $(\cdot)^\perp$ denotes the Euclidean dual code.

Proof. Let $\mathbf{w} = (w_0, w_1, \dots, w_{n-1})$ and $\mathbf{v} = (v_0, v_1, \dots, v_{n-1}) \in R^n$. Since $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$ is an R -module basis of R^n , we can write $\mathbf{w} = \sum_{j=0}^{n-1} w_j \epsilon_{j+1}$ and $\mathbf{v} = \sum_{k=0}^{n-1} v_k \epsilon_{k+1}$. By Lemma 4, $\langle \cdot, \cdot \rangle_{\mathbf{e}}$ is a symmetric R -bilinear form, so we expand the annihilator product as follows:

$$\begin{aligned} \langle \mathbf{w}, \mathbf{v} \rangle_{\mathbf{e}} &= \sum_{j=0}^{n-1} w_j v_0 \langle \epsilon_{j+1}, \epsilon_1 \rangle_{\mathbf{e}} + \sum_{j=0}^{n-1} w_j v_1 \langle \epsilon_{j+1}, \epsilon_2 \rangle_{\mathbf{e}} + \sum_{j=0}^{n-1} w_j v_{n-1} \langle \epsilon_{j+1}, \epsilon_n \rangle_{\mathbf{e}} \\ &= (w_0, w_1, \dots, w_{n-1}) A (v_0, v_1, \dots, v_{n-1})^\top \\ &= \mathbf{w} A \mathbf{v}^\top = \langle \mathbf{w} A, \mathbf{v} \rangle. \end{aligned}$$

By Definition 4, we obtain

$$C^\circ = \{ \mathbf{v} \in R^n \mid \langle \mathbf{w} A, \mathbf{v} \rangle = 0 \text{ for all } \mathbf{w} \in C \}.$$

By the definition of the Euclidean dual code, the right-hand side is exactly $(C \cdot A)^\perp$. Thus, $C^\circ = (C \cdot A)^\perp$. \square

Theorem 4. Let $C = \eta_1 C_1 \oplus \eta_2 C_2$ be an \mathbf{e} -polycyclic code of length n over R . Then $C^\circ = \eta_1 C_1^\circ \oplus \eta_2 C_2^\circ$.

Proof. Let A be the matrix of the annihilator product defined in Lemma 5. By the idempotent decomposition of R , $A = \eta_1 A_1 \oplus \eta_2 A_2$, where each A_i is an $n \times n$ matrix over \mathbb{F}_q .

By Lemma 5, $C^\circ = (C \cdot A)^\perp$. Substituting the decompositions of C and A , we obtain

$$C \cdot A = (\eta_1 C_1 \oplus \eta_2 C_2) \cdot (\eta_1 A_1 \oplus \eta_2 A_2).$$

Using the idempotent properties $\eta_1^2 = \eta_1$, $\eta_2^2 = \eta_2$, and $\eta_1 \eta_2 = 0$, this simplifies to

$$C \cdot A = \eta_1 (C_1 A_1) \oplus \eta_2 (C_2 A_2).$$

The Euclidean dual of a direct sum is the direct sum of the Euclidean duals, thus,

$$(C \cdot A)^\perp = \eta_1 (C_1 A_1)^\perp \oplus \eta_2 (C_2 A_2)^\perp.$$

By Lemma 5, $(C_i A_i)^\perp = C_i^\circ$, and so

$$C^\circ = \eta_1 C_1^\circ \oplus \eta_2 C_2^\circ.$$

This completes the proof. \square

Theorem 5. Let $C = \eta_1 C_1 \oplus \eta_2 C_2$ be an \mathbf{e} -polycyclic code of length n over R . Then,

1. C is annihilator self-orthogonal if and only if both C_1 and C_2 are annihilator self-orthogonal over \mathbb{F}_q ;
2. C is annihilator LCD if and only if both C_1 and C_2 are annihilator LCD over \mathbb{F}_q .

Proof.

1. Assume C is annihilator self-orthogonal, so $C \subseteq C^\circ$. By Theorem 4, $C^\circ = \eta_1 C_1^\circ \oplus \eta_2 C_2^\circ$; hence,

$$\eta_1 C_1 \oplus \eta_2 C_2 \subseteq \eta_1 C_1^\circ \oplus \eta_2 C_2^\circ.$$

Multiplying both sides by η_1 , we obtain $\eta_1 C_1 \subseteq \eta_1 C_1^\circ$. Since $\eta_1 C_1$ and $\eta_1 C_1^\circ$ are disjoint from $\eta_2 R^n$, this implies $C_1 \subseteq C_1^\circ$. Similarly, multiplying by η_2 results in $C_2 \subseteq C_2^\circ$. Thus, C_1 and C_2 are annihilator self-orthogonal.

Suppose $C_1 \subseteq C_1^\circ$ and $C_2 \subseteq C_2^\circ$. Then, $\eta_1 C_1 \subseteq \eta_1 C_1^\circ$ and $\eta_2 C_2 \subseteq \eta_2 C_2^\circ$, so their direct sum satisfies the following:

$$\eta_1 C_1 \oplus \eta_2 C_2 \subseteq \eta_1 C_1^\circ \oplus \eta_2 C_2^\circ = C^\circ.$$

Hence, $C \subseteq C^\circ$, so C is annihilator self-orthogonal.

2. If C is annihilator LCD, then $C \cap C^\circ = \{0\}$. By Theorem 4, this implies $(\eta_1 C_1 \cap \eta_1 C_1^\circ) \oplus (\eta_2 C_2 \cap \eta_2 C_2^\circ) = \{0\}$, so $C_1 \cap C_1^\circ = \{0\}$ and $C_2 \cap C_2^\circ = \{0\}$. Conversely, if C_1 and C_2 are annihilator LCD, then $C \cap C^\circ = (\eta_1 C_1 \cap \eta_1 C_1^\circ) \oplus (\eta_2 C_2 \cap \eta_2 C_2^\circ) = \{0\}$, so C is annihilator LCD.

\square

Theorem 6. Let $C = \eta_1 C_1 \oplus \eta_2 C_2$ be an \mathbf{e} -polycyclic code of length n over R , with generator polynomial $g(x) = \eta_1 g_1(x) + \eta_2 g_2(x)$. For each $i = 1, 2$, let $g_i(x) \mid (x^n - e_i(x))$ and write $x^n - e_i(x) = h_i(x)g_i(x)$. Then, the annihilator dual code C° is an \mathbf{e} -polycyclic code over R with generator polynomial $h(x) = \eta_1 h_1(x) + \eta_2 h_2(x)$.

Proof. By Theorem 1, C_i is an \mathbf{e}_i -polycyclic code over \mathbb{F}_q , so $C_i = \langle g_i(x) \rangle$ as an ideal of $\mathbb{F}_q[x]/\langle x^n - e_i(x) \rangle$. According to Lemma 3.2 in [33], the annihilator dual of a polycyclic code over a finite field is also polycyclic, and its generator polynomial is $h_i(x) = \frac{x^n - e_i(x)}{g_i(x)}$, i.e., $C_i^\circ = \langle h_i(x) \rangle$.

By Theorem 4, we obtain $C^\circ = \eta_1 C_1^\circ \oplus \eta_2 C_2^\circ$. Following the same reasoning as in Theorem 3, the direct sum of the ideals $\eta_1 \langle h_1(x) \rangle$ and $\eta_2 \langle h_2(x) \rangle$ is the ideal generated by $h(x) = \eta_1 h_1(x) + \eta_2 h_2(x)$. Thus,

$$C^\circ = \langle \eta_1 h_1(x) + \eta_2 h_2(x) \rangle = \langle h(x) \rangle.$$

Moreover, since $h(x) \mid (x^n - e(x))$, as shown in the proof of Theorem 3, C° is an \mathbf{e} -polycyclic code over R . \square

Using the notation established in Theorem 6, we now characterize annihilator self-orthogonal and LCD polycyclic codes.

Theorem 7. Let $C = \langle \eta_1 g_1(x) + \eta_2 g_2(x) \rangle$ be an \mathbf{e} -polycyclic code of length n over R , with $x^n - e_i(x) = h_i(x)g_i(x)$ for $i = 1, 2$. Then,

1. C is annihilator self-orthogonal if and only if $h_i(x) \mid g_i(x)$ for each $i = 1, 2$;
2. C is annihilator LCD if and only if $\gcd(h_i(x), g_i(x)) = 1$ for each $i = 1, 2$.

Proof.

1. By Theorem 5, C is annihilator self-orthogonal if and only if C_1 and C_2 are annihilator self-orthogonal over \mathbb{F}_q . By [33] (Theorem 2.6), a polycyclic code $C_i = \langle g_i(x) \rangle$ over \mathbb{F}_q is annihilator self-orthogonal if and only if $h_i(x) \mid g_i(x)$. Hence, the result follows.
2. Similarly, by Theorem 5, C is annihilator LCD if and only if C_1 and C_2 are annihilator LCD over \mathbb{F}_q . By [33] (Theorem 2.8), C_i is annihilator LCD if and only if $\gcd(h_i(x), g_i(x)) = 1$. Thus, the result holds.

□

At the end of this section, using Lemma 1 and Theorems 3, 6, and 7, we construct explicit examples of annihilator LCD codes from polycyclic codes over R via the computational algebra system Magma [34].

Example 1. Let $R = \mathbb{F}_{11^2} + v\mathbb{F}_{11^2}$ with $v^2 = 1$, where $\mathbb{F}_{11^2} = \mathbb{F}_{11}[w]$ and w satisfies the irreducible polynomial $w^2 + 7w + 2 = 0$ over \mathbb{F}_{11} . Recall the idempotent decomposition $\eta_1 = \frac{1-v}{2}$ and $\eta_2 = \frac{1+v}{2}$. Let the vector $\mathbf{e} \in R^5$ have the polynomial representation $e(x) = \eta_1 e_1(x) + \eta_2 e_2(x)$, where $e_1(x) = 5x^4 + 10x + 5$ and $e_2(x) = 9x^4 + 10x^2 + 8x + 9 \in \mathbb{F}_{11^2}[x]$. We first factor the polynomials $x^5 - e_i(x)$ over $\mathbb{F}_{11^2}[x]$:

$$\begin{aligned} x^6 - e_1(x) &= x^5 - 5x^4 - 10x - 5 = (x^4 + 1)(x + 6), \\ x^6 - e_2(x) &= x^5 - 9x^4 - 10x^2 - 8x - 9 = (x^4 + x + 1)(x + 2). \end{aligned}$$

Let $C = \langle g(x) \rangle$ be an \mathbf{e} -polycyclic code over R , where the generator polynomial is $g(x) = \eta_1 g_1(x) + \eta_2 g_2(x)$ with $g_1(x) = x + 6$ and $g_2(x) = x + 2$. By Theorem 6, the annihilator dual codes of $C_1 = \langle g_1(x) \rangle$ and $C_2 = \langle g_2(x) \rangle$ are

$$C_1^\circ = \left\langle \frac{x^6 - e_1(x)}{g_1(x)} \right\rangle = \langle x^4 + 1 \rangle, \quad C_2^\circ = \left\langle \frac{x^6 - e_2(x)}{g_2(x)} \right\rangle = \langle x^4 + x + 1 \rangle.$$

The following computes the greatest common divisors over $\mathbb{F}_{11^2}[x]$:

$$\gcd(x^4 + 1, x + 6) = 1, \quad \gcd(x^4 + x + 1, x + 2) = 1.$$

By Theorem 7, C_1 and C_2 are annihilator LCD codes over \mathbb{F}_{11^2} , so $C = \eta_1 C_1 \oplus \eta_2 C_2$ is an annihilator LCD code of length six over R .

Let $M = \begin{pmatrix} 10 & 2 \\ 2 & 1 \end{pmatrix} \in GL_2(\mathbb{F}_{11})$. Applying the Gray map Φ (defined in Lemma 1) and Magma computations [34], the image $\Phi(C)$ is a linear code over \mathbb{F}_{11^2} , with parameters $[10, 8, \text{and } 3]_{11^2}$. This code is an MDS code.

Remark 1. Tables 1 and 2 present additional examples of annihilator LCD codes constructed from \mathbf{e} -polycyclic codes $C = \langle \eta_1 g_1(x) + \eta_2 g_2(x) \rangle$ over rings $\mathbb{F}_{5^2} + v\mathbb{F}_{5^2}$ and $\mathbb{F}_{7^2} + v\mathbb{F}_{7^2}$, respectively. For notation simplicity, 0^n denotes the string of n consecutive zeros, a polynomial $a_t x^t + a_{t-1} x^{t-1} + \dots + a_0 \in \mathbb{F}_{q^2}[x]$ is abbreviated as $a_t a_{t-1} \dots a_0$, where the coefficients are represented by their field elements (e.g., $x^5 + wx^3 + 1$ is written as $10w0^21$, with 0^2 indicating two consecutive zero coefficients), and $f_i(x) = x^n - e_i(x)$ for $i = 1, 2$ (polynomials defining the quotient rings for C_1 and C_2).

For Table 1, we use the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \in GL_2(\mathbb{F}_5)$, which satisfies $MM^\top = 2I_2$.

For Table 2, the matrix $M = \begin{pmatrix} 6 & 2 \\ 2 & 1 \end{pmatrix} \in GL_2(\mathbb{F}_7)$ satisfies $MM^\top = 5I_2$. All the constructed annihilator LCD codes in these tables are MDS codes.

It is worth noting that the construction of MDS LCD codes over the ring R is significant in two respects: on one hand, the MDS property ensures optimal error correction performance for a given dimension; on the other hand, the LCD structure endows these codes with inherent suitability for secure coding applications resistant to side channel attacks.

Table 1. Annihilator LCD codes from polycyclic codes over $\mathbb{F}_{5^2} + v\mathbb{F}_{5^2}$.

| (f_1, g_1) | (f_2, g_2) | $\Phi(C)$ | Type |
|--------------------------------|----------------------------------|---------------------|------|
| $(1w1w, 1w)$ | $(14w^7w^4, 1w^{10}w^3)$ | $[6, 3, 4]_{5^2}$ | MDS |
| $(1w^{13}01w^{13}, 1w^{13})$ | $(1w^5w^{13}w^{11}w^9, 1w^5w^3)$ | $[8, 5, 4]_{5^2}$ | MDS |
| $(1w^{13}0w^{13}w^1, 1w^{13})$ | $(1w^31w^3w^4, 1w^3)$ | $[10, 8, 3]_{5^2}$ | MDS |
| $(140^3w^{13}, 1w^{12})$ | $(1w^70^21w^{16}w^9, 1w^7)$ | $[12, 10, 3]_{5^2}$ | MDS |
| $(1w^40^41w^4, 1w^4)$ | $(1w^30^41w^3, 1w^3)$ | $[14, 12, 3]_{5^2}$ | MDS |
| $(1w^40^51w^4, 1w^4)$ | $(1w^90^52w^{15}, 1w^9)$ | $[16, 14, 3]_{5^2}$ | MDS |

Table 2. Annihilator LCD codes from polycyclic codes over $\mathbb{F}_{7^2} + v\mathbb{F}_{7^2}$.

| (f_1, g_1) | (f_2, g_2) | $\Phi(C)$ | Type |
|--------------------------------|-------------------------------------|---------------------|------|
| $(1515, 15)$ | $(1633, 13)$ | $[6, 4, 3]_{7^2}$ | MDS |
| $(1w^{13}0^21w^{13}, 1w^3)$ | $(1w^{19}1w^41w^4, 1w^4)$ | $[10, 8, 3]_{7^2}$ | MDS |
| $(1w^2w^{18}w^3, 1w)$ | $(1w^92w^{19}w^5w^{20}w, 1w^{19}1)$ | $[12, 9, 4]_{7^2}$ | MDS |
| $(1w^2w^{18}w^3, 1w)$ | $(110^322, 11)$ | $[12, 10, 3]_{7^2}$ | MDS |
| $(110^411, 11)$ | $(1w^70^41w^7, 1w^7)$ | $[14, 12, 3]_{7^2}$ | MDS |
| $(1w^{13}0^51w^{14}, 1w^{13})$ | $(1w^{14}1w^{14}, 1w^{14})$ | $[16, 14, 3]_{7^2}$ | MDS |

5. AEAQECCs from (ℓ, λ) -LCP of Polycyclic Codes

In this section, we focus on constructing AEAQECCs from \mathbf{e} -polycyclic codes, where $\mathbf{e} = (\lambda, 0, \dots, 0)$ and λ is a unit in R . For brevity, we henceforth refer to such codes as λ -constacyclic codes. Building on Definition 1 of ℓ -LCP for linear codes, we specialize this notion to λ -constacyclic codes and introduce the following definition.

Definition 5. Let C and C' be λ -constacyclic codes of length n over R . If (C, C') forms an ℓ -LCP of codes over R , then (C, C') is called an (ℓ, λ) -LCP of constacyclic codes of length n over R .

Lemma 6. Let $\lambda = \lambda_1 + \lambda_2v$ be an element of R . Then, $\lambda = \lambda_1 + \lambda_2v$ is a unit over R if and only if $\lambda_1 - \lambda_2$ and $\lambda_1 + \lambda_2$ are units of \mathbb{F}_q^* .

Proof. Let $\lambda = \lambda_1 + \lambda_2v = (\lambda_1 - \lambda_2)\eta_1 + (\lambda_1 + \lambda_2)\eta_2$. If λ is a unit, there exists $\mu = a\eta_1 + b\eta_2 \in R$ such that $\lambda\mu = 1 = \eta_1 + \eta_2$. By $\eta_i\eta_j = 0$ for $i \neq j$, $\lambda\mu = (\lambda_1 - \lambda_2)a\eta_1 + (\lambda_1 + \lambda_2)b\eta_2$. According to the uniqueness of decomposition, we obtain $(\lambda_1 - \lambda_2)a = 1$ and $(\lambda_1 + \lambda_2)b = 1$, which implies $\lambda_1 \pm \lambda_2 \in \mathbb{F}_q^*$.

Conversely, let $(\lambda_1 - \lambda_2)^{-1}, (\lambda_1 + \lambda_2)^{-1} \in \mathbb{F}_q^*$. Define $\mu = (\lambda_1 - \lambda_2)^{-1}\eta_1 + (\lambda_1 + \lambda_2)^{-1}\eta_2$. Then, $\lambda\mu = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2)^{-1}\eta_1 + (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2)^{-1}\eta_2 = \eta_1 + \eta_2 = 1$.

Thus, λ is a unit if and only if $\lambda_1 \pm \lambda_2 \in \mathbb{F}_q^*$. \square

Lemma 7. Let $C = \eta_1 C_1 \oplus \eta_2 C_2$ be a linear code of length n over R . Then, C is a $(\lambda_1 + \lambda_2 v)$ -constacyclic code of length n over R if and only if C_1 and C_2 are $(\lambda_1 - \lambda_2)$ -constacyclic codes and $(\lambda_1 + \lambda_2)$ -constacyclic codes of length n over \mathbb{F}_q , respectively.

Proof. Since $\lambda = \lambda_1 + \lambda_2 v = (\lambda_1 - \lambda_2)\eta_1 + (\lambda_1 + \lambda_2)\eta_2$, then the result follows immediately from Theorem 1. \square

From the Theorem 3 and Lemma 7, the following conclusion is readily obtained.

Theorem 8. Let $C = \eta_1 C_1 \oplus \eta_2 C_2$ be a $(\lambda_1 + \lambda_2 v)$ -constacyclic code of length n over R . Then there exists a polynomial $g(x) \in R[x]$ with $g(x) \mid (x^n - (\lambda_1 + \lambda_2 v))$ such that $C = \langle g(x) \rangle$, where $g(x) = \eta_1 g_1(x) + \eta_2 g_2(x)$; $g_1(x)$ is the generator polynomial of the $(\lambda_1 - \lambda_2)$ -constacyclic code C_1 of length n over \mathbb{F}_q and $g_2(x)$ is the generator polynomial of the $(\lambda_1 + \lambda_2)$ -constacyclic code C_2 of length n over \mathbb{F}_q . Moreover, $|C| = q^{2n - \sum_{i=1}^2 \deg(g_i(x))}$.

Lemma 8 ([35]). Let $C = \eta_1 C_1 \oplus \eta_2 C_2$ be a linear code of length n over R . Then, $C^\perp = \eta_1 C_1^\perp \oplus \eta_2 C_2^\perp$.

Let $h_1(x) = \frac{x^n - (\lambda_1 - \lambda_2)}{g_1(x)}$ and $h_2(x) = \frac{x^n - (\lambda_1 + \lambda_2)}{g_2(x)}$. Let $\tilde{h}(x) = x^{\deg(h(x))} h(x^{-1})$ be the reciprocal polynomial of $h(x)$. In light of Lemmas 6 and 8, and Theorem 8, we obtain the following result.

Theorem 9. Let $C = \langle \eta_1 g_1(x) + \eta_2 g_2(x) \rangle$ be a $(\lambda_1 + \lambda_2 v)$ -constacyclic code of length n over R . Then, C^\perp is also a $(\lambda_1 + \lambda_2 v)^{-1}$ -constacyclic code of length n over R , where $(\lambda_1 + \lambda_2 v)^{-1} = \eta_1 (\lambda_1 - \lambda_2)^{-1} + \eta_2 (\lambda_1 + \lambda_2)^{-1}$. Moreover, $C^\perp = \langle \eta_1 \tilde{h}_1(x) + \eta_2 \tilde{h}_2(x) \rangle$, where $\tilde{h}_1(x)$ and $\tilde{h}_2(x)$ are reciprocal polynomials of $h_1(x)$ and $h_2(x)$, respectively. Moreover, $|C^\perp| = q^{\sum_{i=1}^2 \deg(g_i(x))}$.

Lemma 9. Let $C = \langle \eta_1 g_1(x) + \eta_2 g_2(x) \rangle$ be a $(\lambda_1 + \lambda_2 v)$ -constacyclic code of length n over R . Then,

1. $\Phi(C)$ is a linear code over \mathbb{F}_q with parameters $[2n, k, d]$, where $k = 2n - \sum_{i=1}^2 \deg(g_i(x))$ and $d = \min\{d_H(\langle g_1(x) \rangle), d_H(\langle g_2(x) \rangle)\}$.
2. $\Phi(C^\perp)$ is a linear code over \mathbb{F}_q with parameters $[2n, k', d']$, where $k' = \sum_{i=1}^2 \deg(g_i(x))$ and $d' = \min\{d_H(\langle \tilde{h}_1(x) \rangle), d_H(\langle \tilde{h}_2(x) \rangle)\}$.

Proof. The conclusion follows directly from Lemma 1, and Theorems 8 and 9. \square

Theorem 10. Let $C = \eta_1 C_1 \oplus \eta_2 C_2$ and $C' = \eta_1 C'_1 \oplus \eta_2 C'_2$ be two linear codes of length n over R . Then, (C, C') is an ℓ -LCP of codes of length n over R if and only if $\sum_{i=1}^2 \dim_{\mathbb{F}_q}(C_i \cap C'_i) = 2\ell$ and $\sum_{i=1}^2 \dim_{\mathbb{F}_q}(C_i + C'_i) = 2n$.

Proof. Since $C \cap C' = \eta_1(C_1 \cap C'_1) \oplus \eta_2(C_2 \cap C'_2)$ and $C + C' = \eta_1(C_1 + C'_1) \oplus \eta_2(C_2 + C'_2)$, we obtain

$$\log_{|R|} |C \cap C'| = \sum_{i=1}^2 \log_{|R|} |C_i \cap C'_i|,$$

$$\log_{|R|} |C + C'| = \sum_{i=1}^2 \log_{|R|} |C_i + C'_i|.$$

Thus,

$$\dim_R(C \cap C') = \frac{\sum_{i=1}^2 \log_q |C_i \cap C'_i|}{\log_q |R|} = \frac{\sum_{i=1}^2 \dim_{\mathbb{F}_q}(C_i \cap C'_i)}{2},$$

$$\dim_R(C + C') = \frac{\sum_{i=1}^2 \log_q |C_i + C'_i|}{\log_q |R|} = \frac{\sum_{i=1}^2 \dim_{\mathbb{F}_q}(C_i + C'_i)}{2}.$$

By Lemma 2, (C, C') is an ℓ -LCP of codes of length n over R if and only if

$$\frac{\sum_{i=1}^2 \dim_{\mathbb{F}_q}(C_i \cap C'_i)}{2} = \ell \quad \text{and} \quad \frac{\sum_{i=1}^2 \dim_{\mathbb{F}_q}(C_i + C'_i)}{2} = n,$$

which is equivalent to

$$\sum_{i=1}^2 \dim_{\mathbb{F}_q}(C_i \cap C'_i) = 2\ell \quad \text{and} \quad \sum_{i=1}^2 \dim_{\mathbb{F}_q}(C_i + C'_i) = 2n.$$

□

Lemma 10 ([29], Theorem 3.2). *Let $C = \langle g(x) \rangle$ and $C' = \langle u(x) \rangle$ be two λ -constacyclic codes of length n over \mathbb{F}_q . Then, (C, C') is an (ℓ, λ) -LCP of constacyclic codes if and only if $\deg(\text{lcm}(g(x), u(x))) = n - \ell$ and $\gcd(g(x), u(x)) = 1$ in $\mathbb{F}_q[x]$.*

Theorem 11. *Let $C = \langle \eta_1 g_1(x) + \eta_2 g_2(x) \rangle$ and $C' = \langle \eta_1 g'_1(x) + \eta_2 g'_2(x) \rangle$ be λ -constacyclic codes of length n over R , where $\lambda = \lambda_1 + \lambda_2 v$. The polynomial factorizations are given by*

$$\begin{aligned} x^n - (\lambda_1 - \lambda_2) &= h_1(x)g_1(x) = h'_1(x)g'_1(x), \\ x^n - (\lambda_1 + \lambda_2) &= h_2(x)g_2(x) = h'_2(x)g'_2(x). \end{aligned}$$

For $i = 1, 2$, assume $\deg(\text{lcm}(g_i(x), g'_i(x))) = n - \ell$ and $\gcd(g_i(x), g'_i(x)) = 1$ in $\mathbb{F}_q[x]$, where $\ell = n - \deg(g_i(x)) - \deg(g'_i(x))$. Then, $(\Phi(C), \Phi(C'))$ forms a 2ℓ -LCP of codes of length $2n$ over \mathbb{F}_q .

Proof. Let $C_i = \langle g_i(x) \rangle$ and $C'_i = \langle g'_i(x) \rangle$ for $i = 1, 2$. By the structure of codes over R , we obtain $C = \eta_1 C_1 \oplus \eta_2 C_2$ and $C' = \eta_1 C'_1 \oplus \eta_2 C'_2$. Assume $\deg(\text{lcm}(g_i(x), g'_i(x))) = n - \ell$ and $\gcd(g_i(x), g'_i(x)) = 1$ in $\mathbb{F}_q[x]$. By Lemma 10, (C_1, C'_1) forms an $(\ell, \lambda_1 - \lambda_2)$ -LCP and (C_2, C'_2) forms an $(\ell, \lambda_1 + \lambda_2)$ -LCP of constacyclic codes over \mathbb{F}_q . This implies $\dim_{\mathbb{F}_q}(C_i \cap C'_i) = \ell$ and $\dim_{\mathbb{F}_q}(C_i + C'_i) = n$ for each i . Thus,

$$\sum_{i=1}^2 \dim_{\mathbb{F}_q}(C_i \cap C'_i) = 2\ell, \quad \sum_{i=1}^2 \dim_{\mathbb{F}_q}(C_i + C'_i) = 2n.$$

By Theorem 10, (C, C') is an (ℓ, λ) -LCP of constacyclic codes over R ; hence, $\dim_R(C \cap C') = \ell$ and $C + C' = R^n$. Applying the Gray map Φ , we obtain $\Phi(C + C') = \Phi(C) + \Phi(C') = \mathbb{F}_q^{2n}$ and

$$\dim_R(C \cap C') = \frac{\dim_{\mathbb{F}_q}(\Phi(C) \cap \Phi(C'))}{\log_q |R|} = \frac{\dim_{\mathbb{F}_q}(\Phi(C) \cap \Phi(C'))}{2}.$$

Substituting $\dim_R(C \cap C') = \ell$, we obtain $\dim_{\mathbb{F}_q}(\Phi(C) \cap \Phi(C')) = 2\ell$.

Finally, $(\Phi(C), \Phi(C'))$ is a 2ℓ -LCP of codes of length $2n$ over \mathbb{F}_q . □

Lemma 11 ([36]). *Let C^\perp and C' be two linear codes with parameters $[n, n - k(C), d(C^\perp)]_q$ and $[n, k(C'), d_2(C')]_q$ over \mathbb{F}_q , respectively. If (C, C') is an ℓ -LCP of codes of length n over \mathbb{F}_q , then there exists an AEAQEC code with the parameters $[[n, k(C') - \ell, d_z/d_x; k(C) - \ell]]_q$, where $d_z \geq d(C^\perp)$ and $d_x \geq d(C')$.*

Theorem 12. Let $C = \langle \eta_1 g_1(x) + \eta_2 g_2(x) \rangle$ and $C' = \langle \eta_1 g'_1(x) + \eta_2 g'_2(x) \rangle$ be λ -constacyclic codes of length n over R , where $\lambda = \lambda_1 + \lambda_2 v$, and the following polynomial factorizations hold over $\mathbb{F}_q[x]$

$$\begin{aligned} x^n - (\lambda_1 - \lambda_2) &= h_1(x)g_1(x) = h'_1(x)g'_1(x), \\ x^n - (\lambda_1 + \lambda_2) &= h_2(x)g_2(x) = h'_2(x)g'_2(x). \end{aligned}$$

Assume $0 \leq \ell < n$. If for $i = 1, 2$, $\deg(\text{lcm}(g_i(x), g'_i(x))) = n - \ell$ and $\gcd(g_i(x), g'_i(x)) = 1$ in $\mathbb{F}_q[x]$, then there exists an AEAQEC code with the parameters $[[2n, 2n - b - 2\ell; d_z/d_x; 2n - a - 2\ell]]_q$, where $a = \deg(g_1(x)) + \deg(g_2(x))$, $b = \deg(g'_1(x)) + \deg(g'_2(x))$, $d_z \geq \min\{d_H(\tilde{h}_1(x)), d_H(\tilde{h}_2(x))\}$, and $d_x \geq \min\{d_H(g'_1(x)), d_H(g'_2(x))\}$.

Proof. By Theorem 8 and Lemma 9, the images $\Phi(C)$ and $\Phi(C')$ under the map Φ are linear codes over \mathbb{F}_q , with the parameters $[2n, 2n - a, d_1]$ and $[2n, 2n - b, d_2]$, respectively, where $d_1 = \min\{d_H(\langle g_1(x) \rangle), d_H(\langle g_2(x) \rangle)\}$ and $d_2 = \min\{d_H(\langle g'_1(x) \rangle), d_H(\langle g'_2(x) \rangle)\}$. Since $\Phi(C^\perp) = \Phi(C)^\perp$, the dual code $\Phi(C)^\perp$ is also a linear code over \mathbb{F}_q with parameters $[2n, a, d_1^\perp]$, where $d_1^\perp = \min\{d_H(\tilde{h}_1(x)), d_H(\tilde{h}_2(x))\}$.

For $i = 1, 2$, assume $\deg(\text{lcm}(g_i(x), g'_i(x))) = n - \ell$ and $\gcd(g_i(x), g'_i(x)) = 1$ in $\mathbb{F}_q[x]$. By Lemma 10, the pair $(\Phi(C), \Phi(C'))$ forms a 2ℓ -LCP of linear code of length $2n$ over \mathbb{F}_q . Finally, invoking Lemma 11, we conclude that there exists an AEAQEC code with the parameters $[[2n, 2n - b - 2\ell; d_z/d_x; 2n - a - 2\ell]]_q$, where $d_z \geq d_1^\perp$ and $d_x \geq d_2$. \square

We now recall the BCH bound for constacyclic codes, which is stated as follows.

Lemma 12 ([37]). Assume $\gcd(q, n) = 1$. Let C be a γ -constacyclic code of length n over \mathbb{F}_q and let its generator polynomial $g(x)$ have $\{\delta^{1+t_j} \mid 0 \leq j \leq d - 2\}$ as its roots (where δ is a primitive sn -th root of unity). Then the minimum Hamming distance of C is at least d .

Combining the results of Theorem 12 and Lemma 12, we obtain the following corollary.

Corollary 1. Let q be an odd prime power with $n \mid q - 1$, $1 \leq s$ and $0 \leq \ell \leq s + \ell \leq \frac{n}{2}$. Then,

1. If n is even, there exists an AEAQEC code: $[[2n, n + 2s - 2, d_z/d_x; n - 2s - 2\ell + 2]]_q$, where $d_z \geq \frac{n}{2} - s + 2$ and $d_x \geq \frac{n}{2} - s - \ell + 2$;
2. If n is odd, there exists an AEAQEC code: $[[2n, n + 2s - 3, d_z/d_x; n - 2s - 2\ell + 3]]_q$, where $d_z \geq \frac{n-1}{2} - s + 3$ and $d_x \geq \frac{n-1}{2} - s - \ell + 3$.

Proof. Let $n \mid q - 1$ and let ζ be a primitive n -th root of unity in \mathbb{F}_q ; then, the factorization $x^n - 1 = \prod_{k=0}^{n-1} (x - \zeta^k)$ holds in $\mathbb{F}_q[x]$. Suppose that C_1 and C_2 are λ -constacyclic codes defined in Theorem 12.

1. Set $g_1(x) = g_2(x) = \prod_{k=1}^{\frac{n}{2}+s-1} (x - \zeta^k)$ and $g'_1(x) = g'_2(x) = \prod_{k=\frac{n}{2}+s+\ell}^n (x - \zeta^k)$. It is straightforward to verify that $\deg(\text{lcm}(g_i(x), g'_i(x))) = n - \ell$ and $\gcd(g_i(x), g'_i(x)) = 1$ in $\mathbb{F}_q[x]$ for $i = 1, 2$. Define $a = \sum_{i=1}^2 \deg(g_i(x)) = n + 2s - 2$, $b = \sum_{i=1}^2 \deg(g'_i(x)) = n - 2s - 2\ell + 2$. Let $h_i(x)g_i(x) = x^n - 1$. Then the reciprocal polynomial is given by $\tilde{h}_i(x) = \prod_{k=0}^{\frac{n}{2}-s} (x - \zeta^k)$ for $i = 1, 2$. According to Lemma 12, we derive the Hamming distance as follows: $d_H(g'_i(x)) = \frac{n}{2} - s - \ell + 2$ and $d_H(\tilde{h}_i(x)) = \frac{n}{2} - s + 2$ for $i = 1, 2$. Finally, substituting these into Theorem 12, we obtain an AEAQEC code with the parameters $[[2n, 2n - b - 2\ell; d_z/d_x; 2n - a - 2\ell]]_q$, where $2n - b - 2\ell = 2n - (n - 2s - 2\ell + 2) - 2\ell = n + 2s - 2$, $2n - a - 2\ell = 2n - (n + 2s - 2) - 2\ell = n - 2s - 2\ell + 2$, $d_z \geq \frac{n}{2} - s + 2$ and $d_x \geq \frac{n}{2} - s - \ell + 2$.

2. Set $g_1(x) = g_2(x) = \prod_{k=1}^{\frac{n-1}{2}+s-1} (x - \zeta^k)$; $g'_1(x) = g'_2(x) = \prod_{k=\frac{n-1}{2}+s+\ell}^n (x - \zeta^k)$. Similarly, $\deg(\text{lcm}(g_i(x), g'_i(x))) = n - \ell$ and $\gcd(g_i(x), g'_i(x)) = 1$ in $\mathbb{F}_q[x]$, for $i = 1, 2$.

Define $a = \sum_{i=1}^2 \deg(g_i(x)) = n + 2s - 3$, $b = \sum_{i=1}^2 \deg(g'_i(x)) = n - 2s - 2\ell + 3$. For $h_i(x)g_i(x) = x^n - 1$, the reciprocal polynomial is given by $\tilde{h}_i(x) = \prod_{k=0}^{\frac{n-1}{2}-s+1} (x - \zeta^k)$ for $i = 1, 2$. By Lemma 12, we obtain $d_H(g'_i(x)) = \frac{n-1}{2} - s - \ell + 3$ and $d_H(\tilde{h}_i(x)) = \frac{n-1}{2} - s + 3$ for $i = 1, 2$. Substituting these into Theorem 12, we obtain an AEAQEC code with the parameters $[[2n, n + 2s - 3; d_z/d_x; n - 2s - 2\ell + 3]]_q$, where $d_z \geq \frac{n-1}{2} - s + 3$ and $d_x \geq \frac{n-1}{2} - s - \ell + 3$. \square

Remark 2. We list some new AEAQECCs in Table 3, which can be constructed from Corollary 1 over the ring $R = \mathbb{F}_q + v\mathbb{F}_q$ with $v^2 = 1$, where $q = 3^4, 5^2, 7^2, 11^4, 13^2, 17^2, 19$, and 23. The second column denotes the length n of polycyclic codes over R ; the third and fourth columns denote the parameters s and ℓ in the ℓ -LCP construction; the fifth column lists the parameters of the new AEAQECCs obtained from our method; and the sixth column gives known EAQECCs from [30] for comparison.

Compared to the EAQECCs available in [30], we find that our derived codes exhibit improved performance in terms of either code rate or minimum distance. Recall that the code rate of an AEAQECC is defined as $\frac{k}{n}$ for any EAQECCs with parameters $[[n, k, d; c]]_q$ (see [29]). Several of our codes have higher code rate. For instance,

- $[[20, 10, \geq 6 / \geq 6; 10]]_{3^4}$ has a higher rate $10/20 = 0.5$ than $[[22, 10, 6; 10]]_{3^4}$ ($10/22 \approx 0.455$);
- $[[32, 20, \geq 7 / \geq 3; 4]]_{7^2}$ has a higher rate $20/32 = 0.625$ than $[[40, 20, 3; 4]]_{7^2}$ ($20/40 = 0.5$);
- $[[40, 34, \geq 4 / \geq 4; 6]]_{11^4}$ has a higher rate $34/40 = 0.85$ than $[[42, 34, 3; 6]]_{11^4}$ ($34/42 \approx 0.81$);
- $[[48, 24, \geq 13 / \geq 11; 20]]_{13^2}$ has a higher rate $24/48 = 0.5$ than $[[50, 24, 4; 20]]_{13^2}$ ($24/50 = 0.48$).

Several of our codes match the code rate of those in [30] but achieve larger minimum distances. For instance,

- $[[16, 10, \geq 4 / \geq 4; 6]]_{3^4}$ has the same rate $10/16 = 0.625$ as $[[16, 10, 4; 2]]_{3^4}$, but improves both d_z and d_x from $(4, 2)$ to $(\geq 4, \geq 4)$;
- $[[12, 6, \geq 4 / \geq 3; 4]]_{7^2}$ has the same rate $6/12 = 0.5$ as $[[12, 6, 4; 2]]_{7^2}$, yet enhances d_x from 2 to ≥ 4 ;
- $[[24, 20, \geq 3 / \geq 3; 4]]_{13^2}$ has the same rate $20/24 \approx 0.833$ as $[[24, 20, 3; 2]]_{13^2}$, yet enhances d_x from 2 to ≥ 4 ;

Additionally, entries marked as “new” correspond to parameters not previously reported in the literature, demonstrating the novelty and effectiveness of our construction.

Table 3. New EAQECCs from Corollary 1.

| q | n | s | ℓ | New EAQECCs | EAQECCs from Ref. | Ref. |
|--------|-----|-----|--------|--|----------------------------|------|
| 3^4 | 8 | 2 | 0 | $[[16, 10, \geq 4 / \geq 4; 6]]_{3^4}$ | $[[16, 10, 4; 2]]_{3^4}$ | [30] |
| 3^4 | 10 | 1 | 0 | $[[20, 10, \geq 6 / \geq 6; 10]]_{3^4}$ | $[[22, 10, 6; 10]]_{3^4}$ | [30] |
| 5^2 | 4 | 1 | 0 | $[[8, 4, \geq 3 / \geq 3; 2]]_{5^2}$ | $[[8, 2, 3; 2]]_{5^2}$ | [30] |
| 7^2 | 6 | 1 | 1 | $[[12, 6, \geq 4 / \geq 3; 4]]_{7^2}$ | — | new |
| 7^2 | 6 | 1 | 0 | $[[12, 6, \geq 4 / \geq 4; 6]]_{7^2}$ | $[[12, 6, 4; 2]]_{7^2}$ | [30] |
| 7^2 | 16 | 3 | 4 | $[[32, 20, \geq 7 / \geq 3; 4]]_{7^2}$ | $[[40, 20, 3; 4]]_{7^2}$ | [30] |
| 11^4 | 20 | 8 | 0 | $[[40, 34, \geq 4 / \geq 4; 6]]_{11^4}$ | $[[42, 34, 3; 6]]_{11^4}$ | [30] |
| 13^2 | 12 | 5 | 0 | $[[24, 20, \geq 3 / \geq 3; 4]]_{13^2}$ | $[[24, 20, 3; 2]]_{13^2}$ | [30] |
| 13^2 | 24 | 1 | 2 | $[[48, 24, \geq 13 / \geq 11; 20]]_{13^2}$ | $[[50, 24, 4; 20]]_{13^2}$ | [30] |
| 13^2 | 24 | 11 | 0 | $[[48, 44, \geq 3 / \geq 3; 4]]_{13^2}$ | — | new |
| 13^2 | 28 | 13 | 0 | $[[56, 52, \geq 3 / \geq 3; 4]]_{13^2}$ | — | new |

Table 3. Cont.

| q | n | s | ℓ | New EAQECCs | EAQECCs from Ref. | Ref. |
|--------|-----|-----|--------|--|----------------------------|------|
| 17^2 | 16 | 7 | 0 | $[[32, 28, \geq 3 / \geq 3; 4]]_{17^2}$ | $[[32, 28, 3; 2]]_{17^2}$ | [30] |
| 17^2 | 32 | 12 | 0 | $[[64, 54, \geq 6 / \geq 6; 10]]_{17^2}$ | $[[66, 54, 3; 10]]_{17^2}$ | [30] |
| 19 | 9 | 4 | 0 | $[[18, 14, \geq 3 / \geq 3; 4]]_{19}$ | — | new |
| 23 | 11 | 5 | 0 | $[[22, 18, \geq 3 / \geq 3; 4]]_{23}$ | — | new |

6. Conclusions

In this paper, we investigate the algebraic structure of polycyclic codes over R and characterize their annihilator dual, self-orthogonal, and LCD codes. As an application, we construct AEAQECCs via the (ℓ, λ) -LCP of constacyclic codes—a special class of polycyclic codes. It would be interesting to investigate whether the proposed construction can be optimized to yield AEAQECCs with larger minimum distances or higher rates.

Author Contributions: Conceptualization, J.L. and Y.T.; methodology, Y.T.; validation, F.M.; writing—original draft preparation, J.L.; and writing—review and editing, J.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research is supported by Shandong Provincial Natural Science Foundation (Nos. ZR2022QA067, ZR2021QA047).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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