



Planck length in classical and quantum Hamiltonian formulations of general relativity

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Abstract The physical meaning of the Planck length (ℓ_P) is investigated in the framework of the unconstrained synchronous variational formulation of classical general relativity (GR). This theoretical setting permits the establishment of manifestly-covariant Lagrangian and Hamiltonian theories for the Einstein field equations of the continuum gravitational field. It is shown that such a formulation is distinguished by the existence of a novel variational contribution expressed by an infinite series summation of suitable 4-scalar terms in which the coupling coefficients are even powers of the Planck length. However, the requirement of realization of a classical GR Hamiltonian theory places stringent constraints on the admissible Planck-length power terms to be retained. In fact, excluding the trivial gauge constant, it is proved that only the $O(\ell_P^0)$ contribution of the series is ultimately permitted, namely the unique one which is independent of ℓ_P . Therefore, the Planck length is effectively not allowed to appear at the classical level for consistency with the Hamiltonian principle. This places important consequences on the mathematical establishment of the corresponding canonical quantum gravity theory, which is then found to be correct through $O(\ell_P^2)$. Additional implications concern the physical significance of related quantum momenta and their meaning in the semi-classical limit, as well as the role of the Planck length in the same quantum-gravity realm.

1 Introduction

From a mathematical point of view, the Planck length (ℓ_P) is a unit of length introduced in the system of Planck units. It is defined out of three universal constants as

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}}, \quad (1)$$

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where \hbar is the reduced Planck constant, G is the gravitational constant and c is the speed of light in vacuum. In SI units where lengths are expressed in meters (m), the value of the Planck length is of the order of 1.6×10^{-35} m, which represents by far a much smaller length than any known or estimated size in atomic and subatomic physics. Because of its invariant character and due to the combined dependence on Planck constant and gravitational constant in its definition, one might expect that the Planck length can acquire a physical meaning in the framework of quantum gravity. More precisely, it is now commonly believed that ℓ_P should express the invariant scale at which the laws of the theory of Einstein classical gravity in its currently-accepted formulation fail and the structure of continuum space-time predicted by the same theory of general relativity (GR) breaks down. Thus, it is conjectured that at Planck-length scales quantum-gravity phenomena should appear or become relevant, demanding adoption of a theory of quantum gravity for their correct understanding [1–3]. Pushing forward this conceptual argument, it has been also claimed that ℓ_P should not be only a characteristic scale-length of quantum gravity, but it should also represent at the same time the minimum length of the theory. More precisely, according to this postulate, ℓ_P should be viewed itself as a quantum of the geometry of space-time, namely the minimum length of discretization building up the space-time grid and having the continuum space-time as classical or macroscopic limit. For example, this pertains minimum-length quantum-gravity theories focused on modification of the Heisenberg indeterminacy principle [4–9] and related predictions of the manifestation of quantum phenomena for particle physics [10, 11] or on classical GR solutions, like black-hole [12–16] or cosmological ones [17–19]. Studies on the Planck length gain transversal popularity among scientific and philosophical communities of research, partly related to the enthusiasm behind the

possibility of shedding light, in this way, on still-obscure mathematical and theoretical topics and, ultimately, make concrete progress on the search of a comprehensive theory of quantum gravitational field [20]. However, it is clear that only rigorous approaches that are formulated within established conceptual frameworks can have the hope to be considered as plausible ones. In this regard, a fundamental open question that remains to be properly addressed concerns the very nature of the Planck length and its meaning for quantum gravity. In particular, this pertains the epistemology of whether the Planck length can truly represent an intrinsic aspect of the theory of quantum gravity, to be based on first-principle mathematical and physical approaches that overcome common beliefs. Namely, we raise here the question if the Planck length can be proved to be intrinsically rooted on the same conceptual framework on which quantum gravity is founded and how its role can be possibly connected in a consistent way with the postulates of classical theory of GR.

In order to disclose some advances on the meaning of the Planck length consistent with the mathematical set-up of both classical and quantum descriptions, the appropriate framework is provided by the Lagrangian and Hamiltonian variational theories for classical GR and the canonical quantization scheme yielding the corresponding covariant quantum gravity theory. In fact, if ℓ_P is a foundational constant of the theory, then either of the two following possibilities could be admitted:

- (1) ℓ_P could be introduced from first principles at the level of variational treatment of classical GR, under the condition of leaving unaltered classical GR theory. It means that the role of the Planck length should be disclosed in the Lagrangian formulation of GR and it should be such to admit a consistent Hamiltonian formulation of the same theory. The latter should be expressed by a Hamiltonian structure in terms of a Hamiltonian state and Hamiltonian function related to the Lagrangian theory by customary definition of canonical momenta and Legendre transform. Such an occurrence then would warrant that ℓ_P is inherited consistently by the corresponding quantum counterpart theory built from the classical Hamiltonian formulation of Einstein field equations (EFE). The physical meaning of ℓ_P in this case is understood in connection with both classical and quantum Hamiltonian theories of GR.
- (2) Alternatively, ℓ_P could arise consistently in the same canonical quantization procedure, and therefore be a unique quantity of the quantum theory of GR, without any apparent role with its classical variational formulation. In this case the physical meaning of the Planck

length should be purely a quantum one and possibly be specific of the quantization method implemented.

The purpose of the following mathematical derivation is to prove that the statement of case (1) provides an appropriate conceptual framework which can be given a concrete realization. In order to approach the problem, we shall assume validity of two basic principles, denoted in the following as *Postulate A* and *Postulate B* respectively. The first one (*Postulate A*) concerns the principle of manifest covariance (PMC), which states that all classical and quantum dynamical variables and observables should be represented exclusively in 4-tensor form and be endowed with tensor properties with respect to a suitable group of coordinate transformations [21, 22]. We assume for definiteness that the space-time is represented by a Lorentzian differential manifold of the type $\{\mathbf{Q}^4, \widehat{g}(r)\}$, with \mathbf{Q}^4 being the 4-dimensional real vector space \mathbb{R}^4 representing the space-time and $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$ being a real and symmetric metric tensor which is parametrized with respect to a coordinate system (or GR-frame) $r \equiv \{r^\mu\} \in \mathbf{Q}^4$. Then, the same coordinate transformations, denoted as local point transformations (LPT), must be realized by local and differentiable bijections of the form

$$r \rightarrow r' = r'(r), \quad (2)$$

referred to as LPT-group, with inverse

$$r' \rightarrow r = r(r'), \quad (3)$$

being characterized by a non-singular Jacobian matrix $M \equiv \{M_\mu^k(r)\} \equiv \left\{ \frac{\partial r^k(r)}{\partial r'^\mu} \right\}$. Thus, $r \equiv \{r^\mu\}$ and $r' \equiv \{r'^\mu\}$ are arbitrary points belonging to the initial and transformed space-time structures $\{\mathbf{Q}^4, \widehat{g}(r)\}$ and $\{\mathbf{Q}^4, \widehat{g}'(r')\}$ respectively. The space-time structure is preserved under the LPT-group, so that $\{\mathbf{Q}^4, \widehat{g}(r)\} \equiv \{\mathbf{Q}^4, \widehat{g}'(r')\}$, while the Riemann distance in the two space-times is the same, namely it is realized by means of the 4-scalar $ds^2 = \widehat{g}_{\mu\nu}(r) dr^\mu dr^\nu = \widehat{g}'_{\mu\nu}(r') dr'^\mu dr'^\nu$. Instead, any other 4-tensor, including the Ricci and Riemann tensors, transform in accordance with appropriate 4-tensor transformation laws [21].

The second principle to be set (*Postulate B*) concerns the requirement that, whatever theory for ℓ_P is disclosed, it must preserve the postulates of classical GR and space-time dynamical equations, namely be such that the Einstein field equations for the gravitational field remain unaltered [23, 24]. In particular, we require the extremal field equations to take the customary form

$$\widehat{R}_{\mu\nu} - \frac{1}{2} \widehat{R} \widehat{g}_{\mu\nu} + \Lambda \widehat{g}_{\mu\nu} = \kappa \widehat{T}_{\mu\nu}, \quad (4)$$

where $\widehat{R}_{\mu\nu} = R_{\mu\nu}(\widehat{g}(r))$ and $\widehat{R} = \widehat{g}^{\mu\nu}(r) \widehat{R}_{\mu\nu} \equiv R(\widehat{g}(r))$ denote respectively the background Ricci 4-tensor and Ricci 4-scalar, Λ is the cosmological constant, $\widehat{T}_{\mu\nu} = T_{\mu\nu}(\widehat{g}(r))$

is the background stress-energy tensor associated with the external source fields and κ denotes the dimensional constant $\kappa = 8\pi G/c^4$. In addition, the background metric tensor $\widehat{g}_{\mu\nu}$ is subject to so-called metric compatibility condition $\widehat{\nabla}^\mu \widehat{g}_{\mu\nu} = 0$ with respect to the background covariant derivative operator $\widehat{\nabla}^\mu$, which in turn yields the representation for the Christoffel symbols in Riemann and Ricci 4-tensors.

The scheme of the paper is as follows. In Sects. 2 and 3 the Hilbert–Einstein and the synchronous unconstrained Lagrangian variational principles of classical GR are recalled respectively in view of the targets of the present research. It is pointed out that only the synchronous setting warrants the validity of the principle of manifest covariance of physical laws and it affords the consequent realization of deDonder–Weyl representations of Lagrangian and Hamiltonian variational theories for the continuum gravitational field. In Sect. 4 the physical meaning of the Planck length (ℓ_P) is investigated in the framework of the unconstrained synchronous Lagrangian variational formulation of classical GR. It is shown that the Planck length arises as a coupling constant associated with novel variational 4-scalar terms depending on the covariant derivative of the variational metric tensor, which would be otherwise excluded according to the prescriptions of the asynchronous Hilbert–Einstein principle. Altogether, these contributions can be collected and expressed in compact form as a series summation in which the coupling coefficients are even powers of the Planck length. Remarkably, these terms leave unaffected the resulting extremal Einstein field equations, namely their inclusion in the Lagrangian formulation does not change classical GR theory. In Sect. 5 the relationship between the Planck length and existence of a Hamiltonian theory for the gravitational field is addressed. It is shown that the requirements of realization of a classical GR Hamiltonian theory and existence of canonical momenta place stringent constraints on the admissible Planck-length power terms to be retained in the novel Lagrangian series contribution. In particular, excluding the trivial gauge constant, it is proved that only the $O(\ell_P^0) \equiv O(1)$ contribution of the series is permitted, namely the unique one which is independent of ℓ_P . Therefore, ultimately the Planck length is effectively not allowed to appear at the classical level for consistency with the Hamiltonian principle. Finally, in Sect. 6 concluding remarks are pointed out. These concern important consequences placed on the mathematical establishment of the corresponding canonical quantum gravity theory, which is then found to be correct through $O(\ell_P^2)$. Additional implications discussed here concern the physical significance of related quantum momenta and their meaning in the semi-classical limit, as well as the role of the Planck length in the same quantum-gravity realm.

2 Hilbert–Einstein variational principle

Given these premises, we consider briefly the mathematical setting identified by the original Hilbert–Einstein (HE) variational theory [24], which can be regarded as the prototype of the majority of variational theories leading to GR equations. The purpose of the review is to prove that the HE theory is not convenient for the target of the present study, the reasons being the violation of both fundamental postulates displayed above. More precisely, historically the HE theory is based on the action functional

$$S_{HE}(g(r)) \equiv \int_{\mathbf{Q}^4} d\Omega L_{HE}(g), \tag{5}$$

where $d\Omega \equiv d^4r \delta\sqrt{-|g(r)|}$ is the invariant 4-volume element of the Riemann space-time $\{\mathbf{Q}^4, g(r)\}$, with $d^4r \equiv \prod_{i=0,3} dr^i$ being the canonical measure of \mathbf{Q}^4 , and $|g(r)|$ denoting here the determinant of $g(r)$. Furthermore, $L_{HE}(g)$ denotes the HE Lagrangian 4-scalar function defined as

$$L_{HE}(g) = V_{HE}(g) + V_F, \tag{6}$$

where

$$V_{HE}(g) = \kappa (g^{\mu\nu} R_{\mu\nu}(g) - 2\Lambda), \tag{7}$$

with $\Lambda > 0$ and where $\kappa \equiv -\frac{c^3}{16\pi G}$. Instead, $V_F \equiv V_F(g, r)$ is the non-vacuum contribution due to possible external fields to be prescribed in terms of the field Lagrangian L_F as $V_F = \frac{1}{c} L_F$. Hence, the quantity $\mathcal{L}_{HE} \equiv \sqrt{-|g|} L_{HE}(g)$ identifies the corresponding variational Lagrangian density. According to the HE theory, the action $S_{HE}(g(r))$ is considered dependent only on the variational field $g(r) \equiv \{g_{\mu\nu}\}$, whose independent 4-tensor components represent the generalized Lagrangian coordinates. Each $g(r)$ belongs to a suitably-constrained functional setting $\{g(r)\}_C$ [25]. In fact, any 4-tensor $g(r) \in \{g(r)\}_C$ realizes also a metric tensor, so that its contravariant and covariant components respectively raise and lower tensor indices and thus necessarily must satisfy the orthogonality condition $g_{\mu\nu} g^{\mu k} = \delta_\nu^k$, implying in turn the “normalization” condition $g_{\mu\nu}(r) g^{\mu\nu}(r) = 4$. For the same reason, in the functional setting $\{g(r)\}_C$, the tensor $g(r)$ must determine also the Christoffel symbols $\Gamma(g(r))$ and the Ricci tensor $R_{\mu\nu}(g)$, so that $g(r)$ satisfies the metric compatibility condition with vanishing covariant derivative.

The HE variational principle is expressed by the requirement that for arbitrary variations $\delta g(r)$ it must be

$$\delta S_{HE}(g(r))|_{g=\widehat{g}(r)} = \left. \frac{d}{d\theta} S_{HE}(\widehat{g}(r) + \theta \delta g(r)) \right|_{\theta=0} = 0, \tag{8}$$

with the symbol δ denoting the Frechet derivative and $\widehat{g}(r)$ being the extremal classical metric tensor, to be identified “a posteriori” with the solution of EFE. A character-

istic feature of the HE variational theory is that $d\Omega$ yields non-vanishing variational contributions to $\delta S_{HE}(g(r))$, since $\delta d\Omega = d^4r \delta \sqrt{-|g|}$, where $\delta \sqrt{-|g|} = \frac{1}{2} \sqrt{-|g|} g^{\mu\nu} \delta g_{\mu\nu}$. This means that the variation of the functional $S_{HE}(g(r))$ does not preserve the space-time volume element. Because of formal analogies of this property with the analogous occurrence arising in non-relativistic classical mechanics and adopting a similar nomenclature, the HE in variational principles can be referred to as asynchronous [26]. As shown in Ref. [27], this belongs more generally to a class of constrained variational principles. One obtains that

$$\delta S_{HE}(g(r))|_{g=\hat{g}(r)} = \delta S_{HE}(g)|_{\text{expl}} + \delta S_{HE}(g)|_{\text{impl}}, \quad (9)$$

where the implicit contribution is

$$\delta S_{HE}(g)|_{\text{impl}} = \int_{\mathcal{Q}^4} d\Omega \left[\kappa \hat{g}^{\alpha\beta} \frac{\delta R_{\alpha\beta}}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu}, \quad (10)$$

while the explicit contributions can be written as

$$\delta S_{HE}(g)|_{\text{expl}} = \int_{\mathcal{Q}^4} d^4r [A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu}] \delta g^{\mu\nu}, \quad (11)$$

where $A_{\mu\nu}$, $B_{\mu\nu}$ and $C_{\mu\nu}$ are tensor densities defined as

$$A_{\mu\nu} \equiv L_{HE} \frac{\delta \sqrt{-|g|}}{\delta g^{\mu\nu}}, \quad (12)$$

$$B_{\mu\nu} \equiv \kappa \sqrt{-|g|} R_{\alpha\beta} \frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}}, \quad (13)$$

$$C_{\mu\nu} \equiv \frac{1}{c} \sqrt{-|g|} \frac{\delta L_F}{\delta g^{\mu\nu}}. \quad (14)$$

As shown in Ref. [25], in order to recover the correct form of EFE, the condition $\delta S_{HE}(g)|_{\text{impl}} = 0$ must hold for arbitrary variations $\delta g^{\mu\nu}(r)$, so that the explicit contributions are sufficient to yield the Einstein equations in the correct form (4).

The asynchronous HE approach is inconsistent with *Posulates A* and *B* set above. In fact, the following critical aspects arise:

1. *Violation of the principle of manifest covariance* The HE variational Lagrangian density is intrinsically not a 4-scalar because of the dependence on the determinant $|g(r)|$. As a result, also the generalized Lagrangian coordinates, i.e., the continuum Lagrangian fields, cannot be identified with tensors of prescribed order, but only with tensor densities.
2. *The non-standard character of $L_{HE}(g)$* The HE Lagrangian density depends on second-order partial derivatives of Lagrangian coordinate $g_{\mu\nu}(r)$ carried by the Ricci tensor (see Eq. (10)). This means that the HE variational principle cannot be cast in so-called first-order Lagrangian formalism. As a consequence, an appropriate treatment of differential fixed-point boundary terms generated in this way is required (see for example the

attempts proposed in Refs. [21, 25, 28, 29]). The related missing canonical structure of the HE Lagrangian that is not expressed as a customary sum of “kinetic” and “potential” terms prevents in turn also the possibility of determining a canonical Hamiltonian structure consistent with the same principle of manifest covariance. In fact, in the framework of the HE theory only non-manifestly covariant approaches can at most necessarily follow, like the Dirac approach, the ADM theory and generally any 3+1 formulation based on preliminary decomposition of space-time into the product of one time-like dimension and a three-space slice [30–32]. However, the missing canonical structure prevents also any manifestly-covariant canonical quantization of the theory, and therefore the possibility of displaying the role of the invariant Planck length through this procedure or disclosing its nature in the quantum gravitational theory.

3. *Planck length and violation of GR equations* The HE Lagrangian principle does not contain the Planck length. On the other hand, one might think about introducing additional variational contributions which carry the Planck length explicitly. However, any attempt of this kind pursued in the framework of HE theory inevitably leads to a modification of classical GR equations. The reason is intrinsic in the property of the variational field $g_{\mu\nu}(r)$ to be also a metric tensor and the fact that the HE principle treats Lagrangian densities rather than 4-scalar Lagrangian functions. This feature is also a consequence of the violation by the HE variational theory of the fundamental gauge invariance properties which characterize the variational theory of other continuum classical fields [25, 26].

To prove the statement, first of all we notice that inclusion of contributions depending on the covariant derivative of $g_{\mu\nu}(r)$ are always vanishing in such a setting, since by definition $\nabla^\mu g_{\mu\nu}(r) = \hat{\nabla}^\mu g_{\mu\nu}(r) = \hat{\nabla}^\mu \hat{g}_{\mu\nu}(r) = 0$. These terms remain therefore excluded “a priori”. Let us now assume to add a new variational contribution to the HE Lagrangian density $\mathcal{L}_{HE} \equiv \sqrt{-|g|} L_{HE}(g)$, namely letting

$$\mathcal{L}_{HE} \rightarrow \mathcal{L}_{HE} + \mathcal{L}_{\hbar}. \quad (15)$$

Here, \mathcal{L}_{\hbar} denotes a generic Lagrangian density to be properly assigned and such that it carries explicitly a dependence on the Planck length to be prescribed based on dimensional analysis and in order to warrant the correct dimension of the action integral. We can therefore set in full generality

$$\mathcal{L}_{\hbar} = \sqrt{-|g|} L_{\hbar}(\ell_P, g, [g]), \quad (16)$$

where the symbol $[g]$ denotes possible implicit dependences on g , like those contained through the Ricci scalar $R = R(g)$. It is then immediate to verify that \mathcal{L}_{\hbar} yields non-vanishing

variational contributions according to the asynchronous variational theory, which ultimately enter the extremal field equations causing modification of EFE. In fact, for a Lagrangian density of this type we have that the asynchronous variation gives

$$\frac{\delta \mathcal{L}_h}{\delta g^{\mu\nu}} = L_h(\ell_P, g, [g]) \frac{\delta \sqrt{-|g|}}{\delta g^{\mu\nu}} + \sqrt{-|g|} \frac{\delta L_h(\ell_P, g, [g])}{\delta g^{\mu\nu}} \neq 0. \tag{17}$$

When evaluated for the extremal curve $g = \widehat{g}$ this implies as well necessarily that

$$\left. \frac{\delta \mathcal{L}_h}{\delta g^{\mu\nu}} \right|_{g=\widehat{g}} \neq 0. \tag{18}$$

For example, \mathcal{L}_h can indicate a polynomial of the Ricci scalar to be postulated in the framework of so-called $f(R)$ – theories, so that in such a case $\mathcal{L}_h = \mathcal{L}_h(\ell_P, g, R(g))$ can be written in the general form

$$\mathcal{L}_h(\ell_P, g, R(g)) = \sqrt{-|g|} \sum_{n=2}^{\infty} \ell_P^{2n-2} R^n(g). \tag{19}$$

An alternative is provided by a so-called non-local function of the same Ricci scalar, namely letting $\mathcal{L}_h = \mathcal{L}_h(\ell_P, g, \square R)$, where \square is the invariant D’Alembertian operator. An explicit realization is for example

$$\mathcal{L}_h(\ell_P, g, \square R) = \sqrt{-|g|} \ell_P^2 \square R. \tag{20}$$

In both cases the contributions given by the variational derivative $\frac{\delta \mathcal{L}_h}{\delta g^{\mu\nu}} \neq 0$ effectively change EFE, both through the variation of the determinant $\sqrt{-|g|}$ as well as because of the explicit and implicit dependences on $g_{\mu\nu}$ carried by the Ricci 4-scalar and the differential operator \square . Therefore, in the HE theory the explicit occurrence of the Planck length can only be realized at the expenses of modification of classical GR, namely by change of gravitational EFE. In view of these considerations, the asynchronous (or constrained) HE theoretical setting must be discarded as it does not represent an appropriate variational approach for the study of the physical meaning of the Planck length.

3 Synchronous variational principle

In contrast, a convenient formulation for the purpose of the current research is provided by the synchronous unconstrained variational principle for the gravitational field outlined in Refs. [26,33], which realizes a deDonder-Weyl representations of Lagrangian variational theory for the continuum gravitational field that is naturally consistent with PMC [34–36]. In fact, the synchronous variational approach is characterized by a 4-scalar Lagrangian function expressed

in terms of superabundant variables $g_{\mu\nu}$ and $\widehat{g}_{\mu\nu}$. In this setting, the variational tensor $g \equiv \{g_{\mu\nu}\}$ is distinguished from a non-variational background metric tensor $\widehat{g} \equiv \{\widehat{g}_{\mu\nu}\}$, which defines the covariance properties of the theory and is ultimately assumed to be determined “a posteriori” by the extremal EFE. Hence, \widehat{g} expresses the geometric character of the metric tensor, namely it satisfies the orthogonality condition $\widehat{g}_{\mu\nu} \widehat{g}^{\mu k} = \delta_\nu^k$, so that it raises/lowers tensor indices, as well the metric compatibility condition $\widehat{\nabla}_\alpha \widehat{g}_{\mu\nu} = 0$, so that it defines the standard Christoffel connections and curvature tensors of space-time. On the contrary, in this framework the variational tensor g is such that $g_{\mu\nu} g^{\mu k} \neq \delta_\nu^k$. The distinction between g and \widehat{g} holds only at the variational level, since in the (extremal) EFE the identity $g = \widehat{g}$ is restored. In the synchronous setting, hatted quantities depend on the background metric tensor \widehat{g} and do not contribute to the variational calculus. Thus, denoting in particular the synchronous volume element as $d\widehat{\Omega} = d^4r \sqrt{-|\widehat{g}|}$, its variation vanishes by construction so that $\delta d\widehat{\Omega} = 0$. This volume-preserving property under the action of the operator δ justifies the name given to this approach as the synchronous variational principle, in contrast to the asynchronous theory. For completeness, it must be also noticed that the same feature suggests similarities between the synchronous setting and relevant literature approaches known as non-metric volume forms, or modified measures, defined for example in Refs. [37,38], or the so-called non-Riemannian space-time volume elements [39]. Contrary to the customary HE assumption, also these works propose variational models for the GR equations in which the volume elements of integration in the action principles are metric independent and are determined dynamically through additional degrees of freedom, like the inclusion of additional scalar fields. Therefore, both synchronous and non-metric approaches do not treat the volume element of integration as a variational quantity depending on variational metric tensor. This feature certainly represents a breakthrough in the variational approach to EFE with respect to other literature models. However, the synchronous setting remains distinguished because it does not rely on inclusion nor it predicts the onset of additional fields, but only use of superabundant field variables which nevertheless coincide with the unique observable space-time metric tensor in the extremal Einstein equations.

The synchronous Lagrangian action functional is defined as

$$S_s(g(r), \widehat{g}(r)) = \int_{\mathbf{Q}^4} d\widehat{\Omega} L_s(g, \widehat{g}), \tag{21}$$

where $S_s(g(r), \widehat{g}(r))$ is considered a functional dependent only on the variational tensor (not a metric tensor) $g(r) \equiv \{g_{\mu\nu}\}$. Here, $L_s(g, \widehat{g}) \equiv L_s(g(r), \widehat{g}(r))$ is the variational Lagrangian and, in contrast to the asynchronous action functional (5), the volume element takes the form $d\widehat{\Omega}$. The vari-

ational Lagrangian is written as

$$L_s(g, \widehat{g}) \equiv h(g, \widehat{g})L(g, \widehat{g}), \quad (22)$$

where the 4-scalar

$$h(g, \widehat{g}) = 2 - \frac{1}{4}g^{\eta\beta}(r)g^{\mu\nu}(r)\widehat{g}_{\eta\mu}(r)\widehat{g}_{\beta\nu}(r) \quad (23)$$

identifies the variational weight-factor defined so that $h(\widehat{g}, \widehat{g}) = 1$. Instead, the 4-scalar Lagrangian $L(g, \widehat{g})$ takes the form

$$L(g, \widehat{g}) = V_G(g, \widehat{g}) + V_F(g, \widehat{g}), \quad (24)$$

where now

$$V_G(g, \widehat{g}) = -\frac{c^3}{16\pi G} (g^{\mu\nu}\widehat{R}_{\mu\nu} - 2\Lambda), \quad (25)$$

with $\widehat{R}_{\mu\nu} \equiv R_{\mu\nu}(\widehat{g})$ and $V_F(g, \widehat{g}) = \frac{1}{c}L_F(g, \widehat{g})$. Then, the synchronous Lagrangian action principle follows by prescribing

$$\delta S_s(g(r), \widehat{g}(r))|_{g=\widehat{g}} = 0, \quad (26)$$

for arbitrary variations $\delta g(r)$, while noting that $\delta\widehat{g}(r) \equiv 0$. Here, the symbol δ denotes the variation operator, i.e., the Frechet derivative

$$\delta S_s(g(r), \widehat{g}(r))|_{g=\widehat{g}} \equiv \frac{d}{d\theta} S_L(\widehat{g}(r) + \theta\delta g(r), \widehat{g}(r)) \Big|_{\theta=0} \quad (27)$$

By noting that $\delta h(g, \widehat{g}) = -\frac{1}{2}\widehat{g}^{\mu\nu}(r)\delta g_{\mu\nu}$, the evaluation of $\delta S_s(g(r), \widehat{g}(r))|_{g=\widehat{g}(r)}$ is straightforward. In fact, in the synchronous setting only explicit dependences on g give a contribution to EFE, while the implicit ones carried by the Ricci tensor are now excluded. Hence, from Eq. (26) one recovers EFE in the correct form (4). We notice that the variational contribution determined by the function $h(g, \widehat{g})$ in the synchronous principle corresponds to the one provided by $\sqrt{-|g|}$ in the asynchronous principle. Remarkably, the expression of $h(g, \widehat{g})$ given by Eq. (23) is unique, as proved in Ref. [27]. In the synchronous setting the same function $h(g, \widehat{g})$ represents a 4-scalar and permits to deal exclusively with 4-scalar Lagrangian functions, rather than non-manifestly covariant Lagrangian densities as those arising in the asynchronous HE theory.

4 Planck length and synchronous Lagrangian principle

Let us now investigate the connection between the synchronous variational theory and the nature of the Planck length. In this regard we first notice that in the original representation of the synchronous action integral by Eq. (21) the Lagrangian $L_s(g, \widehat{g})$ is independent of the Planck length. The same character of independence from ℓ_P is then shared by the synchronous variational principle and the corresponding Lagrangian derivation of EFE. On the other hand, the

implementation of superabundant variables in terms of distinction between background metric tensor \widehat{g} and variational field tensor g represents a unique peculiar aspect of the synchronous variational theory. It is precisely this feature that yields the required mathematical freedom to investigate the variational role of the Planck length, marking in this way a notable sign of distinction from the asynchronous HE theory.

In order to carry out the derivation, we first notice that in such a framework the transformation (15) must be replaced with the following one:

$$L_s \rightarrow L_s + L_{\hbar}, \quad (28)$$

where now L_{\hbar} denotes a generic 4-scalar Lagrangian function rather than a Lagrangian density (*Postulate A*). The corresponding action integral then becomes

$$S_s \rightarrow S_s + S_{\hbar},$$

where by definition

$$S_{\hbar} \equiv \int_{\mathbf{Q}^4} d\widehat{\Omega} L_{\hbar}. \quad (29)$$

Again, L_{\hbar} must be such to display an explicit dependence on the Planck length to be prescribed consistent with the dimensional analysis of the action integral. According to the superabundant variable formalism and consistent with PMC we can therefore set in full generality

$$L_{\hbar} = L_{\hbar}(\ell_P, g, \widehat{g}). \quad (30)$$

The new contribution L_{\hbar} must acquire a role in the classical Lagrangian theory of GR, so that $\delta S_{\hbar} \neq 0$, where

$$\delta S_{\hbar} = \int_{\mathbf{Q}^4} d\widehat{\Omega} \delta L_{\hbar} \equiv \int_{\mathbf{Q}^4} d\widehat{\Omega} \left[\frac{\delta L_{\hbar}}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu}. \quad (31)$$

Accordingly, this implies the validity of the tensorial equation

$$\frac{\delta L_{\hbar}(\ell_P, g, \widehat{g})}{\delta g^{\mu\nu}} \neq 0. \quad (32)$$

On the other hand, in order to warrant that the resulting extremal field equations must correspond to EFE (*Postulate B*), it is necessary to require also validity of the following condition applying for extremal curves

$$\frac{\delta L_{\hbar}(\ell_P, g, \widehat{g})}{\delta g^{\mu\nu}} \Big|_{g=\widehat{g}} = 0. \quad (33)$$

Without additional specific constraints, the simultaneous requirements of the previous conditions exclude also in the synchronous setting any dependence of $L_{\hbar}(\ell_P, g, \widehat{g})$ on the Ricci scalar, either in the form of local polynomial function or as a non-local differential form. On the contrary, we argue that the choice which is at the same time peculiar of the synchronous variational principle must be through a 4-scalar constructed in terms of both variational and extremal tensors $g_{\mu\nu}$ and $\widehat{g}_{\mu\nu}$ as well as the differential covariant derivative

operator $\widehat{\nabla}_\mu$ defined in terms of extremal Christoffel symbols. Hence, the conjecture is that $L_{\hbar}(\ell_P, g, \widehat{g})$ should be proportional to a non-local (i.e., differential) function of the type

$$L_{\hbar}(\ell_P, g, \widehat{g}) \sim f(\ell_P, g, \widehat{g}) \widehat{\nabla}_\mu g^{\mu\nu}. \tag{34}$$

Based on these considerations, we state that the sought expression for the 4-scalar Lagrangian $L_{\hbar}(\ell_P, g, \widehat{g})$ is not provided by a single term, but rather can be identified with a series summation whose coefficients are even powers of the Planck length. In compact form this can be written as $L_{\hbar}(\ell_P, g, \widehat{g}) \equiv L_{\hbar}^{(n)}(\ell_P, g, \widehat{g})$, where

$$L_{\hbar}^{(n)}(\ell_P, g, \widehat{g}) = \kappa \sum_{n=0}^{\infty} \frac{\ell_P^{2(n-1)}}{2^n} (\widehat{\nabla}_\alpha g^{\mu\nu} \widehat{\nabla}^\alpha g_{\mu\nu})^n, \tag{35}$$

which is homogeneous with L_S . It must be stressed again that a function of this type would be identically vanishing in the framework of the asynchronous HE variational theory. This characterizes this choice as a unique feature of the synchronous principle. From the physical point of view, in the previous summation the Planck length represents a coupling constant associated with novel variational 4-scalar terms depending on the background covariant derivative of the variational metric tensor. In fact, apart from the universal constant κ , from a dimensional analysis one can easily verify that the coefficient in front of the differential variational term $(\widehat{\nabla}_\alpha g^{\mu\nu} \widehat{\nabla}^\alpha g_{\mu\nu})^n$ must be a suitable power of an invariant 4-scalar having the dimension of a length in SI units. In order to assure that classical GR is left unchanged, we must exclude the possibility of introducing new ad hoc coupling constants of unknown physical meaning as well as coupling coefficients associated with existence of new particles or fields. Then, the only left possibility is to invoke the Planck length. The choice of a coupling coefficient depending on the Planck constant \hbar for a classical Lagrangian function remains physically motivated because, as demonstrated below, the novel term $L_{\hbar}(\ell_P, g, \widehat{g})$ does not contribute effectively to the classical GR equations, while it acquires a physical meaning in the formulation of classical Hamiltonian theory and consequently within the realm of quantum gravity.

Let us first prove that the condition (32) is verified. To this aim we notice that the first term of the series corresponding to the choice $n = 0$ is a constant, so that its variational derivative is necessarily null (see also discussion below). Therefore, when evaluating the variational derivative $\frac{\delta L_{\hbar}}{\delta g^{\mu\nu}}$ the relevant summation is for $\sum_{n=1}^{\infty}$. Explicit calculation of the synchronous variation then yields

$$\begin{aligned} \frac{\delta L_{\hbar}^{(n)}}{\delta g^{\mu\nu}} &= \kappa \sum_{n=1}^{\infty} \frac{\ell_P^{2(n-1)}}{2^{n-1}} n (\widehat{\nabla}_\zeta g^{\beta\gamma} \widehat{\nabla}^\zeta g_{\beta\gamma})^{n-1} \\ &\quad \times \frac{\widehat{\nabla}^\alpha g_{\xi\rho} \delta \widehat{\nabla}_\alpha g^{\xi\rho}}{\delta g^{\mu\nu}}. \end{aligned} \tag{36}$$

Replacing this result into Eq. (31) and considering that the covariant derivative is left unaffected by the synchronous variational operator δ gives

$$\begin{aligned} \frac{\delta S_{\hbar}}{\delta g^{\mu\nu}} &= \kappa \sum_{n=1}^{\infty} \frac{\ell_P^{2(n-1)}}{2^{n-1}} n \int_{Q^4} d\widehat{\Omega} (\widehat{\nabla}_\zeta g^{\beta\gamma} \widehat{\nabla}^\zeta g_{\beta\gamma})^{n-1} \\ &\quad \times \widehat{\nabla}^\alpha g_{\xi\rho} \frac{\widehat{\nabla}_\alpha \delta g^{\xi\rho}}{\delta g^{\mu\nu}}. \end{aligned} \tag{37}$$

At this point we notice that the derivative operator $\widehat{\nabla}_\alpha$ is defined on the same background space-time as the invariant 4-volume element $d\widehat{\Omega}$. This property is fundamental in order to permit a consistent integration by parts of the variational term $\widehat{\nabla}_\alpha \delta g^{\beta\gamma}$, yielding in this way

$$\begin{aligned} \frac{\delta S_{\hbar}}{\delta g^{\mu\nu}} &= -\kappa \sum_{n=1}^{\infty} \frac{\ell_P^{2(n-1)}}{2^{n-1}} n \int_{Q^4} d\widehat{\Omega} \\ &\quad \times \widehat{\nabla}_\alpha \left[(\widehat{\nabla}_\zeta g^{\beta\gamma} \widehat{\nabla}^\zeta g_{\beta\gamma})^{n-1} \widehat{\nabla}^\alpha g_{\mu\nu} \right]. \end{aligned} \tag{38}$$

In conclusion, we obtain that

$$\frac{\delta S_{\hbar}}{\delta g^{\mu\nu}} \neq 0 \Rightarrow \frac{\delta L_{\hbar}^{(n)}}{\delta g^{\mu\nu}} \neq 0, \tag{39}$$

where

$$\begin{aligned} \frac{\delta L_{\hbar}^{(n)}}{\delta g^{\mu\nu}} &= -\kappa \sum_{n=1}^{\infty} \frac{\ell_P^{2(n-1)}}{2^{n-1}} n \widehat{\nabla}^\alpha \\ &\quad \times \left[(\widehat{\nabla}_\zeta g^{\beta\gamma} \widehat{\nabla}^\zeta g_{\beta\gamma})^{n-1} \widehat{\nabla}^\alpha g_{\mu\nu} \right], \end{aligned} \tag{40}$$

which proves that the variational contribution of the ℓ_P -dependent Lagrangian L_{\hbar} has the desirable character of being non-vanishing. On the other hand, validity of the second constraint condition (33) readily follows. In fact, when evaluated for the background metric tensor letting $g_{\mu\nu} = \widehat{g}_{\mu\nu}$, Eq. (40) yields $\frac{\delta L_{\hbar}^{(n)}}{\delta g^{\mu\nu}} = 0$ identically due to metric compatibility condition, i.e., $\widehat{\nabla}^\alpha \widehat{g}_{\mu\nu} = 0$. Therefore, as a remarkable feature, the new series terms leave unaffected the resulting extremal Einstein field equations, namely their inclusion in the Lagrangian formulation does not change classical GR theory. This completes the proof about the existence of the Lagrangian term $L_{\hbar}(\ell_P, g, \widehat{g})$ exhibiting dependence on ℓ_P but such that it does not show up in the extremal field equations. Notice that the presence of $L_{\hbar}(\ell_P, g, \widehat{g})$ might remain hidden to classical Lagrangian formulation. However, as discussed below, the inclusion of such term is fundamental in order to obtain a corresponding classical Hamiltonian formulation of GR which can acquire a physical realization as a canonical quantum gravity theory.

Before concluding this section, it is instructive at this point to evaluate explicitly the relevant terms of $L_{\hbar}^{(n)}(\ell_P, g, \widehat{g})$ from the series summation in Eq. (35). We have in particular:

- *Case $n = 0$* The first term of the series obtained for $n = 0$ is purely constant and is given by

$$L_h^{(0)}(\ell_P, g, \widehat{g}) = \kappa \frac{1}{\ell_P^2}. \tag{41}$$

We can see therefore that $L_h^{(0)}(\ell_P, g, \widehat{g}) \sim O\left(\frac{1}{\ell_P^2}\right)$ which is the leading-order term in dimensional Planck-length units. This represents however effectively a gauge constant term which does not contribute to the variational calculus.

- *Case $n = 1$* The second term of the series obtained for $n = 1$ gives

$$L_h^{(1)}(\ell_P, g, \widehat{g}) = \kappa \frac{1}{2} (\widehat{\nabla}_\alpha g^{\mu\nu} \widehat{\nabla}^\alpha g_{\mu\nu}), \tag{42}$$

which brings a quadratic dependence on the covariant derivative of the variational tensor $g_{\mu\nu}$. Remarkably, this is the only term that does not depend on the Planck length. As discussed below, this property has a fundamental physical meaning and it assigns to $L_h^{(1)}(\ell_P, g, \widehat{g})$ a particular role in the theory. We can therefore write that $L_h^{(1)}(\ell_P, g, \widehat{g}) \sim O(1)$ in dimensional Planck-length units.

- *Case $n \geq 2$* The rest of infinite series terms obtained for $n > 1$ depend explicitly on ℓ_P , so that $L_h^{(n \geq 2)}(\ell_P, g, \widehat{g}) \sim O(\ell_P^{2(n-1)})$ in dimensional Planck-length units.

5 Planck length and Hamiltonian theory

The last issue to address concerns the understanding of the relation between the novel Lagrangian contribution $L_h^{(n)}(\ell_P, g, \widehat{g})$ given by Eq. (35) and the possibility of establishing a Hamiltonian theory of EFE on the same basis. More precisely, the goal is to prove that the requirement of realization of a manifestly-covariant Hamiltonian theory of GR and existence of an invertible relationship between canonical momenta and field generalized velocities select the only admissible power terms of the series (35), which must be retained correct through $O(\ell_P^2)$. In this reference, we first notice that, given the validity of PMC for the unconstrained synchronous Lagrangian theory, also the associated Hamiltonian theory should be characterized by the same manifest-covariance character. In the case of continuum fields, the appropriate formalism is again provided by the DeDonder–Weyl Lagrangian and Hamiltonian treatments [40–44]. Such an approach was originally formulated for fields defined on the Minkowski space-time. The setting provided by the unconstrained synchronous Lagrangian principle for EFE, together with the intrinsic validity of manifest covariance,

permit the straightforward extension of the DeDonder–Weyl formalism to include also the gravitational field dynamics [33,45].

Before treating the concrete application to the present case, it is instructive to recall briefly the fundamentals of such an approach. Thus, let us assume to have a generic 4-scalar Lagrangian function denoted by $L(Z, \widehat{\nabla}Z, \widehat{Z}) \equiv L(Z_{\mu\nu}, \widehat{\nabla}_\alpha Z_{\mu\nu}, \widehat{Z}_{\mu\nu})$, which is taken for completeness to depend on the tensorial variational field $Z_{\mu\nu}$, its covariant derivative $\widehat{\nabla}_\alpha Z_{\mu\nu}$ and a set of extremal tensor fields $\widehat{Z}_{\mu\nu}$. The corresponding canonical momenta are formally defined as in classical mechanics and identify 3rd-order tensors:

$$\Pi \equiv \Pi_{\mu\nu}^\alpha = \frac{\partial L(Z, \widehat{\nabla}Z, \widehat{Z})}{\partial (\widehat{\nabla}_\alpha Z^{\mu\nu})}, \tag{43}$$

so that the canonical state can be represented as $\{x\} = \{Z, \Pi\}$. The Hamiltonian density $H = H(x, \widehat{x})$ associated with the Lagrangian $L(Z, \widehat{\nabla}Z, \widehat{Z})$ is then provided by the Legendre transform

$$L(Z, \widehat{\nabla}Z, \widehat{Z}) \equiv \Pi_{\mu\nu}^\alpha \widehat{\nabla}_\alpha Z^{\mu\nu} - H(x, \widehat{x}). \tag{44}$$

Then, given the definition of a suitable functional class of variations for the canonical state variables, the Hamiltonian action functional is written as

$$\begin{aligned} S_H(x, \widehat{x}) &= \int d\Omega L(x, \widehat{x}) \\ &= \int d\Omega [\Pi_{\mu\nu}^\alpha \widehat{\nabla}_\alpha Z^{\mu\nu} - H(x, \widehat{x})], \end{aligned} \tag{45}$$

while the corresponding synchronous Hamiltonian variational principle becomes

$$\delta S_H(x, \widehat{x}) \equiv \frac{d}{d\alpha} \Psi(\alpha) \Big|_{\alpha=0} = 0, \tag{46}$$

which is defined in terms of the Frechet derivative and is required to hold for arbitrary independent variations $\delta Z^{\mu\nu}$ and $\delta \Pi_{\mu\nu}^\alpha$ in the respective functional classes. The corresponding variational derivatives then yield the continuum Hamilton equations

$$\frac{\delta S_H(x, \widehat{x})}{\delta Z^{\mu\nu}} \equiv -\frac{\partial H(x, \widehat{x})}{\partial Z^{\mu\nu}} - \widehat{\nabla}_\alpha \Pi_{\mu\nu}^\alpha = 0, \tag{47}$$

$$\frac{\delta S_H(x, \widehat{x})}{\delta \Pi_{\mu\nu}^\alpha} \equiv \widehat{\nabla}_\alpha Z^{\mu\nu} - \frac{\partial H(x, \widehat{x})}{\partial \Pi_{\mu\nu}^\alpha} = 0. \tag{48}$$

Written explicitly, these become

$$\widehat{\nabla}_\alpha \Pi_{\mu\nu}^\alpha = -\frac{\partial H(x, \widehat{x})}{\partial Z^{\mu\nu}}, \tag{49}$$

$$\widehat{\nabla}_\alpha Z^{\mu\nu} = \frac{\partial H(x, \widehat{x})}{\partial \Pi_{\mu\nu}^\alpha}, \tag{50}$$

which are equivalent to the Euler–Lagrange equation provided by the Lagrangian principle. The DeDonder–Weyl formalism is simultaneously consistent with classical mechanics

and PMC, while the absence of constraints makes the theory easy to implement and understandable from the physical point of view.

In the present context we identify the variational field with $Z \equiv g_{\mu\nu}$, its extremal counterpart with $\widehat{Z} \equiv \widehat{g}_{\mu\nu}$ and the differential term with $\widehat{\nabla}Z \equiv \widehat{\nabla}_\alpha g^{\mu\nu}$, so that the Lagrangian function $L(Z, \widehat{\nabla}Z, \widehat{Z})$ is written as

$$L(Z, \widehat{\nabla}Z, \widehat{Z}) \equiv L_s(g, \widehat{g}) + L_h(\ell_P, g, \widehat{g}), \tag{51}$$

where the only term carrying the covariant derivative contribution is $L_h(\ell_P, g, \widehat{g})$ expressed by Eq. (35). Adopting a language of classical mechanics, it is easily understood that $L_s(g, \widehat{g})$ plays the role of a ‘‘potential’’ term, while $L_h(\ell_P, g, \widehat{g})$ is equivalent to the ‘‘kinetic’’ term of the Lagrangian function. The crucial issue is the calculation of canonical momenta, which at this point are necessarily uniquely related to $L_h(\ell_P, g, \widehat{g}) \equiv L_h^{(n)}(\ell_P, g, \widehat{g})$. Invoking the general series representation (35) we have that

$$\begin{aligned} \Pi_{\mu\nu}^\alpha &= \frac{\partial L_h^{(n)}(\ell_P, g, \widehat{g})}{\partial (\widehat{\nabla}_\alpha g^{\mu\nu})} \\ &= \kappa \sum_{n=0}^\infty \frac{\ell_P^{2(n-1)}}{2^{n-1}} n (\widehat{\nabla}_\zeta g^{\beta\gamma} \widehat{\nabla}^\zeta g_{\beta\gamma})^{n-1} \widehat{\nabla}^\alpha g_{\mu\nu}. \end{aligned} \tag{52}$$

In this way the canonical momentum remains defined through an infinite series. To exemplify the expression, the first two terms are written explicitly as

$$\begin{aligned} \Pi_{\mu\nu}^\alpha &= \kappa \widehat{\nabla}^\alpha g_{\mu\nu} + \kappa \ell_P^2 (\widehat{\nabla}_\zeta g^{\beta\gamma} \widehat{\nabla}^\zeta g_{\beta\gamma}) \widehat{\nabla}^\alpha g_{\mu\nu} \\ &\quad + O(\ell_P^4). \end{aligned} \tag{53}$$

In order to proceed and admit existence of a Legendre transform we must impose the requirement of having invertible canonical momenta. Namely, there must exist an invertible bijection of the type

$$\Pi_{\mu\nu}^\alpha = \Pi_{\mu\nu}^\alpha(\widehat{\nabla}^\alpha g_{\mu\nu}) \Leftrightarrow \widehat{\nabla}^\alpha g_{\mu\nu} = \widehat{\nabla}^\alpha g_{\mu\nu}(\Pi_{\mu\nu}^\alpha). \tag{54}$$

It means that we must represent the generalized field velocity $\widehat{\nabla}^\alpha g_{\mu\nu}$ as a function of the canonical momentum $\Pi_{\mu\nu}^\alpha$ by inverting Eq. (52). This yields the formal solution

$$\widehat{\nabla}^\alpha g_{\mu\nu} = \frac{\Pi_{\mu\nu}^\alpha}{\kappa \sum_{n=0}^\infty \frac{\ell_P^{2(n-1)}}{2^{n-1}} n (\widehat{\nabla}_\zeta g^{\beta\gamma} \widehat{\nabla}^\zeta g_{\beta\gamma})^{n-1}}. \tag{55}$$

However, we notice that in general the inversion cannot be explicitly completed, so that Eq. (55) represents an implicit relation of the type $\widehat{\nabla}^\alpha g_{\mu\nu} = \widehat{\nabla}^\alpha g_{\mu\nu}(\Pi_{\mu\nu}^\alpha, \widehat{\nabla}^\alpha g_{\mu\nu})$. The full series solution (55) is not admitted for the validity of the Legendre transform. The invertible relationship can only be established for the single term corresponding to $n = 1$. In fact, in such a case one obtains the linear relation

$$\widehat{\nabla}^\alpha g_{\mu\nu} = \frac{1}{\kappa} \Pi_{\mu\nu}^\alpha, \tag{56}$$

which remarkably is independent of ℓ_P , namely it corresponds to the $O(1)$ -term in dimensional Planck-length units. Any other series contribution of $O(\ell_P^{2(n-1)})$ with $n > 1$ must consequently be discarded. We therefore conclude that the requirements set by the validity of a manifestly-covariant Hamiltonian theory are more stringent than those introduced initially for the Lagrangian formulation. A compelling representation is inferred in turn ‘‘a posteriori’’ also for the variational Lagrangian function $L_h(\ell_P, g, \widehat{g})$. Excluding the trivial case of $n = 0$ generating a purely gauge-type constant factor, the only surviving term of the series (35) is the $O(1)$ contribution which is independent of ℓ_P . Therefore, in this sense the Planck length is effectively not allowed to appear at the classical level for consistency with the Hamiltonian principle. Hence, $L_h(\ell_P, g, \widehat{g})$ is found to be necessarily of the form $L_h(\ell_P, g, \widehat{g}) = L_h(g, \widehat{g})$, where

$$L_h(g, \widehat{g}) = \kappa \frac{1}{2} \widehat{\nabla}_\alpha g^{\mu\nu} \widehat{\nabla}^\alpha g_{\mu\nu} \tag{57}$$

carries a quadratic dependence on the generalized field velocities, namely the background covariant derivatives of the variational tensor $g_{\mu\nu}$.

The conclusion is remarkable for the following reasons:

1. The resulting expression for the Lagrangian function $L_h(g, \widehat{g})$ in Eq. (57) is consistent with the manifestly-covariant Lagrangian and Hamiltonian treatments of classical GR formulated initially in Refs. [33,45].
2. In addition, the present mathematical derivation proves the uniqueness of the expression (57), and therefore in turn the uniqueness of the novel Lagrangian and Hamiltonian theories determined in the framework of unconstrained synchronous variational treatment provided by Refs. [33,45].
3. This places important consequences on the mathematical establishment of the corresponding canonical quantum gravity theory from the Hamiltonian principle, which has been developed in Refs. [45,46]. In fact, from one side we have seen that a canonical formulation of classical and quantum Hamiltonian theories can only be established based on the validity of PMC in the framework of the synchronous variational principle. On the other side, both classical and quantum Hamiltonian structures are endowed with a uniqueness character, namely the form of the variational Hamiltonian function is uniquely determined by the requirement of existence of the same Hamiltonian theoretical framework.
4. The physical significance of classical and therefore also related quantum momenta arise perspicuously. These are identified with the covariant derivative defined in terms of extremal Christoffel symbols and acting on the variational gravitational field tensor $g_{\mu\nu}$. In the variational theory the latter is generally different from the extremal

space-time metric tensor $\widehat{g}_{\mu\nu}$, while both coincide in the extremal EFE.

5. The final expression of $L_{\hbar}(g, \widehat{g})$ in Eq. (57) is independent of ℓ_P , which supports the conclusion that the Planck length does not arise as a foundational constant of the theory for the establishment of Lagrangian and Hamiltonian formulations of classical GR. As a consequence, the canonical quantum gravity theory established on this basis and reported in Refs. [45, 46] does share this feature, so that ℓ_P is not inherited by the quantum theory as a constituent part of the theory itself.

6 Conclusions

The outcomes of the research represent notable theoretical conclusions on the role of the Planck length (ℓ_P) also for the theory of quantum gravity. The subject can certainly have relevant implications in quantum gravity theory for the problem of mathematical and physical quantization of gravitational field and its connection with classical general relativity (GR) theory. In fact, having disclosed the existence of the ℓ_P -dependent series summation (35) to which $L_{\hbar}(g, \widehat{g})$ belongs, we can infer the following conclusions for the corresponding manifestly-covariant quantum gravity theory:

- (A) First, the classical and quantum canonical Hamiltonian structures of GR, as well as corresponding definitions of canonical momenta, are composed only of $O(\ell_P^0) = O(1)$ terms and higher-order terms in dimensional Planck-length units remain excluded. This means that the Hamiltonian structure by itself retains a significance in the classical domain, and it does not vanish in the semi-classical limit $\hbar \rightarrow 0$. This aspect raises intriguing scenarios on the possibility of detection and macroscopic evidence of quantum-gravity phenomena, in analogy with quantum electrodynamics theory.
- (B) Second, based on the results established above, it is possible to establish also the level of accuracy of the Hamiltonian formulation associated with $L_{\hbar}(g, \widehat{g})$ given by Eq. (57). In fact, the Hamiltonian theory is of $O(\ell_P^0) = O(1)$ in dimensional Planck-length units, and therefore it is correct through $O(\ell_P^2)$. It means that we can estimate the level of accuracy of the current theory to be of $O(\ell_P^2)$, which represents an extremely small value.

In conclusion, the analysis performed in this paper proves on a mathematical basis the relationship existing between the nature of the Planck length and the establishment of manifestly-covariant classical Lagrangian and Hamiltonian theories of GR, with remarkable conceptual implications also on the formulation of the corresponding canonical quantum-gravity theory. The results proved in this paper are peculiar

of the unconstrained synchronous Lagrangian formulation of Einstein field equations and the requirement of consistency with the principle of manifest covariance and canonical Hamiltonian theory for continuum fields. The conceptual results determined in this way support the candidacy of the present theory to be a promising framework for the establishment of self-consistent classical variational formulations of GR and related canonical quantization yielding a quantum gravity theory. This can shed light on the meaning of Planck length and its role as universal constant and establish a useful framework for the analytical study of quantum gravity field dynamics, its semi-classical limit and the connection with the continuum description of space-time emerging in general relativity.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: The paper reports theoretical analytical study. All data pertinent to this study are contained in the paper.]

Declarations

Conflict of interest The author declares that to his knowledge there are no competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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