On solutions of the Yang-Mills equations in the algebra of h-forms

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Abstract. We study the Yang-Mills equations in the algebra of h-forms, which is developed in the works of N. G. Marchuk and the author. The algebra of h-forms is a special geometrization of the Clifford algebra and is a generalization of the Atiyah-Kähler algebra. We discuss an invariant subspace of the constant Yang-Mills operator in the algebra of h-forms and present particular classes of solutions of the Yang-Mills equations.

1. Introduction

The algebra of h-forms is developed by N. G. Marchuk [1, 2]. The algebra of h-forms is a generalization of the Atiyah-Kähler algebra [3, 4, 5, 6] and the Clifford algebra. We use the algebra of h-forms in the works [7, 8, 9, 10] related to the spin connection of general form and the Yang-Mills equations.

In this paper, we discuss an invariant subspace of the constant Yang-Mills operator in the algebra of h-forms and present particular classes of solutions of the Yang-Mills equations.

2. Yang-Mills equations in pseudo-Euclidean space

Let us consider n-dimensional pseudo-Euclidean space $\mathbb{R}^{p,q}$, $p+q=n\geq 1$, with Cartesian coordinates x^μ, $\mu=1,\ldots,n$. The metric tensor of $\mathbb{R}^{p,q}$ is given by the diagonal matrix

$$\eta=(\eta^{\mu\nu})=(\eta_{\mu\nu})=\operatorname{diag}(\underbrace{1,\ldots,1}_{p},\underbrace{-1,\ldots,-1}_{q}) \tag{1}$$

with its first p entries equal to 1 and the last q entries equal to -1 on the diagonal. We can raise or lower indices of components of tensor fields with the aid of the metric tensor. For example, $F^{\mu\nu}=\eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$. We denote partial derivatives by $\partial_\mu=\frac{\partial}{\partial x^\mu}$.

Let G be a semisimple Lie group and $\mathfrak{g}$ be the real Lie algebra of the Lie group G. Multiplication of elements of $\mathfrak{g}$ is given by the Lie bracket $[U,V]=-[V,U]$. Consider the Yang-Mills equations

$$\partial_\mu A_\nu-\partial_\nu A_\mu-[A_\mu,A_\nu] = F_{\mu\nu}, \tag{2}$$
$$\partial_\mu F^{\mu\nu}-[A_\mu,F^{\mu\nu}] = J^\nu, \tag{3}$$

where $A_\mu : \mathbb{R}^{p,q} \to \mathfrak{g}$ is the potential of the Yang-Mills field, $F_{\mu\nu} : \mathbb{R}^{p,q} \to \mathfrak{g}$ is the strength of the Yang-Mills field, and $J^\nu : \mathbb{R}^{p,q} \to \mathfrak{g}$ is the (non-Abelian) current. The equation (2) can be considered as a definition of the strength $F_{\mu\nu}$. We can substitute $F_{\mu\nu}$ from (2) into (3) and obtain

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu - [A^\mu, A^\nu]) - [A_\mu, \partial^\mu A^\nu - \partial^\nu A^\mu - [A^\mu, A^\nu]] = J^\nu. \tag{4}$$

One suggests that A_μ (and $F_{\mu\nu}$) are unknown and J^ν is known. The current (3) satisfies the (non-Abelian) conservation law

$$\partial_\mu J^\mu - [A_\mu, J^\mu] = 0. \tag{5}$$

The equations (2) - (5) are gauge invariant w.r.t. the transformations

$$A_\mu \to S^{-1} A_\mu S - S^{-1}\partial_\mu S, \qquad F_{\mu\nu} \to S^{-1} F_{\mu\nu} S, \qquad J^\mu \to S^{-1} J^\mu S, \tag{6}$$

where $S = S(x) : \mathbb{R}^{p,q} \to \mathrm{G}$.

3. The algebra of h-forms

Let us consider the real Clifford algebra (or geometric algebra) $C\!\ell_{p,q}$, $p+q=n$ [11, 12, 13], with the generators e^a, $a = 1, \ldots, n$, which satisfy

$$e^a e^b + e^b e^a = 2\eta^{ab} e, \tag{7}$$

where $\eta = (\eta^{ab}) = (\eta_{ab})$ is the diagonal matrix (1) and e is the identity element. The basis elements of $C\!\ell_{p,q}$ are enumerated by ordered multi-indexes of length from 0 to n:

$$e^{a_1\ldots a_k} = e^{a_1} \cdots e^{a_k}, \qquad a_1 < \cdots < a_k, \qquad k = 0, 1, \ldots, n.$$

An arbitrary element $U \in C\!\ell_{p,q}$ of the Clifford algebra has the form

$$U = ue + u_a e^a + \sum_{a_1<a_2} u_{a_1a_2} e^{a_1a_2} + \cdots + u_{1\ldots n} e^{1\ldots n}, \qquad u, u_a, u_{a_1a_2}, \ldots, u_{1\ldots n} \in \mathbb{R}.$$

We denote the subspaces of grade k by

$$C\!\ell^k_{p,q} := \{ \sum_{a_1<a_2<\cdots<a_k} u_{a_1\ldots a_k} e^{a_1\ldots a_k} \}, \qquad k = 0, 1, \ldots, n.$$

We have

$$C\!\ell_{p,q} = \bigoplus_{k=0}^{n} C\!\ell^k_{p,q}.$$

The projection of an arbitrary element $U \in C\!\ell_{p,q}$ onto the subspace $C\!\ell^0_{p,q}$ is denoted by $\langle U \rangle_0$.

Let us consider a vector field with values in the Clifford algebra $h^\mu = h^\mu(x)$: $\mathbb{R}^{p,q} \to C\!\ell_{p,q}$

$$h^\mu(x) = y^\mu(x) e + y^\mu_a(x) e^a + \sum_{a<b} y^\mu_{ab}(x) e^{ab} + \cdots + y^\mu_{1\ldots n}(x) e^{1\ldots n}, \tag{8}$$

which satisfy the same conditions as generators of Clifford algebra (7) in any point of pseudo-Euclidean space:

$$h^\mu h^\nu + h^\nu h^\mu = 2\eta^{\mu\nu} e, \qquad \forall x \in \mathbb{R}^{p,q}. \tag{9}$$

In the case of odd n, the condition $\langle h^1(x)h^2(x)\cdots h^n(x)\rangle_0 = 0$ is also required (see the details in [8]). The expression h^μ is called a Clifford field vector. The expression

$$U = ue + u_\mu h^\mu + \sum_{\mu_1<\mu_2} u_{\mu_1\mu_2}h^{\mu_1\mu_2} + \cdots + u_{1\ldots n}h^{1\ldots n} \tag{10}$$

$$= ue + u_\mu h^\mu + \frac{1}{2!}u_{\mu_1\mu_2}h^{\mu_1}\wedge h^{\mu_2} + \cdots + \frac{1}{n!}u_{\mu_1\cdots\mu_n}h^{\mu_1}\wedge\cdots\wedge h^{\mu_n}, \tag{11}$$

where $u_{\mu_1\ldots\mu_j} = u_{[\mu_1\ldots\mu_j]}$ are skewsymmetric tensor fields of rank j and $\wedge$ is the wedge or exterior product [2], is called an h-form. The set of such h-forms is called an algebra of h-forms $C\ell[h]_{p,q}$. In the Atiyah-Kähler algebra, we have differentials dx^μ instead of Clifford field vectors h^μ. The subspaces of grades k are denoted by

$$C\ell[h]^k_{p,q} := \{\sum_{\mu_1<\mu_2<\cdots<\mu_k} u_{\mu_1\ldots\mu_k}h^{\mu_1\cdots\mu_k}\} = \{\frac{1}{k!}u_{\mu_1\ldots\mu_k}h^{\mu_1}\wedge\cdots\wedge h^{\mu_k}\}, \qquad k = 0, 1, \ldots, n.$$

We have

$$C\ell[h]_{p,q} = \bigoplus_{k=0}^{n} C\ell[h]^k_{p,q}.$$

4. The invariant subspace of the constant Yang-Mills operator

The algebra of h-forms $C\ell[h]_{p,q}$ can be considered as a Lie algebra with respect to the commutator $[U, V] = UV - VU$. Particular classes of solutions of the Yang-Mills equations in the case of the Lie algebra $C\ell[h]_{p,q}$ are considered in [1, 8].

Let us consider the following system of equations

$$[A_\mu, [A^\mu, A^\nu]] = J^\nu, \tag{12}$$

$$\partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu + \partial_\mu[A^\mu, A^\nu] + [A_\mu, \partial^\mu A^\nu - \partial^\nu A^\mu] = 0. \tag{13}$$

Solutions of the system of equations (12) - (13) are also solutions of the Yang-Mills system of equations (4). All constant (which do not depend on $x \in \mathbb{R}^{p,q}$) solutions of the system (4) are solutions of the system (12) - (13). In some sense, the system (12) - (13) models certain aspects of the system of the Yang-Mills equations (4), see the details in [14].

Let us consider the operator

$$\mathrm{Q}(A^\nu) := [A_\mu, [A^\mu, A^\nu]]. \tag{14}$$

We call the operator Q the constant Yang-Mills operator because the system (12) can be also interpreted as the system for constant solutions of the Yang-Mills equations. However, the system (12) (or the system (12) - (13)) may also have nonconstant solutions.

Let us consider the subspace of grade 1 of the algebra of h-forms

$$C\ell[h]^1_{p,q} = \{u_\mu h^\mu\} \subset C\ell[h]_{p,q}.$$

Lemma 4.1 *If* $A^\nu \in C\ell[h]^1_{p,q}$, *then* $\mathrm{Q}(A^\nu) \in C\ell[h]^1_{p,q}$.

Proof. The statement follows from

$$[C\ell[h]^1_{p,q}, C\ell[h]^1_{p,q}] \subset C\ell[h]^2_{p,q} \tag{15}$$

and

$$[C\ell[h]^1_{p,q}, C\ell[h]^2_{p,q}] \subset C\ell[h]^1_{p,q}, \tag{16}$$

see, for example, [15]. ■

We call the subspace $C\ell[h]^1_{p,q}$ an invariant subspace of the constant Yang-Mills operator Q.

Let us consider the algebraic system of equations (12) in the invariant subspace $C\ell[h]^1_{p,q}$. Suppose $A^\mu=\sigma^\mu_\alpha h^\alpha\in C\ell[h]^1_{p,q}$, $J^\nu=\epsilon^\nu_\beta h^\beta\in C\ell[h]^1_{p,q}$, where $\sigma^\mu_\alpha,\epsilon^\nu_\beta\in\mathbb{R}$. The operator (14) takes the form

$$\mathrm{Q}(A^\nu)=[A_\mu,[A^\mu,A^\nu]=\sigma_{\mu\alpha}\sigma^\mu_\beta\sigma^\nu_\lambda[h^\alpha,[h^\beta,h^\lambda]]=4(\sigma_{\mu\lambda}\sigma^{\mu\lambda}\sigma^\nu_\omega-\sigma_{\mu\lambda}\sigma^\mu_\omega\sigma^{\nu\lambda})h^\omega. \qquad (17)$$

In the case of the identity matrix $\Sigma=(\sigma^\mu_\nu)$ (i.e. $\sigma^\mu_\alpha=\delta^\mu_\alpha$), we get

$$[A_\mu,[A^\mu,A^\nu]]=4(n-1)h^\nu.$$

In the case of the diagonal matrices $\Sigma=(\sigma^\mu_\nu)$ and $E=(\epsilon^\mu_\nu)$ with the diagonal elements σ_k, $k=1,\ldots,n$, and ϵ_k, $k=1,\ldots,n$, we get the following system of equations

$$4\sigma_k(S-\sigma_k^2)=\epsilon_k,\qquad S:=\sigma_1^2+\cdots+\sigma_n^2,\qquad k=1,2,\ldots,n \qquad (18)$$

with known ϵ_k, $k=1,\ldots,n$, and unknown σ_k, $k=1,\ldots,n$.

From our point of view, the system (18) deserves attention. In the case $n=3$, the equations (18) are the SU(2) Yang-Mills equations for constant solutions because the element e^{123} lies in the center of the Clifford algebra $C\ell_{3,0}$ and the elements e^ke^{123}, $k=1,2,3$ constitute a basis of the subspace $C\ell^2_{3,0}$, which is a Lie algebra of the spin group $\mathrm{Spin}(3)\cong\mathrm{SU}(2)$. We use this fact and the method of the hyperbolic SVD [16] to present all constant solutions of the SU(2) Yang-Mills equations with arbitrary current in [17, 18].

In the next section, we study the system (18) in the case of an arbitrary natural number n.

5. General solution to the corresponding system of cubic equations

Let us consider the algebraic system of equations

$$4\sigma_k(S-\sigma_k^2)=\epsilon_k,\qquad S=\sigma_1^2+\cdots+\sigma_n^2,\qquad k=1,2,\ldots,n \qquad (19)$$

with known ϵ_k, $k=1,\ldots,n$, and unknown σ_k, $k=1,\ldots,n$.

The general solution to this system in the cases $n=2,3$ is given in [17].

Note that in the case $n=3$, the system (19) has the following symmetry. If the system (19) has a solution $(\sigma_1,\sigma_2,\sigma_3)$ with all nonzero σ_k, $k=1,2,3$, then the system (19) has also a solution of the form $(\frac{K}{\sigma_1},\frac{K}{\sigma_2},\frac{K}{\sigma_3})$, where $K=(\sigma_1\sigma_2\sigma_3)^{\frac{2}{3}}$.

Now let us consider the system with all the same $\epsilon:=\epsilon_1=\cdots=\epsilon_n$ (this is condition for the Yang-Mills current) but in the case of an arbitrary natural number n:

$$4\sigma_k(S-\sigma_k^2)=\epsilon,\qquad S=\sigma_1^2+\cdots+\sigma_n^2,\qquad k=1,2,\ldots,n. \qquad (20)$$

Lemma 5.1 *The system (20), $n\geq4$, has the following symmetry. If the system (20) has a solution with all the same $\sigma:=\sigma_1=\cdots=\sigma_k$:*

$$(\sigma,\sigma,\ldots,\sigma), \qquad (21)$$

then it has also the following n solutions

$$(\sigma(n-2)^{\frac{2}{3}},\frac{\sigma}{(n-2)^{\frac{1}{3}}},\ldots,\frac{\sigma}{(n-2)^{\frac{1}{3}}}),\quad (\frac{\sigma}{(n-2)^{\frac{1}{3}}},\sigma(n-2)^{\frac{2}{3}},\ldots,\frac{\sigma}{(n-2)^{\frac{1}{3}}}),$$
$$\ldots,\quad (\frac{\sigma}{(n-2)^{\frac{1}{3}}},\ldots,\frac{\sigma}{(n-2)^{\frac{1}{3}}},\sigma(n-2)^{\frac{2}{3}}). \qquad (22)$$

Proof. The proof is by direct substitution. ■

Note that in the case $n = 3$ the symmetry is trivial: the system has a unique solution (not four) because all solutions (22) coincide with (21) in this case. In the cases $n \geq 4$, the symmetry is not trivial.

Theorem 5.1 *In the cases $n = 2, 3$, the system (20) with $\epsilon \neq 0$ has a unique solution of the form*

$$\sigma = \sigma_k = \sqrt[3]{\frac{\epsilon}{4(n-1)}}, \qquad k = 1, 2, \ldots, n. \tag{23}$$

In the cases $n \geq 4$, the system (20) with $\epsilon \neq 0$ has $n + 1$ solutions: the solution (23) and n solutions of the form

$$\sigma_1 = \sqrt[3]{\frac{\epsilon(n-2)^2}{4(n-1)}}, \qquad \sigma_k = \sqrt[3]{\frac{\epsilon}{4(n-1)(n-2)}}, \qquad k = 2, \ldots, n, \tag{24}$$

with circular permutation.

The system (20) with $\epsilon = 0$ has the following solutions in the case of an arbitrary $n \geq 2$:

$$(a, 0, \ldots, 0), \quad (0, a, \ldots, 0), \quad \ldots, \quad (0, \ldots, 0, a), \qquad \forall a \in \mathbb{R}. \tag{25}$$

Proof. The proof is given in Appendix A. ■

Note that the results of this paper can be generalized to the case of unitary and pseudo-unitary groups in the formalism of the algebra of h-forms (see about the realization of different classical matrix Lie groups in Clifford algebras in [15, 19, 20, 21, 22]).

In this paper, we discussed mathematical structures. The relationship of the proposed mathematical constructions with objects of the real world (elementary particles) is beyond the scope of this study. The explicit formulas (21) - (25) can have physical consequences.

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Appendix A. The proof of Theorem 5.1

The case of $\epsilon = 0$ is trivial, we have the solutions (25).

Let us consider the case $\epsilon \neq 0$. In this case, $\sigma_k \neq 0, k = 1, \ldots, n$. Note that if $\epsilon > 0$, then $\sigma_k > 0$, $k = 1, \ldots, n$. If we change the sign of ϵ, then we must change the sign of all σ_k, $k = 1, \ldots, n$: if $\epsilon < 0$, then $\sigma_k < 0$, $k = 1, \ldots, n$. Without loss of generality, we can assume that $\epsilon > 0$ and $\sigma_k > 0$, $k = 1, \ldots, n$.

We use the following change of variables $x_1 = \sigma_1 > 0$, $x_k = \frac{\sigma_k}{\sigma_1} > 0$, $k = 2, \ldots, n$. The system takes the form

$$4x_1^3(x_2^2 + \cdots + x_n^2) = \epsilon, \qquad 4x_k x_1^3(1 + x_2^2 + \cdots + x_n^2) = \epsilon, \qquad k = 2, \ldots, n.$$

We obtain the following expression for σ_1

$$\sigma_1 = x_1 = \sqrt[3]{\frac{\epsilon}{4(x_2^2 + \cdots + x_n^2)}} \tag{A.1}$$

and the following system of $n-1$ equations for $x_2, \ldots, x_n$:

$$x_2^2+\cdots+x_n^2 = x_2(1+x_3^2+\cdots+x_n^2) = x_3(1+x_2^2+x_4^2+x_5^2+\cdots+x_n^2)$$
$$=\cdots = x_n(1+x_2^2+x_3^2+\cdots+x_{n-1}^2).$$

Equating the first expression with the second expression, we get

$$x_3^2(1-x_2) = x_2(1+x_4^2+\cdots+x_n^2)-(x_2^2+x_4^2+\cdots+x_n^2) = (1-x_2)(x_2-x_4^2-\cdots-x_n^2)$$

i.e.

$$(1-x_2)(x_2-x_3^2-x_4^2-\cdots-x_n^2)=0.$$

Proceeding in the same way with the rest of the equations, we obtain the system of equations

$$(1-x_k)(x_k+x_k^2-T)=0, \qquad k=2,\ldots,n, \qquad T=x_2^2+\cdots+x_n^2. \tag{A.2}$$

Let the expressions in the first brackets of all equations (A.2) are equal to zero, i.e. $x_k=1$, $k=2,\ldots,n$. Using (A.1) and $\sigma_k = x_1x_k$, $k=2,\ldots,n$, we get the solution (23).

Let the expressions in the second brackets of all equations (A.2) are equal to zero. If we have $x_i+x_i^2-T=0$ and $x_j+x_j^2-T=0$ for $i\neq j$, then subtracting one equation from the other, we get $(x_i-x_j)(x_i+x_j+1)=0$ and $x_i=x_j$. We obtain $x_2=\cdots=x_n$. Denoting it by x, we get $x-(n-2)x^2=0$, i.e. $x=\frac{1}{n-2}$. Finally, $x_1=\sigma_1=\sqrt[3]{\frac{\epsilon}{4(\frac{1}{n-2})^2(n-1)}}$, $\sigma_k=x_kx_1$, $k=2,\ldots,n$, and we get the solution (24).

Let the expressions in the first brackets of all equations (A.2), except one, are equal to zero and the expression in the second brackets of one of the equations (for example, for $k=n$) is equal to zero. Then $x_k=1$, $k=2,\ldots,n-1$ and $x_n-x_2^2-\cdots-x_{n-1}^2=0$, i.e. $x_n=n-2$. Using (A.1), we get the solutions of type (24) with circular permutation.

Let the expressions in the second brackets of $2\leq r\leq n-2$ equations (for example, for $k=2,\ldots,r+1$) (A.2) are equal to zero and the expressions in the first brackets of the rest $n-r-1$ equations (for $k=r+2,\ldots,n$) (A.2) are equal to zero. Then $x_2=\cdots=x_{r+1}=:x$, $x_{r+2}=\cdots=x_n=1$. We get $x+x^2-x^2r-(n-r-1)=0$, i.e. $(1-r)x^2+x+(r+1-n)=0$. The discriminant of this quadratic equation $D=1+4(r-1)(r+1-n)$ is negative, because $r\geq 2$ and $r\leq n-2$. There are no solutions of this type.

The theorem is proved.

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