New deformations of N=2 supergravity

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New deformations of N=2 supergravity

Nieuwe deformaties van N=2 supergravitatie

(met een samenvatting in het Nederlands)

Proefschrift

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Publications

This thesis is based on the following publications:

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Contents

Chapter 1

Introduction

Supergravity is a field theory that combines general relativity with local supersymmetry. It serves as a framework for studying a large variety of phenomena, such as black holes and cosmology. Supergravity also describes the low-energy degrees of freedom associated with string theory. String theory, or its conjectured extension M-theory, has been proposed as a consistent theory of quantum gravity. It describes the dynamics of extended objects, such as strings and membranes, which propagate in 10 (string theory) or 11 (M-theory) space-time dimensions. To make contact with the four-dimensional world we see around us, one usually assumes that the extra dimensions are compactified. This means that they are curled up in a compact space, which is so small that it is not directly observable in present-day experiments.

Symmetries play an important role in applications of supergravity. Therefore we start this chapter with an overview of some of the properties of these symmetries, which gradually leads us to supergravity. After discussing some of the general properties of supergravity, we will turn more specifically to the subject of this thesis: deformations of supergravity. In sections 1.2 and 1.3 we introduce, by means of simple examples, two concepts that are central in this thesis: gauge equivalence and electric/magnetic duality. Finally, in section 1.4 we will summarize the content of this thesis.

1.1 Symmetries and deformations

It is useful to distinguish between two types of symmetries, namely space-time and internal symmetries. Let us first consider space-time symmetries. In the absence of gravity, i.e. when space-time is flat, a relativistic theory should be invariant under Lorentz transformations and translations, collectively called Poincaré transformations. When gravitational forces become important we know that special relativity is replaced by general relativity. Space-time can become curved and the transformations that play a role are general coordinate transformations, also called space-time diffeomorphisms, which are arbitrary reparametrizations of the space-time coordinates.

Internal symmetries, on the other hand, act on an internal space, which is not related to space-time. Internal symmetries transform a set of fields into each other. For example, a number of non-interacting fields, all of the same mass, can rotate into each other. As a result, their equations of motion rotate into an equivalent set of equations. The Lagrangian from which the field equations can be derived is then invariant under these internal rotations.

When transformations are the same at every point in space-time, they are called rigid and when they are allowed to differ at different points in space-time, local. There is a well-defined procedure for promoting a rigid symmetry to a local symmetry, which is sometimes called gauging. It requires the introduction of so-called gauge fields, which can propagate the information of these transformations from one space-time point to another. A well-known example of a theory with local internal symmetry is Maxwell's theory of electrodynamics. The photon field acts as the gauge field associated with local phase transformations. Also space-time symmetries can be divided into rigid and local. For instance, the Poincaré transformations of special relativity are rigid, whereas general coordinate transformations of general relativity are local space-time symmetries.

Supersymmetry

Supersymmetry is different from the symmetries above. In some sense it acts like an internal symmetry, since it transforms fields into each other. What makes supersymmetry special is that it relates fields of half-integer spin to fields of integer spin. Fields of integer spin describe bosons, which are particles with the property that they can occupy the same quantum state. Fields of half-integer spin describe fermions, which, in contrast to bosons, cannot occupy the same quantum state. In relativistic field theories, fermions are described by anti-commuting spinors. A spinor is a representation of the Lorentz group, and hence it transforms non-trivially under Lorentz transformations. Since supersymmetry relates bosons to fermions, the generators and parameters of supersymmetry transformations must also be spinors. This is in contrast to the internal symmetries we discussed before, where the generators and transformation parameters are Lorentz scalars. Another defining property of supersymmetry is that two successive supersymmetry transformations lead to a space-time translation. Therefore supersymmetry is in some sense both an internal and a space-time symmetry.

Just like any symmetry, supersymmetry can be realized rigidly or locally. The fact that supersymmetry and space-time symmetries are related has an important and intriguing consequence. When a theory is invariant under rigid supersymmetry, it must also be invariant under rigid space-time symmetries. On the other hand, when a theory is invariant under local supersymmetry, it must necessarily be invariant under local space-time transformations, i.e. general coordinate transformations, the symmetries of general relativity. Hence space-time can be curved. Accordingly, theories that are invariant under local supersymmetry are called supergravity. Among the fields of supergravity is the spin-2 metric field, associated with the graviton. Its supersymmetric partner is the gauge field of supersymmetry, called the gravitino field, which has spin $\frac{3}{2}$.

Historically, supergravity was first developed as a four-dimensional theory [1], but it was soon generalized to other space-time dimensions. It was also realized that it is possible to have more than one kind of supersymmetry transformation, which is referred to as extended supersymmetry. The supersymmetry generators, also called supercharges, then transform reducibly under the Lorentz group and comprise N irreducible Lorentz-spinors. The number of components of such a spinor depends on the dimension (see e.g. [2]). When there are N copies of such a spinor, the number of supercharges is N times the number of spinor components. For instance, in four space-time dimensions, an irreducible (Majorana) spinor has 4 real components, and hence four-dimensional $N = 2$ supergravity has $4 \cdot 2 = 8$ supercharges. A supergravity theory can therefore be characterized by two numbers, the number of supersymmetries N and the space-time dimension d. The bosonic and fermionic fields that transform among each other by the N supersymmetry transformations are called a supermultiplet. It is clear that, the higher the number N, the more restricted a theory is, as more fields will be related to each other. Moreover, with increasing N, fields of higher and higher spin occur. In particular, theories with more than 32 supercharges contain fields with spin greater than 2. These fields cannot consistently couple to other fields or to themselves if one insists on having a finite number of fields (for a review, see [3]). Therefore conventional supergravities have at most 32 supercharges. Since an irreducible spinor in eleven dimensions has exactly 32 components, conventional supergravity can not be realized in dimensions higher than eleven.

Deformations

The fields of supergravity usually include a set of matter fields (e.g. scalar fields) and vector gauge fields, which transform under some internal symmetry group. When this symmetry group is realized rigidly, the vector gauge fields transform under a trivial local abelian symmetry $[\mathrm{U}(1)]^n$, where n is the number of vector fields, under which no fields are charged, i.e. the matter fields do not couple directly to the vector fields. These theories are sometimes referred to as ungauged supergravity theories. They arise as effective field theories of string theory, or M-theory, compactified on a flat or Ricci-flat manifold, such as an higher-dimensional torus, or a Calabi-Yau manifold.

One class of deformations we study in this thesis are gauge deformations. Starting with an ungauged theory, one can assign charges to a subset of the matter fields. As a result, some of the vector gauge fields will couple to the matter fields, consistent with the internal symmetry group. This is referred to as gauged supergravity [4, 5]. To preserve supersymmetry the theory typically needs to be extended with a scalar potential, which can have important consequences. Depending on its form, this scalar potential can for instance generate (partial) spontaneous breaking of supersymmetry, it can give masses to the scalar fields, and it can give rise to an effective cosmological constant. All of these features are relevant for many applications. Just like ungauged supergravity is the low-energy limit of flat string theory compactifications, gauged supergravity is the low-energy limit of so-called flux compactifications of string theory [6, 7]. Here the word flux refers either to a generalization of the electric and magnetic fluxes known from Maxwell's theory, induced by fields in the internal manifold, or to so-called geometric fluxes, which twist the geometry of the internal manifold.

In four space-time dimensions, which is what we consider in this thesis, gauging internal symmetries in the manner described above is subtle due to the presence of electric/magnetic duality. This duality is a generalization of the duality rotations in Maxwell's theory, under which the electric and magnetic fields and inductions are rotated into each other according to,

$$\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} \longrightarrow \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\,, \qquad \begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} \longrightarrow \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix}\,. \tag{1.1}$$

Under such a duality rotation the Maxwell equations are transformed into an equivalent set. The duality can be extended in the presence of charges, provided both electric and magnetic charges occur. Under the duality these charges are then rotated into each other.

In that case, it is only a matter of convention to specify a charge as electric, since one can always make a duality rotation to a different frame where the charge would be magnetic.

The situation changes when one writes down a Lagrangian, from which the Maxwell equations can be derived,

$$\mathcal{L} = \tfrac{1}{2}\left(\mathbf{E}^2 - \mathbf{H}^2\right) , \tag{1.2}$$

which, as one can check, is not invariant under the rotations (1.1). Hence, electric/magnetic duality is not preserved at the level of the Lagrangian. This is in contrast to the Maxwell equations, which are preserved under electric/magnetic duality, as discussed above. Nevertheless, the Lagrangian has some special properties under electric/magnetic duality, as the latter relates different Lagrangians with equivalent field equations. In section 1.3 we illustrate these issues in more detail in the context of an example.

Furthermore, the presence of charges in the Lagrangian requires the introduction of a vector potential, or gauge field, from which the electric and magnetic fields can be derived. However, whereas it is well-known how to couple this gauge field to the electric charge, it is less trivial to couple it to the magnetic charge. In fact, generically magnetic charges lead to the presence of non-local expressions in the Lagrangian [8]. This is reflected by the electric/magnetic duality rotations, under which the gauge field is rotated into a dual gauge field, which is not locally related to the old gauge field.

We now return to the issue of gauging internal symmetries of four-dimensional Lagrangians with a set of matter and abelian vector gauge fields. As mentioned before, electric/magnetic duality transformations relate equivalent Lagrangians. A subgroup of the electric/magnetic duality group may constitute an *invariance* of the generalized Maxwell equations, which means that the electric/magnetic rotation is induced by transformations of the fields in the theory. Gauging this invariance group would in general require coupling to magnetic charges, which, as we have argued above, is problematic. One way to deal with this is to apply an appropriate electric/magnetic duality rotation which converts all the relevant charges to electric ones and in this frame carry out the gauging according to the standard procedure. This can, however, be cumbersome in practice. There is an alternative approach that avoids this and generalizes to gauge groups including magnetic charges. This is the so-called embedding tensor approach introduced in [9]. We will use this approach to study general gauge deformations of four-dimensional $N=2$ supergravity.

Other deformations we consider in this thesis are supersymmetric higher-derivative couplings. These couplings play an important role as next-to-leading order corrections to low-energy effective actions of string theory [10]. For applications and a better understanding of this fundamental theory, knowledge of the possible higher-derivative invariants in supergravity theories is desired.

The importance of higher-derivative couplings can, for instance, be illustrated in the context of black holes. As is well-known, there is a close analogy between the laws of black hole mechanics and the laws of thermodynamics [11–13]. According to this analogy, the area of the event horizon of a black hole (with a specific proportionality constant) plays the role of a thermodynamic entropy. If string theory is indeed a consistent theory of quantum gravity it should provide a statistical interpretation of this black hole entropy. This can be checked explicitly for certain supersymmetric black holes, also called BPS black holes. In string theory, these black holes are given by strings and so-called D-branes that wrap around the compactified dimensions. The entropy of such a black hole is then given by the logarithm of the number of D-brane configurations that lead to the same macroscopic black hole [14]. A description of the corresponding macroscopic black hole is provided by suitable effective four-dimensional supergravity theories, and, as mentioned before, its macroscopic entropy is given by the area of its event horizon. Comparing results from the microscopic and the macroscopic description of the entropy thus provides a highly non-trivial test on string theory. Such a comparison was performed for the first time in [14] and agreement was found in the limit that certain charges are large. Since higher-derivative couplings arise as sub-leading corrections to the low energy effective action of string theory they are needed for a more precise matching [15–17]. Including higher-derivative terms in supergravity turns out to be complicated, but considerable progress has been made.

1.2 Gauge equivalence

In this thesis we make use of the concept of gauge equivalence to describe $N = 2$ supergravity in a setting which has a larger local symmetry group. We will illustrate this idea with a simple example of the gauge equivalence between a massive vector field and a massless vector field together with a scalar field. The Lagrangian of a vector field V_μ with mass m is as follows,

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 - \tfrac{1}{2}m^2 V_\mu^2\,. \tag{1.3}$$

The first term in the Lagrangian is invariant under the abelian gauge transformation $\delta V_\mu = \partial_\mu \Lambda$ familiar from Maxwell's theory. However, the mass-term does not respect this invariance. Hence the four field components of a massive vector describe three physical degrees of freedom, corresponding to two transversal polarizations and one longitudinal polarization.

We now make the following redefinition, introduced by Stueckelberg [18], in terms of a new vector field W_μ and a scalar field ϕ,

$$V_\mu = W_\mu - m^{-1}\partial_\mu\phi \,. \tag{1.4}$$

This redefinition is not unique, as it is invariant under,

$$\delta W_\mu(x) = \partial_\mu\Lambda(x)\,, \qquad \delta\phi(x) = m\,\Lambda(x)\,. \tag{1.5}$$

Therefore also the Lagrangian is invariant under (1.5), which we write in terms of the new fields,

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu W_\nu - \partial_\nu W_\mu)^2 - \tfrac{1}{2}\mathcal{D}_\mu\phi\,\mathcal{D}^\mu\phi\,. \tag{1.6}$$

Here $\mathcal{D}_\mu\phi$ is the covariant derivative under (1.5) given by,

$$\mathcal{D}_\mu\phi = \partial_\mu\phi - mW_\mu\,. \tag{1.7}$$

The Lagrangian (1.6) takes the form of a conventional gauge invariant Lagrangian for a scalar field coupled to a (massless) abelian gauge field. The gauge field describes two physical degrees of freedom, corresponding to two transversal polarizations, and the scalar field one, which adds up to the three physical degrees of freedom of the original massive vector field. Although we have introduced a local gauge symmetry, the presence of the scalar field ensures that the total number of degrees of freedom remains the same. Imposing a gauge condition such as $\phi = 0$ leads back to the original Lagrangian (1.3), and hence the two models are gauge equivalent. The scalar field ϕ is called a compensating field.

In a second example, which is particularly relevant to this thesis, we discuss a gauge equivalent form of Einstein gravity, which, besides the usual invariance under general coordinate transformations, admits invariance under local scale transformations, or dilatations. Under these dilatations the metric transforms as follows,

$$\delta g_{\mu\nu} = -2\Lambda_{\rm D}(x) g_{\mu\nu}\,. \tag{1.8}$$

From this one can derive,

$$\begin{aligned} \delta\sqrt{-g} &= -\,4\Lambda_{\rm D}\sqrt{-g}\,, \\ \delta R &= 2\Lambda_{\rm D} R - 6\,\Box\Lambda_{\rm D}\,. \end{aligned} \tag{1.9}$$

where R is the Ricci scalar and g the determinant of the metric. Using a scalar field ϕ that transforms under dilatations as,

$$\delta\phi = \Lambda_{\rm D}\phi\,, \tag{1.10}$$

one can write down the following Lagrangian which is invariant under local coordinate transformations and dilatations,

$$\mathcal{L} = \sqrt{-g}\,\left(\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{6}\,R\,\phi^2\right)\,. \tag{1.11}$$

By choosing a gauge in which $\phi = 1$ we fix the dilatational invariance and we find the Einstein-Hilbert action,

$$\mathcal{L} = -\tfrac{1}{6}\,\sqrt{-g}\,R(e)\,. \tag{1.12}$$

Therefore (1.11) is gauge equivalent to the Einstein-Hilbert action. The scalar field ϕ compensates for the extra gauge invariance present in the conformal action.

In a similar fashion we describe supergravity with a gauge-equivalent theory, called superconformal gravity. This theory has extra conformal invariances, but the presence of compensating fields ensures that the total number of physical degrees of freedom remain the same. Notice that in order to have scale invariance, the sign in front of the kinetic term for the scalar field in (1.11) is necessarily opposite to what it is for a physical scalar. A similar situation also occurs in superconformal gravity, as we will see.

Although in these examples the benefits of taking one approach over the other are not so obvious, in the case of supergravity there are clear advantages to using the superconformal description. Since there are many more fields involved, the presence of the extra symmetry puts welcome restrictions on the model that make it easier to construct Lagrangians. In particular the superconformal multiplets are smaller because they are subject to more symmetries. Also for the construction of higher-derivative invariants, the superconformal method is superior.

1.3 Electric/magnetic duality

As already discussed, the concept of electric/magnetic duality plays an important role in four-dimensional supergravity models, and in fact in many effective field theories, with or without supersymmetry. In this section we illustrate a few properties of this duality using a simple field theory. These properties will come back in chapter 3 in a more complicated setting.

As mentioned before, Maxwell's theory of electrodynamics in four space-time dimensions without charges is the simplest example of a theory exhibiting electric-magnetic duality. We will slightly generalize the action such that it shows more similarities to a generic effective action,

$$\mathcal{L} = -\tfrac{1}{4} I F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{8} \mathrm{i} R\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \,. \tag{1.13}$$

Here $F_{\mu\nu} = 2\partial_{[\mu} W_{\nu]}$ is the field strength written in terms of the gauge field W_μ. The parameters I and R are just real constants, but in an effective field theory they may take the form of field-dependent matrices. Hence, whereas the last term in this action is equal to a total derivative, in a more general setting this might no longer be the case. Therefore it is important to keep this term to see its role in the duality transformation.

Using the definitions in appendix A, we can split the field strength into a selfdual and anti-selfdual part and rewrite the Lagrangian as,

$$\mathcal{L} = \tfrac{1}{4} \mathrm{i} \left(\tau F^-_{\mu\nu} F^{-\mu\nu} - \bar{\tau} F^+_{\mu\nu} F^{+\mu\nu} \right) , \tag{1.14}$$

where τ is given by,

$$\tau = R + \mathrm{i} I \,. \tag{1.15}$$

The Lagrangian (1.13) or (1.14) is invariant under abelian transformations of which W_μ is the gauge field,

$$\delta W_\mu = \partial_\mu \Lambda \,. \tag{1.16}$$

Since no fields are present that are charged under the abelian gauge symmetry, the gauge fields only appear in the field strength. Therefore, the Bianchi identity and equation of motion for the field strength can be written in a nicely symmetric form,

$$\partial_{[\mu}F_{\nu\rho]} = 0 = \partial_{[\mu}G_{\nu\rho]}\,, \tag{1.17}$$

where we have defined,

$$G_{\mu\nu} \equiv \mathrm{i}\,\varepsilon_{\mu\nu\rho\sigma}\,\frac{\partial\mathcal{L}}{\partial F_{\rho\sigma}}\,. \tag{1.18}$$

From the Lagrangian in (1.14) we derive,

$$G^-_{\mu\nu} = \tau F^-_{\mu\nu}\,. \tag{1.19}$$

It is immediately apparent from (1.17) that the two equations are rotated to an equivalent set of equations under real 2-dimensional transformations,

$$\begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde F_{\mu\nu} \\ \tilde G_{\mu\nu} \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix}\,, \tag{1.20}$$

with parameters that satisfy $UV - WZ = 1$. Notice that this is a generalization of (1.1). As before, this is called an electric/magnetic duality transformation, as it rotates the electric and magnetic fields derived from the field strength $F_{\mu\nu}$. After performing a transformation, $\tilde F_{\mu\nu}$ satisfies a Bianchi identity, so it can be assigned to be the field strength of a new gauge field, $\tilde A_\mu$. One can check that this new gauge field is non-locally related to the old gauge field A_μ, except for transformations with $Z = 0$. Next, the equation for $\tilde G_{\mu\nu}$ can be interpreted as the equation of motion derived from some new Lagrangian $\tilde{\mathcal{L}}$ depending on the new field strength $\tilde F_{\mu\nu}$, via $\tilde G_{\mu\nu} = \mathrm{i}\,\varepsilon_{\mu\nu\rho\sigma}\partial\tilde{\mathcal{L}}/\partial\tilde F_{\rho\sigma}$. In the generic case, where the Lagrangian is an arbitrary function of n field strengths, the duality transformations must belong to the group $\mathrm{Sp}(2n;\mathbb{R})$ in order for $\tilde{\mathcal{L}}$ to exist, as will be discussed in chapter 3.

The new Lagrangian can be written in the same form as the old Lagrangian,

$$\tilde{\mathcal{L}} = \tfrac{1}{4}\mathrm{i}\left(\tilde\tau\tilde F^-_{\mu\nu}\tilde F^{-\mu\nu} - \bar{\tilde\tau}\tilde F^+_{\mu\nu}\tilde F^{+\mu\nu}\right)\,, \tag{1.21}$$

where,

$$\tilde\tau = \frac{W + V\tau}{U + Z\tau}\,. \tag{1.22}$$

The Lagrangian does not transform as a function, since $\tilde{\mathcal{L}}(\tilde{F}) \neq \mathcal{L}(F)$.[1] Now let us assume that τ is field-dependent, as it will be in a generic effective field theory. Hence, $\tau = \tau(X)$ for some field X. When the transformation (1.22) is induced by a transformation of this field X, i.e. when $\tilde{\tau}(\tilde{X}) = \tau(\tilde{X})$, then the duality is an *invariance* of the theory. This means that the Lagrangian $\mathcal{L}$ remains unchanged under the duality transformation, i.e.,

$$\tilde{\mathcal{L}}(\tilde{F}, \tilde{X}) = \mathcal{L}(\tilde{F}, \tilde{X})\,. \tag{1.23}$$

Note that in the literature the word duality is used both for equivalence and for invariance transformations. In chapter 3 we are interested in duality *invariances*, as these are the ones that can be gauged.

To conclude, let us consider electric/magnetic duality transformations (1.20) with $Z = 0$, so $UV = 1$. This is called the electric subgroup of the electric/magnetic duality group. We already noted that in this case the transformed gauge field is locally related to the old gauge field. Under transformations with $Z = 0$ we find that the Lagrangian transforms as,

$$\tilde{\mathcal{L}}(UF) = \mathcal{L}(F) - \tfrac{1}{8}\mathrm{i}WU\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,. \tag{1.24}$$

So up to a total derivative $\mathcal{L}$ is invariant under the electric subgroup. This is the reason why using conventional methods only the electric subgroup of the duality group can be gauged. In chapter 3 we will see how more general subgroups of the electric/magnetic duality group can be gauged.

1.4 The content of this thesis

In this thesis we focus on four-dimensional $N = 2$ supergravity. We derive new deformations related to general gaugings and higher-derivative couplings, and we study their consequences in several applications.

This thesis is organized as follows. In chapter 2 we present the basics of $N = 2$ supergravity. We introduce the $N=2$ supermultiplets that we consider in this thesis, and their corresponding supersymmetry transformations. We also present the corresponding (ungauged) Lagrangians.

In chapter 3 we review electric/magnetic duality and study general gauge deformations of $N=2$ supergravity theories, using the embedding tensor approach introduced in

[1] In fact, one can show that the combination $\mathcal{L}(F) + \frac{1}{8}\mathrm{i}\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\,G_{\rho\sigma}$ does transform as a function.

[9]. In this approach, one introduces, from the start, both electric and magnetic gauge fields. To avoid extra degrees of freedom, tensor gauge fields must be included with corresponding gauge symmetries. The charges are encoded in a so-called embedding tensor. The gauge group is only restricted by two constraints on the embedding tensor. One of these constraints implies that the charges are mutually local. This means that there exists always an electric/magnetic duality frame in which all charges are electric.

Two interesting applications of the embedding tensor approach are considered in chapter 4. First we briefly review partial supersymmetry breaking in maximally symmetric space-times in the presence of general gaugings. In flat Minkowski space, it was established that residual supersymmetry is only possible in the presence of magnetic charges [19–25]. We therefore briefly review the situation in the context of the embedding tensor approach, where it is natural to have both electric and magnetic charges. As a new application we study possible supersymmetric solutions in $\mathrm{AdS}_2 \times S^2$ space-times. We find two classes of solutions. One is fully supersymmetric. It contains for instance the near-horizon solution of ungauged supergravity that appears for BPS black holes. The other class exhibits four supersymmetries. It contains the near-horizon solutions of BPS black holes in $N=2$ gauged supergravity. The spinor parameters associated with the four supersymmetries are AdS_2 Killing spinors that are constant on S^2, so that they carry no spin. Nevertheless the bosonic background is rotationally invariant. These two examples illustrate how the embedding tensor formalism can be used to obtain rather general results about the realizations of $N=2$ gauged supergravities.

In chapter 5 we introduce a systematic procedure to construct a large variety of new higher-derivative deformations in $N=2$ supergravity. As an explicit example, many of the bosonic terms of the supergravity-coupled invariants that contain F^4-, R^2F^2-, and R^4-terms, will be discussed. Here F denotes the abelian vector multiplet field strengths and R the Riemann tensor. We study the possible contribution of these new couplings to the entropy and the electric charges of BPS black holes. As it turns out we can derive a 'non-renormalization' theorem according to which these contributions vanish. This result is not entirely unexpected, in view of the fact that there was already a good agreement for the subleading contributions to the BPS entropy obtained from microstate counting and from supergravity, in which the new couplings had so far not been incorporated. Hence the existence of the non-renormalization theorem offers a partial explanation for this agreement.

Chapter **2**

Supermultiplets and Lagrangians of N=2 superconformal gravity

In section 1.2 we have seen how the concept of gauge equivalence allows one to rewrite a theory so that it has a larger local symmetry group, using so-called compensating fields. These compensating fields ensure that the total number of degrees of freedom remains unchanged in the formulation with the extra local symmetry. We showed in an example how the Einstein-Hilbert action could be constructed in terms of an action which, besides the usual diffeomorphism invariance, is also invariant under local scale transformations, using a scalar field as compensator.

Based on a similar construction, we will describe supergravity, sometimes referred to as Poincaré supergravity, by superconformal gravity with suitable compensating fields [26–28]. These compensating fields are now contained in supermultiplets. Superconformal gravity combines local conformal space-time symmetries with local supersymmetry. Upon gauge-fixing the compensating fields, the local conformal space-time symmetries are reduced to diffeomorphisms.

As preparation, we will start this chapter by reconsidering conformal gravity in a more systematic manner, which paves the way for the construction of $N=2$ superconformal gravity presented in section 2.2. We will first introduce the gauge fields corresponding to the conformal symmetries, and show that one can consistently impose constraints, such that the number of independent gauge fields is reduced. Using a compensating field, which couples to the conformal gauge fields, one can write down Lagrangians that are invariant under conformal symmetries. We will illustrate this by re-deriving the Lagrangian (1.11), which was shown to be gauge-equivalent to the Einstein-Hilbert action, upon gauge-fixing the compensating field.

In section 2.2 we will then indicate how a similar constrained gauge theory can be set up for $N=2$ superconformal gravity, and we will introduce the Weyl multiplet, which contains the superconformal gauge fields. In the remainder of the chapter we introduce matter supermultiplets, and we end in section 2.6 with a Lagrangian that describes $N=2$ superconformal gravity. For introductory texts on supersymmetry we refer to e.g. [29, 30].

2.1 Conformal gravity

Any relativistic theory is invariant under the Poincaré group, which consists of translations (P) and Lorentz rotations (M). For a theory without intrinsic scale, such as mass or coupling constants, this group is extended to the conformal group, which is the group of transformations that leave the light-cone invariant. Next to the symmetries of the Poincaré group it consists of dilatations, or scale transformations (D) and conformal boosts, or special conformal transformations (K). In four dimensions it is given by the group SO(4,2). To each generator we associate a gauge field and a transformation parameter,

$$\begin{array}{lcccc} \text{generators:} & P^a & M^{ab} & D & K^a \\ \text{gauge fields:} & e_\mu{}^a & \omega_\mu{}^{ab} & b_\mu & f_\mu{}^a \\ \text{parameters:} & \xi^a & \varepsilon^{ab} & \Lambda_{\rm D} & \Lambda_{\rm K}{}^a\,, \end{array} \tag{2.1}$$

where ε^{ab} is an antisymmetric tensor. The indices $a, b, \dots = 0, \dots 3$ label the coordinates of a flat manifold with Minkowski signature, which at this point is still an abstract internal space. In a moment we will see how it can be related to the tangent bundle of space-time. The infinitesimal transformations, which follow from the Lie algebra of SO(4,2), are given by,

$$\begin{aligned} \delta e_\mu{}^a &= \mathcal{D}_\mu \xi^a - \Lambda_{\rm D} e_\mu{}^a + \varepsilon^{ab} e_{\mu b}\,, \\ \delta \omega_\mu{}^{ab} &= \mathcal{D}_\mu \varepsilon^{ab} + 2\,\Lambda_{\rm K}^{[a} e_\mu{}^{b]} + 2\,\xi^{[a} f_\mu{}^{b]}\,, \\ \delta b_\mu &= \partial_\mu \Lambda_{\rm D} + \Lambda_{\rm K}{}^a e_{\mu a} - \xi^a f_{\mu a}\,, \\ \delta f_\mu{}^a &= \mathcal{D}_\mu \Lambda_{\rm K}{}^a + \Lambda_{\rm D} f_\mu{}^a + \varepsilon^{ab} f_{\mu b}\,. \end{aligned} \tag{2.2}$$

The derivative $\mathcal{D}_\mu$ is covariant with respect to dilatations and Lorentz transformations, for instance,

$$\mathcal{D}_\mu \xi^a = \partial_\mu \xi^a + b_\mu \xi^a - \omega_\mu{}^{ab} \xi_b \,. \tag{2.3}$$

Again we stress that at this point the conformal transformations are not space-time transformations, but are treated as internal transformations. The gauge fields transform separately as vectors under general coordinate transformations.

Using the transformation rules (2.2) it is easy to construct the curvature tensors. We list two of them that we will need below,

$$\begin{aligned} R(P)_{\mu\nu}{}^a &= 2\,\partial_{[\mu}\, e_{\nu]}{}^a + 2\, b_{[\mu}\, e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\, e_{\nu]b} \,, \\ R(M)_{\mu\nu}{}^{ab} &= 2\,\partial_{[\mu} \omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac} \omega_{\nu]c}{}^b - 4 f_{[\mu}{}^{[a} e_{\nu]}{}^{b]} \,. \end{aligned} \tag{2.4}$$

It is well known that by imposing so-called conventional (algebraic) constraints on the curvatures of the superconformal fields one can relate the transformations (2.2) to space-time transformations [31, 32]. Here the gauge field of the translations $e_\mu{}^a$ is assumed to be invertible and identified as the vielbein. As a result of the constraints, the local translations are effectively replaced by general coordinate transformations of space-time. To see this, note that one can rewrite a P-transformation of the vielbein $e_\mu{}^a$ as follows,

$$\begin{aligned} \delta_{\rm P}\, e_\mu{}^a = \mathcal{D}_\mu \xi^a &= \xi^\nu \partial_\nu e_\mu{}^a + \partial_\mu \xi^\nu e_\nu{}^a + \xi^\nu b_\nu e_\mu{}^a - \xi^\nu \omega_\nu{}^{ab} e_{\mu b} + \xi^\nu R(P)_{\mu\nu}{}^a \\ &= \delta^{\rm cov}(\xi)\, e_\mu{}^a + \xi^\nu R(P)_{\mu\nu}{}^a \,. \end{aligned} \tag{2.5}$$

Upon imposing the constraint $R(P)_{\mu\nu}{}^a = 0$, the right hand side reduces to a general coordinate transformation with appropriate covariantization terms, i.e. a covariant general coordinate transformation, which we denoted in the second line by $\delta^{\rm cov}(\xi)$. Hence, after imposing this constraint the P-transformations will be ignored.

The constraint $R(P)_{\mu\nu}{}^a = 0$ can be solved for $\omega_\mu{}^{ab}$,

$$\omega_\mu{}^{ab} = -\,2 e^{\nu[a} \partial_{[\mu} e_{\nu]}{}^{b]} - e^{\nu[a} e^{b]\sigma} e_{\mu c} \partial_\sigma e_\nu{}^c - 2 e_\mu{}^{[a} e^{b]\nu} b_\nu \,,$$

which identifies the gauge field $\omega_\mu{}^{ab}$ with the spin connection. It differs from the standard spin-connection of general relativity by the term proportional to b_μ. A second constraint that is imposed is given by $e^{\nu}{}_{b}R(M)_{\mu\nu a}{}^{b}=0$ and can be solved for $f_{\mu}{}^{a}$,

$$f_{\mu}{}^{a}=\tfrac{1}{2}R(\omega,e)_{\mu}{}^{a}-\tfrac{1}{12}R(\omega,e)e_{\mu}{}^{a}\,, \tag{2.6}$$

where $R(\omega,e)_{\mu}{}^{a}=R(\omega)_{\mu\nu}{}^{ab}e_{b}{}^{\nu}$ is the non-symmetric Ricci tensor, and $R(\omega,e)$ the corresponding Ricci scalar. The curvature $R(\omega)_{\mu\nu}{}^{ab}$ is associated with the spin connection field $\omega_{\mu}{}^{ab}$. It coincides with the Riemann tensor of general relativity upon setting $b_{\mu}=0$. Thus the two constraints can be solved algebraically, making $\omega_{\mu}{}^{ab}$ and $f_{\mu}{}^{a}$ dependent on the vielbein and the dilatational gauge field. Only the vielbein $e_{\mu}{}^{a}$ and the dilatational gauge field b_{μ} are left as independent fields, and we will see below that b_{μ} can be eliminated by gauge-fixing.

Next we will illustrate how one can write down a Lagrangian that is invariant under conformal symmetries, using a compensating field which couples to the conformal gauge fields. For that purpose we consider a scalar field ϕ that is invariant under conformal boosts, and has Weyl weight w. The Weyl weight w of a field characterizes how a field transforms under dilatations,

$$\delta_{\mathrm{D}}\phi=w\,\phi\,. \tag{2.7}$$

Consequently the first and second covariant derivative of ϕ are given by,

$$\begin{aligned}\mathcal{D}_{\mu}\phi&=(\partial_{\mu}-w\,b_{\mu})\phi\,,\\ \mathcal{D}_{\mu}\mathcal{D}^{a}\phi&=(\partial_{\mu}-(w+1)b_{\mu})\,\mathcal{D}^{a}\phi-\omega_{\mu}{}^{ab}\mathcal{D}_{b}\phi+wf_{\mu}{}^{a}\,.\end{aligned} \tag{2.8}$$

Notice that the Weyl weight of $\mathcal{D}_{\mu}\phi$ is raised by one unit by the presence of the inverse vielbein. The occurrence of the gauge field of conformal boosts $f_{\mu}{}^{a}$ in the second derivative might be surprising, since ϕ was assumed to be invariant under conformal boosts. However, the presence of the dilatational gauge field b_{μ} in the covariant derivative makes the latter transform under K, since $\delta_{\mathrm{K}}b_{\mu}=\Lambda_{\mathrm{K}\mu}$. Now one can check the following variation,

$$\delta\mathcal{D}_{a}\mathcal{D}^{a}\phi=2(1-w)\Lambda_{\mathrm{K}}{}^{a}\mathcal{D}_{a}\phi+(2+w)\Lambda_{\mathrm{D}}\mathcal{D}_{a}\mathcal{D}^{a}\phi\,. \tag{2.9}$$

Thus for a scalar field with $w=1$ we can write down an invariant Lagrangian,

$$\mathcal{L}=-e\,\phi\,\mathcal{D}_{a}\mathcal{D}^{a}\phi\,, \tag{2.10}$$

where e is the determinant of the vielbein. Note that b_μ is the only independent field that transforms under K-transformations. Since (2.10) is invariant under the latter we can conclude that it does not depend on b_μ. Indeed upon substituting the expressions for ${\omega_\mu}^{ab}$ and ${f_\mu}^a$ in terms of b_μ and ${e_\mu}^a$ we find,

$$\mathcal{L} = -e\,\phi\Box^{\rm grav}\phi - \tfrac{1}{6}\,e\,R(e)\phi^2 = e\,\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{6}\,e\,R(e)\phi^2\,, \tag{2.11}$$

where $\Box^{\rm grav}$ is the d'Alembertian in which only the standard spin-connection of general relativity appears (i.e. (2.6) without the term proportional to b_μ). In the second step we performed a partial integration, which leads us back to the action in (1.11).

The above approach can be summarized as follows. First one constructs a constrained gauge theory associated with the conformal algebra. Then by coupling a compensating field to the conformal gauge fields, one finds a conformally invariant action that is gauge equivalent to the Einstein-Hilbert action. In the next section we will generalize the above analysis by adding supersymmetry generators, yielding a constrained gauge theory for the $N=2$ superconformal group. The corresponding gauge fields will be contained in a supermultiplet, called the Weyl multiplet.

2.2 The Weyl multiplet

In this section we introduce the Weyl supermultiplet, which contains the gauge fields of the $N = 2$ superconformal algebra [26, 27, 33]. The $N = 2$ superconformal group is given by the supergroup $\mathrm{SU}(2,2|2)$.[1] The generators of the latter include, besides the generators of the conformal group that we introduced in the previous section, two supersymmetry generators Q^i, which carry indices $i = 1,2$ [34]. They are Majorana spinors and satisfy the following anti-commutation relation,

$$\{Q^i, \bar{Q}^j\} = 2\gamma^a P_a \delta^{ij}\,. \tag{2.12}$$

Where the $N=2$ Poincaré superalgebra consist of translations, Lorentz transformations and supersymmetry transformations (generated by Q^i), the $N = 2$ superconformal algebra requires additional generators. Among these generators are two more Majorana spinors S^i, which correspond to the so-called the S-supersymmetries. Similar to (2.12), they satisfy an anti-commutation relation that closes into the generator of the special

[1] Note that $\mathrm{SU}(2,2)$ is the double cover group of $\mathrm{SO}(4,2)$, the conformal group in four space-time dimensions. Hence spinors form a representation of this double cover group.

conformal transformations,

$$\{S^i, \bar{S}^j\} = -\gamma^a K_a \delta^{ij} \,. \tag{2.13}$$

Notice that the anti-commutators are invariant under U(2)≃U(1)×SU(2) transformations, which are called the automorphism, or R-symmetry, transformations. They are part of the superconformal algebra. Since the supersymmetry generators Q^i and S^i are Majorana spinors, one can show that the U(2) transformations act in a chiral fashion, i.e. the positive (left) and negative (right) chirality components transform in conjugate representations. Therefore we introduce so-called chiral notation [35, 36], where one writes the SU(2) index as an upper index when it transforms in the fundamental representation and with a lower index when it transforms in the anti-fundamental representation. This implies that upper and lower SU(2) indices have a specific chirality, and for each spinor it is a matter of definition whether one associates an upper index with left or with right chirality. The relevant assignments are listed in various tables in this thesis, see e.g. table B.1 in appendix B for the chirality of the fields in the Weyl multiplet. Note that hermitian conjugation is always accompanied by raising or lowering of the SU(2) indices. We refer to appendix A for more information on this chiral notation.

Thus in order to form the $N=2$ superconformal algebra, the generators of the conformal group and their corresponding parameters and gauge fields written in (2.1) are extended by,

$$\begin{array}{lcccc} \text{generators:} & Q^i & S^i & V^i{}_j & A \\ \text{gauge fields:} & \psi_\mu{}^i & \phi_\mu{}^i & \mathcal{V}_\mu{}^i{}_j & A_\mu \\ \text{parameters:} & \epsilon^i & \eta^i & \Lambda_{\mathrm{SU}(2)}{}^i{}_j & \Lambda_{\mathrm{U}(1)}\,, \end{array} \tag{2.14}$$

where $\mathcal{V}_\mu{}^i{}_j$ is the anti-hermitian and traceless gauge field of the chiral SU(2), and A_μ the gauge field of the U(1). Just as in the previous section, conventional constraints are imposed on the curvatures, which determine the fields $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and $\phi_\mu{}^i$ in terms of the other fields of the multiplet [4, 27, 28]. In order to balance the bosonic and fermionic degrees of freedom three additional fields are needed: a Majorana spinor doublet χ^i, a scalar D, and a selfdual Lorentz tensor T_{abij}, which is anti-symmetric in $[ab]$ and $[ij]$. The resulting Weyl multiplet consists of 24+24 degrees of freedom and forms an off-shell representation of the $N=2$ superconformal algebra. This means that the commutators of the algebra, of which we will consider the most non-trivial one in the next paragraph, close on the fields without the use of field equations. We refer to appendix B for an extended summary of the superconformal transformations of the Weyl multiplet fields, the expressions for the curvatures and other useful identities.

As a result of the constraints the local translations are again discarded and effectively replaced by covariant general coordinate transformations [37]. Thus the anti-commutator (2.12) now closes into such a covariant general coordinate transformation. The presence of the auxiliary fields χ^i, D, and T_{abij} further modify the algebra. We present the decomposition of the commutator of two infinitesimal Q-supersymmetry transformations, with parameters ϵ_1 and ϵ_2,

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta^{\rm cov}(\xi) + \delta_M(\varepsilon) + \delta_K(\Lambda_K) + \delta_S(\eta) + \delta_{\rm gauge}\,, \tag{2.15}$$

where the parameters of the various infinitesimal transformations on the right-hand side are given by,[2]

$$\begin{aligned}
\xi^\mu &= 2\,\bar\epsilon_2{}^i\gamma^\mu\epsilon_{1i} + \text{h.c.}\,,\\
\varepsilon^{ab} &= \bar\epsilon_1{}^i\epsilon_2{}^j\,T^{ab}{}_{ij} + \text{h.c.}\,,\\
\Lambda_K^a &= \bar\epsilon_1^i\epsilon_2{}^j\,D_bT^{ba}{}_{ij} - \tfrac{3}{2}\,\bar\epsilon_2^i\gamma^a\epsilon_{1i}\,D + \text{h.c.}\,,\\
\eta^i &= 6\,\bar\epsilon_{[1}{}^i\epsilon_{2]}{}^j\,\chi_j\,,
\end{aligned} \tag{2.16}$$

The variation $\delta_{\rm gauge}$ denotes an additional, internal gauge symmetry, which commutes with the superconformal algebra. It is not relevant for the fields of the Weyl multiplet, since they do not transform under such a gauge symmetry. It will play a role for the vector multiplets, which we will introduce in section 2.4. As in (2.5), $\delta^{\rm cov}(\xi)$ denotes the infinitesimal covariant general coordinate transformation, which includes contributions from all the field-dependent gauge transformations such as a Q- and S-supersymmetry transformation with parameters $-\frac{1}{2}\xi^\rho\psi_\rho{}^i$ and $-\frac{1}{2}\xi^\rho\phi_\rho{}^i$, such that the combined result takes a supercovariant form.

The Weyl multiplet provides the necessary gauge fields that are needed to find an action that is invariant under $N=2$ superconformal gauge transformations. However, just as in the case with only conformal symmetry, one cannot write down an action that is gauge equivalent to Poincaré supergravity without the use of compensating fields. These compensating fields must be provided for by other supermultiplets, i.e. other representations of the superconformal algebra. Different choices are possible, leading to inequivalent versions of Poincaré supergravity [27, 38]. In this thesis we take the

[2]Full supercovariant derivatives are denoted by D_μ, while $\mathcal{D}_\mu$ denotes a covariant derivative with respect to Lorentz, dilatation, chiral U(1), and SU(2) transformations (see appendix B).

compensating fields to be provided for by a vector multiplet and a hypermultiplet, which we will introduce in section 2.4 and 2.5, respectively. In general we will introduce more than one vector and hypermultiplet, so that there will be additional matter fields present. Vector multiplets can be derived from a more general multiplet, called a chiral multiplet. These chiral multiplets are also very useful for the construction of higher derivatives, as we will see in chapter 5. Therefore we will start by presenting these multiplets in the next section.

2.3 Chiral multiplets

Chiral superfields in flat $N = 2$ superspace were first discussed in [39]. Subsequently they were derived in a conformal supergravity background [27, 36]. The latter result was formulated in components and the same approach is followed in this thesis, although it is convenient to make use of superfield notions at the same time. $N = 2$ superspace is obtained upon supplementing the four bosonic coordinates of space-time x^μ with four chiral and four anti-chiral anti-commuting coordinates, θ^i and θ_i. The concept of a field is extended to a superfield, which in general is a function of x^μ, θ^i and θ_i. Since the fermionic coordinates are anti-commuting, a Taylor expansion in terms of these coordinates is finite. The x^μ-dependent components of the expansion define the field components of a superfield. A general complex scalar superfield $\Phi(x, \theta^i, \theta_i)$ gives rise to $256 + 256$ degrees of freedom. One can however impose the constraint that the superfield does not depend on the anti-chiral coordinates θ_i. This defines a scalar chiral superfield, which contains $16 + 16$ components. These multiplets carry a Weyl weight w and a chiral U(1) weight c, which is opposite to the Weyl weight, i.e. $c = -w$. The weights indicate how the lowest-θ component of the superfield scales under Weyl and chiral U(1) transformations. Anti-chiral multiplets can be obtained from chiral ones by complex conjugation, so that anti-chiral multiplets will have equal Weyl and chiral weights, hence $w = c$.

The components of a generic scalar chiral multiplet are a complex scalar A, a Majorana spinor Ψ_i, a complex symmetric scalar B_{ij}, an anti-selfdual tensor F^-_{ab}, a Majorana spinor Λ_i, and a complex scalar C. The assignment of their Weyl and chiral weights is shown in table 2.1. The spinors Ψ_i and Λ_i transform as doublets under the R-symmetry group SU(2), which is realized locally with gauge fields $\mathcal{V}_\mu{}^i{}_j$ belonging to the superconformal background. The Q- and S-supersymmetry transformations for a

	A	Ψ_i	B_{ij}	F_{ab}^-	Λ_i	C
w	w	$w+\frac{1}{2}$	$w+1$	$w+1$	$w+\frac{3}{2}$	$w+2$
c	$-w$	$-w+\frac{1}{2}$	$-w+1$	$-w+1$	$-w+\frac{3}{2}$	$-w+2$
γ_5		$+$			$+$	

Table 2.1: Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the chiral multiplet component fields.

scalar chiral multiplet of weight w, are as follows,

$$\begin{aligned}
\delta A =&\, \bar\epsilon^i\Psi_i \,,\\
\delta\Psi_i =&\, 2\,\not{D}A\epsilon_i + B_{ij}\,\epsilon^j + \tfrac{1}{2}\gamma^{ab}F_{ab}^-\,\varepsilon_{ij}\epsilon^j + 2\,wA\,\eta_i \,,\\
\delta B_{ij} =&\, 2\,\bar\epsilon_{(i}\not{D}\Psi_{j)} - 2\,\bar\epsilon^k\Lambda_{(i}\,\varepsilon_{j)k} + 2(1-w)\,\bar\eta_{(i}\Psi_{j)} \,,\\
\delta F_{ab}^- =&\, \tfrac{1}{2}\varepsilon^{ij}\,\bar\epsilon_i\not{D}\gamma_{ab}\Psi_j + \tfrac{1}{2}\bar\epsilon^i\gamma_{ab}\Lambda_i - \tfrac{1}{2}(1+w)\,\varepsilon^{ij}\,\bar\eta_i\gamma_{ab}\Psi_j \,,\\
\delta\Lambda_i =&\, -\tfrac{1}{2}\gamma^{ab}\not{D}F_{ab}^-\epsilon_i - \not{D}B_{ij}\varepsilon^{jk}\epsilon_k + C\varepsilon_{ij}\,\epsilon^j + \tfrac{1}{4}\big(\not{D}A\,\gamma^{ab}T_{abij} + w\,A\,\not{D}\gamma^{ab}T_{abij}\big)\varepsilon^{jk}\epsilon_k\\
&\, - 3\,\gamma_a\varepsilon^{jk}\epsilon_k\,\bar\chi_{[i}\gamma^a\Psi_{j]} - (1+w)\,B_{ij}\varepsilon^{jk}\,\eta_k + \tfrac{1}{2}(1-w)\,\gamma^{ab}\,F_{ab}^-\eta_i \,,\\
\delta C =&\, -2\,\varepsilon^{ij}\bar\epsilon_i\not{D}\Lambda_j - 6\,\bar\epsilon_i\chi_j\,\varepsilon^{ik}\varepsilon^{jl}B_{kl}\\
&\, - \tfrac{1}{4}\varepsilon^{ij}\varepsilon^{kl}\big((w-1)\,\bar\epsilon_i\gamma^{ab}\not{D}T_{abjk}\Psi_l + \bar\epsilon_i\gamma^{ab}T_{abjk}\not{D}\Psi_l\big) + 2\,w\varepsilon^{ij}\,\bar\eta_i\Lambda_j \,.
\end{aligned} \tag{2.17}$$

The transformation rules are linear in the chiral multiplet fields, and contain other fields associated with the superconformal background, such as the self-dual tensor field T_{abij} and the spinor χ^i. Other superconformal fields are contained in the superconformal derivatives D_μ. Like the Weyl multiplet, the chiral supermultiplet forms an off-shell representation of the superconformal algebra.

Products of chiral superfields constitute again a chiral superfield, whose Weyl weight is equal to the sum of the Weyl weights of the separate multiplets. Also functions of chiral superfields may describe chiral superfields, assuming that they can be assigned a proper Weyl weight. For instance, homogeneous functions of chiral superfields of the same Weyl weight w define a chiral supermultiplet whose Weyl weight equals the product of w and the degree of homogeneity. This is called supermultiplet calculus and the relevant formulae are presented in appendix D.1.

Chiral multiplets of $w = 1$ are special, because they are reducible [36, 39]. Some details about these multiplets are given in appendix D.2. For a scalar chiral multiplet with $w = 1$ the tensor $F^-_{ab} + F^+_{ab}$ is subject to a Bianchi identity, which can be solved in terms of a vector gauge field. The reduced scalar chiral multiplet thus describes the covariant fields and field strength of a vector multiplet, which encompasses $8+8$ bosonic and fermionic components: a complex scalar X, a Majorana doublet spinor Ω_i, a vector gauge field W_μ, and a triplet of auxiliary fields Y_{ij}. In the next section we will discuss the vector multiplet in more detail.

There also exists an anti-selfdual tensor version of the chiral multiplet with $w = 1$ that is reducible. This multiplet, which comprises $24 + 24$ off-shell degrees of freedom, contains all the covariant fields and curvatures of $N = 2$ superconformal gravity. It is especially useful for the construction of higher-derivative invariants [16, 17, 40], as we will demonstrate in chapter 5. It is also called the Weyl supermultiplet, since it is based on the same fields as the Weyl multiplet introduced in the previous section. It will be clear from the context whether we refer to the multiplet of superconformal gauge fields or to the corresponding chiral multiplet.

Another special chiral multiplet is the so-called 'kinetic' multiplet, which has Weyl weight $w = 2$. This multiplet is constructed from an anti-chiral multiplet with $w = 0$. It will be discussed in detail in chapter 5.

Finally, scalar chiral multiplets with $w = 2$ lead to superconformal actions when including a conformal supergravity background. Their highest θ-component C has Weyl weight 4, and chiral weight 0. To define a Lagrangian that is invariant under local superconformal transformations one makes use of a density formula [36],

$$\begin{aligned} e^{-1}\mathcal{L} = {} & C - \varepsilon^{ij}\,\bar\psi_{\mu i}\gamma^\mu\Lambda_j - \tfrac{1}{8}\bar\psi_{\mu i}T_{ab\,jk}\gamma^{ab}\gamma^\mu\Psi_l\,\varepsilon^{ij}\varepsilon^{kl} - \tfrac{1}{16}A(T_{ab\,ij}\varepsilon^{ij})^2 \\ & - \tfrac{1}{2}\bar\psi_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}\,B_{kl}\;\varepsilon^{ik}\varepsilon^{jl} + \varepsilon^{ij}\bar\psi_{\mu i}\psi_{\nu j}(F^{-\mu\nu} - \tfrac{1}{2}A\,T^{\mu\nu}{}_{kl}\,\varepsilon^{kl}) \\ & - \tfrac{1}{2}\varepsilon^{ij}\varepsilon^{kl}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu i}\psi_{\nu j}(\bar\psi_{\rho k}\gamma_\sigma\Psi_l + \bar\psi_{\rho k}\psi_{\sigma j}\,A)\,. \end{aligned} \tag{2.18}$$

As such it is not yet a sensible Lagrangian[3], since it does not contain any kinetic terms. We will indicate in the next section how it can be used to construct a Lagrangian for vector multiplets. It will also play a central role in the construction of higher-derivative invariants in chapter 5.

[3]Notice that one should add the complex conjugate of the density formula in order to obtain a real-valued Lagrangian.

	X^Λ	${\Omega_i}^\Lambda$	${W_\mu}^\Lambda$	${Y_{ij}}^\Lambda$
w	1	$\frac{3}{2}$	0	2
c	-1	$-\frac{1}{2}$	0	0
γ_5		$+$		

Table 2.2: Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the vector multiplet component fields.

2.4 Vector multiplets

In the previous section we introduced the vector multiplet as a reduced chiral multiplet. In this section we will elaborate further on vector supermultiplets in a $N = 2$ superconformal background [26, 28]. Consider $n + 1$ of these multiplets, labeled by indices $\Lambda = 0, 1, \ldots, n$. Vector supermultiplets comprise complex scalar fields X^Λ, gauge fields ${W_\mu}^\Lambda$, and Majorana spinors ${\Omega_i}^\Lambda$. These spinors transform as doublets under the chiral R-symmetry group SU(2), which is realized locally with gauge fields ${\mathcal{V}_\mu}^i{}_j$ belonging to the superconformal background. Furthermore there are auxiliary fields ${Y_{ij}}^\Lambda$, which satisfy the pseudo-reality constraint $({Y_{ij}}^\Lambda)^* = \varepsilon^{ik}\varepsilon^{jl}{Y_{kl}}^\Lambda$, so that they transform as real vectors under SU(2). The tensors $F^{\pm}_{\mu\nu}{}^\Lambda$ are the (anti-)selfdual (complex) components of the field strengths, which will be expressed in terms of vector fields ${W_\mu}^\Lambda$. These vector fields are subject to abelian gauge transformations,

$$\delta {W_\mu}^\Lambda = \partial_\mu \Lambda^\Lambda \,. \tag{2.19}$$

The transformations of the vector multiplet fields under dilatations and chiral transformations are given in table 2.2. Under local Q- and S-supersymmetry they are as follows [27],

$$\begin{aligned}
\delta X^\Lambda &= \bar{\epsilon}^i {\Omega_i}^\Lambda \,, \\
\delta {W_\mu}^\Lambda &= \varepsilon^{ij}\bar{\epsilon}_i(\gamma_\mu {\Omega_j}^\Lambda + 2\,\psi_{\mu j} X^\Lambda) + \varepsilon_{ij}\bar{\epsilon}^i(\gamma_\mu \Omega^{j\,\Lambda} + 2\,{\psi_\mu}^j \bar{X}^\Lambda) \,, \\
\delta {\Omega_i}^\Lambda &= 2\not{D} X^\Lambda \epsilon_i + \tfrac{1}{2}\gamma^{\mu\nu}\hat{F}^{-}_{\mu\nu}{}^\Lambda \varepsilon_{ij}\epsilon^j + {Y_{ij}}^\Lambda \epsilon^j + 2\,X^\Lambda \eta_i \,, \\
\delta {Y_{ij}}^\Lambda &= 2\,\bar{\epsilon}_{(i}\not{D}{\Omega_{j)}}^\Lambda + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\not{D}\Omega^{l)\Lambda} \,.
\end{aligned} \tag{2.20}$$

The field strengths $F_{\mu\nu}{}^{\Lambda} = 2\,\partial_{[\mu}W_{\nu]}{}^{\Lambda}$ are contained in the supercovariant combination,

$$\begin{aligned}\hat{F}_{\mu\nu}{}^{\Lambda} = & F^{+\,\Lambda}_{\mu\nu} + F^{-\,\Lambda}_{\mu\nu} - \varepsilon^{ij}\bar{\psi}_{[\mu\,i}(\gamma_{\nu]}\Omega_j{}^{\Lambda} + \psi_{\nu]j}X^{\Lambda}) - \varepsilon_{ij}\bar{\psi}_{[\mu}{}^{i}(\gamma_{\nu]}\Omega^{j\,\Lambda} + \psi_{\nu]}{}^{j}\bar{X}^{\Lambda}) \\ & - \tfrac{1}{4}(X^{\Lambda}\,T_{\mu\nu ij}\,\varepsilon^{ij} + \bar{X}^{\Lambda}\,T_{\mu\nu}{}^{ij}\,\varepsilon_{ij})\,. \end{aligned} \tag{2.21}$$

As before, the full superconformally covariant derivatives are denoted by D_μ, while $\mathcal{D}_\mu$ will denote a covariant derivative with respect to Lorentz, dilatation, chiral U(1), and SU(2) transformations. As an example of the latter, we note the definitions,

$$\begin{aligned}\mathcal{D}_\mu X^{\Lambda} &= \big(\partial_\mu - b_\mu + \mathrm{i}A_\mu\big)X^{\Lambda}\,, \\ \mathcal{D}_\mu\Omega_i{}^{\Lambda} &= \big(\partial_\mu - \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \tfrac{3}{2}b_\mu + \tfrac{1}{2}\mathrm{i}A_\mu\big)\Omega_i{}^{\Lambda} - \tfrac{1}{2}\mathcal{V}_\mu{}^{j}{}_{i}\,\Omega_j{}^{\Lambda}\,. \end{aligned} \tag{2.22}$$

Just like any chiral multiplet, the vector multiplet is an off-shell representation of the superconformal algebra. However, since the vector field is subject to abelian gauge transformations (2.19), the commutator of two infinitesimal Q-supersymmetry transformations (2.15) contains a gauge transformation $\delta_{\rm gauge}(\Lambda^{\Lambda})$ with parameter,

$$\Lambda^{\Lambda} = 4\,\bar{X}^{\Lambda}\,\bar{\epsilon}_2{}^{i}\epsilon_1{}^{j}\,\varepsilon_{ij} + \text{h.c.}\,, \tag{2.23}$$

in addition to the other terms specified in (2.16). To see this, let us evaluate the supersymmetry commutator on the vector fields $W_\mu{}^{\Lambda}$,

$$\begin{aligned}[\delta(\epsilon_1),\delta(\epsilon_2)]W_\mu{}^{\Lambda} = & \,\xi^{\rho}F_{\rho\mu}{}^{\Lambda} + \partial_\mu\Lambda^{\Lambda} \\ & - \xi^{\rho}\left(\tfrac{1}{2}\,\varepsilon_{ij}\,\bar{\psi}_{\rho}{}^{i}\gamma_\mu\Omega^{j\Lambda} + \varepsilon_{ij}\bar{X}^{\Lambda}\bar{\psi}_{\rho}{}^{i}\psi_\mu{}^{j} + \text{h.c.}\right)\,, \end{aligned} \tag{2.24}$$

where the parameters ξ^{μ} and Λ^{Λ} are as in (2.16) and (2.23). Now we use the following equality,

$$\xi^{\rho}F_{\rho\mu}{}^{\Lambda} = \xi^{\rho}\partial_{\rho}W_\mu{}^{\Lambda} + \partial_\mu\xi^{\rho}W_{\rho}{}^{\Lambda} - \partial_\mu\left(\xi^{\rho}W_{\rho}{}^{M}\right)\,. \tag{2.25}$$

Substituting this identity into (2.24) shows that the ξ^{μ}-dependent terms decompose into a general coordinate transformation with parameter ξ^{μ}, an abelian gauge transformation with parameter $-\xi^{\mu}W_\mu{}^{\Lambda}$ and a supersymmetry transformation with parameter $-\frac{1}{2}\xi^{\mu}\psi_{\mu i}$. Together they constitute a covariant general coordinate transformation with parameter ξ^{μ}. Consequently the supersymmetry commutator closes on $W_\mu{}^{\Lambda}$ according to (2.15).

We now assume a holomorphic function $F(X)$ of the fields X^{Λ}, which is homogeneous of second degree, i.e.,

$$F(\lambda X) = \lambda^2 F(X)\,, \tag{2.26}$$

for any complex parameter λ [4, 41]. As explained in the previous section, this defines a chiral multiplet of Weyl weight 2 according to (D.2). Therefore one can use the highest component of this multiplet based on $F(X)$ in the density formula (2.18) to write down a consistent action for the vector multiplets in the superconformal background provided by the Weyl multiplet fields. We will refrain from doing this explicitly here. In section 2.6 we will give the complete Lagrangian for vector multiplets.

We will end this section with some identities and definitions concerning the function $F(X)$, also called the pre-potential. From (2.26) one can show that,

$$\begin{aligned} F(X) &= \tfrac{1}{2} F_\Lambda X^\Lambda\,, \\ F_\Lambda &= F_{\Lambda\Sigma} X^\Sigma\,, \\ F_{\Lambda\Sigma\Gamma} X^\Gamma &= 0\,, \end{aligned} \tag{2.27}$$

where $F_\Lambda = \partial F/\partial X^\Lambda$ and similarly for higher derivatives.

As we will see more explicitly in section 2.6, when we introduce the Lagrangian for vector multiplets, the scalar fields parameterize a so-called Kähler manifold. This implies that the metric $N_{\Lambda\Sigma}$ that encodes the coupling of the kinetic terms for the scalar fields can be derived from a scalar function K, called the Kähler potential,

$$K = \mathrm{i}(X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda) = N_{\Lambda\Sigma} X^\Lambda \bar{X}^\Sigma\,. \tag{2.28}$$

The metric $N_{\Lambda\Sigma}$, whose inverse will be denoted by $N^{\Lambda\Sigma}$ is then given by,

$$N_{\Lambda\Sigma} = \partial_\Lambda \partial_{\bar{\Sigma}} K = -\mathrm{i} F_{\Lambda\Sigma} + \mathrm{i}\bar{F}_{\Lambda\Sigma}\,. \tag{2.29}$$

This metric is not of definite sign, due to fact that one of the vector multiplets is a compensating multiplet. This is familiar from the discussion at the end of section (1.2).

Before we present the Lagrangian for the vector multiplets, we will first discuss superconformal hypermultiplets in the next section.

2.5 Hypermultiplets

Another representation of the superconformal algebra that is important for this thesis is the hypermultiplet. In this section we give a rather technical synopsis of superconformal hypermultiplets and their superconformal transformations, following the framework of [42]. The $n_{\rm H}+1$ hypermultiplets are described by $4(n_{\rm H}+1)$ real scalars ϕ^A, $2(n_{\rm H}+1)$ positive-chirality spinors $\zeta^{\bar\alpha}$ and $2(n_{\rm H}+1)$ negative-chirality spinors ζ^α. Hence target-space indices $A, B, \ldots$ take values $1, 2, \ldots, 4(n_{\rm H}+1)$, and the indices $\alpha, \beta, \ldots$ and $\bar\alpha, \bar\beta, \ldots$ run from 1 to $2(n_{\rm H}+1)$. The chiral and anti-chiral spinors are related by complex conjugation (as we are dealing with $2(n_{\rm H}+1)$ Majorana spinors) under which indices are converted according to $\alpha \leftrightarrow \bar\alpha$. For superconformally invariant Lagrangians, the scalar fields of the hypermultiplets parametrize a $4(n_{\rm H}+1)$-dimensional hyperkähler cone [42–45]. Such a cone has a homothetic conformal Killing vector χ^A,

$$D_A\chi^B = \delta_A{}^B\,, \tag{2.30}$$

which, locally, can be expressed in terms of a hyperkähler potential χ,

$$\chi_A = \partial_A\chi\,. \tag{2.31}$$

The cone metric can thus be written as $g_{AB} = D_A\partial_B\chi$. This relation does not define the metric directly, because of the presence of the covariant derivative which contains the Christoffel connection. We also note the relation,

$$\chi = \tfrac{1}{2} g_{AB}\, \chi^A\chi^B\,. \tag{2.32}$$

Just like the Kähler metric $N_{\Lambda\Sigma}$ for the scalar fields of the vector multiplets, the hyperkähler metric g_{AB} is not of definite sign, due to fact that one of the hypermultiplets is a compensating multiplet.

Hyperkähler spaces have three hermitian, complex structures $J_{ij} = J_{ji}$, that are covariantly constant and satisfy the algebra of quaternions,

$$\begin{aligned} J_{ij\,AB} &\equiv (J^{ij}{}_{AB})^* = \varepsilon_{ik}\varepsilon_{jl}J^{kl}{}_{AB}\,,\\ J^{ij}{}_A{}^C\, J^{kl}{}_{CB} &= \tfrac{1}{2}\varepsilon^{i(k}\varepsilon^{l)j}\, g_{AB} + \varepsilon^{(i(k}\, J^{l)j)}{}_{AB}\,. \end{aligned} \tag{2.33}$$

As it turns out, the hyperkähler potential serves as a Kähler potential for each of the complex structures.

Hyperkähler cones have SU(2) isometries; the corresponding Killing vectors are expressed in terms of the complex structures and the homothetic Killing vector,

$$k_{ij}{}^{A} = J_{ij}{}^{AB}\,\chi_B\,, \tag{2.34}$$

from which it follows that,

$$D_A k^{ij}{}_B = -J^{ij}{}_{AB}\,. \tag{2.35}$$

From the above results, it follows that the homothetic Killing vector χ^A and the three SU(2) Killing vectors k^{ijA} are mutually orthogonal,

$$\chi^A\chi_A = 2\chi\,, \qquad k_{ij}{}^{A}\,k^{kl}{}_A = \delta_{(i}{}^{k}\,\delta_{j)}{}^{l}\,\chi\,, \qquad \chi^A\,k^{ij}{}_A = 0\,. \tag{2.36}$$

The hypermultiplet fields transform under dilations, associated with the homothetic Killing vector, and the SU(2) × U(1) transformations of the superconformal group, with parameters $\Lambda_{\rm D}$, $\Lambda_{\rm SU(2)}$ and $\Lambda_{\rm U(1)}$, respectively,

$$\begin{aligned} \delta\phi^A &= \Lambda_{\rm D}\,\chi^A + \Lambda_{\rm SU(2)}{}^{i}{}_{k}\,\varepsilon^{jk}\,k_{ij}{}^{A}\,, \\ \delta\zeta^\alpha + \delta\phi^A\,\Gamma_A{}^{\alpha}{}_{\beta}\,\zeta^\beta &= \big(\tfrac{3}{2}\Lambda_{\rm D} - \tfrac{1}{2}{\rm i}\Lambda_{\rm U(1)}\big)\zeta^\alpha\,. \end{aligned} \tag{2.37}$$

Here $\Gamma_A{}^{\alpha}{}_{\beta}$ denote the connections associated with field-dependent reparametrizations of the fermions of the form $\zeta^\alpha \to S^\alpha{}_\beta(\phi)\,\zeta^\beta$. Naturally the conjugate connections $\bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}$ are associated with the reparametrizations $\zeta^{\bar\alpha} \to \bar S^{\bar\alpha}{}_{\bar\beta}(\phi)\,\zeta^{\bar\beta}$. These tangent-space reparametrizations act on all quantities carrying indices α and $\bar\alpha$. The corresponding curvatures $R_{AB}{}^{\alpha}{}_{\beta}$ and $\bar R_{AB}{}^{\bar\alpha}{}_{\bar\beta}$ take their values in ${\rm sp}(n_{\rm H}+1) \cong {\rm usp}(2n_{\rm H}+2;\mathbb{C})$. These curvatures are linearly related to the Riemann curvature $R_{ABC}{}^{D}$ of the target space, as we shall see later.

To define the supersymmetry transformations one needs the notion of quaternionic vielbeine, which can convert the $4(n_{\rm H}+1)$ target-space indices $A, B, \ldots$ to the tangent-space indices $\alpha, \beta, \ldots, \bar\alpha, \bar\beta \ldots$ carried by the fermions. All quantities of interest can be expressed in terms of these vielbeine. For instance, the scalar fields transform as follows under supersymmetry,

$$\delta\phi^A = 2(\gamma^A_{i\bar\alpha}\,\bar\epsilon^i\zeta^{\bar\alpha} + \bar\gamma^{Ai}_{\alpha}\,\bar\epsilon_i\zeta^\alpha)\,, \tag{2.38}$$

where the pseudoreal quantity $\gamma^A_{i\bar\alpha}(\phi)$ corresponds to the $(4n_{\rm H}+4)\times(4n_{\rm H}+4)$ inverse quaternionic vielbein. Its inverse is the vielbein denoted by $\bar V^{i\bar\alpha}_{A}$, which is needed for writing down the supersymmetry transformation of the fermions. So we have,

$$\begin{aligned} \bar{V}_A^{i\bar\alpha}\,\gamma^A_{j\bar\beta} &= \delta^i{}_j\,\delta^{\bar\alpha}{}_{\bar\beta}\,, \\ \gamma^A_{i\bar\alpha}\bar{V}_B^{j\bar\alpha} + \bar\gamma^{Aj}_\alpha V^\alpha_{Bi} &= \delta_i{}^j\,\delta^A{}_B\,. \end{aligned} \tag{2.39}$$

As before, SU(2) indices are raised and lowered by complex conjugation. The quaternionic vielbeine are covariantly constant, e.g.,

$$D_A\gamma^B_{i\bar\alpha} = \partial_A\gamma^B_{i\bar\alpha} + \Gamma_{AC}{}^B\gamma^C_{i\bar\alpha} - \bar\Gamma_A{}^{\bar\beta}{}_{\bar\alpha}\,\gamma^B_{i\bar\beta} = 0\,. \tag{2.40}$$

Observe that it is not necessary to introduce a SU(2) connection here. When coupling to the superconformal fields, the SU(2) symmetry will be realized locally and a connection will be provided by the gauge field $\mathcal{V}_\mu{}^i{}_j$ of the Weyl multiplet. The fact that the vielbeine are covariantly constant provides a relation between the Riemann curvature $R_{ABC}{}^D$ and the tangent-space curvature $\bar{R}_{AB}{}^{\bar\alpha}{}_{\bar\beta}$,

$$R_{ABC}{}^D\,\gamma^C_{i\bar\alpha} - \bar{R}_{AB}{}^{\bar\beta}{}_{\bar\alpha}\,\gamma^D_{i\bar\beta} = 0\,. \tag{2.41}$$

Both curvatures can actually be written in terms of,

$$W_{\bar\alpha\beta\bar\gamma\delta} = \tfrac{1}{2}R_{ABCD}\,\gamma^A_{i\bar\alpha}\,\bar\gamma^{iB}_\beta\,\gamma^C_{j\bar\gamma}\,\bar\gamma^{jD}_\delta\,, \tag{2.42}$$

which appears as the coefficient of the four-spinor term in the supersymmetric Lagrangian (cf. (2.56)).

A typical feature of the superconformal hypermultiplets is that they can be formulated in terms of local sections $A_i{}^\alpha(\phi)$ of an $\mathrm{Sp}(n_\mathrm{H}{+}1)\times\mathrm{Sp}(1)$ bundle.[4] This section is provided by,

$$A_i{}^\alpha(\phi) \equiv \chi^B(\phi)\,V^\alpha_{Bi}(\phi)\,. \tag{2.43}$$

Obviously the vielbeine can be re-obtained from these sections, as we easily derive,

$$D_BA_i{}^\alpha = V^\alpha_{Bi}\,. \tag{2.44}$$

[4]The existence of such an associated quaternionic bundle was established based on a general analysis of quaternion-Kähler manifolds [46]. Here $\mathrm{Sp}(1)\cong\mathrm{SU}(2)$ denotes the corresponding R-symmetry subgroup of the $N=2$ superconformal group.

	$A_i{}^\alpha$	ζ^α
w	1	$\frac{3}{2}$
c	0	$-\frac{1}{2}$
γ_5		$-$

Table 2.3: Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the hypermultiplet fields.

We note a few relevant equations,

$$
\begin{aligned}
g^{AB}\, D_A A_i{}^\alpha\, D_B A_j{}^\beta &= \varepsilon_{ij}\, \Omega^{\alpha\beta}\,, \\
g^{AB}\, D_A A_i{}^\alpha\, D_B A^{j\bar\beta} &= \delta_i{}^j\, G^{\alpha\bar\beta}\,,
\end{aligned}
\tag{2.45}
$$

which defines two tensors, $\Omega^{\alpha\beta}$ and $G^{\alpha\bar\beta}$, which are skew symmetric and hermitian, respectively. Obviously both tensors are covariantly constant. We also note the following relations,

$$
\begin{aligned}
G_{\bar\alpha\beta}\, V_{A\,i}^{\beta} &= \varepsilon_{ij}\, \Omega_{\bar\alpha\bar\beta}\, \bar V_A^{j\bar\beta} = g_{AB}\, \gamma^B_{i\bar\alpha}\,, \\
G_{\bar\gamma\alpha}\bar\Omega^{\bar\gamma\bar\delta} G_{\bar\delta\beta} &= \bar\Omega_{\alpha\beta}\,, \\
\Omega_{\bar\alpha\bar\beta}\bar\Omega^{\bar\beta\bar\gamma} &= -\,\delta_{\bar\alpha}{}^{\bar\gamma}\,, \\
\bar\Omega_{\alpha\beta}\, A_i{}^\alpha A_j{}^\beta &= \varepsilon_{ij}\chi\,.
\end{aligned}
\tag{2.46}
$$

The first one establishes the fact that the quaternionic vielbein V^α_{Ai} is pseudoreal. Furthermore we note,

$$
\begin{aligned}
\bar\Omega_{\alpha\beta} A_i{}^\alpha\, D_B A_j{}^\beta &= \tfrac{1}{2}\varepsilon_{ij}\chi_B + k_{ijB}\,, \\
\bar\Omega_{\alpha\beta}\, D_A A_i{}^\alpha\, D_B A_j{}^\beta &= \tfrac{1}{2}\varepsilon_{ij}\, g_{AB} - J_{ij\,AB}\,, \\
A^{i\bar\alpha} \equiv (A_i{}^\alpha)^* &= \varepsilon^{ij}\, \bar\Omega^{\bar\alpha\bar\beta}\, G_{\bar\beta\gamma}\, A_j{}^\gamma\,.
\end{aligned}
\tag{2.47}
$$

For additional relations we refer to [42].

Let us now introduce the local Q- and S-supersymmetry transformations of the hypermultiplet fields, employing the sections $A_i{}^\alpha$,

$$\begin{aligned}
\delta A_i{}^\alpha + \delta\phi^B \Gamma_B{}^\alpha{}_\beta A_i{}^\beta &= 2\,\bar\epsilon_i \zeta^\alpha + 2\,\varepsilon_{ij} G^{\alpha\bar\beta} \Omega_{\bar\beta\bar\gamma}\, \bar\epsilon^j \zeta^{\bar\gamma}\,, \\
\delta\zeta^\alpha + \delta\phi^A\, \Gamma_A{}^\alpha{}_\beta\, \zeta^\beta &= \not{D} A_i{}^\alpha\, \epsilon^i + A_i{}^\alpha\, \eta^i\,, \\
\delta\zeta^{\bar\alpha} + \delta\phi^A\, \bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}\, \zeta^{\bar\beta} &= \not{D} A^{i\bar\alpha}\, \epsilon_i + A^{i\bar\alpha}\, \eta_i\,.
\end{aligned} \tag{2.48}$$

The Weyl and chiral weights of these sections and the fermion fields are listed in table 2.3. The reader can easily verify that these weight assignments are consistent with the above supersymmetry transformations. The bosonic part of the covariant derivative on the scalar and fermion fields is given by,

$$\begin{aligned}
\mathcal{D}_\mu \phi^A &= \partial_\mu \phi^A - b_\mu\, \chi^A + \tfrac{1}{2} \mathcal{V}_\mu{}^i{}_k\, \varepsilon^{jk}\, k_{ij}^A\,, \\
\mathcal{D}_\mu A_i{}^\alpha &= \partial_\mu A_i{}^\alpha - b_\mu A_i{}^\alpha + \tfrac{1}{2} \mathcal{V}_{\mu i}{}^j A_j{}^\alpha + \partial_\mu \phi^A \Gamma_A{}^\alpha{}_\beta A_i{}^\beta\,, \\
\mathcal{D}_\mu \zeta^\alpha &= \partial_\mu \zeta^\alpha - \tfrac{1}{4} \omega_\mu{}^{ab} \gamma_{ab}\, \zeta^\alpha - \tfrac{3}{2} b_\mu \zeta^\alpha + \tfrac{1}{2} \mathrm{i} A_\mu \zeta^\alpha + \partial_\mu \phi^A\, \Gamma_A{}^\alpha{}_\beta\, \zeta^\beta\,,
\end{aligned} \tag{2.49}$$

where we have now introduced the superconformal gauge fields, in addition to the target-space connections. The covariantization of the above derivatives with respect to Q- and S-supersymmetry follows immediately from (2.48). We note that, in contrast to the vector and the Weyl multiplet, the hypermultiplets form an on-shell representation of the superconformal algebra. This is inevitable for hypermultiplets based on a finite number of fields.

2.6 Superconformal Lagrangians

In this section we consider the superconformally invariant Lagrangians for the vector and hypermultiplets. These Lagrangians can be found in the literature (see, e.g., [4, 27, 28, 42]), including certain terms quartic in the fermions that we will neglect here. We have not eliminated any auxiliary fields, so that the results pertain to fully off-shell couplings, with the exception of the hypermultiplets. In the formula below, we have substituted the explicit expressions for the dependent gauge fields associated with Lorentz transformations, conformal boosts and S-supersymmetry written in (B.5).

All Lagrangians given below can be viewed as matter Lagrangians in a given superconformal supergravity background. However, for the Lagrangian of the vector multiplets, one of the vector multiplets acts as a compensating field: its scalar and spinor degrees of freedom are not physical and only the vector field and the corresponding triplet of auxiliary fields remain. Physical fields can be identified that are invariant under scale transformations and S-supersymmetry, so that effectively we will be dealing with supergravity coupled to only n vector supermultiplets. For the hypermultiplet Lagrangian, a similar rearrangement of degrees of freedom will take place. One of the hypermultiplets will play the role of a compensator with respect to the local SU(2). The precise choice of the compensator multiplets is irrelevant, and the resulting theories remain gauge equivalent. Therefore it is best to not make any particular choice for the compensating multiplets at this stage and keep the formulae in their most symmetric form. At the end one may then select fields that are invariant under certain local superconformal transformations, so that the compensating fields decouple from the Lagrangian, or one may simply adopt a convenient gauge choice.

We decompose the Lagrangian for the vector multiplets into four separate parts,

$$\mathcal{L}_{\text{vector}} = \mathcal{L}^{(1)}_{\text{kin}} + \mathcal{L}^{(2)}_{\text{kin}} + \mathcal{L}_{\text{aux}} + \mathcal{L}_{\text{conf}}\,. \tag{2.50}$$

The first term in (2.50) contains the kinetic terms of the scalar and spinor fields,

$$\begin{aligned} e^{-1}\mathcal{L}^{(1)}_{\text{kin}} = &- N_{\Lambda\Sigma}\,\mathcal{D}_\mu X^\Lambda\,\mathcal{D}^\mu \bar X^\Sigma - \tfrac14 N_{\Lambda\Sigma}\left[\bar\Omega^{i\Lambda}\not{\mathcal{D}}\Omega_i{}^\Sigma + \bar\Omega_i{}^\Lambda\not{\mathcal{D}}\Omega^{i\Sigma}\right]\\ &- \tfrac14\mathrm{i}\left[F_{\Lambda\Sigma\Gamma}\bar\Omega_i{}^\Lambda\not{\mathcal{D}}X^\Sigma\Omega^{i\Gamma} - \bar F_{\Lambda\Sigma\Gamma}\bar\Omega^{i\Lambda}\not{\mathcal{D}}\bar X^\Sigma\Omega_i{}^\Gamma\right]\\ &+ \tfrac12 N_{\Lambda\Sigma}\left[\bar\psi_\mu{}^i\not{\mathcal{D}}\bar X^\Lambda\gamma^\mu\,\Omega_i{}^\Sigma - \bar\psi_{\mu i}\not{\mathcal{D}}X^\Lambda\gamma^\mu\,\Omega^{i\Sigma}\right]\,. \end{aligned} \tag{2.51}$$

The kinetic terms for the vector fields and their moment couplings to the tensor and fermion fields are contained in $\mathcal{L}^{(2)}_{\text{kin}}$,

$$\begin{aligned} e^{-1}\mathcal{L}^{(2)}_{\text{kin}} = &\tfrac14\mathrm{i}\left[F_{\Lambda\Sigma}\,F^{-\,\Lambda}_{\mu\nu}F^{-\,\mu\nu\Sigma} - \bar F_{\Lambda\Sigma}\,F^{+\,\Lambda}_{\mu\nu}F^{+\mu\nu\Sigma}\right]\\ &+\left[\mathcal{O}^-_{\mu\nu\Lambda}F^{-\mu\nu\Lambda} - N^{\Lambda\Sigma}\,\mathcal{O}^-_{\mu\nu\Lambda}\mathcal{O}^{-\mu\nu}{}_\Sigma + \text{h.c.}\right]\,, \end{aligned} \tag{2.52}$$

where $\mathcal{O}^-_{\mu\nu\Lambda}$ is defined by,

$$\begin{aligned} \mathcal{O}^-_{\mu\nu\Lambda} = &-\tfrac{1}{16}\mathrm{i}F_{\Lambda\Sigma\Gamma}\,\bar\Omega_i{}^\Sigma\gamma_{\mu\nu}\Omega_j{}^\Gamma\,\varepsilon^{ij} - \tfrac18 N_{\Lambda\Sigma}\varepsilon_{ij}\bar\psi_\rho{}^i\gamma_{\mu\nu}\gamma^\rho\Omega^{j\Sigma}\\ &- \tfrac18 N_{\Lambda\Sigma}\bar X^\Sigma\,\varepsilon_{ij}\bar\psi_\rho{}^i\gamma^{\rho\sigma}\gamma_{\mu\nu}\psi_\sigma{}^j + \tfrac18 N_{\Lambda\Sigma}\bar X^\Sigma\,T_{\mu\nu}{}^{ij}\varepsilon_{ij}\,. \end{aligned} \tag{2.53}$$

The reason for writing the terms in (2.52) in this particular form, including a term quadratic in the tensors $\mathcal{O}$, has to do with electric/magnetic duality. The first line is of the form (1.14), with the field-dependent matrix $F_{\Lambda\Sigma}$ replacing the constant parameter τ.

The second line behaves under electric/magnetic duality in a similar way [9]. We will discuss electric/magnetic duality in detail in section 3.1.

The terms associated with the auxiliary fields ${Y_{ij}}^\Lambda$ are given in $\mathcal{L}_{\rm aux}$ [47],

$$e^{-1}\mathcal{L}_{\rm aux} = \tfrac{1}{8}N^{\Lambda\Sigma}\,\big(N_{\Lambda\Gamma}{Y_{ij}}^\Gamma + \tfrac{1}{2}{\rm i}(F_{\Lambda\Gamma\Omega}\,{\bar\Omega_i}{}^\Gamma{\Omega_j}{}^\Omega - \bar F_{\Lambda\Gamma\Omega}\,\bar\Omega^{k\Gamma}\Omega^{l\Omega}\varepsilon_{ik}\varepsilon_{jl})\big) \\ \times\big(N_{\Sigma\Xi}Y^{ij\Xi} + \tfrac{1}{2}{\rm i}(F_{\Sigma\Xi\Delta}\,{\bar\Omega_m}{}^\Xi{\Omega_n}{}^\Delta\varepsilon^{im}\varepsilon^{jn} - \bar F_{\Sigma\Xi\Delta}\,\bar\Omega^{i\Xi}\Omega^{j\Delta})\big)\,. \tag{2.54}$$

Again, this particular combination of terms is convenient in the light of electric/magnetic duality, as we will discuss in due course. The last part of the Lagrangian describes the remaining couplings of the vector multiplet fields to conformal supergravity,

$$e^{-1}\mathcal{L}_{\rm conf} = \tfrac{1}{6}K\,\Big[R + (e^{-1}\varepsilon^{\mu\nu\rho\sigma}{\bar\psi_\mu}{}^i\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i} - {\bar\psi_\mu}{}^i{\psi_\nu}{}^j\,{T^{\mu\nu}}_{ij} + \text{h.c.})\Big] \\ - K\,\Big[D + \tfrac{1}{2}{\bar\psi_\mu}{}^i\gamma^\mu\chi_i + \tfrac{1}{2}\bar\psi_{\mu i}\gamma^\mu\chi^i\Big] \\ - \Big(K_\Lambda\Big[\tfrac{1}{4}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu i}\gamma_\nu{\psi_\rho}{}^i\,\mathcal{D}_\sigma X^\Lambda + \tfrac{1}{48}\,\bar\psi_{i\mu}\gamma^\mu\gamma_{\rho\sigma}{\Omega_j}{}^\Lambda\,T^{ij\rho\sigma}\Big] + \text{h.c.}\Big) \\ - \Big(K_\Lambda\Big[\tfrac{1}{3}{\bar\Omega_i}{}^\Lambda\gamma^{\mu\nu}\mathcal{D}_\mu{\psi_\nu}{}^i - {\bar\Omega_i}{}^\Lambda\chi^i\Big] + \text{h.c.}\Big)\,, \tag{2.55}$$

where K is defined in (2.28). In this part of the Lagrangian we suppress terms quartic in the fermion fields. Note that (2.52) and (2.54) do contain terms quartic in the fermion fields, due to their significance to electric/magnetic duality, as we mentioned before.

We now exhibit the superconformal Lagrangian for hypermultiplets [42, 45],

$$e^{-1}\mathcal{L}_{\rm hyper} = \tfrac{1}{6}\,\chi\,\Big[R + (e^{-1}\varepsilon^{\mu\nu\rho\sigma}{\bar\psi_\mu}{}^i\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i} - \tfrac{1}{4}{\bar\psi_\mu}{}^i{\psi_\nu}{}^j\,{T^{\mu\nu}}_{ij} + \text{h.c.})\Big] \\ + \tfrac{1}{2}\,\chi\,\Big[D + \tfrac{1}{2}{\bar\psi_\mu}{}^i\gamma^\mu\chi_i + \tfrac{1}{2}\bar\psi_{\mu i}\gamma^\mu\chi^i\Big] \\ - \tfrac{1}{2}G_{\bar\alpha\beta}\,\mathcal{D}_\mu{A_i}^\beta\,\mathcal{D}^\mu A^{i\bar\alpha} - G_{\bar\alpha\beta}(\bar\zeta^{\bar\alpha}\not{\mathcal{D}}\zeta^\beta + \bar\zeta^\beta\not{\mathcal{D}}\zeta^{\bar\alpha}) - \tfrac{1}{4}W_{\bar\alpha\beta\bar\gamma\delta}\,\bar\zeta^{\bar\alpha}\gamma_\mu\zeta^\beta\,\bar\zeta^{\bar\gamma}\gamma^\mu\zeta^\delta \\ - \chi_A\Big({\gamma^A}_{i\bar\alpha}\Big[\tfrac{2}{3}\bar\zeta^{\bar\alpha}\gamma^{\mu\nu}\mathcal{D}_\mu{\psi_\nu}{}^i + \bar\zeta^{\bar\alpha}\chi^i - \tfrac{1}{6}\,\bar\zeta^{\bar\alpha}\gamma_\mu\psi_{\nu j}\,T^{\mu\nu ij}\Big] + \text{h.c.}\Big) \\ + \Big[\tfrac{1}{16}\,\bar\Omega_{\alpha\beta}\,\bar\zeta^\alpha\gamma^{\mu\nu}T_{\mu\nu ij}\varepsilon^{ij}\zeta^\beta - \tfrac{1}{2}\,\bar\zeta^\alpha\gamma^\mu\gamma^\nu\psi_{\mu i}\big({\bar\psi_\nu}{}^i\,G_{\alpha\bar\beta}\,\zeta^{\bar\beta} + \varepsilon^{ij}\,\bar\Omega_{\alpha\beta}\,\bar\psi_{\nu j}\zeta^\beta\big) \\ + G_{\bar\alpha\beta}\,\bar\zeta^\beta\gamma^\mu\not{\mathcal{D}}A^{i\bar\alpha}\psi_{\mu i} - \tfrac{1}{4}e^{-1}\epsilon^{\mu\nu\rho\sigma}G_{\bar\alpha\beta}\,{\bar\psi_\mu}{}^i\gamma_\nu\psi_{\rho j}\,{A_i}^\beta\mathcal{D}_\sigma A^{j\bar\alpha} + \text{h.c.}\Big]\,, \tag{2.56}$$

where $W_{\bar\alpha\beta\bar\gamma\delta}$ was defined in (2.42), and the hyperkähler potential was introduced in section 2.5. As mentioned in section 2.5, the target-space geometry is that of a hyperkähler cone. This hyperkähler cone is a cone over a so-called tri-Sasakian manifold [42, 48]. The latter is a fibration of Sp(1) over a $4(n_{\rm H}-1)$-dimensional quaternion-Kähler manifold $\mathbb{Q}^{4(n_{\rm H}-1)}$. Hence the hyperkähler cone can be written as $R^+ \times (\mathrm{Sp}(1) \times \mathbb{Q}^{4(n_{\rm H}-1)})$.

We have now introduced all the necessary ingredients to study the subject of the thesis: deformations of $N = 2$ supergravity. We will study these deformations in a superconformal setting, it is never necessary to gauge-fix the extra conformal symmetries. In the next chapter we will study general gaugings of $N = 2$ superconformal gravity theories based on vector multiplets and hypermultiplets. The vector fields contained in the vector multiplets will play the role of gauge fields for the internal symmetry group of the theory. In chapter 5 we will introduce deformations in the form of higher-derivative couplings. These couplings will be based on vector multiplets, the Weyl multiplet and possible other multiplets based on chiral multiplets.

Chapter **3**

General gauge deformations of N=2 superconformal gravity

As discussed in the introduction of this thesis, in four space-time dimensions, Lagrangians with abelian gauge fields have generically less symmetry than their corresponding equations of motion. The full invariance group of the combined field equations and Bianchi identities in principle involves a subgroup of the electric/magnetic duality group, $\mathrm{Sp}(2n,\mathbb{R})$ for n vector fields, suitably combined with transformations of the matter fields. Subgroups of the symmetry group of the Lagrangian can be gauged in the conventional way by introducing covariant derivatives and covariant field strengths. Introducing gauge groups which involve elements of the electric/magnetic duality group that do not belong to the symmetry group of the Lagrangian, are not possible in this way.

To circumvent this problem, one may therefore first convert the Lagrangian by an electric/magnetic equivalence transformation to a different, but equivalent, Lagrangian that has the desired gauge group as a symmetry. However, this procedure is cumbersome. One reason for this is that the gauge fields in the old and in the new electric/magnetic duality frame are not generically related by local field redefinitions. The effect of changing the duality frame is therefore not straightforward, and it is by no means trivial to explicitly obtain the new Lagrangian (see e.g. [49]). A related aspect is that, when the gauge fields belong to supermultiplets, their relation with other fields of the multiplet will be affected by changes of the duality frame, unless one simultaneously performs corresponding redefinitions of these fields as well.[1] The embedding tensor approach circumvents all these problems by introducing, from the start, both electric and magnetic gauge fields as well as tensor gauge fields. In this approach the gauge group is not restricted to a subgroup of the invariance group of the Lagrangian, but it must only be a subgroup of the symmetry group of field equations and Bianchi identities. The formalism is straightforwardly applicable to any given Lagrangian, and the gauge group is only restricted by two group-theoretical constraints on the embedding tensor [9].

In this chapter we study general gaugings of $N = 2$ superconformal gravity theories based on vector supermultiplets and hypermultiplets, using the embedding tensor formalism. This study is facilitated by the fact that the embedding tensor framework has already been considered for rigid $N=2$ supersymmetric gauge theories [47], without paying particular attention to the class of superconformally invariant models. The present chapter fills this gap by presenting a complete treatment of the embedding tensor method in the context of locally superconformal $N=2$ theories.

Theories with $N = 2$ supersymmetry are special with respect to electric/magnetic duality. For $N=1$ supersymmetry the transformations of the matter fields under electric/magnetic duality, and thus under the gauge group, are not a priori defined, and will depend on the details of the model. On the other hand, in theories with $N > 2$ supersymmetries all of the matter fields are closely linked to the vector fields, because they belong to common supermultiplets. Theories with $N=2$ supersymmetries are exceptional in that they exhibit both of these characteristic features. The complex scalars belonging to the vector multiplets transform in a well-defined way under electric/magnetic duality so that the Lagrangian will retain its standard form expressed in terms of a holomorphic function, while the scalars of the hypermultiplets have no a priori defined transformations under electric/magnetic duality. Prior to switching on the gauging, the hypermultiplets are invariant under some rigid symmetry group that is independent of the electric/magnetic duality group. Once the gauge group has been embedded in the latter group, then one has to separately specify its embedding into the symmetry group associated with the hypermultiplets.

The embedding tensor approach of [9] makes use of both electric and magnetic charges and their corresponding gauge fields. The charges are encoded in terms of an embedding tensor, which specifies the embedding of the gauge group into the full rigid invariance group. This embedding tensor is treated as a spurionic object (a quantity that is treated as a dynamical field, but that is frozen to a constant at the end of the calculation), so

[1] One way to circumvent this is by describing the scalar fields in terms of sections whose parametrization is linked to a specific frame (see, for instance, [50]).

that the electric/magnetic duality structure of the ungauged theory is preserved when the charges are turned on. Besides introducing a set of dual magnetic gauge fields, also tensor gauge fields are required transforming in the adjoint representation of the rigid invariance group. These extra fields carry additional off-shell degrees of freedom, but the number of physical degrees of freedom remains the same owing to extra gauge transformations. Prior to [9] it had already been discovered that magnetic charges tend to be accompanied by tensor fields. An early example of this was presented in [51], and subsequently more theories with magnetic charges and tensor fields were constructed, for instance, in [52–54], mostly in the context of abelian gauge groups. The embedding tensor approach has already been explored for many supersymmetric theories in four space-time dimensions. For instance, it was successfully applied to $N = 4$ supergravity [55] and to $N = 8$ supergravity [56]. More recently it has also been discussed for $N=1$ supergravity [57]. In [47] some applications to $N=2$ supergravity were already presented, under the assumption that the conformal multiplet calculus [4, 27, 28] is applicable. As it turned out, the results of the embedding tensor approach confirm and/or clarify various previous results in the literature, especially for abelian gaugings [58, 59]. The embedding tensor is ideally suited for the study of flux compactifications in string theory (for a review, see [6]). It has also been used to construct stable de Sitter vacua [60–62], where the presence of magnetic charges is crucial [63]. Recently it was successfully employed in a study of partial breaking of $N=2$ to $N=1$ supersymmetry [24, 25].

This chapter is organized as follows. In section 3.1 we review the relevant features of electric/magnetic duality in the context of $N = 2$ superconformal vector multiplets, and discuss the electric and magnetic gauge fields. Isometries of hypermultiplets are introduced in a superconformal setting in section 3.2. Section 3.3 contains a discussion of the possible gauge transformations, the electric and magnetic charges, and the embedding tensor. In section 3.4 we describe the introduction of tensor fields, needed in the presence of general charge assignments. Section 3.5 deals with the algebra of superconformal transformations in the presence of a gauging. It presents the extra masslike terms and the scalar potential in the vector multiplet and hypermultiplet Lagrangians that are induced by these gaugings.

3.1 Vector multiplets and electric/magnetic duality

In section 2.4 we introduced vector multiplets and their supersymmetry transformations. Their corresponding Lagrangian was given in section 2.6, where we already alluded to the presence of electric/magnetic duality. In this section we will consider electric/magnetic duality transformations on vector multiplets. Parts of this discussion will generalize the analysis in section 1.3. We will consider an extension of the field representation of the vector multiplet that will facilitate the treatment of electric/magnetic duality in the presence of non-zero gauge charges.

In the absence of charged fields, abelian gauge fields $W_\mu{}^\Lambda$ appear exclusively through the field strengths, $F_{\mu\nu}{}^\Lambda = 2\,\partial_{[\mu} W_{\nu]}{}^\Lambda$. The field equations for these fields and the Bianchi identities for the field strengths comprise $2(n+1)$ equations,

$$\partial_{[\mu} F_{\nu\rho]}{}^\Lambda = 0 = \partial_{[\mu} G_{\nu\rho]\,\Lambda}\,, \tag{3.1}$$

where,

$$G_{\mu\nu\,\Lambda} = \mathrm{i}e\,\varepsilon_{\mu\nu\rho\sigma}\,\frac{\partial\mathcal{L}}{\partial F_{\rho\sigma}{}^\Lambda}\;. \tag{3.2}$$

From the Lagrangian in (2.52) we derive the following decomposition for $G^-_{\mu\nu\Lambda}$ (and likewise for $G^+_{\mu\nu\Lambda}$),

$$G^-_{\mu\nu\Lambda} = F_{\Lambda\Sigma} F^-_{\mu\nu}{}^\Sigma - 2\mathrm{i}\mathcal{O}^-_{\mu\nu\Lambda}\,, \tag{3.3}$$

with $\mathcal{O}^-_{\mu\nu\Lambda}$ as in (2.53).

It is convenient to combine the tensors $F_{\mu\nu}{}^\Lambda$ and $G_{\mu\nu\Lambda}$ into a $2(n+1)$-dimensional vector,

$$G_{\mu\nu}{}^M = \begin{pmatrix} F_{\mu\nu}{}^\Lambda \\ G_{\mu\nu\Lambda} \end{pmatrix}\,, \tag{3.4}$$

so that (3.1) reads $\partial_{[\mu} G_{\nu\rho]}{}^M = 0$. Obviously these $2(n+1)$ equations are invariant under real $2(n+1)$-dimensional electric/magnetic duality rotations of the tensors $G_{\mu\nu}{}^M$,

$$\begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} \longrightarrow \begin{pmatrix} U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma \end{pmatrix} \begin{pmatrix} F^\Sigma \\ G_\Sigma \end{pmatrix}\,, \tag{3.5}$$

which generalizes (1.20). Half of the rotated tensors can be adopted as new field strengths defined in terms of new gauge fields, and the Bianchi identities on the remaining tensors can then be interpreted as field equations belonging to some new Lagrangian expressed in terms of the new field strengths. In order that such a Lagrangian exists, the real matrix in (3.5) must belong to the group $\mathrm{Sp}(2n+2;\mathbb{R})$ [64]. This group consists of real matrices that leave the skew-symmetric tensor Ω_{MN} invariant,

$$\Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} . \tag{3.6}$$

The conjugate matrix Ω^{MN} is defined by $\Omega^{MN}\Omega_{NP} = -\delta^M{}_P$. Here we employ an $\mathrm{Sp}(2n+2;\mathbb{R})$ covariant notation for the $2(n+1)$-dimensional symplectic indices $M, N, \ldots$, such that $Z^M = (Z^\Lambda, Z_\Sigma)$. Likewise we use vectors with lower indices according to $Y_M = (Y_\Lambda, Y^\Sigma)$, transforming according to the conjugate representation so that $Z^M\, Y_M$ is invariant.

The Lagrangian depends on the electric/magnetic duality frame and is therefore not unique. Different Lagrangians related by electric/magnetic duality lead to equivalent field equations and thus belong to the same equivalence class. These alternative Lagrangians remain supersymmetric but because the field strengths (and thus the underlying gauge fields) have been redefined, the standard relation between the various fields belonging to the vector supermultiplet, encoded in (2.20), is lost. However, upon a suitable redefinition of the other vector multiplet fields (possibly up to terms that will vanish subject to equations of motion) this relation can be preserved. It is to be expected that the new Lagrangian is again encoded in terms of a holomorphic homogeneous function, expressed in terms of the redefined scalar fields. Just as the Lagrangian changes, this function will change as well. Hence, different functions $F(X)$ can belong to the same equivalence class. The new function is such that the vector $X^M = (X^\Lambda, F_\Lambda)$ transforms under electric/magnetic duality according to,

$$\begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{X}^\Lambda \\ \tilde{F}_\Lambda \end{pmatrix} = \begin{pmatrix} U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma \end{pmatrix} \begin{pmatrix} X^\Sigma \\ F_\Sigma \end{pmatrix} . \tag{3.7}$$

The new function $\tilde{F}(\tilde{X})$ of the new scalars $\tilde{X}^\Lambda$ follows from integration of (3.7) and takes the form,

$$\begin{aligned} \tilde{F}(\tilde{X}) = {} & F(X) - \tfrac{1}{2} X^\Lambda F_\Lambda(X) + \tfrac{1}{2}(U^{\mathrm{T}}W)_{\Lambda\Sigma} X^\Lambda X^\Sigma \\ & + \tfrac{1}{2}(U^{\mathrm{T}}V + W^{\mathrm{T}}Z)_\Lambda{}^\Sigma X^\Lambda F_\Sigma(X) + \tfrac{1}{2}(Z^{\mathrm{T}}V)^{\Lambda\Sigma} F_\Lambda(X) F_\Sigma(X) . \end{aligned} \tag{3.8}$$

There are no integration constants because the function must remain homogeneous of second degree.

In general it is not easy to determine $\tilde{F}(\tilde{X})$ from (3.8) as it involves the inversion of

$\tilde{X}^\Lambda = U^\Lambda{}_\Sigma X^\Sigma + Z^{\Lambda\Sigma} F_\Sigma(X)$. As we emphasized in the beginning of this chapter, this is the reason why one prefers to avoid changing the electric/magnetic duality frame. The duality transformations on higher derivatives of $F(X)$ follow by differentiation and we note the results,

$$\begin{aligned}\tilde{F}_{\Lambda\Sigma}(\tilde{X}) &= (V_\Lambda{}^\Gamma F_{\Gamma\Xi} + W_{\Lambda\Xi})\,[\mathcal{S}^{-1}]^\Xi{}_\Sigma\,,\\ \tilde{F}_{\Lambda\Sigma\Gamma}(\tilde{X}) &= F_{\Xi\Delta\Omega}\,[\mathcal{S}^{-1}]^\Xi{}_\Lambda\,[\mathcal{S}^{-1}]^\Delta{}_\Sigma\,[\mathcal{S}^{-1}]^\Omega{}_\Gamma\,,\end{aligned} \tag{3.9}$$

where,

$$\mathcal{S}^\Lambda{}_\Sigma = \frac{\partial\tilde{X}^\Lambda}{\partial X^\Sigma} = U^\Lambda{}_\Sigma + Z^{\Lambda\Gamma} F_{\Gamma\Sigma}\,. \tag{3.10}$$

The symmetric real matrix $N_{\Lambda\Sigma}$ that we introduced in (2.29) transforms under electric/magnetic duality according to,

$$\tilde{N}_{\Lambda\Sigma}(\tilde{X},\bar{\tilde{X}}) = N_{\Gamma\Delta}\,[\mathcal{S}^{-1}]^\Gamma{}_\Lambda\,[\bar{\mathcal{S}}^{-1}]^\Delta{}_\Sigma\,. \tag{3.11}$$

To determine the action of the dualities on the fermion fields, we consider supersymmetry transformations of the symplectic vector $X^M = (X^\Lambda, F_\Lambda)$, according to (2.20), which can be written as $\delta X^M = \bar{\epsilon}^i\Omega_i{}^M$, thus defining an $\mathrm{Sp}(2n{+}2;\mathbb{R})$ covariant fermionic vector, $\Omega_i{}^M$,

$$\Omega_i{}^M = \begin{pmatrix} \Omega_i{}^\Lambda \\ F_{\Lambda\Sigma}\,\Omega_i{}^\Sigma \end{pmatrix}\,. \tag{3.12}$$

Complex conjugation leads to a second vector, Ω^{iM}, of opposite chirality. From (3.12) one derives that, under electric/magnetic duality,

$$\tilde{\Omega}_i{}^\Lambda = \mathcal{S}^\Lambda{}_\Sigma\,\Omega_i{}^\Sigma\,. \tag{3.13}$$

Another useful transformation rule that one can now check is,

$$\tilde{\mathcal{O}}^-_{\mu\nu\Lambda} = \mathcal{O}^-_{\mu\nu\Sigma}\,[\mathcal{S}^{-1}]^\Sigma{}_\Lambda\,. \tag{3.14}$$

Note the identity,

$$\Omega_{MN}\,X^M\Omega_i{}^N = 0\,, \tag{3.15}$$

which implies that supersymmetry variations of $\Omega_i{}^M$ are subject to $\Omega_{MN}\,X^M\,\delta\Omega_i{}^N = 0$ as well, up to terms quadratic in the vector multiplet spinors. This observation explains some of the identities that we will encounter in due course, especially in the next chapter.

The supersymmetry transformation of ${\Omega_i}^M$ also follows from (2.20), and we decompose it into the following form,

$$\delta{\Omega_i}^M = 2\not{D}X^M\epsilon_i + \tfrac{1}{2}\gamma^{\mu\nu}\hat{G}^{-}_{\mu\nu}{}^M\varepsilon_{ij}\,\epsilon^j + {Z_{ij}}^M\epsilon^j + 2\,X^M\eta_i\,. \tag{3.16}$$

Here the quantities ${Z_{ij}}^M$ are defined by,

$${Z_{ij}}^M = \begin{pmatrix} {Y_{ij}}^\Lambda \\ F_{\Lambda\Sigma}\,{Y_{ij}}^\Sigma - \tfrac{1}{2}F_{\Lambda\Sigma\Gamma}\,\bar{\Omega}_i{}^\Sigma\Omega_j{}^\Gamma \end{pmatrix}\,, \tag{3.17}$$

which suggests that ${Z_{ij}}^M$ transforms under electric/magnetic duality as a symplectic vector. However, this is only possible provided we impose a pseudo-reality condition on $Z_{ij\Lambda}$. As one can check, this constraint can be understood as the result of the field equation for ${Y_{ij}}^\Lambda$ associated with the Lagrangian presented in the section 2.6.

From (3.16) we also find a symplectic array of anti-selfdual supercovariant field strengths,

$$\hat{G}^{-}_{\mu\nu}{}^M = \begin{pmatrix} \hat{G}^{-}_{\mu\nu}{}^\Lambda \\ \hat{G}^{-}_{\mu\nu\Lambda} \end{pmatrix}\,. \tag{3.18}$$

where $\hat{G}^{-}_{\mu\nu}{}^\Lambda = \hat{F}^{-}_{\mu\nu}{}^\Lambda$, with $\hat{F}^{-}_{\mu\nu}{}^\Lambda$ defined in (2.21), and $\hat{G}^{-}_{\mu\nu\Lambda}$ is defined by,

$$\hat{G}^{-}_{\mu\nu\Lambda} = F_{\Lambda\Sigma}\hat{F}^{-}_{\mu\nu}{}^\Sigma - \tfrac{1}{8}F_{\Lambda\Sigma\Gamma}\,\bar{\Omega}_i{}^\Sigma\gamma_{\mu\nu}\Omega_j{}^\Gamma\,\varepsilon^{ij}\,. \tag{3.19}$$

A similar symplectic vector of the field strengths was given in (3.3) and by comparing with (3.4) we can make the identification (which generalizes (2.21)),

$$\begin{aligned}\hat{G}_{\mu\nu}{}^M = {} & G^{+}_{\mu\nu}{}^M + G^{-}_{\mu\nu}{}^M - \varepsilon^{ij}\bar{\psi}_{[\mu\,i}(\gamma_{\nu]}\Omega_j{}^M + \psi_{\nu]j}X^M) - \varepsilon_{ij}\bar{\psi}_{[\mu}{}^i(\gamma_{\nu]}\Omega^{j\,M} + \psi_{\nu]}{}^j\bar{X}^M) \\ & - \tfrac{1}{4}(X^M\,T_{\mu\nu ij}\,\varepsilon^{ij} + \bar{X}^M\,T_{\mu\nu}{}^{ij}\,\varepsilon_{ij})\,. \end{aligned} \tag{3.20}$$

The homogeneity of $F(X)$ is crucial for deriving these results. The relation (3.20) shows that also $\hat{G}^{-}_{\mu\nu}{}^M$ transforms as a symplectic vector under electric/magnetic duality.

The field strengths $G_{\mu\nu}{}^M$ satisfy a Bianchi identity. For $G_{\mu\nu}{}^\Lambda = F_{\mu\nu}{}^\Lambda$ this is obvious, and it implies that $F_{\mu\nu}{}^\Lambda$ can be expressed in terms of a vector potential $W_\mu{}^\Lambda$. For the field strengths $G_{\mu\nu\Lambda}$ the Bianchi identity is provided by the field equation for the vector fields (3.1).[2] This is similar to the situation with $Z_{ij}{}^M$, where the pseudo-reality of $Z_{ij\Lambda}$ is implied by the field equation of $Y_{ij}{}^\Lambda$. From the fact that the field strengths $G_{\mu\nu\Lambda}$ are subject to a Bianchi identity, it follows that they can be expressed in terms of magnetic duals $W_{\mu\Lambda}$. Hence we introduce these magnetic gauge fields, whose role will eventually become clear in the context of the embedding tensor formalism which will be introduced in due course.[3] Together with the electric gauge fields $W_\mu{}^\Lambda$, the magnetic duals constitute a symplectic vector, $W_\mu{}^M = (W_\mu{}^\Lambda, W_{\mu\Lambda})$, where $G_{\mu\nu}{}^M = 2\,\partial_{[\mu}W_{\nu]}{}^M$. As we shall see, this relationship is, however, not exact and the identification is subject to terms that depend on equations of motion. The supersymmetry transformations of $W_\mu{}^M$ are conjectured to take a duality covariant form,

$$\delta W_\mu{}^M = \varepsilon^{ij}\bar\epsilon_i(\gamma_\mu\Omega_j{}^M + 2\,\psi_{\mu j}X^M) + \varepsilon_{ij}\bar\epsilon^i(\gamma_\mu\Omega^{j\,M} + 2\,\psi_\mu{}^j\bar X^M)\,. \tag{3.21}$$

Observe that, with this transformation rule, the field strengths $\hat G_{\mu\nu}{}^M$ are supercovariant. As mentioned above, $G_{\mu\nu\Lambda}$ and $2\,\partial_{[\mu}W_{\nu]\Lambda}$ are not identical! This can be seen by calculating the supersymmetry variation of $2\,\partial_{[\mu}W_{\nu]\Lambda}$ and showing that it only coincides with the supersymmetry variation of (3.3) up to equations of motion.

The consistency, up to equations of motion, of introducing dual gauge fields $W_{\mu\Lambda}$ is also confirmed when considering the closure of the supersymmetry algebra, based on (3.21). Although we started with an off-shell definition of the vector multiplets, so that all superconformal transformations will close under commutation without the use of field equations, this is not necessarily the case for the newly introduced gauge field $W_{\mu\Lambda}$. The validity of (2.15) on $W_{\mu\Lambda}$ can be derived in direct analogy with the calculation of the commutation relation on $W_\mu{}^\Lambda$ in (2.24), upon replacing $G_{\mu\nu\Lambda}$ by $2\,\partial_{[\mu}W_{\nu]\Lambda}$. The abelian gauge transformation $\delta_{\rm gauge}$ contained in the commutation relation acts on both the electric and the magnetic gauge fields, and its parameter is given by (compare with (2.23)),

$$\Lambda^M = 4\,\bar X^M\,\bar\epsilon_2{}^i\epsilon_1{}^j\,\varepsilon_{ij} + \text{h.c.}\,. \tag{3.22}$$

We now turn once more to the Lagrangian for the vector multiplets (2.50). The kinetic terms of the scalar and spinor fields (2.51) can now be rewritten in a symplectic

[2] It should be obvious that also the field strengths $\hat G_{\mu\nu}{}^M$ satisfy a Bianchi-type identity of a more complicated form. Identities of this type have been presented in [27] for $\hat G_{\mu\nu}{}^\Lambda$.

[3] In the presence of gauge charges in the context of embedding tensor formalism, the Lagrangian can depend simultaneously on electric and magnetic gauge fields, as is described in later sections.

form,

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm kin}^{(1)} = & -\mathrm{i}\Omega_{MN}\,\mathcal{D}_\mu X^M\,\mathcal{D}^\mu \bar X^N + \tfrac14 \mathrm{i}\Omega_{MN}\left[\bar\Omega^{iM}\not{\mathcal{D}}\Omega_i{}^N - \bar\Omega_i{}^M\not{\mathcal{D}}\Omega^{iN}\right] \\
& - \tfrac12 \mathrm{i}\Omega_{MN}\left[\bar\psi_\mu{}^i\not{\mathcal{D}}\bar X^M\gamma^\mu\,\Omega_i{}^N - \bar\psi_{\mu i}\not{\mathcal{D}} X^M\gamma^\mu\,\Omega^{iN}\right] .
\end{aligned}
\tag{3.23}
$$

Also the Kähler potential (2.28) can be written in a symplectic form, $K = \mathrm{i}\Omega_{MN}X^M\bar X^N$. The four parts of the Lagrangian are each separately consistent with electric/magnetic duality, as was already hinted to.[4] We stress that this is not a invariance property. As mentioned before, the electric/magnetic duality transformations define equivalence classes of Lagrangians. A subgroup thereof may constitute an invariance of the theory, meaning that the Lagrangian and its underlying function $F(X)$ do not change [4, 65]. More specifically, an invariance implies,

$$
\tilde F(\tilde X) = F(\tilde X)\,, \tag{3.24}
$$

so that the result of the duality leads to a Lagrangian based on $\tilde F(\tilde X)$ which is identical to the original Lagrangian. Because $\tilde F(\tilde X) \neq F(X)$, as is obvious from (3.8), $F(X)$ is not an invariant *function*. Instead the above equation implies that the substitution $X^\Lambda \to \tilde X^\Lambda$ into the function $F(X)$ and its derivatives, induces precisely the duality transformations.[5] For example, we obtain,

$$
\begin{aligned}
F_\Lambda(\tilde X) &= V_\Lambda{}^\Sigma F_\Sigma(X) + W_{\Lambda\Sigma}X^\Sigma\,, \\
F_{\Lambda\Sigma}(\tilde X) &= (V_\Lambda{}^\Gamma F_{\Gamma\Xi} + W_{\Lambda\Xi})\,[\mathcal{S}^{-1}]^\Xi{}_\Sigma\,, \\
F_{\Lambda\Sigma\Gamma}(\tilde X) &= F_{\Xi\Delta\Omega}\,[\mathcal{S}^{-1}]^\Xi{}_\Lambda\,[\mathcal{S}^{-1}]^\Delta{}_\Sigma\,[\mathcal{S}^{-1}]^\Omega{}_\Gamma\,.
\end{aligned}
\tag{3.25}
$$

In section 3.3 we are precisely interested in this subclass of electric/magnetic duality transformations, as these are the ones that can be gauged.

This concludes the discussion about the transformation of vector multiplets under electric/magnetic duality. In the next section we will turn to the isometries of superconformal hypermultiplets.

---

[4] We note that (2.52) can be written as,

$$
e^{-1}\mathcal{L}_{\rm kin}^{(2)} = \tfrac14\mathrm{i}\big[F_{\mu\nu}^{-\Lambda}\,G^{-\mu\nu}{}_\Lambda + \text{h.c.}\big] - \mathrm{i}\big[\mathcal{O}^{-\mu\nu}{}_\Sigma\,N^{\Sigma\Lambda}\big(G^-_{\mu\nu\Lambda} - \bar F_{\Lambda\Gamma}\,F_{\mu\nu}^-{}^\Lambda\big) + \text{h.c.}\big]\,.
$$

Modulo the field equation of the vector fields, the first term can be written as a total derivative, whereas the second term is manifestly consistent with electric/magnetic duality as follows from (3.11), (3.25) and (3.14).

[5] This discussion can be compared to the discussion below (1.22).

3.2 Isometries of hyperkähler cones

As mentioned in the beginning of this chapter, hypermultiplets have no a priori defined transformations under electric/magnetic duality. Before switching on the gauging, the hypermultiplets are invariant under some rigid symmetry group that is independent of the electric/magnetic duality group. In section 3.3 we will consider general gaugings of the invariance group of the electric/magnetic dualities and the symmetry group of the hypermultiplets. Once the gauge group has been embedded in the electric/magnetic duality group, then one has to separately specify its embedding into the symmetry group associated with the hypermultiplets.

In this section we will discuss possible isometries of hyperkähler cones that commute with supersymmetry. Again, we follow the framework of [42]. The isometries are characterized by Killing vectors $k^A{}_{\mathsf{m}}(\phi)$, labeled by indices $\mathsf{m}, \mathsf{n}, \mathsf{p}$, etcetera. They generate a group of motions, denoted by $\mathrm{G}_{\mathrm{hyper}}$, that leaves the complex structures invariant so that they are called tri-holomorphic. Furthermore, they commute with SU(2) R-symmetry and dilatations. These three properties are reflected in the following equations,

$$\begin{aligned} k^C{}_{\mathsf{m}}\,\partial_C J^{ij}{}_{AB} - 2\partial_{[A}k^C{}_{\mathsf{m}}\,J^{ij}{}_{B]C} &= 0\,, \\ k_{ij}{}^B\,D_B k^A{}_{\mathsf{m}} = D_B k_{ij}{}^A\,k^B{}_{\mathsf{m}} &= J_{ij}{}^A{}_B\,k^B{}_{\mathsf{m}}\,, \\ \chi_A\,k^A{}_{\mathsf{m}} &= 0\,. \end{aligned} \tag{3.26}$$

Such tri-holomorphic isometries can be gauged by coupling to the (electric and/or magnetic) gauge fields belonging to the vector multiplets, as we shall discuss in due course. The structure constants of $\mathrm{G}_{\mathrm{hyper}}$ are denoted by $f_{\mathsf{mn}}{}^{\mathsf{p}}$, and follow from the Lie bracket relation,

$$k^B{}_{\mathsf{m}}\,\partial_B k^A{}_{\mathsf{n}} - k^B{}_{\mathsf{n}}\,\partial_B k^A{}_{\mathsf{m}} = -f_{\mathsf{mn}}{}^{\mathsf{p}}\,k^A{}_{\mathsf{p}}\,. \tag{3.27}$$

We note that derivatives of Killing vectors are constrained by the Killing equation, which induces constraints on multiple derivatives, as is shown below,

$$D_A k_B + D_B k_A = 0\,, \qquad D_A D_B k_C = R_{BCAE}\,k^E\,. \tag{3.28}$$

The infinitesimal transformations act on the hypermultiplet fields according to,

$$\begin{aligned} \delta\phi^A &= g\,\Lambda^{\mathsf{m}}\,k^A{}_{\mathsf{m}}(\phi)\,, \\ \delta\zeta^\alpha + \delta\phi^A\Gamma_A{}^\alpha{}_\beta\,\zeta^\beta &= g\,\Lambda^{\mathsf{m}}\,t_{\mathsf{m}}{}^\alpha{}_\beta(\phi)\,\zeta^\beta\,, \end{aligned} \tag{3.29}$$

where we introduced a generic coupling constant g and ϕ-dependent matrices $t_{\mathsf{m}}{}^{\alpha}{}_{\beta}(\phi)$ which take values in $\mathrm{sp}(n_{\mathrm{H}}+1)$, and are proportional to $D_A k^B{}_{\mathsf{m}}$. Explicit definitions will be given later, but we already note that they satisfy the following relations,

$$\begin{aligned} D_A t_{\mathsf{m}}{}^{\alpha}{}_{\beta} &= R_{AB}{}^{\alpha}{}_{\beta}\, k^B{}_{\mathsf{m}}\,, \\ [\,t_{\mathsf{m}},\, t_{\mathsf{n}}\,]^{\alpha}{}_{\beta} &= f_{\mathsf{mn}}{}^{\mathsf{p}}\,(t_{\mathsf{p}})^{\alpha}{}_{\beta} + k^A{}_{\mathsf{m}}\, k^B{}_{\mathsf{n}}\, R_{AB}{}^{\alpha}{}_{\beta}\,. \end{aligned} \tag{3.30}$$

This result is consistent with the Jacobi identity. The above results can be summarized by noting that the linear combinations, $X_{\mathsf{m}}{}^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta}\, k^A{}_{\mathsf{m}} D_A - t_{\mathsf{m}}{}^{\alpha}{}_{\beta}$, close under commutation according to,[6]

$$[X_{\mathsf{m}}, X_{\mathsf{n}}]^{\alpha}{}_{\beta} = -f_{\mathsf{mn}}{}^{\mathsf{p}}\, X_{\mathsf{p}}{}^{\alpha}{}_{\beta}\,. \tag{3.31}$$

One can show that the curl of $J^{ij}{}_{AB}\, k^B{}_{\mathsf{m}}$ vanishes, so that these vectors can be solved in terms of the derivative of the so-called Killing potentials, or moment maps, denoted by $\mu^{ij}{}_{\mathsf{m}}$. On the hyperkähler cone there are no integration constants, and one can explicitly determine these potentials,

$$\mu^{ij}{}_{\mathsf{m}} = -\tfrac{1}{2} k^{ij}{}_A\, k^A{}_{\mathsf{m}}\,. \tag{3.32}$$

This can easily be verified by showing that $\partial_A \mu^{ij}{}_{\mathsf{m}} = J^{ij}{}_{AB}\, k^B{}_{\mathsf{m}}$, making use of (3.26) and the Killing equation given in (3.28). Using also (3.27) one derives the so-called equivariance condition,

$$J^{ij}{}_{AB}\, k^A{}_{\mathsf{m}}\, k^B{}_{n} = -f_{\mathsf{mn}}{}^{\mathsf{p}}\, \mu^{ij}{}_{\mathsf{p}}\,. \tag{3.33}$$

The Killing potentials scale with weight $w=2$ under dilatations and transform covariantly under the isometries and SU(2) transformations,

$$\begin{aligned} \delta\mu^{ij}{}_{\mathsf{m}} &= \big(g\,\Lambda^{\mathsf{n}}\, k^A{}_{\mathsf{n}} + \Lambda_{\mathrm{SU}(2)}{}^k{}_m\, \varepsilon^{lm}\, k_{kl}{}^A\big)\, \partial_A \mu^{ij}_{\mathsf{m}} \\ &= \big(-g\,\Lambda^n\, f_{\mathsf{nm}}{}^{\mathsf{p}}\, \mu^{ij}{}_p + 2\,\Lambda_{\mathrm{SU}(2)}{}^{(i}{}_k\, \mu^{j)k}{}_{\mathsf{m}}\big)\,. \end{aligned} \tag{3.34}$$

An expression for the generators t_{m} associated with the tri-holomorphic Killing vectors follows from requiring the invariance of the quaternionic vielbeine V^{α}_{Ai} up to a target-space rotation,

$$(t_{\mathsf{m}})^{\alpha}{}_{\beta} = \tfrac{1}{2} V^{\alpha}_{Ai}\, \bar\gamma^{Bi}_{\beta}\, D_B k^A{}_{\mathsf{m}}\,. \tag{3.35}$$

[6] To be precise, the X_{m} are the generators acting of ϕ-dependent tangent-space tensors (provided the matrix t_{m} is replaced by the appropriate generator for the corresponding tensor representation).

The invariance implies that target-space scalars satisfy algebraic identities such as,

$$\bar{t}_{\mathsf{m}}{}^{\bar{\gamma}}{}_{\bar{\alpha}}\, G_{\bar{\gamma}\beta} + t_{\mathsf{m}}{}^{\gamma}{}_{\beta}\, G_{\bar{\alpha}\gamma} = \bar{t}_{\mathsf{m}}{}^{\bar{\gamma}}{}_{[\bar{\alpha}}\, \Omega_{\bar{\beta}]\bar{\gamma}} = 0\,, \tag{3.36}$$

which confirm that the matrices $t_{\mathsf{m}}{}^{\alpha}{}_{\beta}$ take values in $\mathrm{sp}(n_{\mathrm{H}}+1)$. Furthermore we note the relations,

$$\begin{aligned} k^{A}{}_{\mathsf{m}}\, V_{Ai}^{\alpha} &= k^{A}{}_{\mathsf{m}}\, D_{A} A_{i}{}^{\alpha} = t_{\mathsf{m}}{}^{\alpha}{}_{\beta}\, A_{i}{}^{\beta}\,, \\ \mu_{ij\mathsf{m}} &= -\tfrac{1}{2} k_{Aij}\, k^{A}{}_{\mathsf{m}} = -\tfrac{1}{2}\bar{\Omega}_{\alpha\beta}\, A_{i}{}^{\alpha}\, t_{\mathsf{m}}{}^{\beta}{}_{\gamma} A_{j}{}^{\gamma}\,. \end{aligned} \tag{3.37}$$

For a more complete list of identities we refer to [42].

3.3 Gauge invariance, electric and magnetic charges, and the embedding tensor

Possible gauge groups must be embedded into the rigid invariance group $\mathrm{G}_{\mathrm{rigid}}$ of the theory. Since we consider both vector and hypermultiplets, we are in principle dealing with a product group, $\mathrm{G}_{\mathrm{rigid}} = \mathrm{G}_{\mathrm{symp}} \times \mathrm{G}_{\mathrm{hyper}}$, where $\mathrm{G}_{\mathrm{symp}}$ refers to the invariance group of the electric/magnetic dualities, which acts exclusively on the vector multiplets, and $\mathrm{G}_{\mathrm{hyper}}$ refers to the possible invariance group of the hypermultiplet sector generated by the tri-holomorphic Killing vectors. Here we first concentrate on the gauge group embedded into $\mathrm{G}_{\mathrm{symp}}$, which constitutes a subgroup of the electric/magnetic duality group $\mathrm{Sp}(2n+2;\mathbb{R})$ related to the matrices considered in (3.5). The corresponding gauge group generators thus take the form of $(2n+2)$-by-$(2n+2)$ matrices T_M. Since we are assuming the presence of both electric and magnetic gauge fields, these generators decompose according to $T_M = (T_\Lambda, T^\Lambda)$. Obviously the gauge-group generators $T_{MN}{}^{P}$ must generate a subalgebra of the Lie algebra associated with $\mathrm{Sp}(2n+2;\mathbb{R})$, which implies,

$$T_{M[N}{}^{Q}\, \Omega_{P]Q} = 0\,, \tag{3.38}$$

or, in components,

$$T_{M\Lambda}{}^{\Sigma} = -T_{M}{}^{\Sigma}{}_{\Lambda}\,, \qquad T_{M[\Lambda\Sigma]} = 0 = T_{M}{}^{[\Lambda\Sigma]}\,. \tag{3.39}$$

Denoting the gauge group parameters by Λ^M, infinitesimal variations of generic $2(n+1)$-dimensional $\mathrm{Sp}(2n+2;\mathbb{R})$ vectors Y^M and Z_M thus take the form,

$$\delta Y^M = -g\Lambda^N\, T_{NP}{}^M\, Y^P\,, \qquad \delta Z_M = g\Lambda^N\, T_{NM}{}^P\, Z_P\,, \tag{3.40}$$

where g denotes a universal gauge coupling constant.[7] Covariant derivatives can easily be constructed, and read,[8]

$$\begin{aligned}\mathcal{D}_\mu Y^M &= \partial_\mu Y^M + gW_\mu{}^N\, T_{NP}{}^M\, Y^P \\ &= \partial_\mu Y^M + gW_\mu{}^\Lambda\, T_{\Lambda P}{}^M\, Y^P + gW_{\mu\Lambda}\, T^\Lambda{}_P{}^M\, Y^P\,,\end{aligned} \tag{3.41}$$

and similarly for $\mathcal{D}_\mu Z_M$. The gauge fields then transform according to,

$$\delta W_\mu{}^M = \mathcal{D}_\mu\Lambda^M = \partial_\mu\Lambda^M + g\, T_{PQ}{}^M W_\mu{}^P\, \Lambda^Q\,. \tag{3.42}$$

Note that, for constant parameters Λ^M, $W_\mu{}^M$ should transform according to (3.40). Consistency with (3.42) then requires that $T_{MN}{}^P$ is antisymmetric in $[MN]$. Nevertheless, as we shall see, antisymmetry of $T_{MN}{}^P$ is not necessary in the general case. Rather, it is sufficient that the $T_{MN}{}^P$ are subject to the so-called representation constraint [9],

$$T_{(MN}{}^Q\, \Omega_{P)Q} = 0 \Longrightarrow \begin{cases} T^{(\Lambda\Sigma\Gamma)} = 0\,, \\ 2\, T^{(\Gamma\Lambda)}{}_\Sigma = T_\Sigma{}^{\Lambda\Gamma}\,, \\ T_{(\Lambda\Sigma\Gamma)} = 0\,, \\ 2\, T_{(\Gamma\Lambda)}{}^\Sigma = T^\Sigma{}_{\Lambda\Gamma}\,. \end{cases} \tag{3.43}$$

which does not imply antisymmetry of $T_{MN}{}^P$ in $[M,N]$. However, for the conventional electric gaugings, where the magnetic gauge fields $A_{\mu\Lambda}$ decouple and where $T^\Lambda{}_N{}^P = 0$ and $T_\Lambda{}^{\Sigma\Gamma} = 0$, (3.43) does imply that $T_{\Gamma\Sigma}{}^\Lambda$ is antisymmetric in $[\Gamma\Sigma]$.

Note that full covariance of the derivative defined in (3.41) has not yet been established to order g^2, since we have not discussed the closure of the gauge group generators. This point will be addressed later in this section.

[7] The generators follow by expanding the symplectic matrix appearing in (3.5) and (3.7) about the identity. Comparing with (3.40), one establishes the correspondence, $U^\Lambda{}_\Sigma \approx \delta^\Lambda{}_\Sigma - g\Lambda^M T_{M\Sigma}{}^\Lambda$, $V_\Lambda{}^\Sigma \approx \delta_\Lambda{}^\Sigma + g\Lambda^M T_{M\Lambda}{}^\Sigma$, $Z^{\Lambda\Sigma} \approx -g\Lambda^M T_M{}^{\Lambda\Sigma}$, $W_{\Lambda\Sigma} \approx -g\Lambda^M T_{M\Lambda\Sigma}$.

[8] In this section and in section 3.4, we suppress the covariantization with respect to superconformal symmetries. Starting with section 3.5 the derivative $\mathcal{D}_\mu$ will indicate covariantization with respect to Lorentz, dilatation, and chiral symmetries, and with the newly introduced gauge symmetries associated with the fields $W_\mu{}^M$.

Let us first consider some generic features of the infinitesimal transformations (3.40). Combining the two equations (3.8) and (3.24) leads to an expression for $F(\tilde{X}) - F(X)$, which, for an infinitesimal symmetry transformation $\delta X^\Lambda = -g\,\Lambda^M T_{MN}{}^\Lambda\, X^N$, yields,

$$F_\Lambda\, \delta X^\Lambda = -\tfrac{1}{2} g \Lambda^M \Big(T_{M\Lambda\Sigma} X^\Lambda X^\Sigma + T_M{}^{\Lambda\Sigma} F_\Lambda F_\Sigma \Big)\,. \tag{3.44}$$

Substituting the expression for δX^Λ then leads to the condition [4],

$$T_{MN}{}^Q \Omega_{PQ}\, X^N X^P = T_{M\Lambda\Sigma} X^\Lambda X^\Sigma - 2 T_{M\Lambda}{}^\Sigma X^\Lambda F_\Sigma - T_M{}^{\Lambda\Sigma} F_\Lambda F_\Sigma = 0\,. \tag{3.45}$$

which must hold for general X^Λ. The solution of this condition will specify all continuous symmetries of the vector Lagrangian (2.50). There are two more useful identities that follow from it. First one takes the derivative of (3.45) with respect to X^Λ,

$$T_{MN\Lambda} X^N = F_{\Lambda\Sigma}\, T_{MN}{}^\Sigma X^N\,, \tag{3.46}$$

and subsequently applies a supersymmetry transformation leading to,

$$T_{MN\Lambda} \Omega_i{}^N = F_{\Lambda\Sigma}\, T_{MN}{}^\Sigma \Omega_i{}^N + F_{\Lambda\Sigma\Gamma}\, \Omega_i{}^\Sigma\, T_{MN}{}^\Gamma X^N\,. \tag{3.47}$$

The latter two identities show that the gauge covariantization of the kinetic term for the scalars and spinors in (3.23) will not involve $T_{M\Lambda\Sigma}$.

By introducing a vector $U^M = (U^\Lambda, F_{\Lambda\Sigma} U^\Sigma)$, it is possible to cast (3.46) in the symplectically covariant form, $T_{MN}{}^Q\, \Omega_{PQ}\, X^N U^P = 0$. This equation can be rewritten by making use of the representation constraint (3.43). Note, for instance, the following identities,

$$\begin{aligned} T_{(MN)}{}^P\, X^M\, U^N &= 0\,, \\ T_{MN}{}^Q\, \Omega_{PQ}\, \bar{X}^M X^N \bar{X}^P = \mathrm{i}\, T_{MN}{}^\Lambda\, \bar{X}^M X^N\, N_{\Lambda\Sigma}\, \bar{X}^\Sigma &= 0\,. \end{aligned} \tag{3.48}$$

As a side remark we note that the Killing potential (or moment map) associated with the isometries considered above, is related to,

$$\nu_M = g\, T_{MN}{}^Q \Omega_{PQ} \bar{X}^N X^P\,. \tag{3.49}$$

Its derivative takes the form $\partial_\Lambda \nu_M = \mathrm{i} N_{\Lambda\Sigma}\, \delta\bar{X}^\Sigma$, as follows from making use of (3.46).

Finally we return to the gauge transformations of the auxiliary fields $Y_{ij}{}^\Lambda$, which can be derived by requiring that $\mathcal{L}_{\rm aux}$ written in (2.54) is gauge invariant. A straightforward calculation leads to the following result,

$$\delta Y_{ij}{}^{\Lambda} = -\tfrac{1}{2} g \Lambda^M T_{MN}{}^{\Lambda} (Z_{ij}{}^N + \varepsilon_{ik}\varepsilon_{jl}\, Z^{klN}) \,, \tag{3.50}$$

where $Z_{ij}{}^M$ was defined in (3.17). Note that this result is in accord with the electric/magnetic dualities suggested for $Z_{ij}{}^M$.

In the remainder of this section we consider the gauge group embedding in more detail. The embedding into the rigid invariance group $\mathrm{G}_{\rm rigid} = \mathrm{G}_{\rm symp} \times \mathrm{G}_{\rm hyper}$ is encoded in a so-called embedding tensor. This tensor must be specified separately for the vector multiplet and for the hypermultiplet sector, so that we have the following definitions,

$$\begin{aligned} T_{MN}{}^P &= \Theta_M{}^{\mathsf{a}}\, t_{\mathsf{a}N}{}^P \,, \\ k^A{}_M &= \Theta_M{}^{\mathsf{m}}\, k^A{}_{\mathsf{m}} \,, \qquad T_M{}^{\alpha}{}_{\beta} = \Theta_M{}^{\mathsf{m}}\, t_{\mathsf{m}}{}^{\alpha}{}_{\beta} \,, \end{aligned} \tag{3.51}$$

where the t_{a} denote the generators of $\mathrm{G}_{\rm symp}$, and $k^A{}_{\mathsf{m}}$ and t_{m} the tri-holomorphic Killing vectors and the corresponding matrices of the group $\mathrm{G}_{\rm hyper}$. Because these generators belong to different groups and act on different multiplets, they carry different indices (namely, indices $M, N, \ldots$ for the vector multiplets and indices $\alpha, \beta, \ldots$ for the hypermultiplets). The embedding tensor can be further decomposed into electric and magnetic components, according to $\Theta_M{}^{\mathsf{a}} = (\Theta_\Lambda{}^{\mathsf{a}}, \Theta^{\Lambda\,\mathsf{a}})$, and $\Theta_M{}^{\mathsf{m}} = (\Theta_\Lambda{}^{\mathsf{m}}, \Theta^{\Lambda\,\mathsf{m}})$. With these definitions, we can now also present the gauge-covariant derivatives on the hypermultiplet fields (we remind the reader that in this section and in the next one, we suppress the covariantization with respect to the superconformal symmetries),

$$\begin{aligned} \mathcal{D}_\mu \phi^A &= \partial_\mu \phi^A - g W_\mu{}^M\, k^A{}_M \,, \\ \mathcal{D}_\mu A_i{}^\alpha &= \partial_\mu A_i{}^\alpha - g W_\mu{}^M\, T_M{}^\alpha{}_\beta A_i{}^\beta \,, \\ \mathcal{D}_\mu \zeta^\alpha &= \partial_\mu \zeta^\alpha + \partial_\mu \phi^A\, \Gamma_A{}^\alpha{}_\beta\, \zeta^\beta - g W_\mu{}^M T_M{}^\alpha{}_\beta\, \zeta^\beta \,. \end{aligned} \tag{3.52}$$

In particular the covariant derivative of the spinor field is not entirely straightforward, in view of the fact that matrices $t_{\mathsf{m}}{}^\alpha{}_\beta$ depend on the fields ϕ^A. However, because the Jacobi identity is satisfied on these matrices, there are no further complications associated with this feature (see (3.30)).

The gauge group generators T_M should close under commutation for both representations. This leads to two equations that depend quadratically on the embedding tensor [66],

$$\begin{aligned} f_{\mathsf{ab}}{}^{\mathsf{c}}\,\Theta_M{}^{\mathsf{a}}\,\Theta_N{}^{\mathsf{b}} + (t_{\mathsf{a}})_N{}^P\,\Theta_M{}^{\mathsf{a}}\Theta_P{}^{\mathsf{c}} &= 0\,,\\ f_{\mathsf{mn}}{}^{\mathsf{p}}\,\Theta_M{}^{\mathsf{m}}\,\Theta_N{}^{\mathsf{n}} + (t_{\mathsf{a}})_N{}^P\,\Theta_M{}^{\mathsf{a}}\Theta_P{}^{\mathsf{p}} &= 0\,, \end{aligned} \tag{3.53}$$

where $f_{\mathsf{ab}}{}^{\mathsf{c}}$ and $f_{\mathsf{mn}}{}^{\mathsf{p}}$ are the structure constants of G_{symp} and $\mathrm{G}_{\text{hyper}}$, respectively.[9] The above equations imply that the gauge algebra generators close according to,

$$[T_M, T_N] = -T_{MN}{}^P\,T_P\;, \qquad k^B{}_M\partial_B k^A{}_N - k^B{}_N\partial_B k^A{}_M = T_{MN}{}^P\,k^A{}_P\;, \tag{3.54}$$

so that the structure constants of the gauge group are contained in $-T_{MN}{}^P \equiv -\Theta_M{}^{\mathsf{a}}\,(t_{\mathsf{a}})_N{}^P$, as is required by the gauge group embedding in G_{symp}. This observation was in fact used as input when deriving (3.53). Note, however, that the gauge group structure constants are not necessarily identical to $-T_{MN}{}^P$, as they may differ by terms that vanish upon contraction with the embedding tensor $\Theta_P{}^{\mathsf{a}}$ or $\Theta_P{}^{\mathsf{m}}$. This explains why the $T_{MN}{}^P$ are not necessarily antisymmetric in M, N.

Here and henceforth, the embedding tensor will be regarded as a spurionic object which we allow to transform under the rigid invariance group $\mathrm{G}_{\text{rigid}}$, so that the Lagrangian and transformation rules will remain formally invariant. Therefore the embedding tensor can be assigned to a (not necessarily irreducible) representation of $\mathrm{G}_{\text{rigid}}$. Eventually the embedding tensor will be frozen to a constant, so that the invariance under $\mathrm{G}_{\text{rigid}}$ will be broken. In this context, it is relevant to note that (3.53) implies that the embedding tensor is invariant under the gauge group. The gauge group is thus contained in the corresponding stability subgroup of $\mathrm{G}_{\text{rigid}}$. From symmetrizing the first constraint (3.53) in (MN) and making use of the linear conditions (3.43) and (3.38), one further derives that $\Omega^{MN}\,\Theta_M{}^{\mathsf{a}}\Theta_N{}^{\mathsf{b}}\,(t_{\mathsf{b}})_P{}^Q$ must vanish. Hence,

$$\Omega^{MN}\,\Theta_M{}^{\mathsf{a}}\Theta_N{}^{\mathsf{b}} \;=\; 0 \iff \Theta^{\Lambda\,[\mathsf{a}}\Theta_\Lambda{}^{\mathsf{b}]} = 0\;, \tag{3.55}$$

which implies that the charges in the vector multiplet sector are mutually local, so that an electric/magnetic duality must exist that converts all the charges to electric ones. Likewise, one derives from the second constraint (3.53),

$$\Omega^{MN}\,\Theta_M{}^{\mathsf{a}}\Theta_N{}^{\mathsf{m}} \;=\; 0 \iff \Theta^{\Lambda\,[\mathsf{a}}\Theta_\Lambda{}^{\mathsf{m}]} = 0\;, \tag{3.56}$$

[9] For convenience we have ignored that the matrices t_{m} depend on the scalar fields (see (3.31) and the preceding text).

which implies that the charges in the hypermultiplet sector are mutually local with the vector multiplet charges. It is clear that gauge fields that couple exclusively to charges associated to hypermultiplets are not restricted by (3.55) and (3.56). Their corresponding gauge groups are necessarily abelian. To ensure that those charges are also mutually local, we must impose an additional constraint,

$$\Omega^{MN}\,\Theta_M{}^{\mathsf{m}}\Theta_N{}^{\mathsf{n}} \;=\; 0 \quad\Longleftrightarrow\quad \Theta^{\Lambda\,[\mathsf{m}}\Theta_\Lambda{}^{\mathsf{n}]} = 0\;, \tag{3.57}$$

which is obviously not related to the closure of the gauge algebra. As it turns out, the relations (3.55), (3.56) and (3.57) play an crucial role when discussing the Lagrangian.

Generically only a subset of the gauge fields will be involved in the gauging, so that the embedding tensor will project out a restricted set of (linear combinations of) gauge fields; the rank of the tensor determines the dimension of the gauge group, up to possible central extensions associated with abelian factors.

As stressed before, the generators $T_{MN}{}^P$ are not required to be antisymmetric in M,N. The symmetric part can be written as follows,

$$T_{(MN)}{}^P = Z^{P,\mathsf{a}}\, d_{\mathsf{a}\,MN}\;, \tag{3.58}$$

with,

$$\begin{aligned} d_{\mathsf{a}\,MN} &\equiv (t_{\mathsf{a}})_M{}^P\,\Omega_{NP}\,,\\ Z^{M,\mathsf{a}} &\equiv \tfrac{1}{2}\Omega^{MN}\Theta_N{}^{\mathsf{a}} \quad\Longrightarrow\quad \left\{\begin{array}{rl} Z^{\Lambda\mathsf{a}} = & \tfrac{1}{2}\Theta^{\Lambda\mathsf{a}}\,,\\ Z_\Lambda{}^{\mathsf{a}} = & -\tfrac{1}{2}\Theta_\Lambda{}^{\mathsf{a}}\,, \end{array}\right. \end{aligned} \tag{3.59}$$

so that $d_{\mathsf{a}\,MN}$ defines an $\mathrm{Sp}(2n+2,\mathbb{R})$-invariant tensor symmetric in (MN). Likewise one can introduce a similar tensor $Z^{M,\mathsf{m}}$, which is relevant for the hypermultiplets,

$$Z^{M,\mathsf{m}} \equiv \tfrac{1}{2}\Omega^{MN}\Theta_N{}^{\mathsf{m}} \quad\Longrightarrow\quad \left\{\begin{array}{rl} Z^{\Lambda\mathsf{m}} = & \tfrac{1}{2}\Theta^{\Lambda\mathsf{m}}\,,\\ Z_\Lambda{}^{\mathsf{m}} = & -\tfrac{1}{2}\Theta_\Lambda{}^{\mathsf{m}}\,. \end{array}\right. \tag{3.60}$$

Subsequently we note that the constraints (3.55), (3.56) and (3.57) can now be written as,

$$Z^{M,\mathsf{a}}\,\Theta_M{}^{\mathsf{b}} = 0 = Z^{M,\mathsf{a}}\,\Theta_M{}^{\mathsf{m}}\,,\qquad Z^{M,\mathsf{m}}\,\Theta_M{}^{\mathsf{a}} = 0 = Z^{M,\mathsf{m}}\,\Theta_M{}^{\mathsf{n}}\,. \tag{3.61}$$

This implies that $Z^{M,\mathsf{a}}$ and $Z^{M,\mathsf{m}}$ vanish when contracted with the gauge-group generators T_M. Because of these constraints, only the antisymmetric part of $T_{MN}{}^P$ will appear in the commutation relation (3.54). What remains is to consider the Jacobi identity on the generators T_M. Explicit calculation based on (3.54) leads to,

$$T_{[NP}{}^R\, T_{Q]R}{}^M = \tfrac{2}{3} Z^{M,\mathsf{a}}\, d_{\mathsf{a}R[N}\, T_{PQ]}{}^R\,, \tag{3.62}$$

which shows that the Jacobi identity holds up to terms that vanish upon contraction with the embedding tensor. In the following section we will describe how to introduce a consistent gauging in this non-standard situation.

3.4 The gauge hierarchy

To compensate for the lack of closure noted in the previous section, and, at the same time, to avoid unwanted degrees of freedom, the strategy is to introduce an extra gauge invariance for the gauge fields, in addition to the usual non-abelian gauge transformations,

$$\delta W_\mu{}^M = \mathcal{D}_\mu \Lambda^M - g\big[Z^{M,\mathsf{a}}\, \Xi_{\mu\,\mathsf{a}} + Z^{M,\mathsf{m}}\, \Xi_{\mu\,\mathsf{m}}\big]\,, \tag{3.63}$$

where the Λ^M are the gauge transformation parameters and the covariant derivative reads, $\mathcal{D}_\mu\Lambda^M = \partial_\mu\Lambda^M + g\, T_{PQ}{}^M\, W_\mu{}^P\Lambda^Q$. The transformations proportional to $\Xi_{\mu\,\mathsf{a}}$ and $\Xi_{\mu\,\mathsf{m}}$ enable one to gauge away those vector fields that are in the sector where the Jacobi identity is not satisfied (this sector is perpendicular to the embedding tensor by virtue of (3.61)). Note that the covariant derivative is invariant under the transformations parametrized by $\Xi_{\mu\,\mathsf{a}}$ and $\Xi_{\mu\,\mathsf{m}}$, because of the contraction of the gauge fields $W_\mu{}^M$ with the generators T_M. However, gauge transformations do no longer form a group by themselves, as is reflected in the commutation relation,

$$[\delta(\Lambda_1), \delta(\Lambda_2)] = \delta(\Lambda_3) + \delta(\Xi_{\mathsf{a}\,3})\,, \tag{3.64}$$

where,

$$\begin{aligned} \Lambda_3{}^M &= g\, T_{[NP]}{}^M \Lambda_1^N \Lambda_2^P\,, \\ \Xi_{3\mu\,\mathsf{a}} &= d_{\mathsf{a}NP}(\Lambda_1^N \mathcal{D}_\mu \Lambda_2^P - \Lambda_2^N \mathcal{D}_\mu \Lambda_1^P)\,, \end{aligned} \tag{3.65}$$

with $T_{M\mathsf{a}}{}^{\mathsf{b}} = -\Theta_M{}^{\mathsf{c}} f_{\mathsf{ca}}{}^{\mathsf{b}}$ the gauge group generators in the adjoint representation of $\mathrm{G}_{\mathrm{symp}}$. As it turns out, this commutation relation forms the beginning of a full hierarchy of vector and tensor gauge fields that form a closed algebra [67, 68]. Other commutators involving $\delta(\Lambda)$, $\delta(\Xi_{\mathsf{a}})$ and $\delta(\Xi_{\mathsf{m}})$ vanish on the gauge fields $W_\mu{}^\Lambda$, so that those can only be uncovered for the higher-rank tensor gauge fields that we will introduce shortly.

Non-abelian field strengths associated with the gauge fields $W_\mu{}^M$ follow from the Ricci identity, $[D_\mu, D_\nu] = -g\mathcal{F}_{\mu\nu}{}^M\, T_M$, and depend only on the antisymmetric part of $T_{MN}{}^P$,

$$\mathcal{F}_{\mu\nu}{}^M = \partial_\mu W_\nu{}^M - \partial_\nu W_\mu{}^M + g\, T_{[NP]}{}^M\, W_\mu{}^N W_\nu{}^P\,. \tag{3.66}$$

Because of the lack of closure expressed by (3.62), these field strengths do not satisfy the Palatini identity,

$$\delta\mathcal{F}_{\mu\nu}{}^M = 2\,\mathcal{D}_{[\mu}\delta W_{\nu]}{}^M - 2g\, T_{(PQ)}{}^M\, W_{[\mu}{}^P\, \delta W_{\nu]}{}^Q\,, \tag{3.67}$$

under arbitrary variations $\delta W_\mu{}^M$, because of the last term, which cancels upon multiplication with the generators T_M. The result (3.67) shows in particular that $\mathcal{F}_{\mu\nu}{}^M$ transforms under the combined gauge transformations (3.63) as,

$$\begin{aligned}\delta\mathcal{F}_{\mu\nu}{}^M =&\, g\,\Lambda^P T_{NP}{}^M\, \mathcal{F}_{\mu\nu}{}^N - 2g\, Z^{M,\mathsf{a}}\big(\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{a}} + d_{\mathsf{a}PQ}\, W_{[\mu}{}^P\, \delta W_{\nu]}{}^Q\big)\\ &- 2g\, Z^{M,\mathsf{m}}\, \mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{m}}\,,\end{aligned} \tag{3.68}$$

and is therefore not covariant. In deriving this one makes use of the fact that the tensors $Z^{M,\mathsf{a}}$ and $Z^{M,\mathsf{m}}$ are invariant under the gauge group. The covariant derivative on $\Xi_{\nu\mathsf{a}}$ is defined by $\mathcal{D}_\mu\Xi_{\nu\mathsf{a}} = \partial_\mu\Xi_{\nu\mathsf{a}} - gW_\mu{}^M T_{M\mathsf{a}}{}^{\mathsf{b}}\Xi_{\nu\mathsf{b}}$, and similarly for $\Xi_{\nu\mathsf{m}}$. These tensor fields belong to the adjoint representation of the group $G_{\rm symp}$.

The standard strategy is therefore to define modified field strengths,

$$\mathcal{H}_{\mu\nu}{}^M = \mathcal{F}_{\mu\nu}{}^M + g\big[Z^{M,\mathsf{a}}\, B_{\mu\nu\,\mathsf{a}} + Z^{M,\mathsf{m}}\, B_{\mu\nu\,\mathsf{m}}\big]\,, \tag{3.69}$$

by introducing new tensor fields $B_{\mu\nu\,\mathsf{a}}$ and $B_{\mu\nu\,\mathsf{m}}$ with suitably chosen gauge transformation rules, so that covariant results are obtained. This implies that the variation of the tensor fields should in any case absorb the unwanted non-covariant terms in (3.68). At this point we recall that the invariance transformations in the ungauged case transform on the field strengths $G_{\mu\nu}{}^M$, defined in (3.4), according to a subgroup of $\mathrm{Sp}(2n+2,\mathbb{R})$ (cf. (3.5)). The field strengths $G_{\mu\nu}{}^M$ consist of the abelian field strengths $F_{\mu\nu}{}^\Lambda$ and the dual field strengths $G_{\mu\nu\Lambda}$. The latter were decomposed in (3.3) in the form $G^-_{\mu\nu\Lambda} = F_{\Lambda\Sigma}\, F^{-}_{\mu\nu}{}^\Sigma - 2\mathrm{i}\mathcal{O}^-_{\mu\nu\Lambda}$. Obviously, in the presence of the non-abelian gauge interactions, the abelian field strengths $F_{\mu\nu}{}^\Lambda$ should now be replaced by $\mathcal{H}_{\mu\nu}{}^\Lambda$, defined in (3.69). Hence it is natural to define new covariant field strengths according to,

$$\mathcal{G}_{\mu\nu}{}^{M} = \begin{pmatrix} \mathcal{H}_{\mu\nu}{}^{\Lambda} \\ \mathcal{G}_{\mu\nu\Lambda} \end{pmatrix} , \tag{3.70}$$

with,

$$\begin{aligned} \mathcal{G}^{-}_{\mu\nu}{}^{\Lambda} &= \mathcal{H}^{-}_{\mu\nu}{}^{\Lambda} , \\ \mathcal{G}^{-}_{\mu\nu\Lambda} &= F_{\Lambda\Sigma}\, \mathcal{H}^{-}_{\mu\nu}{}^{\Sigma} - 2\mathrm{i}\mathcal{O}^{-}_{\mu\nu\Lambda} . \end{aligned} \tag{3.71}$$

Just as in section 3.1, there exist corresponding supercovariant field strengths $\hat{\mathcal{G}}_{\mu\nu}{}^{M}$ that will appear in the supersymmetry transformations of the vector multiplet fermion fields. Those will be discussed in the next section. As before, the field strengths $\hat{\mathcal{G}}_{\mu\nu}{}^{M}$ and $\mathcal{G}_{\mu\nu}{}^{M}$ will only differ by fermionic bilinears and by terms proportional to the tensor field of the Weyl multiplet.

Following [9] we subsequently introduce the following transformation rule for $B_{\mu\nu\mathsf{a}}$ and $B_{\mu\nu\mathsf{m}}$ (contracted with $Z^{M,\mathsf{a}}$ and $Z^{M,\mathsf{m}}$, respectively, because only these combinations will appear in the Lagrangian),

$$\begin{aligned} Z^{M,\mathsf{a}}\,\delta B_{\mu\nu\,\mathsf{a}} &= 2\, Z^{M,\mathsf{a}}\big(\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{a}} + d_{\mathsf{a}\,NP} W_{[\mu}{}^{N}\delta W_{\nu]}{}^{P}\big) - 2\, T_{(NP)}{}^{M}\Lambda^{P}\mathcal{G}_{\mu\nu}{}^{N} , \\ Z^{M,\mathsf{m}}\,\delta B_{\mu\nu\,\mathsf{m}} &= 2\, Z^{M,\mathsf{m}}\, \mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{m}} . \end{aligned} \tag{3.72}$$

Note that $B_{\mu\nu\,\mathsf{a}}$ has variations proportional to $\Xi_{\mu\mathsf{m}}$ through the term $\delta W_{\mu}{}^{M}$ (cf. (3.63)). As a result of (3.72) the modified field strengths (3.69) are invariant under tensor gauge transformations. Under the vector gauge transformations we derive the following result,

$$\begin{aligned} \delta\mathcal{G}^{-}_{\mu\nu}{}^{\Lambda} &= -\, g\,\Lambda^{P}T_{PN}{}^{\Lambda}\,\mathcal{G}^{-}_{\mu\nu}{}^{N} - g\,\Lambda^{P}T^{\Gamma}{}_{P}{}^{\Lambda}\,(\mathcal{G}^{-}_{\mu\nu} - \mathcal{H}^{-}_{\mu\nu})_{\Gamma} , \\ \delta\mathcal{G}^{-}_{\mu\nu\Lambda} &= -\, g\,\Lambda^{P}T_{PN\Lambda}\,\mathcal{G}^{-}_{\mu\nu}{}^{N} - g\,F_{\Lambda\Sigma}\,\Lambda^{P}T^{\Gamma}{}_{P}{}^{\Sigma}\,(\mathcal{G}^{-}_{\mu\nu} - \mathcal{H}^{-}_{\mu\nu})_{\Gamma} , \\ \delta(\mathcal{G}^{-}_{\mu\nu} - \mathcal{H}^{-}_{\mu\nu})_{\Lambda} &= g\,\Lambda^{P}(T^{\Gamma}{}_{P\Lambda} - T^{\Gamma}{}_{P}{}^{\Sigma}\,F_{\Sigma\Lambda})\,(\mathcal{G}^{-}_{\mu\nu} - \mathcal{H}^{-}_{\mu\nu})_{\Gamma} . \end{aligned} \tag{3.73}$$

Hence $\delta\mathcal{G}_{\mu\nu}{}^{M} = -g\,\Lambda^{P}T_{PN}{}^{M}\,\mathcal{G}_{\mu\nu}{}^{N}$, just as the variation of the abelian field strengths $G_{\mu\nu}{}^{M}$ in the absence of charges, up to terms proportional to $\Theta^{\Lambda,\mathsf{a}}(\mathcal{G}_{\mu\nu}-\mathcal{H}_{\mu\nu})_{\Lambda}$. According to [9], the latter terms represent a set of field equations, as we will verify later (cf. (3.98)), and so the last equation of (3.73) expresses the well-known fact that, under a symmetry, field equations transform into field equations. As a result the gauge algebra on the tensors $\mathcal{G}_{\mu\nu}{}^{M}$ closes according to (3.64), up to the same field equations.

In order that the Lagrangians for the vector multiplets (2.50) and the hypermultiplets (2.56) become invariant under vector and tensor gauge transformations, we have to make a number of changes. First of all, we replace the covariant derivatives on the scalars and spinors by gauge-covariant derivatives. This ensures the invariance of $\mathcal{L}^{(1)}_{\text{kin}}$, $\mathcal{L}_{\text{conf}}$ and $\mathcal{L}_{\text{hyper}}$, given in (2.51), (2.55) and (2.56), respectively. The Lagrangian for the auxiliary fields (2.54) is already gauge-invariant. In the following we therefore concentrate on $\mathcal{L}^{(2)}_{\text{kin}}$ (2.52) which depends on the abelian field strengths $F_{\mu\nu}{}^{\Lambda}$. These abelian field-strengths are now replaced by $\mathcal{H}_{\mu\nu}{}^{\Lambda}$, so that,

$$\mathcal{G}_{\mu\nu\,\Lambda} = \mathrm{i}e\,\varepsilon_{\mu\nu\rho\sigma}\,\frac{\partial\mathcal{L}_{\text{vector}}}{\partial\mathcal{H}_{\rho\sigma}{}^{\Lambda}}\;. \tag{3.74}$$

The Lagrangian $\mathcal{L}^{(2)}_{\text{kin}}$ therefore reads,

$$\begin{aligned} e^{-1}\mathcal{L}^{(2)}_{\text{kin}} = {}& \tfrac{1}{4}\mathrm{i}\left[F_{\Lambda\Sigma}\,\mathcal{H}^{-\Lambda}_{\mu\nu}\mathcal{H}^{-\Sigma\,\mu\nu} - \bar{F}_{\Lambda\Sigma}\,\mathcal{H}^{+\Lambda}_{\mu\nu}\mathcal{H}^{+\mu\nu\Sigma}\right] \\ & + \left[\mathcal{O}^{-}_{\mu\nu\Lambda}\mathcal{H}^{-\mu\nu\Lambda} - N^{\Lambda\Sigma}\mathcal{O}^{-}_{\mu\nu\Lambda}\mathcal{O}^{-\mu\nu}{}_{\Sigma} + \text{h.c.}\right]. \end{aligned} \tag{3.75}$$

It is separately invariant under the tensor gauge transformations, because the tensors $\mathcal{H}_{\mu\nu}{}^{\Lambda}$ are invariant under those transformations.

However, the Lagrangian (2.50) is not invariant under the vector gauge transformations. To establish this, one has to take into account that also the other fields of the vector multiplets transform under the gauge group. For instance, there are contributions from infinitesimal gauge transformations of $F_{\Lambda\Sigma}$ and $\mathcal{O}_{\mu\nu\Lambda}$, which follow from (3.25) and (3.14),

$$\begin{aligned} \delta F_{\Lambda\Sigma} &= g\Lambda^{M}\big(-T_{M\Lambda\Sigma} + 2\,T_{M(\Lambda}{}^{\Gamma}F_{\Sigma)\Gamma} + F_{\Lambda\Gamma}T_{M}{}^{\Gamma\Xi}F_{\Xi\Sigma}\big)\,, \\ \delta\mathcal{O}^{-}_{\mu\nu\Lambda} &= g\Lambda^{M}\,\mathcal{O}^{-}_{\mu\nu\Sigma}\big(T_{M\Lambda}{}^{\Sigma} + T_{M}{}^{\Sigma\Gamma}F_{\Gamma\Lambda}\big)\,. \end{aligned} \tag{3.76}$$

Nevertheless, it was shown in [9] that this is still not sufficient for gauge invariance, and it is necessary to introduce an additional, universal, term to the Lagrangian, equal to,

$$\begin{aligned} \mathcal{L}_{\text{top}} = {}& \tfrac{1}{8}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}\,\big(\Theta^{\Lambda\mathsf{a}}\,B_{\mu\nu\,\mathsf{a}} + \Theta^{\Lambda\mathsf{m}}\,B_{\mu\nu\,\mathsf{m}}\big) \\ & \qquad\times\big(2\,\partial_\rho W_{\sigma\,\Lambda} + gT_{MN\,\Lambda}\,W_\rho{}^{M}W_\sigma{}^{N} - \tfrac{1}{4}g\Theta_{\Lambda}{}^{\mathsf{b}}B_{\rho\sigma\,\mathsf{b}} - \tfrac{1}{4}g\Theta_{\Lambda}{}^{\mathsf{n}}B_{\rho\sigma\,\mathsf{n}}\big) \\ & + \tfrac{1}{3}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}T_{MN\,\Lambda}\,W_\mu{}^{M}W_\nu{}^{N}\big(\partial_\rho W_\sigma{}^{\Lambda} + \tfrac{1}{4}gT_{PQ}{}^{\Lambda}W_\rho{}^{P}W_\sigma{}^{Q}\big) \\ & + \tfrac{1}{6}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}T_{MN}{}^{\Lambda}\,W_\mu{}^{M}W_\nu{}^{N}\big(\partial_\rho W_{\sigma\Lambda} + \tfrac{1}{4}gT_{PQ\Lambda}W_\rho{}^{P}W_\sigma{}^{Q}\big)\,. \end{aligned} \tag{3.77}$$

The first term represents a topological coupling of the anti-symmetric tensor fields with the magnetic gauge fields; the last two terms are a generalization of the Chern-Simons-like terms that were first found in [28].

Under arbitrary variations of the vector and tensor fields, (3.75) and (3.77) yield (up to total derivative terms),

$$\begin{aligned} e^{-1}\left(\delta\mathcal{L}_{\rm kin}^{(2)}+\delta\mathcal{L}_{\rm top}\right) = &-\tfrac{1}{4}{\rm i}g\,\left(\mathcal{G}^{+\mu\nu M}-\mathcal{H}^{+\mu\nu M}\right)\,\Theta_M{}^{\mathsf a}(\delta B_{\mu\nu\mathsf a}-2d_{\mathsf a PQ}W_\mu{}^P\delta W_\nu{}^Q) \\ &-\tfrac{1}{4}{\rm i}g\,\left(\mathcal{G}^{+\mu\nu M}-\mathcal{H}^{+\mu\nu M}\right)\,\Theta_M{}^{\mathsf m}\,\delta B_{\mu\nu\mathsf m} \\ &+{\rm i}\mathcal{G}^{+\mu\nu M}\Omega_{MN}\,\mathcal{D}_\mu\delta W_\nu{}^N+\text{h.c.}\,. \end{aligned} \tag{3.78}$$

Under the tensor gauge transformations this variation becomes equal to,

$$e^{-1}\big(\delta\mathcal{L}_{\rm kin}^{(2)}+\delta\mathcal{L}_{\rm top}\big)={\rm i}g\mathcal{H}^{+\mu\nu M}\big[\Theta_M{}^{\mathsf a}\mathcal{D}_\mu\Xi_{\nu\mathsf a}+\Theta_M{}^{\mathsf m}\mathcal{D}_\mu\Xi_{\nu\mathsf m}\big]+\text{h.c.}\,. \tag{3.79}$$

We already demonstrated that $\mathcal{L}_{\rm kin}^{(2)}$ is separately invariant under these transformations, so that the above terms originate exclusively from the variation of $\mathcal{L}_{\rm top}$. The expression (3.79) turns out to be equal to a total derivative. To see this, note that the embedding tensor is gauge invariant. Also there exists a Bianchi identity,

$$\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho]}{}^M=\tfrac{1}{3}g\big[Z^{M,\mathsf a}\,\mathcal{H}_{\mu\nu\rho\,\mathsf a}+Z^{M,\mathsf m}\,\mathcal{H}_{\mu\nu\rho\,\mathsf m}\big]\,. \tag{3.80}$$

Here the gauge-covariant field strengths of the tensor fields are defined as,

$$\begin{aligned} \mathcal{H}_{\mu\nu\rho\,\mathsf a} &= 3\,\mathcal{D}_{[\mu}B_{\nu\rho]\,\mathsf a}+6\,d_{\mathsf a\,NP}\,W_{[\mu}{}^N\left(\partial_\nu W_{\rho]}{}^P+\tfrac{1}{3}gT_{[RS]}{}^PW_\nu{}^RW_{\rho]}{}^S+(\mathcal{G}-\mathcal{H})_{\nu\rho]}{}^P\right)\,, \\ \mathcal{H}_{\mu\nu\rho\,\mathsf m} &= 3\,\mathcal{D}_{[\mu}B_{\nu\rho]\,\mathsf m}\,, \end{aligned} \tag{3.81}$$

where $\mathcal{D}_\mu B_{\nu\rho\mathsf a}=\partial_\mu B_{\nu\rho\mathsf a}-gW_\mu{}^MT_{M\mathsf a}{}^{\mathsf b}B_{\nu\rho\mathsf b}$, and likewise for $\mathcal{D}_\mu B_{\nu\rho\mathsf m}$. The fully gauge-covariant derivative of $\mathcal{H}_{\mu\nu}{}^M$ takes the form,

$$\begin{aligned} \mathcal{D}_\rho\mathcal{H}_{\mu\nu}{}^M &= \partial_\rho\mathcal{H}_{\mu\nu}{}^M+gW_\rho{}^P\,T_{PN}{}^M\,\mathcal{G}_{\mu\nu}{}^N+gW_\rho{}^P\,T_{NP}{}^M\,(\mathcal{G}-\mathcal{H})_{\mu\nu}{}^N \\ &= \partial_\rho\mathcal{H}_{\mu\nu}{}^M+gW_\rho{}^P\,T_{PN}{}^M\,\mathcal{H}_{\mu\nu}{}^N+2\,gW_\rho{}^P\,Z^{M,\mathsf a}d_{\mathsf a PN}\,(\mathcal{G}-\mathcal{H})_{\mu\nu}{}^N\,, \end{aligned} \tag{3.82}$$

Observe that the covariantization proportional to $(\mathcal{G}-\mathcal{H})_{\mu\nu}{}^N$ is not generated by partially integrating the right-hand side of (3.79), but it vanishes upon contraction with the embedding tensor. So does the right-hand side of (3.80), so that (3.79) is indeed a total derivative.

As was mentioned before, the combined gauge invariance of the vector and tensor gauge fields are important to ensure that the number of physical degrees of freedom will not change by the introduction of the magnetic vector gauge fields and the tensor gauge fields [9]. The combined gauge algebra is consistent for the tensor fields upon projection with the embedding tensor, which is sufficient as the action depends only on these projected fields. If this were not the case, new tensor fields of higher rank would have been required [67]. The projection with the embedding tensor will determine in which fields the physical degrees of freedom can reside. The precise way in which the number of physical degrees is described of freedom is therefore rather subtle. From (3.78) it is indeed clear that the components of the tensor fields that are projected to zero by multiplication with $\Theta^{\Lambda\mathsf{a}}$ or $\Theta^{\Lambda\mathsf{m}}$, are simply not present in the action. Their absence can be regarded as the result of an additional gauge invariance. In addition, there are transformations of the tensor fields linear in $(\mathcal{G}-\mathcal{H})_{\mu\nu\Lambda}$ that leave the Lagrangian invariant [47, 56],

$$\begin{aligned}\Theta^{\Lambda\mathsf{a}}\delta B_{\mu\nu\mathsf{a}} &= \Delta_1^{[\Lambda\Sigma]}\,(\mathcal{G}-\mathcal{H})^+_{\mu\nu\Sigma} + \text{h.c.}\,,\\ \Theta^{\Lambda\mathsf{a}}\delta B_{\mu\nu\mathsf{a}} &= \Delta_2^{(\Lambda\Sigma)\rho}{}_{[\mu}\,(\mathcal{G}-\mathcal{H})_{\nu]\rho\Sigma}\,,\end{aligned} \qquad (3.83)$$

where $\Delta_1^{\Lambda\Sigma}$ is an arbitrary complex parameter, and $\Delta_2^{\Lambda\Sigma\rho}{}_{\mu}$ is real and traceless. Similar transformations exists for variations contracted with $\Theta^{\Lambda\mathsf{m}}$. Often these transformations emerge when verifying the validity of the supersymmetry algebra.

A similar situation arises with the magnetic gauge fields $W_{\mu\Lambda}$. Under variations of the gauge fields $W_\mu{}^M$ one derives,

$$\delta\mathcal{L}^{(2)}_{\text{kin}} + \delta\mathcal{L}_{\text{top}} = \tfrac{1}{2}\mathrm{i}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{D}_\nu\mathcal{G}_{\rho\sigma}{}^M\Omega_{MN}\delta W_\mu{}^N\,, \qquad (3.84)$$

up to a total derivative and up to terms that vanish as a result of the field equation for $B_{\mu\nu\,\mathsf{a}}$. Substituting (3.80) we can rewrite (3.84) as follows,

$$\begin{aligned}\delta\mathcal{L}^{(2)}_{\text{kin}} + \delta\mathcal{L}_{\text{top}} = \tfrac{1}{2}\mathrm{i}\,\varepsilon^{\mu\nu\rho\sigma}\big[&-\mathcal{D}_\nu\mathcal{G}_{\rho\sigma\Lambda}\,\delta W_\mu{}^\Lambda\\ &+\tfrac{1}{6}g\big(\mathcal{H}_{\nu\rho\sigma\mathsf{a}}\,\Theta^{\Lambda\mathsf{a}} + \mathcal{H}_{\nu\rho\sigma\mathsf{m}}\,\Theta^{\Lambda\mathsf{m}}\big)\delta W_{\mu\Lambda}\big]\,.\end{aligned} \qquad (3.85)$$

Because the minimal coupling of the gauge fields to matter fields is always proportional to the embedding tensor, the full Lagrangian does not change under variations of the magnetic gauge fields that are projected to zero by the embedding tensor components $\Theta^{\Lambda\mathsf{a}}$ or $\Theta^{\Lambda\mathsf{m}}$, up to terms that are generated by the variations of the tensor fields through the 'universal' variation, $\delta B_{\mu\nu\mathsf{a}} = 2\, d_{\mathsf{a}PQ} W_{[\mu}{}^P \delta W_{\nu]}{}^Q$.

All these gauge symmetries have a role to play in balancing the degrees of freedom. Observe that not all these symmetries have a bearing on the dynamical modes of the theory as they also act on fields that only play an auxiliary role.

3.5 The superconformal algebra and the Lagrangian with general gaugings

When switching on a gauging there are several qualitative changes that are of interest. First of all, the superconformal algebra will no longer be realized off-shell (i.e. without using the equations of motion) in the vector multiplet sector, at least for gaugings with magnetic charges. Only for the Weyl multiplet the closure remains realized off-shell. Naturally a generic gauging induces the presence of vector multiplet fields into the hypermultiplet supersymmetry transformations. It is therefore not surprising that also the vector multiplet transformations will generically acquire terms proportional to the hypermultiplet fields. In this section we will present the full transformation rules that include new terms of order g, and subsequently we will re-establish the closure for general gaugings. As it turns out, additional symmetries such as (3.83), are relevant for the closure. This feature is well known from previous applications of the embedding tensor formalism.

A second, not unrelated, feature is that the Lagrangian must be modified by including masslike terms for the fermions proportional to g, and a scalar potential proportional to g^2. The explicit expressions for these terms, which are relevant for many applications, will be presented at the end of this section. These modifications are familiar from $N=2$ supergravity theories with purely electric charges [27, 28, 42].

Rigid $N=2$ supersymmetric theories with both electric and magnetic charges, have been presented in [47], and it remains to complete these results in a fully superconformal setting. It is clear that the modification of the results derived in [47] must be relatively minor. The supersymmetry transformations of the matter fields will now become covariant with respect to the superconformal symmetries, while at the same time they should remain in accord with the known results for rigid theories. Modifications that supersede previous work will therefore mainly involve terms proportional to the gravitino fields. The most conspicuous ones are those appearing in the supersymmetry transformations of the tensor fields $B_{\mu\nu\mathsf{a}}$ and $B_{\mu\nu\mathsf{m}}$.

To exhibit this in more detail, let us first present the full Q- and S-supersymmetry transformations for the hypermultiplet fields. They follow straightforwardly upon super-covariantizing the rules presented in section 2.5, including the terms of order g that were already found in [47],

$$\begin{aligned}
\delta\phi^A &= 2\,(\gamma^A_{i\bar\alpha}\,\bar\epsilon^i\zeta^{\bar\alpha} + \bar\gamma^{Ai}_{\alpha}\,\bar\epsilon_i\zeta^\alpha)\,,\\
\delta A_i{}^\alpha + \delta\phi\Gamma_A{}^\alpha{}_\beta A_i{}^\beta &= 2\,\bar\epsilon_i\zeta^\alpha + 2\,\varepsilon_{ij}G^{\alpha\bar\beta}\Omega_{\bar\beta\bar\gamma}\,\bar\epsilon^j\zeta^{\bar\gamma}\,,\\
\delta\zeta^\alpha + \delta\phi^A\,\Gamma_A{}^\alpha{}_\beta\,\zeta^\beta &= \not{D}A_i{}^\alpha\,\epsilon^i + 2gX^M\,T_M{}^\alpha{}_\beta A_i{}^\beta\,\varepsilon^{ij}\epsilon_j + A_i{}^\alpha\,\eta^i\,.
\end{aligned} \tag{3.86}$$

where D_μ denotes the derivative fully covariantized with respect to all the superconformal transformations and the gauge symmetries. Likewise we present the full Q- and S-supersymmetry transformations for the vector multiplet fields,

$$\begin{aligned}
\delta X^M &= \bar\epsilon^i\Omega_i{}^M\,,\\
\delta\Omega_i{}^M &= 2\not{D}X^M\epsilon_i + \hat Z_{ij}{}^M\epsilon^j + \tfrac12\gamma^{\mu\nu}\hat{\mathcal{G}}^{-}_{\mu\nu}{}^M\varepsilon_{ij}\epsilon^j\\
&\quad - 2g\,T_{PN}{}^M\bar X^P X^N\varepsilon_{ij}\epsilon^j + 2\,\mathrm{i}g\Omega^{MN}\mu_{ijN}\epsilon^j + 2X^M\eta_i\,,\\
\delta W_\mu{}^M &= \varepsilon^{ij}\bar\epsilon_i(\gamma_\mu\Omega_j{}^M + 2\,\psi_{\mu j}X^M) + \varepsilon_{ij}\bar\epsilon^i(\gamma_\mu\Omega^{j\,M} + 2\,\psi_\mu{}^j\bar X^M)\,,\\
\delta Y_{ij}{}^\Lambda &= 2\,\bar\epsilon_{(i}\not{D}\Omega_{j)}{}^\Lambda + 2\,\varepsilon_{ik}\varepsilon_{jl}\bar\epsilon^{(k}\not{D}\Omega^{l)\Lambda}\\
&\quad - 4g\,T_{MN}{}^\Lambda\big[\bar\Omega_{(i}{}^M\epsilon^k\varepsilon_{j)k}\,\bar X^N - \bar\Omega^{kM}\epsilon_{(i}\varepsilon_{j)k}\,X^N\big]\\
&\quad + 4\,\mathrm{i}g\,k^{A\Lambda}\big[\varepsilon_{k(i}\,\gamma_{j)\bar\alpha A}\bar\epsilon^k\zeta^{\bar\alpha} + \varepsilon_{k(i}\,\bar\epsilon_{j)}\zeta^\alpha\,\bar\gamma^k_{\alpha A}\big]\,.
\end{aligned} \tag{3.87}$$

Here the moment maps are defined by,

$$\mu_{ijM} = \Theta_M{}^{\mathsf{m}}\mu_{ij\mathsf{m}}\,, \tag{3.88}$$

with $\mu_{ij\mathsf{m}}$ defined in (3.32). The symplectic vector $\hat Z_{ij}{}^M$ appearing in $\delta\Omega_i{}^M$ is given by,

$$\hat Z_{ij}{}^M = \begin{pmatrix} Y_{ij}{}^\Lambda \\ F_{\Lambda\Sigma}\,Y_{ij}{}^\Sigma - \tfrac12 F_{\Lambda\Sigma\Gamma}\,\bar\Omega_i{}^\Sigma\Omega_j{}^\Gamma + 2\,\mathrm{i}g[\mu_{ij\Lambda} + F_{\Lambda\Sigma}\,\mu_{ij}{}^\Sigma]\end{pmatrix}\,. \tag{3.89}$$

This expression differs from the previous one for the ungauged theory, given in (3.17), by the presence of the moment maps originating from the hypermultiplet sector. This implies that the original pseudo-reality condition on $Z_{ij\Lambda}$ must be replaced by a pseudo-reality condition on $\hat Z_{ij\Lambda}$. As this condition was previously imposed by invoking the field equations for the auxiliary fields, it follows that those field equations must now receive modifications proportional to the moment maps, as we shall confirm later in this section. Note that, in (3.87), we refrained from giving the supersymmetry transformation of $\hat{Z}_{ij\Lambda}$, which is not an independent field.

Another tensor appearing in $\delta\Omega_i{}^M$ is the supercovariant field strength $\hat{\mathcal{G}}_{\mu\nu}{}^M$, which is the non-abelian version of (3.18). These supercovariant field strengths are defined by,

$$\begin{aligned}\hat{\mathcal{G}}^{-}_{\mu\nu}{}^{\Lambda} &= \hat{\mathcal{H}}^{-}_{\mu\nu}{}^{\Lambda}\,,\\ \hat{\mathcal{G}}^{-}_{\mu\nu\Lambda} &= F_{\Lambda\Sigma}\,\hat{\mathcal{H}}^{-}_{\mu\nu}{}^{\Sigma} - \tfrac{1}{8}F_{\Lambda\Sigma\Gamma}\,\bar{\Omega}_i{}^{\Sigma}\gamma_{\mu\nu}\Omega_j{}^{\Gamma}\,\varepsilon^{ij}\,. \end{aligned}\tag{3.90}$$

where $\hat{\mathcal{H}}_{\mu\nu}{}^{\Lambda}$ is the supercovariant extension of (3.69). In view of (2.21), we expect the following decomposition for $\hat{\mathcal{H}}_{\mu\nu}{}^{\Lambda}$,

$$\begin{aligned}\hat{\mathcal{H}}_{\mu\nu}{}^{\Lambda} ={}& \mathcal{H}_{\mu\nu}{}^{\Lambda} - \varepsilon^{ij}\bar{\psi}_{[\mu\,i}(\gamma_{\nu]}\Omega_j{}^{\Lambda} + \psi_{\nu]j}X^{\Lambda}) - \varepsilon_{ij}\bar{\psi}_{[\mu}{}^{i}(\gamma_{\nu]}\Omega^{j\,\Lambda} + \psi_{\nu]}{}^{j}\bar{X}^{\Lambda})\\ &- \tfrac{1}{4}(X^{\Lambda}\,T_{\mu\nu ij}\,\varepsilon^{ij} + \bar{X}^{\Lambda}\,T_{\mu\nu}{}^{ij}\,\varepsilon_{ij})\,. \end{aligned}\tag{3.91}$$

However, in the presence of a gauging, the supersymmetry variation of this expression leads to terms proportional to the gravitini fields induced by the terms in $\delta\Omega_i{}^{\Lambda}$ of order g. As it turns out, by suitably adjusting the supersymmetry transformations of the tensor fields, $\delta B_{\mu\nu\mathsf{a}}$ and $\delta B_{\mu\nu\mathsf{m}}$, one can ensure that the $\hat{\mathcal{H}}_{ab}{}^{\Lambda}$ will still transform covariantly under Q- and S-supersymmetry,

$$\begin{aligned}\delta\hat{\mathcal{H}}_{ab}{}^{\Lambda} ={}& -2\,\varepsilon_{ij}\,\bar{\epsilon}^{i}\gamma_{[a}D_{b]}\Omega^{j\Lambda} - 2g\,T_{(NP)}{}^{\Lambda}\bar{X}^{N}\,\bar{\Omega}_i{}^{P}\gamma_{ab}\epsilon^{i}\\ &- 2\mathrm{i}g\,k^{A\Lambda}\,\gamma_{Ai\bar{\alpha}}\,\bar{\zeta}^{\bar{\alpha}}\gamma_{ab}\epsilon^{i} - \varepsilon^{ij}\bar{\eta}_i\gamma_{ab}\Omega_j{}^{\Lambda} + \text{h.c.}\,. \end{aligned}\tag{3.92}$$

As a result the combined transformations of the tensor fields $B_{\mu\nu\mathsf{a}}$ and $B_{\mu\nu\mathsf{m}}$ under tensor and vector gauge transformations and Q- and S-supersymmetry now read as follows,

$$\begin{aligned}Z^{M,\mathsf{a}}\,\delta B_{\mu\nu\,\mathsf{a}} ={}& 2\,Z^{M,\mathsf{a}}\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{a}} + 2\,T_{(NP)}{}^{M}\big[W_{[\mu}{}^{N}\delta W_{\nu]}{}^{P} - \Lambda^{N}\mathcal{G}_{\mu\nu}{}^{P}\big]\\ &- 2\,T_{(NP)}{}^{M}\big[\bar{X}^{N}\bar{\Omega}_i{}^{P}\gamma_{\mu\nu}\epsilon^{i} + X^{N}\bar{\Omega}^{iP}\gamma_{\mu\nu}\epsilon_i\\ &+ 2\,\bar{X}^{N}X^{P}\big(\bar{\epsilon}^{i}\gamma_{[\mu}\psi_{\nu]i} + \bar{\epsilon}_i\gamma_{[\mu}\psi_{\nu]}{}^{i}\big)\big]\,,\\ Z^{M,\mathsf{m}}\,\delta B_{\mu\nu\,\mathsf{m}} ={}& 2\,Z^{M,\mathsf{m}}\,\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{m}} - 2\mathrm{i}\Omega^{MN}k^{A}{}_{N}\,\big[\gamma_{Ai\bar{\alpha}}\,\bar{\zeta}^{\bar{\alpha}}\gamma_{\mu\nu}\epsilon^{i} - \bar{\gamma}^{i}_{A\alpha}\,\bar{\zeta}^{\alpha}\gamma_{\mu\nu}\epsilon_i\big]\\ &+ 4\,\mathrm{i}\,\Omega^{MN}\mu_{jkN}\,\varepsilon^{ij}\,\big[\bar{\psi}_{i[\mu}\gamma_{\nu]}\epsilon^{k} + \bar{\psi}^{k}{}_{[\mu}\gamma_{\nu]}\epsilon_i\big]\,. \end{aligned}\tag{3.93}$$

Note that the tensors transform covariantly under diffeomorphisms, and are scale invariant. As was already alluded to, the moment maps μ_{ijM} enter the transformation rules of the vector multiplet fields. In fact, only the magnetic moment maps $\mu_{ij}{}^{\Lambda}$ appear in these transformation rules.[10] For purely electric charges and corresponding moment maps $\mu_{ij\Lambda}$, the supersymmetry transformations (3.86) and (3.87) reduce to the transformations presented in [28] and [42]. The latter transformations still realize the supersymmetry algebra for the vector multiplet fields (but not for the hypermultiplet fields) without the need for imposing equations of motion.

Now that the full supersymmetry transformations have been established, we consider the superconformal algebra. Its most non-trivial commutation relation is the one of two Q-supersymmetries. This commutation relation, which was already specified in (2.15), must now be extended with tensor gauge transformations. Hence,

$$\begin{aligned}[\delta(\epsilon_1),\delta(\epsilon_2)] = {}& \delta^{\rm cov}(\xi) + \delta_M(\varepsilon) + \delta_K(\Lambda_K) + \delta_S(\eta) + \delta_{\rm gauge}(\Lambda^M) \\ & + \delta_{\rm tensor}(\Xi_{\mu\,\mathsf{a}}) + \delta_{\rm tensor}(\Xi_{\mu\,\mathsf{m}})\,, \end{aligned} \tag{3.94}$$

and it should hold modulo field equations and some of the spurious symmetries that we discussed in the previous section. The various parameters in (3.94) have already been specified in (2.16) and (3.22), except for the parameters of the tensor gauge transformations, which read,

$$\begin{aligned} \Xi_{\mu\,\mathsf{a}} &= -\,2\,d_{\mathsf{a}\,NP}\bar X^N X^P \xi_\mu\,, \\ \Xi_{\mu\mathsf{m}} &= -\,8\,\mathrm{i}\,\varepsilon^{ij}\mu_{jk\mathsf{m}}\big(\bar\epsilon_{2i}\gamma_\mu\epsilon_1{}^k + \bar\epsilon_2{}^k\gamma_\mu\epsilon_{1i}\big)\,, \end{aligned} \tag{3.95}$$

up to terms that vanish upon contraction with the embedding tensor.[11] As before, $\delta^{\rm cov}(\xi)$ denotes an infinitesimal covariant general coordinate transformation, which now includes contributions from the various gauge transformations such that the combined result takes a supercovariant form. For the vector gauge transformations the parameters take the form $\Lambda^M = -\xi^\rho W_\rho{}^M$. For the corresponding field-dependent tensor gauge transformations, the parameters take a slightly more complicated form [56],

$$\begin{aligned} \Xi_{\mu\,\mathsf{a}} &= -\,\xi^\rho\left(B_{\rho\mu\,\mathsf{a}} + d_{\mathsf{a}\,NP}W_\rho{}^N W_\mu{}^P\right)\,, \\ \Xi_{\mu\,\mathsf{m}} &= -\,\xi^\rho B_{\rho\mu\,\mathsf{m}}\,. \end{aligned} \tag{3.96}$$

[10] The reader may verify that the contribution to $\Omega_i{}^M$ proportional to $\mu_{ij\Lambda}$ vanishes against a similar contribution contained in $\hat Z_{ij}{}^M$.

[11] The result for $\Xi_{\mu\mathsf{m}}$ given in (3.95) is new compared to previous work. It is determined by verifying the commutator (3.94) on the vector and tensor gauge fields, as will be discussed in some detail below.

In what follows we will verify the validity of (3.94) on the auxiliary fields $Y_{ij}{}^{\Lambda}$, $W_\mu{}^M$ and the tensor fields $B_{\mu\nu\mathsf{a}}$ and $B_{\mu\nu\mathsf{m}}$, as these are most susceptible to the presence of the new gauge transformations, thereby exhibiting a variety of subtleties that play a role. Many aspects of this evaluation have their counterpart in a similar evaluation of $N=8$ supergravity, which appeared in [56]. At this point we mention two general identities that are relevant in the present calculations. They follow from (3.46), (3.47) and (3.48),

$$\begin{aligned} T_{(MN)}{}^P X^M \hat{Z}_{ij}{}^N &= \tfrac{1}{2}\, T_{(MN)}{}^P\, \bar{\Omega}_i{}^M \Omega_j{}^N - 2\mathrm{i}g T_{(MN)}{}^P X^M \Omega^{NQ} \mu_{ijQ}\,, \\ T_{(MN)}{}^P X^M \hat{\mathcal{G}}^{-}_{\mu\nu}{}^N &= \tfrac{1}{8}\, T_{(MN)}{}^P\, \varepsilon^{ij}\, \bar{\Omega}_i{}^M \gamma_{\mu\nu} \Omega_j{}^N\,. \end{aligned} \tag{3.97}$$

Of course, in the calculations we must also take into account that the superconformal gauge fields, $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and $\phi_\mu{}^i$, depend on the other superconformal fields, as given in (B.5).

Let us first consider the supersymmetry commutator (3.94) on the auxiliary fields $Y_{ij}{}^{\Lambda}$. As it turns out, its validity requires to impose the field equations associated with the tensor fields, which take the following form,

$$\Theta^{\Lambda\mathsf{a}}\, \mathcal{G}_{\mu\nu\Lambda} = \Theta^{\Lambda\mathsf{a}}\, \mathcal{H}_{\mu\nu\Lambda}\,, \qquad \Theta^{\Lambda\mathsf{m}}\, \mathcal{G}_{\mu\nu\Lambda} = \Theta^{\Lambda\mathsf{m}}\, \mathcal{H}_{\mu\nu\Lambda}\,, \tag{3.98}$$

and the field equations associated with the magnetic gauge fields,

$$\begin{aligned} 0 = {} & \tfrac{1}{6} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \left(Z^{\Lambda,\mathsf{a}} \mathcal{H}_{\nu\rho\sigma\,\mathsf{a}} + Z^{\Lambda,\mathsf{m}} \mathcal{H}_{\nu\rho\sigma\,\mathsf{m}} \right) \\ & + T_{(MN)}{}^{\Lambda} \big(-2\, \bar{X}^M \overset{\leftrightarrow}{\mathcal{D}}{}^\mu X^N + \bar{\Omega}^{iM} \gamma^\mu \Omega_i{}^N \\ & + \bar{X}^M \bar{\psi}_\nu{}^i \gamma^\mu \gamma^\nu \Omega_i{}^N - X^M \bar{\psi}_{\nu i} \gamma^\mu \gamma^\nu \Omega^{iN} - \tfrac{1}{2} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_{\nu i} \gamma_\rho \psi_\sigma{}^i\, \bar{X}^M X^N \big) \\ & + \mathrm{i} G_{\bar{\alpha}\beta} T^{\Lambda\beta}{}_\gamma \big(\tfrac{1}{2} A^{i\bar{\alpha}} \overset{\leftrightarrow}{\mathcal{D}}{}^\mu A_i{}^\gamma - 2\bar{\zeta}^{\bar{\alpha}} \gamma^\mu \zeta^\gamma + \bar{\psi}_\nu{}^i \gamma^\mu \gamma^\nu \zeta^{\bar{\alpha}} A_i{}^\gamma - \bar{\psi}_{\nu i} \gamma^\mu \gamma^\nu \zeta^\gamma A^{i\bar{\alpha}} \big) \\ & - \mathrm{i} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\nu{}^i \gamma_\rho \psi_{\sigma j} \varepsilon^{jk} \mu_{ik}{}^{\Lambda}\,, \end{aligned} \tag{3.99}$$

where we made use of the Bianchi identity (3.80).

Secondly we evaluate the supersymmetry commutator on the vector fields $W_\mu{}^M$,

$$\begin{aligned} [\delta(\epsilon_1), \delta(\epsilon_2)] W_\mu{}^M = {} & \xi^\rho \mathcal{G}_{\rho\mu}{}^M + \mathcal{D}_\mu \Lambda^M - g\, Z^{M,\mathsf{a}}\, \Xi_{\mu\,\mathsf{a}} - g\, Z^{M,\mathsf{m}}\, \Xi_{\mu\,\mathsf{m}} \\ & - \xi^\rho \left(\tfrac{1}{2}\, \varepsilon_{ij}\, \bar{\psi}_\rho{}^i \gamma_\mu \Omega^{jM} + \varepsilon_{ij} \bar{X}^M \bar{\psi}_\rho{}^i \psi_\mu{}^j + \text{h.c.} \right)\,, \end{aligned} \tag{3.100}$$

where the parameters ξ^μ, Λ^M, $\Xi_{\mu\,\mathsf{a}}$ and $\Xi_{\mu\,\mathsf{m}}$ are as in (3.94). In this result one can replace $\mathcal{G}_{\mu\nu}{}^M$ by $\mathcal{H}_{\mu\nu}{}^M$. For the electric gauge fields this is trivial as $\mathcal{G}_{\mu\nu}{}^{\Lambda}$ and $\mathcal{H}_{\mu\nu}{}^{\Lambda}$ are identical. For the magnetic gauge fields the replacement is effectively allowed because $W_{\mu\Lambda}$ appear in the Lagrangian contracted with the embedding tensor, as can be seen from (3.85). Therefore, without loss of generality, one can safely contract (3.100) for the magnetic gauge fields with the embedding tensors, $\Theta^{\Lambda\mathsf{a}}$ or $\Theta^{\Lambda\mathsf{m}}$, upon which one can replace $\mathcal{G}_{\mu\nu\Lambda}$ with $\mathcal{H}_{\mu\nu\Lambda}$ by virtue of (3.98). Finally one uses the following equality,

$$\begin{aligned}\xi^\rho\mathcal{H}_{\rho\mu}{}^M =&\,\xi^\rho\partial_\rho W_\mu{}^M + \partial_\mu\xi^\rho W_\rho{}^M - \mathcal{D}_\mu\left(\xi^\rho W_\rho{}^M\right)\\ &+ gZ^{M,\mathsf{a}}\xi^\rho\left(B_{\rho\mu\,\mathsf{a}} + d_{\mathsf{a}\,NP}W_\rho{}^N W_\mu{}^P\right) + gZ^{M,\mathsf{m}}\xi^\rho B_{\rho\mu\,\mathsf{m}}\,.\end{aligned} \tag{3.101}$$

Substituting this identity into (3.100) shows that the ξ^μ-dependent terms decompose into a general coordinate transformation with parameter ξ^μ, a non-abelian gauge transformation with parameter $-\xi^\mu W_\mu{}^M$, tensor gauge transformations with parameters $-\xi^\rho\big(B_{\rho\mu\,\mathsf{a}} + d_{\mathsf{a}\,NP}W_\rho^N W_\mu^P\big)$ and $-\xi^\rho B_{\rho\mu\,\mathsf{m}}$ and a supersymmetry transformation with parameter $-\frac{1}{2}\xi^\mu\psi_{\mu i}$. Together they constitute a covariant general coordinate transformation with parameter ξ^μ. Consequently the supersymmetry commutator closes according to (3.94).

Subsequently we turn to the supersymmetry commutator on the tensor fields $B_{\mu\nu\,\mathsf{a}}$. Here it suffices to consider those fields contracted with $Z^{\Lambda,\mathsf{a}}$ because no other components of the tensor field appear in the Lagrangian according to (3.78). Hence, we first evaluate,

$$\begin{aligned}Z^{\Lambda,\mathsf{a}}\,[\delta(\epsilon_1),\delta(\epsilon_2)]B_{\mu\nu\,\mathsf{a}} =&\,2\,Z^{\Lambda,\mathsf{a}}\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{a}} - 2\,T_{(MN)}{}^\Lambda\Lambda^M\mathcal{G}_{\mu\nu}{}^N\\ &+ 2\,T_{(MN)}{}^\Lambda\,W_{[\mu}{}^M\,[\delta(\epsilon_1),\delta(\epsilon_2)]\,W_{\nu]}{}^N\\ &+ T_{(MN)}{}^\Lambda\xi^\rho\left(\bar X^M\bar\Omega_i{}^N\gamma_{\mu\nu}\psi_\rho{}^i - 2\bar\psi_\rho{}^i\gamma_{[\mu}\psi_{\nu]i}\,\bar X^M X^N + \text{h.c.}\right)\\ &+ 16\,\mathrm{i}\,g\,T_{(MN)}{}^\Lambda\Omega^{MP}\big(X^N\,\mu^{ij}{}_P\,\bar\epsilon_{2i}\gamma_{\mu\nu}\epsilon_{1j} - \bar X^N\,\mu_{ijP}\,\bar\epsilon_2^i\gamma_{\mu\nu}\epsilon_1^j\big)\\ &+ e\,\varepsilon_{\mu\nu\rho\sigma}\,T_{(MN)}{}^\Lambda\xi^\rho\big(-2\,\bar X^M\,\overleftrightarrow{\mathcal{D}}^\sigma X^N + \bar\Omega^{iM}\gamma^\sigma\Omega_i{}^N\\ &+ \bar X^M\bar\psi_\lambda{}^i\gamma^\sigma\gamma^\lambda\Omega_i{}^N - X^M\bar\psi_{\lambda i}\gamma^\sigma\gamma^\lambda\Omega^{iN}\\ &- \tfrac{1}{2}e^{-1}\varepsilon^{\sigma\lambda\tau\omega}\bar\psi_{\lambda i}\gamma_\tau\psi_\omega{}^i\,\bar X^M X^N\big)\,,\end{aligned} \tag{3.102}$$

with the parameters ξ^μ, Λ^M and $\Xi_{\mu\,\mathsf{a}}$ as in (3.94). The first four terms can straightforwardly be compared to the variation of $B_{\mu\nu\mathsf{a}}$ given in the first formula of (3.93). However, there is a subtlety regarding the commutator on $W_\nu{}^N$ in the third term, because this supersymmetry commutator only closes on the gauge fields, up to a term $\xi^\rho(\mathcal{G}-\mathcal{H})_{\rho\nu}{}^N$. Therefore the commutator yields the transformations indicated on the right-hand side of (3.94) plus this extra term.[12] Obviously the commutator on ${W_\nu}^N$ generates also a diffeomorphism, which will play a role later on in the calculation. Finally the fourth term represents precisely a supersymmetry transformation with parameter $\epsilon^i = -\tfrac{1}{2}\xi^\rho{\psi_\rho}^i$.

The remaining terms in (3.102), however, do not seem to have a role to play. At this point we note that the Lagrangian does not depend separately on $Z^{\Lambda,\mathsf{a}}B_{\mu\nu\,\mathsf{a}}$ and $Z^{\Lambda,\mathsf{m}}B_{\mu\nu\,\mathsf{m}}$, but depends only on the linear combination $Z^{\Lambda,\mathsf{a}}\,B_{\mu\nu\,\mathsf{a}} + Z^{\Lambda,\mathsf{m}}\,B_{\mu\nu\,\mathsf{m}}$. Consequently, the algebra is required to close only on this linear combination. Therefore we also evaluate the commutator on $Z^{\Lambda,\mathsf{m}}\,B_{\mu\nu\,\mathsf{m}}$,

$$
\begin{aligned}
Z^{\Lambda,\mathsf{m}}\,[\delta(\epsilon_1),\delta(\epsilon_2)]B_{\mu\nu\,\mathsf{m}} = {}& 2\,Z^{\Lambda,\mathsf{m}}\mathcal{D}_{[\mu}\Xi_{\nu]\mathsf{m}} \\
&+ \mathrm{i}\,\xi^\rho\big(k^{A\Lambda}\,\gamma_{Ai\bar\alpha}\,\bar\zeta^{\bar\alpha}\gamma_{\mu\nu}{\psi_\rho}^i - 2\,\varepsilon^{ij}{\mu_{jk}}^\Lambda\bar\psi_{i[\mu}\gamma_{\nu]}{\psi_\rho}^k - \text{h.c.}\big) \\
&- 16\,\mathrm{i}g{T_{(MN)}}^\Lambda\Omega^{MP}\big(X^N\,{\mu^{ij}}_P\,\bar\epsilon_{2i}\gamma_{\mu\nu}\epsilon_{1j} - \bar X^N\,\mu_{ijP}\,\bar\epsilon_2^i\gamma_{\mu\nu}\epsilon_1^j\big) \\
&+ \mathrm{i}e\,\varepsilon_{\mu\nu\rho\sigma}\xi^\rho\big[\,G_{\bar\alpha\beta}{T^{\Lambda\beta}}_\gamma\,\big(\tfrac{1}{2}A^{i\bar\alpha}\,\overleftrightarrow{\mathcal{D}}^\sigma {A_i}^\gamma - 2\,\bar\zeta^{\bar\alpha}\gamma^\sigma\zeta^\gamma \\
&+ {\bar\psi_\lambda}{}^i\gamma^\sigma\gamma^\lambda\zeta^{\bar\alpha}{A_i}^\gamma - \bar\psi_{\lambda i}\gamma^\sigma\gamma^\lambda\zeta^\gamma A^{i\bar\alpha}\big) \\
&- e^{-1}\varepsilon^{\sigma\lambda\tau\omega}\,{\bar\psi_\lambda}{}^i\gamma_\tau\psi_{\omega j}\varepsilon^{jk}{\mu_{ik}}^\Lambda\big]\,,
\end{aligned}
\tag{3.103}
$$

with the parameters ξ^μ and $\Xi_{\mu\,\mathsf{m}}$ as in (3.94). The first line establishes closure with respect to $\Xi_{\mu\,\mathsf{m}}$. Furthermore, the next line correctly reproduces a supersymmetry transformation with parameter $\epsilon^i = -\tfrac{1}{2}\xi^\rho{\psi_\rho}^i$.

When considering the sum of the two variations (3.102) and (3.103) there are some cancelations, and on the remaining terms we can impose the field equation (3.99). This leaves the following terms,

$$
\begin{aligned}
[\delta(\epsilon_1),\delta(\epsilon_2)]\big(Z^{\Lambda,\mathsf{a}}B_{\mu\nu\,\mathsf{a}} + Z^{\Lambda,\mathsf{m}}B_{\mu\nu\,\mathsf{m}}\big) = {}& Z^{\Lambda,\mathsf{a}}\,\xi^\rho\,\big(\mathcal{H}_{\rho\mu\nu\,\mathsf{a}} - 2\,d_{\mathsf{a}\,MN}{W_{[\mu}}^M(\mathcal{G}-\mathcal{H})_{\nu]\rho}{}^N\big) \\
&+ Z^{\Lambda,\mathsf{m}}\,\xi^\rho\,\mathcal{H}_{\rho\mu\nu\,\mathsf{m}} + \cdots\,,
\end{aligned}
\tag{3.104}
$$

where the dots refer to terms that have already been accounted for in the context of (3.94). The explicit terms in (3.104) contribute to the (covariant) general coordinate transformation, as follows from the following identities, which can be derived straight-

[12] Upon contraction with $Z^{N\,\mathsf{a}}$ this term vanishes by virtue of (3.98) and we have argued that it could therefore be suppressed in the commutator on the gauge fields on ${W_\nu}^N$. See the text preceding (3.101). However, in the case at hand, the term is not contracted with $Z^{N\,\mathsf{a}}$, and thus the extra term has to be retained.

forwardly from (3.81),

$$\begin{aligned} Z^{\Lambda,\mathsf{a}}\,\xi^\rho\,\mathcal{H}_{\rho\mu\nu\,\mathsf{a}} =&\, Z^{\Lambda,\mathsf{a}}\left(\xi^\rho\partial_\rho B_{\mu\nu\,\mathsf{a}} - 2\,\partial_{[\mu}\xi^\rho B_{\nu]\rho\,\mathsf{a}}\right)\\ &+ 2\,Z^{\Lambda,\mathsf{a}}\mathcal{D}_{[\mu}\left(\xi^\rho B_{\nu]\rho\,\mathsf{a}} - \xi^\rho d_{\mathsf{a}\,MN}W_{\nu]}{}^M W_\rho{}^N\right)\\ &+ 2\,T_{(MN)}{}^\Lambda\xi^\rho W_\rho{}^M\mathcal{G}_{\mu\nu}{}^N\\ &- 2\,T_{(MN)}{}^\Lambda W_{[\mu}{}^M\left(\xi^\rho\partial_{|\rho|}W_{\nu]}{}^N + \partial_{\nu]}\xi^\rho W_\rho{}^N - 2\,\xi^\rho(\mathcal{G}-\mathcal{H})_{\nu]\rho}{}^N\right)\\ &- 2\,g\,T_{(MN)}{}^\Lambda Z^{M,\mathsf{m}}\,\xi^\rho\,W_\rho{}^N B_{\mu\nu\,\mathsf{m}}\,,\\ Z^{\Lambda,\mathsf{m}}\,\xi^\rho\,\mathcal{H}_{\rho\mu\nu\,\mathsf{m}} =&\, Z^{\Lambda,\mathsf{m}}\left(\xi^\rho\partial_\rho B_{\mu\nu\,\mathsf{m}} - 2\,\partial_{[\mu}\xi^\rho B_{\nu]\rho\,\mathsf{m}}\right)\\ &+ 2\,Z^{\Lambda,\mathsf{m}}\mathcal{D}_{[\mu}(\xi^\rho B_{\nu]\rho\,\mathsf{m}})\\ &+ 2\,g\,T_{(MN)}{}^\Lambda Z^{M,\mathsf{m}}\,\xi^\rho\,W_\rho{}^N B_{\mu\nu\,\mathsf{m}}\,. \end{aligned} \tag{3.105}$$

The first two lines in the equations (3.105) denote the expected general coordinate transformation, and the tensor gauge transformations with parameters given in (3.96). The third term in the first equations represents the appropriate gauge transformation. The last terms in the two equations cancel directly, so that the only terms in (3.104) that are still unaccounted for, are given by,

$$\begin{aligned} [\delta(\epsilon_1),\delta(\epsilon_2)]\big(Z^{\Lambda,\mathsf{a}}B_{\mu\nu\,\mathsf{a}} + Z^{\Lambda,\mathsf{m}}B_{\mu\nu\,\mathsf{m}}\big) =&\, -2\,T_{(MN)}{}^\Lambda W_{[\mu}{}^M\big(\xi^\rho\partial_{|\rho|}W_{\nu]}{}^N + \partial_{\nu]}\xi^\rho W_\rho{}^N\big)\\ &+ 2\,T_{(MN)}{}^\Lambda W_{[\mu}{}^M\xi^\rho(\mathcal{G}-\mathcal{H})_{\nu]\rho}{}^N\\ &+ \cdots\,. \end{aligned} \tag{3.106}$$

The first of these terms cancels against the general coordinate transformation induced by the supersymmetry commutator on $W_\nu{}^N$ in (3.102), which we already referred to earlier. The second term can be suppressed by virtue of the special invariance noted in (3.83). To see this, we note that, up to the first equation of motion (3.98), we can write the induced variation of $B_{\mu\nu\mathsf{a}}$ as,

$$\begin{aligned} Z^{\Lambda,\mathsf{a}}\,\delta B_{\mu\nu\mathsf{a}} \propto&\, T^{(\Lambda}{}_M{}^{\Sigma)}\,[4\,\xi^\rho W_{[\mu}{}^M - \xi^\sigma W_\sigma{}^M\,\delta^\rho_{[\mu}](\mathcal{G}-\mathcal{H})_{\nu]\rho\Sigma}\\ &- T^{[\Lambda}{}_M{}^{\Sigma]}\,\xi^\sigma W_\sigma{}^M\,(\mathcal{G}-\mathcal{H})_{\mu\nu\Sigma}\,. \end{aligned} \tag{3.107}$$

This completes our discussion of the supersymmetry algebra.

Finally we summarize the modifications to the Lagrangian that are required by the general gaugings. As usual these concern both masslike terms for the fermions, which are proportional to the gauge coupling g, and a scalar potential proportional to g^2. The masslike terms independent of the gravitini follow directly from the rigid theory in the presence of both electric and magnetic charges [47]. The terms that involve gravitini are generalizations of the known results for the superconformal theory in the presence of electric charges [27, 28, 42]. The result includes also a non-fermionic term which describes the coupling of the auxiliary fields ${Y_{ij}}^{\Lambda}$ to the moments μ_{ijM},

$$\begin{aligned} e^{-1}\mathcal{L}_g = & -\tfrac{1}{2}\mathrm{i}g\,\Omega_{MQ}{T_{PN}}^{Q}\,\varepsilon^{ij}\,\bar{X}^N\bar{\Omega}_i{}^M\big(\Omega_j{}^P+\gamma^\mu\psi_{\mu j}X^P\big)+\text{h.c.}\\ & +2g\,k_{AM}\gamma^A_{i\bar\alpha}\varepsilon^{ij}\,\bar\zeta^{\bar\alpha}\big(\Omega_j{}^M+\gamma^\mu\psi_{\mu j}X^M\big)+\text{h.c.}\\ & +g\,\mu^{ij}{}_M\,\bar\psi_{\mu i}\left(\gamma^\mu\Omega_j{}^M+\gamma^{\mu\nu}\psi_{\nu j}X^M\right)+\text{h.c.}\\ & +2g\left[\bar X^M T_M{}^\gamma{}_\alpha\,\bar\Omega_{\beta\gamma}\,\bar\zeta^\alpha\zeta^\beta+X^M T_M{}^{\bar\gamma}{}_{\bar\alpha}\,\Omega_{\bar\beta\bar\gamma}\,\bar\zeta^{\bar\alpha}\zeta^{\bar\beta}\right]\\ & -\tfrac14 g\,\left[F_{\Lambda\Sigma\Gamma}\,\mu^{ij\Lambda}\,\bar\Omega_i{}^\Sigma\Omega_j{}^\Gamma+\bar F_{\Lambda\Sigma\Gamma}\,\mu_{ij}{}^\Lambda\,\bar\Omega^{i\Sigma}\Omega^{j\Gamma}\right]\\ & +g\,Y^{ij\Lambda}\left[\mu_{ij\Lambda}+\tfrac12(F_{\Lambda\Sigma}+\bar F_{\Lambda\Sigma})\,\mu_{ij}{}^\Sigma\right]\,. \end{aligned} \tag{3.108}$$

Upon solving the auxiliary fields ${Y_{ij}}^I$ one obtains an additional contribution to the scalar potential of order g^2. Without this contribution the scalar potential reads,

$$\begin{aligned} e^{-1}\mathcal{L}_{g^2} = & \,\mathrm{i}g^2\,\Omega_{MN}\,T_{PQ}{}^M X^P\bar X^Q\;T_{RS}{}^N\bar X^R X^S\\ & -2g^2k^A{}_M\,k^B{}_N\,g_{AB}\,X^M\bar X^N-\tfrac12 g^2\,N_{\Lambda\Sigma}\,\mu_{ij}{}^\Lambda\,\mu^{ij\Sigma}\,. \end{aligned} \tag{3.109}$$

Upon eliminating the auxiliary fields, the last term in this expression changes into,

$$-\tfrac12 g^2\,N_{\Lambda\Sigma}\,\mu_{ij}{}^\Lambda\,\mu^{ij\Sigma}\longrightarrow -2\,g^2\left[\mu^{ij}{}_\Lambda+F_{\Lambda\Gamma}\,\mu^{ij\Gamma}\right]\,N^{\Lambda\Sigma}\left[\mu_{ij\Sigma}+\bar F_{\Sigma\Xi}\,\mu_{ij}{}^\Xi\right]\,. \tag{3.110}$$

The above expressions are not of definite sign. From the Lagrangians in section 2.6 one can deduce that K, χ and the metrics that appear in the kinetic terms of the physical scalar fields should be negative. The latter metrics are proportional to two matrices, $M_{\Lambda\Sigma}$ and G_{AB}, that should therefore be negative definite. They are defined by,

$$\begin{aligned} M_{\Lambda\bar\Sigma} = & \,K^{-2}\,(N_{\Lambda\Sigma}N_{\Gamma\Xi}-N_{\Lambda\Gamma}N_{\Sigma\Xi})\,\bar X^\Gamma X^\Xi\,,\\ G_{AB} = & \,\chi^{-1}\left(g_{AB}-\chi^{-1}(\tfrac12\chi_A\chi_B+k_{Aij}k_B{}^{ij})\right)\,. \end{aligned} \tag{3.111}$$

With these observations we can separate the terms in the potential in positive and negative ones,

$$
\begin{aligned}
e^{-1}\mathcal{L}_{g^2} = & - g^2\, K\, M_{\bar{\Lambda}\Sigma}\, (T_{PQ}{}^{\Lambda} X^P \bar{X}^Q)\, (T_{RS}{}^{\Sigma} \bar{X}^R X^S) \\
& - 4\, g^2 K\, k^A{}_M\, k^B{}_N\, G_{AB}\, X^M \bar{X}^N \\
& - 2\, g^2\, K\, M_{\bar{\Lambda}\Sigma}\, N^{\Lambda\Gamma} \left[\mu^{ij}{}_{\Gamma} + F_{\Gamma\Omega}\, \mu^{ij\Omega}\right]\, N^{\Sigma\Xi} \left[\mu_{ij\Xi} + \bar{F}_{\Xi\Delta}\, \mu_{ij}{}^{\Delta}\right] \\
& - 6\, g^2 K^{-1}\, X^M \bar{X}^N\, \mu_{ijM}\, \mu^{ij}{}_N\,,
\end{aligned}
\qquad (3.112)
$$

where we used that $\chi = 2\,K$, as is implied by the field equation associated with the field D. It then follows that all contributions to $\mathcal{L}_{g^2}$ are negative, with the exception of the last term which is positive. This decomposition generalizes a similar decomposition known for purely electric charges [45].

The supersymmetric Lagrangians derived in this chapter incorporate gaugings in both the vector and hypermultiplet sectors. The vector multiplets are initially defined as off-shell multiplets, but the presence of the magnetic charges causes a breakdown of off-shell supersymmetry. Of course, conventional hypermultiplets based on a finite number of fields do not constitute an off-shell representation of the supersymmetry algebra irrespective of the presence of charges. We refer to a more in-depth discussion of the off-shell aspects of the embedding tensor method in [47], where a construction was presented in which the tensor fields associated with the magnetic charges were contained in a tensor supermultiplet.

Chapter 4

Two applications of the embedding tensor formalism

In the previous chapter we presented Lagrangians and supersymmetry transformations for general superconformal systems of vector multiplets and hypermultiplets in the presence of both electric and magnetic charges. The results were verified to all orders and are consistent with results known in the literature based on both rigidly supersymmetric theories and on superconformal systems without magnetic charges. In the presence of magnetic charges the off-shell closure of the superconformal algebra is only realized on the Weyl multiplet. The results establish a general framework for studying gauge interactions in matter-coupled $N=2$ supergravity.

In this chapter we present two applications to illustrate how the embedding tensor formalism can be used to obtain rather general results about realizations of $N=2$ gauged supergravities. One concerns the supersymmetric realizations in maximally symmetric spaces. In flat Minkowski space, it was established that residual supersymmetry is only possible in the presence of magnetic charges [19–23]. Here, we therefore briefly review the situation in the context of the embedding tensor approach, where it is natural to have both electric and magnetic charges.

A second application deals with supersymmetric solutions in $\mathrm{AdS}_2 \times S^2$ space-times. Here we establish that there exist only two classes of supersymmetric solutions. One concerns fully supersymmetric solutions. It contains the solutions described in [69] as well as the near-horizon solution of ungauged supergravity that appears for BPS black holes. The other class exhibits four supersymmetries and these solutions may appear as near-horizon geometries of BPS black holes in $N=2$ gauged supergravity. Interestingly enough, solutions in $\mathrm{AdS}_2 \times S^2$ with only two supersymmetries are excluded. The spinor parameters associated with the four supersymmetries are AdS_2 Killing spinors that are constant on S^2, so that they carry no spin. Nevertheless the bosonic background is rotationally invariant. The spin assignments change in this background, because the spin rotations associated with the S^2 isometries become entangled with R-symmetry transformations, a phenomenon that is somewhat similar to what happens for magnetic monopole solutions where the rotational symmetry becomes entangled with gauge transformations [70]. In the superconformal perspective, these solutions have R-symmetry connections living on S^2, and this explains the geometric origin of the entanglement. It is to be expected that the near-horizon geometry of a recently presented static, spherically symmetric, black hole solution [71, 72] will coincide with one of the solutions described in this chapter. The results of this chapter then imply that this black hole solution must exhibit supersymmetry enhancement at the horizon.

4.1 Maximally symmetric space-times and supersymmetry

In this application we briefly consider the question of full or partial supersymmetry in a maximally symmetric space-time. Hence one evaluates the supersymmetry variations of the fermion fields in the maximally symmetric background, where only $g_{\mu\nu}$, $A_i{}^{\alpha}$, X^{Λ} and $Y_{ij}{}^{\Lambda}$ can take non-zero values, taking into account that the fermion fields transform under both Q- and S-supersymmetry. In this particular background, it turns out that the gravitino field strength, $R(Q)_{\mu\nu}{}^i$ (and the related spinor χ^i) is S-invariant. Since its Q-supersymmetry variation is proportional to the field D, it immediately follows that $D = 0$, so that the special conformal gauge field takes the value (we assume the gauge choice $b_\mu = 0$, which leaves a residual invariance under constant scale transformations),

$$f_\mu{}^a = \tfrac{1}{2} R(e,\omega)_\mu{}^a - \tfrac{1}{12} e_\mu{}^a\, R(e,\omega)\,, \tag{4.1}$$

where $R(e,\omega)_{\mu\nu}{}^{ab}$ denotes the space-time curvature.

In what follows it thus suffices to concentrate on the fermions belonging to the vector multiplets and the hypermultiplets. We first present their variations in the background, which follow directly from (3.86) and (3.87),

$$\begin{aligned}\delta\zeta^\alpha &= 2gX^M\, T_M{}^\alpha{}_\beta A_i{}^\beta\, \varepsilon^{ij}\epsilon_j + A_i{}^\alpha\,\eta^i\,,\\ \delta\Omega_i{}^M &= \hat{Z}_{ij}{}^M\epsilon^j - 2g\,T_{PN}{}^M\bar{X}^P X^N\varepsilon_{ij}\epsilon^j + 2\,\mathrm{i}g\Omega^{MN}\mu_{ijN}\epsilon^j + 2X^M\eta_i\,. \end{aligned}\qquad (4.2)$$

Substituting the equations of motion for the auxiliary fields $Y_{ij}{}^\Lambda$, the variation of the independent fermion fields $\delta\Omega_i{}^\Lambda$ takes the following form,

$$\delta\Omega_i{}^\Lambda = -2g\,T_{NP}{}^\Lambda\,\bar{X}^N X^P\,\varepsilon_{ij}\,\epsilon^j - 4\,gN^{\Lambda\Sigma}\big(\mu_{ij\Sigma} + \bar{F}_{\Sigma\Gamma}\,\mu_{ij}{}^\Gamma\big)\epsilon^j + 2\,X^\Lambda\eta_i\,. \qquad (4.3)$$

Following the strategy adopted by [17], we consider only combinations of fermion fields that are invariant under S-supersymmetry. To construct S-invariant combinations of these fermions, it is convenient to define the following two spinor fields,

$$\begin{aligned}\zeta_i^{\mathrm{H}} &= \chi^{-1}\bar{\Omega}_{\alpha\beta}A_i{}^\alpha\,\zeta^\beta\,,\\ \Omega_i^{\mathrm{V}} &= -\tfrac{1}{2}\mathrm{i}K^{-1}\Omega_{MN}\bar{X}^M\Omega_i{}^N = \tfrac{1}{2}K^{-1}\,\bar{X}^\Lambda N_{\Lambda\Sigma}\Omega_i{}^\Sigma\,,\end{aligned}\qquad (4.4)$$

which are both formally invariant under electric/magnetic duality when treating the embedding tensor as a spurion. Under supersymmetry these two spinors transform equivalently in this background, provided we also use the field equation of the field D, which yields $\chi = 2\,K$. Indeed one easily derives,

$$\delta\Omega_i^{\mathrm{V}} = A_{ij}\,\epsilon^j + \eta_i = -\varepsilon_{ij}\,\delta\zeta^{\mathrm{H}\,j}\,, \qquad (4.5)$$

where the symmetric matrix A_{ij} is given by,

$$A_{ij} = -2\,g\,K^{-1}\,\bar{X}^M\mu_{ijM}\,. \qquad (4.6)$$

Here we made use of the second equation of (3.48).

To make contact with the terms appearing in the potential (3.109) (combined with (3.110), since we eliminated the auxiliary fields $Y_{ij}{}^\Lambda$), we consider the variations of three other spinors, which are S-invariant and consistent with duality. As it turns out, considering such variations gives important information regarding the possible supersymmetric realizations, although it will not yet fully determine whether the corresponding solutions will actually be realized. The first two variations are,

$$\begin{aligned}
g\big(\mu^{ij}{}_\Lambda + F_{\Lambda\Sigma}\,\mu^{ij\Sigma}\big)\,\delta[\Omega_j{}^\Lambda - 2\,X^\Lambda\Omega^{\rm V}_j]
&= -\,2\,g^2\,\bar X^M X^N T_{MN}{}^P \mu^{ij}{}_P\,\varepsilon_{jk}\,\epsilon^k \\
&\quad -\,2\,g^2(\mu^{kl}{}_\Lambda + F_{\Lambda\Sigma}\,\mu^{kl\Sigma})N^{\Lambda\Gamma}(\mu_{kl\Gamma} + \bar F_{\Gamma\Xi}\,\mu^{kl\Xi})\epsilon^i \\
&\quad +K\,A^{ij}A_{jk}\epsilon^k\,, \\
gN_{\Lambda\Sigma}\,T_{MN}{}^\Sigma X^M\bar X^N\,\delta[\Omega_i{}^\Lambda - 2\,X^\Lambda\Omega^{\rm V}_i]
&= 2\,{\rm i}\,g^2\Omega_{MN}(T_{PQ}{}^M X^P\bar X^Q)\,(T_{RS}{}^N\bar X^R X^S)\,\varepsilon_{ij}\epsilon^j \\
&\quad -\,4\,g^2\,X^M\bar X^N T_{MN}{}^P\,\mu_{ijP}\,\epsilon^j\,.
\end{aligned}\tag{4.7}$$

In deriving this result we made use of identities such as (3.46) and (3.48). Furthermore we used $\Omega^{MN}\mu_{ijM}\,\mu_{klN} = \mu_{ij\Lambda}\,\mu_{kl}{}^\Lambda - \mu_{ij}{}^\Lambda\,\mu_{kl\Lambda} = 0$, which follows directly from (3.57). The third spinor variation is based on hypermultiplets,

$$\begin{aligned}
g\,\bar X^M T_M{}^\alpha{}_\beta A_i{}^\beta\,\bar\Omega_{\alpha\gamma}\,\delta\big[\zeta^\gamma + \varepsilon^{jk}A_j{}^\gamma\,\zeta^{\rm H}_k\big] =&\ -\,g^2\bar X^M X^N\,k^A{}_M\,k^B{}_N\,g_{AB}\,\epsilon_i \\
&-\,2\,g^2\bar X^M X^N\,T_{MN}{}^P\,\mu_{ijP}\,\varepsilon^{jk}\,\epsilon_k \\
&+K\,A_{ij}A^{jk}\,\epsilon_k\,.
\end{aligned}\tag{4.8}$$

Here we made use of the identity,

$$T_M{}^\alpha{}_\beta A_i{}^\beta\,\bar\Omega_{\alpha\gamma}\,T_N{}^\gamma{}_\delta A_j{}^\delta = \tfrac12\varepsilon_{ij}\,k^A{}_M\,k_{AN} + T_{MN}{}^P\,\mu_{ijP}\,, \tag{4.9}$$

which follows from (2.47), (3.33), (3.37) and (3.54). Combining (4.8) with the two previous identities gives,

$$\big[e^{-1}\mathcal{L}_{g^2}\,\delta^i{}_j + 3\,KA^{ik}A_{kj}\big]\,\epsilon^j = 0\,. \tag{4.10}$$

This relation requires $e^{-1}\mathcal{L}_{g^2}$ to be non-negative, confirming the known result that de Sitter space-times cannot be supersymmetric.

According to [17] one must also consider the symmetry variation of the supercovariant derivative of at least one of the spinor fields. Let us, for instance, consider $D_\mu\Omega^{\rm V}_i$, which transforms also under S-supersymmetry. The following combination is then again S-invariant, and changes under Q-symmetry according to,

$$\delta\big[D_\mu\Omega^{\rm V}_i - \tfrac12 A_{ij}\gamma_\mu\Omega^{{\rm V}\,j}\big] = f_\mu{}^a\gamma_a\epsilon_i - \tfrac12 A_{ij}A^{jk}\,\gamma_\mu\epsilon_k\,. \tag{4.11}$$

Therefore we must require that the supersymmetry parameters are subject to the eigenvalue condition,

$$\big[\delta^i{}_j\,\big(R(e,\omega)_\mu{}^a - \tfrac{1}{6}e_\mu{}^a R(e,\omega)\big) - e_\mu{}^a\,A^{ik}A_{kj}\big]\epsilon^j = 0\,. \tag{4.12}$$

Combining this result with (4.10) reproduces the Einstein equation for the maximally symmetric space-time, irrespective of whether supersymmetry is realized fully or partially. Observe that full supersymmetry requires that $A^{ik}A_{kj} = \frac{1}{12}R(e,\omega)\delta^i{}_j$.

The result (4.10) can also be written as,

$$\big[A^{ik}A_{kj} - \tfrac{1}{2}A^{kl}A_{kl}\,\delta^i{}_j\big]\epsilon^j = -\frac{e^{-1}\mathcal{L}^-_{g^2}}{3\,K}\,\epsilon^i\,, \tag{4.13}$$

where $\mathcal{L}^-_{g^2}$ pertains to the negative terms in $\mathcal{L}_{g^2}$. For full supersymmetry we thus find that $\mathcal{L}^-_{g^2}$ must vanish, while partial supersymmetry is associated with the smallest eigenvalue of $A^{ik}A_{kj}$ and $\mathcal{L}^-_{g^2} \neq 0$. We refrain from giving more explicit details here, but we briefly consider the special case of Minkowski space-time.

For partial supersymmetry, the unbroken supersymmetry parameter is subject to the condition $A_{ij}\epsilon^j = 0$. In this context one can consider the variation of yet another spinor, which is invariant under S-supersymmetry, but no longer under duality,

$$\begin{aligned} 0 = X^\Lambda N_{\Lambda\Sigma}\,\delta[\Omega_i{}^\Sigma - 2\,X^\Sigma\,\Omega_i^{\rm V}] = &-2\,gX^\Lambda N_{\Lambda\Sigma}\,\big[T_{MN}{}^\Sigma\,\bar X^M X^N\,\varepsilon_{ij} - 2{\rm i}\mu_{ij}{}^\Sigma\big]\epsilon^j \\ &+ 2\,X^\Lambda N_{\Lambda\Sigma}\,\big[\bar X^\Sigma\,\varepsilon_{ik}\varepsilon_{jl}\,A^{kl} - X^\Sigma\,A_{ij}\big]\epsilon^j\,. \end{aligned} \tag{4.14}$$

In the absence of magnetic charges, the moment map $\mu_{ij}{}^\Sigma$ vanishes. Also the first term on the right-hand side vanishes because $T_{MN}{}^\Sigma\bar X^M X^N$ can be replaced by $T_{(MN)}{}^\Sigma\bar X^M X^N$ by virtue of the second equation of (3.48). The latter vanishes without magnetic charges. Therefore both $A_{ij}\epsilon^j$ and $A^{ij}\varepsilon_{jk}\epsilon^k$ vanish, which implies that A_{ij} vanishes. To show this first note that one can write an (anti-)hermitian 2×2 matrix as $\vec x\cdot\vec\sigma^i{}_j$, where $\vec\sigma^i{}_j$ denotes the three sigma-matrices and $\vec x$ denotes a Euclidian three-vector, which is real in the case of a hermitian matrix, and purely imaginary in the case of an anti-hermitian matrix. Now suppose that such a matrix has a zero eigenvalue for the eigenvector ϵ^i, then one can show,

$$0 = (\vec x\cdot\vec\sigma)^i{}_k\,(\vec x\cdot\vec\sigma)^k{}_j\,\epsilon^j = \vec x\cdot\vec x\,\epsilon^i \quad\Longrightarrow\quad \vec x = 0\,, \tag{4.15}$$

and hence the matrix itself vanishes. We can use these observations, by noting that the combination $\varepsilon^{ik}A_{kj} - A^{ik}\varepsilon_{kj}$ is a hermitian, traceless 2×2 matrix, while $\varepsilon^{ik}A_{kj} + A^{ik}\varepsilon_{kj}$ is an anti-hermitian, traceless 2×2 matrix. From the fact that $A_{ij}\epsilon^j$ and $A^{ij}\varepsilon_{jk}\epsilon^k$ vanish in the absence of magnetic charges, it then follows that A_{ij} vanishes, so that supersymmetry must be fully realized. This is in accord with a known theorem according to which $N=2$ supersymmetry can only be broken to $N=1$ supersymmetry in Minkowski space in the presence of magnetic charges [19–24]. For abelian gaugings the situation simplifies, and one can show that Minkowski solutions with residual $N = 1$ supersymmetry are possible provided that,

$$\begin{aligned} \bar{X}^M\, T_M{}^\alpha{}_\beta\, A_i{}^\beta \epsilon^i &= 0\,, \\ (\mu_{ij\Lambda} + \bar{F}_{\Lambda\Sigma}\, \mu_{ij}{}^\Sigma)\, \epsilon^j &= 0\,, \end{aligned} \tag{4.16}$$

with the two terms of the abelian potential vanishing separately (this follows from the first equation of (4.7) and from (4.8)),

$$\begin{aligned} \bar{X}^M X^N\, k^A{}_M\, k^B{}_N\, g_{AB} &= 0\,, \\ (\mu^{kl}{}_\Lambda + F_{\Lambda\Sigma}\, \mu^{kl\Sigma}) N^{\Lambda\Gamma} (\mu_{kl\Gamma} + \bar{F}_{\Gamma\Xi}\, \mu_{kl}{}^\Xi) &= 0\,. \end{aligned} \tag{4.17}$$

Without magnetic charges, one can easily verify that residual $N = 1$ supersymmetric solutions are not possible.

Apart from this latter result, the above analysis only indicates which supersymmetric solutions can, in principle, exist. To confirm that they are actually realized, one has to also examine the supersymmetry variations of the remaining fermion fields. This can be done, but we prefer not to demonstrate this here. Instead we will discuss this explicitly in the application presented in the next section, which is less straightforward, and where we will follow the same set-up as in this section.

4.2 Supersymmetry in $\mathrm{AdS}_2 \times S^2$

In this second application we consider an $\mathrm{AdS}_2 \times S^2$ space-time background and analyze possible supersymmetric solutions. Hence the space-time metric can be chosen equal to,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = v_1\Big(- r^2\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2}\Big) + v_2\Big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\Big)\,, \tag{4.18}$$

whose non-vanishing Riemann curvature components are equal to,

$$R_{\underline{ab}}{}^{\underline{cd}} = 2\,v_1^{-1}\delta_{\underline{ab}}{}^{\underline{cd}}\,, \qquad R_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}} = -2\,v_2^{-1}\delta_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}\,, \tag{4.19}$$

so that the four-dimensional Ricci scalar equals $R = 2(v_1^{-1} - v_2^{-1})$. Observe that we used tangent-space indices above, where $\underline{a}, \underline{b}, \ldots$ label the flat AdS_2 indices $(0,1)$ associated with (t,r), and $\hat{a}, \hat{b}, \ldots$ label the flat S^2 indices $(2,3)$ associated with (θ,φ). Furthermore the non-vanishing components of the auxiliary tensor field are parametrized by a complex scalar w,

$$-\,T_{01}{}^{ij}\varepsilon_{ij} = -\mathrm{i}\,T_{23}{}^{ij}\varepsilon_{ij} = w\,. \tag{4.20}$$

Using the previous results one finds the following expressions for the bosonic part of the special conformal gauge field $f_a{}^b$,

$$\begin{aligned} f_{\underline{a}}{}^{\underline{b}} &= \big(\tfrac{1}{6}(2\,v_1^{-1} + v_2^{-1}) - \tfrac{1}{4}D - \tfrac{1}{32}|w|^2\big)\delta_{\underline{a}}{}^{\underline{b}} + \tfrac{1}{2}R(A)_{23}\,\varepsilon_{\underline{a}}{}^{\underline{b}}\,,\\ f_{\hat{a}}{}^{\hat{b}} &= \big(-\tfrac{1}{6}(v_1^{-1} + 2\,v_2^{-1}) - \tfrac{1}{4}D + \tfrac{1}{32}|w|^2\big)\delta_{\hat{a}}{}^{\hat{b}} + \tfrac{1}{2}R(A)_{01}\,\varepsilon_{\hat{a}}{}^{\hat{b}}\,, \end{aligned} \tag{4.21}$$

where the two-dimensional Levi-Civita symbols are normalized by $\varepsilon^{01} = \varepsilon^{23} = 1$. The non-zero components of the modified curvature $\mathcal{R}(M)_{ab}{}^{cd}$ are given by,

$$\begin{aligned} \mathcal{R}(M)_{\underline{ab}}{}^{\underline{cd}} &= (D + \tfrac{1}{3}R)\,\delta_{\underline{ab}}{}^{\underline{cd}}\,,\\ \mathcal{R}(M)_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}} &= (D + \tfrac{1}{3}R)\,\delta_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}\,,\\ \mathcal{R}(M)_{\underline{a}\hat{b}}{}^{\underline{c}\hat{d}} &= \tfrac{1}{2}(D - \tfrac{1}{6}R)\,\delta_{\underline{a}}{}^{\underline{c}}\,\delta_{\hat{b}}{}^{\hat{d}} - \tfrac{1}{2}R(A)_{23}\,\varepsilon_{\underline{a}}{}^{\underline{c}}\,\delta_{\hat{b}}{}^{\hat{d}} - \tfrac{1}{2}R(A)_{01}\,\delta_{\underline{a}}{}^{\underline{c}}\,\varepsilon_{\hat{b}}{}^{\hat{d}}\,. \end{aligned} \tag{4.22}$$

We refer to appendix B for the general definitions of these quantities, which appear in the superconformal transformation rules of the Weyl multiplet fields and are therefore needed below.

Motivated by the maximal symmetry of the two two-dimensional subspaces, we expect the various fields to be invariant under the same symmetry. Therefore we will assume that the scalars X^M and $A_i{}^\alpha$ are covariantly constant (for other fields the covariant constancy will be discussed in due course). The corresponding integrability condition then requires that the U(1) and SU(2) R-symmetry curvatures are not necessarily vanishing, and are related to the curvatures of the vector multiplet gauge fields. This result is consistent with the field equations for the R-symmetry gauge fields, A_μ and $\mathcal{V}_\mu{}^i{}_j$, which lead to the expressions (we again choose the gauge $b_\mu = 0$),

$$\begin{aligned} R(A)_{\mu\nu} &= g\, K^{-1} \mathcal{H}_{\mu\nu}{}^{M} T_{MQ}{}^{N} \Omega_{PN} \bar{X}^{Q} X^{P} \,, \\ R(\mathcal{V})_{\mu\nu}{}^{i}{}_{j} &= -\, 4 g \chi^{-1} \mathcal{H}_{\mu\nu}{}^{M} \mu^{ik}{}_{M}\, \varepsilon_{kj} \,. \end{aligned} \tag{4.23}$$

Observe that the above equations only contribute for $\mu, \nu = t, r$, or $\mu, \nu = \theta, \varphi$, in view of the space-time symmetry. We can rewrite these equations in a different form, which is convenient later on,

$$\begin{aligned} R(A)^{-}_{\mu\nu} &= g\, K^{-1} \hat{\mathcal{H}}^{-}_{\mu\nu}{}^{\Lambda} \big[T_{\Lambda Q}{}^{N} + F_{\Lambda\Sigma}\, T^{\Sigma}{}_{Q}{}^{N} \big] \Omega_{PN} \bar{X}^{Q} X^{P} \,, \\ R(\mathcal{V})^{-}_{\mu\nu}{}^{i}{}_{j} &= -\, 4 g \chi^{-1} \hat{\mathcal{H}}^{-}_{\mu\nu}{}^{\Lambda} \big[\mu^{ik}{}_{\Lambda} + F_{\Lambda\Sigma}\, \mu^{ik\Sigma} \big]\, \varepsilon_{kj} + \tfrac{1}{4}\, \varepsilon^{ik} A_{kj}\, T_{\mu\nu}{}^{mn} \varepsilon_{mn} \,, \end{aligned} \tag{4.24}$$

where we suppressed all the fermionic terms which vanish in the background and made use of the field equations (3.98) of the tensor fields $B_{\mu\nu\,\mathsf{a}}$ and $B_{\mu\nu\,\mathsf{m}}$, and of (3.48).

To study supersymmetry in this background, we present the non-vanishing terms in the supersymmetry transformations of the spinors $\Omega_i{}^{\Lambda}$ and ζ^{α},

$$\begin{aligned} \delta\Omega_i{}^{\Lambda} &= \tfrac{1}{2} \gamma^{\mu\nu} \hat{\mathcal{H}}^{-}_{\mu\nu}{}^{\Lambda}\, \varepsilon_{ij} \epsilon^{j} - 2 g\, T_{NP}{}^{\Lambda}\, \bar{X}^{N} X^{P}\, \varepsilon_{ij}\, \epsilon^{j} \\ &\quad - 4\, g N^{\Lambda\Sigma} \big(\mu_{ij\Sigma} + \bar{F}_{\Sigma\Gamma}\, \mu_{ij}{}^{\Gamma} \big) \epsilon^{j} + 2\, X^{\Lambda} \eta_i \,, \\ \delta\zeta^{\alpha} &= 2 g X^{M}\, T_{M}{}^{\alpha}{}_{\beta} A_i{}^{\beta}\, \varepsilon^{ij} \epsilon_j + A_i{}^{\alpha}\, \eta^{i} \,. \end{aligned} \tag{4.25}$$

Note that $\delta\Omega_i{}^{\Lambda}$ has changed as compared to (4.3) by the presence of the field strength (3.91) (suppressing the fermionic terms, so that $\hat{\mathcal{H}}^{-}_{\mu\nu}{}^{\Lambda} = \mathcal{H}^{-}_{\mu\nu}{}^{\Lambda} - \tfrac{1}{4} \bar{X}^{\Lambda} T_{\mu\nu}{}^{ij} \varepsilon_{ij}$), while the expression for $\delta\zeta^{\alpha}$ is identical to the one given in (4.2). Just as before, we make use of the two spinors Ω_i^{V} and ζ_i^{H} defined in (4.4). The supersymmetry variation of these fields in the given background are,

$$\begin{aligned} \delta\Omega_i^{\mathrm{V}} &= \tfrac{1}{4} K^{-1}\, \bar{X}^{\Lambda} N_{\Lambda\Sigma} \hat{\mathcal{H}}^{-}_{\mu\nu} \gamma^{\mu\nu} \varepsilon_{ij} \epsilon^{j} + A_{ij} \epsilon^{j} + \eta_i \,, \\ \delta\zeta_i^{\mathrm{H}} &= \varepsilon_{ij} \big(A^{jk} \epsilon_k + \eta^{j} \big) \,, \end{aligned} \tag{4.26}$$

where A_{ij} was defined in (4.6). Supersymmetry therefore implies that the terms proportional to $\gamma^{\mu\nu}$ must vanish. As it turns out, this condition is just the field equation for $T_{ab}{}^{ij}$,

$$\bar{X}^{\Lambda} N_{\Lambda\Sigma}\, \hat{\mathcal{H}}^{-\Sigma}_{ab} = 0 \,. \tag{4.27}$$

Two additional fermionic variations are,

$$\begin{aligned}\delta[R(Q)_{ab}{}^i - \tfrac{1}{8}T_{cd}{}^{ij}\gamma^{cd}\gamma_{ab}\Omega_j^{\rm V}] &= R(\mathcal{V})_{ab}^{-}{}^i{}_j\epsilon^j - \tfrac{1}{2}\mathcal{R}(M)_{ab}{}^{cd}\gamma_{cd}\epsilon^i - \tfrac{1}{8}T_{cd}{}^{ij}\,\gamma^{cd}\gamma_{ab}A_{jk}\,\epsilon^k\,,\\ \delta\big[D_a\Omega_i^{\rm V} - \tfrac{1}{2}A_{ij}\gamma_a\Omega^{{\rm V}\,j}\big] &= f_a{}^b\gamma_b\epsilon_i + \tfrac{1}{4}{\rm i}R(A)^{-}_{cd}\gamma^{cd}\gamma_a\epsilon_i - \tfrac{1}{8}R(\mathcal{V})^{-}_{bc i}{}^j\gamma^{bc}\gamma_a\epsilon_j\\ &\quad + \tfrac{1}{16}A_{ij}T_{bc}{}^{jk}\gamma^{bc}\gamma_a\epsilon_k - \tfrac{1}{2}A_{ij}A^{jk}\,\gamma_a\,\epsilon_k\,. \end{aligned} \tag{4.28}$$

The variation of $R(Q)_{ab}{}^i$ is given in (D.7) and the variations of the superconformal gauge fields that are contained in the supercovariant derivative of $\Omega_i^{\rm V}$ are given in (B.1) and (B.7). Observe that we have assumed, motivated by the maximal symmetry of the two-dimensional subspaces, that also $T_{ab}{}^{ij}$ and A_{ij} are covariantly constant.

The consequences of (4.28) can be expressed as follows,[1]

$$\begin{aligned}(D+\tfrac{1}{12}R)\epsilon^i + \big[R(\mathcal{V})_{23}^{-}{}^i{}_j - {\rm i}R(A)^{-}_{23}\,\delta^i{}_j\big]\gamma^{23}\,\epsilon^j &= 0\,,\\ (D-\tfrac{1}{6}R)\epsilon^i - \big[2{\rm i}R(A)^{-}_{23}\,\delta^i{}_j + \tfrac{1}{2}{\rm i}w\,\varepsilon^{ik}A_{kj}\big]\gamma^{23}\,\epsilon^j &= 0\,,\\ \big[A^{ik}A_{kj}\epsilon^j + \tfrac{1}{4}{\rm i}w\,\varepsilon^{ik}A_{kj}\,\gamma^{23}\big]\,\epsilon^j &= 0\,,\\ (v_1^{-1}+v_2^{-1} - \tfrac{1}{8}|w|^2)\epsilon^i - \big[\tfrac{1}{2}{\rm i}\bar w A^{ik}\varepsilon_{kj} + 2\,R(\mathcal{V})_{23}^{+}{}^i{}_j + 2{\rm i}R(A)^{+}_{23}\,\delta^i{}_j\big]\gamma^{23}\epsilon^j &= 0\,. \end{aligned} \tag{4.29}$$

Furthermore we note that the covariant constancy of $T_{ab}{}^{ij}$ and A_{ij} implies the conditions,

$$w\,R(A)_{\mu\nu} = 0\,, \qquad R(\mathcal{V})_{\mu\nu}{}^k{}_{(i}\,A_{j)k} = -{\rm i}R(A)_{\mu\nu}\,A_{ij}\,. \tag{4.30}$$

We now turn to possible supersymmetric solutions for this background. We proceed in two steps. First we analyze the conditions for supersymmetry, ignoring the transformations (4.26). This will reveal the possible existence of three distinct classes of supersymmetric solutions, with four or eight supersymmetries, depending on the values of $R(\mathcal{V})_{\mu\nu}{}^i{}_j$ and A_{ij}. The corresponding information is summarized in table 4.1. As a last step we then analyze the transformations (4.26), which lead to additional constraints. It then follows that one of the classes listed in table 4.1 is actually not realized. In what follows we will decompose the equations (4.29) in eigenstates of ${\rm i}\gamma^{23}$, denoted by $\epsilon^i_\pm = \tfrac{1}{2}(1\pm{\rm i}\gamma^{23})\epsilon^i$. Observe that these spinors transform as a product representation of the SU(2) isometry group associated with S^2 and the SU(2) R-symmetry. This

[1] There are also charge conjugated equations. For instance, the first equation reads,

$$(D+\tfrac{1}{12}R)\epsilon_i + \big[R(\mathcal{V})_{23}^{+}{}_i{}^j + {\rm i}R(A)^{+}_{23}\,\delta_i{}^j\big]\gamma^{23}\,\epsilon_j = 0\,.$$

observation will be relevant shortly. Note also that the spinors transform according to $\epsilon_\pm{}^i \to \epsilon_{i\mp}$ under charge conjugation.

We start by considering solutions with $w = 0$. In that case the equations (4.29) yield,

$$\begin{aligned} R(A)^\pm_{23} &= \tfrac{1}{2} R(A)_{23}\,, \\ D &= \tfrac{1}{6} R \pm R_{23}(A)\,, \\ \mathrm{i}R(\mathcal{V})^-_{23}{}^i{}_j\,\epsilon^j_\pm &= \big[\pm\tfrac{1}{4}R + \tfrac{1}{2}R_{23}(A)\big]\,\epsilon^i_\pm\,, \\ \mathrm{i}R(\mathcal{V})^+_{23}{}^i{}_j\,\epsilon^j_\pm &= \big[\mp\tfrac{1}{2}(v_1^{-1} + v_2^{-1}) + \tfrac{1}{2}R_{23}(A)\big]\,\epsilon^i_\pm\,. \end{aligned} \tag{4.31}$$

Since $\mathrm{i}R(\mathcal{V})^+_{23} - \mathrm{i}R(\mathcal{V})^-_{23}$ is an anti-hermitian matrix, its eigenvalues should be imaginary. However, from (4.31), we find an eigenvalue given by $\mp\frac{1}{2}(v_1^{-1}+v_2^{-1})\mp\frac{1}{4}R = \mp v_1^{-1}$, which is real. Therefore, consistency demands that $v_1^{-1} = 0$. Hence taking $w = 0$ will only lead to a supersymmetric solution provided $v_1^{-1} = 0$. Discarding this singular solution, we thus assume $R(A)_{\mu\nu} = 0$. Then we consider two classes of solutions, denoted by A and B in table 4.1, depending on whether $D - \frac{1}{6}R$ vanishes or not.

For $R(A)_{\mu\nu} = 0$ and $D - \frac{1}{6}R = 0$, the equations (4.29) imply,

$$\begin{aligned} w\,A_{ij}\,\epsilon^j_\pm &= 0\,, \\ \mathrm{i}R(\mathcal{V})^-_{23}{}^i{}_j\,\epsilon^j_\pm &= \pm\tfrac{1}{4}R\,\epsilon^i_\pm\,, \\ \big[\mathrm{i}R(\mathcal{V})^+_{23}{}^i{}_j - \tfrac{1}{4}\bar{w}A^{ik}\varepsilon_{kj}\big]\,\epsilon^j_\pm &= \mp\tfrac{1}{2}(v_1^{-1} + v_2^{-2} - \tfrac{1}{8}|w|^2)\,\epsilon^i_\pm\,. \end{aligned} \tag{4.32}$$

Let us now assume that $A_{ij} \neq 0$. In that case $\varepsilon^{ik}A_{kj}$ must have a single null vector in order that a supersymmetric solution exists. On the other hand, it must commute with the SU(2) curvatures, which in this case implies that the $R(\mathcal{V})_{\mu\nu}{}^i{}_j$ must vanish. Supersymmetry then requires that $v_1 = v_2$ and,

$$w\,A_{ij}\,\epsilon^j_\pm = 0\,, \qquad \bar{w}A^{ik}\varepsilon_{kj}\,\epsilon^j_\pm = \pm(4\,v_1^{-1} - \tfrac{1}{4}|w|^2)\,\epsilon^i_\pm\,. \tag{4.33}$$

Again, since $w\,\varepsilon^{ik}A_{kj} + \bar{w}A^{ik}\varepsilon_{kj}$ is an anti-hermitian matrix, it should have imaginary eigenvalues, and hence from (4.33) it follows that $4v_1^{-1} - \frac{1}{4}|w|^2 = 0$. Then, from the argument around (4.15), it is clear that these equations have no solution unless $A_{ij} = 0$. When $A_{ij} = 0$ and the SU(2) curvatures are non-vanishing, one can show that (4.32) implies,

$$\mathrm{i}R(\mathcal{V})_{23}{}^i{}_j\,\epsilon^j_\pm = \pm\tfrac{1}{2}R\,\epsilon^i_\pm\,, \qquad v_1^{-1} = \tfrac{1}{16}|w|^2\,. \tag{4.34}$$

This solution, denoted by $A_{[2]}$, has generically four supersymmetries, two associated with two of the spinor parameters $\epsilon^i_\pm$, and two related with the charge-conjugated spinors $\epsilon_{i\mp}$. The two spinors of the $\epsilon^i_\pm$ must be eigenspinors of both $\mathrm{i}\gamma^{23}$ and $\mathrm{i}R(\mathcal{V})_{23}{}^i{}_j$ with related eigenvalues. Therefore the supersymmetries of class $A_{[2]}$ (and also of class B, as we shall see later) cannot transform consistently under the SU(2) isometry group. We will return to this aspect shortly.

In the special case where both A_{ij} and the SU(2) curvatures vanish, we have $v_1^{-1} = v_2^{-1} = \frac{1}{16}|w|^2$. Generically we then have eight supersymmetries. This class is denoted by $A_{[1]}$. Here the supersymmetries act consistently under the action of both SU(2) groups. This completes the discussion of the type-A solutions.

Subsequently we turn to the solutions of class B, where $D - \frac{1}{6}R \neq 0$ and $R(A)_{\mu\nu} = 0$. This class is denoted by B. In that case the first two equations (4.29) imply,

$$\begin{aligned}
\mathrm{i}R(\mathcal{V})^{-}_{23}{}^i{}_j\epsilon^j_\pm &= \pm\,(D + \tfrac{1}{12}R)\,\epsilon^i_\pm\,,\\
\tfrac{1}{2}w\,\varepsilon^{ik}A_{kj}\,\epsilon^j_\pm &= \pm\,(D - \tfrac{1}{6}R)\,\epsilon^i_\pm\,.
\end{aligned} \tag{4.35}$$

With this result, the last two equations then yield the eigenvalue equations,

$$\begin{aligned}
\mathrm{i}R(\mathcal{V})^{+}_{23}{}^i{}_j\epsilon^j_\pm &= \mp\,\tfrac{1}{2}(v_1^{-1} + v_2^{-1} - \tfrac{1}{4}|w|^2)\,\epsilon^i_\pm\,,\\
\tfrac{1}{2}\bar{w}\,A^{ik}\varepsilon_{kj}\,\epsilon^j_\pm &= \pm\,\tfrac{1}{8}|w|^2\,\epsilon^i_\pm\,.
\end{aligned} \tag{4.36}$$

Again, the same strategy can be followed: first construct anti-hermitian matrices and demand that the eigenvalues are purely imaginary or zero. If they are zero, the matrix itself vanishes by the argument given in (4.15). In this way (4.35) and (4.36) lead to,

$$\begin{aligned}
\bar{w}\,A^{ij} &= -\,w\,\varepsilon^{ik}\,\varepsilon^{jl}\,A_{kl}\,,\\
R(\mathcal{V})^{-}_{23}{}^i{}_j = R(\mathcal{V})^{+}_{23}{}^i{}_j &= \tfrac{1}{2}R(\mathcal{V})_{23}{}^i{}_j = -\frac{2\mathrm{i}}{v_2\,\bar{w}}\,\varepsilon^{ik}\,A_{kj}\,,\\
\mathrm{i}R(\mathcal{V})_{23}{}^i{}_j\epsilon^j_\pm &= \mp\,v_2^{-1}\,\epsilon^i_\pm\,,\\
D &= -\,\tfrac{1}{6}\big(v_1^{-1} + 2v_2^{-1}\big)\,,\\
v_1^{-1} &= \tfrac{1}{4}|w|^2\,.
\end{aligned} \tag{4.37}$$

Just as in class $A_{[2]}$, these solution have generically four supersymmetries, which cannot transform consistently under the action of the SU(2) isometry group. Furthermore, note that the solutions become singular in the limit where $\mathcal{V}_{\mu\nu}{}^i{}_j$ and A_{ij} vanish, so that this class is really distinct from the type-A class.

class	$R(\mathcal{V})$	A_{ij}	v_1, v_2	susy
$A_{[1]}$	$R(\mathcal{V}) = 0$	$A_{ij} = 0$	$v_1^{-1} = v_2^{-1} = \frac{1}{16}\lvert w\rvert^2$	$\mathbf{4} + \bar{\mathbf{4}}$
$A_{[2]}$	$R(\mathcal{V})_{23} = \mathcal{O}(v_1^{-1} - v_2^{-1})$	$A_{ij} = 0$	$v_1^{-1} = \frac{1}{16}\lvert w\rvert^2 \neq v_2^{-1}$	$\mathbf{2} + \bar{\mathbf{2}}$
B	$R(\mathcal{V})_{23}{}^i{}_j = -\frac{4\mathrm{i}}{v_2\,\bar{w}}\,\varepsilon^{ik}A_{kj} = \mathcal{O}(v_2^{-1})$		$v_1^{-1} = \frac{1}{4}\lvert w\rvert^2$	$\mathbf{2} + \bar{\mathbf{2}}$

Table 4.1: Three classes of supersymmetric solutions. As shown in due course, only the classes $A_{[1]}$ and B are actually realized.

In view of the fact that the supersymmetry spinors do not always seem to transform consistently under the action of the SU(2) transformations associated with the S^2 isometries, let us now first clarify this issue and turn to a discussion of the Killing spinor equations (in gauge $b_\mu = 0$) for each of the three classes. These equations take the following form,

$$\delta\big(\psi_\mu{}^i + \gamma_\mu\,\Omega^{\mathrm{V}i}\big) = 2\mathring{\nabla}_\mu\epsilon^i + \mathrm{i}A_\mu\,\epsilon^i + \mathcal{V}_\mu{}^i{}_j\,\epsilon^j - \varepsilon^{ik}\big[\tfrac{1}{4}\mathrm{i}w\,\gamma^{23}\delta_k{}^j + \varepsilon_{kl}A^{lj}\big]\gamma_\mu\,\epsilon_j\,. \tag{4.38}$$

where $\mathring{\nabla}_\mu$ denotes the $\mathrm{AdS}_2\times S^2$ covariant derivative. Obviously we may set A_μ and $\mathcal{V}_{\underline{a}} = 0$.

For class-A solutions (4.38) leads to,

$$\begin{aligned} \mathring{\nabla}_{\underline{a}}\epsilon^i_\pm \mp \tfrac{1}{8}w\,\varepsilon^{ij}\gamma_{\underline{a}}\,\epsilon_{j\pm} &= 0\,,\\ \mathring{\nabla}_{\hat{a}}\epsilon^i_\pm + \tfrac{1}{2}\mathcal{V}_{\hat{a}}{}^i{}_j\,\epsilon^j_\pm \mp \tfrac{1}{8}w\,\varepsilon^{ij}\gamma_{\hat{a}}\,\epsilon_{j\mp} &= 0\,, \end{aligned} \tag{4.39}$$

where $v_1^{-1} = \frac{1}{16}|w|^2$. For the solution of class $A_{[1]}$, we may take $\mathcal{V}_{\hat{a}}{}^i{}_j = 0$, so that we obtain the standard Killing spinor equations for $\mathrm{AdS}_2\times S^2$. For the $A_{[2]}$ solutions, the Killing spinor equation on S^2 is somewhat unusual, because of the presence of the R-symmetry connection whose strength is not related to the size of the S^2. Since we will show later that the type-$A_{[2]}$ solutions are in fact not realized, we refrain from further discussion concerning these solutions.

Hence we proceed to the class-B solutions. In this case, the Killing spinor equation (4.38) decomposes into,

$$\mathring{\nabla}_{\underline{a}}\epsilon^i_\pm \mp \tfrac{1}{4}w\,\varepsilon^{ij}\gamma_{\underline{a}}\,\epsilon_{j\pm} = 0\,,$$

$$\overset{\circ}{\nabla}_{\hat a}\epsilon^i_\pm + \tfrac{1}{2}\mathcal{V}_{\hat a}{}^i{}_j\,\epsilon^j_\pm = 0\,. \tag{4.40}$$

Because $v_1^{-1} = \frac{1}{4}|w|^2$, the first equation is the standard AdS_2 Killing spinor equation. However, the second equation does not coincide with the standard Killing spinor equation on S^2. We note that the strength of the R-symmetry connection is proportional to v_2^{-1}, and is therefore also determined by the S^2 radius. To elucidate the situation, let us briefly discuss the relevant equations for the unit sphere ($v_2 = 1$).

We use the standard coordinates θ and φ on S^2, with zweibeine $e^2 = \mathrm{d}\theta$ and $e^3 = \sin\theta\,\mathrm{d}\varphi$, and gamma matrices γ_2 and γ_3 that satisfy the standard Clifford algebra relation with positive signature. The spin connection field in our convention equals $\omega = \omega^{23} = -\omega^{32} = \cos\theta\,\mathrm{d}\varphi$. Consequently we have that $\overset{\circ}{\nabla}_\theta = \partial_\theta$ and $\overset{\circ}{\nabla}_\varphi = \big(\partial_\varphi - \frac{1}{2}\cos\theta\,\gamma^{23}\big)$. Now we adopt an R-symmetry transformation to bring $R(\mathcal{V})_{23}{}^i{}_j$ in diagonal form. In that case we can assume $\mathcal{V}^i{}_j = -\mathrm{i}\lambda\,(\sigma_3)^i{}_j\,\cos\theta\,\mathrm{d}\varphi$ with λ some real constant and σ_3 the diagonal Pauli matrix. This leads to the corresponding field strength $R(\mathcal{V})_{23}{}^i{}_j = \mathrm{i}\lambda\,(\sigma_3)^i{}_j$. From the third equation of (4.37) we conclude that $|\lambda| = 1$ and by an additional R-symmetry transformation we can ensure that $\lambda = 1$. In that case (remember that we put $v_2 = 1$) the supersymmetries are parametrized by the parameters ϵ^1_+ and ϵ^2_-. It is now straightforward to verify that these spinors do not depend on the S^2 coordinates as a result of the second equation (4.40).

Consequently the supersymmetries do not transform under the isometries of S^2, which implies that they carry no spin! Along the same lines one expects that also the fields in this background will change their spin assignment. The reason that the spin assignments change in this background, is that the spin rotations associated with the isometries of S^2 become entangled with R-symmetry transformations, in a similar way as in magnetic monopole solutions, where the rotational symmetry becomes entangled with gauge transformations [70]. In the superconformal context, where one has R-symmetry connections (which in this solution live on S^2), the geometric origin of the entanglement is clear. While such conditions on the supersymmetry spinor have been obtained previously in the literature for a variety of four- and five-dimensional supersymmetric solutions (see, e.g. [71–75], this phenomenon seems not to have received special attention.

Finally we must investigate the remaining variations based on (4.25). Consider first the variation for the fields $\Omega_i{}^\Lambda$, which we parametrize as $\delta\Omega_i{}^\Lambda = A_{ij}{}^\Lambda\epsilon^j - 2X^\Lambda\eta_i$, so that,

$$A_{ij}{}^\Lambda = 2\,\hat{\mathcal{H}}^{-}_{23}{}^\Lambda\varepsilon_{ij}\gamma^{23} - 2g\,T_{NP}{}^\Lambda\,\bar X^N X^P\,\varepsilon_{ij} - 4\,gN^{\Lambda\Sigma}\big(\mu_{ij\Sigma} + \bar F_{\Sigma\Gamma}\,\mu_{ij}{}^\Gamma\big)\,. \tag{4.41}$$

Then we consider the variation of two S-invariant combinations, $\Omega_i{}^\Lambda - 2X^\Lambda\Omega_i^{\rm V}$, and $D_a(\Omega^{i\,\Lambda} - 2\bar X^\Lambda\Omega^{i{\rm V}}) - \frac{1}{2}(A^{ij\Lambda} - 2\,\bar X^\Lambda A^{ij})\gamma_a\Omega_j^{\rm V}$, whose vanishing under supersymmetry imply the following identities,

$$
\begin{aligned}
\big[A_{ij}{}^\Lambda - 2\,X^\Lambda A_{ij}\big]\,\epsilon^j &= 0\,,\\
\big(A^{ik\Lambda} - 2\,\bar X^\Lambda A^{ik}\big)\,\big(A_{kj} - \tfrac{1}{8}T_{bckj}\,\gamma^{bc}\big)\,\gamma_a\epsilon^j &= 0\,,
\end{aligned}
\tag{4.42}
$$

where we assumed that $\mathcal{D}_\mu A^\Lambda = 0$ in line with our earlier ansätze. Likewise we obtain two equations for the hypermultiplets,

$$
\begin{aligned}
\big[2\,g\bar X^M\bar T_M{}^{\bar\alpha}{}_{\bar\beta}\,A^{i\bar\beta}\,\varepsilon_{ij} - A^{i\bar\alpha}A_{ij}\big]\,\epsilon^j &= 0\,,\\
\big(2\,gX^MT_M{}^\alpha{}_\beta\,A_i{}^\beta\,\varepsilon^{ik} - A_i{}^\alpha A^{ik}\big)\,\big(A_{kj} - \tfrac{1}{8}T_{bckj}\,\gamma^{bc}\big)\,\gamma_a\epsilon^j &= 0\,.
\end{aligned}
\tag{4.43}
$$

We note the presence of a universal factor on the right-hand side of the equation in (4.42) and (4.43), proportional to,

$$
A_{kj} - \tfrac{1}{8}T_{bckj}\,\gamma^{bc} = -\varepsilon_{kl}\big(\varepsilon^{lm}A_{mj} - \tfrac{1}{4}{\rm i}\bar w\,\gamma^{23}\,\delta^l{}_j\big)\,,
\tag{4.44}
$$

which is the hermitian conjugate of the term that appears at the right-hand side of (4.38).

The equations (4.42) and (4.43) lead to the following six conditions,

$$
\begin{aligned}
&\big[2\,gN^{\Lambda\Sigma}\,\varepsilon^{ik}\big(\mu_{kj\Sigma} + \bar F_{\Sigma\Gamma}\,\mu_{kj}{}^\Gamma\big) + X^\Lambda\,\varepsilon^{ik}A_{kj}\big]\,\epsilon^j_\pm\\
&\qquad\qquad -g\,T_{NP}{}^\Lambda\,\bar X^N X^P\,\epsilon^i_\pm = \pm\,{\rm i}\hat{\mathcal{H}}^{-}_{23}{}^\Lambda\,\epsilon^i_\pm\,,\\
&\big[2\,gN^{\Lambda\Sigma}\big(\mu^{ik}{}_\Sigma + F_{\Sigma\Gamma}\,\mu^{ik\Gamma}\big) + \bar X^\Lambda\,A^{ik}\big]\,A_{kj}\,\epsilon^j_\pm\\
&\qquad\qquad +g\,T_{NP}{}^\Lambda\,X^N\bar X^P\,\varepsilon^{ik}\,A_{kj}\,\epsilon^j_\pm = \tfrac{1}{4}{\rm i}\bar w\,\hat{\mathcal{H}}^{+}_{23}{}^\Lambda\,\epsilon^i_\pm\,,\\
&\bar w\big[-2\,gN^{\Lambda\Sigma}\varepsilon^{ik}\big(\mu_{kj\Sigma} + F_{\Sigma\Gamma}\,\mu_{kj}{}^\Gamma\big) - \bar X^\Lambda\,A^{ik}\varepsilon_{kj}\big]\,\epsilon^j_\pm\\
&\qquad\qquad +\bar w\,g\,T_{NP}{}^\Lambda\,X^N\bar X^P\,\epsilon^i_\pm = 4{\rm i}\,\hat{\mathcal{H}}^{+}_{23}{}^\Lambda\,\varepsilon^{ik}A_{kj}\,\epsilon^j_\pm\,,\\
&\big[2\,g\bar X^M\bar T_M{}^{\bar\alpha}{}_{\bar\beta}\,A^{i\bar\beta}\,\varepsilon_{ij} - A^{i\bar\alpha}A_{ij}\big]\,\epsilon^j_\pm = 0\,,\\
&\big[2\,gX^MT_M{}^\alpha{}_\beta\,A_i{}^\beta\,\varepsilon^{ik} - A_i{}^\alpha A^{ik}\big]\,A_{kj}\,\epsilon^j_\pm = 0\,,\\
&\big[2\,gX^MT_M{}^\alpha{}_\beta\,A_i{}^\beta\,\varepsilon^{ik} - A_i{}^\alpha A^{ik}\big]\,\varepsilon_{kj}\,\epsilon^j_\pm = 0\,.
\end{aligned}
\tag{4.45}
$$

Let us now consider the various classes of solutions shown in table 4.1. First of all the solutions of type A, characterized by $A_{ij} = 0$. From the second equation of (4.45) it then follows that $\hat{\mathcal{H}}_{\mu\nu}{}^\Lambda = 0$. Combining this result with the equations (4.24) shows that both $R(A)_{\mu\nu}$ and $R(\mathcal{V})_{\mu\nu}{}^i{}_j$ must vanish. This implies that solution $A_{[2]}$ is not realized. Hence we are left with the fully supersymmetric solution $A_{[1]}$. Therefore we proceed by determining the additional restrictions for this solution.

The first, third, fourth and sixth equations of (4.45) can be written as follows,

$$
\begin{aligned}
\mathrm{i}\varepsilon^{ik}\mu_{kj}{}^{\Lambda}\,\epsilon^j_{\pm} &= -\tfrac{1}{2}T_{NP}{}^{\Lambda}(\bar{X}^N X^P - X^N\bar{X}^P)\,\epsilon^i_{\pm}\,,\\
\mathrm{i}N^{\Lambda\Sigma}\varepsilon^{ik}\big(2\,\mu_{kj\Sigma} + (F_{\Sigma\Gamma} + \bar{F}_{\Sigma\Gamma})\mu_{kj}{}^{\Gamma}\big)\,\epsilon^j_{\pm} &= \tfrac{1}{2}\mathrm{i}T_{NP}{}^{\Lambda}(\bar{X}^N X^P + X^N\bar{X}^P)\,\epsilon^i_{\pm}\,,\\
\bar{X}^M\bar{T}_M{}^{\bar{\alpha}}{}_{\bar{\beta}}\,A^{i\bar{\beta}}\,\varepsilon_{ij}\,\epsilon^j_{\pm} &= 0\,,\\
X^M T_M{}^{\alpha}{}_{\beta}\,A_i{}^{\beta}\,\epsilon^i_{\pm} &= 0\,.
\end{aligned}
\tag{4.46}
$$

Since a hermitian matrix must have real eigenvalues, it follows that both sides of the first two equations should vanish. Also the factors in the last two equations should vanish, so that,

$$
\begin{aligned}
\mu_{ij\Lambda} = \mu_{ij}{}^{\Lambda} &= 0\,,\\
T_{NP}{}^{\Lambda}X^N\bar{X}^P &= 0\,,\\
X^M T_M{}^{\alpha}{}_{\beta}\,A_i{}^{\beta} &= 0 = \bar{X}^M T_M{}^{\alpha}{}_{\beta}\,A_i{}^{\beta}\,.
\end{aligned}
\tag{4.47}
$$

Note that $\mathcal{L}_{g^2}$ is now vanishing. For electric charges these solutions have already been identified in [69]. Without charges this is the well-known solution that arises as a near-horizon geometry of BPS black holes. The fact that the moment maps and certain combinations of Killing vectors are vanishing does not warrant the conclusion that there is no gauging. One can only conclude that the field equations require some of these quantities to vanish for these solutions.

Now consider the type-B solution where A_{ij} is non-vanishing. In that case the first three equations of (4.45) lead to two independent equations,

$$
\begin{aligned}
&-2\,gN^{\Lambda\Sigma}\,\varepsilon^{ik}\big(\mu_{kj\Sigma} + \bar{F}_{\Sigma\Gamma}\,\mu_{kj}{}^{\Gamma}\big)\,\epsilon^j_{\pm}\\
&\qquad\qquad + g\,T_{NP}{}^{\Lambda}\,\bar{X}^N X^P\,\epsilon^i_{\pm} = \mp\big(\mathrm{i}\hat{\mathcal{H}}^{-}_{23}{}^{\Lambda} + \tfrac{1}{4}\bar{w}X^{\Lambda}\big)\,\epsilon^i_{\pm}\,,\\
&-2\,gN^{\Lambda\Sigma}\varepsilon^{ik}\big(\mu_{kj\Sigma} + F_{\Sigma\Gamma}\,\mu_{kj}{}^{\Gamma}\big)\,\epsilon^j_{\pm}\\
&\qquad\qquad + g\,T_{NP}{}^{\Lambda}\,X^N\bar{X}^P\,\epsilon^i_{\pm} = \mp\big(\mathrm{i}\,\hat{\mathcal{H}}^{+}_{23}{}^{\Lambda} - \tfrac{1}{4}w\bar{X}^{\Lambda}\big)\,\epsilon^i_{\pm}\,.
\end{aligned}
\tag{4.48}
$$

These equations can be analyzed in a similar way as the corresponding equations in

(4.46). The results are as follows,

$$\begin{aligned} &T_{NP}{}^{\Lambda}\bar{X}^{N}X^{P} = 0\,, \\ &g\varepsilon^{ik}\mu_{kj}{}^{\Lambda}\,\epsilon^{j}_{\pm} = \mp\,\tfrac{1}{2}\big[\big(\hat{\mathcal{H}}^{-\Lambda}_{23} - \tfrac{1}{4}\mathrm{i}\bar{w}X^{\Lambda}\big) - \big(\hat{\mathcal{H}}^{+\Lambda}_{23} + \tfrac{1}{4}\mathrm{i}w\bar{X}^{\Lambda}\big)\big]\,\epsilon^{i}_{\pm}\,, \\ &g\varepsilon^{ik}\mu_{kj\Lambda}\,\epsilon^{j}_{\pm} = \pm\,\tfrac{1}{2}\big[F_{\Lambda\Sigma}\big(\hat{\mathcal{H}}^{-\Sigma}_{23} - \tfrac{1}{4}\mathrm{i}\bar{w}X^{\Sigma}\big) - \bar{F}_{\Lambda\Sigma}\big(\hat{\mathcal{H}}^{+\Sigma}_{23} + \tfrac{1}{4}\mathrm{i}w\bar{X}^{\Sigma}\big)\big]\,\epsilon^{i}_{\pm}\,. \end{aligned} \tag{4.49}$$

From (3.46), it follows that the first constraint of (4.49) can be generalized to $T_{MN}{}^{P}\bar{X}^{M}X^{N} = 0$. Using also the representation constraint (3.43), one reconfirms that $R(A)_{\mu\nu}$, as given in (4.24), vanishes. The same argument applies to solutions of type $A_{[1]}$. Furthermore, as a check one may also reconstruct the eigenvalue equation for A_{ij} which shows once more that (4.27) must be valid.

One can use the same strategy and determine $R(\mathcal{V})_{23}{}^{i}{}_{j}$ from (4.24), making use of (4.49) with $T_{MN}{}^{P}\bar{X}^{M}X^{N} = 0$. Evaluating this curvature on the supersymmetry parameters, making use of the eigenvalue condition for this curvature presented in (4.37) as well as of (4.27), it follows that,

$$v_2^{-1} = -2K^{-1}N_{\Lambda\Sigma}\hat{\mathcal{H}}^{-\Lambda}_{23}\hat{\mathcal{H}}^{+\Sigma}_{23} - \tfrac{1}{8}|w|^2\,. \tag{4.50}$$

In the first expression on the right-hand side, one can verify, replacing $N_{\Lambda\Sigma}$ by the negative definite metric $M_{\Lambda\bar{\Sigma}}$ defined in (3.111) and using (4.27), that this expression must be positive, which yields an upper bound on $|w|^2$ for given field strengths $\hat{\mathcal{H}}_{23}{}^{\Lambda}$.

The last three equations of (4.45) lead to two equations,

$$\begin{aligned} X^{M}\big[T_{M}{}^{\alpha}{}_{\beta}A_{i}{}^{\beta} + K^{-1}\varepsilon_{ij}\,\mu^{jk}{}_{M}\,A_{k}{}^{\alpha}\big] = 0\,, \\ \bar{X}^{M}\big[T_{M}{}^{\alpha}{}_{\beta}A_{i}{}^{\beta} + K^{-1}\varepsilon_{ij}\,\mu^{jk}{}_{M}\,A_{k}{}^{\alpha}\big] = 0\,. \end{aligned} \tag{4.51}$$

From these equations, one derives, upon using (4.9),

$$g^2\bar{X}^{M}X^{N}\,k^{A}{}_{M}\,k_{AN} = \tfrac{1}{16}K\,|w|^2\,. \tag{4.52}$$

The scalar potential in the type-B solutions thus takes the form,

$$\begin{aligned} e^{-1}\mathcal{L}_{g^2} = &-2\,g^2\,K\,M_{\bar{\Lambda}\Sigma}\,N^{\Lambda\Gamma}\,\big[\mu^{ij}{}_{\Gamma} + F_{\Gamma\Omega}\,\mu^{ij\Omega}\big]\;N^{\Sigma\Xi}\,\big[\mu_{ij\Xi} + \bar{F}_{\Xi\Delta}\,\mu_{ij}{}^{\Delta}\big] \\ &-\tfrac{3}{16}K\,|w|^2\,, \end{aligned} \tag{4.53}$$

where the first term is negative and the second one positive. We refrain from giving further results.

For a single (compensating) hypermultiplet, which can only have abelian gaugings, we expect that one of these type-B solutions describes the near-horizon geometry of the spherically symmetric static black hole solution presented in [71, 72]. The result of this section then ensures that this black hole solution has supersymmetry enhancement at the horizon.

Chapter 5

Higher-derivatives couplings in N=2 superconformal gravity

The Lagrangians we have seen so far, with or without gauge deformations, were all restricted to contain at most two derivatives of the fields. In this chapter we will consider a rather large class of higher-derivative couplings in theories without gaugings. As we will see, their construction is based on chiral multiplets. Therefore, the higher-derivative couplings will only pertain to vector multiplets, the Weyl multiplet and possible other multiplets based on chiral multiplets, as we will discuss in detail. Consequently, higher-derivative couplings of hypermultiplets are not considered. The higher-derivative terms are coupled to conformal supergravity and are realized off-shell. This feature greatly facilitates their construction, which is based on previous work on $N=2$ supergravity (in particular, on [27, 36]).

Supersymmetric invariants with higher-derivative couplings play a role in many applications. The first higher-derivative couplings that were considered in $N=2$ supergravity involve the square of the Weyl tensor coupled to vector supermultiplets [76]. This particular class of invariants is based on an integration over a chiral subspace of $N=2$ superspace. It is relevant for the topological string [77, 78], and furthermore, it has important implications for BPS black hole entropy [16]. Another class of invariants for vector multiplets that involve terms quartic in the field strengths, was derived in terms of $N=1$ superfields, both for the abelian [79] and for the non-abelian case [80]. Unlike the previous class, this one is based on an integral over full superspace. It yields important contributions to the effective action of $N=2$ supersymmetric gauge theories (for some additional references, see e.g., [81–84]). A related class of locally supersymmetric higher-derivative couplings was considered in [85, 86]. Those couplings, which involve both the Weyl tensor and higher-order coupling of the vector field strengths, were conjectured to describe certain deformations of the topological string partition function. This chapter deals with an explicit construction of this rather large class of invariant couplings based on full superspace integrals.

This chapter is organized as follows. Section 5.1 describes the general strategy for the construction of the higher-derivative couplings, based on the use of the so-called 'kinetic supermultiplet', which can be constructed from an anti-chiral supermultiplet of zero Weyl weight. The components of this multiplet are given in considerable detail, fully taking into account the presence of the superconformal background. The construction of the bosonic terms of the higher-derivative couplings is presented in section 5.2, together with explicit examples based on a class of Lagrangians that involves terms such as F^4, R^2F^2 and R^4. Here F denotes the abelian vector multiplet field strengths and R the Riemann tensor. As we mentioned in the introduction, an important application of this work is to study the possible contribution of these new couplings to the entropy and the electric charges of BPS black holes. In section 5.3 a non-renormalization theorem is proven, according to which these contributions vanish. Some concluding remarks are presented in section 5.4.

5.1 The kinetic chiral multiplet

General chiral multiplets were presented in section 2.3. We briefly mentioned the existence of a so-called 'kinetic' multiplet, which we will introduce in this section.

The term kinetic multiplet was first used in the context of the $N=1$ tensor calculus [87], because this is the chiral multiplet that enables the construction of the kinetic terms, conventionally described by a real superspace integral, in terms of a chiral superspace integral. In flat $N=1$ superspace, this construction is simply effected by the conversion,[1]

$$\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\; \Phi\, \bar\Phi' \approx \int \mathrm{d}^2\theta\; \Phi\, \mathbb{T}(\bar\Phi')\,, \tag{5.1}$$

up to space-time boundary terms. Here Φ and Φ' are two chiral superfields and $\bar\Phi'$ is the anti-chiral field obtained from Φ' by complex conjugation. The kinetic multiplet equals

[1] In this chapter we will sometimes make use of superfield notions, such as superspace integrals like (5.1), but they are always used for illustrative purposes. Actual calculations are only made in the component approach used throughout this thesis.

$\mathbb{T}(\bar\Phi') = \bar D^2\bar\Phi'$, where $\bar D$ denotes the supercovariant $\bar\theta$-derivative. Obviously the kinetic multiplet contains terms linear and quadratic in space-time derivatives, so that, upon identifying Φ and Φ', the right-hand side of (5.1) does indeed give rise to the kinetic terms of an $N=1$ chiral multiplet.

In [27] a corresponding kinetic multiplet was identified for $N = 2$ supersymmetry, which now involves four rather than two covariant $\bar\theta$-derivatives, i.e. $\mathbb{T}(\bar\Phi) \propto \bar D^4\bar\Phi$. As a result, $\mathbb{T}(\bar\Phi)$ contains now up to four space-time derivatives, so that the expression,

$$\int \mathrm{d}^4\theta\, \mathrm{d}^4\bar\theta\; \Phi\, \bar\Phi' \approx \int \mathrm{d}^4\theta\, \Phi\, \mathbb{T}(\bar\Phi')\,, \tag{5.2}$$

does not correspond to a kinetic term, but to a higher-order derivative coupling. Furthermore, for $N = 2$ supersymmetry one has the option of expressing the chiral multiplets in terms of (products of) reduced chiral multiplets. In that case, expressions such as (5.2) will correspond to higher-derivative couplings of vector multiplets. Since we are considering the kinetic multiplets in a conformal supergravity background, their Weyl weight is relevant. Both in $N=1,2$ supergravity the kinetic multiplet carries Weyl weight $w = 2$. The conversion starts from a $w = 1$ chiral multiplet for $N=1$ and from a $w = 0$ chiral multiplet for $N=2$ supersymmetry, respectively.

To demonstrate this in more detail, consider an anti-chiral $N=2$ supermultiplet in the presence of the superconformal background. Its supersymmetry transformations follow from taking the complex conjugate of (2.17). Precisely for $w = 0$ we note that the field $\bar C$ is invariant under S-supersymmetry and transforms under Q-supersymmetry as the lowest component of a chiral supermultiplet with $w = 2$. This observation proves that we are dealing with a $w = 2$ chiral supermultiplet, as is also confirmed by the weight assignments specified in table 2.1. What remains is to identify the various components of this multiplet in terms of the underlying $w = 0$ multiplet. This can be done by applying successive Q-supersymmetry transformations on $\bar C$, something that requires rather tedious calculations in the presence of a superconformal background.

Denoting the components of $\mathbb{T}(\bar\Phi_{w=0})$ by $(A,\Psi,B,F^-,\Lambda,C)|_{\mathbb{T}(\bar\Phi)}$, while (A,Ψ,B,F^-,Λ,C) will denote the components of the original $w = 0$ chiral multiplet, we have established the following relation,

$$\begin{aligned}
A|_{\mathbb{T}(\bar\Phi)} &= \bar C\,,\\
\Psi_i|_{\mathbb{T}(\bar\Phi)} &= -\,2\,\varepsilon_{ij}\not{D}\Lambda^j - 6\;\varepsilon_{ik}\varepsilon_{jl}\chi^j B^{kl} - \tfrac14\varepsilon_{ij}\varepsilon_{kl}\,\gamma^{ab}T_{ab}{}^{jk}\,\overleftrightarrow{\not{D}}\,\Psi^l\,,\\
B_{ij}|_{\mathbb{T}(\bar\Phi)} &= -\,2\,\varepsilon_{ik}\varepsilon_{jl}\big(\Box_{\mathrm{c}} + 3\,D\big)B^{kl} - 2\,F^+_{ab}\,R(\mathcal{V})^{ab\,k}{}_i\,\varepsilon_{jk}
\end{aligned}$$

$$
\begin{aligned}
&\quad - 6\,\varepsilon_{k(i}\,\bar\chi_{j)}\Lambda^k + 3\,\varepsilon_{ik}\varepsilon_{jl}\bar\Psi^{(k}\not{D}\chi^{l)}\,,\\
F^-_{ab}|_{\mathbb{T}(\bar\Phi)} =&\, - \big(\delta_a{}^{[c}\delta_b{}^{d]} - \tfrac12\varepsilon_{ab}{}^{cd}\big)\\
&\quad \times \big[4\,D_cD^eF^+_{ed} + (D^e\bar A\,D_cT_{de}{}^{ij} + D_c\bar A\,D^eT_{ed}{}^{ij})\varepsilon_{ij}\big]\\
&\quad + \Box_{\rm c}\bar A\,T_{ab}{}^{ij}\varepsilon_{ij} - R(\mathcal{V})^-{}_{ab}{}^i{}_k\,B^{jk}\,\varepsilon_{ij} + \tfrac18 T_{ab}{}^{ij}\,T_{cdij}F^{+cd} - \varepsilon_{kl}\,\bar\Psi^k\,\overleftrightarrow{\not{D}}\,R(Q)_{ab}{}^l\\
&\quad - \tfrac94\varepsilon_{ij}\,\bar\Psi^i\gamma^c\gamma_{ab}D_c\chi^j + 3\,\varepsilon_{ij}\bar\chi^i\gamma_{ab}\not{D}\Psi^j + \tfrac38 T_{ab}{}^{ij}\varepsilon_{ij}\,\bar\chi_k\Psi^k\,,\\
\Lambda_i|_{\mathbb{T}(\bar\Phi)} =&\, 2\,\Box_{\rm c}\not{D}\Psi^j\varepsilon_{ij} + \tfrac14\gamma^c\gamma_{ab}(2\,D_cT^{ab}{}_{ij}\,\Lambda^j + T^{ab}{}_{ij}\,D_c\Lambda^j)\\
&\quad - \tfrac12\varepsilon_{ij}\big(R(\mathcal{V})_{ab}{}^j{}_k + 2{\rm i}\,R(A)_{ab}\delta^j{}_k\big)\,\gamma^c\gamma^{ab}D_c\Psi^k\\
&\quad + \tfrac12\,\varepsilon_{ij}\big(3\,D_bD - 4{\rm i}D^aR(A)_{ab} + \tfrac14 T_{bc}{}^{ij}\,\overleftrightarrow{D}_a\,T^{ac}{}_{ij}\big)\,\gamma^b\Psi^j\\
&\quad - 2\,F^{+ab}\,\not{D}R(Q)_{abi} + 6\,\varepsilon_{ij}D\,\not{D}\Psi^j\\
&\quad + 3\,\varepsilon_{ij}\,\big(\not{D}\chi_k\,B^{kj} + \not{D}\bar A\,\not{D}\chi^j\big)\\
&\quad + \tfrac32\big(2\,\not{D}B^{kj}\varepsilon_{ij} + \not{D}F^+_{ab}\gamma^{ab}\,\delta^k_i + \tfrac14\varepsilon_{mn}T_{ab}{}^{mn}\,\gamma^{ab}\,\not{D}\bar A\,\delta_i{}^k\big)\chi_k\\
&\quad + \tfrac94\,(\bar\chi^l\gamma_a\chi_l)\,\varepsilon_{ij}\gamma^a\Psi^j - \tfrac92\,(\bar\chi_i\gamma_a\chi^k)\,\varepsilon_{kl}\gamma^a\Psi^l\,,\\
C|_{\mathbb{T}(\bar\Phi)} =&\, 4(\Box_{\rm c} + 3\,D)\Box_{\rm c}\bar A - \tfrac12 D_a\big(T^{ab}{}_{ij}\,T_{cb}{}^{ij}\big)\,D^c\bar A + \tfrac1{16}(T_{abij}\varepsilon^{ij})^2\bar C\\
&\quad + D_a\big(\varepsilon^{ij}D^aT_{bcij}\,F^{+bc} + 4\,\varepsilon^{ij}T^{ab}{}_{ij}\,D^cF^+_{cb} - T_{bc}{}^{ij}\,T^{ac}{}_{ij}\,D^b\bar A\big)\\
&\quad + \big(6\,D_bD - 8{\rm i}D^aR(A)_{ab}\big)\,D^b\bar A + \cdots\,,
\end{aligned}
\qquad (5.3)
$$

where in the last expression we suppressed terms quadratic in the covariant fermion fields. Obviously terms involving the fermionic gauge fields, $\psi_\mu{}^i$ and $\phi_\mu{}^i$, are already contained in the superconformal derivatives. Observe that the right-hand side of these expressions is always linear in the conjugate components of the $w=0$ chiral multiplet, i.e. in $(\bar A, \Psi^i, B^{ij}, F^+_{ab}, \Lambda^i, \bar C)$. As an extra test of the correctness of (5.3) we verified that these expressions satisfy the correct transformation behaviour under S-supersymmetry. This test cannot be performed on the last component $C|_{\mathbb{T}(\bar\Phi)}$, because we refrained from collecting the fermionic contributions. As an extra check we have therefore verified that the bosonic terms of $C|_{\mathbb{T}(\bar\Phi)}$ are invariant under special conformal boosts.

The definition of the superconformal D'Alembertian $\Box_{\rm c}$, defined by the contraction of two superconformal derivatives D_a, as well as multiple superconformal derivatives in general, may require further comment. Therefore we have presented some relevant material in appendix C. Below we give the most non-trivial transformation rules under special conformal boosts that are needed in this chapter,

$$
\delta_{\rm K}\Box_{\rm c}\Box_{\rm c}A = -\,2\,\Lambda_{\rm K}^a\big([D_a,D_b]D^b + D^b[D_a,D_b]\big)A
$$

$$
\begin{aligned}
&= \tfrac{1}{4}\Lambda_{\rm K}^{a}\, T_{ac}{}^{ij}\, T^{bc}{}_{ij}\, D_b A - 3\,\Lambda_{\rm K}{}^{a} D\, D_a A - 2\,\Lambda_{\rm K}{}^{a}\, D^b(\bar R(Q)_{ba}\Psi_i) \\
&\quad - \tfrac{3}{4}\Lambda_{\rm K}{}^{a}\, \bar\chi_i T_{ab}{}^{ij}\gamma^b\Psi_j + \tfrac{3}{4}\Psi_i \not\Lambda_{\rm K} \not D \chi^i\,, \\
\delta_{\rm K}\Box_{\rm c}\not D\Psi_i &= \not\Lambda_{\rm K}\big[\tfrac{1}{4}\big(R(\mathcal{V})_{ab}{}^j{}_i + 2{\rm i}R(A)_{ab}\,\delta^j{}_i\big)\gamma^{ab}\Psi_j - \tfrac{3}{2}D\,\Psi_i\big] \\
&\quad + \not\Lambda_{\rm K}\big[\tfrac{3}{2}B_{ij}\,\chi^j - \varepsilon_{ij}F^{-\,ab}\,R(Q)^j_{ab} - \tfrac{3}{4}\varepsilon_{ij}F^-_{ab}\,\gamma^{ab}\chi^j\big]\,. \qquad (5.4)
\end{aligned}
$$

These results follow from (C.6), upon making use of the relevant curvatures.

5.2 Invariant higher-derivative couplings

Using the results of the previous section one can construct a large variety of superconformal invariants for chiral multiplets with higher-derivative couplings. For unrestricted chiral supermultiplets one cannot write down Lagrangians that are at most quadratic in derivatives, so they usually play a role as composite fields that are expressed in terms of reduced chiral multiplets, such as the vector multiplets and the Weyl multiplet. The construction of the higher-order Lagrangians therefore proceeds in two steps. First one constructs the Lagrangian in terms of unrestricted chiral multiplets of the appropriate Weyl weights, and subsequently one expresses the unrestricted supermultiplets in terms of reduced supermultiplets. In these expressions it is natural to introduce a variety of arbitrary homogeneous functions.

The invariants are expressed as chiral superspace integrals, because all possible anti-chiral fields are contained in the kinetic multiplets that we have introduced in section 5.1. A simple example of this approach was already exhibited in (5.2). The fact that these invariants are actually based on full superspace integrals implies that they must vanish whenever all the chiral (or, alternatively, all the anti-chiral) fields are put equal to a constant. In the chiral formulation of the integral, this phenomenon is reflected in the fact that the kinetic multiplet of a *constant* anti-chiral multiplet vanishes. This result can easily be deduced from (5.3). Invariants can be substantially more complicated than (5.2). The integrand does not have to be linear in a kinetic multiplet, and can depend on a function of kinetic multiplets. One can also consider 'nested' situations, where a kinetic multiplet is constructed starting from an expression of superfields among which there are other kinetic multiplets, thus leading to even higher multiple derivatives.

The above approach is a constructive one and in general it will be hard to classify all these invariant couplings, say, in terms of a limited number of functions, as is often possible for supersymmetric theories. For definiteness, we henceforth restrict attention to invariants proportional to a single kinetic multiplet, as given in (5.2). In that case, expressing the composite chiral multiplets in terms of vector multiplets, one obtains the supergravity-coupled invariants corresponding to the actions derived in [79, 80] in the abelian limit, which contain F^4-couplings. By including the Weyl multiplet, one also obtains R^2F^2- and R^4-couplings. The R^2F^2-couplings will in principle overlap with part of a subclass of invariants discussed in [85, 86] in connection with certain deformations of the topological string partition function. These couplings are encoded in terms of a single function of holomorphic and anti-holomorphic fields. In a rigid supersymmetry background these actions exhibit Kähler geometry with this function playing the role of a Kähler potential. As we will demonstrate below, this feature survives in the presence of the superconformal background. Other examples of higher-derivative couplings based on more than a single kinetic multiplet will be discussed in section 5.4.

Hence we start by writing down the bosonic terms of the Lagrangian (5.2). It is convenient to first note the following relation,

$$
\begin{aligned}
C|_{\mathbb{T}(\bar\Phi)} = {}& \tfrac{1}{16}(T_{abij}\varepsilon^{ij})^2\bar C + 4\left(\mathcal{D}^\mu\mathcal{D}_\mu\right)^2\bar A \\
&- 8\,\mathcal{D}^\mu\big[\big(R_\mu{}^a(\omega,e) - \tfrac13 R(\omega,e)\,e_\mu{}^a - D\,e_\mu{}^a + \mathrm{i}R(A)_\mu{}^a\big)\mathcal{D}_a\bar A\big] \\
&+ \mathcal{D}_\mu\big[\varepsilon^{ij}\mathcal{D}^\mu T_{bcij}\,F^{+bc} + 4\,\varepsilon^{ij}T^{\mu b}{}_{ij}\,\mathcal{D}^c F^+_{cb} - 2\,T_{bc}{}^{ij}\,T^{\mu c}{}_{ij}\,\mathcal{D}^b\bar A\big] \\
&+ \cdots\,,
\end{aligned}
\tag{5.5}
$$

where we suppressed all fermionic contributions. In deriving this result we made use of (B.6). Subsequently we derive the bosonic part of the Lagrangian corresponding to (5.2), up to total derivatives, by making use of the density formula (2.18) and of the product rule (D.1),

$$
\begin{aligned}
e^{-1}\mathcal{L} = {}& 4\,\mathcal{D}^2A\,\mathcal{D}^2\bar A + 8\,\mathcal{D}^\mu A\,\big[R_\mu{}^a(\omega,e) - \tfrac13 R(\omega,e)\,e_\mu{}^a\big]\mathcal{D}_a\bar A + C\,\bar C \\
&- \mathcal{D}^\mu B_{ij}\,\mathcal{D}_\mu B^{ij} + (\tfrac16 R(\omega,e) + 2\,D)\,B_{ij}B^{ij} \\
&- \big[\varepsilon^{ik}\,B_{ij}\,F^{+\mu\nu}\,R(\mathcal{V})_{\mu\nu}{}^j{}_k + \varepsilon_{ik}\,B^{ij}\,F^{-\mu\nu}R(\mathcal{V})_{\mu\nu j}{}^k\big] \\
&- 8\,D\,\mathcal{D}^\mu A\,\mathcal{D}_\mu\bar A + \big(8\,\mathrm{i}R(A)_{\mu\nu} + 2\,T_\mu{}^{cij}\,T_{\nu cij}\big)\mathcal{D}^\mu A\,\mathcal{D}^\nu\bar A \\
&- \big[\varepsilon^{ij}\mathcal{D}^\mu T_{bcij}\mathcal{D}_\mu A\,F^{+bc} + \varepsilon_{ij}\mathcal{D}^\mu T_{bc}{}^{ij}\mathcal{D}_\mu\bar A\,F^{-bc}\big] \\
&- 4\big[\varepsilon^{ij}T^{\mu b}{}_{ij}\,\mathcal{D}_\mu A\,\mathcal{D}^c F^+_{cb} + \varepsilon_{ij}T^{\mu bij}\,\mathcal{D}_\mu\bar A\,\mathcal{D}^c F^-_{cb}\big] \\
&+ 8\,\mathcal{D}_aF^{-ab}\,\mathcal{D}^cF^+{}_{cb} + 4\,F^{-ac}\,F^+{}_{bc}\,R(\omega,e)_a{}^b + \tfrac14 T_{ab}{}^{ij}\,T_{cdij}F^{-ab}F^{+cd}\,.
\end{aligned}
\tag{5.6}
$$

Note that we suppressed the prime on the second chiral multiplet indicated in (5.2). In general, however, we will not always identify the two multiplets, so that the complex conjugated components in the above formula do not have to correspond to the same supermultiplet. However, upon making this identification, the above Lagrangian is manifestly real, which provides an additional check on the correctness of our result. The reason is that the corresponding Lagrangian (5.2) is also real in that case (up to total derivatives). Note also that the Lagrangian (5.6) vanishes whenever either one of the multiplets is equal to a constant, thus confirming the analysis presented at the beginning of this section.

We will now use the above results to write down the extension to local supersymmetry of the class of vector multiplet Lagrangians constructed in [79, 80]. Just as above we concentrate on the purely bosonic terms. The extension follows by writing the $w = 0$ chiral multiplets Φ and Φ' as composite multiplets expressed in terms of vector multiplets. In (5.2), and correspondingly in (5.6), one thus performs the following substitutions,

$$\Phi \to f(\Phi^\Lambda)\,, \qquad \bar{\Phi}' \to \bar{g}(\bar{\Phi}^\Lambda)\,, \tag{5.7}$$

where Φ^Λ denote the (reduced) chiral multiplets associated with vector multiplets, and the functions f and g are homogeneous of zeroth degree. Upon expanding Φ and $\bar{\Phi}'$ in terms of the vector supermultiplets, making use of the material presented in appendices D.1 and D.2, one obtains powers of the vector multiplet components multiplied by derivatives of $f(X)$ and $\bar{g}(\bar{X})$, where the X^Λ denote the complex scalars of the vector multiplets. Homogeneity implies that $X^\Lambda f_\Lambda(X) = 0 = \bar{X}^\Lambda \bar{g}_{\bar{\Lambda}}(\bar{X})$, where f_Λ and $\bar{g}_{\bar{\Lambda}}$ denote the first derivatives of the two functions with respect to X^Λ and $\bar{X}^\Lambda$, respectively. Here we recall that the expression (5.6) vanishes whenever $f(X)$ or $\bar{g}(\bar{X})$ are constant. As noted previously, the origin of this phenomenon can be traced back to the fact that the full superspace integral of a chiral or an anti-chiral field vanishes (up to total derivatives). Therefore the Lagrangian will depend exclusively on mixed holomorphic/anti-holomorphic derivatives of the product function $f(X)\,\bar{g}(\bar{X})$. By summing over an arbitrary set of pairs of functions $f^{(n)}(X)\,\bar{g}^{(n)}(\bar{X})$, we can further extend this function to a general function $\mathcal{H}(X,\bar{X})$ that is separately homogeneous of zeroth degree in X and $\bar{X}$. Because $\mathcal{H}(X,\bar{X})$ is only defined up to a purely holomorphic or anti-holomorphic function, it is thus subject to Kähler transformations,

$$\mathcal{H}(X,\bar{X}) \to \mathcal{H}(X,\bar{X}) + \Lambda(X) + \bar{\Lambda}(\bar{X})\,. \tag{5.8}$$

Hence $\mathcal{H}(X,\bar{X})$ can be regarded as a Kähler potential, which may be taken real (so that $\bar{\Lambda}(\bar{X}) = [\Lambda(X)]^*$).

Carrying out the various substitutions leads directly to the following bosonic contribution to the supersymmetric Lagrangian (for convenience, we assume $\mathcal{H}$ to be real, unless stated otherwise),

$$
\begin{aligned}
e^{-1}\mathcal{L} = &\,\mathcal{H}_{\Lambda\Sigma\bar\Gamma\bar\Xi}\Big[\tfrac14\big(\hat F_{ab}^{-\Lambda}\,\hat F^{-ab\,\Sigma} - \tfrac12 Y_{ij}{}^{\Lambda}\,Y^{ij\Sigma}\big)\big(\hat F_{ab}^{+\Gamma}\,\hat F^{+ab\,\Xi} - \tfrac12 Y^{ij\Gamma}\,Y_{ij}{}^{\Xi}\big)\\
&\qquad + 4\,\mathcal{D}_aX^\Lambda\,\mathcal{D}_b\bar X^\Gamma\big(\mathcal{D}^aX^\Sigma\,\mathcal{D}^b\bar X^\Xi + 2\,\hat F^{-\,ac\,\Sigma}\,\hat F^{+\,b}{}_c{}^{\Xi} - \tfrac14\eta^{ab}\,Y_{ij}^{\Sigma}\,Y^{\Xi\,ij}\big)\Big]\\
&+\Big\{\mathcal{H}_{\Lambda\Sigma\bar\Gamma}\Big[4\,\mathcal{D}_aX^\Lambda\,\mathcal{D}^aX^\Sigma\,\mathcal{D}^2\bar X^\Gamma - \mathcal{D}_aX^\Lambda\,Y_{ij}^{\Sigma}\,\mathcal{D}^aY^{\Gamma\,ij}\\
&\qquad - \big(\hat F^{-ab\,\Lambda}\,\hat F_{ab}^{-\,\Sigma} - \tfrac12 Y_{ij}^{\Lambda}\,Y^{\Sigma ij}\big)\big(\Box_{\rm c}X^\Gamma + \tfrac18\hat F_{ab}^{-\,\Gamma}\,T^{abij}\varepsilon_{ij}\big)\\
&\qquad + 8\,\mathcal{D}^aX^\Lambda\hat F_{ab}^{-\,\Sigma}\big(\mathcal{D}_c\hat F^{+\,cb\,\Gamma} - \tfrac12\mathcal{D}_c\bar X^\Gamma T^{ij\,cb}\varepsilon_{ij}\big)\Big] + \text{h.c.}\Big\}\\
&+\mathcal{H}_{\Lambda\bar\Sigma}\Big[4\big(\Box_{\rm c}\bar X^\Lambda + \tfrac18\hat F_{ab}^{+\,\Lambda}\,T^{ab}{}_{ij}\varepsilon^{ij}\big)\big(\Box_{\rm c}X^\Sigma + \tfrac18\hat F_{ab}^{-\,\Sigma}\,T^{abij}\varepsilon_{ij}\big) + 4\,\mathcal{D}^2X^\Lambda\,\mathcal{D}^2\bar X^\Sigma\\
&\qquad + 8\,\mathcal{D}_a\hat F^{-\,ab\Lambda}\,\mathcal{D}_c\hat F^{+c}{}_b{}^{\Sigma} - \mathcal{D}_aY_{ij}{}^{\Lambda}\,\mathcal{D}^aY^{ij\,\Sigma} + \tfrac14 T_{ab}{}^{ij}\,T_{cdij}\,\hat F^{-ab\,\Lambda}\hat F^{+cd\,\Sigma}\\
&\qquad + \big(\tfrac16R(\omega,e) + 2\,D\big)Y_{ij}{}^{\Lambda}\,Y^{ij\,\Sigma} + 4\,\hat F^{-ac\,\Lambda}\,\hat F^{+}{}_{bc}{}^{\Sigma}\,R(\omega,e)_a{}^b\\
&\qquad + 8\big(R^{\mu\nu}(\omega,e) + \tfrac14 T^{\mu}{}_b{}^{ij}\,T^{\nu b}{}_{ij} + \mathrm{i}R(A)^{\mu\nu}\big)\mathcal{D}_\mu X^\Lambda\,\mathcal{D}_\nu\bar X^\Sigma\\
&\qquad - 8\big(D + \tfrac13R(\omega,e)\big)\mathcal{D}_\mu X^\Lambda\,\mathcal{D}^\mu\bar X^\Sigma\\
&\qquad - \big[\mathcal{D}_c\bar X^\Sigma\big(\mathcal{D}^cT_{ab}{}^{ij}\,\hat F^{-\,\Lambda\,ab} + 4\,T^{ij\,cb}\,\mathcal{D}^a\hat F_{ab}^{-\,\Lambda}\big)\varepsilon_{ij} + [\text{h.c.};\Lambda\leftrightarrow\Sigma]\big]\\
&\qquad - \big[\varepsilon^{ik}\,Y_{ij}{}^{\Lambda}\,\hat F^{+ab\,\Sigma}\,R(\mathcal{V})_{ab}{}^j{}_k + [\text{h.c.};\Lambda\leftrightarrow\Sigma]\big]\Big]\,,
\end{aligned}
\tag{5.9}
$$

where (we suppress fermionic contributions),

$$
\begin{aligned}
\hat F_{ab}^{-\Lambda} &= \big(\delta_{ab}{}^{cd} - \tfrac12\varepsilon_{ab}{}^{cd}\big)e_c{}^\mu e_d{}^\nu\,\partial_{[\mu}W_{\nu]}{}^\Lambda - \tfrac14\,\bar X^\Lambda\,T_{ab}{}^{ij}\varepsilon_{ij}\,,\\
\Box_{\rm c}X^\Lambda &= \mathcal{D}^2X^\Lambda + \big(\tfrac16R(\omega,e) + D\big)\,X^\Lambda\,.
\end{aligned}
\tag{5.10}
$$

In view of the Kähler equivalence transformations (5.8), the mixed derivative $\mathcal{H}_{\Lambda\bar\Sigma}$ can be identified as a Kähler metric. Hence we have the following results for the metric, connection, and the curvature of the corresponding Kähler space,

$$
\begin{aligned}
g_{\Lambda\bar\Sigma} &= \mathcal{H}_{\Lambda\bar\Sigma}\,,\\
\Gamma^\Lambda{}_{\Sigma\Gamma} &= g^{\Lambda\bar\Xi}\,\mathcal{H}_{\Sigma\Gamma\bar\Xi}\,,\\
R_{\Lambda\bar\Sigma\Gamma\bar\Xi} &= \mathcal{H}_{\Lambda\Gamma\bar\Sigma\bar\Xi} - g_{\Pi\bar\Upsilon}\,\Gamma^\Pi{}_{\Lambda\Gamma}\,\Gamma^{\bar\Upsilon}{}_{\bar\Sigma\bar\Xi}\,.
\end{aligned}
\tag{5.11}
$$

The Lagrangian (5.9) can then be written in a Kähler covariant form,

$$
\begin{aligned}
e^{-1}\mathcal{L} = R_{\Lambda\bar\Gamma\Sigma\bar\Xi}\Big[&\tfrac{1}{4}\big(\hat F^{-\Lambda}_{ab}\,\hat F^{-ab\,\Sigma} - \tfrac12 Y_{ij}{}^{\Lambda}\,Y^{ij\Sigma}\big)\big(\hat F^{+\Gamma}_{ab}\,\hat F^{+ab\,\Xi} - \tfrac12 Y^{ij\Gamma}\,Y_{ij}{}^{\Xi}\big)\\
&+ 4\,\mathcal{D}_a X^\Lambda\,\mathcal{D}_b\bar X^\Gamma\big(\mathcal{D}^a X^\Sigma\,\mathcal{D}^b\bar X^\Xi + 2\,\hat F^{-\,ac\,\Sigma}\,\hat F^{+\,b}{}_c{}^{\Xi} - \tfrac14\eta^{ab}\,Y^{\Sigma}_{ij}\,Y^{\Xi\,ij}\big)\Big]\\
+ g_{\Lambda\bar\Sigma}\Big[&4\big(\Box_{\rm c}\bar X^\Lambda + \tfrac18\hat F^{+\Lambda}_{ab}\,T^{ab}{}_{ij}\varepsilon^{ij} - \tfrac14\Gamma^\Lambda{}_{\Gamma\Xi}(\hat F^{-\Gamma}_{ab}\,\hat F^{-ab\,\Xi} - \tfrac12 Y^{ij\Gamma}\,Y_{ij}{}^{\Xi})\big)\\
&\times\big(\Box_{\rm c}X^\Sigma + \tfrac18\hat F^{-\Sigma}_{ab}\,T^{abij}\varepsilon_{ij} - \tfrac14\Gamma^{\bar\Sigma}{}_{\bar\Gamma\bar\Xi}(\hat F^{+\Gamma}_{ab}\,\hat F^{+ab\,\Xi} - \tfrac12 Y^{ij\Gamma}\,Y_{ij}{}^{\Xi})\big)\\
&+ 4\,\big(\mathcal{D}^2 X^\Lambda + \Gamma^\Lambda{}_{\Gamma\Xi}\,\mathcal{D}_b X^\Gamma\,\mathcal{D}^b X^\Xi\big)\,\big(\mathcal{D}^2\bar X^\Sigma + \Gamma^{\bar\Sigma}{}_{\bar\Gamma\bar\Xi}\,\mathcal{D}_b\bar X^\Gamma\,\mathcal{D}^b\bar X^\Xi\big)\\
&+ 8\,\big(\mathcal{D}_a\hat F^{-\,ab\Lambda} + \Gamma^\Lambda{}_{\Gamma\Xi}\,\mathcal{D}_a X^\Gamma\,\hat F^{-\,ab\Xi}\big)\,\big(\mathcal{D}_c\hat F^{+c}{}_b{}^\Sigma + \Gamma^{\bar\Sigma}{}_{\bar\Gamma\bar\Xi}\,\mathcal{D}_c\bar X^\Gamma\,\hat F^{+c}{}_b{}^\Xi\big)\\
&- \big(\mathcal{D}_a Y_{ij}{}^\Lambda + \Gamma^\Lambda{}_{\Gamma\Xi}\,\mathcal{D}_b X^\Gamma\,Y_{ij}{}^\Xi\big)\,\big(\mathcal{D}^a Y^{\Sigma\,ij} + \Gamma^{\bar\Sigma}{}_{\bar\Gamma\bar\Xi}\,\mathcal{D}_b\bar X^\Gamma\,Y^{ij\Xi}\big)\\
&+ \tfrac14 T_{ab}{}^{ij}\,T_{cdij}\,\hat F^{-ab\,\Lambda}\,\hat F^{+cd\,\Sigma}\\
&+ \big(\tfrac16 R(\omega,e) + 2\,D\big)Y_{ij}{}^\Lambda\,Y^{ij\,\Sigma} + 4\,\hat F^{-ac\,\Lambda}\,\hat F^{+}{}_{bc}{}^\Sigma\,R(\omega,e)_a{}^b\\
&+ 8\big(R^{\mu\nu}(\omega,e) + \tfrac14 T^\mu{}_b{}^{ij}\,T^{\nu b}{}_{ij} + \mathrm{i}R(A)^{\mu\nu}\big)\mathcal{D}_\mu X^\Lambda\,\mathcal{D}_\nu\bar X^\Sigma\\
&- 8\big(D + \tfrac13 R(\omega,e)\big)\mathcal{D}_\mu X^\Lambda\,\mathcal{D}^\mu\bar X^\Sigma\\
&- \big[\mathcal{D}_c\bar X^\Sigma\big(\mathcal{D}^c T_{ab}{}^{ij}\,\hat F^{-\,\Lambda\,ab} + 4\,T^{ij\,cb}\,(\mathcal{D}^a\hat F^{-\Lambda}_{ab} + \Gamma^\Lambda{}_{\Gamma\Xi}\mathcal{D}^a X^\Gamma\,\hat F^{-\Xi}_{ab})\big)\varepsilon_{ij}\\
&\quad + \varepsilon^{ik}\,Y_{ij}{}^\Lambda\,\hat F^{+ab\,\Sigma}\,R(\mathcal{V})_{ab}{}^j{}_k + [\text{h.c.};\Lambda\leftrightarrow\Sigma]\big]\Big]\,. \qquad (5.12)
\end{aligned}
$$

The covariantizations in the various combinations can be understood systematically by rewriting the chiral multiplet components of the vector multiplets such that they are covariant with respect to the complex reparametrizations of the Kähler space (in the limit where the fermions are suppressed). An easy way to appreciate these covariantizations is by reorganizing the expansion of a composite chiral multiplet into vector multiplets according to (D.2) by replacing the ordinary derivatives of the function $\mathcal{G}$ by covariant derivatives.

The Lagrangians (5.9) and/or (5.12) can also be used in the context of rigidly supersymmetric theories upon suppressing all the superconformal fields. The resulting Lagrangian is then superconformally invariant in flat Minkowski space. This invariance can be further reduced to ordinary Poincaré supersymmetry by replacing one of the vector multiplets by a constant.

As an extension of the previous results we return to (5.6), and consider composite chiral multiplets that depend on both vector multiplets and on the Weyl multiplet. Hence we replace (5.7) by,

$$
\Phi \to f(\Phi^\Lambda, W^2)\,, \qquad \bar\Phi' \to \bar g(\bar\Phi^\Lambda, \bar W^2)\,, \qquad (5.13)
$$

where W^2 refers to the square of the Weyl multiplet. The components of this reduced chiral multiplet are given in (D.8). Upon expanding these functions and substituting the results into (5.6), one obtains a Lagrangian that contains R^4-, R^2F^2- and F^4-terms. All terms are proportional to mixed holomorphic/anti-holomorphic derivatives of a function $\mathcal{H}(X, T^2, \bar{X}, \bar{T}^2)$, where $T^2 = (T_{ab}{}^{ij}\varepsilon_{ij})^2$ and $\bar{T}^2 = (T_{abij}\varepsilon^{ij})^2$, and where $\mathcal{H}$ is constructed from pairs of products of functions $f(X, T^2)$ and $\bar{g}(\bar{X}, \bar{T}^2)$. The fact that the composite multiplets have $w = 0$ implies a modified homogeneity property,

$$X^\Lambda \mathcal{H}_\Lambda(X, T^2, \bar{X}, \bar{T}^2) + 2\, T^2 \mathcal{H}_{T^2}(X, T^2, \bar{X}, \bar{T}^2) = 0\,, \qquad (5.14)$$

and likewise for the anti-holomorphic derivatives.

The Lagrangian consists of the Lagrangian (5.9) plus a large number of terms that involve multiple derivatives of $\mathcal{H}$ with respect to T^2, $\bar{T}^2$, X^Λ and $\bar{X}^\Lambda$. Below we concentrate on terms proportional to multiple derivatives of $\mathcal{H}$ with respect to only T^2 and $\bar{T}^2$. Among others those contain contributions of fourth order in $\mathcal{R}(M)$, whose leading contribution is equal to the Weyl tensor,

$$\begin{aligned}
&(64)^{-2}\, e^{-1}\mathcal{L} = \\
&\quad 4\,\mathcal{H}_{T^2T^2\bar{T}^2\bar{T}^2}\; T^{abij}\varepsilon_{ij}\, T^{cdkl}\varepsilon_{kl}\, T^{ef}{}_{mn}\varepsilon^{mn}\, T^{gh}{}_{pq}\varepsilon^{pq} \\
&\qquad \times \big[\mathcal{R}(M)_{aba'b'}\mathcal{R}(M)_{cd}{}^{a'b'} + \tfrac{1}{2}R(\mathcal{V})_{ab}{}^i{}_j\, R(\mathcal{V})_{cd}{}^j{}_i\big] \\
&\qquad \times \big[\mathcal{R}(M)_{efe'f'}\mathcal{R}(M)_{gh}{}^{e'f'} + \tfrac{1}{2}R(\mathcal{V})_{ef}{}^i{}_j\, R(\mathcal{V})_{gh}{}^j{}_i\big] \\
&\quad + 2\,\Big\{\mathcal{H}_{T^2T^2\bar{T}^2}\; T^{abij}\varepsilon_{ij}\, T^{cdkl}\varepsilon_{kl} \\
&\qquad \times \big[\mathcal{R}(M)_{aba'b'}\mathcal{R}(M)_{cd}{}^{a'b'} + \tfrac{1}{2}R(\mathcal{V})_{ab}{}^i{}_j\, R(\mathcal{V})_{cd}{}^j{}_i\big] \\
&\qquad \times \big[\mathcal{R}(M)^+_{efgh}\mathcal{R}(M)^{+efgh} + \tfrac{1}{2}R(\mathcal{V})^+_{ef}{}^i{}_j\, R(\mathcal{V})^{+efj}{}_i - \tfrac{1}{2}T^{ef}{}_{mn}D_eD^hT_{hf}{}^{mn}\big] + [\text{h.c.}]\Big\} \\
&\quad + \mathcal{H}_{T^2\bar{T}^2}\Big\{\big|\mathcal{R}(M)^+_{abcd}\mathcal{R}(M)^{+abcd} + \tfrac{1}{2}R(\mathcal{V})^+_{ab}{}^i{}_j\, R(\mathcal{V})^{+abj}{}_i \\
&\qquad\qquad - \tfrac{1}{2}T^{ab}{}_{mn}D_aD^eT_{eb}{}^{mn}\big|^2 + \cdots\Big\}\,.
\end{aligned} \qquad (5.15)$$

Besides the terms quartic in $\mathcal{R}(M)$ we have retained some of the terms that come with them as part of the basic building blocks that emerge in the calculation (similar blocks appear in (5.9)). Besides giving a little more information in this way, this has the advantage that the origin of the various term will be easier to track down.

In addition to the above terms there are mixed terms which lead to explicit contributions from the vector multiplets (i.e. beyond the X and $\bar{X}$ dependence in the function

$\mathcal{H}$). Those include, for instance, terms proportional to $[\mathcal{R}(M)]^2$ times the product of two vector multiplet field strengths, $\hat{F}_{\mu\nu}{}^{\Lambda}$. We will not exhibit those terms here (they can in principle be deduced from (5.6) along the same lines as for the previous contributions). Some of these terms will be shown in the equation below.

A special case, which is worth mentioning in view of the work of [85, 86], corresponds to functions $\mathcal{H}(X, T^2, \bar{X})$ that do not depend on $\bar{T}^2$. Hence the function $\mathcal{H}$ is not real. Again we do not present all the terms, but we give all the terms that contain $\mathcal{R}(M)$ (with some completions), with the exception of terms proportional to derivatives of X^{Λ} and $T_{ab}{}^{ij}$ or their complex conjugates,

$$
\begin{aligned}
&(64)^{-1} e^{-1}\mathcal{L} = \\
&\quad \mathcal{H}_{T^2T^2\bar{\Gamma}\bar{\Xi}}\Big\{T^{abij}\varepsilon_{ij}\,T^{cdkl}\varepsilon_{kl}\,\big[\mathcal{R}(M)_{aba'b'}\mathcal{R}(M)_{cd}{}^{a'b'} + \tfrac{1}{2}R(\mathcal{V})_{ab}{}^{i}{}_{j}\,R(\mathcal{V})_{cd}{}^{j}{}_{i}\big] \\
&\qquad\qquad \times\big[\hat{F}^{+\Gamma}_{ef}\hat{F}^{+ef\Xi} - \tfrac{1}{2}Y^{mn\Gamma}Y_{mn}{}^{\Xi}\big] + \cdots\Big\} \\
&\quad - 4\,\mathcal{H}_{T^2T^2\bar{\Gamma}}\Big\{T^{abij}\varepsilon_{ij}\,T^{cdkl}\varepsilon_{kl}\,\big[\mathcal{R}(M)_{aba'b'}\mathcal{R}(M)_{cd}{}^{a'b'} + \tfrac{1}{2}R(\mathcal{V})_{ab}{}^{i}{}_{j}\,R(\mathcal{V})_{cd}{}^{j}{}_{i}\big] \\
&\qquad\qquad \times\big[\Box_{\mathrm{c}}X^{\Gamma} + \tfrac{1}{8}\hat{F}_{ef}{}^{\Gamma}T^{efij}\varepsilon_{ij}\big] + \cdots\Big\} \\
&\quad + \tfrac{1}{2}\mathcal{H}_{T^2\Lambda\bar{\Gamma}}\Big\{T^{cdlm}\varepsilon_{lm}\big[\hat{F}^{-\Lambda}_{ab}\mathcal{R}(M)_{cd}{}^{ab} - \tfrac{1}{2}Y^{ij\Lambda}\varepsilon_{ki}\,R(\mathcal{V})_{cd}{}^{k}{}_{j}\big] \\
&\qquad\qquad \times\big[\Box_{\mathrm{c}}X^{\Gamma} + \tfrac{1}{8}\hat{F}_{ef}{}^{\Gamma}T^{efij}\varepsilon_{ij}\big] + \cdots\Big\} \\
&\quad - \tfrac{1}{8}\mathcal{H}_{T^2\Lambda\bar{\Gamma}\bar{\Xi}}\Big\{T^{cdlm}\varepsilon_{lm}\big[\hat{F}_{ab}{}^{\Lambda}\mathcal{R}(M)_{cd}{}^{ab} - \tfrac{1}{2}Y^{ij\Lambda}\varepsilon_{ki}\,R(\mathcal{V})_{cd}{}^{k}{}_{j}\big] \\
&\qquad\qquad \times\big[\hat{F}^{+\Gamma}_{ab}\hat{F}^{+ab\Xi} - \tfrac{1}{2}Y^{ij\Gamma}Y_{ij}{}^{\Xi}\big] + \cdots\Big\} \\
&\quad + \tfrac{1}{2}\mathcal{H}_{T^2\bar{\Gamma}\bar{\Xi}}\Big\{\big[\mathcal{R}(M)^{-}_{cdef}\mathcal{R}(M)^{-cdef} + \tfrac{1}{2}R(\mathcal{V})^{-}_{cd}{}^{i}{}_{j}\,R(\mathcal{V})^{-cdj}{}_{i} - \tfrac{1}{2}T^{cdmn}D_cD^eT_{edmn}\big] \\
&\qquad\qquad \times\big[\hat{F}^{+\Gamma}_{ab}\hat{F}^{+ab\Xi} - \tfrac{1}{2}Y^{ij\Gamma}Y_{ij}{}^{\Xi}\big] + \cdots\Big\} \\
&\quad - 2\,\mathcal{H}_{T^2\bar{\Gamma}}\Big\{\big[\mathcal{R}(M)^{-}_{abcd}\mathcal{R}(M)^{-abcd} + \tfrac{1}{2}R(\mathcal{V})^{-}_{ab}{}^{i}{}_{j}\,R(\mathcal{V})^{-abj}{}_{i} - \tfrac{1}{2}T^{abmn}D_aD^cT_{cbmn}\big] \\
&\qquad\qquad \times\big[\Box_{\mathrm{c}}X^{\Gamma} + \tfrac{1}{8}\hat{F}_{ef}{}^{\Gamma}T^{efij}\varepsilon_{ij}\big] \\
&\qquad\quad + \big[\tfrac{1}{32}T_{ab}{}^{kl}T_{efkl}\,\hat{F}^{ef\Gamma} + \tfrac{1}{2}\hat{F}^{+\Gamma}_{eb}\,R(\omega,e)_{a}{}^{e} - \tfrac{1}{8}\varepsilon_{km}\,Y^{kl\Gamma}R(\mathcal{V})_{abl}{}^{m}\big] \\
&\qquad\qquad \times T_{cd}{}^{ij}\varepsilon_{ij}\,\mathcal{R}(M)^{cdab} \\
&\qquad\quad + T_{cd}{}^{ij}\varepsilon_{ij}\,\mathcal{D}_a\mathcal{R}(M)^{cdab}\,\mathcal{D}^e\hat{F}^{+\Gamma}_{eb} + \cdots\Big\}\,.
\end{aligned}
\tag{5.16}
$$

5.3 A non-renormalization theorem for BPS black hole entropy

The results of this chapter can be used in the study of black holes. Based on any linear combination of the various $N=2$ locally supersymmetric Lagrangians, one can evaluate the corresponding expressions for the Wald entropy and the electric charges in terms of the values of the fields taken at the black hole horizon. In the case of BPS black holes, the horizon values of the fields are highly restricted due to full supersymmetry enhancement at the horizon, and therefore the resulting expressions for the entropy and the charges will simplify. To explore this one must determine the possible supersymmetric field configurations, preferably in an off-shell formulation so that the results do not depend on the specific Lagrangian. This has already been done in [17], which provided a generalization of the attractor equations found in [88–90]. So far, generic chiral supermultiplets were not considered, but it is convenient to do so as well. As it will turn out, it suffices to restrict oneself to chiral multiplets of Weyl weight $w = 0$, for which results are rather straightforward to obtain.

The first relevant observation is that a constant chiral superfield (i.e. a supermultiplet with constant A and all other components vanishing) is only supersymmetric provided it has $w = 0$. In fact there exist no other supersymmetric values of the chiral superfield. All this can be derived directly from the transformation rules (2.17). The second observation is that the kinetic multiplet constructed from a $w = 0$ anti-chiral multiplet, vanishes when the latter multiplet is equal to a constant. This follows by inspection of (5.3). These two observations prove immediately that any invariant proportional to a kinetic multiplet, must vanish for supersymmetric field configurations. This fact can immediately be verified from (5.6), because when the fields A and $\bar{A}'$ are constant and all other chiral multiplet component fields are vanishing, the expression (5.6) indeed vanishes.

The above result is interesting in its own right, but we are also interested in the first-order variation of the action induced by a change of some of the fields, evaluated for a supersymmetric background. Given the fact that all the invariants discussed in this chapter will contain at least one kinetic multiplet, we thus consider,

$$\delta\mathcal{L} \propto \int \mathrm{d}^4\theta \left[\delta\Phi\,\mathbb{T}(\bar{\Phi}') + \Phi\,\delta\mathbb{T}(\bar{\Phi}')\right], \qquad (5.17)$$

where Φ and Φ' are composite chiral fields, which are themselves expressed in various chiral fields, including possible kinetic multiplets. They are not necessarily uniquely defined, and it is also possible to consider linear combinations of such terms. Since we will be evaluating the variation at supersymmetric values of the fields, the first term in (5.17) vanishes, because the kinetic multiplet vanishes, whereas the second term can be evaluated for constant Φ.

However, rather than continuing in this way, we may simply return to (5.6) and consider its variation. Observe that each term is proportional to a product of one component of Φ and another one of $\bar{\Phi}'$ (we remind the reader that in (5.6) we suppressed the prime for notational clarity). All these components will be equal to zero in a supersymmetric background, with the exception of A and $\bar{A}'$, which will take constant values. However, only space-time derivatives of A and $\bar{A}'$ appear, and those will vanish as well. In other words, (5.6) is always quadratic in quantities that are vanishing in the supersymmetry limit. Hence any first-order variation of any Lagrangian of this type must necessarily vanish in a supersymmetric background!

The above result suffices to derive a non-renormalization theorem for electric charges and the Wald entropy [91–93] for BPS black holes. The reason is that these quantities are always expressed in terms of first-order derivatives of the Lagrangian with respect to certain fields, such as the abelian field strengths or the Riemann tensor, or possible derivatives thereof. This concludes the proof of the non-renormalization theorem.

The existence of this non-renormalization theorem is a welcome result. So far good agreement has been established for BPS black hole entropy evaluated on the basis of supergravity and of microstate counting, suggesting that other invariants in supergravity should contribute only marginally, or perhaps not at all, at the subleading level. The result of this section lends support to this idea. Nevertheless the possible existence of alternative supersymmetric invariants that do not belong to the class of invariants discussed in this chapter, cannot be excluded at this stage.

5.4 An infinite hierarchy of higher-derivative invariants

In this chapter we studied a large class of $N = 2$ superconformal invariants involving higher-derivative couplings, based on full superspace integrals. For a special subclass we have presented explicit results for some of the bosonic terms. This is the subclass that contains only a single kinetic multiplet.

As indicated already, there are further options. The most obvious one is to include more kinetic multiplets, based on various composite chiral and anti-chiral multiplets with suitable Weyl weights,

$$\int \mathrm{d}^4\theta\ \Phi_0\, \mathbb{T}(\bar\Phi_1)\cdots\mathbb{T}(\bar\Phi_n)\,, \tag{5.18}$$

where $\bar\Phi_1,\ldots\bar\Phi_n$ are anti-chiral superfields of zero weight and Φ_0 is a chiral superfield of weight $w=-2(n-1)$. This leads to actions that contain four space-time derivatives. However, when treating the chiral multiplets as composites of reduced chiral multiplets, one obtains invariants with terms of $2(1+n)$ powers of field strengths and/or explicit derivatives, i.e. $R^{2m}F^{2p}\mathcal{D}^{2(n+1-m-p)}$. The case of $n=1$ has been dealt with in considerable detail in section 5.2. The expression of the composite chiral multiplets in terms of the reduced ones allows again for the presence of functions $\mathcal{H}^{(n)}$ which are subject to a generalized version of the Kähler transformations noted in section 5.2.

As alluded to before, one can also consider nested situations where the kinetic multiplet is constructed from a combination of (anti)chiral fields that include again other kinetic multiplets. In this way one constructs multiplets with multiple derivatives of arbitrary power. We are then led to introduce quantities of the type,

$$\mathbb{T}^{(2)}=\mathbb{T}(\bar\Phi_2\,\mathbb{T}(\Phi_1))\,,\quad \mathbb{T}^{(3)}=\mathbb{T}(\bar\Phi_3\,\mathbb{T}(\Phi_2\,\mathbb{T}(\bar\Phi_1)))\,,\ \ldots,\ \mathbb{T}^{(n)}=\mathbb{T}(\bar\Phi_n\,\mathbb{T}^{(n-1)})\,, \tag{5.19}$$

which can be part of any superspace integrand, on the same footing as the kinetic multiplets in (5.18). Here Φ_1 has $w=0$ and $\Phi_2,\Phi_3,\cdots$ have $w=-2$. This extends the number of invariants to all possible combinations of the form,

$$\int \mathrm{d}^4\theta\, \Phi_0\, \mathbb{T}^{(n_1)}\,\mathbb{T}^{(n_2)}\ldots\ \mathbb{T}^{(n_k)}\,, \tag{5.20}$$

where Φ_0 has $w=-2(k-1)$ and where we assume $n_k\geq 1$ with $\mathbb{T}(\bar\Phi_1)\equiv\mathbb{T}^{(1)}$. When expressing all the chiral multiplets in terms of reduced ones, then one can show that the maximal number of derivatives of the invariants (5.20) is equal to $2(1+\sum_k n_k)$.

These types of invariants are not necessarily independent in the sense that there can be linear combinations that are equal to a total derivative. For example, at the six-derivative level, one has,

$$\int \mathrm{d}^4\theta\, \Phi_0\, \mathbb{T}(\bar\Phi_2\,\mathbb{T}(\Phi_1)) \approx \int \mathrm{d}^4\bar\theta\, \bar\Phi_2\mathbb{T}(\Phi_0)\,\mathbb{T}(\Phi_1)\,, \tag{5.21}$$

up to total derivatives. Nevertheless it is clear that we are dealing with an infinite hierarchy of higher-derivative invariants.

Of course, a relevant question is whether the invariant couplings presented in this chapter exhaust the possible higher-derivative invariants. Most likely, this will not be the case. From the perspective of BPS black holes the question would then remain whether these conjectured couplings could still contribute to the entropy and electric charges.

Appendix **A**

Conventions and useful identities

Throughout this thesis we use Pauli-Källén conventions and follow the notation used e.g. in [17]. Space-time and Lorentz indices are denoted by $\mu, \nu, \ldots$, and $a, b, \ldots$, respectively, and our space-time metric has signature $-+++$. Our (anti-)symmetrizations are always defined with unit strength. The completely antisymmetric tensor satisfies,

$$\varepsilon^{abcd} = e^{-1} \varepsilon^{\mu\nu\rho\sigma} e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d \,, \qquad \varepsilon^{0123} = i \,. \tag{A.1}$$

The selfdual and anti-selfdual part of an antisymmetric tensor F_{ab} are defined by,

$$F^\pm_{ab} = \tfrac{1}{2}(F_{ab} \pm \tilde{F}_{ab}) \,, \tag{A.2}$$

where,

$$\tilde{F}_{ab} = \tfrac{1}{2} \varepsilon_{abcd} F^{cd} \,. \tag{A.3}$$

Notice that under complex conjugation, the selfdual tensor becomes anti-selfdual and vice versa. We note the following useful identities for products of (anti)selfdual tensors,

$$\begin{aligned}
G^\pm_{[a[c}\, H^\pm_{d]b]} &= \pm\, \tfrac{1}{8} G^\pm_{ef}\, H^{\pm ef}\, \varepsilon_{abcd} - \tfrac{1}{4}(G^\pm_{ab}\, H^\pm_{cd} + G^\pm_{cd}\, H^\pm_{ab}) \,, \\
G^\pm_{ab}\, H^{\mp cd} + G^{\pm cd}\, H^\mp_{ab} &= 4 \delta^{[c}_{[a} G^\pm_{b]e}\, H^{\mp d]e} \,, \\
\tfrac{1}{2} \varepsilon^{abcd}\, G^{\pm e}_{[c}\, H^\pm_{d]e} &= \pm\, G^{\pm[a}{}_e\, H^{\pm b]e} \,, \\
G^{\pm ac}\, H^{\pm b}_c + G^{\pm bc}\, H^{\pm a}_c &= -\, \tfrac{1}{2} \eta^{ab}\, G^{\pm cd}\, H^\pm_{cd} \,,
\end{aligned}$$

$$\begin{aligned} G^{\pm ac}\, H_c^{\mp b} &= G^{\pm bc}\, H_c^{\mp a}\,, \\ G^{\pm ab}\, H^{\mp}_{ab} &= 0\,. \end{aligned} \tag{A.4}$$

SU(2)-indices are denoted by $i, j, \ldots$ and under complex conjugation the indices are raised or lowered. For example,

$$(T_{abij})^* = T_{ab}{}^{ij}\,. \tag{A.5}$$

We make use of the two-dimensional completely anti-symmetric tensor ε^{ij}, which satisfies $\varepsilon^{12} = 1$.

Our gamma-matrices γ^a are unitary and satisfy,

$$\gamma_a\gamma_b = \eta_{ab} + \gamma_{ab}\,, \qquad \gamma_5 = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3\,. \tag{A.6}$$

We note the following useful identities involving gamma matrices,

$$\begin{aligned} \gamma_{ab} &= -\tfrac{1}{2}\varepsilon_{abcd}\gamma^{cd}\gamma_5\,, & \gamma^b\gamma_a\gamma_b &= -2\gamma_a\,, \\ \gamma_{ab}\gamma^{ab} &= -12\,, & \gamma^{cd}\gamma_{ab}\gamma_{cd} &= 4\,\gamma_{ab}\,, \\ \gamma^c\gamma_{ab}\gamma_c &= 0\,, & \gamma^{ab}\gamma_c\gamma_{ab} &= 0\,, \\ [\gamma^c, \gamma_{ab}] &= 4\,\delta_{[a}{}^c\gamma_{b]}\,, & \{\gamma^c, \gamma_{ab}\} &= 2\,\varepsilon_{ab}{}^{cd}\gamma_5\gamma_d\,, \\ [\gamma_{ab}, \gamma^{cd}] &= -8\,\delta_{[a}{}^{[c}\gamma_{b]}{}^{d]}\,, & \{\gamma_{ab}, \gamma^{cd}\} &= -4\,\delta_{[a}{}^c\delta_{b]}{}^d + 2\,\varepsilon_{ab}{}^{cd}\gamma_5\,. \end{aligned} \tag{A.7}$$

We use a charge conjugation matrix C that satisfies,

$$\begin{aligned} C^\dagger &= C^{-1}\,, & C\gamma_5C^{-1} &= \gamma_5^T\,, \\ C^T &= -C\,, & C\gamma_\mu C^{-1} &= -\gamma_\mu^T\,. \end{aligned} \tag{A.8}$$

A Majorana spinor ψ is defined by,

$$\bar\psi = \psi^T C\,, \tag{A.9}$$

where $\bar\psi = \psi^\dagger\gamma_0$ is the Dirac conjugate. Two spinors that do not form a bilinear can be decomposed as a linear combination of bilinears by a Fierz rearrangement,

$$\phi\,\bar\psi = -\tfrac{1}{4}(\bar\psi\phi) - \tfrac{1}{4}(\bar\psi\gamma^a\phi)\gamma_a - \tfrac{1}{4}(\bar\psi\gamma_5\phi)\gamma_5 + \tfrac{1}{4}(\bar\psi\gamma^a\gamma_5\phi)\gamma_a\gamma_5 + \tfrac{1}{8}(\bar\psi\gamma^{ab}\phi)\gamma_{ab}\,. \tag{A.10}$$

We will give some more details about the chiral spinor notation introduced in section 2.2 and used throughout this thesis. Suppose we have two Majorana spinors $\psi^i_{\rm M}$, with $i=1,2$. We decompose $\psi^i_{\rm M}$ into lefthanded and righthanded spinors, which consequently are no longer Majorana spinors themselves,

$$\psi^i_{\rm L} = \tfrac{1}{2}(1+\gamma_5)\psi^i_{\rm M}\,, \qquad \psi^i_{\rm R} = \tfrac{1}{2}(1-\gamma_5)\psi^i_{\rm M}\,. \tag{A.11}$$

The original Majorana spinor is by definition invariant under charge conjugation, defined by $\psi^c \equiv C\,\bar\psi^T$. However, one can show that the left- and righthanded spinors transform into each other under charge conjugation,

$$(\psi^i_{\rm L})^c = \psi^i_{\rm R}\,, \qquad (\psi^i_{\rm R})^c = \psi^i_{\rm L}\,. \tag{A.12}$$

Hence it should be clear that the left- and righthanded fields transform in conjugate representations $\mathbf{2}$ and $\bar{\mathbf{2}}$ of U(2). We therefore change notation such that $\psi^i = \psi^i_{\rm L}$ and $\psi_i = \psi^i_{\rm R}$ (or the other way around, this is just a matter of definition, and must be specified for each fermion separately, see e.g. table B.1) with the upper index transforming in the $\mathbf{2}$ and the lower index in the $\bar{\mathbf{2}}$ representation. Notice that this is consistent with the property that SU(2)-indices are raised or lowered under complex conjugation. For completeness we note that if $\gamma_5\,\psi^i = \psi^i$, i.e. if ψ^i has positive chirality, then $\bar\psi^i\,\gamma_5 = \bar\psi^i$, and similarly for negative chirality. Now we can easily proof the following identities for spinors ψ^i and ϕ^j of equal chirality,

$$\begin{aligned} \bar\psi^i\phi_j &= 0\,, & \bar\psi^i\gamma_\mu\phi^j &= 0\,,\\ \bar\psi^i\phi^j &= \bar\phi^j\psi^i\,, & \big(\bar\psi^i\phi^j\big)^* &= \bar\psi_i\phi_j\,,\\ \bar\psi^i\gamma_\mu\phi_j &= -\,\bar\phi_j\gamma_\mu\psi^i\,, & \big(\bar\psi^i\gamma_\mu\phi_j\big)^* &= \bar\psi_i\gamma_\mu\phi^j\,, \end{aligned} \tag{A.13}$$

and so on for other bilinears. Also the Fierz rearrangement (A.10) simplifies on spinors of a definite chirality, for instance,

$$\begin{aligned} (\bar\chi^k\phi^i)\,\bar\psi^j &= -\,\tfrac{1}{2}(\bar\psi^j\phi^i)\bar\chi^k + \tfrac{1}{8}(\bar\psi^j\gamma^{ab}\phi^i)\bar\chi^k\gamma_{ab}\,,\\ (\bar\chi^k\phi^i)\,\bar\psi_j &= -\,\tfrac{1}{2}(\bar\psi^j\gamma^a\phi_j)\bar\chi^k\gamma_a\,. \end{aligned} \tag{A.14}$$

Appendix B

Superconformal gravity

In this appendix we present the transformation rules of the superconformal fields and their relation to the superconformal algebra, as well as their covariant quantities contained in the so-called Weyl supermultiplet. The superconformal algebra comprises the generators of the general-coordinate, local Lorentz, dilatation, special conformal, chiral SU(2) and U(1), supersymmetry (Q) and special supersymmetry (S) transformations. The gauge fields associated with general-coordinate transformations (${e_\mu}^a$), dilatations (b_μ), chiral symmetry (${\mathcal{V}_\mu}^i{}_j$ and A_μ) and Q-supersymmetry (${\psi_\mu}^i$) are independent fields. The remaining gauge fields associated with the Lorentz (${\omega_\mu}^{ab}$), special conformal (${f_\mu}^a$) and S-supersymmetry transformations (${\phi_\mu}^i$) are dependent fields. They are composite objects, which depend on the independent fields of the multiplet [4, 27, 28]. The corresponding supercovariant curvatures and covariant fields are contained in a tensor chiral multiplet, which comprises 24 + 24 off-shell degrees of freedom. In addition to the independent superconformal gauge fields, it contains three other fields: a Majorana spinor doublet χ^i, a scalar D, and a selfdual Lorentz tensor T_{abij}, which is anti-symmetric in $[ab]$ and $[ij]$. The Weyl and chiral weights have been collected in table B.1.

Under Q-supersymmetry, S-supersymmetry and special conformal transformations the independent fields of the Weyl multiplet transform as follows,

$$\begin{aligned}
\delta {e_\mu}^a &= \bar\epsilon^i\,\gamma^a\psi_{\mu i} + \bar\epsilon_i\,\gamma^a{\psi_\mu}^i\,,\\
\delta {\psi_\mu}^i &= 2\,\mathcal{D}_\mu\epsilon^i - \tfrac{1}{8}{T_{ab}}^{ij}\gamma^{ab}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i\,,\\
\delta b_\mu &= \tfrac{1}{2}\bar\epsilon^i\phi_{\mu i} - \tfrac{3}{4}\bar\epsilon^i\gamma_\mu\chi_i - \tfrac{1}{2}\bar\eta^i\psi_{\mu i} + \text{h.c.} + \Lambda^a_K e_{\mu a}\,,
\end{aligned}$$

	Weyl multiplet											parameters	
	$e_\mu{}^a$	$\psi_\mu{}^i$	b_μ	A_μ	$\mathcal{V}_\mu{}^i{}_j$	$T_{ab}{}^{ij}$	χ^i	D	ω_μ^{ab}	$f_\mu{}^a$	$\phi_\mu{}^i$	ϵ^i	η^i
w	-1	$-\frac{1}{2}$	0	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
c	0	$-\frac{1}{2}$	0	0	0	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
γ_5		$+$					$+$				$-$	$+$	$-$

Table B.1: Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the Weyl multiplet component fields and the supersymmetry transformation parameters.

$$\begin{aligned}
\delta A_\mu &= \tfrac{1}{2}\mathrm{i}\bar\epsilon^i\phi_{\mu i} + \tfrac{3}{4}\mathrm{i}\bar\epsilon^i\gamma_\mu\,\chi_i + \tfrac{1}{2}\mathrm{i}\bar\eta^i\psi_{\mu i} + \text{h.c.}\,,\\
\delta\mathcal{V}_\mu{}^i{}_j &= 2\,\bar\epsilon_j\phi_\mu{}^i - 3\bar\epsilon_j\gamma_\mu\,\chi^i + 2\bar\eta_j\,\psi_\mu{}^i - (\text{h.c.}\ ;\ \text{traceless})\,,\\
\delta T_{ab}{}^{ij} &= 8\,\bar\epsilon^{[i}R(Q)_{ab}{}^{j]}\,,\\
\delta\chi^i &= -\tfrac{1}{12}\gamma^{ab}\,\not{D}T_{ab}{}^{ij}\,\epsilon_j + \tfrac{1}{6}R(\mathcal{V})_{\mu\nu}{}^i{}_j\gamma^{\mu\nu}\epsilon^j - \tfrac{1}{3}\mathrm{i}R_{\mu\nu}(A)\gamma^{\mu\nu}\epsilon^i + D\epsilon^i\\
&\quad + \tfrac{1}{12}\gamma_{ab}T^{abij}\eta_j\,,\\
\delta D &= \bar\epsilon^i\,\not{D}\chi_i + \bar\epsilon_i\,\not{D}\chi^i\,.
\end{aligned}\tag{B.1}$$

Here ϵ^i and ϵ_i denote the spinorial parameters of Q-supersymmetry, η^i and η_i those of S-supersymmetry, and $\Lambda_{\mathrm{K}}{}^a$ is the transformation parameter for special conformal boosts. The full superconformally covariant derivative is denoted by D_μ, while $\mathcal{D}_\mu$ denotes a covariant derivative with respect to Lorentz, dilatation, chiral U(1), and SU(2) transformations,

$$\mathcal{D}_\mu\epsilon^i = \big(\partial_\mu - \tfrac{1}{4}\omega_\mu{}^{cd}\,\gamma_{cd} + \tfrac{1}{2}\,b_\mu + \tfrac{1}{2}\mathrm{i}\,A_\mu\big)\epsilon^i + \tfrac{1}{2}\,\mathcal{V}_\mu{}^i{}_j\,\epsilon^j\,. \tag{B.2}$$

The covariant curvatures of the various gauge symmetries take the following form,

$$\begin{aligned}
R(P)_{\mu\nu}{}^a &= 2\,\partial_{[\mu}\,e_{\nu]}{}^a + 2\,b_{[\mu}\,e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\,e_{\nu]b} - \tfrac{1}{2}(\bar\psi_{[\mu}{}^i\gamma^a\psi_{\nu]i} + \text{h.c.})\,,\\
R(Q)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\psi_{\nu]}{}^i - \gamma_{[\mu}\phi_{\nu]}{}^i - \tfrac{1}{8}\,T^{abij}\,\gamma_{ab}\,\gamma_{[\mu}\psi_{\nu]j}\,,\\
R(A)_{\mu\nu} &= 2\,\partial_{[\mu}A_{\nu]} - \mathrm{i}\,\big(\tfrac{1}{2}\bar\psi_{[\mu}{}^i\phi_{\nu]i} + \tfrac{3}{4}\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_i - \text{h.c.}\big)\,,\\
R(\mathcal{V})_{\mu\nu}{}^i{}_j &= 2\,\partial_{[\mu}\mathcal{V}_{\nu]}{}^i{}_j + \mathcal{V}_{[\mu}{}^i{}_k\,\mathcal{V}_{\nu]}{}^k{}_j + 2(\bar\psi_{[\mu}{}^i\,\phi_{\nu]j} - \bar\psi_{[\mu j}\,\phi_{\nu]}{}^i) - 3(\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_j - \bar\psi_{[\mu j}\gamma_{\nu]}\chi^i)\\
&\quad - \delta_j{}^i(\bar\psi_{[\mu}{}^k\,\phi_{\nu]k} - \bar\psi_{[\mu k}\,\phi_{\nu]}{}^k) + \tfrac{3}{2}\delta_j{}^i(\bar\psi_{[\mu}{}^k\gamma_{\nu]}\chi_k - \bar\psi_{[\mu k}\gamma_{\nu]}\chi^k)\,,\\
R(M)_{\mu\nu}{}^{ab} &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 4f_{[\mu}{}^{[a}e_{\nu]}{}^{b]} + \tfrac{1}{2}(\bar\psi_{[\mu}{}^i\,\gamma^{ab}\,\phi_{\nu]i} + \text{h.c.})\\
&\quad + (\tfrac{1}{4}\bar\psi_\mu{}^i\,\psi_\nu{}^j\,T^{ab}{}_{ij} - \tfrac{3}{4}\bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,\gamma^{ab}\chi_i - \bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,R(Q)^{ab}{}_i + \text{h.c.})\,,
\end{aligned}$$

$$
\begin{aligned}
R(D)_{\mu\nu} &= 2\,\partial_{[\mu}b_{\nu]} - 2f_{[\mu}{}^a e_{\nu]a} - \tfrac{1}{2}\bar\psi_{[\mu}{}^i\phi_{\nu]i} + \tfrac{3}{4}\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_i - \tfrac{1}{2}\bar\psi_{[\mu i}\phi_{\nu]}{}^i + \tfrac{3}{4}\bar\psi_{[\mu i}\gamma_{\nu]}\chi^i\,,\\
R(S)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\phi_{\nu]}{}^i - 2f_{[\mu}{}^a\gamma_a\psi_{\nu]}{}^i - \tfrac{1}{8}\not{D}T_{ab}{}^{ij}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]\,j} - \tfrac{3}{2}\gamma_a\psi_{[\mu}{}^i\,\bar\psi_{\nu]}{}^j\gamma^a\chi_j\\
&\quad + \tfrac{1}{4}R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^j + \tfrac{1}{2}\mathrm{i}R(A)_{ab}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^i\,,\\
R(K)_{\mu\nu}{}^a &= 2\,\mathcal{D}_{[\mu}f_{\nu]}{}^a - \tfrac{1}{4}\big(\bar\phi_{[\mu}{}^i\gamma^a\phi_{\nu]i} + \bar\phi_{[\mu i}\gamma^a\phi_{\nu]}{}^i\big)\\
&\quad + \tfrac{1}{4}\big(\bar\psi_\mu{}^i D_b T^{ba}{}_{ij}\psi_\nu{}^j - 3\,e_{[\mu}{}^a\psi_{\nu]}{}^i\not{D}\chi_i + \tfrac{3}{2}D\,\bar\psi_{[\mu}{}^i\gamma^a\psi_{\nu]j}\\
&\quad - 4\,\bar\psi_{[\mu}{}^i\gamma_{\nu]}D_bR(Q)^{ba}{}_i + \text{h.c.}\big)\,.
\end{aligned}
\tag{B.3}
$$

There are three conventional constraints (which have already been incorporated in (B.3),

$$
\begin{aligned}
&R(P)_{\mu\nu}{}^a = 0\,,\\
&\gamma^\mu R(Q)_{\mu\nu}{}^i + \tfrac{3}{2}\gamma_\nu\chi^i = 0\,,\\
&e^\nu{}_b\,R(M)_{\mu\nu a}{}^b - \mathrm{i}\tilde R(A)_{\mu a} + \tfrac{1}{8}T_{abij}T_\mu{}^{bij} - \tfrac{3}{2}D\,e_{\mu a} = 0\,,
\end{aligned}
\tag{B.4}
$$

which are S-supersymmetry invariant. They determine the fields $\omega_\mu{}^{ab}$, $\phi_\mu{}^i$ and $f_\mu{}^a$ as follows,

$$
\begin{aligned}
\omega_\mu^{ab} &= -\,2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\,\partial_\sigma e_\nu{}^c - 2e_\mu{}^{[a}e^{b]\nu}b_\nu\\
&\quad - \tfrac{1}{4}(2\bar\psi_\mu^i\gamma^{[a}\psi_i^{b]} + \bar\psi^{ai}\gamma_\mu\psi_i^b + \text{h.c.})\,,\\
\phi_\mu{}^i &= \tfrac{1}{2}\left(\gamma^{\rho\sigma}\gamma_\mu - \tfrac{1}{3}\gamma_\mu\gamma^{\rho\sigma}\right)\left(\mathcal{D}_\rho\psi_\sigma{}^i - \tfrac{1}{16}T^{abij}\gamma_{ab}\gamma_\rho\psi_{\sigma j} + \tfrac{1}{4}\gamma_{\rho\sigma}\chi^i\right)\,,\\
f_\mu{}^\mu &= \tfrac{1}{6}R(\omega,e) - D\\
&\quad - \left(\tfrac{1}{12}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu{}^i\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i} - \tfrac{1}{12}\bar\psi_\mu{}^i\psi_\nu{}^jT^{\mu\nu}{}_{ij} - \tfrac{1}{4}\bar\psi_\mu{}^i\gamma^\mu\chi_i + \text{h.c.}\right)\,.
\end{aligned}
\tag{B.5}
$$

We will also need the bosonic part of the expression for the uncontracted connection $f_\mu{}^a$,

$$
f_\mu{}^a = \tfrac{1}{2}R(\omega,e)_\mu{}^a - \tfrac{1}{4}\big(D + \tfrac{1}{3}R(\omega,e)\big)e_\mu{}^a - \tfrac{1}{2}\mathrm{i}\tilde R(A)_\mu{}^a + \tfrac{1}{16}T_{\mu b}{}^{ij}T^{ab}{}_{ij}\,,
\tag{B.6}
$$

where $R(\omega,e)_\mu{}^a = R(\omega)_{\mu\nu}{}^{ab}e_b{}^\nu$ is the non-symmetric Ricci tensor, and $R(\omega,e)$ the corresponding Ricci scalar. The curvature $R(\omega)_{\mu\nu}{}^{ab}$ is associated with the spin connection field $\omega_\mu{}^{ab}$, given in (B.5).

The transformations of $\omega_\mu{}^{ab}$, $\phi_\mu{}^i$ and $f_\mu{}^a$ are induced by the constraints (B.4). We present their Q- and S-supersymmetry variations, as well as the transformations under conformal boosts, below,

$$\begin{aligned}
\delta\omega_\mu{}^{ab} =&\, -\tfrac12\bar\epsilon^i\gamma^{ab}\phi_{\mu i} - \tfrac12\bar\epsilon^i\psi_\mu{}^j\,T^{ab}{}_{ij} + \tfrac34\bar\epsilon^i\gamma_\mu\gamma^{ab}\chi_i \\
&+ \bar\epsilon^i\gamma_\mu\,R^{ab}{}_i(Q) - \tfrac12\bar\eta^i\gamma^{ab}\psi_{\mu i} + \text{h.c.} + 2\,\Lambda_{\rm K}{}^{[a}e_\mu{}^{b]}\,, \\
\delta\phi_\mu{}^i =&\, -2\,f_\mu{}^a\gamma_a\epsilon^i + \tfrac14 R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_\mu\epsilon^j + \tfrac12\mathrm{i}R(A)_{ab}\gamma^{ab}\gamma_\mu\epsilon^i - \tfrac18\not{D}T^{ab\,ij}\gamma_{ab}\gamma_\mu\epsilon_j \\
&+ \tfrac32[(\bar\chi_j\gamma^a\epsilon^j)\gamma_a\psi_\mu{}^i - (\bar\chi_j\gamma^a\psi_\mu{}^j)\gamma_a\epsilon^i] + 2\,\mathcal{D}_\mu\eta^i + \Lambda_{\rm K}{}^a\gamma_a\psi_\mu{}^i\,, \\
\delta f_\mu{}^a =&\, -\tfrac12\bar\epsilon^i\psi_\mu{}^i\,D_bT^{ba}{}_{ij} - \tfrac34 e_\mu{}^a\bar\epsilon^i\not{D}\chi_i - \tfrac34\bar\epsilon^i\gamma^a\psi_{\mu i}\,D \\
&+ \bar\epsilon^i\gamma_\mu\,D_bR^{ba}{}_i(Q) + \tfrac12\bar\eta^i\gamma^a\phi_{\mu i} + \text{h.c.} + \mathcal{D}_\mu\Lambda_{\rm K}{}^a\,.
\end{aligned} \tag{B.7}$$

The transformations under S-supersymmetry and conformal boosts reflect the structure of the underlying SU(2, 2|2) gauge algebra. The presence of curvature constraints and of the non-gauge fields T_{abij}, χ^i and D induce deformations of the Q-supersymmetry algebra, as is manifest in the above results, in particular in (B.3) and (B.7).

Combining the conventional constraints (B.4) with the various Bianchi identities one derives that not all the curvatures are independent. For instance,

$$\varepsilon^{abcd}D_bR(M)_{cd}{}^{ef} = 2\,\varepsilon^{abc[e}\,R(K)_{bc}{}^{f]} + \tfrac92\,\eta^{a[e}\bar\chi^i\gamma^{f]}\chi_i + \tfrac32\big[\bar\chi^i\gamma^aR(Q)_i^{ef} - \text{h.c.}\big]\,. \tag{B.8}$$

Furthermore it is convenient to modify two of the curvatures by including suitable covariant terms,

$$\begin{aligned}
\mathcal{R}(M)_{ab}{}^{cd} =&\, R(M)_{ab}{}^{cd} + \tfrac1{16}\big(T_{abij}\,T^{cdij} + T_{ab}{}^{ij}\,T^{cd}{}_{ij}\big)\,, \\
\mathcal{R}(S)_{ab}{}^i =&\, R(S)_{ab}{}^i + \tfrac34 T_{ab}{}^{ij}\chi_j\,.
\end{aligned} \tag{B.9}$$

where we observe that $\gamma^{ab}\big(\mathcal{R}(S) - R(S)\big)_{ab}{}^i = 0$. The modified curvature $\mathcal{R}(M)_{ab}{}^{cd}$ satisfies the following relations,

$$\begin{aligned}
\mathcal{R}(M)_{\mu\nu}{}^{ab}\,e^\nu{}_b =&\, \mathrm{i}\tilde R(A)_{\mu\nu}e^{\nu a} + \tfrac32 D\,e_\mu{}^a\,, \\
\tfrac14\varepsilon_{ab}{}^{ef}\,\varepsilon^{cd}{}_{gh}\,\mathcal{R}(M)_{ef}{}^{gh} =&\, \mathcal{R}(M)_{ab}{}^{cd}\,, \\
\varepsilon_{cdea}\,\mathcal{R}(M)^{cd\,e}{}_b =&\, \varepsilon_{becd}\,\mathcal{R}(M)_a{}^{e\,cd} = 2\tilde R(D)_{ab} = 2\mathrm{i}R(A)_{ab}\,.
\end{aligned} \tag{B.10}$$

The first of these relations corresponds to the third constraint given in (B.4), while the remaining equations follow from combining the curvature constraints with the Bianchi identities. Note that the modified curvature does not satisfy the pair exchange property;

instead we have,

$$\mathcal{R}(M)_{ab}{}^{cd} = \mathcal{R}(M)^{cd}{}_{ab} + 4\mathrm{i}\delta^{[c}_{[a}\,\tilde{R}(A)_{b]}{}^{d]}\,. \tag{B.11}$$

We now turn to the fermionic constraint given in (B.4) and its consequences for the modified curvature defined in (B.9). First we note that the constraint on $R(Q)_{\mu\nu}{}^i$ implies that this curvature is anti-selfdual, as follows from contracting the constraint with $\gamma^\nu\,\gamma_{ab}$,

$$\tilde{R}(Q)_{\mu\nu}{}^i = -R(Q)_{\mu\nu}{}^i\,. \tag{B.12}$$

Furthermore, combination of the Bianchi identity and the constraint on $R(Q)_{\mu\nu i}$ yields the following condition on the modified curvature $\mathcal{R}(S)_{ab}{}^i$,

$$\gamma^a\tilde{\mathcal{R}}(S)_{ab}{}^i = 2\,D^a\tilde{R}(Q)_{ab}{}^i = -2\,D^a R(Q)_{ab}{}^i\,. \tag{B.13}$$

This identity (upon contraction with $\gamma^b\gamma_{cd}$) leads to the following identity on the anti-selfdual part of $\mathcal{R}(S)_{ab}{}^i$,

$$\mathcal{R}(S)_{ab}{}^i - \tilde{\mathcal{R}}(S)_{ab}{}^i = 2\,\not{D}\big(R(Q)_{ab}{}^i + \tfrac{3}{4}\gamma_{ab}\chi^i\big)\,. \tag{B.14}$$

Appendix C

Covariantization under conformal boosts

In principle covariant (multiple) derivatives are defined by the standard procedure by adding gauge fields to absorb all symmetry variations proportional to derivatives of the transformation parameters. In this procedure the gauge field ${f_\mu}^a$ associated with the conformal boosts (parametrized by ${\Lambda_{\rm K}}^a$) appears somewhat indirectly, because the only other fields that transform under the conformal boosts are the gauge fields b_μ, ${\omega_\mu}^{ab}$ and ${\phi_\mu}^i$. Therefore supercovariant derivatives of fields that are themselves invariant, will transform under these K-transformations, and usually these variations take a relatively simple form. We give some examples for a scalar field ϕ, a spinor field ψ, and a tensor field t_{ab}, each of Weyl weight w,

$$\begin{aligned}\delta_{\rm K} D_a\phi &= -\,w\,\Lambda_{{\rm K}a}\phi\,,\\ \delta_{\rm K} D_a t_{bc} &= -\,w\,\Lambda_{{\rm K}a} t_{bc} + 2\,t_{a[b}\Lambda_{{\rm K}c]} - 2\,\eta_{a[b}t_{c]d}\,{\Lambda_{\rm K}}^d\,,\\ \delta_{\rm K} D_a\psi &= \big[-\,w\,\Lambda_{{\rm K}a} + \tfrac12 {\Lambda_{\rm K}}^b\gamma_{ab}\big]\psi\,.\end{aligned} \tag{C.1}$$

These transformation rules simplify for certain contractions, such as in $D^a t_{ab}$ or $\not{D}\psi$,

$$\begin{aligned}\delta_{\rm K} D^a t_{ab} &= (2-w){\Lambda_{\rm K}}^a t_{ab}\,,\\ \delta_{\rm K} D_{[a} t_{bc]} &= (2-w)\Lambda_{{\rm K}[a} t_{bc]}\,,\\ \delta_{\rm K}\not{D}\psi &= (\tfrac32 - w)\not{\Lambda}_{\rm K}\,\psi\,,\end{aligned} \tag{C.2}$$

showing, for instance, that the Dirac operator on a spinor field of weight $w = \frac{3}{2}$ is invariant.

Applying an extra covariant derivative we explicitly indicate the presence of the K-connection field $f_\mu{}^a$,

$$\begin{aligned} D_\mu D_a\phi &= \mathcal{D}_\mu D_a\phi + w f_{\mu a}\,\phi\,, \\ D_\mu D^a t_{ab} &= \mathcal{D}_\mu D^a t_{ab} + (w-2) f_\mu{}^a\, t_{ab}\,, \\ D_\mu \not{D}\psi &= \mathcal{D}_\mu \not{D}\psi + (w-\tfrac{3}{2}) f_\mu{}^a \gamma_a\psi\,, \end{aligned} \tag{C.3}$$

where $\mathcal{D}_\mu$ denotes the covariant derivative without including the field $f_\mu{}^a$. Under K-transformations these multiple derivatives transform as,

$$\begin{aligned} \delta_{\rm K} D_\mu D_a\phi &= -(w+1)\big[\Lambda_{{\rm K}\mu} D_a + \Lambda_{{\rm K}a} D_\mu\big]\phi + e_{\mu a}\Lambda_{\rm K}{}^b D_b\phi\,, \\ \delta_{\rm K} D_\mu D^a t_{ab} &= -(w+1)\Lambda_{{\rm K}\mu} D^a t_{ab} - \Lambda_{{\rm K}b} D^a t_{a\mu} + e_{\mu b}\Lambda_{\rm K}{}^c D^a t_{ac} + (2-w)\Lambda_{\rm K}{}^a D_\mu t_{ab}\,, \\ \delta_{\rm K} D_\mu \not{D}\psi &= \big[-(w+1)\Lambda_{{\rm K}\mu} + \tfrac{1}{2}\Lambda_{\rm K}{}^a\gamma_{\mu a}\big]\not{D}\psi + (\tfrac{3}{2}-w)\not{\Lambda}_{\rm K} D_\mu\psi\,. \end{aligned} \tag{C.4}$$

Contracting the first equation with $e^{a\mu}$ shows that the conformal D'Alembertian transforms under K-transformations as $\delta_{\rm K}\Box_{\rm c}\phi = -2(w-1)\Lambda_{\rm K}{}^a D_a\phi$, which vanishes for $w = 1$.

This pattern repeats itself when considering even higher derivatives. We present the following results,

$$\begin{aligned} D_\mu\Box_{\rm c}\phi &= \mathcal{D}_\mu\Box_{\rm c}\phi + 2(w-1) f_\mu{}^a D_a\phi\,, \\ \Box_{\rm c}\Box_{\rm c}\phi &= \mathcal{D}_\mu D^\mu\Box_{\rm c}\phi + (w+2) f_\mu{}^\mu\Box_{\rm c}\phi + 2(w-1) f_{\mu a} D^\mu D^a\phi\,, \\ \Box_{\rm c}\not{D}\psi &= \mathcal{D}_\mu D^\mu\not{D}\psi + \big[(w+1) f_\mu{}^\mu - \tfrac{1}{2} f_{\mu a}\gamma^{\mu a}\big]\not{D}\psi + (w-\tfrac{3}{2}) f_{\mu a}\gamma^a D^\mu\psi\,, \end{aligned} \tag{C.5}$$

and,

$$\begin{aligned} \delta_{\rm K}\Box_{\rm c}\Box_{\rm c}\phi &= -2(w-1)\Lambda_{\rm K}{}^a\Box_{\rm c} D_a\phi - 2(w+1)\Lambda_{\rm K}{}^a D_a\Box_{\rm c}\phi \\ &= -2w\Lambda_{\rm K}{}^a\big[\Box_{\rm c} D_a\phi + D_a\Box_{\rm c}\big]\phi + 2\Lambda_{\rm K}{}^a\big[\Box_{\rm c} D_a - D_a\Box_{\rm c}\big]\phi\,, \\ \delta_{\rm K}\Box_{\rm c}\not{D}\psi &= -(2w-1)\Lambda_{\rm K}{}^a D_a\not{D}\psi - \tfrac{1}{2}\not{\Lambda}_{\rm K}\big[(2w-1)\Box_{\rm c} + [\not{D},\not{D}]\big]\psi\,. \end{aligned} \tag{C.6}$$

In order to obtain (5.4) we have evaluated the previous two variations for the fields A and Ψ_i, which have weights $w = 0, \frac{1}{2}$, respectively. In this case all the terms cubic and quadratic in derivatives in (C.6) appear with a certain degree of anti-symmetry, such that they become proportional to curvatures.

Appendix **D**

Chiral multiplets

D.1 Multiplication of chiral multiplets

In this appendix we summarize the product rules for two chiral supermultiplets and the Taylor expansion for functions of these multiplets. In the local supersymmetry setting, we will usually be dealing with homogeneous functions of chiral multiplets with equal Weyl weight so that a scaling weight under Weyl transformations can be assigned to the function.

The product of two chiral multiplets, specified by the component fields $\big(A, \Psi_i, B_{ij}, F^-_{ab}, \Lambda_i, C\big)$ and $\big(a, \psi_i, b_{ij}, f^-_{ab}, \lambda_i, c\big)$, respectively, leads to the following decomposition,

$$\begin{aligned}
&\big(A, \Psi_i, B_{ij}, F^-_{ab}, \Lambda_i, C\big) \otimes \big(a, \psi_i, b_{ij}, f^-_{ab}, \lambda_i, c\big) = \\
&\quad \big(A\, a\,,\ A\,\psi_i + a\,\Psi_i, A\, b_{ij} + a\, B_{ij} - \bar\Psi_{(i}\psi_{j)}\,, \\
&\quad A\, f^-_{ab} + a\, F^-_{ab} - \tfrac{1}{4}\varepsilon^{ij}\bar\Psi_i\gamma_{ab}\psi_j\,, \\
&\quad A\,\lambda_i + a\,\Lambda_i - \tfrac{1}{2}\varepsilon^{kl}(B_{ik}\,\psi_l + b_{ik}\,\Psi_l) - \tfrac{1}{4}(F^-_{ab}\gamma^{ab}\psi_i + f^-_{ab}\gamma^{ab}\Psi_i)\,, \\
&\quad A\, c + a\, C - \tfrac{1}{2}\varepsilon^{ik}\varepsilon^{jl}B_{ij}\, b_{kl} + F^-_{ab}\, f^{-ab} + \varepsilon^{ij}(\bar\Psi_i\lambda_j + \bar\psi_i\Lambda_j)\big)\,.
\end{aligned} \tag{D.1}$$

A function $\mathcal{G}(\Phi)$ of chiral superfields Φ^Λ defines a chiral superfield, whose component fields take the following form,

$$\begin{aligned}
A|_{\mathcal{G}} &= \mathcal{G}(A)\,, \\
\Psi_i|_{\mathcal{G}} &= \mathcal{G}(A)_\Lambda\, \Psi_i{}^\Lambda\,,
\end{aligned}$$

$$
\begin{aligned}
B_{ij}|_{\mathcal{G}} =& \mathcal{G}(A)_\Lambda\, B_{ij}{}^\Lambda - \tfrac{1}{2}\mathcal{G}(A)_{\Lambda\Sigma}\, \bar\Psi_{(i}{}^\Lambda \Psi_{j)}{}^\Sigma\,,\\
F^-_{ab}|_{\mathcal{G}} =& \mathcal{G}(A)_\Lambda\, F^{-\Lambda}_{ab} - \tfrac{1}{8}\mathcal{G}(A)_{\Lambda\Sigma}\, \varepsilon^{ij}\bar\Psi_i{}^\Lambda\gamma_{ab}\Psi_j{}^\Sigma\,,\\
\Lambda_i|_{\mathcal{G}} =& \mathcal{G}(A)_\Lambda\, \Lambda_i{}^\Lambda - \tfrac{1}{2}\mathcal{G}(A)_{\Lambda\Sigma}\big[B_{ij}{}^\Lambda\varepsilon^{jk}\Psi_k{}^\Sigma + \tfrac{1}{2}F^{-\Lambda}_{ab}\gamma^{ab}\Psi_k{}^\Sigma\big]\\
&+ \tfrac{1}{48}\mathcal{G}(A)_{\Lambda\Sigma\Gamma}\,\gamma^{ab}\Psi_i{}^\Lambda\,\varepsilon^{jk}\bar\Psi_j{}^\Sigma\gamma_{ab}\Psi_k{}^\Gamma\,,\\
C|_{\mathcal{G}} =& \mathcal{G}(A)_\Lambda\, C^\Lambda - \tfrac{1}{4}\mathcal{G}(A)_{\Lambda\Sigma}\big[B_{ij}{}^\Lambda B_{kl}{}^\Sigma\,\varepsilon^{ik}\varepsilon^{jl} - 2\,F^{-\Lambda}_{ab}F^{-ab\Sigma} + 4\,\varepsilon^{ik}\bar\Lambda_i{}^\Lambda\Psi_j{}^\Sigma\big]\,,\\
&+ \tfrac{1}{4}\mathcal{G}(A)_{\Lambda\Sigma\Gamma}\big[\varepsilon^{ik}\varepsilon^{jl}B_{ij}{}^\Lambda\Psi_k{}^\Sigma\Psi_l{}^\Gamma - \tfrac{1}{2}\varepsilon^{kl}\bar\Psi_k{}^\Lambda F^{-\Sigma}_{ab}\gamma^{ab}\Psi_l{}^\Gamma\big]\\
&+ \tfrac{1}{192}\mathcal{G}(A)_{\Lambda\Sigma\Gamma\Xi}\,\varepsilon^{ij}\bar\Psi_i{}^\Lambda\gamma_{ab}\Psi_j{}^\Sigma\,\varepsilon^{kl}\bar\Psi_k{}^\Gamma\gamma_{ab}\Psi_l{}^\Xi\,.
\end{aligned}
\tag{D.2}
$$

Here derivatives of the function $\mathcal{G}(A)$ with respect to the scalar fields are denoted with a lower index Λ, e.g. $\mathcal{G}(A)_\Lambda = \partial\mathcal{G}(A)/\partial A^\Lambda$. This result follows straightforwardly from expanding the superfield expression in powers of the fermionic coordinates.

D.2 Reduced chiral multiplets

Chiral multiplets can be consistently reduced by imposing a reality constraint. This usually requires specific values for the Weyl and chiral weights. The two cases that are relevant are the vector multiplet, which arises upon reduction from a scalar chiral multiplet, and the Weyl multiplet, which is a reduced anti-selfdual chiral tensor multiplet. Both reduced multiplets require weight $w = 1$.

We will denote the components of the $w = 1$ multiplet that describes the vector multiplet by $(A, \Psi, B, F^-, \Lambda, C)|_{\text{vector}}$. The constraint for a scalar chiral supermultiplet reads, $\varepsilon^{ij}\,\bar D_i\gamma_{ab}D_j\Phi = [\varepsilon^{ij}\,\bar D_i\gamma_{ab}D_j\Phi]^*$, which implies that $C|_{\text{vector}}$ and $\Lambda_i|_{\text{vector}}$ are expressed in terms of the lower components of the multiplet, and imposes a reality constraint on $B|_{\text{vector}}$ and a Bianchi identity on $F^-|_{\text{vector}}$ [27, 36, 39]. The latter implies that $F^-|_{\text{vector}}$ can be expressed in terms of a gauge field W_μ. This feature is not affected by the presence of the superconformal background field.

Denoting the independent components of the vector multiplet by $(X, \Omega, Y, \hat F^-)$, the identification with the chiral multiplet components is as follows,

$$
\begin{aligned}
A|_{\text{vector}} =& X\,,\\
\Psi_i|_{\text{vector}} =& \Omega_i\,,\\
B_{ij}|_{\text{vector}} =& Y_{ij} = \varepsilon_{ik}\varepsilon_{jl}Y^{kl}\,,\\
F^-_{ab}|_{\text{vector}} =& \hat F^-_{ab} = F^-_{ab} + \tfrac{1}{4}\big[\bar\psi_\rho{}^i\gamma_{ab}\gamma^\rho\Omega^j + \bar X\,\bar\psi_\rho{}^i\gamma^{\rho\sigma}\gamma_{ab}\psi_\sigma{}^j - \bar X\,T_{ab}{}^{ij}\big]\varepsilon_{ij}\,,
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_i|_{\text{vector}} &= -\,\varepsilon_{ij}\not{D}\Omega^j\,,\\
C|_{\text{vector}} &= -\,2\,\Box_{\rm c}\bar{X} - \tfrac{1}{4}F^+_{ab}\,T^{ab}{}_{ij}\varepsilon^{ij} - 3\,\bar{\chi}_i\Omega^i\,,
\end{aligned}
\tag{D.3}
$$

where $F_{\mu\nu} = 2\partial_{[\mu}W_{\nu]}$ is the field strength written in terms of the gauge field W_μ and $\hat{F}_{ab}$ denotes the supercovariant field strength. The Bianchi identity on $\hat{F}_{ab}$ can be written as,

$$
D^b\left(\hat{F}^+_{ab} - \hat{F}^-_{ab} + \tfrac{1}{4}XT_{abij}\varepsilon^{ij} - \tfrac{1}{4}\bar{X}T_{ab}{}^{ij}\varepsilon_{ij}\right) + \tfrac{3}{4}\left(\bar{\chi}_i\gamma_a\Omega_j\varepsilon^{ij} - \bar{\chi}^i\gamma_a\Omega^j\varepsilon_{ij}\right) = 0\,, \tag{D.4}
$$

and the reality constraint on Y_{ij} is included in (D.3).

The Q- and S-supersymmetry transformations for the vector multiplet take the form,

$$
\begin{aligned}
\delta X &= \bar{\epsilon}^i\Omega_i\,,\\
\delta\Omega_i &= 2\not{D}X\epsilon_i + \tfrac{1}{2}\varepsilon_{ij}\hat{F}_{\mu\nu}\gamma^{\mu\nu}\epsilon^j + Y_{ij}\epsilon^j + 2X\eta_i\,,\\
\delta W_\mu &= \varepsilon^{ij}\bar{\epsilon}_i(\gamma_\mu\Omega_j + 2\,\psi_{\mu j}X) + \varepsilon_{ij}\bar{\epsilon}^i(\gamma_\mu\Omega^j + 2\,\psi_\mu{}^j\bar{X})\,,\\
\delta Y_{ij} &= 2\,\bar{\epsilon}_{(i}\not{D}\Omega_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\not{D}\Omega^{l)}\,,
\end{aligned}
\tag{D.5}
$$

and, for $w = 1$, are in clear correspondence with the supersymmetry transformations of generic scalar chiral multiplets given in (2.17).

Subsequently we turn to the Weyl multiplet, which is a chiral anti-selfdual tensor multiplet subject to $\bar{D}_i\gamma^{ab}D_j\,\Phi_{ab}{}^{ij} = [\bar{D}_i\gamma^{ab}D_j\,\Phi_{ab}{}^{ij}]^*$. Its chiral superfield components take the following form,

$$
\begin{aligned}
A_{ab}|_W &= T_{ab}{}^{ij}\varepsilon_{ij}\,,\\
\Psi_{abi}|_W &= 8\,\varepsilon_{ij}R(Q)^j_{ab}\,,\\
B_{abij}|_W &= -\,8\,\varepsilon_{k(i}R(\mathcal{V})^-_{ab}{}^k{}_{j)}\,,\\
\left(F^-_{ab}\right)^{cd}|_W &= -\,8\,\mathcal{R}(M)^{-\,cd}_{ab}\,,\\
\Lambda_{abi}|_W &= 8\left(\mathcal{R}(S)^-_{abi} + \tfrac{3}{4}\gamma_{ab}\not{D}\chi_i\right)\,,\\
C_{ab}|_W &= 4D_{[a}\,D^cT_{b]c\,ij}\varepsilon^{ij} - \text{dual}\,.
\end{aligned}
\tag{D.6}
$$

We give the Q- and S-supersymmetry variations for the first few components,

$$
\begin{aligned}
\delta T_{ab}{}^{ij} &= 8\,\bar{\epsilon}^{[i}R(Q)_{ab}{}^{j]}\,,\\
\delta R(Q)_{ab}{}^i &= -\,\tfrac{1}{2}\not{D}T_{ab}{}^{ij}\,\epsilon_j + R(\mathcal{V})^-{}_{ab}{}^i{}_j\,\epsilon^j - \tfrac{1}{2}\mathcal{R}(M)_{ab}{}^{cd}\,\gamma_{cd}\epsilon^i + \tfrac{1}{8}T_{cd}{}^{ij}\,\gamma^{cd}\gamma_{ab}\,\eta_j\,,\\
\delta R(\mathcal{V})^-{}_{ab}{}^i{}_j &= 2\bar{\epsilon}_j\not{D}R(Q)_{ab}{}^i - 2\bar{\epsilon}^i\left(\mathcal{R}(S)^-_{abj} + \tfrac{3}{4}\gamma_{ab}\not{D}\chi_j\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \bar{\eta}_j(2R(Q)_{ab}{}^i + 3\gamma_{ab}\chi^i) - (\text{traceless})\,,\\
\delta\mathcal{R}(M)_{ab}^{-\,cd} &= \tfrac{1}{2}\bar{\epsilon}_i \not{D}\gamma^{cd}R(Q)_{ab}{}^i - \tfrac{1}{2}\bar{\epsilon}^i\gamma^{cd}\big(\mathcal{R}(S)^-_{abi} + \tfrac{3}{4}\gamma_{ab}\not{D}\chi_i\big)\\
&\quad - \bar{\eta}_i\gamma_{ab}R(Q)^{cdi} - \tfrac{1}{2}\bar{\eta}_i\gamma^{cd}R(Q)_{ab}{}^i - \tfrac{3}{4}\bar{\eta}_i\gamma_{ab}\gamma^{cd}\chi^i\,. \qquad \text{(D.7)}
\end{aligned}
$$

A scalar chiral multiplet with $w = 2$ is obtained by squaring the Weyl multiplet. The various scalar chiral multiplet components are given by,

$$
\begin{aligned}
A|_{W^2} &= (T_{ab}{}^{ij}\varepsilon_{ij})^2\,,\\
\Psi_i|_{W^2} &= 16\,\varepsilon_{ij}R(Q)^j_{ab}\,T^{klab}\,\varepsilon_{kl}\,,\\
B_{ij}|_{W^2} &= -\,16\,\varepsilon_{k(i}R(\mathcal{V})^k{}_{j)ab}\,T^{lmab}\,\varepsilon_{lm} - 64\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{R}(Q)_{ab}{}^k\,R(Q)^{l\,ab}\,,\\
F^{-ab}|_{W^2} &= -\,16\,\mathcal{R}(M)_{cd}{}^{ab}\,T^{klcd}\,\varepsilon_{kl} - 16\,\varepsilon_{ij}\,\bar{R}(Q)^i_{cd}\gamma^{ab}R(Q)^{cd\,j}\,,\\
\Lambda_i|_{W^2} &= 32\,\varepsilon_{ij}\,\gamma^{ab}R(Q)^j_{cd}\,\mathcal{R}(M)^{cd}{}_{ab} + 16\,(\mathcal{R}(S)_{ab\,i} + 3\gamma_{[a}D_{b]}\chi_i)\,T^{klab}\,\varepsilon_{kl}\\
&\quad - 64\,R(\mathcal{V})_{ab}{}^k{}_i\,\varepsilon_{kl}\,R(Q)^{ab\,l}\,,\\
C|_{W^2} &= 64\,\mathcal{R}(M)^{-cd}{}_{ab}\,\mathcal{R}(M)^{-ab}_{cd} + 32\,R(\mathcal{V})^{-ab\,k}{}_l\,R(\mathcal{V})^{-l}_{ab\;k}\\
&\quad - 32\,T^{ab\,ij}\,D_a\,D^cT_{cb\,ij} + 128\,\bar{\mathcal{R}}(S)^{ab}{}_i\,R(Q)_{ab}{}^i + 384\,\bar{R}(Q)^{ab\,i}\gamma_aD_b\chi_i\,. \qquad \text{(D.8)}
\end{aligned}
$$

These components can straightforwardly be substituted in the expression for the higher-derivative couplings.

Nederlandse samenvatting

De Nederlandse titel van dit proefschrift is ‘Nieuwe deformaties van $N = 2$ supergravitatie’. Supergravitatie is een theorie die zwaartekracht (gravitatie) combineert met supersymmetrie.[1] Het woord ‘deformaties’ kan hier losjes opgevat worden als ‘uitbreidingen’ of ‘variaties’. In deze samenvatting zullen we deze begrippen toelichten. We beginnen met het bespreken van enkele kenmerkende verschijnselen die een rol spelen in supergravitatie, zoals zwaartekracht en symmetrieën. Zo zullen we geleidelijk toewerken naar het onderwerp van dit proefschrift: deformaties van supergravitatie.

Zwaartekracht

Zwaartekracht is een kracht waar alles en iedereen aan onderhevig is. Alles met massa trekt elkaar aan. Zwaartekracht zorgt ervoor dat als we iets laten vallen, het op de grond terecht komt, dat de aarde in een baan om de zon draait en de maan in een baan om de aarde. De precieze werking van deze kracht, in situaties zoals we die tegenkomen in het dagelijks leven, is vastgelegd in Newtons wet van de zwaartekracht. Deze wet vertelt ons bijvoorbeeld hoe snel een steen (of een appel) naar beneden valt als deze vanaf een bepaalde hoogte boven het aardoppervlak losgelaten wordt.

Het blijkt dat Newtons wet van de zwaartekracht niet meer toereikend is wanneer de snelheden van objecten de lichtsnelheid benaderen, of wanneer objecten zeer grote massa’s hebben. In deze limieten worden de effecten van een fundamentelere theorie van de

[1] De toevoeging ‘$N = 2$’ duidt de klasse aan binnen supergravitatie en is verder niet relevant voor dit hoofdstuk.

zwaartekracht merkbaar, namelijk Einsteins algemene relativiteitstheorie. Deze situatie is kenmerkend voor de ontwikkeling van de natuurkunde. Een theorie die gangbare situaties goed beschrijft, blijkt bij extremere situaties niet meer consistent te zijn. Dit vormt de drijfveer om een fundamentelere theorie te vinden, die bij algemenere situaties geldt en waarvan de originele theorie een benadering is.

Algemene relativiteitstheorie speelt een belangrijke rol in het beschrijven van de evolutie van het universum. Ook geeft deze theorie een bepaalde klasse van oplossingen genaamd *zwarte gaten*. Een zwart gat is een object dat zo zwaar is dat niets meer aan de zwaartekracht van het object kan ontsnappen, zelfs licht niet. De massa van een zwart gat is geconcentreerd in één punt met oneindige dichtheid, genaamd de singulariteit. Rond een zwart gat bevindt zich een denkbeeldig oppervlak, de waarnemingshorizon, vanwaar licht nog net aan de zwaartekracht van het zwarte gat kan ontsnappen. De singulariteit is dus onzichtbaar voor een waarnemer die zich buiten deze waarnemingshorizon bevindt.

De aanwezigheid van een singulariteit in de oplossingen die zwarte gaten beschrijven, geeft aan dat algemene relativiteitstheorie tekort schiet in deze extreme situatie. De theorie moet dus wederom vervangen worden door een fundamentelere theorie om een systeem te beschrijven met zwaartekracht op hele kleine lengteschalen, net zoals Newtons wet van de zwaartekracht vervangen moet worden door algemene relativiteitstheorie bij grote massa's of hoge snelheden. Op kleine lengteschalen gaan zogenaamde kwantumeffecten een rol spelen en een nieuwe theorie van zwaartekracht moet deze kwantumeffecten incorporeren. Het vinden van zo'n kwantumzwaartekrachttheorie is een belangrijk onderwerp in de huidige theoretische natuurkunde. Zwarte gaten spelen hierbij een grote rol als test voor een mogelijke kwantumzwaartekrachttheorie, omdat in zo'n theorie de singulariteit afwezig zou moeten zijn. We zullen hier later in dit hoofdstuk op terugkomen.

Symmetrieën

In de natuurkunde spelen symmetrieën een belangrijke rol. Men spreekt van een symmetrie, of invariantie, als een eigenschap van een systeem onveranderd blijft na het uitvoeren van een transformatie. Bijvoorbeeld, een perfect ronde bol, zonder opdruk, kan men ronddraaien, zonder dat er een verschil aan de bol te zien is. Het is duidelijk dat de aanwezigheid van een symmetrie eisen legt aan een systeem. Als de bol niet perfect rond is, of als er een tekening op de bol zit, dan is de bol niet symmetrisch onder rotaties.

Vaak ligt er aan een natuurkundige theorie een bepaalde symmetrie ten grondslag. Zo ook bij algemene relativiteitstheorie. Dit heeft te maken met het feit dat zwaartekracht (lokaal) niet te onderscheiden is van de kracht die een object voelt onder een versnellende beweging. Iedereen die wel eens in een hard optrekkende auto heeft gezeten, heeft deze laatste kracht lijfelijk ervaren. Met deze kracht kan zwaartekracht 'gesimuleerd' worden. Stel, je bevindt je in een raket in de ruimte, waar zwaartekracht te verwaarlozen is. Als deze raket versneld wordt, voel je een kracht die vergelijkbaar is aan zwaartekracht en je met beide benen op de grond houdt. Sterker nog, als de beweging van de raket precies goed afgestemd is, en de raket geen ramen heeft, kan je niet bepalen door middel van experimenten of je je in een stilstaande raket op aarde bevindt, of in een versnellende raket in de ruimte. Met andere woorden, de wetten van de natuurkunde zijn hetzelfde in beide situaties - de beide situaties zijn symmetrisch. Dit is een voorbeeld van Einsteins equivalentieprincipe. Einstein heeft dit idee geformaliseerd, waaruit de algemene relativiteitstheorie volgt.

Supergravitatie

Supergravitatie, ook wel superzwaartekracht genoemd, is een theorie die algemene relativiteitstheorie combineert met een bijzonder soort symmetrie, genaamd supersymmetrie. In supersymmetrische theorieën worden twee klassen van deeltjes gerelateerd. Alle elementaire deeltjes, de bouwstenen van alles om ons heen, zijn namelijk op te splitsen in fermionen en bosonen. Fermionen zijn deeltjes die, ruwweg gezegd, niet bij elkaar kunnen zitten, terwijl bosonen dat wel kunnen. In een supersymmetrische theorie heeft elke boson een fermionische 'superpartner', die onder een supersymmetrietransformatie in elkaar roteren. De theorie blijft hetzelfde onder deze rotaties, oftewel, de theorie is invariant onder supersymmetrie.

Toen supergravitatie bedacht werd, hoopte men dat dit een consistente theorie van kwantumzwaartekracht zou zijn. Inmiddels wordt supergravitatie voornamelijk gezien als een benadering van snaartheorie, één van de huidige kandidaten om kwantumzwaartekracht te beschrijven. De fundamentele objecten in snaartheorie zijn uitgebreide objecten, zoals snaren (denk aan touwtjes) en membranen (denk aan een vel papier). Deze objecten blijken in hogere dimensies te leven dan de drie dimensies waarmee we bekend zijn, namelijk lengte, breedte en hoogte. Het wordt aangenomen dat deze extra dimensies niet uitgestrekt zijn, zoals de drie dimensies die wij zien, maar in plaats daarvan een klein pakketje vormen (gecompactificeerd zijn) zodat ze te klein zijn om direct zichtbaar te zijn voor ons en in de huidige experimenten.

De effecten van snaartheorie zijn vooral aanwezig op extreem kleine lengteschalen. Wanneer de relevante lengteschalen niet zo klein zijn geeft supergravitatie de benadering van snaartheorie. Omdat veel van snaartheorie nog onbekend is, of te moeilijk is om uit te rekenen, is het belangrijk om supergravitatie te bestuderen, zoals in dit proefschrift gedaan wordt.

Deformaties

In dit proefschrift worden nieuwe deformaties van een specifieke vierdimensionale supergravitatietheorie afgeleid. Ook worden enkele toepassingen behandeld waarin deze deformaties een rol spelen. Deze hebben met name betrekking op zwarte gaten.

Eén klasse van deformaties heeft te maken met symmetrieën. Naast supersymmetrie en de symmetrieën van algemene relativiteitstheorie, kan supergravitatie nog meer symmetrieën bevatten. Symbolisch kunnen we deze symmetrieën vergelijken met de, al eerder besproken, perfect ronde bol. Laten we nu aannemen dat op elk punt in de ruimte zich zo'n bol bevindt. Dit kunnen we ons voorstellen als een veld vol met (even grote) bollen. Nu laten we de bollen roteren. Dit kan op twee manieren. De eerste manier is dat alle bollen precies tegelijkertijd en op gelijke wijze ronddraaien. Dit is in de praktijk natuurlijk een lastige onderneming, maar in theorie eenvoudig, omdat men alleen maar hoeft te weten hoe één bol roteert, om te weten hoe alle bollen roteren. De tweede manier is dat alle bollen allemaal door elkaar op hun eigen wijze ronddraaien. Dit symboliseert twee verschillende soorten van symmetrie. Als een theorie invariant is onder 'bollen die tegelijkertijd ronddraaien', noemen we dit een *rigide* symmetrie. Als een theorie invariant is onder 'bollen die allemaal apart ronddraaien', noemen we dit een *lokale* symmetrie. Deze laatste eis is sterker dan de eerste, omdat de theorie dan invariant moet zijn onder de rotaties van elke bol afzonderlijk. Daardoor ziet een theorie die invariant is onder een lokale symmetrie er anders uit ziet dan een theorie die invariant is onder een rigide symmetrie. Er bestaat een specifieke procedure om binnen een theorie van een rigide symmetrie een lokale te maken. Dit noemt men het *ijken* van een theorie, en de veranderingen van de theorie ten opzichte van de theorie met de rigide symmetrie worden ijk-deformaties genoemd. Dit proefschrift beschrijft hoe men op de meest algemene manier de symmetrieën van een specifieke supergravitatietheorie systematisch lokaal kan realiseren en de resulterende ijk-deformaties worden afgeleid. In een toepassing worden, in de aanwezigheid van deze ijk-deformaties, de mogelijke supersymmetrische oplossingen bestudeerd in de buurt van de waarnemingshorizon van bepaalde zwarte gaten.

De andere klasse van deformaties die in dit proefschrift wordt behandeld, bestaat uit zogenaamde hogere afgeleide termen. Als men de supergravitatie-benadering van snaartheorie neemt, spelen deze termen een subleidende rol. Dat betekent dat ze niet de leidende termen zijn, maar de termen die daarna het meest belangrijk zijn. Ruwweg geldt, hoe meer we van deze termen in beschouwing nemen, hoe preciezer supergravitatie snaartheorie benadert. Dit is bijvoorbeeld van belang bij het berekenen van bepaalde eigenschappen van zwarte gaten, zoals hun oppervlakte en lading. Voor sommige supersymmetrische zwarte gaten kunnen deze eigenschappen exact uitgerekend worden binnen snaartheorie. Deze resultaten kunnen worden vergeleken met wat verkregen wordt wanneer deze eigenschappen vanuit de supergravitatiebenadering berekend worden. Hieruit blijkt dat in supergravitatie hogere afgeleide termen nodig zijn om een meer precieze overeenkomst met snaartheorie van deze eigenschappen te krijgen.

Het construeren van zulke hogere afgeleide termen binnen supergravitatie blijkt lastig te zijn, omdat ze aan bepaalde voorwaarden moeten voldoen. Zo moeten ze de symmetrieën van supergravitatie respecteren. In dit proefschrift staat een systematische procedure beschreven waarmee het mogelijk is een grote verscheidenheid aan hogere afgeleide termen te construeren. Het bijzondere aan de termen die via deze methode verkregen worden is dat bewezen kan worden dat ze geen van alle bijdragen aan de oppervlakte en lading van supersymmetrische zwarte gaten. Dit resultaat is niet helemaal onverwacht, omdat er al een goede overeenkomst was in de subleidende termen van de oppervlakte en lading van bepaalde supersymmetrische zwarte gaten, berekend met snaartheorie en supergravitatie, zonder dat deze nieuwe hogere afgeleide termen waren meegenomen. Het resultaat in dit proefschrift geeft een gedeeltelijke verklaring voor deze overeenkomst.

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Maaike van Zalk, augustus 2011.

Curriculum Vitae

Ik ben geboren op 25 september 1984 in Harderwijk, waar ik de VWO-opleiding aan het Christelijk College Nassau Veluwe van 1996 tot 2002 heb gevolgd. Hierna ben ik natuurkunde gaan studeren aan de Universiteit Utrecht. In 2005 heb in mijn B.Sc. diploma in natuur- en sterrenkunde gehaald en in 2007 mijn M.Sc. diploma in theoretische natuurkunde (cum laude). Mijn afstudeeronderzoek, getiteld 'On slowly moving solitons', heb ik gedaan onder begeleiding van Prof. dr. B. de Wit. Vanaf oktober 2007 heb ik, in dienst van de stichting FOM, mijn promotieonderzoek gedaan, wederom onder begeleiding van Prof. dr. B. de Wit. De resultaten van mijn onderzoek zijn het onderwerp van dit proefschrift.

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