



Colliding wave solutions in 2 + 1-dimensions

M. Halilsoy^a, S. Habib Mazharimousavi^b, S. Motalebi^c

Department of Physics, Faculty of Arts and Sciences, Eastern Mediterranean University, North Cyprus via Mersin 10, Famagusta, Turkey

Received: 22 January 2022 / Accepted: 1 April 2022
© The Author(s) 2022

Abstract We identified a new metric that describes both colliding electromagnetic shock waves and scalar waves without cosmological constant in 2 + 1-dimensions. To the future of collision point our metric has spacelike singularity so that the electromagnetic solution differs from its 3 + 1-dimensional counterpart. Our interaction region with scalar waves is related through the Kaluza–Klein argument with the 3 + 1-dimensional Schwarzschild spacetime.

1 Introduction

After the discovery of Bañados–Teitelboim–Zanelli (BTZ) black hole [1] for topological gravity with a negative cosmological constant, the 2 + 1-dimensional spacetime gained popularity and paved the way to a vast literature. Physical properties of 2 + 1-dimensions were already studied in details beforehand [2]. The wave collision problem in 2 + 1-dimensions, however, did not attract much attention. For this reason, it is our aim in this paper to fill such a gap. We remind that colliding waves in 3 + 1-dimensions were studied extensively during 1970s and 1980s [3]. Asymptotic solutions for interacting conformal scalar waves was obtained in all dimensions and in particular in 2 + 1-dimensions [4]. Such higher dimensional solutions found application also in string theory [5, 6].

In this paper, we obtain the interaction region (Region IV), resulted from the head-on collision at $u = v = 0$ of two foregoing opposite electromagnetic (em) shock waves that are propagating in the 2 + 1-spacetime. Similarly, we repeat the same procedure for the spinless shock scalar waves. Unlike the case of conformal scalar waves in [4], our scalar field satisfies the massless scalar field equation. Herein, em and scalar wave collisions are considered separately in the common metric ansatz given by

^a e-mail: mustafa.halilsoy@emu.edu.tr

^b e-mail: habib.mazhari@emu.edu.tr (corresponding author)

^c e-mail: sara.motalebi@emu.edu.tr

$$ds^2 = -2 f(u, v) du dv + r^2(u, v) dx^2. \quad (1)$$

The metric functions $f(u, v)$ and $r(u, v)$ are functions of the null coordinates, $u = \frac{1}{\sqrt{2}}(t + z)$ and $v = \frac{1}{\sqrt{2}}(t - z)$. The absence of $du dx$ term in the metric restricts the problem to linearly polarized waves and also we take zero cosmological constant. We present the singularity structure, geodesics in the $u - v$ plane and the 2+1-dimensional cosmological form of our interaction region. Furthermore, by an appropriate extension we show that our sourceful 2+1-dimensional interaction region for colliding scalar waves transforms into the 3+1-dimensional Schwarzschild vacuum spacetime.

2 Collision of em shock waves

We consider two em shock waves moving oppositely along z -direction in the incoming regions (Region II and III) as shown in Fig. 1. For the details of collision problem, see [3]. The em vector potentials are given by

$$A_\mu(u) = \delta_\mu^x A(u), \quad (\text{Region II}) \quad (2)$$

and

$$A_\mu(v) = \delta_\mu^x A(v), \quad (\text{Region III}) \quad (3)$$

where $A(u) = \sin(au \Theta(u))$ and $A(v) = \sin(bv \Theta(v))$, in which (a, b) are constants and $(\Theta(u), \Theta(v))$ stand for the Heaviside step functions. For $(u < 0, v < 0)$, we have the flat Minkowski (vacuum) spacetime (Region I). At $u = v = 0$, the waves collide and develop the interaction region (Region IV) to be determined from the following Maxwell and Einstein–Maxwell equations

$$2 A_{uv} - (\ln r)_u A_v - (\ln r)_v A_u = 0, \quad (4)$$

$$(\ln f)_u r_u - r_{uu} = \frac{1}{r} A_u^2, \quad (5)$$

$$(\ln f)_v r_v - r_{vv} = \frac{1}{r} A_v^2, \quad (6)$$

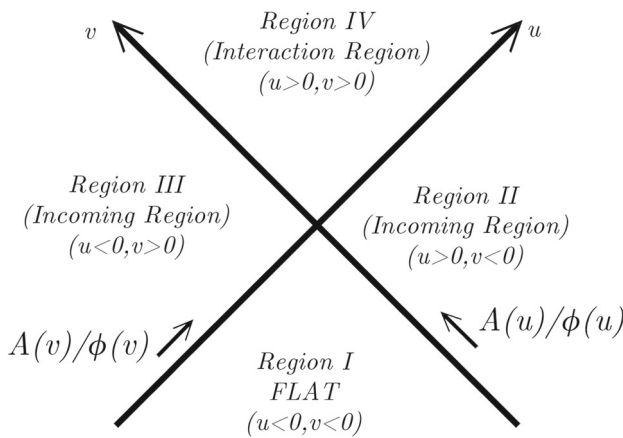


Fig. 1 The $u - v$ picture for colliding waves in 2+1-dimensions. The third dimension $x -$ is orthogonal to the plane so that each point in the plane represents a line

$$r^2 (\ln f)_{uv} = A_u A_v \tag{7}$$

and

$$r_{uv} = 0. \tag{8}$$

Note that our notation is such that $A_u = \frac{\partial A}{\partial u}$, $A_{uv} = \frac{\partial^2 A}{\partial u \partial v}$, etc. The solution describing the collision of em shock waves is given by

$$A(u, v) = \sqrt{2} \sin(au \Theta(u) - bv \Theta(v)), \tag{9}$$

$$f(u, v) = \cos^2(au \Theta(u) + bv \Theta(v)), \tag{10}$$

and

$$r(u, v) = \cos^2(au \Theta(u)) - \sin^2(bv \Theta(v)). \tag{11}$$

The incoming region (Region II) where $v < 0$, is given by the line element

$$ds^2 = \cos^2(au \Theta(u))(-2 du dv + \cos^2(au \Theta(u)) dx^2), \tag{12}$$

which is conformal to the em shock wave metric with the conformal factor, $\cos^2(au \Theta(u))$. That is, the line element of Region II

$$ds^2 = -2 du dv + \cos^2(au \Theta(u)) dx^2 \tag{13}$$

and that of Region III with, $(a \leftrightarrow b, u \leftrightarrow v)$ represent also incoming em shock waves. The collision problem of this latter type shock waves, however, remains open where our solution is valid for the type of waves (12) with the appropriate conformal factor. The incoming region (Region III) is same with the substitutions $a \rightarrow b$ and $u \rightarrow v$. The conformal form of the incoming metric is a feature different from the waves of the 3 + 1-dimensionsal colliding waves [7]. For collision problem in nonlinear electrodynamics, see [8].

3 Collision of scalar waves

In this case, we consider the incoming scalar waves $\phi(u)$ and $\phi(v)$ from regions II and III, respectively, colliding at $u = v = 0$, in the same metric ansatz (1). The Einstein-Scalar equations are

$$2 \phi_{uv} + (\ln r)_u \phi_v + (\ln r)_v \phi_u = 0, \tag{14}$$

$$(\ln f)_u r_u - r_{uu} = r \phi_u^2, \tag{15}$$

$$(\ln f)_v r_v - r_{vv} = r \phi_v^2, \tag{16}$$

$$-(\ln f)_{uv} = \phi_u \phi_v \tag{17}$$

and

$$r_{uv} = 0. \tag{18}$$

Solution of this set of equations is provided by the same $f(u, v)$ and $r(u, v)$ as in part II, where the scalar field is

$$\phi(u, v) = \frac{1}{\sqrt{2}} \ln \left[\frac{1 + \sin(au + bv)}{1 - \sin(au + bv)} \right] \tag{19}$$

We note that the line element (of Region II)

$$ds^2 = -2 du dv + (1 - a^2 u^2 \Theta(u))^2 dx^2 \tag{20}$$

represents also a shock scalar wave with the scalar field

$$\phi(u) = \sqrt{2} \arcsin(au \Theta(u)) \tag{21}$$

It is interesting that by a shift of the null coordinate u , (20) can be expressed alternatively with $f(u) \neq 1$. A similar scalar wave can be considered with $(a \leftrightarrow b, u \leftrightarrow v)$ for Region III. We were unable to obtain a Region IV solution for Einstein-scalar equations that will match with the type (20) of scalar wave. Our solution concerns the type of wave with conformal factor, as in the em case. For colliding scalar waves in 3 + 1-dimensions we refer to [9, 10]. After obtaining separately the collision of em (Sect. 2) and scalar waves (Sect. 3), it is easy to observe that the collision problem of combined (em+scalar) waves is also solved. This follows from the right hand sides of the set of Eqs. (5-7) and (15-17). Namely, if we make the replacements $A(u, v) \rightarrow \frac{1}{\sqrt{2}} A(u, v)$ and $\phi(u, v) \rightarrow \frac{1}{\sqrt{2}} \phi(u, v)$ and add the right hand sides of those mentioned equations with the same $f(u, v)$ and $r(u, v)$ functions, we obtain the collision problem of combined (em + scalar) waves.

4 The singularity structure

It is shown that the line element (with suppressed step functions)

$$ds^2 = -2 \cos^2(au + bv) du dv + (\cos^2 au - \sin^2 bv)^2 dx^2 \tag{22}$$

that solves colliding em/scalar waves has the Ricci scalar

$$\mathcal{R} = R_{ab}g^{ab} = -\frac{4a b \Theta(u)\Theta(v)}{f(u, v)} \tag{23}$$

and the Kretschmann scalar

$$\begin{aligned} \mathcal{K} = \frac{4a^2 b^2 \Theta(u)\Theta(v)}{f(u, v)^3 r(u, v)^2} & \left[\frac{3}{16}(\cos(2au + 6bv)) \right. \\ & + \cos(6au + 2bv) + \frac{9}{8}(\cos(4au) + \cos(4bv)) \\ & + \frac{15}{8} \cos(2au - 2bv) + \frac{35}{16} \cos(2au + 2bv) \\ & \left. + \frac{1}{16} \cos(6au + 6bv) + \frac{1}{2} \cos(4au + 4bv) + \frac{7}{4} \right]. \end{aligned} \tag{24}$$

The conditions $f(u, v) = 0 = r(u, v)$ give

$$au \pm bv = \pm \frac{\pi}{2} \tag{25}$$

as singular surfaces (Fig. 2). In the interaction region $au + bv = \frac{\pi}{2}$, is a spacelike singularity beyond which is not accessible. The timelike singularities $au - bv = \pm \frac{\pi}{2}$ are shown as straight lines whose slopes depend on the constants a and b . The specific points ($u = 0, v = \frac{\pi}{2b}$) and ($v = 0, u = \frac{\pi}{2a}$) are null singularities. Extension of these points to the incoming regions II and III make the line element to vanish due to the conformal factor. Therefore the ranges of the waves must be chosen as $u < \frac{\pi}{2a}$ (Region II) and $v < \frac{\pi}{2b}$ (Region III), in order to have physical incoming waves. For a discussion of 3 + 1-dimensional em problem, which is different from the present case, we refer to [11].

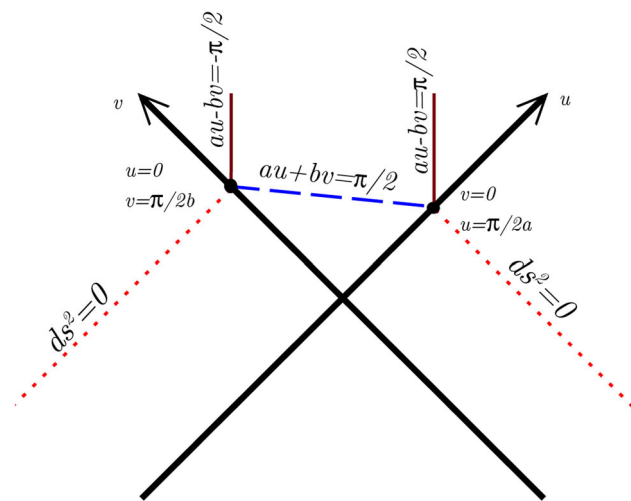


Fig. 2 The singularity structure in the spacetime of colliding waves. The surface $au + bv = \frac{\pi}{2}$ and $au + bv = \pm \frac{\pi}{2}$ are respectively the spacelike and timelike singularities. The null singularities are denoted by heavy dots on the light cone. Extension of these points to the incoming regions give vanishing line element. This is due to the conformal factor

5 Geodesics in the $u - v$ plane

Our metric confined to the (u, v) plane with $x = const$, reads

$$ds^2 = -2 f(u, v) du dv. \tag{26}$$

The geodesics can be obtained upon parametrizing $v = v(u)$, from the action

$$I [u, v, v'] = \int \sqrt{f v'} du \tag{27}$$

where $v' = \frac{dv}{du}$, which yields the geodesic equation

$$v'' = v' ((\ln f)_u - v' (\ln f)_v). \tag{28}$$

We introduce the new variable $y = au + bv$, in which the equation takes the form

$$y'' = 2 (\tan y) (y' - a) (y' - 2a) \tag{29}$$

where a prime denotes $\frac{d}{du}$. An exact solution for the geodesics is available in terms of elliptic functions. We obtain

$$au + c_2 = \frac{1}{2} y \pm F(c_0 y, \frac{2\sqrt{a}}{c_0}) \tag{30}$$

in which c_0 and c_2 are integration constants and the elliptic function F is defined by [12]

$$F(z, k) = \int_0^z \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} \tag{31}$$

The geodesics equation can also be solved perturbatively for $u \ll 1$. To this end we choose

$$y = a_1 u + a_3 u^3 + a_5 u^5 + \dots \tag{32}$$

with the constant coefficients. It turns out that the coefficients are given by

$$a_3 = \frac{1}{3} a_1 (a_1 - a) (a_1 - 2a) \tag{33}$$

$$a_5 = \frac{1}{5} a_3 (4a_1^2 - 6a_1 a + a^2) \tag{34}$$

⋮

in which, a_1 remains arbitrary.

6 Cosmological form of the metric

It is instructive to transform the interaction region of our metric (1) into an explicit time dependent form. For this purpose we apply the coordinate transformation

$$t = \sin (au + bv) \tag{35}$$

$$\cos \theta = \sin (au - bv) \tag{36}$$

$$x = \frac{\varphi}{\sqrt{2ab}} \tag{37}$$

followed by an overall scaling of the metric. We obtain the 2+1-dimensional cosmological metric

$$ds^2 = -dt^2 + (1 - t^2)(d\theta^2 + \sin^2 \theta d\varphi^2) \quad (38)$$

with a topology $T \times S^2$. The scalar field takes the form

$$\phi(t) = \frac{1}{\sqrt{2}} \ln \left(\frac{1+t}{1-t} \right). \quad (39)$$

Starting from $t = 0$, this 2+1-cosmological spacetime with its em/scalar field collapses to the singularity at $t = 1$. This corresponds to the spacelike singularity of colliding waves $au + bv = \frac{\pi}{2}$. Note that this form of the metric can be extended to include a cosmological constant. To this end, one can consider an ansatz metric

$$ds^2 = -dt^2 + B^2(t) (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (40)$$

which contains a scalar field $\phi(t)$ and a cosmological constant Λ , and reduce the field equation to

$$\frac{1}{2} \dot{\phi}^2 = -\frac{\ddot{B}}{B} + \Lambda = \frac{1 + \dot{B}}{B^2} - \Lambda \quad (41)$$

in which a dot denotes time derivative. Among a large class of solutions which we shall ignore the interesting case is when the scalar field vanishes. This results in

$$ds^2 = \frac{1}{|\Lambda|} \left(-dt^2 + \cosh^2 t (d\theta^2 + \sin^2 \theta d\varphi^2) \right) \quad (42)$$

as a regular model shaped entirely by Λ . Participation of Λ in the wave collision problem, however, remains to be seen.

7 Relation with the Schwarzschild metric

The interaction region of colliding scalar waves and also the em waves can be related to the 3 + 1-dimensional vacuum Schwarzschild metric through the Kaluza–Klein formalism [13, 14]. Herein, we shall concentrate only on the scalar field. The sourceful 2 + 1-dimensional geometry (1) is extended to the 3 + 1-dimensional vacuum as follows. We introduce an extra coordinate y , as a new dimension and consider the 3 + 1-dimensional line element

$$ds^2 = e^{\sqrt{2}\phi} \left(-2f du dv + r^2 dx^2 \right) + e^{-\sqrt{2}\phi} dy^2 \quad (43)$$

in which f , r , ϕ are the functions from (10), (11) and (19). Upon substitution of these functions from the solution obtained and by applying the transformation

$$\sin(au + bv) = R - 1 \quad (44)$$

$$\sin(au - bv) = \cos \theta \quad (45)$$

we obtain

$$ds^2 = \frac{1}{2ab} \left(\frac{dR^2}{1 - \frac{2}{R}} + R^2 d\theta^2 \right) + R^2 \sin^2 \theta dx^2 + \frac{2 - R}{R} dy^2 \quad (46)$$

Next, we make the scaling transformations

$$x \rightarrow \frac{\varphi}{\sqrt{2ab}} \quad (47)$$

$$y \rightarrow \frac{t}{\sqrt{2ab} M} \quad (48)$$

$$R \rightarrow \frac{R}{M} \quad (49)$$

$$ds \rightarrow \sqrt{2ab} M ds \quad (50)$$

which yields the Schwarzschild metric

$$ds^2 = - \left(1 - \frac{2M}{R} \right) dt^2 + \frac{dR^2}{\left(1 - \frac{2M}{R} \right)} + R^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (51)$$

That is, as a higher dimensional vacuum spacetime creates source upon dimensional reduction, the opposite process take place in the present colliding wave metric of 2 + 1-dimensions.

8 Conclusion and discussion

Solution for colliding particular em/scalar shock waves are found in 2 + 1-dimensions. This has been possible by choosing an appropriate conformal factor to the colliding waves. Interestingly for both types of fields, the spacetime emerges identical, which is isometric to a cosmological model with $T \times S^2$ topology. In our colliding wave metric, there are spacelike, null and timelike singularities in the interaction region. Finding regular colliding wave solutions in 2 + 1-dimensions is much desirable. We recall for comparison that although scalar wave collisions were also singular in 3 + 1-dimensions, apart from the null singularities, regular colliding em waves in the interaction region was available [7]. It is also shown that by extension from 2 + 1 to 3 + 1-dimensions, we obtain the Schwarzschild metric. Consideration of cosmological constant in colliding waves prior to the collision also remains to be seen. Collision of different fields, such as em with scalar and neutrino fields also deserve to be searched in 2 + 1-dimensions. In conclusion, we expect that the solutions presented will add new momentum to the 2 + 1-dimensions already highlighted by the BTZ black hole.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This theoretical work does not use or produce numerical data.]

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.
Funded by SCOAP³.

References

1. M. Bañados, C. Teitelboim, J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992)
2. S. Deser, R. Jackiw, G. 'tHooft, Ann. Phys. (NY). **152**, 220 (1984)
3. J.B. Griffiths, *Colliding Plane Waves in General Relativity* (Dover Publications Inc., New York, 2016)
4. C.K. Klimcik, P. Kolnik, Phys. Rev. D **48**, 616 (1993)
5. M. Gutperle, B. Pioline, JHEP **09**, 061 (2003)
6. B. Chen, C.-S. Chu, K. Furuta, F.-L. Lin, JHEP **02**, 020 (2004)
7. P. Bell, P. Szekeres, Gen. Relativ. Gravit. **5**, 275 (1974)
8. S.H. Mazharimousavi, M. Halilsoy, Class. Quantum Gravity **37**, 137001 (2020)
9. Z.C. Wu, J. Phys. A **15**, 2429 (1982)
10. M. Halilsoy, Lett. Nuovo Cimento **44**, 544 (1985)
11. C.J.S. Clarke, S.A. Hayward, Class. Quantum Gravity **6**, 615 (1989)
12. I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1965)
13. G.W. Gibbons, G.T. Horowitz, P.K. Townsend, Class. Quantum Gravity **12**, 297 (1995)
14. E. Halilsoy, M. Halilsoy, O. Unver, J. Math. Phys. **47**, 012502 (2006)