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Complete complementarity relations for three-flavor neutrino oscillations

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Abstract. We study quantum correlations encoded in a three-flavor neutrino system by using complete complementarity relations (CCR). Due to the presence of local coherence in two-flavor subsystems, the CCR has an additional contribution not present in the two flavor mixing case. We investigate such coherence for the three possible bipartite subsystems of the global state both for an electron and a muon neutrino system.

1. Introduction

In the last few years, elementary particles as neutrinos have been investigated in the context of quantum information [1]. In fact, the property of neutrinos to interact very weakly and to deeply penetrate into matter makes these particles interesting candidates for applications of quantum information beyond photons[1]. The characterization of quantum correlations in such systems is therefore important for the development of algorithms and protocols that can harness not only quantum entanglement but also other resources, such as quantum discord [3] and coherence [4]. Such quantum correlations have been studied and probed in quantum optics and condensed matter physics, but only recently they have been investigated in relativistic systems of neutrinos and mesons [5]-[19]. In particular, the phenomenon of neutrino oscillations offers a rare example of quantum correlation on macroscopic scale.

The quantum nature of neutrino oscillations has been studied in terms of entanglement [5]–[8], Bell and Leggett-Garg inequalities [11]–[14] and various aspects of quantum coherence such as steering [16, 17] and Non-local Advantage of Quantum Coherence [18]–[21]. Neutrino oscillations have also been considered in the context of entropic uncertainty relations [22, 23].

Complete complementarity relations provide a way to characterize quantum correlations in bi- and multi-partite systems [24] and can be applied to the description of quantum correlations intrinsic to neutrino systems [25]. The concept of complementarity is often associated with wave-particle duality [26]: it is summarized in the statement that a quantum system may possess properties which are equally real but mutually exclusive, in the sense that the more information one has about one aspect of the system, the less information can be obtained about

[1] Classical communication using a neutrino beam was demonstrated in [2].

the other. In the context of the two-slit experiment, CCRs can be formalized [27, 28] by defining a predictability P, associated to the knowledge of the path of the particle, and a visibility V, connected with the capacity to distinguish the interference fringes:

$$P^2 + V^2 \leq 1. \tag{1}$$

Complementarity relation as in Eq.(1) are saturated only for pure single-partite quantum states. In [29] it is shown that for a bipartite state we have to consider a third entry C – representing the correlation between the subsystems – in order to obtain a complete complementarity relation:

$$P_k^2 + V_k^2 + C^2 = 1, \qquad k = 1, 2. \tag{2}$$

The quantities associated with the wave-particle duality generate local, single-partite realities, while C generates an exclusive bipartite nonlocal reality. In [24] it has been shown that the CCR can be efficiently expressed in terms of the elements of the density matrix representing the system.

In this work, we briefly review the formalism of CCR for bipartite states and its application to two-flavor neutrino system in the plane-wave approximation. Then, we consider the extension to the case of three flavors (tripartite system).

The interesting new feature in the three-flavor case with respect to the results of Ref.[25] is the presence of non-vanishing local coherences for the elements of the possible bipartitions of the system. We investigate in detail such coherences for an electron and a muon neutrino system.

2. Formalism of CCR

2.1. CCR for bipartite states

Let us consider [24] a bipartite state represented as a vector in the Hilbert state $\mathcal{H}_A \otimes \mathcal{H}_B$ of dimension $d = d_A d_B$, where d_A and d_B are the dimension of the subsystem A and B, respectively. $\{|i\rangle_A \otimes |j\rangle_B = |i,j\rangle_{AB}\}_{i,j=0}^{d_A-1,d_B-1}$ represents an orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\{|i\rangle_A\}_{i=0}^{d_A-1}$ and $\{|j\rangle_B\}_{j=0}^{d_B-1}$ are the local basis for the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. In this basis, the density matrix of any bipartite state is:

$$\rho_{A,B} = \sum_{i,k=0}^{d_A-1} \sum_{j,l=0}^{d_B-1} \rho_{ij,kl} \, |i,j\rangle_{AB} \langle k,l| \,. \tag{3}$$

The state of subsystem A(B) is obtained by tracing over B(A). For example, for subsystem A, we have:

$$\rho_A = \sum_{i,k=0}^{d_A-1} \left(\sum_{j=0}^{d_B-1} \rho_{ij,kj} \right) |i\rangle_A \langle k| \equiv \sum_{i,k=0}^{d_A-1} \rho_{ik}^A \, |i\rangle_A \langle k| \,, \tag{4}$$

with a similar form for the subsystem B.

In general, even if the joint state $\rho_{A,B}$ is pure, the states of the subsystems A and B are not pure, which implies that some information is missing when the state of a single subsystem is considered. The missing information is being shared via correlations with the subsystem B [30]. Hence, the complete complementarity relation to consider is:

$$P_{\rm hs}(\rho_A) + C_{\rm hs}(\rho_A) + C_{\rm hs}^{nl}(\rho_{A|B}) = \frac{d_A - 1}{d_A} \tag{5}$$

where $P_{\rm hs}(\rho_A) = \sum_{i=0}^{d_A-1} (\rho_{ii}^A)^2 - \frac{1}{d_A}$ is the predictability measure and $C_{\rm hs}(\rho_A) = \sum_{i\neq k}^{d_A-1} |\rho_{ik}^A|^2$ is the Hilbert-Schmidt quantum coherence, a measure of visibility and $C_{\rm hs}^{nl}(\rho_{A|B}) =$

$\sum_{i\neq k,j\neq l}|\rho_{ij,kl}|^2 - 2\sum_{i\neq k,j<l}\mathrm{Re}(\rho_{ij,kj}\rho^*_{il,kl})$ is called non local quantum coherence, that is the coherence shared between A and B.

Another form of CCR can be obtained by defining the predictability and the coherence measures in terms of the von Neumann entropy:

$$C_{\mathrm{re}}(\rho_A) + P_{\mathrm{vn}}(\rho_A) + S_{\mathrm{vn}}(\rho_A) = \log_2 d_A, \tag{6}$$

where $C_{\mathrm{re}}(\rho_A) = S_{\mathrm{vn}}(\rho_{A,\,\mathrm{diag}}) - S_{\mathrm{vn}}(\rho_A)$ is the relative entropy of coherence, with $S_{\mathrm{vn}}(\rho)$ denoting the von Neumann entropy of ρ, and $\rho_{A,\,\mathrm{diag}} = \sum_{i=1}^{d_A}\rho^A_{ii}\,|i\rangle\langle i|$. $P_{\mathrm{vn}}(\rho_A) \equiv \log_2 d_A - S_{\mathrm{vn}}(\rho_{A,\,\mathrm{diag}})$, is a measure of predictability.

2.2. CCR for tripartite states

In [24] the generalization of the CCR for tri-partite pure states is obtained. Let us consider a tri-partite state represented as a vector in the Hilbert state $\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_C$ of dimension $d = d_A d_B d_C$, where d_A, d_B, d_C are the dimension of the subsystem A, B and C, respectively. $\{|i\rangle_A\otimes|j\rangle_B\otimes|k\rangle_C = |i,j,k\rangle_{ABC}\}_{i,j,k=0}^{d_A-1,d_B-1,d_C-1}$ represents an orthonormal basis for $\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_C$, where $\{|i\rangle_A\}_{i=0}^{d_A-1}$, $\{|j\rangle_B\}_{j=0}^{d_B-1}$ and $\{|k\rangle_C\}_{k=0}^{d_C-1}$ are the local basis for the spaces $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively. In this basis, the density matrix of any tri-partite state is:

$$\rho_{A,B,C} = \sum_{i,l=0}^{d_A-1}\sum_{j,m=0}^{d_B-1}\sum_{k,n=0}^{d_C-1}\rho_{ijk,lmn}\,|i,j,k\rangle_{ABC}\,\langle l,m,n|\,. \tag{7}$$

The state of subsystem A is obtained by tracing over B and C:

$$\rho_A = \sum_{i,l=0}^{d_A-1}\left(\sum_{j=0}^{d_B-1}\sum_{k=0}^{d_C-1}\rho_{ijk,ljk}\right)|i\rangle_A\langle l| \equiv \sum_{i,l=0}^{d_A-1}\rho^A_{il}\,|i\rangle_A\langle l|\,, \tag{8}$$

with a similar form for the subsystems B and C.

The complete complementarity relation to consider for subsystem A is:

$$P_{\mathrm{hs}}(\rho_A) + C_{\mathrm{hs}}(\rho_A) + C^{nl}_{\mathrm{hs}}(\rho_{A|BC}) = \frac{d_A-1}{d_A} \tag{9}$$

where, in this case, the non local coherence is given by:

$$\begin{aligned} C^{nl}_{\mathrm{hs}}(\rho_{A|BC}) = &\sum_{i\neq l}\left(\sum_{j\neq m,k\neq n} + \sum_{j=m,k\neq n} + \sum_{j\neq m,k=n}\right)|\rho_{ijk,lmn}|^2 \\ &- 2\sum_{i\neq l}\left(\sum_{j=m,k<n} + \sum_{j<m,k=n} + \sum_{j<m,k\neq n}\right)\mathrm{Re}(\rho_{ijk,ljk}\rho^*_{imn,lmn}). \end{aligned} \tag{10}$$

The other form of the CCR, Eq.(6), is still valid for the single-partite subsystems A, B and C. But the interesting behaviour comes out when we consider the three possible bipartite subsystems AB, AC and BC. Indeed, as it is shown in the next sections, in which we apply this formalism to a neutrino system, the local coherences for bipartite subsystems are non-zero, in contrast to the local coherence of a single-partite subsystem. For the subsystem AB, for example, is valid the following CCR:

$$C_{\mathrm{re}}(\rho_{AB}) + P_{\mathrm{vn}}(\rho_{AB}) + S_{\mathrm{vn}}(\rho_{AB}) = \log_2(d_A d_B). \tag{11}$$

3. The three-flavor neutrino model

Let us consider a three-flavor neutrino oscillation model, in which the flavor states are written in terms of the mass eigenstates:

$$|\nu_\alpha\rangle = \sum_k U_{\alpha k} |\nu_k\rangle , \tag{12}$$

where $k = 1, 2, 3$ and $\alpha = e, \mu, \tau$. $U_{\alpha k}$ are the elements of a 3×3 unitary PMNS matrix, characterized by the three mixing angles and a CP violating phase:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{CP}} & c_{13}c_{23} \end{pmatrix}, \tag{13}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$, $(i, j = 1, 2, 3)$.

The time evolution of the flavor neutrino state $|\nu_\alpha(t)\rangle$ is given by:

$$|\nu_\alpha(t)\rangle = a_{\alpha e}(t) |\nu_e\rangle + a_{\alpha\mu}(t) |\nu_\mu\rangle + a_{\alpha\tau}(t) |\nu_\tau\rangle , \tag{14}$$

where $a_{\alpha\beta}(t) = \sum_k U_{\alpha k} e^{-iE_k t/\hbar} U^*_{\beta k}$ and E_k is the energy of the k-th mass eigenstate.

The transition flavor probability $P_{\alpha\beta} = |\langle\nu_\beta|\nu_\alpha(t)\rangle|^2$ is given by:

$$P_{\alpha\beta} = \sum_{kl} U^*_{\alpha k} U_{\beta k} U_{\alpha l} U^*_{\beta l} e^{-i(E_k - E_l)t}. \tag{15}$$

For ultra-relativistic neutrinos, we can use the approximation $E_k \simeq E + \frac{m_k^2}{2E}$, leading to $E_k - E_l \simeq \frac{\Delta m^2_{kl}}{2E}$ with $\Delta m^2_{kl} = m^2_k - m^2_l$. E and $L \approx ct$ are the energy and the baseline of the neutrino experiment, respectively. Hence, Eq.(15) becomes:

$$P_{\alpha\beta} = \sum_{kl} U^*_{\alpha k} U_{\beta k} U_{\alpha l} U^*_{\beta l} e^{-i\frac{\Delta m^2_{kl}}{2E} L}. \tag{16}$$

It is also possible to write the oscillation probability in a more convenient way that permits us to separate a constant term to the oscillating one by exploiting the unitary relation $UU^\dagger = \mathbf{1}$, i.e. $\sum_\alpha U_{\alpha k} U^*_{\beta k} = \delta_{\alpha\beta}$. From the square of the unitary relation, by separating the real and imaginary parts of $U^*_{\alpha k} U_{\beta k} U_{\alpha l} U^*_{\beta l}$ we obtain:

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4\,\mathrm{Re}\big(U^*_{\alpha k} U_{\beta k} U_{\alpha l} U^*_{\beta l}\big)\sin^2\left(\Delta m^2_{kl}\frac{Lc^3}{4\hbar E}\right) + 2\sum_{k>l} Im\left(U^*_{\alpha k} U_{\beta k} U_{\alpha l} U^*_{\beta l}\right)\sin\left(\Delta m^2_{kl}\frac{Lc^3}{2\hbar E}\right). \tag{17}$$

In what follows, we will use the following oscillation parameters [31]–[34]:

$$\begin{aligned} \Delta m^2_{21} &= 7.50 \times 10^{-5} eV^2, \\ \Delta m^2_{31} &= 2.46 \times 10^{-3} eV^2, \\ \Delta m^2_{32} &= 2.38 \times 10^{-3} eV^2, \\ \theta_{12} = 33.48^\circ, \theta_{23} &= 42.3^\circ, \theta_{13} = 8.50^\circ. \end{aligned} \tag{18}$$

For simplicity, here we consider $\delta_{CP} = 0$. We can then use the following correspondence [5]:

$$\begin{aligned} |\nu_e\rangle &= |1\rangle_e \otimes |0\rangle_\mu \otimes |0\rangle_\tau = |100\rangle , \\ |\nu_\mu\rangle &= |0\rangle_e \otimes |1\rangle_\mu \otimes |0\rangle_\tau = |010\rangle , \\ |\nu_\tau\rangle &= |0\rangle_e \otimes |0\rangle_\mu \otimes |1\rangle_\tau = |010\rangle , \end{aligned} \tag{19}$$

where it is highlighted the composite nature of neutrino flavor states.

4. CCR for neutrino states

In [25] we have analyzed the CCR for a bipartite neutrino state. We briefly recall the principal results. For example, if we consider an initial electronic neutrino Eq.(14) becomes:

$$|\nu_e(t)\rangle = a_{ee}(t)\,|10\rangle + a_{e\mu}(t)\,|01\rangle\,. \tag{20}$$

By constructing the density matrix for the state $\rho_{A,B}$ and by tracing to obtain the density matrices for the subsystems ρ_A and ρ_B, it is simple to check that Eq.(5) is verified. In fact, $P_{\rm hs}(\rho_A) = P_{ee}^2 + P_{e\mu}^2 - \frac{1}{2}$, $C_{\rm hs}(\rho_A) = 0$ and $C_{\rm hs}^{nl}(\rho_{AB}) = 2P_{ee}P_{e\mu}$, where we use $|a_{ee}(t)|^2 = P_{ee}$, $|a_{e\mu}(t)|^2 = P_{e\mu}$ and $P_{ee} + P_{e\mu} = 1$. Furthermore, it is simple to see that $\rho_A = \rho_{A,\,\rm diag}$ and, consequently, $S_{\rm vn}(\rho_A) = S_{\rm vn}(\rho_{A,\,\rm diag})$. As result, $C_{\rm re}(\rho_A) = 0$, $P_{\rm vn}(\rho_A) = |a_{ee}|^2 \log_2 |a_{ee}|^2 + |a_{e\mu}|^2 \log_2 |a_{e\mu}|^2$ and $S_{\rm vn}(\rho_A) = -|a_{ee}|^2 \log_2 |a_{ee}|^2 - |a_{e\mu}|^2 \log_2 |a_{e\mu}|^2$. Since the dimension of subsystem A is $d_A = 2$, $\log_2 d_A = 1$ and Eq.(6) is satisfied.

It is worth to notice that in the case of a bipartite pure neutrino state, for both Eqs.(5) and (6), the local coherence term is zero. It is natural to ask what happens in the case of a tripartite neutrino state, in which there are bipartite subsystems with their own specific internal structure. We will see indeed that in this case the local coherence terms are non vanishing and depend on the chosen bipartition.

Let us suppose to have a neutrino state in a flavor $\alpha = e, \mu, \tau$ at $t = 0$. The time evolved state is given by:

$$|\nu_\alpha(t)\rangle = a_{\alpha e}(t)\,|100\rangle + a_{\alpha\mu}(t)\,|010\rangle + a_{\alpha\tau}(t)\,|001\rangle\,, \tag{21}$$

The density matrix associated to this state is given by:

$$\rho^\alpha_{e\mu\tau} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho^\alpha_{22} & \rho^\alpha_{23} & 0 & \rho^\alpha_{25} & 0 & 0 & 0 \\ 0 & \rho^\alpha_{32} & \rho^\alpha_{33} & 0 & \rho^\alpha_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho^\alpha_{52} & \rho^\alpha_{53} & 0 & \rho^\alpha_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{22}$$

where the matrix elements are written as:

$$\rho^\alpha_{22} = |a_{\alpha\tau}(t)|^2; \qquad \rho^\alpha_{23} = \rho^{\alpha *}_{32} = a_{\alpha\tau}(t) a^*_{\alpha\mu}(t); \qquad \rho^\alpha_{25} = \rho^{\alpha *}_{52} = a_{\alpha\tau}(t) a^*_{\alpha e}(t); \tag{23}$$

$$\rho^\alpha_{33} = |a_{\alpha\mu}(t)|^2; \qquad \rho^\alpha_{35} = \rho^{\alpha *}_{53} = a_{\alpha\mu}(t) a^*_{\alpha e}(t); \qquad \rho^\alpha_{55} = |a_{\alpha e}(t)|^2. \tag{24}$$

The corresponding oscillation probabilities are $P_{\alpha e}(t) = |a_{\alpha e}(t)|^2$, $P_{\alpha\mu}(t) = |a_{\alpha\mu}(t)|^2$, $P_{\alpha\tau}(t) = |a_{\alpha\tau}(t)|^2$. By tracing with respect one of the subsystems we can obtain the reduced density matrix for bipartite subsystems $e\mu$, $e\tau$, $\mu\tau$, which are, respectively:

$$\rho^\alpha_{e\mu} = \begin{pmatrix} \rho^\alpha_{22} & 0 & 0 & 0 \\ 0 & \rho^\alpha_{33} & \rho^\alpha_{35} & 0 \\ 0 & \rho^\alpha_{53} & \rho^\alpha_{55} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho^\alpha_{e\tau} = \begin{pmatrix} \rho^\alpha_{33} & 0 & 0 & 0 \\ 0 & \rho^\alpha_{22} & \rho^\alpha_{25} & 0 \\ 0 & \rho^\alpha_{52} & \rho^\alpha_{55} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho^\alpha_{\mu\tau} = \begin{pmatrix} \rho^\alpha_{55} & 0 & 0 & 0 \\ 0 & \rho^\alpha_{22} & \rho^\alpha_{23} & 0 \\ 0 & \rho^\alpha_{32} & \rho^\alpha_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

By tracing again we can obtain the reduced density matrices of the single-partite subsystems:

$$\rho^\alpha_e = \begin{pmatrix} \rho^\alpha_{22} + \rho^\alpha_{33} & 0 \\ 0 & \rho^\alpha_{55} \end{pmatrix}, \quad \rho^\alpha_\mu = \begin{pmatrix} \rho^\alpha_{22} + \rho^\alpha_{55} & 0 \\ 0 & \rho^\alpha_{33} \end{pmatrix}, \quad \rho^\alpha_\tau = \begin{pmatrix} \rho^\alpha_{55} + \rho^\alpha_{33} & 0 \\ 0 & \rho^\alpha_{22} \end{pmatrix}. \tag{26}$$

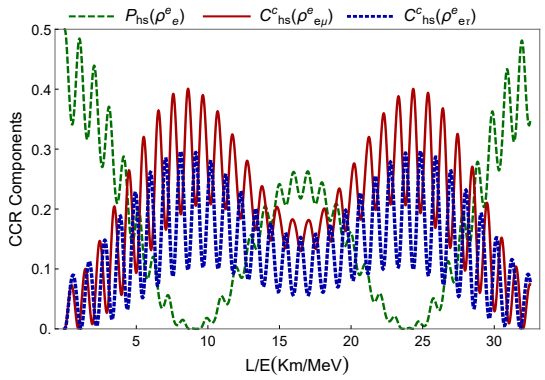


Figure 1: CCR terms, for an initial electronic neutrino, of Eq.(30) as function of L/E.

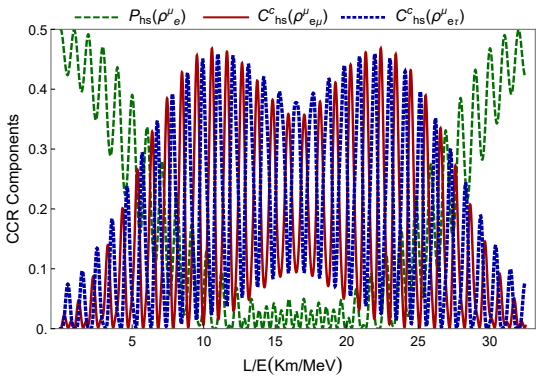


Figure 2: CCR terms, for an initial muonic neutrino, of Eq.(30) as function of L/E.

By following the above prescription, it is simple to evaluate the CCR terms of Eq.(9):

$$P_{\rm hs}(\rho^\alpha_e) = (|a_{\alpha\mu}(t)|^2 + |a_{\alpha\tau}(t)|^2)^2 + |a_{\alpha e}(t)|^2 - \frac{1}{2}, \quad (27)$$

$$C_{\rm hs}(\rho^\alpha_e) = 0, \quad (28)$$

$$C^{nl}_{\rm hs}(\rho^\alpha_{e|\mu\tau}) = 1 - |a_{\alpha e}(t)|^2 - (|a_{\alpha\mu}(t)|^2 + |a_{\alpha\tau}(t)|^2)^2. \quad (29)$$

By summing up all these terms we verify that Eq.(9) is satisfied.

For a state such as in Eq.(21), $C^{nl}_{\rm hs}(\rho^\alpha_{e|\mu\tau}) = C_{hs}(\rho^\alpha_{e\mu}) + C_{hs}(\rho^\alpha_{e\tau})$, i.e. the non-local coherence that the subsystem e shares with $\mu\tau$ is equal to the sum of the bipartite correlations that e shares with μ and τ separately. So, Eq.(9) can be written as:

$$P_{\rm hs}(\rho^\alpha_e) + C_{hs}(\rho^\alpha_{e\mu}) + C_{hs}(\rho^\alpha_{e\tau}) = \frac{1}{2}, \quad (30)$$

with $C_{hs}(\rho^\alpha_{e\mu}) = (a_{\alpha e}(t)a_{\alpha\mu}(t)^*)^2 + (a_{\alpha\mu}(t)a^*_{\alpha e}(t))^2$ and $C_{hs}(\rho^\alpha_{e\tau}) = (a_{\alpha e}(t)a^*_{\alpha\tau}(t))^2 + (a_{\alpha\tau}(t)a^*_{\alpha e}(t))^2$.

Let us now evaluate the terms of Eq.(11) for subsystem $e\mu$. By evaluating the eigenvalues of the reduced density matrices in Eq.(25) we obtain:

$$S_{vn}(\rho^\alpha_{e\mu}) = -(P_{\alpha e} + P_{\alpha\mu})\log_2(P_{\alpha e} + P_{\alpha\mu}) - P_{\alpha\tau}\log_2 P_{\alpha\tau}, \quad (31)$$

$$P_{vn}(\rho^\alpha_{e\mu}) = 2 + P_{\alpha e}\log_2 P_{\alpha e} + P_{\alpha\mu}\log_2 P_{\alpha\mu} + P_{\alpha\tau}\log_2 P_{\alpha\tau}, \quad (32)$$

$$C_{re}(\rho^\alpha_{e\mu}) = -P_{\alpha e}\log_2 P_{\alpha e} - P_{\alpha\mu}\log_2 P_{\alpha\mu} + (P_{\alpha e} + P_{\alpha\mu})\log_2(P_{\alpha e} + P_{\alpha\mu}). \quad (33)$$

For completeness we also evaluate the CCR terms of Eq.(11) for subsystems $e\tau$ and $\mu\tau$. For $e\tau$ bipartition we have:

$$S_{vn}(\rho^\alpha_{e\tau}) = -(P_{\alpha e} + P_{\alpha\tau})\log_2(P_{\alpha e} + P_{\alpha\tau}) - P_{\alpha\mu}\log_2 P_{\alpha\mu}), \quad (34)$$

$$P_{vn}(\rho^\alpha_{e\tau}) = 2 + P_{\alpha e}\log_2 P_{\alpha e} + P_{\alpha\mu}\log_2 P_{\alpha\mu} + P_{\alpha\tau}\log_2 P_{\alpha\tau}, \quad (35)$$

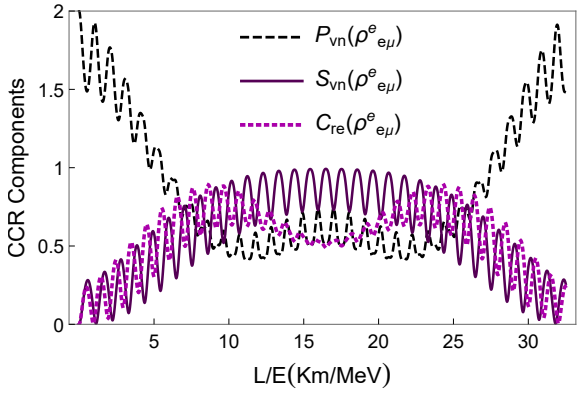
$$C_{re}(\rho^\alpha_{e\tau}) = -P_{\alpha e}\log_2 P_{\alpha e} - P_{\alpha\tau}\log_2 P_{\alpha\tau} + (P_{\alpha e} + P_{\alpha\tau})\log_2(P_{\alpha e} + P_{\alpha\tau}), \quad (36)$$

and for $\mu\tau$ bipartition:

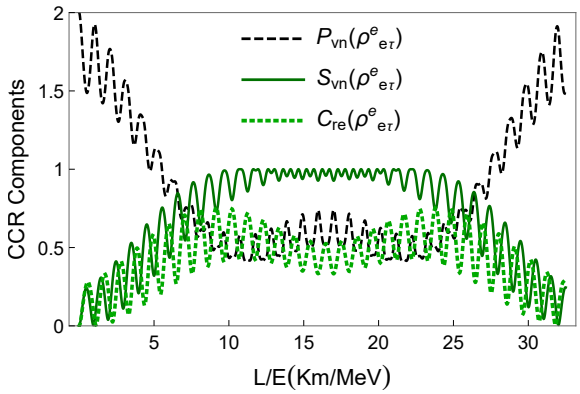
$$S_{vn}(\rho^\alpha_{\mu\tau}) = -(P_{\alpha\mu} + P_{\alpha\tau})\log_2(P_{\alpha\mu} + P_{\alpha\tau}) - P_{\alpha e}\log_2 P_{\alpha e}), \quad (37)$$

$$P_{vn}(\rho^\alpha_{\mu\tau}) = 2 + P_{\alpha e}\log_2 P_{\alpha e} + P_{\alpha\mu}\log_2 P_{\alpha\mu} + P_{\alpha\tau}\log_2 P_{\alpha\tau}, \quad (38)$$

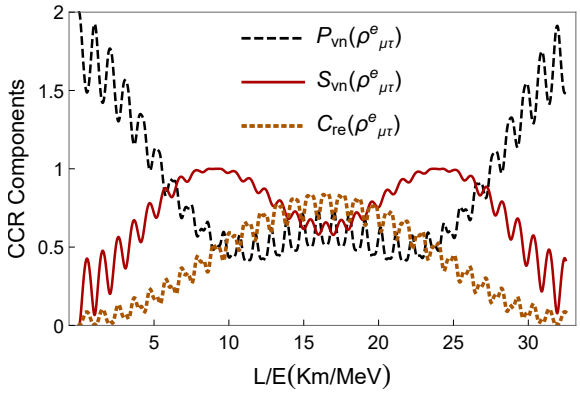
$$C_{re}(\rho^\alpha_{\mu\tau}) = -P_{\alpha\mu}\log_2 P_{\alpha\mu} - P_{\alpha\tau}\log_2 P_{\alpha\tau} + (P_{\alpha\mu} + P_{\alpha\tau})\log_2(P_{\alpha\mu} + P_{\alpha\tau}). \quad (39)$$



(a) $e\mu$ *subsystem*

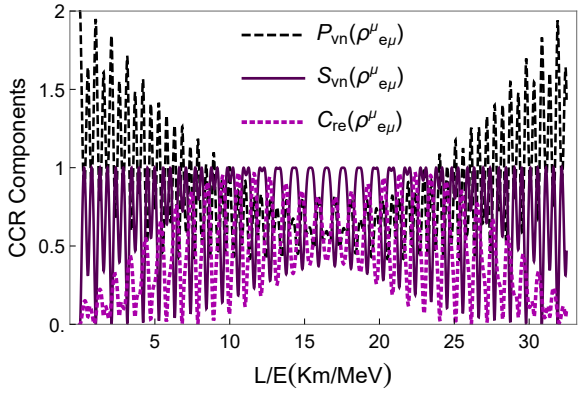


(b) $e\tau$ *subsystem*

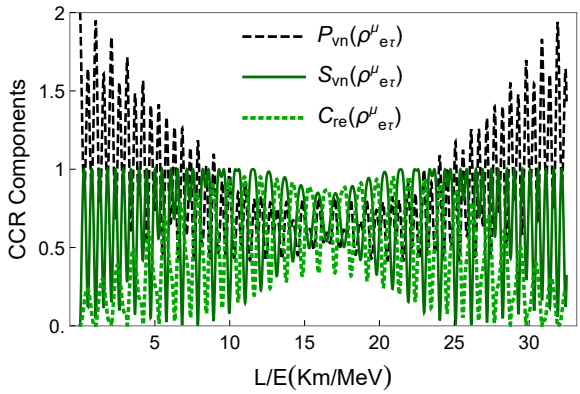


(c) $\mu\tau$ *subsystem*

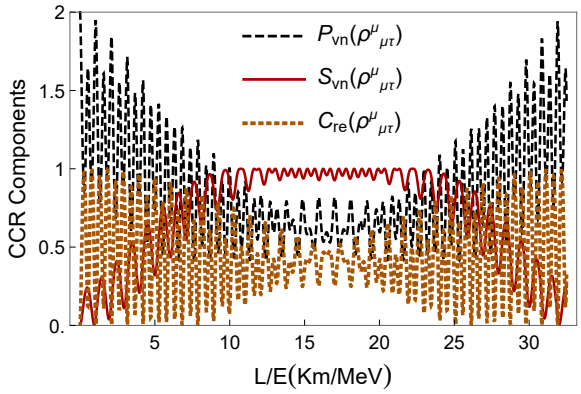
Figure 3: CCR terms for bipartite subsystems $e\mu$, $e\tau$ and $\mu\tau$ as function of L/E in the case of an initial electronic neutrino.



(a) $e\mu$ *subsystem*



(b) $e\tau$ *subsystem*



(c) $\mu\tau$ *subsystem*

Figure 4: CCR terms for bipartite subsystems $e\mu$, $e\tau$ and $\mu\tau$ as function of L/E in the case of an initial muonic neutrino.

4.1. Results for electron neutrino oscillations

Here we show the results for the case of an initial electron neutrino state, i.e. $\alpha = e$. In Fig.1 are plotted the terms of Eq.(30) as function of L/E. We can observe as the bipartite correlations between e and μ is greater than the bipartite correlations between e and τ. So, the term $C_{hs}(\rho^e_{e\mu})$ gives a greater contribution in completing the complementary relation with respect to $C_{hs}(\rho^e_{e\tau})$.

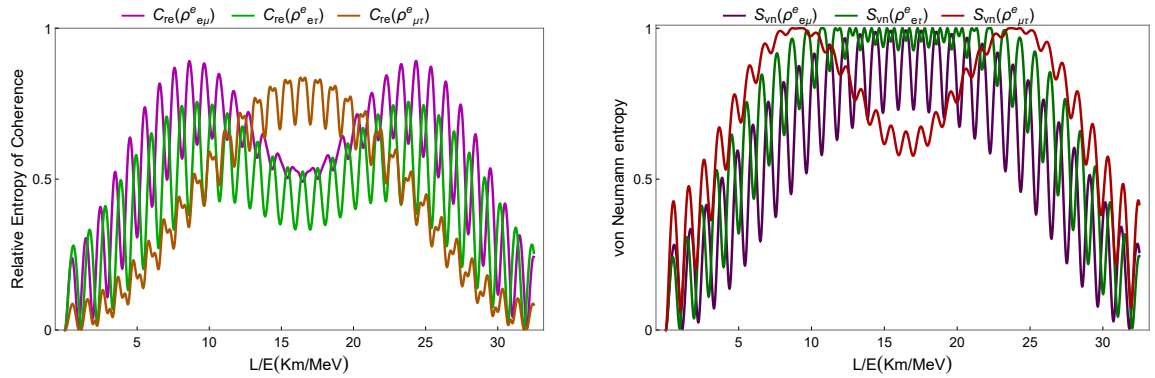


Figure 5: Comparison among $C_{re}(\rho^e_{e\mu})$, $C_{re}(\rho^e_{e\tau})$ and $C_{re}(\rho^e_{\mu\tau})$ (left panel) and $S_{vn}(\rho^e_{e\mu})$, $S_{vn}(\rho^e_{e\tau})$ and $S_{vn}(\rho^e_{\mu\tau})$ (right panel), for an electronic neutrino.

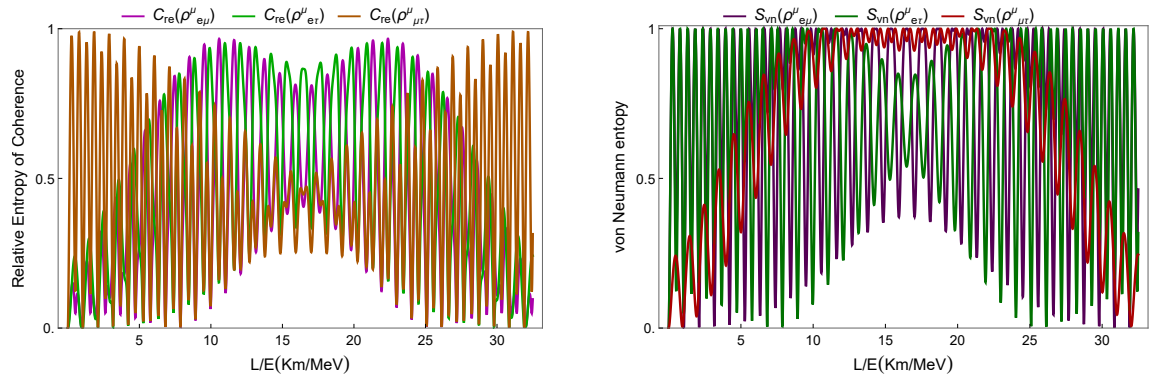


Figure 6: Comparison among $C_{re}(\rho^\mu_{e\mu})$, $C_{re}(\rho^\mu_{e\tau})$ and $C_{re}(\rho^\mu_{\mu\tau})$ (left panel) and $S_{vn}(\rho^\mu_{e\mu})$, $S_{vn}(\rho^\mu_{e\tau})$ and $S_{vn}(\rho^\mu_{\mu\tau})$ (right panel), for a muonic neutrino.

Similar considerations apply to subsystems μ and τ.

In Fig.3, the CCR terms of Eq.(11), written in terms of oscillation probabilities, for subsystems $e\mu$, $e\tau$ and $\mu\tau$ are shown as function of L/E. It is interesting to note the plateau exhibited by the von Neumann entropy in correspondence of its maximum value for $e\tau$ subsystem Fig.3(a), that persists for a relatively large range of L/E. It would be interesting to analyze this aspect for quantum information tasks. On the left panels of Fig.5 we can observe a comparison among the three bipartite local coherences $C_{re}(\rho^e_{e\mu})$, $C_{re}(\rho^e_{e\tau})$ and $C_{re}(\rho^e_{\mu\tau})$. On the right panel of Fig.5 it is shown a comparison among $S_{vn}(\rho^e_{e\mu})$, $S_{vn}(\rho^e_{e\tau})$ and $S_{vn}(\rho^e_{\mu\tau})$, representing the entanglement between subsystems $e\mu - \tau$, $e\tau - \mu$ and $\mu\tau - e$, respectively. It is worth noting how the behaviour of these terms is different depending on the bipartite subsystem considered.

4.2. Results for muon neutrino oscillations

Here we consider the case of an initial muon neutrino state, i.e. $\alpha = \mu$. In Fig.2 are shown the terms of Eq.(30) as function of L/E. We can observe that, differently to the electron case, it is difficult to recognize a dominant contribution of one of the two bipartite correlations, $C_{hs}(\rho^\mu_{e\mu})$ and $C_{hs}(\rho^\mu_{e\tau})$. It is possible to observe that the bigger the one the smaller the other, showing a sort of anti-correlation between them. However, overall, they show the same trend.

In Fig.4 the CCR terms, Eq.(11), written in terms of oscillation probabilities, for subsystems $e\mu$, $e\tau$ and $\mu\tau$ are shown as function of L/E. In this case, the plateau of the von Neumann entropy is exhibited for $\mu\tau$ subsystem (Fig.4(c)), differently to the electron case.

On the left panels of Fig.6 we can observe a comparison among the three bipartite local coherences $C_{re}(\rho^{\mu}_{e\mu})$, $C_{re}(\rho^{\mu}_{e\tau})$ and $C_{re}(\rho^{\mu}_{\mu\tau})$. On the right panel of Fig.6 it is shown a comparison among $S_{vn}(\rho_{e\mu})$, $S_{vn}(\rho_{e\tau})$ and $S_{vn}(\rho_{\mu\tau})$, representing the entanglement between subsystems $e\mu - \tau$, $e\tau - \mu$ and $\mu\tau - e$, respectively. It is worth noting how the behaviour of these terms is different depending on the bipartite subsystem considered.

5. Conclusions

In this paper, we have analyzed the quantumness of a three-flavor pure neutrino state by means of the complete complementarity relations.

In particular, we focused on the quantum coherence, which can be contained either locally or in the correlations. We highlight how, in contrast to the case of a pure bipartite neutrino state, where the local coherences of subsystems are zero, for a tri-partite neutrino system the local coherences are non-vanishing and we investigate them for the three possible bipartite subsystems of the global state both for an electron and a muon neutrino system. We find a dependence of these terms on the particular bipartition chosen.

We plan to extend these consideration by using a wave packet approach for neutrino oscillation. In fact, in this case we expect that at great distances, one of the three local coherences associated with the three bipartitions will dominate the other two.

Our analysis has been performed for the case without CP violating phase, which we plan to include in future work.

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