QUANTIZATION OF A SELF-COUPLED BOSON FIELD (*)

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(presented by L. I. Schiff)

The present work is intended to explore the extent to which the recently observed, multi-pion resonances can be accounted for in terms of a self-coupled pion field, without explicit introduction of a pion-nucleon interaction. A representation is chosen in which the state functions are emphasized, and they are approximated by means of the variational principle. Thus far, the work is confined to the neutral, spin zero, boson field, although it is expected that the same methods will be applied to the pion (unit isospin) field.

We start with the field Hamiltonian

$$H = \int [\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\mu_0^2\phi^2 + \tfrac{1}{4}\lambda_0\phi^4] d^3r\,, \quad (1)$$

and the commutation relations

$$[\phi(\mathbf{r}, t), \pi(\mathbf{r}', t)] = i\delta(\mathbf{r}-\mathbf{r}')\,, \quad (2)$$

which form a covariant system. Units are chosen such that $\hbar = c = 1$; μ_0 and λ_0 are the (unrenormalized) rest mass and coupling constant, respectively. The integration volume in (1) is chosen at first to be a rectangular box of volume V with periodic boundary conditions at the edges; V is later allowed to become arbitrarily large. Apart from the restricted translation invariance associated with V, Eqs. (1) and (2) are also invariant under the substitution $\phi \to -\phi$, $\pi \to -\pi$; thus we expect solutions to possess a quantum number that we shall call amplitude parity.

The field amplitudes are expanded in terms of the normal modes of the box,

$$\phi = V^{-\frac{1}{2}}\sum q_k \exp(i\mathbf{k}\cdot\mathbf{r})\,, \qquad \pi = V^{-\frac{1}{2}}\sum p_k \exp(-i\mathbf{k}\cdot\mathbf{r})\,,$$

from which it follows that $[q_k, p_{k'}] = i\delta_{k,k'}$, with other pairs commuting, and also $q_k^* = q_{-k}$, $p_k^* = p_{-k}$. Rather than express the y's and p's in terms of the usual non-Hermitian creation and destruction operators, we introduce the following Hermitian operators

$$q_k \equiv 2^{-\frac{1}{2}}(x_k + iy_k), \qquad p_k \equiv 2^{-\frac{1}{2}}(X_k - iY_k),$$

$$[x_k, X_{k'}] = [y_k, Y_{k'}] = i\delta_{k,k'}\,,$$

with other pairs commuting. We can then put

$$X_k = -i\partial/\partial x_k, \qquad Y_k = -i\partial/\partial y_k\,,$$

and express H in terms of the x's, y's, and their derivatives. It is slightly more convenient to define cylindrical coordinates

$$x_k \equiv z_k \cos\theta_k\,, \qquad y_k \equiv z_k \sin\theta_k\,,$$

and express H in terms of the z's, θ's, and their derivatives:

$$H = \frac{1}{2}\sum{}' \left\{ -\left[\frac{1}{z_k}\frac{\partial}{\partial z_k}\left(z_k\frac{\partial}{\partial z_k}\right) + \frac{1}{z_k^2}\frac{\partial^2}{\partial\theta_k^2}\right] + \omega_k^2 z_k^2 \right\} +$$

$$+ (\lambda_0/V^2)\int\left[\sum{}' z_k \cos(\mathbf{k}\cdot\mathbf{r} + \theta_k)\right]^4 d^3r\,. \quad (3)$$

Here, the prime on the summation indicates that it extends over half of the $\mathbf{k}$-space, and $\omega_k^2 \equiv \mathbf{k}^2 + \mu_0^2$.

We now apply the variational principle to the approximate determination of the solutions of the equation $H\psi = E\psi$, where E is the quantized field energy. For the lowest state, which represents the physical vacuum, we choose the trial form

$$\psi_0 = \Pi' f_k(z_k, \theta_k)\,, \quad (4)$$

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where the f's are normalized:

$$\int f_k^2 \mathrm{d}\tau_k \equiv \int_0^\infty z_k \mathrm{d}z_k \int_0^{2\pi} (\mathrm{d}\theta_k/2\pi) f_k^2 = 1 .$$

Because of the invariance of H with respect to change of sign of the field amplitude, or replacement of each θ_k by $\theta_k+\pi$, we expect the f's to have definite amplitude parity and the f^2's to be even. Then the expectation value of the Hamiltonian (3) for the trial function (4) is

$$\langle H\rangle = \sum{}' \int f_k H_k^0 f_k \mathrm{d}\tau_k + (3\lambda_0/4V)[\sum{}' \int z_k^2 f_k^2 \mathrm{d}\tau_k]^2 , \quad (5)$$

where H_k^0 is one-half the curly bracket in Eq. (3). In obtaining (5), use was made of the fact that each summation is of order V, and only the leading terms as V becomes infinite were retained. Thus $\langle H\rangle$ depends on a particular f_k only through $\int f_k H_k f_k \mathrm{d}\tau_k$, where

$$H_k = H_k^0 + (3\lambda_0 A/2V) z_k^2 , \quad A \equiv \sum{}' \int z_k^2 f_k^2 \mathrm{d}\tau_k \quad (6)$$

It follows that $\langle H\rangle$ is stationary with respect to variations of the f's when they satisfy the equations $H_k f_k = \varepsilon_k f_k$. Thus the f's are two-dimensional harmonic oscillator functions, and the lowest eigenvalues are given by

$$\varepsilon_k^2 = \omega_k^2 + (3\lambda_0 A/V), \quad A = \sum{}' (1/\varepsilon_k) . \quad (7)$$

These solutions are the best possible of the separable form (4), which combine modes of equal and opposite momentum in an arbitrary way. Our treatment is closely related to the Bogoliubov transformation in the theory of superconductivity, but it is perhaps easier to see by the present method that the solutions obtained are optimal from the point of view of the variational principle.

Mass renormalization may be introduced by replacing μ_0^2 by μ^2-$\delta\mu^2$, where μ is seen below to be the physical particle rest mass and $\delta\mu^2 = 3\lambda_0 A/V$ is the quadratically divergent mass counter term.

The first excited states of the Hamiltonian (3) may be found by using a trial function of the form (4), but allowing one of the f's to be the first excited eigenfunction of the operator (6). The only θ-dependence of this ψ is through a single factor $\exp(\pm i\theta_k)$. Since the total field momentum operator is $\sum' ik(\partial/\partial\theta_k)$, this state has momentum $\mp\mathbf{k}$; it also has odd amplitude parity. The variational principle again shows that this is the optimal state of separable form with this momentum and parity. The energy of the state is found to exceed that of the vacuum by ε_k, where as in (7) we have $\varepsilon_k^2 = \mathbf{k}^2+\mu^2$. Thus the first excited states correspond to single relativistic particles of momentum $\mathbf{k}$ and (renormalized) rest mass μ.

The second excited states are based on functions of the form (4) in which two of the f's are first excited eigenfunctions of (6). The only θ-dependence is through a factor $\exp i(\theta_k - \theta_{K+k})$, so that the state has total momentum $\mathbf{K}$ and even amplitude parity. It is natural to work with a linear combination of such product wave functions, which corresponds to a superposition of states containing particles with momenta $-\mathbf{k}$ and $\mathbf{K}+\mathbf{k}$ for all values of $\mathbf{k}$. Each term in the sum is normalized by itself, and the coefficient is a_k, so that the overall normalization condition is

$$\sum{}' |a_k|^2 = 1 . \quad (8)$$

After some reduction, the expectation value of H for such a two-particle state is found to exceed the vacuum energy by

$$\sum{}' |a_k|^2 (\varepsilon_k + \varepsilon_{K+k}) + (3\lambda_0/2V) \left|\sum{}' a_k/(\varepsilon_k \varepsilon_{K+k})^{\frac{1}{2}}\right|^2 . \quad (9)$$

At this point it is possible to set $\mathbf{K} = 0$, so that the rest of the calculation is performed in the centre-of-mass coordinate system. Variation of the a's to make (9) stationary subject to the normalization restriction (8) is carried out by using an undetermined multiplier E, which subsequently can be shown to be the energy of the two-particle system. The equation that determines E is

$$\sum{}' \frac{1}{\varepsilon_k^2 (E-2\varepsilon_k)} = \frac{2V}{3\lambda_0} ,$$

and the a's are obtained by solving the set

$$(E-2\varepsilon_k) a_k = (3\lambda_0/2V\varepsilon_k) \sum{}' (a_{k'}/\varepsilon_{k'}) .$$

Each a_k may be expanded in spherical harmonics of the direction of $\mathbf{k}$; then since ε_k is independent of direction, it can be shown that the non-spherical parts of a_k correspond to $E = 2\varepsilon_k$ and hence to no scattering. The S-wave scattering phase shift δ can be expressed in terms of the energy shift $E-2\varepsilon_k$ which corresponds to the spherically symmetric part of a_k; the result after some calculation is

$$(\pi k/\varepsilon)\cot\delta = -(16\pi^2/3\lambda_0) - P\int_{\mu}^{\infty}\frac{k'\mathrm{d}\varepsilon'}{\varepsilon'(\varepsilon'-\varepsilon)}, \qquad (10)$$

where P denotes the principal value and the subscript has been dropped from ε_k.

The integral in (10) diverges logarithmically, so we renormalize the coupling constant to absorb the factor $\int_{\mu}^{\infty} k\mathrm{d}\varepsilon/\varepsilon^2$. In terms of the renormalized coupling constant λ, Eq. (10) becomes

$$(\pi k/\varepsilon)\cot\delta = -(16\pi^2/3\lambda) - \varepsilon P\int_{\mu}^{\infty}\frac{k'\mathrm{d}\varepsilon'}{\varepsilon'^2(\varepsilon'-\varepsilon)}; \qquad (11)$$

the last integral is readily evaluated analytically. Eq. (11) may be rewritten in the form

$$e^{i\delta}\sin\delta = \pi r(\varepsilon)/D(\varepsilon), \quad r(\varepsilon) \equiv -(3\lambda/16\pi^2)(k/\varepsilon),$$

$$D(\varepsilon) \equiv 1-\varepsilon\int_{\mu}^{\infty}\frac{r(\varepsilon')\mathrm{d}\varepsilon'}{\varepsilon'(\varepsilon'-\varepsilon-i\eta)},$$

which is closely related to the first approximation of Baker and Zachariasen [1)]. The scattering cross-section is a monotonically decreasing function of the total energy 2ε, which shows a resonance at zero kinetic energy when $\lambda=\lambda_r=-[32\pi^2/3(\pi+2)]=-20.5$. There is also a single bound state ($\varepsilon<\mu$) for $\lambda<\lambda_r$; the energy of this state decreases monotonically to zero as $\lambda\to-\infty$.

Finally, it can be shown that the vacuum energy is not only stationary when the f's in (4) are eigenfunctions of (6), but also that this energy is a minimum with respect to arbitrary second variations of the f's if, and only if, the renormalized coupling constant λ is negative.

Apart from the extension to the pion (unit isospin) field mentioned at the beginning, it is expected that the present methods can be applied to the scattering and bound states of more than two particles.

LIST OF REFERENCES

1. M. Baker and F. Zachariasen, Phys. Rev. *118*, 1659 (1960).

DISCUSSION

BLOKHINTSEV: There is an essential difference in the physical interpretation of the field excitations in the linear and non-linear cases. In the linear case one can consider the n'th excited state of the field $\varphi(x)$ at the point x as representing n identical non-interacting mesons placed at this point. In the non linear case we have to consider this same excitation as representing a single heavy meson placed at x. The energy of this excitation is finite. The possibility of transfer of the excitation to a neighbouring point (movement of the meson) does not change the magnitude.

MARX: Would it be possible to repeat this calculation if μ_0^2 were negative?

SCHIFF: The value of μ_0^2 is unimportant, because it is renormalized away. It could even be zero. The value of the renormalized mass is chosen to equal the experimentally observed mass. It would not be sensible to make this μ_0^2 negative.

MARX: If you have a curve with a double minimum, the situation is similar to that in a hydrogen molecular ion. You will have a degeneracy of the ground state. Then there may or may not be a complete degeneracy. It would be interesting to see the role of the sign.

SCHIFF: I can only say that with a dependence more complicated than φ^4 the analysis would become very difficult.