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How the twofold light cone arises from nonlinear electrodynamics

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Abstract We analyze the propagation of light signals in the context of nonlinear electrodynamics. As a general feature of the nonlinear theories, the superposition principle is no longer satisfied. In the electromagnetic theory, this is due to the self-interactions of the field and light propagation is governed by an effective or optical metric. We present a simple derivation of the two light cones that arise if the Lagrangian depends on the electromagnetic invariants in a nonlinear way. Using the algebraic properties of the electromagnetic tensor $f_{\mu\nu}$, we determine the dispersion relations from the eigenvalues of a Sturm–Liouville equation. It turns out that in the presence of a background field, light propagation can be slower or faster than the one in vacuum. We also derive the corresponding transport vector fields.

1 Introduction

Vacuum polarization appears when strong fields, of the order of the critical field $E_{cr} = B_{cr} = m_e^2 c^3/e\hbar \approx 1.3 \times 10^{18}$ V/m $\approx 4.4 \times 10^{13} G$, are present in a certain region. In this situation, small deviations from the standard results of Classical Electrodynamics arise, leading to the appearance of new phenomena such as photon–photon scattering, vacuum dichroism, photon acceleration and birefringence [1, 2], among others. Moreover, light velocity depends on the polarization [3, 4], and light trajectories follow the null geodesics of an effective or optical metric, which in most cases is non-unique, rather than null geodesics of the geometric spacetime.

These effects are described by Quantum Electrodynamics (QED), however, on a classical level phenomenological features are captured by nonlinear electrodynamics (NLE). Some of these effects, like birefringence, can also be observed in materials with a nonlinear response that can be modeled in an effective way by NLE, where the electromagnetic background plays the role of an optical material. The deviations of light trajectories respect to Maxwell ones can also be described as the propagation in a curved spacetime and in this sense, it resembles light propagating in an effective gravitational field [5]. In other words, gravity emerges from the nonlinearity of the theory, allowing for the establishment of optical analogies in gravitational kinematics [6], effectively curved geometries which guide the propagation of electromagnetic waves in material media whose physical properties depend on an external electric field were derived in [7]. Birefringence as well arises in wave solutions to Maxwell equations in anisotropic backgrounds [8].

Conditions on the NLE Lagrangian that guarantee causality of the optical metrics for all allowed background fields were established in [9]. And in [10], it was shown that NLE is a symmetric hyperbolic theory if and only if light cones these metrics give rise to, have a nonempty intersection, and in that case, the initial-value problem is well posed.

In linear electrodynamics, the algebraic structures of a general electromagnetic field and its energy-momentum tensor in stationary spaces were analyzed in [11]. In [12], was established that shock waves propagate along characteristic surfaces of the Maxwell equations.

NLE theories that are Lorentz invariant and gauge invariant were studied and classified by Plebański [13] with important contributions due to Boillat [14] who obtained the laws of propagation of photons and of charged particles, along with an anisotropic propagation of the wavefronts from a Lagrangian which is an arbitrary nonlinear function of the two electromagnetic invariants.

It is known that for any theory of the Plebański class, the rays are the null geodesics of two optical metrics; this was also pointed out by Novello et al. [5]. Using a different representation Obukhov and Rubilar [15] derived the Fresnel equation for the wave covectors and for the class of local nonlinear Lagrangian nondispersive models, it was demonstrated that the quartic Fresnel equation factorizes, yielding the generic birefringence effect.

So far, there are several ways to derive the eikonal equation for nonlinear fields. In [16] the authors introduce an approximate plane wave Ansatz on an arbitrary general relativistic spacetime and evaluate the "generalized Maxwell" equations; to zeroth order,

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they obtain the eikonal equation an one condition for the polarization plane. The treatment, although being more general, because it is valid in a curved spacetime, it is more involved, and the basic features are not so evident.

In contrast, we present a straightforward derivation of the twofold light cones that, although in a restricted case, captures the essential features on how the twofold light cone emerge. Taking a nonlinear Lagrangian depending on one of the electromagnetic Lorentz invariants we derive the two possible dispersion relations. Our analysis is simpler than others available in the literature [17, 18], which focus on the stress-energy tensor and treat the problem as a propagation in an effective metric, then propose an approximation in terms of the background metric and the stress-energy tensor.

In this work from a general Lagrangian that depends nonlinearly on the electromagnetic invariants F and G and by variation of the action, we obtain the field equations. Then, we restrict to the vacuum case $j^{\alpha} = 0$ and present a simple derivation of the two light cones that arise when the Lagrangian depends nonlinearly on one of the electromagnetic invariants. We analyze the propagation of electromagnetic field discontinuities using Hadamard's description [19] of the first derivative discontinuity of the electromagnetic tensor. Introducing the discontinuity into the electromagnetic field equations, they can be cast as an eigenvalue problem for the vectors on the discontinuity surface. The dispersion relations obeyed by the wave number can be derived imposing the vanishing condition to the eigenvalues, and from the dispersion relation, the transport vector fields are deduced.

This article is organized as follows: In Sect. 2 we review the derivation of the field equations from the NLE Lagrangian, showing the need of introducing the skew symmetric tensor $P_{\mu\nu}$ and its relation with the electromagnetic field tensor $f_{\mu\nu}$. In Sect. 3, the equation for the propagation of the discontinuity of the electromagnetic field is put in the form of an eigenvalue problem whose eigenvalues and eigenvectors are determined; from them, we obtain two possible cones for the propagation of signals, the Maxwell cone and a modified or distorted cone. In Sect. 4, we show that the 4D vector space generated by the eigenvectors of the electromagnetic tensor, can be split into two 2D disjoint vector spaces, and we apply this to the analysis of the modified cone. In Sect. 5, we connect our results with the more familiar formalism of effective or optical metrics and determine the constraint on the polarization of an electromagnetic wave as well as the refraction index of an equivalent material medium. Finally, in Sect. 6, we present some conclusions.

2 Nonlinear electrodynamics

For the sake of completeness and to emphasize how the NLE equations arise, pointing out their structure and the need of introducing a new skew symmetric tensor, $P_{\mu\nu}$, in addition to the electromagnetic field tensor $f_{\mu\nu}$, we present in this section the derivation of the field equations by varying the NLE action.

We utilize a general formalism that can be applied to any locally constructed gauge-invariant field theory based on the two invariants of the Maxwell field, namely F and G. Nonlinear electrodynamics Lagrangians can be written in the form $\tilde{L} = L_f + L_i + L_{NL}$, where the subscripts stand for free, interaction, and nonlinear terms, respectively. The nonlinear terms depend explicitly on the invariants G and F, defined by

$$F = \frac{1}{4} f_{\mu\nu} f^{\mu\nu} = \frac{1}{2}(B^2 - E^2), \quad G = \frac{1}{4} f_{\mu\nu} \tilde{f}^{\mu\nu} = -\vec{E} \cdot \vec{B}, \tag{1}$$

where $f_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ is the field-strength of the electromagnetic field, and A_{ν} is the electromagnetic potential. $\tilde{f}^{\mu\nu} = (1/2)\varepsilon^{\mu\nu\alpha\beta} f_{\alpha\beta}$ is the dual stress-tensor, and $\varepsilon^{\mu\nu\alpha\beta}$ represents the Levi-Civita symbol. We start with the action written as

$$W = \int_{\Sigma_1}^{\Sigma_2} \left[-m_0 c^2 - \frac{1}{c^2} j^{\alpha} A_{\alpha} - \frac{1}{4\pi c} L(F, G) \right] ds, \tag{2}$$

denoting $L_{NL}(F, G) = L(F, G)$ for short, m_0 is the mass of a charged test particle; j^{α} represents the sources of electrical or magnetic fields, charges and currents, that in the discrete case is a sum of charges and currents, while it is an integral of charge and current densities in the continuous case, and Σ_1, Σ_2 are arbitrary surfaces crossed by the particle world sheet on its trajectory.

2.1 The Lorentz force equation

By varying the action respect to the coordinates x^{α} and imposing the least action principle $\delta W = 0$, we obtain the Euler–Lagrange equations,

$$\frac{d}{ds} \left(\frac{\partial \tilde{L}}{\partial \dot{x}_{\alpha}} \right) - \frac{\partial \tilde{L}}{\partial x_{\alpha}} = 0,$$

where $\tilde{L} = -m_0 c^2 - \frac{1}{c^2} j^{\alpha} A_{\alpha} - \frac{1}{4\pi c} L(F, G)$ and $\dot{x}^{\alpha} = u^{\alpha}$, is the 4-velocity of the test charge. The Euler–Lagrange equations, with the notation $dv/dx^{\alpha} = v_{,\alpha}$ and $dv/du = v_{,u}$ are explicitly,

$$\frac{d}{ds} \left[\frac{\partial}{\partial u_{\alpha}} \left(-m_0 c^2 - \frac{1}{c^2} j^{\beta} A_{\beta} - \frac{1}{4\pi c} L(F, G) \right) \right] - \frac{\partial}{\partial x_{\alpha}} \left[-m_0 c^2 - \frac{1}{c^2} j^{\beta} A_{\beta} - \frac{1}{4\pi c} L(F, G) \right] = 0.$$

Using that $u_\alpha u^\alpha = c^2$, and $\partial u_\beta / \partial u_\alpha = \delta_\beta{}^\alpha$ we obtain,

$$m_0 \frac{d}{ds}(u^\alpha) - \frac{e}{c^2}\frac{d}{ds}(A^\alpha) - \frac{1}{4\pi c}\left[\frac{d}{ds}\left(\frac{\partial L(F, G)}{\partial u_\alpha}\right) - \frac{\partial L(F, G)}{\partial x^\alpha}\right] + \frac{e}{c^2}u^\beta A_{\beta,}{}^\alpha = 0.$$

Demanding that $L(F, G)$ satisfies the Euler–Lagrange equations, the third term is zero and using $d/ds = (dx^\beta/ds)\,(d/dx_\beta) = u^\beta\,d/dx_\beta$, we arrive at the Lorentz force equation,

$$m_0 c \frac{du^\alpha}{ds} = -\frac{e}{c}u^\beta f^\alpha{}_\beta. \tag{3}$$

This equation describes the kinematics followed by a particle with mass m_0 in the presence of the electromagnetic field $f^\alpha{}_\beta$ [20]. No changes are introduced by the inclusion of $L(F, G)$, not as long as it complies with the Euler–Lagrange equations.

2.2 The electromagnetic field equations

The electromagnetic field equations are obtained by a variation of the action (2) respect to the potential A_α. Note that only the second and third term of the action depend on A_α, the dependence of the second term is linear and the one of the third term is through the electromagnetic tensor $f_{\mu\nu}$ from the invariants F and G. Let's us compute this variation,

$$\delta_{A_\alpha} L(F, G) = L_F \delta F + L_G \delta G = L_F \delta(\frac{1}{4} f_{\mu\nu} f^{\mu\nu}) + L_G \delta(\frac{1}{4} f_{\mu\nu} \tilde{f}^{\mu\nu}) = \frac{1}{2}L_F f^{\mu\nu}\delta f_{\mu\nu} + \frac{1}{2}L_G \tilde{f}^{\mu\nu}\delta f_{\mu\nu},$$

where $dL/dF = L_F$ and using that $f^{\mu\nu}\delta f_{\mu\nu} = f^{\mu\nu}(\delta A_{\nu,\mu} - \delta A_{\mu,\nu}) = (-f^{\nu\mu})\delta A_{\nu,\mu} - f^{\mu\nu}\delta A_{\mu,\nu} = -2f^{\mu\nu}\delta A_{\mu,\nu}$, then

$$\delta_{A_\alpha} L(F, G) = -\left[L_F f^{\mu\nu} + L_G \tilde{f}^{\mu\nu}\right]\delta A_{\mu,\nu}.$$

Using the previous relations, the variation of the action Eq. (2) respect to A_α is given by

$$\delta_{A_\alpha} W = -\frac{1}{c^2}\int_\Omega j^\alpha \delta A_\alpha d^4 x + \frac{1}{4\pi c}\int_\Omega \left[L_F f^{\mu\nu} + L_G \tilde{f}^{\mu\nu}\right]\delta A_{\mu,\nu} d^4 x;$$

at this point, it is convenient to define the skew symmetric tensor $P^{\mu\nu}$ as,

$$P^{\mu\nu} = L_F f^{\mu\nu} + L_G \tilde{f}^{\mu\nu}, \tag{4}$$

then the variation of the action respect to the potential can be written as

$$\delta_{A_\alpha} W = -\frac{1}{c^2}\int_\Omega j^\alpha \delta A_\alpha d^4 x - \frac{1}{4\pi c}\int_\Omega \delta A_\mu \left(\frac{\partial}{\partial x^\nu}P^{\mu\nu}\right)d^4 x + \frac{1}{4\pi c}\int_\Omega \frac{\partial}{\partial x^\nu}\left(\delta A_\mu P^{\mu\nu}\right)d^4 x,$$

by applying the Gauss's theorem to the third term, passing from a volume integral $\int_\Omega d^4 x$ to a surface integral $\int_{\partial\Omega} d^3 x$ of $\delta A_\mu P^{\mu\nu}$ over the surface $\partial\Omega$ that encloses the volume Ω and considering that $P^{\mu\nu}$ is localized in a finite region, then, when the surface that encloses the volume tends to infinity, the integral vanishes and we have

$$\delta_{A_\alpha} W = -\frac{1}{c}\int_\Omega \left[\frac{1}{c}j^\alpha \delta A_\alpha + \frac{1}{4\pi}\delta A_\mu \frac{\partial}{\partial x^\nu}P^{\mu\nu}\right]d^4 x$$

$$= -\frac{1}{c}\int_\Omega \left[\frac{1}{c}j^\mu + \frac{1}{4\pi}\frac{\partial}{\partial x^\nu}P^{\mu\nu}\right]\delta A_\mu d^4 x,$$

imposing that $\delta_{A_\alpha} W = 0$, then, the fields equations result in

$$P^{\mu\nu}{}_{,\nu} = -\frac{4\pi}{c}j^\mu. \tag{5}$$

Equation (5) has the same form than Maxwell's equations for $f^{\mu\nu}$, i.e., $f^{\mu\nu} \to P^{\mu\nu}$, then, we can identify $P^{\mu\nu}$ as the field produced by the source j^μ that obeys the continuity equation $j^\mu{}_{,\mu} = 0$. Meanwhile, $f^{\mu\nu}$ can be treated like an effective field associated with $P^{\mu\nu}$ by the structure equations (4) and being the field that drives the particle motion through (3). The Faraday-Maxwell equations express the necessary and sufficient conditions that $f_{\mu\nu}$ is a curl, by means of $f_{[\mu\nu,\rho]} = O \iff f_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$, and that complete the set of the electrodynamical equations in NLE. At this point, we have derived the dynamics of the field and kinematics of a charged test particles using the least action principle, following mainly [21].

3 The eigenvalue problem for the propagation of discontinuities

We restrict ourselves to Lagrangians depending on only one of the invariants, $L = L(F)$, since it is sufficient to exhibit the main features of the emergence of the twofold light cone.

Substituting Eq. (4) into (5), and considering that $j^\mu = 0$, we obtain

$$P^{\mu\nu}{}_{,\nu} = \frac{\partial}{\partial x^\nu}\big(L_F f^{\mu\nu}\big) = L_{FF}\frac{\partial F}{\partial x^\nu} f^{\mu\nu} + L_F f^{\mu\nu}{}_{,\nu} = 0.$$

Using that

$$F = \frac{1}{4} f^{\mu\nu} f_{\mu\nu}, \quad \frac{\partial F}{\partial x^\alpha} = \frac{1}{2} f^{\mu\nu} f_{\mu\nu,\alpha},$$

the field equation (3) becomes,

$$\frac{1}{2} L_{FF} f^{\alpha\beta} f_{\alpha\beta,\nu} f^{\mu\nu} + L_F f^{\mu\nu}{}_{,\nu} = 0. \tag{6}$$

Note that Eq. (6) depends on the first and second derivatives of the Lagrangian, L_{FF}, L_F. We shall introduce in Eq. (6), the propagation of the electromagnetic field discontinuities, through the characteristic surfaces.

We shall analyze the propagation of an electromagnetic discontinuity based on the Hadamard setting [19]. Let a hypersurface Σ be defined by $\xi(x^\alpha) = 0$; the spacetime is divided into the half-spaces $U^-(\xi < 0)$ and $U^+(\xi > 0)$. The discontinuity of a function $f(x^\alpha)$ across Σ is defined by

$$[f]|_\Sigma = \lim_{P^-,P^+ \to P} \big\{ f(P^-) - f(P^+) \big\}, \tag{7}$$

where $P^- \in U^-$ and $P^+ \in U^+$ and $P \in \Sigma$.

Let us consider that the electromagnetic potential A_α is discontinuous in its second derivatives at the characteristic surface Σ, where $\xi(x) = 0$. We shall analyze the propagation of the discontinuity assuming that the potential A_α and its first derivative $A_{\alpha,\mu}$ are continuous, and that at least some of its second derivatives are not. Then, we assume that

$$[A_\alpha] = 0, \quad [A_{\alpha,\beta}] = 0, \quad [A_{\alpha,\beta\gamma}] \neq 0. \tag{8}$$

Following Hadamard's lemma [19], we can write the jump of the second derivatives of A_α in the form

$$[A_{\alpha,\beta\gamma}(x)]|_\Sigma = \xi_\alpha k_\beta k_\gamma, \tag{9}$$

where ξ_α is a vector related to the polarization that is located on the surface Σ where the discontinuity of the field lies; k_β is the wave 4-vector, which is normal to Σ, $\partial_\beta \Sigma = k_\beta$. Since $f_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, then, $f_{\alpha\beta}$ is discontinuous in its first derivatives, such that

$$[f_{\alpha\beta,\gamma}]|_\Sigma = \xi_\beta k_\alpha k_\gamma - \xi_\alpha k_\beta k_\gamma = \big(\xi_\beta k_\alpha - \xi_\alpha k_\beta\big)k_\gamma. \tag{10}$$

To analyze the propagation of the discontinuity, we substitute (10) into (6) and obtain an equation for the surface of discontinuity,

$$\frac{1}{2} L_{FF} f^{\alpha\beta}\big(\xi_\beta k_\alpha - \xi_\alpha k_\beta\big)k_\nu f^{\mu\nu} + L_F\big(\xi^\nu k^\mu - \xi^\mu k^\nu\big)k_\nu = 0. \tag{11}$$

Permuting the subscripts α and β Eq. (11) can be cast as a linear operation over ξ^μ

$$\Big[-L_{FF}\big(f_\nu{}^\alpha k_\alpha\big)\big(f^{\mu\beta} k_\beta\big) + L_F k^\mu k_\nu - L_F k^2 \delta^\mu{}_\nu\Big]\xi^\nu = 0, \tag{12}$$

where we have denoted $k^2 = k_\nu k^\nu$. Equation (12) is known as the eikonal equation for Σ and as well can be considered as an algebraic condition on the polarization ξ^ν. Defining $\Theta^\alpha = f^{\alpha\beta} k_\beta$ and the operator $Y^\mu{}_\nu$ as

$$Y^\mu{}_\nu = -L_{FF}\Theta^\mu\Theta_\nu + L_F k^\mu k_\nu - L_F k^2 \delta^\mu{}_\nu, \tag{13}$$

Eq.(12) becomes the eigenvalue equation for the operator $Y^\mu{}_\nu$ acting on ξ^ν,

$$Y^\mu{}_\nu \xi^\nu = 0; \tag{14}$$

therefore, ξ^ν is an eigenvector with zero eigenvalue of the symmetric operator $Y^\mu{}_\nu$. In what follows we show that imposing the vanishing condition to the eigenvalues of $Y^\mu{}_\nu$ we arrive at the dispersion relations.

3.1 Eigenvectors and eigenvalues

In this subsection, we find the eigenvectors in the kernel of $Y^\mu{}_\nu$; and then imposing the vanishing condition to their eigenvalues we determine the twofold light cone.

Let us consider a vector ξ_1^μ that is parallel to the wave 4-vector, $\xi_1^\mu = \alpha k^\mu$, with α being an arbitrary constant. Applying $Y^\mu{}_\nu$ to ξ_1^μ we have that, since $k^\mu \Theta_\mu = k^\mu\big(f_{\mu\nu} k^\nu\big) = 0$, then, $Y\xi_1 = 0$ and ξ_1^μ is an eigenvector of Y with zero eigenvalue, that we call $\lambda_1 = 0$. However, $\xi_1^\mu = \alpha k^\mu$ does not give us new information on the discontinuity of the first derivative of $f_{\mu\nu}$. Therefore, for having a nontrivial jump on Σ, we look for eigenvectors with vanishing eigenvalues such that $\xi_i^\mu \neq \alpha k^\mu$; i.e., we should consider

eigenvectors that are linearly independent of k^μ. We shall now determine such eigenvectors and eigenvalues of Y and require the latter to vanish.

Let us consider ξ_2^μ that is orthogonal to both k and Θ, i.e., $k_\nu \xi_2^\nu = 0$ and $\Theta_\nu \xi_2^\nu = 0$, then,

$$\begin{aligned} Y^\mu{}_\nu \xi_2^\nu &= \left[-L_{FF} \Theta^\mu \Theta_\nu + L_F k^\mu k_\nu - L_F k^2 \delta^\mu{}_\nu \right] \xi_2^\nu \\ &= -L_{FF} \Theta^\mu (\Theta_\nu \xi_2^\nu) + L_F k^\mu (k_\nu \xi_2^\nu) - L_F (k_\alpha k^\alpha) \delta^\mu{}_\nu \xi_2^\nu \\ &= -(L_F k^2) \xi_2^\mu \end{aligned}$$

Then, ξ_2^μ is an eigenvector of Y with eigenvalue $\lambda_2 = -L_F k^2$.

Suppose now a vector ξ_3^ν proportional to Θ, $\xi_3^\nu = \beta \Theta^\nu$, β is an arbitrary constant; note that ξ_3^ν is orthogonal to k_μ, $\xi_3^\nu k_\mu = 0$; applying the operator Y on ξ_3^ν,

$$\begin{aligned} Y^\mu{}_\nu \xi_3^\nu &= \left[-L_{FF} \Theta^\mu \Theta_\nu + L_F k^\mu k_\nu - L_F k^2 \delta^\mu{}_\nu \right] \xi_3^\nu \\ &= -L_{FF} \Theta^\mu (\Theta_\nu \xi_3^\nu) + L_F k^\mu (k_\nu \xi_3^\nu) - L_F k^2 \delta^\mu{}_\nu \xi_3^\nu \\ &= -\beta (L_{FF} \Theta^2 + L_F k^2) \Theta^\mu = -(L_{FF} \Theta^2 + L_F k^2) \xi_3^\mu, \end{aligned}$$

then, ξ_3^ν is an eigenvector of Y with eigenvalue $\lambda_3 = -(L_{FF} \Theta^2 + L_F k^2)$.

Therefore, the eigenvalues of Y are $\lambda_1 = 0$, $\lambda_2 = -L_F k^2$, $\lambda_3 = -L_{FF} \Theta^2 - L_F k^2$. Thus, the characteristic equation of Y is

$$Y \left[Y + L_F k^2 \mathbb{I} \right] \left[Y + (L_F k^2 + L_{FF} \Theta^2) \mathbb{I} \right] = 0, \tag{15}$$

where $\mathbb{I}$ denotes the identity matrix. If we impose the condition that $\lambda_i = 0$, $i = 1, 2, 3$; the set $\{\xi_1^\mu, \xi_2^\mu, \xi_3^\mu\}$ spans the kernel of the operator Y. Demanding that $\lambda_2 = 0$, we obtain,

$$\lambda_2 = -L_F k^2 = 0 \Rightarrow k^2 = k_\alpha k^\alpha = 0, \tag{16}$$

recalling that k_α is the normal directions to the characteristic surface, Σ, i.e., k_α is the null directions that generate the light cone in Minkowski space that we call the Maxwell light cone; the wavefronts or characteristics surfaces are normal to the light cone.

From the condition that $\lambda_3 = 0$, we obtain,

$$\lambda_3 = -(L_{FF} \Theta^2 + L_F k^2) = 0 \Rightarrow L_{FF} k_\nu f^{\nu\mu} f_{\mu\sigma} k^\sigma - L_F k^2 = 0, \tag{17}$$

Equation (17) being a dispersion relation for k; this second possibility for the signal propagation given by the equation of a cone different from the Maxwell cone, we call it the modified cone; assuming that $L_F \neq 0$, the modified cone is described by

$$k^2 - \frac{L_{FF}}{L_F} k f^2 k = 0, \tag{18}$$

where we are denoting $k f^2 k = -\Theta^2 = k_\nu f^{\nu\mu} f_{\mu\sigma} k^\sigma$. The twofold cone, known as *birefringence*, arises by the two possibilities of propagation of the discontinuity, one way is along the Maxwell cone, Eq. (16), and the other one is along the modified cone, Eq. (18), that besides is a first order differential equation for the characteristic surface Σ ($k_\alpha = \partial_\alpha \Sigma$).

In the next section, we examine the algebraic structure of f^β_α that directly determines the geometry of the modified cone.

4 The geometry of the modified cone

The geometry of the modified cone in Eq. (18) is determined by the algebraic structure of $f^2 = f^{\mu\sigma} f_{\sigma\nu}$, as well as the properties of the quadratic form ($k f^2 k$). Let us analyze the structure of f^2; $f = f^\mu{}_\nu$ is given by

$$f^\mu{}_\nu = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{19}$$
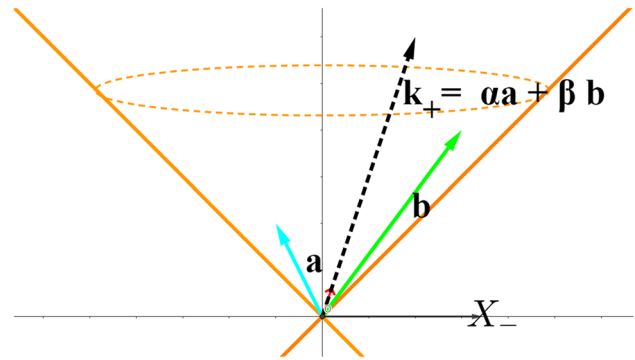
where we denote the components of the electric and magnetic fields, respectively, by E_i and B_i, $i = 1, 2, 3$. The analysis that follows is completely general for the electromagnetic field tensor and independent of the electromagnetic Lagrangian. In terms of the invariants $F = (B^2 - E^2)/2$ and $G = -\vec{E} \cdot \vec{B}$, the tensor $f = f^\sigma{}_\nu$ has the eigenvalues $\kappa_1 = -\kappa_2 = \sqrt{\kappa_-}$, $\kappa_3 = -\kappa_4 = \sqrt{\kappa_+}$, where

$$\kappa_\pm = -F \pm \sqrt{F^2 + G^2} = -\frac{1}{2}(B^2 - E^2) \pm \frac{1}{2} \sqrt{(B^2 - E^2)^2 + 4(\vec{E} \cdot \vec{B})^2}.$$

It can be shown, by straightforward calculation, that the tensor f^2 satisfies

$$f^4 + 2F f^2 - (f^2)(\tilde{f}^2) = 0, \quad \text{or} \quad f^4 + (B^2 - E^2) f^2 - (\vec{E} \cdot \vec{B})^2 \mathbb{I} = 0,$$

Fig. 1 It is illustrated the light cone of Maxwell as the lines at 45° (in yellow); a vector k_+ in X_+ is shown as the dashed arrow; it is a linear combination of **a**, **b**, $k_+ = \alpha\mathbf{a} + \beta\mathbf{b}$, α and β are constants. **a**, **b** lie on the Maxwell cone; X_- is the orthogonal complement to X_+

such that it can be factorized as

$$(f^2 - \kappa_+\mathbb{I})(f^2 - \kappa_-\mathbb{I}) = 0,$$

and thus f^2 can be diagonalized as,

$$f^2 = \kappa_\pm\mathbb{I} = \left(-F \pm \sqrt{F^2 + G^2}\right)\mathbb{I}. \tag{20}$$

Then, it is possible to split the 4-dimensional vector space X of f^2 into two 2-dimensional disjoint proper subspaces X_+ and X_-, i.e., $X = X_+ \oplus X_-$, where X_+ is the subspace of vectors with positive or null norm, and X_- is the subspace of spacelike vectors with negative norm.

In the case $\vec{E} \cdot \vec{B} \neq 0$, it turns out that $\kappa_+ > 0$ and

$$(f^2 - \kappa_+\mathbb{I}) = 0.$$

Since there are not multiple roots, f^2 can be factorized as

$$(f - \sqrt{\kappa_+}\,\mathbb{I})(f + \sqrt{\kappa_+}\,\mathbb{I}) = 0,$$

such that f can be diagonalized in X_+.

4.1 Light cones

Let us consider two eigenvectors of f, **a** and **b**, corresponding to different eigenvalues $f\mathbf{a} = \sqrt{\kappa_+}\mathbf{a}$ and $f\mathbf{b} = -\sqrt{\kappa_+}\mathbf{b}$, where $\pm\sqrt{\kappa_+} \in R$, and then, **a** and **b** are real. Due to the antisymmetry of $f_{\alpha\beta}$, **a** and **b** are null vectors,

$$f_{\alpha\beta}\mathbf{a}^\alpha\mathbf{a}^\beta = 0, \quad f_{\alpha\beta}\mathbf{b}^\alpha\mathbf{b}^\beta = 0. \tag{21}$$

We can then assume that both **a** and **b** are in the future cone and that **a** and **b** span X_+, i.e., a vector k_+ in X_+ is a linear combination of **a** and **b**,

$$k_+ = \alpha\mathbf{a} + \beta\mathbf{b}, \tag{22}$$

where α and β are constants. **a** and **b** lie on the Maxwell cone.

Any vector $k_- \in X_-$ is such that $k_-^2 < 0$, i.e., is spacelike, then, X_- is the orthogonal complement to X_+. An illustrative sketch is shown in Fig. 1.

Since $X = X_+ \oplus X_-$ any vector in X can be written as a linear combination of one vector in X_+ ($k_+ = \alpha\mathbf{a} + \beta\mathbf{b}$) and one vector in X_- (k_-),

$$k = k_+ + k_- = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma k_-, \tag{23}$$

where α, β and γ are constants. If the norm of k is positive, then, k is inside the Maxwell cone; while if the norm of k is negative, it is outside the Maxwell cone; the null vectors $k^2 = 0$ are located on the Maxwell cone.

Regarding the quadratic form $kf^2k = k_\nu f^{\nu\mu} f_{\mu\sigma} k^\sigma$, it is easy to show that it is non-negative, substituting $k = k_+ + k_-$ we show that it cannot be negative IN or ON the cone,

$$\begin{aligned} kf^2k &= (k_+ + k_-)f^2(k_+ + k_-) = (k_+ + k_-)(\kappa_+k_+ + \kappa_-k_-) \\ &= (\kappa_+k_+^2 + \kappa_-k_-^2) \geq 0, \end{aligned}$$

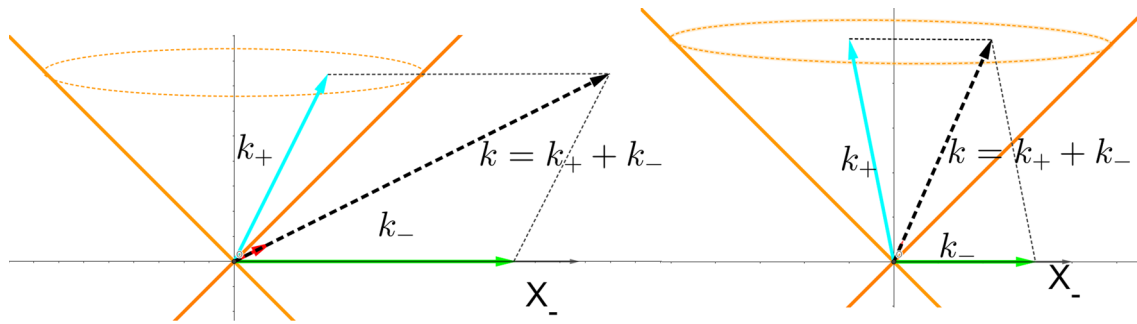and is zero only when $k_+^2 = k_-^2 = 0$.

Fig. 2 To the left the dashed arrow illustrates a spacelike vector, $k^2 < 0$, corresponding to $\Lambda = \frac{L_{FF}}{L_F} < 0$; in this case Eq. (18) cannot be satisfied neither INSIDE nor ON the light cone. To the right the dashed arrow illustrates a timelike vector, $k^2 > 0$, for $\Lambda = \frac{L_{FF}}{L_F} > 0$; in this case the vector k lies inside the Maxwell cone

Defining $\Lambda = L_{FF}/L_F$, from Eq. (18), $k^2 - \Lambda k f^2 k = 0$, and since $k f^2 k \geq 0$, then, the norm of the vector k depends on the sign of Λ. In case $\Lambda < 0$, Eq. (18) cannot be satisfied unless $k^2 < 0$, then, k is not IN (inside) or ON the light cone but belongs to the subspace X_-.

In the case $\Lambda > 0$, the norm of k is positive and k lies inside the Maxwell cone. Recall that the vectors **a** and **b** are null, then, they lie on the Maxwell cone. The two cases are illustrated in Fig. 2.

This is how the structure of the modified cone depends on the sign of $\Lambda = L_{FF}/L_F$. The cone in the Minkowski spacetime corresponds to the Maxwell Lagrangian $L_M = -F$, with $\Lambda = 0$.

4.2 The Monge cone

For NLE, the direction of the vectors k^α in general does not coincide with the propagation direction. The propagating surfaces are orthogonal to the vectors $k^\alpha = \partial_\alpha \Sigma$, while the propagating direction vectors, also called transport vector fields, denoted by V^σ, can be deduced from the dispersion relations for k^α as follows,

We define the vector,

$$V^\sigma = \left(g^{\sigma\mu} - \Lambda f^{\mu\nu} f_\nu^{\cdot\sigma}\right)k_\mu, \tag{24}$$

and introducing the "modified metric" $A^{\alpha\beta}$,

$$A^{\alpha\beta} = g^{\alpha\beta} - \Lambda f^{\alpha\nu} f_\nu^{\cdot\beta}; \tag{25}$$

using Eq. (18), it is easy to show that vectors k^α satisfy,

$$A_{\alpha\beta} k^\alpha k^\beta = 0, \tag{26}$$

while the vectors V^α satisfy

$$B_{\alpha\beta} V^\alpha V^\beta = 0, \tag{27}$$

where the matrix $B_{\alpha\beta}$ is given by

$$B^{\alpha\beta} = g^{\alpha\beta} + \Lambda f^{\alpha\nu} f_\nu^{\cdot\beta}; \tag{28}$$

and it is easy to check that $B_{\alpha\gamma} A^{\gamma\beta} = \delta_\alpha^\beta$, i.e., $B_{\alpha\gamma}$ is the inverse matrix of $A_{\alpha\gamma}$ (in some treatments B is called the *adjugate* of the matrix A [22]). Moreover, it is easy to check that V^α is orthogonal to k^α at each point on Σ,
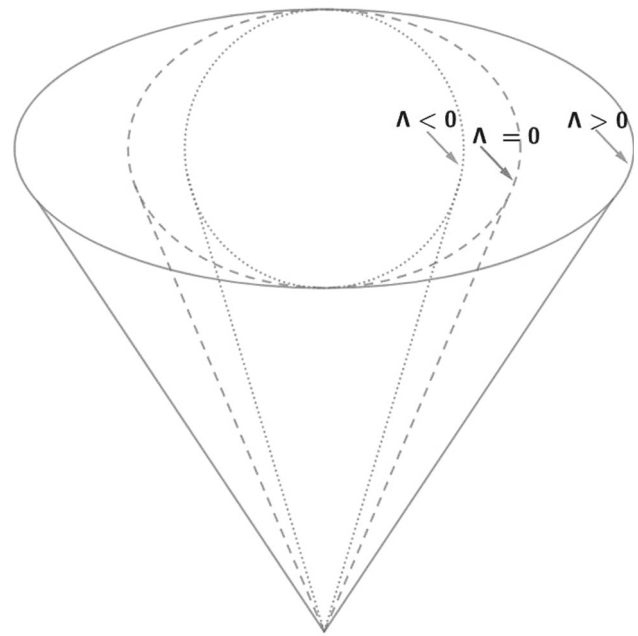
$$V^\alpha k_\alpha = A^{\alpha\mu} k_\mu k_\alpha = 0. \tag{29}$$

From Eqs. (26) and (27), we see that k^α and V^α constitute two complementary cones in $\mathcal{R}^4$; at each point on Σ, Eq. (26) is the cone of the directions that are normal to Σ, while Eq. (27) is the cone of the propagating signals; the latter is also known as the Monge Cone or Monge patch [23].

A Monge surface is a surface obtained by sweeping a generating plane curve along a trajectory that is orthogonal to the moving plane containing the curve. Locally, they are characterized as being foliated by a family of planar geodesic lines of curvature.

In the case, we are analyzing; these are the geodesics of the modified "metric" $A_{\alpha\beta}$. The geometric relevance of the modified "metric" $A_{\alpha\beta}$ is that it is related to the integral curves of k^α that in fact satisfy the null geodesic equation with $A_{\alpha\beta}$ as the metric: Since the propagator vector $k^\alpha = \partial_\alpha \Sigma$, is and exact gradient, then $\nabla^\mu k_\nu = \nabla^\nu k_\mu$, it implies it satisfies the geodesic equation, $k_\mu \nabla^\mu k_\nu = 0$; where this covariant derivative is calculated assuming that $A_{\alpha\beta}$ is the metric. The same applies to V^α with the "metric" $B_{\alpha\beta}$; these geodesics folliate the Monge surface.

Fig. 3 Sketch of the different
cones generated by the transport
vectors V^α; the cone in the middle
is Maxwell's, the inner cone
corresponds to the case
$\Lambda = L_{FF}/L_F < 0$, and the
external cone is the $\Lambda > 0$ case

As we deduced above, the cone of the normals to Σ, generated by k^α, lies inside (outside) the Maxwell cone if the sign of Λ is positive (negative); accordingly, the complementary cone of the propagating signals, V^α lies outside (inside) the Maxwell cone if Λ is positive (negative); the sketch in Fig. 3 illustrates the situation. The inner cone of V^α ($\Lambda < 0$) corresponds to a subluminal signal; while the external cone ($\Lambda > 0$) corresponds to a superluminal signal.

5 The two effective metrics

The information of the two light cones can be comprised in the equation

$$(\tilde{k}^2 - \Lambda\Theta^2)k^2 = 0, \tag{30}$$

where $\Lambda = L_{FF}/L_F$ and $-\Theta^2 = \tilde{k}f^2\tilde{k}$; the modified cone corresponds to the vanishing of the first factor, and $\tilde{k}$ is the vector orthogonal to Σ. While $k^2 = 0$ describes the null Maxwell's cone. In each case, this is the eikonal equation and each solution k determines a family of light rays, in the same way as in Hamiltonian mechanics each solution to the Hamilton-Jacobi equation determines a family of trajectories. It can also be considered as an algebraic condition on the covector k_μ. This equation is also known as the dispersion relation, the characteristic equation or the Fresnel equation.

Equivalently we can set Eq. (30) defining the two effective metrics g_+ and g_-,

$$
\begin{aligned}
(\tilde{k}^2 - \Lambda\Theta^2)k^2 &= \left[\tilde{k}^\mu\tilde{k}_\mu - \Lambda\tilde{k}_\alpha f^{\alpha\mu}f_{\mu\beta}\tilde{k}^\beta\right]k^\nu k_\nu \\
&= \left[\eta^{\mu\alpha}\tilde{k}_\alpha\tilde{k}_\mu - \Lambda f^{\mu\alpha}\tilde{k}_\alpha f_\mu{}^\beta\tilde{k}_\beta\right](\eta^{\nu\sigma}k_\sigma k_\nu) = \left(\left[\eta^{\mu\alpha} - \Lambda f^{\beta\alpha}f_\beta{}^\mu\right]\tilde{k}_\alpha\tilde{k}_\mu\right)(\eta^{\nu\sigma}k_\sigma k_\nu) \\
&= \left(g_+^{\mu\alpha}\tilde{k}_\mu\tilde{k}_\alpha\right)\left(g_-^{\nu\sigma}k_\nu k_\sigma\right) = 0
\end{aligned}
\tag{31}
$$

where $\eta = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. The two possible effective metrics are (1) the metric of the Minkowski spacetime $g_-^{\nu\sigma} = \eta^{\nu\sigma}$ and (2) an effective metric $g_+^{\mu\alpha} = \eta^{\mu\alpha} - \Lambda f^{\beta\alpha}f_\beta{}^\mu$. Equation (31) comprises the non-uniqueness of the propagating surface, i.e., the birefringence effect. Depending on the light polarization, a part of the signals (discontinuities) will propagate along the isotropic (usual) surfaces of g_-, while the other part will follow the modified surfaces that are normal to the null vectors of g_+, $\tilde{k}$.

5.1 Connection with the wave polarization

In order to relate the previous treatment with a more familiar one, we will show the connection with the polarization. Considering that the propagating discontinuities are plane waves characterized by the electromagnetic 4-potential, $A_\mu = \epsilon_\mu e^{i\Sigma}$, where ϵ_μ and Σ represent the polarization and phase of the wave, respectively. We can choose both quantities to be real functions of time and space. The wave vector was previously defined,

$$k_\mu = \nabla_\mu \Sigma = \partial_\mu \Sigma,$$

where k_0 is the frequency of the wave, and k_i represents the spatial components of the wave vector. As we showed earlier, the nature of the propagation depends on the sign of $k^\mu k_\mu = k^2$. Substituting $A_\mu = \epsilon_\mu e^{i\Sigma}$ into $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

$$f_{\mu\nu} = \left(\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu\right)e^{i\Sigma}, \tag{32}$$

and then substituting in the equation for the propagating discontinuity, Eq. (12), we arrive at,

$$\left[L_{FF}\left(f^{\ \nu\alpha}k_\alpha\right)\left(f^{\mu\beta}k_\beta\right) - L_F\left(k^\mu k^\nu - k^2\delta^{\ \nu\mu}\right)\right]\epsilon_\nu = 0, \tag{33}$$

this equation is the dispersion relation for the modified cone, and it restricts the allowed polarization modes ϵ_ν; cf. [24].

5.2 The refraction index of the effective metric

The effect of NLE on light propagation can be modeled as well as a material medium characterized by tensors of electric permittivity, ϵ^{ij}, and magnetic permeability μ_j. Thus, the electric displacement **D** is related to the electric field **E** through the electric permittivity tensor, ϵ^{ij}, and the magnetic field **B** is related to the field intensity **H** through the magnetic permeability tensor, μ_j, in this way [25],

$$D^i = \epsilon^{ij} E_j - \varepsilon^{0ijk}\mu_j H_k$$
$$B^i = \epsilon^{ij} H_j - \varepsilon^{0ijk}\mu_j E_k$$

with

$$\epsilon^{ij} = -\sqrt{-g}\frac{g^{ij}}{g_{00}}, \quad \mu_i = -\frac{g_{0i}}{g_{00}}, \tag{34}$$

Latin subscripts i, j run for the spatial coordinates, from 1 to 3 and g^{ij} refers to the effective metric.

We present explicitly the permittivity and permeability for the effective metric $g_+^{\mu\nu} = \eta^{\mu\nu} - \Lambda f_\beta^{\ \mu} f^{\beta\nu}$. In matrix form, the effective metric g_+ can be written as

$$g_+^{\mu\nu} = \begin{pmatrix} -\Lambda E^2 - 1 & \Lambda(B_2 E_3 - B_3 E_2) & -\Lambda(B_1 E_3 - B_3 E_1) & \Lambda(B_1 E_2 - B_2 E_1) \\ \Lambda(B_2 E_3 - B_3 E_2) & 1 - \Lambda(B_2^2 + B_3^2 - E_1^2) & \Lambda(B_1 B_2 + E_1 E_2) & \Lambda(B_1 B_3 + E_1 E_3) \\ -\Lambda(B_1 E_3 - B_3 E_1) & \Lambda(B_1 B_2 + E_1 E_2) & 1 - \Lambda(B_1^2 + B_3^2 - E_2^2) & \Lambda(B_2 B_3 + E_2 E_3) \\ \Lambda(B_1 E_2 - B_2 E_1) & \Lambda(B_1 B_3 + E_1 E_3) & \Lambda(B_2 B_3 + E_2 E_3) & 1 - \Lambda(B_1^2 + B_2^2 - E_3^2) \end{pmatrix} \tag{35}$$

the determinant of the effective metric is

$$\det[g] = g = -\left(1 - 2\Lambda F + \Lambda^2 G^2\right)^2 = -\left(1 - \Lambda(B^2 - E^2) + \Lambda^2(B \cdot E)^2\right)^2$$

the inverse of the effective metric is

$$g_{\mu\nu}^+ = g^{-1}\begin{pmatrix} -\Lambda B^2 + 1 & -\Lambda(B_2 E_3 - B_3 E_2) & \Lambda(B_1 E_3 - B_3 E_1) & -\Lambda(B_1 E_2 - B_2 E_1) \\ -\Lambda(B_2 E_3 - B_3 E_2) & -\Lambda(E_2^2 + E_3^2 - B_1^2) - 1 & \Lambda(B_1 B_2 + E_1 E_2) & +\Lambda(B_1 B_3 + E_1 E_3) \\ \Lambda(B_1 E_3 - B_3 E_1) & \Lambda(B_1 B_2 + E_1 E_2) & -\Lambda(E_1^2 + E_3^2 - B_2^2) - 1 & \Lambda(B_2 B_3 + E_2 E_3) \\ -\Lambda(B_1 E_2 - B_2 E_1) & \Lambda(B_1 B_3 + E_1 E_3) & \Lambda(B_2 B_3 + E_2 E_3) & -\Lambda(E_1^2 + E_2^2 - B_3^2) - 1 \end{pmatrix} \tag{36}$$

Substituting the effective metric covariant and contravariant expressions into Eq. (34), the indexes of permittivity ϵ and permeability μ are given by

$$\epsilon_{ij} = g^2(1 - \Lambda B^2)^{-1}\begin{pmatrix} \Lambda(E_1^2 - B_2^2 - B_3^2) - 1 & -\Lambda(B_1 B_2 + E_1 E_2) & -\Lambda(B_1 B_3 + E_1 E_3) \\ -\Lambda(B_1 B_2 + E_1 E_2) & \Lambda(E_2^2 - B_1^2 - B_3^2) - 1 & -\Lambda(B_2 B_3 + E_2 E_3) \\ -\Lambda(B_1 B_3 + E_1 E_3) & -\Lambda(B_2 B_3 + E_2 E_3) & \Lambda(E_3^2 - B_1^2 - B_2^2) - 1 \end{pmatrix} \tag{37}$$

$$\mu_i = -\Lambda(1 - \Lambda B^2)^{-1}\left(B_3 E_2 - B_2 E_3, \ B_1 E_3 - B_3 E_1, \ B_2 E_1 - B_1 E_2\right) \tag{38}$$

While in vacuum, spacetime is flat and $k^\mu k_\mu = 0$, $\epsilon^{ij} = \delta^{ij}$ and $\mu_i = 0$.

6 Conclusions

In this article, we have shown that the twofold light cones (related to the birefringence effect) naturally arise from adding to the classical electromagnetic Lagrangian nonlinear terms that depend nonlinearly on the electromagnetic invariants. We did not specify the nonlinear electromagnetic theory, apart from the fact that it is of the Plebanski class and that $L(F)$. The field equations obtained for the propagation of field discontinuities can be settled as a Sturm–Liouville equation, whose eigenvalues and eigenvectors are

related to the different paths followed by the light; to be precise, from the vanishing eigenvalues, the two different dispersion relations can be identified; one of them corresponds to the classical Maxwell light cone and the other one to a modified cone. The algebraic structure of the electromagnetic field tensor is discussed in order to understand, the modified cone depends on structure. For the modified light cone, it is not possible to describe the light trajectories using global-oriented cones because the factor $\Lambda = L_{FF}/L_f$ can change from point to point. The different situations are illustrated with light cone diagrams.

We obtained that the velocity of the signal propagation depends on the signs of the derivatives of the Lagrangian, and superluminic signals arise as a possibility. This result tells us that not all theories should be considered as physically meaningful, because probably some of them violate causality in the sense that the light cones of the optical metrics are not inside the light cone of the spacetime metric [26]. The signs of L and its derivatives may be constrained if we additionally demand the fulfillment of one or more energy conditions [27]. For instance, if we impose the Dominant Energy Condition (DEC), we need $L_F \leq 0$ and $T \leq 0$, in our case, $T = -L - FL_F$, where T is the trace of the energy-momentum tensor. On the other hand, imposing Strong Energy Condition (SEC), we need $L_F \leq 0$ and $T \geq 0$. For each particular NLE, it has to be checked that the initial-value problem is well posed [28]. Finally, we connect our results with the polarization of a plane wave, which is restricted by the dispersion relations and also present the permeability and permittivity of an equivalent material medium. In this way, we have illustrated the form in which light signals propagate in NLE; the scheme can be applied as well to anisotropic media.

Data Availability Statement No Data associated in the manuscript.

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