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Research Article

Affine Connection Representation of Gauge Fields

Zhao-Hui Man

10-2-1102, Hua Yang Nian Hua Bei Qu, Yinchuan, China

Correspondence should be addressed to Zhao-Hui Man; shetcslion@163.com

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There are two ways to unify gravitational field and gauge field. One is to represent gravitational field as principal bundle connection, and the other is to represent gauge field as affine connection. Poincaré gauge theory and metric-affine gauge theory adopt the first approach. This paper adopts the second. In this approach, (i) gauge field and gravitational field can both be represented by affine connection; they can be described by a unified spatial frame. (ii) Time can be regarded as the total metric with respect to all dimensions of internal coordinate space and external coordinate space. On-shell can be regarded as gradient direction. Quantum theory can be regarded as a geometric theory of distribution of gradient directions. Hence, gauge theory, gravitational theory, and quantum theory all reflect intrinsic geometric properties of manifold. (iii) Coupling constants, chiral asymmetry, PMNS mixing, and CKM mixing arise spontaneously as geometric properties in affine connection representation, so they are not necessary to be regarded as direct postulates in the Lagrangian anymore. (iv) The unification theory of gauge fields that are represented by affine connection can avoid the problem that a proton decays into a lepton in theories such as $SU(5)$. (v) There exists a geometric interpretation to the color confinement of quarks. In the affine connection representation, we can get better interpretations to the above physical properties; therefore, to represent gauge fields by affine connection is probably a necessary step towards the ultimate theory of physics.

1. Introduction

1.1. Background and Purpose. We know that in gauge theory, the field strength and the gauge-covariant derivative

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu,$$
$$D_\mu = \partial_\mu - igT^a A^a_\mu, \tag{1}$$

both contain a coupling constant g, which measures the strength of interaction. A problem is that why is there a coupling constant g?

If we represent gauge fields by affine connection, we can obtain a nice interpretation. For example, if we use Γ_{MNP} to represent gauge potentials, it is not hard to find some specific conditions to turn the curvature tensor R^M_{NPQ} to

$$\begin{aligned} R_{MNPQ} &= \partial_P \Gamma_{MNQ} - \partial_Q \Gamma_{MNP} + \Gamma_{MHP}\Gamma^H_{NQ} - \Gamma^H_{NP}\Gamma_{MHQ} \\ &= \partial_P \Gamma_{MNQ} - \partial_Q \Gamma_{MNP} + G^{RH}\left(\Gamma_{MHP}\Gamma_{RNQ} - \Gamma_{RNP}\Gamma_{MHQ}\right). \end{aligned} \tag{2}$$

Thus, R_{MNPQ} can be used to represent field strength. In addition, for any ρ_M, we see that

$$\rho_{M;P} = \partial_P \rho_M - \Gamma^H_{MP}\rho_H = \partial_P \rho_M - G^{RH}\Gamma_{RMP}\rho_H. \tag{3}$$

Equations (2) and (3) mean that the coupling constant g may have a geometric meaning, which originates from G^{RH}.

This implies that only when affine connection is adopted to represent gauge field can some physical properties be better interpreted. On the other hand, in the general relativity theory, gravitational field is also described by affine connection, so it is convenient to describe gravitational field and gauge field uniformly by affine connection. Therefore, it is necessary to study the affine connection representation of gauge fields. This is the basic motivation of this paper.

There are the following two ways to unify gravitational field and gauge field.

One way is to represent gravitational field as principal bundle connection. We can take the transformation group $\text{Gravi}(3,1)$ of gravitational field as the structure group of principal bundle to establish a gauge theory of gravitational field, the local transformation group of which is in the form of $\text{Gravi}(3,1) \otimes \text{Gauge}(n)$, e.g., Poincaré gauge theory [1–11] and metric-affine gauge theory [12–23]. This way can be interpreted intuitively as

$$\boxed{\text{gravitation theory}} \xrightarrow{\text{be incorporated into}} \boxed{\text{the framework of gauge theory.}} \tag{4}$$

The other way is to represent gauge field as affine connection. This is the approach adopted by this paper. Gravitational field and gauge field can both be described by affine connection. Besides, we will also establish an affine connection representation of elementary particles. This way can be interpreted intuitively as

$$\boxed{\text{gauge theory}} \xrightarrow{\text{be incorporated into}} \boxed{\text{the framework of gravitation theory.}} \tag{5}$$

1.2. Ideas and Methods. We divide the problem of establishing affine connection representation of gauge fields into three parts as follows.

(i) Which affine connection is suitable for describing not only gravitational field, but also gauge field and elementary particle field?

(ii) How to describe the evolution of these fields in affine connection representation?

(iii) What are the concrete forms of electromagnetic, weak, and strong interaction fields in affine connection representation?

For the problem (i). On a Riemannian manifold (M, G), the metric tensor can be expressed as $G_{MN} = \delta_{AB} B_M^A B_N^B$ and $G^{MN} = \delta^{AB} C_A^M C_B^N$, where B_M^A and C_A^M are semimetrics or to say frame fields. It is evident that semimetric is more fundamental than metric, so we hope B_M^A or C_A^M is regarded as a unified frame field of gravitational field and gauge field, and the frame transformation of B_M^A or C_A^M is regarded as gauge transformation. Hence, we need a more general manifold (M, B_M^A) rather than the Riemannian manifold (M, G).

Next, we put metric and semimetric together to construct a new connection, which is not only an affine connection, but also a connection on a fibre bundle. In this way, gravitational field and various gauge fields can be unified on a manifold (M, B_M^A) that is defined by semimetric.

In addition, we notice that in the theories based on principal bundle connection representation,

(1) Several complex-valued functions, which satisfy the Dirac equation, are sometimes used to refer to a charged lepton field l and sometimes a neutrino field ν. It is not clear how to distinguish these field functions l and ν by inherent geometric constructions

(2) Gauge potentials are abstract; they have no inherent geometric constructions. In other words, the Levi-Civita connection $\Gamma_{\nu\rho}^{\mu}$ of gravity is constructed by the metric $g_{\mu\nu}$; however, it is not explicit what geometric quantity the connection A_μ^a of gauge field is constructed by

By contrast, in the affine connection representation of this paper, we are able to use the semimetrics B_M^A and C_A^M of internal coordinate space to endow particle fields l and ν and gauge field A_μ^a with geometric constructions. Thus, they are not only irreducible representations of group but also possessed of concrete geometric entities.

For the problem (ii). There is a fundamental difficulty that time is effected by gravitational field, but not effected by gauge field. This leads to an essential difference between the description of evolution of gravitational field and that of gauge field. In this case, it seems difficult to obtain a unified theory of evolution in affine connection representation. Nevertheless, we find that we can define time as the total metric with respect to all dimensions of internal coordinate space and external coordinate space and define evolution as one-parameter group of diffeomorphism, to overcome the above difficulty.

Now that gauge field and gravitational field are both represented as affine connection, then the properties that are related to gauge field, such as charge, current, mass, energy, momentum, and action, must have corresponding affine representations. Thus, Yang-Mills equation, energy-momentum equation, and Dirac equation are turned into geometric properties in gradient direction; in other words, on-shell evolution is characterized by gradient direction. Correspondingly, quantum theory can be interpreted as a geometric theory of distribution of gradient directions.

For the problem (iii). The basic idea is that on a $\mathfrak{D}$-dimensional manifold, the components B_m^a and C_a^m of semimetrics B_M^A and C_A^M with $m, a \in \{4, 5, \cdots, \mathfrak{D}\}$ are regarded as the frame field of electromagnetic, weak, and strong interactions. The other components of B_M^A and C_A^M are regarded as the frame field of gravitation.

We take the affine connection as

$$\begin{aligned}
\Gamma_{NP}^{M} &\triangleq \frac{1}{2}\left(\left[{}_{NP}^{M}\right] + \left\{{}_{NP}^{M}\right\}\right) = \frac{1}{2}\left[C_A^M \left(D_P B_N^A\right) + \left\{{}_{NP}^{M}\right\}\right] \\
&= \frac{1}{2}\left[C_A^M \left(D_C B_N^A\right) b_P^C + \left\{{}_{NP}^{M}\right\}\right] \\
&= \frac{1}{2}\left[C_A^M \left(\frac{\partial B_N^A}{\partial \zeta^C} + \left({}_{BC}^{A}\right) B_N^B\right) b_P^C + \frac{1}{2} G^{MQ} \left(\frac{\partial G_{NQ}}{\partial x^P} + \frac{\partial G_{PQ}}{\partial x^N} - \frac{\partial G_{NP}}{\partial x^Q}\right)\right] \\
&= \frac{1}{2}\left[\left(C_A^M \frac{\partial B_N^A}{\partial x^P} + C_A^M \left({}_{BP}^{A}\right) B_N^B\right) + \frac{1}{2} G^{MQ} \left(\frac{\partial G_{NQ}}{\partial x^P} + \frac{\partial G_{PQ}}{\partial x^N} - \frac{\partial G_{NP}}{\partial x^Q}\right)\right],
\end{aligned} \tag{6}$$

where $b_P^C \triangleq \partial\zeta^C/\partial x^P$ is a local coordinate transformation, $\left\{ {M \atop NP} \right\}$ is Christoffel symbol, $G_{MN} = \delta_{AB} B_M^A B_N^B$,

$$\left[{M \atop NP} \right] \triangleq C_A^M \left(D_P B_N^A \right) = C_A^M \frac{\partial B_N^A}{\partial x^P} + C_A^M \left({A \atop BP} \right) B_N^B \tag{7}$$

is said to be a gauge connection, and Γ_{NP}^M is said to be a holonomic connection. $\left({A \atop BP} \right) \triangleq \left({A \atop BC} \right) b_P^C$.

$$\left({A \atop BC} \right) \triangleq \frac{1}{2} C_{A'}^A \left(\frac{\partial B_B^{A'}}{\partial \zeta^C} + \frac{\partial B_C^{A'}}{\partial \zeta^B} \right) \tag{8}$$

is said to be a torsion-free simple connection. Thus,

$$\begin{aligned} \Gamma_{MNP} &= \frac{1}{2} \left([MNP] + \{MNP\} \right) \\ &= \frac{1}{2} \left[\delta_{AD} B_M^D \left(\frac{\partial B_N^A}{\partial x^P} + \left({A \atop BP} \right) B_N^B \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{\partial G_{NM}}{\partial x^P} + \frac{\partial G_{PM}}{\partial x^N} - \frac{\partial G_{NP}}{\partial x^M} \right) \right]. \end{aligned} \tag{9}$$

For the sake of simplicity, we firstly consider the affine connection representation of gauge fields without gravitation. That is to say, let

$$s, i, j = 1, 2, 3\ ;\ a, m, n, l, q = 4, 5, \cdots, \mathfrak{D}\ ;\ A, B, M, N, P = 1, 2, \cdots, \mathfrak{D}, \tag{10}$$

and consider a $\mathfrak{D}$-dimensional manifold (M, B_M^A) that satisfies the following conditions:

(i) $B_i^s = \delta_i^s, B_i^a = 0, B_m^s = 0$

(ii) $G_{ij} = \delta_{ij}, G_{mn} = const, G_{mi} = 0$

(iii) When $m \neq n$, $G_{mn} = 0$

Thus, $\{MNP\} = 0$, $[MNP] \neq 0$ in general. The components Γ_{mnP} of $\Gamma_{MNP} = 1/2[MNP]$ with $m, n \in \{4, 5, \cdots, \mathfrak{D}\}$ describe gauge potentials of electromagnetic, weak, and strong interactions. We also use the affine connection Γ_{NP}^M to construct elementary particle fields ρ_{MN}. The components ρ_{mn} of ρ_{MN} with $m, n \in \{4, 5, \cdots, \mathfrak{D}\}$ describe field functions of leptons and quarks.

The components G^{mn} of G^{MN} with $m, n \in \{4, 5, \cdots, \mathfrak{D}\}$ describe coupling constants of particle fields ρ_{mn} and gauge potentials Γ_{mnP}. The other components of G^{MN} are the metrics of gravitational field. The other components of ρ_{MN} and Γ_{MNP} provide possible candidates for dark matters and their interactions.

1.3. Content and Organization. In this paper, we are going to show how to construct the affine connection representation of gauge fields. Sections are organized as follows.

Corresponding to the problem (i), in Section 2, we make some necessary mathematical preparations and discuss the coordinate transformation and frame transformation of the above connection. Meanwhile, in order to make the languages that are used to describe gauge field and gravitational field unified and harmonized, we generalize the notion of reference-system and give it a strict mathematical definition. The reference-system in conventional sense is just only defined on a local coordinate neighborhood, and it has only $(1+3)$ dimensions. But in this paper, we define the concept of reference-system over the entire manifold. It is possessed of more dimensions but different from Kaluza-Klein theory [24–26] and string theories [27–39]. Thus, both of gravitational field and gauge field are regarded as special cases of such a concept of reference-system.

Corresponding to the problem (ii), in Section 3, we establish the general theory of evolution in affine connection representation of gauge fields, and in Section 4, we discuss the application of this general theory of evolution to $(1+3)$-dimensional classical spacetime.

Corresponding to the problem (iii), in Sections 5–7, we show concrete forms of affine connection representations of electromagnetic, weak, and strong interaction fields.

Some important topics are organized as follows.

(1) Time is regarded as the total metric with respect to all spatial dimensions including external coordinate space and internal coordinate space (see Definition 2 and Remark 35 for detail). The CPT inversion is interpreted as the composition of full inversion of coordinates and full inversion of metrics (see Section 3.7 for detail). The conventional $(1+3)$-dimensional Minkowski coordinate x^μ originates from the general $\mathfrak{D}$-dimensional coordinate x^M. The construction method of extra dimensions is different from those of Kaluza-Klein theory and string theory (see Section 4.2 for detail)

(2) On-shell evolution is characterized by gradient direction field (see Sections 3.4–3.6 and 4.3 for detail). Quantum theory is regarded as a geometric theory of distribution of gradient directions. We show two dual descriptions of gradient direction. They just exactly correspond to the Schrödinger picture and the Heisenberg picture. In these points of view, the gravitational theory and quantum theory become coordinated. They have a unified description of evolution, and the definition of Feynman propagator is simplified to a stricter form (see Sections 3.8 and 3.9 for detail)

(3) Yang-Mills equation originates from a geometric property of gradient direction. We show the affine connection representation of Yang-Mills equation (see Sections 3.5 and 4.5 for detail)

(4) Energy-momentum equation originates from a geometric property of gradient direction. We show the affine connection representation of mass, energy, momentum, and action (see Section 3.6, Definition 37, and Discussion 38 for detail). Furthermore, we also show the affine connection representation of Dirac equation (see Section 4.4 for detail)

(5) Why do not neutrinos participate in the electromagnetic interactions? And why do not right-handed neutrinos participate in the weak interactions with W bosons? In the theory of this paper, they are natural and geometric results of affine connection representation of gauge fields; therefore, they are not necessary to be regarded as postulates anymore (see Propositions 52 and 63 for detail)

(6) In Section 7, we give new interpretations to PMNS mixing of leptons, CKM mixing of quarks, and color confinement. That is to say, in affine connection representation of gauge fields, these physical properties can be interpreted as geometric properties on manifold

2. Mathematical Preparations

2.1. Geometric Manifold. In order to make the languages that are used to describe gauge field and gravitational field unified and harmonized, we adopt the following definition.

Definition 1. Let M be a $\mathfrak{D}$-dimensional connected smooth real manifold. $\forall p \in M$, take a coordinate chart (U_p, φ_{Up}) on a neighborhood U_p of p. They constitute a coordinate covering

$$\varphi \triangleq \left\{ \left(U_p, \varphi_{Up} \right) \right\}_{p\in M}, \tag{11}$$

which is said to be a point-by-point covering. For the sake of simplicity, U_p can be denoted by U and φ_{Up} by φ_U.

Let φ and ψ be two point-by-point coverings. For the two coordinate frames φ_U and ψ_U on the neighborhood U of point p, if

$$f_p \triangleq \varphi_U \circ \psi_U^{-1} : \psi_U(U) \longrightarrow \varphi_U(U), \quad \xi^A \mapsto x^M \tag{12}$$

is a smooth homeomorphism, f_p is called a **local reference-system**.

If every $p \in M$ is endowed with a local reference-system $f(p)$ and we require the semimetrics B_M^A and C_A^M in Equation (15) to be smooth real functions on M, then

$$f : M \longrightarrow REF, \quad p \mapsto f(p) \tag{13}$$

is said to be a **reference-system** on M, and (M, f) is said to be a **geometric manifold**.

2.2. Metric and Semimetric. In the absence of a special declaration, the indices take values as $A, B, C, D, E = 1, 2, \cdots, \mathfrak{D}$ and $M, N, P, Q, R = 1, 2, \cdots, \mathfrak{D}$. The derivative functions

$$b_M^A \triangleq \frac{\partial \xi^A}{\partial x^M}, \quad c_A^M \triangleq \frac{\partial x^M}{\partial \xi^A} \tag{14}$$

of $f(p)$ on U_p define the semimetrics (or to say frame field) B_M^A and C_A^M of f on the manifold M that are

$$\begin{aligned} B_M^A &: M \longrightarrow \mathbb{R}, \quad p \mapsto B_M^A(p) \triangleq \left(b_{f(p)} \right)_M^A (p), \\ C_A^M &: M \longrightarrow \mathbb{R}, \quad p \mapsto C_A^M(p) \triangleq \left(c_{f(p)} \right)_A^M (p). \end{aligned} \tag{15}$$

Let $\delta_{AB} = \delta^{AB} = \delta_B^A = \text{Kronecker}(A, B)$ and $\varepsilon_{MN} = \varepsilon^{MN} = \varepsilon_N^M = \text{Kronecker}(M, N)$. The metric tensors of f are

$$\begin{aligned} G_{MN} &= \delta_{AB} B_M^A B_N^B, \\ H_{AB} &= \varepsilon_{MN} C_A^M C_B^N. \end{aligned} \tag{16}$$

Similarly, it can also be defined that $\bar{b}_A^M \triangleq \partial \xi_A / \partial x_M$, $\bar{c}_M^A \triangleq \partial x_M / \partial \xi_A$ and corresponding $\bar{B}_A^M$, $\bar{C}_M^A$.

2.3. Gauge Transformation in Affine Connection Representation. $\forall p \in M$, $f(p) \triangleq \rho_U \circ \psi_U^{-1}$ induces local reference-system transformations

$$\begin{aligned} L_{f(p)} &: k(p) \triangleq \psi_U \circ \varphi_U^{-1} \mapsto \rho_U \circ \varphi_U^{-1} = f(p) \circ k(p), \\ R_{f(p)} &: h(p) \triangleq \varphi_U \circ \rho_U^{-1} \mapsto \varphi_U \circ \psi_U^{-1} = h(p) \circ f(p), \end{aligned} \tag{17}$$

and reference-system transformations on the manifold M

$$L_f : p \mapsto L_{f(p)}, \quad R_f : p \mapsto R_{f(p)}. \tag{18}$$

We also speak of L_f and R_f as **(affine) gauge transformations**.

(i) L_f and R_f are identical transformations if and only if $[B_M^A]$ of f is an identity matrix

(ii) L_f and R_f are flat transformations if and only if $\forall p_1, p_2 \in M$, $B_M^A(p_1) = B_M^A(p_2)$

(iii) L_f and R_f are orthogonal transformations if and only if $\delta_{AB} B_M^A B_N^B = \varepsilon_{MN}$

The totality of all reference-system transformations on M is denoted by $GL(M)$, which is a subgroup of $\otimes_{p\in M} GL(\mathfrak{D}, \mathbb{R})_p$, where $\otimes$ represents external direct product.

2.4. Coordinate Transformation of Holonomic Connection and Frame Transformation of Gauge Connection. Suppose there are reference-systems g and $\mathfrak{g}$ on the manifold M, denote $\mathscr{G} \triangleq \mathfrak{g} \circ g$, and $\forall p \in M$, on the neighborhood U of p, $g(p)$ and $\mathfrak{g}(p)$ satisfy

$$\left(U, x^M\right) \xleftarrow{g(p)} \left(U, \zeta^A\right) \xleftarrow{\mathfrak{g}(p)} \left(U, \beta^{A'}\right). \tag{19}$$

On the geometric manifold $(M, \mathfrak{g})$, we define torsion-free simple connection D and its coefficients $\genfrac{(}{)}{0pt}{}{A}{BC}_{\mathfrak{g}}$ by

$$D\frac{\partial}{\partial \zeta^B} \triangleq \left(\omega_{\mathfrak{g}}\right)^A_B \otimes \frac{\partial}{\partial \zeta^A} = \genfrac{(}{)}{0pt}{}{A}{BC}_{\mathfrak{g}} d\zeta^C \otimes \frac{\partial}{\partial \zeta^A} = \frac{1}{2}\left(C_{\mathfrak{g}}\right)^A_{A'}\left(\frac{\partial \left(B_{\mathfrak{g}}\right)^{A'}_B}{\partial \zeta^C} + \frac{\partial \left(B_{\mathfrak{g}}\right)^{A'}_C}{\partial \zeta^B}\right) d\zeta^C \otimes \frac{\partial}{\partial \zeta^A}. \tag{20}$$

Then, we can compute the absolute derivative of the frame field $\partial/\partial x^{\mathrm{N}}$

$$\begin{aligned} D\frac{\partial}{\partial x^N} &= D\left(\left(B_g\right)^B_N \frac{\partial}{\partial \zeta^B}\right) = d\left(B_g\right)^B_N \otimes \frac{\partial}{\partial \zeta^B} + \left(B_g\right)^B_N D\frac{\partial}{\partial \zeta^B} \\ &= \frac{\partial \left(B_g\right)^B_N}{\partial \zeta^C} d\zeta^C \otimes \frac{\partial}{\partial \zeta^B} + \left(B_g\right)^B_N \genfrac{(}{)}{0pt}{}{A}{BC}_{\mathfrak{g}} d\zeta^C \otimes \frac{\partial}{\partial \zeta^A} \\ &= \left(\frac{\partial \left(B_g\right)^A_N}{\partial \zeta^C} + \left(B_g\right)^B_N \genfrac{(}{)}{0pt}{}{A}{BC}_{\mathfrak{g}}\right) d\zeta^C \otimes \frac{\partial}{\partial \zeta^A}. \end{aligned} \tag{21}$$

Thus, it is obtained that

$$D_C\left(B_g\right)^A_N = \frac{\partial \left(B_g\right)^A_N}{\partial \zeta^C} + \left(B_g\right)^B_N \genfrac{(}{)}{0pt}{}{A}{BC}_{\mathfrak{g}}. \tag{22}$$

Denote $D_P \triangleq \left(b_{g(p)}\right)^C_P D_C$; thus, we can define on $(M, \mathscr{G})$ the required **gauge connection**, which is

$$\genfrac{[}{]}{0pt}{}{M}{NP}_{\mathscr{G}} \triangleq \left(C_g\right)^M_A D_P \left(B_g\right)^A_N = \left(C_g\right)^M_A \frac{\partial \left(B_g\right)^A_N}{\partial x^P} + \left(C_g\right)^M_A \genfrac{(}{)}{0pt}{}{A}{BP}_{\mathfrak{g}} \left(B_g\right)^B_N. \tag{23}$$

It is important that $\genfrac{[}{]}{0pt}{}{M}{NP}_{\mathscr{G}}$ is not only an affine connection on $(M, \mathscr{G})$, but also a connection on frame bundle.

(i) $\genfrac{[}{]}{0pt}{}{M}{NP}_{\mathscr{G}}$ *as an Affine Connection*. Under the coordinate transformation $L_{k(p)} : (U, x^M) \longrightarrow (U, x^{M'})$, $b^M_{M'} \triangleq \partial x^M / \partial x^{M'}$, $c^{M'}_M \triangleq \partial x^{M'}/\partial x^M$, $\left(B_g\right)^A_M \mapsto \left(B_g\right)^A_{M'} = b^M_{M'}\left(B_g\right)^A_M$, $\left(C_g\right)^M_A \mapsto \left(C_g\right)^{M'}_A = c^{M'}_M \left(C_g\right)^M_A$. Consequently, the gauge connection $\genfrac{[}{]}{0pt}{}{M}{NP}_{\mathscr{G}}$ is transformed according to

$$L_{k(p)} : \genfrac{[}{]}{0pt}{}{M}{NP}_{\mathscr{G}} \mapsto \genfrac{[}{]}{0pt}{}{M'}{N'P'}_{\mathscr{G}} = c^{M'}_M \genfrac{[}{]}{0pt}{}{M}{NP}_{\mathscr{G}} b^N_{N'} b^P_{P'} + c^{M'}_M \frac{\partial b^M_{N'}}{\partial x^{P'}}. \tag{24}$$

Due to Equation (24), under the coordinate transformation, the holonomic connection

$$\begin{aligned} \left(\Gamma_{\mathscr{G}}\right)^M_{NP} &\triangleq \frac{1}{2}\left(\genfrac{[}{]}{0pt}{}{M}{NP}_{\mathscr{G}} + \genfrac{\{}{\}}{0pt}{}{M}{NP}_{\mathscr{G}}\right) \\ &= \frac{1}{2}\left[\left(\left(C_g\right)^M_A \frac{\partial \left(B_g\right)^A_N}{\partial x^P} + \left(C_g\right)^M_A \genfrac{(}{)}{0pt}{}{A}{BP}_{\mathfrak{g}} \left(B_g\right)^B_N\right) \right. \\ &\quad \left. + \frac{1}{2}\left(G_{\mathscr{G}}\right)^{MQ}\left(\frac{\partial \left(G_{\mathscr{G}}\right)_{NQ}}{\partial x^P} + \frac{\partial \left(G_{\mathscr{G}}\right)_{PQ}}{\partial x^N} - \frac{\partial \left(G_{\mathscr{G}}\right)_{NP}}{\partial x^Q}\right)\right] \end{aligned} \tag{25}$$

is transformed according to

$$L_{k(p)} : \left(\Gamma_{\mathscr{G}}\right)^M_{NP} \mapsto \left(\Gamma_{\mathscr{G}}\right)^{M'}_{N'P'} = c^{M'}_M \left(\Gamma_{\mathscr{G}}\right)^M_{NP} b^N_{N'} b^P_{P'} + c^{M'}_M \frac{\partial b^M_{N'}}{\partial x^{P'}}. \tag{26}$$

(ii) $\genfrac{[}{]}{0pt}{}{M}{NP}_{\mathscr{G}}$ *as a Connection on Frame Bundle*. Under the frame transformation $L_k : (M, \mathscr{G}) \mapsto (M, \mathscr{G}')$, $\partial/\partial x^M \mapsto \partial/\partial x^{M'} = \left(B_k\right)^M_{M'} \partial/\partial x^M$,

$\left(B_g\right)^A_M \mapsto \left(B_{g'}\right)^A_{M'} = \left(B_k\right)^M_{M'} \left(B_g\right)^A_M$, $\left(C_g\right)^M_A \mapsto \left(C_{g'}\right)^{M'}_A = \left(C_k\right)^{M'}_M \left(C_g\right)^M_A$. Consequently, the gauge connection $\genfrac{[}{]}{0pt}{}{M}{NP}_{\mathscr{G}}$ is tranformed according to

$$\begin{aligned} L_k : \genfrac{[}{]}{0pt}{}{M}{NP}_{\mathscr{G}} \mapsto \genfrac{[}{]}{0pt}{}{M'}{N'P'}_{\mathscr{G}'} &= \genfrac{[}{]}{0pt}{}{M'}{N'P}_{\mathscr{G}'} b^P_{P'} \\ &= \left(\left(C_k\right)^{M'}_M \genfrac{[}{]}{0pt}{}{M}{NP}_{\mathscr{G}} \left(B_k\right)^N_{N'} + \left(C_k\right)^{M'}_M \frac{\partial \left(B_k\right)^M_{N'}}{\partial x^P}\right) b^P_{P'}. \end{aligned} \tag{27}$$

Equations (24) and (27) show that $\genfrac{[}{]}{0pt}{}{M}{NP}_{\mathscr{G}}$ is not only an affine connection, but also a connection on frame bundle.

Apply Equations (24)–(27) to the curvature tensors

$$\begin{aligned} \genfrac{[}{]}{0pt}{}{M}{NPQ} &\triangleq \frac{\partial \genfrac{[}{]}{0pt}{}{M}{NQ}}{\partial x^P} - \frac{\partial \genfrac{[}{]}{0pt}{}{M}{NP}}{\partial x^Q} + \genfrac{[}{]}{0pt}{}{M}{HP}\genfrac{[}{]}{0pt}{}{H}{NQ} - \genfrac{[}{]}{0pt}{}{H}{NP}\genfrac{[}{]}{0pt}{}{M}{HQ}, \\ \genfrac{\{}{\}}{0pt}{}{M}{NPQ} &\triangleq \frac{\partial \genfrac{\{}{\}}{0pt}{}{M}{NQ}}{\partial x^P} - \frac{\partial \genfrac{\{}{\}}{0pt}{}{M}{NP}}{\partial x^Q} + \genfrac{\{}{\}}{0pt}{}{M}{HP}\genfrac{\{}{\}}{0pt}{}{H}{NQ} - \genfrac{\{}{\}}{0pt}{}{H}{NP}\genfrac{[}{]}{0pt}{}{M}{HQ}, \\ R^M_{NPQ} &\triangleq \frac{\partial \Gamma^M_{NQ}}{\partial x^P} - \frac{\partial \Gamma^M_{NP}}{\partial x^Q} + \Gamma^M_{HP}\Gamma^H_{NQ} - \Gamma^H_{NP}\Gamma^M_{HQ}, \end{aligned} \tag{28}$$

and then, it is obtained that

$$\begin{aligned} L_k : \genfrac{[}{]}{0pt}{}{M}{NPQ}_{\mathscr{G}} \mapsto \genfrac{[}{]}{0pt}{}{M'}{N'P'Q'}_{\mathscr{G}'} &= \genfrac{[}{]}{0pt}{}{M'}{N'PQ}_{\mathscr{G}'} b^P_{P'} b^Q_{Q'} \\ &= \left(\left(C_k\right)^{M'}_M \genfrac{[}{]}{0pt}{}{M}{NPQ}_{\mathscr{G}} \left(B_k\right)^N_{N'}\right) b^P_{P'} b^Q_{Q'}, \\ L_{k(p)} : \genfrac{[}{]}{0pt}{}{M}{NPQ}_{\mathscr{G}} \mapsto \genfrac{[}{]}{0pt}{}{M'}{N'P'Q'}_{\mathscr{G}} &= c^{M'}_M \genfrac{[}{]}{0pt}{}{M}{NPQ}_{\mathscr{G}} b^N_{N'} b^P_{P'} b^Q_{Q'}, \\ L_{k(p)} : \genfrac{[}{]}{0pt}{}{M}{NPQ}_{\mathscr{G}} \mapsto \genfrac{[}{]}{0pt}{}{M'}{N'P'Q'}_{\mathscr{G}} &= c^{M'}_M \genfrac{[}{]}{0pt}{}{M}{NPQ}_{\mathscr{G}} b^N_{N'} b^P_{P'} b^Q_{Q'}, \\ L_{k(p)} : \left(R_{\mathscr{G}}\right)^M_{NPQ} \mapsto \left(R_{\mathscr{G}}\right)^{M'}_{N'P'Q'} &= c^{M'}_M \left(R_{\mathscr{G}}\right)^M_{NPQ} b^N_{N'} b^P_{P'} b^Q_{Q'}. \end{aligned} \tag{29}$$

We see from Equation (29) that the $\left[{}^{M}_{NPQ}\right]_{\mathscr{G}}$ without gravitation is both a curvature tensor of affine connection and a curvature tensor on frame bundle, and that the $(R_{\mathscr{G}})^{M}_{NPQ}$ with gravitation is a curvature tensor of affine connection, but not a curvature tensor on frame bundle. In other words, under the gauge transformation L_k, $\left[{}^{M}_{NPQ}\right]_{\mathscr{G}}$ and $\left[{}^{M'}_{N'PQ}\right]_{\mathscr{G}'}$ represent the same physical state, while $(R_{\mathscr{G}})^{M}_{NPQ}$ and $(R_{\mathscr{G}'})^{M'}_{N'PQ}$ represent different physical states. This shows that the gravitational field in $(R_{\mathscr{G}})^{M}_{NPQ}$ makes the gauge frames B^{A}_{M} and C^{M}_{A} have physical effects.

3. The Evolution in Affine Connection Representation of Gauge Fields

Now that we have the required affine connection, next we have to solve the problem that how to describe the evolution in affine connection representation.

In the existing theories, time is effected by gravitational field, but not effected by gauge field. This leads to an essential difference between the description of evolution of gravitational field and that of gauge field. In this case, it is difficult to obtain a unified theory of evolution in affine connection representation. We adopt the following way to overcome this difficulty.

3.1. The Relation between Time and Space

Definition 2. Suppose $M = P \times N$ and $r \triangleq \dim P = 3$. Let

$$A, B, M, N = 1, \cdots, \mathfrak{D}; \quad s, i = 1, \cdots, r; \quad a, m = r+1, \cdots, \mathfrak{D}. \tag{30}$$

On a geometric manifold (M, f), the $d\xi^0$ and dx^0 which are defined by

$$\begin{aligned} \left(d\xi^0\right)^2 &\triangleq \sum_{A=1}^{\mathfrak{D}} \left(d\xi^A\right)^2 = \delta_{AB} d\xi^A d\xi^B = G_{MN} dx^M dx^N, \\ \left(dx^0\right)^2 &\triangleq \sum_{M=1}^{\mathfrak{D}} \left(dx^M\right)^2 = \varepsilon_{MN} dx^M dx^N = H_{AB} d\xi^A d\xi^B, \end{aligned} \tag{31}$$

are said to be total space metrics or **time metrics**. We also suppose

$$\begin{aligned} \left(d\xi^{(P)}\right)^2 &\triangleq \sum_{s=1}^{r} \left(d\xi^s\right)^2, \quad \left(d\xi^{(N)}\right)^2 \triangleq \sum_{a=r+1}^{\mathfrak{D}} \left(d\xi^a\right)^2, \\ \left(dx^{(P)}\right)^2 &\triangleq \sum_{i=1}^{r} \left(dx^i\right)^2, \quad \left(dx^{(N)}\right)^2 \triangleq \sum_{m=r+1}^{\mathfrak{D}} \left(dx^m\right)^2. \end{aligned} \tag{32}$$

$d\xi^{(N)}$ and $dx^{(N)}$ are regarded as proper-time metrics. For convenience, P is said to be **external space** and N is said to be **internal space**.

Remark 3. The above definition implies a new viewpoint about time and space. The relation between time and space in this way is different from the Minkowski coordinates $x^\mu (\mu = 0, 1, 2, 3)$. Time and space are not the components on an equal footing anymore, but have a relation of total to component. It can be seen later that time reflects the total evolution in the full-dimensional space, while a specific spatial dimension reflects just a partial evolution in a specific direction.

3.2. Evolution Path as a Submanifold

Definition 4. Let there be reference-systems f, g, $\mathfrak{f}$, and $\mathfrak{g}$ on a manifold M, such that $\forall p \in M$, on the neighborhood U of p,

$$\left(U, \alpha^{A'}\right) \xrightarrow{\mathfrak{f}(p)} \left(U, \xi^A\right) \xrightarrow{f(p)} \left(U, x^M\right) \xleftarrow{g(p)} \left(U, \zeta^A\right) \xleftarrow{\mathfrak{g}(p)} \left(U, \beta^{A'}\right). \tag{33}$$

Denote $\mathscr{F} \triangleq \mathfrak{f} \circ f$ and $\mathscr{G} \triangleq \mathfrak{g} \circ g$; then, we say $\mathscr{F}$ and $\mathscr{G}$ **move relatively and interact mutually**, and also we say that $\mathscr{F}$ evolves in $\mathscr{G}$, or $\mathscr{F}$ evolves on the geometric manifold $(M, \mathscr{G})$. Meanwhile, $\mathscr{G}$ evolves in $\mathscr{F}$, or we say $\mathscr{G}$ evolves on $(M, \mathscr{F})$.

From Equation (23), we know that in $\mathscr{F}$ and $\mathscr{G}$, gauge fields originate from $\mathfrak{f}$ and $\mathfrak{g}$, and gravitational fields $(G_{\mathscr{F}})_{MN}$ and $(G_{\mathscr{G}})_{MN}$ are effected by $\mathfrak{f}$ and $\mathfrak{g}$, respectively. We are going to describe their evolutions step by step in the following sections.

Let there be a one-parameter group of diffeomorphisms

$$\varphi_X : M \times \mathbb{R} \longrightarrow M, \tag{34}$$

acting on M, such that $\varphi_X(p, 0) = p$. Thus, φ_X determines a smooth tangent vector field X on M. If X is nonzero everywhere, we say φ_X is **a set of evolution paths**, and X is an evolution direction field. Let $T \subseteq \mathbb{R}$ be an interval; then, the regular imbedding

$$L_p \triangleq \varphi_{X,p} : T \longrightarrow M, t \mapsto \varphi_X(p, t) \tag{35}$$

is said to be an evolution path through p. The tangent vector $d/dt \triangleq [L_p] = X(p)$ is called an evolution direction at p. For the sake of simplicity, we also denote $L_p \triangleq L_p(T) \subset M$; then,

$$\pi : L_p \longrightarrow M, q \mapsto q \tag{36}$$

is also a regular imbedding. If it is not necessary to emphasize the point p, L_p is denoted by L concisely.

In order to describe physical evolution, next we are going to strictly describe the mathematical properties of the reference-systems f and g which are sent onto the evolution path L.

Definition 5. Let the time metrics of (U, ξ^A), (U, x^M), and (U, ζ^A) be $d\xi^0$, dx^0, and $d\zeta^0$, respectively. On $U_L \triangleq U \cap L_p$, we have parameter equations

$$\begin{aligned} \xi^A &= \xi^A\left(x^0\right), \quad x^M = x^M\left(\xi^0\right), \quad \zeta^A = \zeta^A\left(x^0\right), \\ \xi^0 &= \xi^0\left(x^0\right), \quad x^0 = x^0\left(\xi^0\right), \quad \zeta^0 = \zeta^0\left(x^0\right). \end{aligned} \tag{37}$$

Take f for example, according to Equation (37), on U_L we define

$$\begin{aligned} b_0^A &\triangleq \frac{d\xi^A}{dx^0}, \quad b_0^0 \triangleq \frac{d\xi^0}{dx^0}, \quad \varepsilon_0^M \triangleq \frac{dx^M}{dx^0} = b_0^0 c_0^M = b_0^A c_A^M, \\ c_0^M &\triangleq \frac{dx^M}{d\xi^0}, \quad c_0^0 \triangleq \frac{dx^0}{d\xi^0}, \quad \delta_0^A \triangleq \frac{d\xi^A}{d\xi^0} = c_0^0 b_0^A = c_0^M b_M^A. \end{aligned} \tag{38}$$

Define $d\xi_0 \triangleq dx^0/d\xi^0 dx^0$ and $dx_0 \triangleq d\xi^0/dx^0 d\xi^0$, which induce $d/d\xi_0$ and d/dx_0, such that $\langle d/d\xi_0, d\xi_0\rangle = 1$ and $\langle d/dx_0, dx_0\rangle = 1$. So we can also define

$$\begin{aligned} \bar{b}_A^0 &\triangleq \frac{d\xi_A}{dx_0}, \quad \bar{b}_0^0 \triangleq \frac{d\xi_0}{dx_0}, \quad \bar{\varepsilon}_M^0 \triangleq \frac{dx_M}{dx_0} = \bar{b}_0^0 \bar{c}_M^0 = \bar{b}_A^0 \bar{c}_M^A, \\ \bar{c}_M^0 &\triangleq \frac{dx_M}{d\xi_0}, \quad \bar{c}_0^0 \triangleq \frac{dx_0}{d\xi_0}, \quad \bar{\delta}_A^0 \triangleq \frac{d\xi_A}{d\xi_0} = \bar{c}_0^0 \bar{b}_A^0 = \bar{c}_M^0 \bar{b}_A^M. \end{aligned} \tag{39}$$

They determine the following smooth functions on the entire L, similar to Section 2.2, that

$$\begin{aligned} B_0^A &: L \longrightarrow \mathbb{R}, p \mapsto B_0^A(p) \triangleq \left(b_{f(p)}\right)_0^A(p), \quad C_0^M : L \longrightarrow \mathbb{R}, p \mapsto C_0^M(p) \triangleq \left(c_{f(p)}\right)_0^M(p), \\ \bar{B}_A^0 &: L \longrightarrow \mathbb{R}, p \mapsto \bar{B}_A^0(p) \triangleq \left(\bar{b}_{f(p)}\right)_A^0(p), \quad \bar{C}_M^0 : L \longrightarrow \mathbb{R}, p \mapsto \bar{C}_M^0(p) \triangleq \left(\bar{c}_{f(p)}\right)_M^0(p), \\ B_0^0 &: L \longrightarrow \mathbb{R}, p \mapsto B_0^0(p) \triangleq \left(b_{f(p)}\right)_0^0(p), \quad C_0^0 : L \longrightarrow \mathbb{R}, p \mapsto C_0^0(p) \triangleq \left(c_{f(p)}\right)_0^0(p), \\ \bar{B}_0^0 &: L \longrightarrow \mathbb{R}, p \mapsto \bar{B}_0^0(p) \triangleq \left(\bar{b}_{f(p)}\right)_0^0(p), \quad \bar{C}_0^0 : L \longrightarrow \mathbb{R}, p \mapsto \bar{C}_0^0(p) \triangleq \left(\bar{c}_{f(p)}\right)_0^0(p). \end{aligned} \tag{40}$$

For convenience, we still use the notations ε and δ and have the following smooth functions.

$$\begin{aligned} \varepsilon_0^M &\triangleq B_0^0 C_0^M = B_0^A C_A^M, \quad \delta_0^A \triangleq C_0^0 B_0^A = C_0^M B_M^A, \quad G_{00} \triangleq B_0^0 B_0^0 = G_{MN} \varepsilon_0^M \varepsilon_0^N, \\ \bar{\varepsilon}_M^0 &\triangleq \bar{B}_0^0 \bar{C}_M^0 = \bar{B}_A^0 \bar{C}_M^A, \quad \bar{\delta}_A^0 \triangleq \bar{C}_0^0 \bar{B}_A^0 = \bar{C}_M^0 \bar{B}_A^M, \quad G^{00} \triangleq C_0^0 C_0^0 = G^{MN} \bar{\varepsilon}_M^0 \bar{\varepsilon}_N^0. \end{aligned} \tag{41}$$

It is easy to verify that $dx_0 = G_{00} dx^0$ and $d/dx_0 = G^{00} d/dx^0$ are both true on L by a simple calculation.

3.3. Evolution Lemma. We have the following two evolution lemmas. The affine connection representations of Yang-Mills equation, energy-momentum equation, and Dirac equation are dependent on them.

Definition 6. $\forall p \in L$, suppose $T_p(M)$ and $T_p(L)$ are tangent spaces, $T_p^*(M)$ and $T_p^*(L)$ are cotangent spaces. The regular imbedding $\pi : L \longrightarrow M, q \mapsto q$ induces the tangent map and the cotangent map

$$\begin{aligned} \pi_* &: T_p(L) \longrightarrow T_p(M), \quad [\gamma_L] \mapsto [\pi \circ \gamma_L], \\ \pi^* &: T_p^*(M) \longrightarrow T_p^*(L), \quad df \mapsto d(f \circ \pi). \end{aligned} \tag{42}$$

Evidently, π_* is an injection, and π^* is a surjection. $\forall d/dt_L \in T_p(L), d/dt \in T_p(M), df \in T_p^*(M), df_L \in T_p^*(L)$, if and only if

$$\begin{aligned} \frac{d}{dt} &= \pi_*\left(\frac{d}{dt_L}\right), \\ df_L &= \pi^*(df) \end{aligned} \tag{43}$$

are true, we denote

$$\begin{aligned} \frac{d}{dt} &\cong \frac{d}{dt_L}, \\ df &\simeq df_L. \end{aligned} \tag{44}$$

Then, we have the following two propositions that are evidently true.

Proposition 7. *If $d/dt \cong d/dt_L$ and $df \simeq df_L$, then*

$$\left\langle \frac{d}{dt}, df \right\rangle = \left\langle \frac{d}{dt_L}, df_L \right\rangle. \tag{45}$$

Proposition 8. *The following conclusions are true.*

$$\begin{cases} w^M \dfrac{\partial}{\partial x^M} \cong w^0 \dfrac{d}{dx^0} & \Longleftrightarrow w^M = w^0 \varepsilon_0^M, \\ w_M dx^M \simeq w_0 dx^0 & \Longleftrightarrow w_M \varepsilon_0^M = w_0, \end{cases}$$
$$\begin{cases} \bar{w}_M \dfrac{\partial}{\partial x_M} \cong \bar{w}_0 \dfrac{d}{dx_0} & \Longleftrightarrow \bar{w}_M = \bar{w}_0 \bar{\varepsilon}_M^0, \\ \bar{w}^M dx_M \simeq \bar{w}^0 dx_0 & \Longleftrightarrow \bar{w}^M \bar{\varepsilon}_M^0 = \bar{w}^0. \end{cases} \tag{46}$$

3.4. On-Shell Evolution as a Gradient. Let $\mathbf{T}$ be a smooth n-order tensor field. The restriction on (U, x^M) is $\mathbf{T} \triangleq t\{\partial/\partial x \otimes dx\}$, where $\{\partial/\partial x \otimes dx\}$ represents the tensor basis generated by several $\partial/\partial x^M$ and dx^M, and the tensor coefficients of $\mathbf{T}$ are concisely denoted by $t : U \longrightarrow \mathbb{R}$.

Let D be a holonomic connection. Consider $D\mathbf{T} \triangleq t_{;Q} dx^Q \otimes \{\partial/\partial x \otimes dx\}$. Denote

$$Dt \triangleq t_{;Q} dx^Q, \quad \nabla t \triangleq t_{;Q} \frac{\partial}{\partial x_Q}. \tag{47}$$

$\forall p \in M$, the integral curve of ∇t, that is, $L_p \triangleq \varphi_{\nabla t, p}$, is a gradient line of $\mathbf{T}$. It can be seen later that the above gradient operator ∇ characterizes the on-shell evolution.

For any evolution path L, let $U_L \triangleq U \cap L$. Denote $t_L \triangleq t|_{U_L}$ and $t_{L;0} \triangleq t_{;Q} \varepsilon_0^Q$, as well as

$$D_L t_L \triangleq t_{L;0} dx^0, \quad \nabla_L t_L \triangleq t_{L;0} \frac{d}{dx_0}. \tag{48}$$

Proposition 9. *The following conclusions are evidently true.*

(i) *$Dt \simeq D_L t_L$ if and only if L is an arbitrary evolution path*

(ii) *$\nabla t \cong \nabla_L t_L$ if and only if L is a gradient line of $\mathbf{T}$*

Remark 10. More generally, suppose there is a tensor $\mathbf{U} \triangleq u_Q dx^Q \otimes \{\partial/\partial x \otimes dx\}$. In such a notation, all the indices are concisely ignored except Q. $u_Q dx^Q$ uniquely determines a characteristic direction $u_Q \partial/\partial x_Q$.

If the system of 1-order linear partial differential equations $t_{;Q} = u_Q$ has a solution t, then it is true that $Dt = u_Q dx^Q$ and $\nabla t = u_Q \partial/\partial x_Q$. Thus, in the evolution direction $[L] = u_Q \partial/\partial x_Q$, the following conclusions are true.

$$Dt \simeq D_L t_L, \quad \nabla t \cong \nabla_L t_L, \tag{49}$$

where $D_L t_L \triangleq u_0 dx^0$, $\nabla_L t_L \triangleq u_0 d/dx_0$, and $u_0 \triangleq u_Q \varepsilon_0^Q$.

Now for any geometric property in the form of tensor $\mathbf{U}$, we are able to express its on-shell evolution in the form of ∇t.

Next, two important on-shell evolutions are discussed in the following two sections. One is the on-shell evolution of the potential field of a reference-system. The other is the one that a general charge of a reference-system evolves in the potential field of another reference-system.

3.5. On-Shell Evolution of Potential Field and Affine Connection Representation of Yang-Mills Equation. Table I of article [40] proposes a famous correspondence between gauge field terminologies and fibre bundle terminologies. However, it does not find out the corresponding mathematical object to the source J_μ^K. In this section, we give an answer to this problem and show the affine connection representation of Yang-Mills equation.

In order to obtain the general Yang-Mills equation with gravitation, we have to adopt holonomic connection to construct it. Suppose $\mathscr{F}$ evolves in $\mathscr{G}$ according to Definition 4, that is, $\forall p \in M$,

$$\left(U, \alpha^{A'}\right) \xrightarrow{\mathfrak{f}(p)} \left(U, \xi^A\right) \xrightarrow{f(p)} \left(U, x^M\right) \xleftarrow{g(p)} \left(U, \zeta^A\right) \xleftarrow{\mathfrak{g}(p)} \left(U, \beta^{A'}\right). \tag{50}$$

We always take the following notations in the coordinate frame (U, x^M).

(i) Let the holonomic connections, which are defined by Equation (25), of geometric manifolds $(M, \mathscr{F})$ and $(M, \mathscr{G})$ be $(\Gamma_{\mathscr{F}})_{NP}^M$ and $(\Gamma_{\mathscr{G}})_{NP}^M$, respectively. The colon ":" and the semicolon ";" are used to express the covariant derivatives on $(M, \mathscr{F})$ and $(M, \mathscr{G})$, respectively, e.g.,

$$u^Q{}_{:P} = \frac{\partial u^Q}{\partial x^P} + (\Gamma_{\mathscr{F}})_{HP}^Q u^H, \quad u^Q{}_{;P} = \frac{\partial u^Q}{\partial x^P} + (\Gamma_{\mathscr{G}})_{HP}^Q u^H. \tag{51}$$

(ii) Let the coefficients of curvature tensor of $(M, \mathscr{F})$ and $(M, \mathscr{G})$ be K_{NPQ}^M and R_{NPQ}^M, respectively, i.e.,

$$\begin{aligned} K_{NPQ}^M &\triangleq \frac{\partial (\Gamma_{\mathscr{F}})_{NQ}^M}{\partial x^P} - \frac{\partial (\Gamma_{\mathscr{F}})_{NP}^M}{\partial x^Q} + (\Gamma_{\mathscr{F}})_{NQ}^H (\Gamma_{\mathscr{F}})_{HP}^M - (\Gamma_{\mathscr{F}})_{NP}^H (\Gamma_{\mathscr{F}})_{HQ}^M, \\ R_{NPQ}^M &\triangleq \frac{\partial (\Gamma_{\mathscr{G}})_{NQ}^M}{\partial x^P} - \frac{\partial (\Gamma_{\mathscr{G}})_{NP}^M}{\partial x^Q} + (\Gamma_{\mathscr{G}})_{NQ}^H (\Gamma_{\mathscr{G}})_{HP}^M - (\Gamma_{\mathscr{G}})_{NP}^H (\Gamma_{\mathscr{G}})_{HQ}^M. \end{aligned} \tag{52}$$

Denote $K_{NPQ}^M{}^{:P} \triangleq (G_{\mathscr{F}})^{PP'} K_{NPQ:P'}^M$. On an arbitrary evolution path L, we define

$$\rho_{N0}^M dx^0 \triangleq \pi^* \left(K_{NPQ}^M{}^{:P} dx^Q \right) \in T^*(L). \tag{53}$$

Then, according to Definition 6 and the evolution lemma of Proposition 8, we obtain $\rho_{N0}^M = K_{NPQ}^M{}^{:P} \varepsilon_0^Q$ and

$$K_{NPQ}^M{}^{:P} dx^Q \simeq \rho_{N0}^M dx^0. \tag{54}$$

Let $\nabla t = K_{NPQ}^{M\ \ :P}\partial/\partial x_Q$. Then, according to Proposition 9, if and only if $\forall p \in M$, $[L_p] = \nabla t|_p$, we have

$$K_{NPQ}^{M\ \ :P}\frac{\partial}{\partial x_Q} \cong \rho_{N0}^{M}\frac{d}{dx_0}. \tag{55}$$

Applying the evolution lemma of Proposition 8 again, we obtain

$$K_{NPQ}^{M\ \ :P} = \rho_{N0}^{M}\bar{\varepsilon}_Q^0. \tag{56}$$

Denote $j_{NQ}^M \triangleq \rho_{N0}^M\bar{\varepsilon}_Q^0$; then, if and only if $[L_p] = \nabla t|_p$, we have

$$K_{NPQ}^{M\ \ :P} = j_{NQ}^M, \tag{57}$$

which is said to be (affine) Yang-Mills equation of $\mathscr{F}$. It contains effects of gravitation, which makes the gauge frames $(B_f)_M^A$ and $(C_f)_A^M$ have physical effects. According to Equation (29), we know Equation (57) is coordinate covariant, and if gravitation is removed, it is also gauge covariant.

Thus, we have the following two results.

(i) The Yang-Mills equation originates from a geometric property in the direction ∇t. In other words, the on-shell evolution of gauge field is described by the direction field ∇t

(ii) We obtain the mathematical origination of charge and current. We know that the evolution path L is an imbedding submanifold of M. Thus, the charge ρ_{N0}^M originates from the pull-back π^* from M to L, and the current j_{NQ}^M originates from ∇t that is associated to ρ_{N0}^M

If we let (M, f) be completely flat, i.e., $(B_f)_M^A = \delta_M^A$, $(C_f)_A^M = \delta_A^M$, then by calculation, we find ρ_{N0}^M can still be nonvanishing. This shows that ρ_{N0}^M originates from $(M, \mathfrak{f})$ ultimately.

Definition 11. We speak of the real-valued

$$\rho_{MN0} \triangleq G_{MH}\rho_{N0}^H \tag{58}$$

as the field function of a general charge or speak of it as a charge of $\mathscr{F}$ for short.

3.6. On-Shell Evolution of General Charge and Affine Connection Representation of Mass, Energy, Momentum, and Action. In order to be compatible with the affine connection representation of gauge fields, we also have to define mass, energy, momentum, and action in the form associated to affine connection. We are going to show them in this section and Section 4.3.

Let $\mathbf{F}_0 \triangleq \rho_{MN0}dx^M \otimes dx^N$. For the sake of simplicity, denote the charge ρ_{MN0} of $\mathscr{F}$ by ρ_{MN} concisely. Let D be the holonomic connection of $(M, \mathscr{G})$; then,

$$D\mathbf{F}_0 \triangleq D\rho_{MN} \otimes dx^M \otimes dx^N, \quad \nabla\mathbf{F}_0 \triangleq \nabla\rho_{MN} \otimes dx^M \otimes dx^N, \tag{59}$$

where $D\rho_{MN} \triangleq \rho_{MN;Q}dx^Q$ and $\nabla\rho_{MN} \triangleq \rho_{MN;Q}\partial/\partial x_Q$. According to Proposition 9, if and only if $\forall p \in L$, the evolution direction is taken as $[L_p] = \nabla\rho_{MN}|_p$, we have

$$D\rho_{MN} \simeq D_L\rho_{MN}, \quad \nabla\rho_{MN} \cong \nabla_L\rho_{MN}, \tag{60}$$

that is,

$$\rho_{MN;Q}dx^Q \simeq \rho_{MN;0}dx^0, \quad \rho_{MN;Q}\frac{\partial}{\partial x_Q} \cong \rho_{MN;0}\frac{d}{dx_0}. \tag{61}$$

Definition 12. For more convenience, the notation ρ_{MN} is further abbreviated as ρ. In affine connection representation, energy and momentum of ρ are defined as

$$\begin{aligned} &E_0 \triangleq \rho_{;0} \triangleq \rho_{;Q}\varepsilon_0^Q, \quad p_Q \triangleq \rho_{;Q}, \quad H_0 \triangleq \frac{d\rho}{dx^0}, \quad P_Q \triangleq \frac{\partial\rho}{\partial x^Q}, \\ &E^0 \triangleq \rho^{;0} \triangleq \rho^{;Q}\bar{\varepsilon}_Q^0, \quad p^Q \triangleq \rho^{;Q}, \quad H^0 \triangleq \frac{d\rho}{dx_0}, \quad P^Q \triangleq \frac{\partial\rho}{\partial x_Q}. \end{aligned} \tag{62}$$

Proposition 13. *At any point p on M, the equation*

$$E_0E^0 = p_Qp^Q \tag{63}$$

holds if and only if the evolution direction $[L_p] = \nabla\rho|_p$. Equation (63) is the (affine) energy-momentum equation of ρ.

Proof. According to the above discussion, $\forall p \in M$, $[L_p] = \nabla\rho|_p$ is equivalent to

$$p_Qdx^Q \simeq E_0dx^0, \quad p_Q\frac{\partial}{\partial x_Q} \cong E_0\frac{d}{dx_0}. \tag{64}$$

Then, due to Proposition 7, we obtain the directional derivative in the gradient direction $\nabla\rho$:

$$\left\langle p_Q\frac{\partial}{\partial x_Q}, p_Mdx^M\right\rangle = \left\langle E_0\frac{d}{dx_0}, E_0dx^0\right\rangle, \tag{65}$$

i.e., $G^{QM}p_Qp_M = G^{00}E_0E_0$, or $p_Qp^Q = E_0E^0$. □

Proposition 14. *At any point p on M, the equations*

$$p^Q = E^0 \frac{dx^Q}{dx^0}, \quad p_Q = E_0 \frac{dx_Q}{dx_0} \tag{66}$$

hold if and only if the evolution direction $[L_p] = \nabla\rho|_p$.

Proof. Due to the evolution lemma of Proposition 8, we immediately obtain Equation (66) from Equation (64). □

Remark 15. In the gradient direction $\nabla\rho$, Equation (66) is consistent with the conventional formula

$$p = mv. \tag{67}$$

Thus, in affine connection representation, the energy-momentum equation and the conventional definition of momentum both originate from a geometric property in gradient direction. In other words, the on-shell evolution of the particle field ρ is described by the gradient direction field $\nabla\rho$.

Definition 16. Let $\mathscr{P}(b, a)$ be the totality of paths from a to b. And suppose $L \in \mathscr{P}(b, a)$, and the evolution parameter x^0 satisfies $t_a \triangleq x^0(a) < x^0(b) \triangleq t_b$. The elementary affine action of ρ is defined as

$$\mathfrak{s}(L) \triangleq \int_L D\rho = \int_L p_Q dx^Q = \int_{t_a}^{t_b} E_0 dx^0. \tag{68}$$

Thus, $\delta\mathfrak{s}(L) = 0$ if and only if L is a gradient line of ρ.

In particular, in the case where $\mathscr{G}$ is orthogonal, we can also define action in the following way.

On $(M, \mathscr{G})$, let there be Dirac algebras γ^M and γ_N such that

$$\gamma^M\gamma^N + \gamma^N\gamma^M = 2G^{MN}, \quad \gamma_M\gamma_N + \gamma_N\gamma_M = 2G_{MN}, \quad \gamma_M\gamma^M = 1. \tag{69}$$

In a gradient direction of ρ, from Equation (63), we obtain that

$$\begin{aligned} p_Q p^Q = E_0 E^0 &\Longleftrightarrow \rho_{;Q}\rho^{;Q} = \rho_{;0}\rho^{;0} \Longleftrightarrow G^{PQ}\rho_{;P}\rho_{;Q} \\ &= G^{00}\rho_{;0}\rho_{;0} \Longleftrightarrow \left(\gamma^P\gamma^Q + \gamma^Q\gamma^P\right)\rho_{;P}\rho_{;Q} \\ &= 2\rho_{;0}\rho_{;0} \Longleftrightarrow \left(\gamma^P\rho_{;P}\right)\left(\gamma^Q\rho_{;Q}\right) + \left(\gamma^Q\rho_{;Q}\right)\left(\gamma^P\rho_{;P}\right) \\ &= 2\rho_{;0}\rho_{;0} \Longleftrightarrow \left(\gamma^P\rho_{;P}\right)^2 = \left(\rho_{;0}\right)^2. \end{aligned} \tag{70}$$

Take $\gamma^P\rho_{;P} = \rho_{;0}$ without loss of generality, and then, in the gradient direction of ρ, we have

$$\gamma^P\rho_{;P}dx^0 = \rho_{;0}dx^0 = \varepsilon_0^P\rho_{;P}dx^0 = D\rho. \tag{71}$$

So we can take

$$\begin{aligned} s(L) \triangleq \int_L \left(\gamma^P\rho_{;P}dx^0 + D\rho\right) &= \int_{t_a}^{t_b} \left(\gamma^P\rho_{;P} + \varepsilon_0^P\rho_{;P}\right)dx^0 \\ &= \int_{t_a}^{t_b} \left(\gamma^P\rho_{;P} + E_0\right)dx^0. \end{aligned} \tag{72}$$

Remark 17 and Remark 41 explain the rationality of this definition. We have $s(L) = 2\mathfrak{s}(L)$ in the gradient direction of ρ, so $\mathfrak{s}(L)$ and $s(L)$ are consistent.

Remark 17. In the Minkowski coordinate frame of Section 4.2, the evolution parameter x^0 is replaced by $\tilde{x}^\tau$; then, there still exists a concept of gradient direction $\tilde{\nabla}\tilde{\rho}$. Correspondingly, Equations (68) and (72) present as

$$\tilde{\mathfrak{s}}(L) \triangleq \int_L \tilde{D}\tilde{\rho} = \int_L \tilde{p}_\mu d\tilde{x}^\mu = \int_{\tau_a}^{\tau_b} \tilde{m}_\tau d\tilde{x}^\tau, \quad \tilde{s}(L) = \int_{\tau_a}^{\tau_b} \left(\gamma^\mu\tilde{\rho}_{;\mu} + \tilde{m}_\tau\right)d\tilde{x}^\tau, \tag{73}$$

where $\tilde{m}_\tau$ is the rest-mass and $\tilde{x}^\tau$ is the proper-time.

Remark 18. Define the following notations.

$$\begin{aligned} \left[\rho\Gamma_G\right] \triangleq \frac{\partial\rho_{MN}}{\partial x^G} - \rho_{MN;G} = \rho_{MH}\Gamma^H_{NG} \\ + \rho_{HN}\Gamma^H_{MG}, \quad \left[\rho R_{PQ}\right] \triangleq \rho_{MH}R^H_{NPQ} + \rho_{HN}R^H_{MPQ}. \end{aligned} \tag{74}$$

Then, through some calculations, we can obtain that

$$f_P \triangleq p_{P;0} = E_{0;P} - p_Q\varepsilon^Q_{0;P} + \left[\rho R_{PQ}\right]\varepsilon^Q_0, \tag{75}$$

which is the affine connection representation of general Lorentz force equation (see Discussion 38 for further illustrations).

3.7. Inversion Transformation in Affine Connection Representation. In affine connection representation, *CPT* inversion is interpreted as a full inversion of coordinates and metrics. Let $i, j = 1, 2, 3$ and $m, n = 4, 5, \cdots, \mathfrak{D}$.

Let the local coordinate representation of reference-system k be $x'^j = -\delta^j_i x^i$, $x'^n = \delta^n_m x^m$; then, parity inversion can be represented as

$$P \triangleq L_k : x^i \longrightarrow -x^i, x^m \longrightarrow x^m. \tag{76}$$

Let the local coordinate representation of reference-system h be $x'^j = \delta^j_i x^i$, $x'^n = -\delta^n_m x^m$; then, charge conjugate inversion can be represented as

$$C \triangleq L_h : x^i \longrightarrow x^i, x^m \longrightarrow -x^m. \tag{77}$$

Time coordinate inversion can be represented as

$$T_0 : x^0 \longrightarrow -x^0. \tag{78}$$

Full inversion of coordinates can be represented as

$$CPT_0 : x^Q \longrightarrow -x^Q, x^0 \longrightarrow -x^0. \tag{79}$$

The positive or negative sign of metric marks two opposite directions of evolution. Let N be a closed submanifold of M, and let its metric be $dx^{(N)}$. Denote the totality of closed submanifolds of M by $\mathfrak{B}(M)$; then, full inversion of metrics can be expressed as

$$T^{(M)} \triangleq \prod_{N\in\mathfrak{B}(M)} \left(dx^{(N)} \longrightarrow -dx^{(N)}\right). \tag{80}$$

Denote time inversion by

$$T \triangleq T^{(M)} T_0, \tag{81}$$

and then, the joint transformation of the full inversion of coordinates CPT_0 and the full inversion of metrics $T^{(M)}$ is

$$(CPT_0)\left(T^{(M)}\right) = CPT. \tag{82}$$

Summarize the above discussions; then, we have

$$\begin{aligned} CPT_0 &: x^Q \longrightarrow -x^Q, x^0 \longrightarrow -x^0, dx^Q \longrightarrow dx^Q, dx^0 \longrightarrow dx^0, \\ T^{(M)} &: x^Q \longrightarrow x^Q, x^0 \longrightarrow x^0, dx^Q \longrightarrow -dx^Q, dx^0 \longrightarrow -dx^0, \\ CPT &: x^Q \longrightarrow -x^Q, x^0 \longrightarrow -x^0, dx^Q \longrightarrow -dx^Q, dx^0 \longrightarrow -dx^0. \end{aligned} \tag{83}$$

The CPT invariance in affine connection representation is very clear. Concretely, on $(M, \mathscr{G})$, we consider the CPT transformation acting on $\mathscr{G}$. Denote $s \triangleq \int_L D\rho$ and $D_P e^{is} \triangleq (\partial/\partial x^P - i[\rho\Gamma_P])e^{is}$; then, through simple calculations, we obtain that

$$CPT : D\rho \longrightarrow D\rho, \quad D_P e^{is} \longrightarrow -D_P e^{-is}. \tag{84}$$

Remark 19. In quantum mechanics, there is a complex conjugation in the time inversion of wave function $T : \psi(x, t) \longrightarrow \psi^*(x, -t)$. In affine connection representation, we know the complex conjugation can be interpreted as a straightforward mathematical result of the full inversion of metrics $T^{(M)}$.

3.8. Two Dual Descriptions of Gradient Direction Field

Discussion 20. Let X and Y be nonvanishing smooth tangent vector fields on the manifold M. And let L_Y be the Lie derivative operator induced by the one-parameter group of diffeomorphism φ_Y. Then, according to a well-known theorem [41], we obtain the Lie derivative equation

$$[X, Y] = L_Y X. \tag{85}$$

Suppose $\forall p \in M$, $Y(p)$ is a unit-length vector, i.e., $\|Y(p)\| = 1$. Let the parameter of φ_Y be x^0. Then, on the evolution path $L \triangleq \varphi_{Y,p}$, we have

$$Y \cong \frac{d}{dx^0}. \tag{86}$$

Thus, Equation (85) can also be represented as

$$[X, Y] = \frac{d}{dx^0} X. \tag{87}$$

On the other hand, $\forall df \in T(M)$ and $df_L \triangleq \pi^*(df)$, and due to (86) and Proposition 7, we have $\langle Y, df\rangle = \langle d/dx^0, df_L\rangle$, that is,

$$Yf = \frac{d}{dx^0} f_L. \tag{88}$$

Definition 21. Let $H \triangleq \|\nabla\rho\|^{-1}\nabla\rho = \varepsilon_0^M \partial/\partial x^M \cong d/dx^0$. It is evident that $\forall p \in M$, $\|H(p)\| = 1$. If and only if taking $Y = H$, we speak of (87) and (88) as real-valued (affine) Heisenberg equation and (affine) Schrödinger equation, respectively, that is,

$$[X, H] = \frac{d}{dx^0} X, \quad Hf = \frac{d}{dx^0} f_L. \tag{89}$$

Discussion 22. The above two equations both describe the gradient direction field and thereby reflect on-shell evolution. Such two dual descriptions of gradient direction show the real-valued affine connection representation of Heisenberg picture and Schrödinger picture.

It is not hard to find out several different kinds of complex-valued representations of gradient direction. For examples, one is the affine Dirac equation in Section 4.4, and another is as follows.

Let $\psi \triangleq f e^{is_L}$, where it is fine to take either $s_L \triangleq s(L)$ or $s_L \triangleq \mathfrak{s}(L)$ from Definition 16. According to Equation (89), it is easy to obtain on L that

$$[X, H] = \frac{d}{dx^0} X, \quad H\psi = \frac{d\psi}{dx^0}. \tag{90}$$

This is consistent with the conventional Heisenberg equation and Schrödinger equation (taking the natural units that $\hbar = 1, c = 1$)

$$[X, -iH] = \frac{\partial}{\partial t} X, \quad -iH\psi = \frac{\partial\psi}{\partial t}, \tag{91}$$

and they have a coordinate correspondence

$$\frac{\partial}{\partial(ix^k)} \longleftrightarrow \frac{\partial}{\partial x^k}, \quad \frac{\partial}{\partial t} \leftrightarrow \frac{d}{dx^0}. \tag{92}$$

We know that $\partial/\partial t \longleftrightarrow d/dx^0$ originates from the difference that the evolution parameter is x^τ or x^0. The imaginary unit i originates from the difference between the regular coordinates x^1, x^2, x^3, x^τ and the Minkowski coordinates x^1, x^2, x^3, x^0. That is to say, the regular coordinates satisfy

$$\left(dx^0\right)^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^\tau\right)^2, \tag{93}$$

and the Minkowski coordinates satisfy

$$\begin{aligned}\left(dx^\tau\right)^2 &= \left(dx^0\right)^2 - \left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 \\ &= \left(dx^0\right)^2 + \left(d\left(ix^1\right)\right)^2 + \left(d\left(ix^2\right)\right)^2 + \left(d\left(ix^3\right)\right)^2.\end{aligned} \tag{94}$$

This causes the appearance of the imaginary unit i in the correspondence

$$ix^k \longleftrightarrow x^k. \tag{95}$$

So Equations (90) and (91) have exactly the same essence, and their differences only come from different coordinate representations.

The differences between coordinate representations have nothing to do with the geometric essence and the physical essence. We notice that the value of a gradient direction is dependent on geometry, but independent of that the equations are real-valued or complex-valued. Therefore, it is unnecessary for us to confine to such algebraic forms as real-valued or complex-valued forms, but we should focus on such geometric essence as gradient direction.

The essential virtue of complex-valued form is that it is applicable for describing the coherent superposition of propagator. However, this is independent of the above discussions, and we are going to discuss it in Section 3.9.

3.9. Quantum Evolution as a Distribution of Gradient Directions. From Proposition 13, we see that, in affine connection representation, the classical on-shell evolution is described by gradient direction. Then, naturally, quantum evolution should be described by the distribution of gradient directions.

The distribution of gradient directions on a geometric manifold $(M, \mathscr{G})$ is effected by the bending shape of $(M, \mathscr{G})$; in other words, the distribution of gradient directions can be used to reflect the shape of $(M, \mathscr{G})$. This is the way that the quantum theory in affine connection representation describes physical reality.

In order to know the full picture of physical reality, it is necessary to fully describe the shape of the geometric manifold. For a single observation,

(1) It is the reference-system, not a point, that is used to describe the physical reality, so the coordinate of an individual point is not enough to fully describe the location information about the physical reality

(2) Through a single observation of momentum, we can only obtain information about an individual gradient direction; this cannot reflect the full picture of the shape of the geometric manifold

Quantum evolution provides us with a guarantee that we can obtain the distribution of gradient directions through multiple observations, so that we can describe the full picture of the shape of the geometric manifold.

Next, we are going to carry out strict mathematical descriptions for the quantum evolution in affine connection representation.

Definition 23. Let ρ be a geometric property on M, such as a charge of $\mathscr{F}$. Then, $H \triangleq \nabla\rho$ is a gradient direction field of ρ on $(M, \mathscr{G})$.

Let $\mathfrak{T}$ be the totality of all flat transformations L_k defined in Section 2.3. $\forall T \in \mathfrak{T}$, the flat transformation $T : f \mapsto Tf$ induces a transformation $T^* : \rho \mapsto T^*\rho$. Denote

$$|\rho| \triangleq \{\rho_T \triangleq T^*\rho \mid T \in \mathfrak{T}\}, \quad |H| \triangleq \{H_T \triangleq \nabla\rho_T \mid T \in \mathfrak{T}\}. \tag{96}$$

$\forall a \in M$, the restriction of $|H|$ at a are denoted by $|H(a)| \triangleq \{H_T(a) \mid T \in \mathfrak{T}\}$.

We say $|H|$ is the total distribution of the gradient direction field H.

Remark 24. When T is fixed, H_T can reflect the shape of $(M, \mathscr{G})$. When a is fixed, the extension to $|H(a)|$ can reflect the shape of $(M, \mathscr{G})$.

However, when T and a are both fixed, $H_T(a)$ is a fixed individual gradient direction, which cannot reflect the shape of $(M, \mathscr{G})$. In other words, if the momentum p_T and the position x_a of ρ are both definitely observed, the physical reality $\mathscr{G}$ would be unknowable; therefore, this is unacceptable. This is an embodiment of quantum uncertainty in affine connection representation.

Definition 25. Let φ_H be the one-parameter group of diffeomorphisms corresponding to H. The parameter of φ_H is x^0. $\forall a \in M$, according to Definition 4, let $\varphi_{H,a}$ be the evolution path through a, such that $\varphi_{H,a}(0) = a$. $\forall t \in \mathbb{R}$, denote

$$\varphi_{|H|,a} \triangleq \left\{\varphi_{X,a} \mid X \in |H|\right\}, \quad \varphi_{|H|,a}(t) \triangleq \left\{\varphi_{X,a}(t) \mid X \in |H|\right\}. \tag{97}$$

$\forall \Omega \subseteq \mathfrak{T}$, we also denote $|H_\Omega| \triangleq \{H_T \mid T \in \Omega\} \subseteq |H|$ and

$$\begin{aligned}\varphi_{|H_\Omega|,a} &\triangleq \left\{\varphi_{X,a} \mid X \in |H_\Omega|\right\} \subseteq \varphi_{|H|,a}, \\ \varphi_{|H_\Omega|,a}(t) &\triangleq \left\{\varphi_{X,a}(t) \mid X \in |H_\Omega|\right\} \subseteq \varphi_{|H|,a}(t).\end{aligned} \tag{98}$$

$\forall a \in M$, the restriction of $|H_\Omega|$ at a is denoted by $|H_\Omega(a)| \triangleq \{H_T(a) \mid T \in \Omega\}$.

Remark 26. At the beginning $t = 0$, intuitively, the gradient directions $|H(a)|$ of $|\rho|$ start from a and point to all directions around a uniformly. If $(M, \mathcal{G})$ is not flat, when evolving to a certain time $t > 0$, the distribution of gradient directions on $\varphi_{|H|,a}(t)$ is no longer as uniform as beginning. The following definition precisely characterizes this kind of ununiformity.

Definition 27. Let the transformation $L_{\mathcal{G}^{-1}}$ act on $\mathcal{G}$; then, we obtain the trivial $e \triangleq L_{\mathcal{G}^{-1}}(\mathcal{G})$. Now $(M, \mathcal{G})$ is sent to a flat (M, e), and the gradient direction field $|H|$ of $|\rho|$ on $(M, \mathcal{G})$ is sent to a gradient direction field $|O|$ of $|\rho|$ on (M, e). Correspondingly, $\forall t \in \mathbb{R}$, $\varphi_{|H|,a}(t)$ is sent to $\varphi_{|O|,a}(t)$. In a word, $L_{\mathcal{G}^{-1}}$ induces the following two maps:

$$\begin{aligned} \mathcal{G}_*^{-1} &: |H| \longrightarrow |O|, \\ \mathcal{G}_{**}^{-1} &: \varphi_{|H|,a} \longrightarrow \varphi_{|O|,a}. \end{aligned} \tag{99}$$

$\forall T \in \mathfrak{T}$, deonte $\mathfrak{N} \triangleq \{N \in \mathfrak{T} \mid \det N = \det T\}$. Due to $\mathfrak{T} \cong GL(\mathfrak{D}, \mathbb{R})$, let $\mathfrak{U}$ be a neighborhood of T, with respect to the topology of $GL(\mathfrak{D}, \mathbb{R})$.

Take $\Omega = \mathfrak{N} \cap \mathfrak{U}$; then,

$$|O_\Omega| = \mathcal{G}_*^{-1}(|H_\Omega|), \quad \varphi_{|O_\Omega|,a} = \mathcal{G}_{**}^{-1}\left(\varphi_{|H_\Omega|,a}\right). \tag{100}$$

Let μ be a Borel measure on the manifold M. We know $\forall t \in \mathbb{R}$,

$$\varphi_{|H_{\mathfrak{N}}|,a}(t) \simeq \varphi_{|O_{\mathfrak{N}}|,a}(t) \simeq \mathbb{S}^{\mathfrak{D}-1}. \tag{101}$$

Thus, $\varphi_{|H_\Omega|,a}(t) \subseteq \varphi_{|H_{\mathfrak{N}}|,a}(t)$ and $\varphi_{|O_\Omega|,a}(t) \subseteq \varphi_{|O_{\mathfrak{N}}|,a}(t)$ are Borel sets, so they are measurable. Denote

$$\mu_a\left(\varphi_{|H_\Omega|,a}(t)\right) \triangleq \mu\left(\mathcal{G}_{**}^{-1}\left(\varphi_{|H_\Omega|,a}(t)\right)\right) = \mu\left(\varphi_{|O_\Omega|,a}(t)\right). \tag{102}$$

When $\mathfrak{U} \longrightarrow T$, we have $\Omega \longrightarrow T$, $|H_\Omega| \longrightarrow H_T$, $|H_\Omega(a)| \longrightarrow H_T(a)$, and $\varphi_{|H_\Omega|,a}(t) \longrightarrow b \triangleq \varphi_{H_T,a}(t)$.

For the sake of simplicity, denote $L \triangleq \varphi_{H_T,a}$. Thus, we have $a = L(0), b = L(t)$, and denote $p_a \triangleq [L_a] = H_T(a), p_b \triangleq [L_b] = H_T(b)$.

Because μ_a is absolutely continuous with respect to μ, Radon-Nikodym theorem [42] ensures the existence of the following limit. The Radon-Nikodym derivative

$$\begin{aligned} W_L(b, a) &\triangleq \frac{d\mu_a}{d\mu_b} \triangleq \lim_{\mathfrak{U} \longrightarrow T} \frac{\mu_a\left(\varphi_{|H_\Omega|,a}(t)\right)}{\mu\left(\varphi_{|H_\Omega|,a}(t)\right)} \\ &= \lim_{\mathfrak{U} \longrightarrow T} \frac{\mu\left(\mathcal{G}_{**}^{-1}\left(\varphi_{|H_\Omega|,a}(t)\right)\right)}{\mu\left(\varphi_{|H_\Omega|,a}(t)\right)} \\ &= \lim_{\mathfrak{U} \longrightarrow T} \frac{\mu\left(\varphi_{|O_\Omega|,a}(t)\right)}{\mu\left(\varphi_{|H_\Omega|,a}(t)\right)} \end{aligned} \tag{103}$$

is said to be the **distribution density** of $|H|$ along L **in position representation**.

On a neighborhood U of a, $\forall T \in \mathfrak{T}$, denote the normal section of $H_T(a)$ by $N_{H_T,a}$, that is,

$$\begin{aligned} N_{H_T,a} &\triangleq \{n \in U \mid H_T(a) \cdot (n - a) = 0\}, \\ N_{H_T,a}(t) &\triangleq \left\{\varphi_{H_T,x}(t) \mid x \in N_{H_T,a}\right\}. \end{aligned} \tag{104}$$

Thus, $N_{H_T,a} = N_{H_T,a}(0)$ and $N_{H_T,b} \triangleq N_{H_T,a}(t)$. If $U \longrightarrow a$, we have $N_{H_T,a} \longrightarrow a$ and $N_{H_T,a}(t) \longrightarrow b \triangleq \varphi_{H_T,a}(t)$. The Radon-Nikodym derivative

$$Z_L(b, a) \triangleq \frac{d\mu(a)}{d\mu(b)} \triangleq \lim_{U \longrightarrow a} \frac{\mu(N_{H_T,a})}{\mu(N_{H_T,b})} = \lim_{U \longrightarrow a} \frac{\mu(N_{H_T,a})}{\mu(N_{H_T,a}(t))} \tag{105}$$

is said to be the **distribution density** of $|H|$ along L **in momentum representation**.

In a word, $W_L(b, a)$ and $Z_L(p_b, p_a)$ describe the density of the gradient lines that are adjacent to b in two different ways. They have the following property that is evidently true.

Proposition 28. *Let L be a gradient line. $\forall a, b, c \in L$ such that $L(x_a^0) = a$, $L(x_b^0) = b$, $L(x_c^0) = c$, and $x_b^0 > x_c^0 > x_a^0$; then,*

$$W_L(b, a) = W_L(b, c) W_L(c, a), \quad Z_L(b, a) = Z_L(b, c) Z_L(c, a). \tag{106}$$

Definition 29. If L is a gradient line of some $\rho' \in |\rho|$, we also say L is a gradient line of $|\rho|$.

Remark 30. For any a and b, it anyway makes sense to discuss the gradient line of $|\rho|$ from a to b. It is because even if the gradient line of ρ starting from a does not pass through b, it just only needs to carry out a certain flat transformation T defined in Section 2.3 to obtain a $\rho' \triangleq T_* \rho$; thus, the gradient line of ρ' starting from a can just exactly pass through b. Due to $\rho, \rho' \in |\rho|$, we do not distinguish them, and it is just fine to uniformly use $|\rho|$. Intuitively speaking, when $|\rho|$ takes two different initial momentums, $|\rho|$ presents as ρ and ρ', respectively.

Discussion 31. With the above preparations, we obtain a new way to describe the construction of the propagator strictly.

For any path L that starts at a and ends at b, we denote $\|L\| \triangleq \int_L dx^0$ concisely. Let $\mathscr{P}(b, a)$ be the totality of all the paths from a to b. Denote

$$\mathscr{P}\left(b, x_b^0 ; a, x_a^0\right) \triangleq \left\{L \mid L \in \mathscr{P}(b, a), \quad \|L\| = x_b^0 - x_a^0\right\}. \quad (107)$$

$\forall L \in \mathscr{P}(b, x_b^0 ; a, x_a^0)$, we can let $L(x_a^0) = a$ and $L(x_b^0) = b$ without loss of generality. Thus, $\mathscr{P}(b, x_b^0 ; a, x_a^0)$ is the totality of all the paths from $L(x_a^0) = a$ to $L(x_b^0) = b$.

Abstractly, the propagator is defined as the Green function of the evolution equation. Concretely, the propagator still needs a constructive definition. One method is the Feynman path integral

$$K\left(b, x_b^0 ; a, x_a^0\right) \triangleq \int_{\mathscr{P}\left(b, x_b^0 ; a, x_a^0\right)} e^{\mathrm{i}s} dL. \quad (108)$$

However, there are so many redundant paths in $\mathscr{P}(b, x_b^0 ; a, x_a^0)$ that (i) it is difficult to generally define a measure dL on $\mathscr{P}(b, x_b^0 ; a, x_a^0)$, and (ii) it may cause unnecessary infinities when carrying out some calculations.

In order to solve this problem, we try to reduce the scope of summation from $\mathscr{P}(b, x_b^0 ; a, x_a^0)$ to $H(b, x_b^0 ; a, x_a^0)$, where $H(b, x_b^0 ; a, x_a^0)$ is the totality of all the gradient lines of $|\rho|$ from $L(x_a^0) = a$ to $L(x_b^0) = b$. Thus, Equation (108) is turned into

$$K\left(b, x_b^0 ; a, x_a^0\right) = \int_{H\left(b, x_b^0 ; a, x_a^0\right)} \Psi(L) e^{\mathrm{i}s} dL. \quad (109)$$

We notice that as long as we take the probability amplitude $\Psi(L)$ of the gradient line L such that $[\Psi(L)]^2 = W_L(b, a)$ in position representation or take $[\Psi(L)]^2 = Z_L(b, a)$ in momentum representation, it can exactly be consistent with the Copenhagen interpretation. This provides the following new constructive definition for the propagator.

Definition 32. Suppose $|\rho|$ is defined as Definition 23, and denote $H \triangleq \nabla\rho$.

Let $\mathscr{L}(b, a)$ be the totality of all the gradient lines of $|\rho|$ from a to b. Denote

$$H\left(b, x_b^0 ; a, x_a^0\right) \triangleq \left\{L \mid L \in \mathscr{L}(b, a), \quad \|L\| = x_b^0 - x_a^0\right\}. \quad (110)$$

Let $\mathscr{L}(p_b, p_a)$ be the totality of all the gradient lines of $|\rho|$, whose starting direction is p_a and ending direction is p_b. Denote

$$H\left(p_b, x_b^0 ; p_a, x_a^0\right) \triangleq \left\{L \mid L \in \mathscr{L}(p_b, p_a), \quad \|L\| = x_b^0 - x_a^0\right\}. \quad (111)$$

Let dL be a Borel measure on $H(b, x_b^0 ; a, x_a^0)$. In consideration of Remark 41, we let s be the affine action $s(L)$ in Definition 16. We say the geometric property

$$K\left(b, x_b^0 ; a, x_a^0\right) \triangleq \int_{H\left(b, x_b^0 ; a, x_a^0\right)} \sqrt{W_L(b, a)} e^{\mathrm{i}s} dL \quad (112)$$

is the **propagator** of $|\rho|$ from (a, x_a^0) to (b, x_b^0) **in position representation**. If we let dL be a Borel measure on $H(p_b, x_b^0 ; p_a, x_a^0)$, then we say

$$\mathscr{K}\left(p_b, x_b^0 ; p_a, x_a^0\right) \triangleq \int_{H\left(p_b, x_b^0 ; p_a, x_a^0\right)} \sqrt{Z_L(b, a)} e^{\mathrm{i}s} dL \quad (113)$$

is the **propagator** of $|\rho|$ from (p_a, x_a^0) to (p_b, x_b^0) **in momentum representation**.

Discussion 33. Now (112) and (113) are strictly defined, but the Feynman path integral (108) has not been possessed of a strict mathematical definition until now. This makes it impossible at present to obtain (e.g., in position representation) a strict mathematical proof of

$$\int_{\mathscr{P}\left(b, x_b^0 ; a, x_a^0\right)} e^{\mathrm{i}s} dL = \int_{H\left(b, x_b^0 ; a, x_a^0\right)} \sqrt{W_L(b, a)} e^{\mathrm{i}s} dL. \quad (114)$$

Fortunately, the following two reasons make us believe that Equation (114) is expected to be regarded as a strict definition of Feynman path integral; that is to say, the integral on the right-hand side of "=" can be regarded as the strict definition of the notation on the left-hand side of "=".

On the one hand, we notice that the distribution densities $W_L(b, a)$ and $Z_L(b, a)$ of gradient directions establish an association between probability interpretation and geometric interpretation of quantum evolution. Therefore, we can base on probability interpretation to intuitively consider both sides of "=" in Equation (114) as the same thing.

On the other hand, on the condition of Proposition 28, denote $H(x_c^0) \triangleq \{L(x_c^0) | L \in H(b, x_b^0 ; a, x_a^0)\}$; then,

$$K\left(b, x_b^0 ; a, x_a^0\right) = \int_{H\left(x_c^0\right)} K\left(b, x_b^0 ; c, x_c^0\right) K\left(c, x_c^0 ; a, x_a^0\right) dc \quad (115)$$

is expected to be provable according to Equations (106) and (112). However, to obtain a strict proof of Equation (115) from Equations (106) and (112) is not a trivial mathematical problem, which is necessary but not easy, and needs more mathematical research.

Discussion 34. The quantization methods of QFT are successful, and they are also applicable in affine connection representation, but in this paper, we do not discuss them. We try to propose some more ideas to understand the quantization of field in affine connection representation.

(1) If we take

$$\mathfrak{s} = \int_L D\rho = \int_L p_Q dx^Q = \int_L E_0 dx^0, \tag{116}$$

according to Definition 16, where D is the holonomic connection of $(M, \mathscr{G})$, then consider the distribution of $H \triangleq \nabla \rho$, and we know that

$$\begin{aligned} K\left(b, x_b^0; a, x_a^0\right) &\triangleq \int_{\nabla\rho\left(b, x_b^0; a, x_a^0\right)} \sqrt{W_L(b,a)} e^{i\mathfrak{s}} dL, \\ \mathscr{K}\left(p_b, x_b^0; p_a, x_a^0\right) &\triangleq \int_{\nabla\rho\left(p_b, x_b^0; p_a, x_a^0\right)} \sqrt{Z_L(b,a)} e^{i\mathfrak{s}} dL \end{aligned} \tag{117}$$

describes the quantization of energy-momentum. Every gradient line in $\nabla\rho(b, x_b^0; a, x_a^0)$ corresponds to a set of eigenvalues of energy and momentum. This is consistent with conventional theories, and this inspires us to consider the following new ideas to carry out the quantization of charge and current of gauge field.

(2) In an analogous manner, if we take

$$\mathfrak{s} = \int_L Dt = \int_L K_{NPQ}^{M\ \ :P} dx^Q = \int_L \rho_{N0}^M dx^0, \tag{118}$$

according to Section 3.5, where D is the holonomic connection of $(M, \mathscr{F})$, then consider the distribution of $H \triangleq \nabla t$,

$$\begin{aligned} K\left(b, x_b^0; a, x_a^0\right) &\triangleq \int_{\nabla t\left(b, x_b^0; a, x_a^0\right)} \sqrt{W_L(b,a)} e^{i\mathfrak{s}} dL, \\ \mathscr{K}\left(p_b, x_b^0; p_a, x_a^0\right) &\triangleq \int_{\nabla t\left(p_b, x_b^0; p_a, x_a^0\right)} \sqrt{Z_L(b,a)} e^{i\mathfrak{s}} dL. \end{aligned} \tag{119}$$

Denote $H(b, x_b^0; x_c^0) \triangleq \{c \in M | \forall L \in H(b, x_b^0; c, x_c^0), \|L\| = x_b^0 - x_c^0\}$ and take $H = \nabla t$; then, the wave function $\psi(b, x_b^0)$ that is defined by the equation

$$\psi\left(b, x_b^0\right) = \int_{H\left(b, x_b^0; x_c^0\right)} K\left(b, x_b^0; c, x_c^0\right) \psi\left(c, x_c^0\right) dc \tag{120}$$

describes the quantization of charge and current. It should be emphasized that this is not the quantization of the energy-momentum of the field, but the quantization of the field itself, which presents as quantized (e.g., discrete) charges and currents.

4. Affine Connection Representation of Gauge Fields in Classical Spacetime

The new framework established in Section 3 is discussed in the $\mathfrak{D}$-dimensional general coordinate x^M, which is more general than the $(1+3)$-dimensional conventional Minkowski coordinate x^μ. $(dx^0)^2 = \sum_{M=1}^{\mathfrak{D}} (dx^M)^2$ is the total metric of internal space and external space, and $(dx^\tau)^2 = \sum_{m=4}^{\mathfrak{D}} (dx^m)^2$ is the metric of internal space.

(i) The evolution parameter of the $\mathfrak{D}$-dimensional general coordinate $x^M (M = 1, 2, \cdots, \mathfrak{D})$ is x^0. The parameter equation of an evolution path L is represented as $x^M = x^M(x^0)$

(ii) The evolution parameter of the $(1+3)$-dimensional Minkowski coordinate $x^\mu (\mu = 0, 1, 2, 3)$ is x^τ. The parameter equation of L is represented as $x^\mu = x^\mu(x^\tau)$

The coordinate x^μ works on the $(1+3)$-dimensional classical spacetime submanifold defined as follows.

4.1. Classical Spacetime Submanifold. Let there be a smooth tangent vector field X on (M, f). If $\forall p \in M$, $X(p) = b^A \partial/\partial\xi^A|_p = c^M \partial/\partial x^M|_p$ satisfies that b^a are not all zero and c^m are not all zero, where $a, m = r+1, \cdots, \mathfrak{D}$; then, we say X is internal-directed. For any evolution path $L \triangleq \varphi_{X,p}$, we also say L is internal-directed.

Suppose $M = P \times N$, $\mathfrak{D} \triangleq dimM$, and $r \triangleq dimP = 3$. X is a smooth tangent vector field on M. Fix a point $o \in M$. If X is internal-directed; then, there exist a unique $(1+3)$-dimensional imbedding submanifold $\gamma : \tilde{M} \longrightarrow M, p \mapsto p$ and a unique smooth tangent vector field $\tilde{X}$ on $\tilde{M}$ such that

(i) $P \times \{o\}$ is a closed submanifold of $\tilde{M}$

(ii) The tangent map $\gamma_* : T(\tilde{M}) \longrightarrow T(M)$ satisfies that $\forall q \in \tilde{M}$, $\gamma_* : \tilde{X}(q) \mapsto X(q)$

Such an $\tilde{M}$ is said to be a classical spacetime submanifold.

Let $\varphi_X : M \times \mathbb{R} \longrightarrow M$ and $\varphi_{\tilde{X}} : \tilde{M} \times \mathbb{R} \longrightarrow \tilde{M}$ be the one-parameter groups of diffeomorphisms corresponding to X and $\tilde{X}$, respectively. Thus, we have

$$\varphi_{\tilde{X}} = \varphi_X|_{\tilde{M}\times\mathbb{R}}. \tag{121}$$

So the evolution in classical spacetime can be described by $\varphi_{\tilde{X}}$. It should be noticed that

(i) $\tilde{M}$ inherits a part of geometric properties of M, but not all. The physical properties reflected by $\tilde{M}$ are incomplete

(ii) The correspondence between $\tilde{X}$ and the restriction of X to $\tilde{M}$ is one-to-one. For convenience, next we are not going to distinguish the notations X and $\tilde{X}$ on $\tilde{M}$ but uniformly denote them by X

(iii) An arbitrary path $\tilde{L} : T \longrightarrow \tilde{M}, t \mapsto p$ on $\tilde{M}$ uniquely corresponds to a path $L \triangleq \gamma \circ \tilde{L} : T \longrightarrow M, t \mapsto p$ on M. Evidently, the image sets of L and $\tilde{L}$ are the same, that is, $L(T) = \tilde{L}(T)$. For convenience, later we are not going to distinguish the notations L and $\tilde{L}$ on $\tilde{M}$ but uniformly denote them by L

4.2. Classical Spacetime Reference-System. Let there be a geometric manifold (M, f) and its classical spacetime submanifold $\tilde{M}$. And let $L \triangleq \varphi_{\tilde{X},a}$ be an evolution path on $\tilde{M}$. Suppose $p \in L$ and U is a coordinate neighborhood of p. According to Definition 5, suppose the $f(p)$ on U and the $f_L(p)$ on $U_L \triangleq U \cap L$ satisfy that

$$f(p)\colon \xi^A = \xi^A(x^M) = \xi^A(x^0), \quad \xi^0 = \xi^0(x^0), \quad A, M = 1, 2, \cdots, \mathfrak{D}. \tag{122}$$

Thus, it is true that

(1) There exists a unique local reference-system $\tilde{f}(p)$ on $\tilde{U} \triangleq U \cap \tilde{M}$ such that

$$\tilde{f}(p)\colon \xi^U = \xi^U(x^K) = \xi^U(x^0), \quad \xi^0 = \xi^0(x^0), \quad U, K = 1, 2, 3, \tau. \tag{123}$$

(2) If L is internal-directed, then the above coordinate frames $(\tilde{U}, \xi^U)$ and $(\tilde{U}, x^K)$ of $\tilde{f}(p)$ uniquely determine the coordinate frames $(\tilde{U}, \tilde{\xi}^\alpha)$ and $(\tilde{U}, \tilde{x}^\mu)$ such that

$$\tilde{f}(p)\colon \tilde{\xi}^\alpha = \tilde{\xi}^\alpha(\tilde{x}^\mu) = \tilde{\xi}^\alpha(\tilde{x}^\tau), \quad \tilde{\xi}^\tau = \tilde{\xi}^\tau(\tilde{x}^\tau), \quad \alpha, \mu = 0, 1, 2, 3, \tag{124}$$

and the coordinates satisfy

$$\tilde{\xi}^s = \xi^s, \quad \tilde{\xi}^\tau = \xi^\tau, \quad \tilde{\xi}^0 = \xi^0, \quad \tilde{x}^i = x^i, \quad \tilde{x}^\tau = x^\tau, \quad \tilde{x}^0 = x^0. \tag{125}$$

That is to say, $\tilde{f}(p)$ is just exactly the reference system in conventional sense, which has two different coordinate representations (123) and (124).

We speak of

$$\tilde{f} : \tilde{M} \longrightarrow REF_{\tilde{M}}, \quad p \mapsto \tilde{f}(p) \in REF_p \tag{126}$$

as a classical spacetime reference-system. Thus, inertial system can be strictly defined as follows, no need for Newton's first law. Suppose we have a geometric manifold $(\tilde{M}, \tilde{g})$. $F_{\tilde{g}}$ is a transformation induced by $\tilde{g}$.

(1) If $\tilde{\delta}_{\alpha\beta}\tilde{B}^\alpha_\mu\tilde{B}^\beta_\nu = \tilde{\varepsilon}_{\mu\nu}$, then $\tilde{g}$ is said to be (Lorentz) orthogonal. In this case, $F_{\tilde{g}}$ is just exactly a local Lorentz transformation

(2) If $\tilde{B}^\alpha_\mu$ and $\tilde{C}^\mu_\alpha$ are constants on $\tilde{M}$, then $\tilde{g}$ is said to be flat

(3) If $\tilde{g}$ is both orthogonal and flat, then $\tilde{g}$ is said to be an inertial-system. In this case, $F_{\tilde{g}}$ is just exactly a Lorentz transformation

Remark 35. Due to

$$\begin{aligned}\left(d\tilde{\xi}^\tau\right)^2 &= \left(d\xi^0\right)^2 - \sum_{s=1}^{3}\left(d\xi^s\right)^2 = \tilde{\delta}_{\alpha\beta}d\tilde{\xi}^\alpha d\tilde{\xi}^\beta \\ &= \tilde{G}_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu, \tilde{G}_{\mu\nu} \triangleq \tilde{\delta}_{\alpha\beta}\tilde{B}^\alpha_\mu\tilde{B}^\beta_\nu,\end{aligned}$$

$$\begin{aligned}\left(d\tilde{x}^\tau\right)^2 &= \left(dx^0\right)^2 - \sum_{i=1}^{3}\left(dx^i\right)^2 = \tilde{\varepsilon}_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu \\ &= \tilde{H}_{\alpha\beta}d\tilde{\xi}^\alpha d\tilde{\xi}^\beta, \tilde{H}_{\alpha\beta} \triangleq \tilde{\varepsilon}_{\mu\nu}\tilde{C}^\mu_\alpha\tilde{C}^\nu_\beta,\end{aligned} \tag{127}$$

it is easy to know that $\tilde{g}$ is orthogonal if and only if $d\tilde{\xi}^\tau = d\tilde{x}^\tau$, i.e., $\tilde{G}_{\tau\tau} \triangleq \tilde{B}^\tau_\tau\tilde{B}^\tau_\tau = 1$, $\tilde{G}^{\tau\tau} \triangleq \tilde{C}^\tau_\tau\tilde{C}^\tau_\tau = 1$. It is only in this case that we can denote $d\tilde{\xi}^\tau$ and $d\tilde{x}^\tau$ uniformly by $d\tau$; otherwise, we should be aware of the difference between $d\tilde{\xi}^\tau$ and $d\tilde{x}^\tau$ in nontrivial gravitational field. No matter whether $\tilde{g}$ is an inertial system or not, and whether there is a nontrivial gravitation field or not, $(d\tilde{\xi}^\tau)^2 = (d\xi^0)^2 - \sum_{s=1}^{3}(d\xi^s)^2$ and $(d\tilde{x}^\tau)^2 = (dx^0)^2 - \sum_{i=1}^{3}(dx^i)^2$ are always both true in their respective coordinate frames.

Remark 36. The evolution lemmas in Section 3.3 can be expressed in Minkowski coordinate as follows:

(i) If $d/d\tilde{t} \cong d/d\tilde{t}_L$ and $d\tilde{f} \simeq d\tilde{f}_L$, then $\langle d/d\tilde{t}, d\tilde{f}\rangle = \langle d/d\tilde{t}_L, d\tilde{f}_L\rangle$

(ii) The following conclusions are true

$$\begin{aligned}&w^\mu \frac{\partial}{\partial\tilde{x}^\mu} \cong w^\tau \frac{d}{d\tilde{x}^\tau} \Longleftrightarrow w^\mu = w^\tau\tilde{\varepsilon}^\mu_\tau, \quad \bar{w}_\mu \frac{\partial}{\partial\tilde{x}_\mu} \cong \bar{w}_\tau \frac{d}{d\tilde{x}_\tau} \Longleftrightarrow \bar{w}_\mu = \bar{w}_\tau\tilde{\bar{\varepsilon}}^\tau_\mu, \\ &w_\mu d\tilde{x}^\mu \simeq w_\tau d\tilde{x}^\tau \Longleftrightarrow \tilde{\varepsilon}^\mu_\tau w_\mu = w_\tau, \quad \bar{w}^\mu d\tilde{x}_\mu \simeq \bar{w}^\tau d\tilde{x}_\tau \Longleftrightarrow \tilde{\bar{\varepsilon}}^\tau_\mu\bar{w}^\mu = \bar{w}^\tau.\end{aligned} \tag{128}$$

4.3. Affine Connection Representation of Classical Spacetime Evolution. Let $\tilde{D}$ be the holonomic connection on $(\tilde{M}, \tilde{\mathscr{G}})$, and denote $\tilde{t}_{L;\tau} \triangleq \tilde{t}_{;\sigma}\tilde{\varepsilon}^\sigma_\tau$; then, the absolute differential and gradient of Section 3.4 can be expressed on $\tilde{M}$ in Minkowski coordinate as

$$\begin{aligned}&\tilde{D}\tilde{t} \triangleq \tilde{t}_{;\sigma}d\tilde{x}^\sigma, \quad \tilde{D}_L\tilde{t}_L \triangleq \tilde{t}_{L;\tau}d\tilde{x}^\tau, \\ &\nabla^{\sim}\tilde{t} \triangleq \tilde{t}_{;\sigma}\frac{\partial}{\partial\tilde{x}_\sigma}, \quad \nabla^{\sim}{}_L\tilde{t}_L \triangleq \tilde{t}_{L;\tau}\frac{d}{d\tilde{x}_\tau}.\end{aligned} \tag{129}$$

Evidently, $\tilde{D}\tilde{t} \simeq \tilde{D}_L \tilde{t}_L$ if and only if L is an arbitrary path. $\nabla^{\sim}\tilde{t} \cong \nabla^{\sim}{}_L \tilde{t}_L$ if and only if L is the gradient line.

Definition 37. Similar to Section 3.6, suppose a charge $\tilde{\rho}$ of $\tilde{\mathscr{F}}$ evolves on $(\tilde{M}, \tilde{\mathscr{G}})$. We have the following definitions.

(1) The geometric properties $\tilde{m}^\tau \triangleq \tilde{\rho}^{;\tau}$ and $\tilde{m}_\tau \triangleq \tilde{\rho}_{;\tau}$ are said to be the rest mass of $\tilde{\rho}$

(2) $\tilde{p}^\mu \triangleq -\tilde{\rho}^{;\mu}$ and $\tilde{p}_\mu \triangleq -\tilde{\rho}_{;\mu}$ are said to be the energy-momentum of $\tilde{\rho}$, and $\tilde{E}^0 \triangleq \tilde{\rho}^{;0}$ and $\tilde{E}_0 \triangleq \tilde{\rho}_{;0}$ are said to be the energy of $\tilde{\rho}$

(3) $\tilde{M}^\tau \triangleq d\tilde{\rho}/d\tilde{x}_\tau$ and $\tilde{M}_\tau \triangleq d\tilde{\rho}/d\tilde{x}^\tau$ are said to be the canonical rest mass of $\tilde{\rho}$

(4) $\tilde{P}^\mu \triangleq -\partial\tilde{\rho}/\partial\tilde{x}_\mu$ and $\tilde{P}_\mu \triangleq -\partial\tilde{\rho}/\partial\tilde{x}^\mu$ are said to be the canonical energy-momentum of $\tilde{\rho}$, and $\tilde{H}^0 \triangleq \partial\tilde{\rho}/\partial\tilde{x}_0$, $\tilde{H}_0 \triangleq \partial\tilde{\rho}/\partial\tilde{x}^0$ are said to be the canonical energy of $\tilde{\rho}$

Discussion 38. Similar to Proposition 13, $\forall p \in \tilde{M}$, if and only if the evolution direction $[L_p] = \nabla^{\sim}\tilde{\rho}|_p$, the directional derivative is

$$\left\langle \tilde{m}_\tau \frac{d}{d\tilde{x}_\tau}, \tilde{m}_\tau d\tilde{x}^\tau \right\rangle = \left\langle \tilde{p}_\mu \frac{\partial}{\partial\tilde{x}_\mu}, \tilde{p}_\mu d\tilde{x}^\mu \right\rangle, \tag{130}$$

that is, $\tilde{G}^{\tau\tau}\tilde{m}_\tau\tilde{m}_\tau = \tilde{G}^{\mu\nu}\tilde{p}_\mu\tilde{p}_\nu$, or

$$\tilde{m}_\tau\tilde{m}^\tau = \tilde{p}_\mu\tilde{p}^\mu, \tag{131}$$

which is the affine connection representation of energy-momentum equation.

Similar to Proposition 14, according to the evolution lemma, $\forall p \in \tilde{M}$, if and only if the evolution direction $[L_p] = \nabla^{\sim}\tilde{\rho}|_p$, we have $\tilde{p}_\mu = -\tilde{m}_\tau d\tilde{x}_\mu/d\tilde{x}_\tau$, that is $\tilde{E}_0 = \tilde{m}_\tau d\tilde{x}_0/d\tilde{x}_\tau = \tilde{m}_\tau dx_0/dx_\tau$ and $\tilde{p}_i = -\tilde{m}_\tau d\tilde{x}_i/d\tilde{x}_\tau = \tilde{m}_\tau(-d\tilde{x}_i)/d\tilde{x}_\tau = \tilde{m}_\tau dx_i/dx_\tau = \tilde{E}_0 dx_i/dx_0$. This can also be regarded as the origin of $p = mv$.

Similar to Remark 18, denote

$$\left[\tilde{\rho}\tilde{\Gamma}_\omega\right] \triangleq \frac{\partial\tilde{\rho}_{\mu\nu}}{\partial\tilde{x}^\omega} - \tilde{\rho}_{\mu\nu;\omega} = \tilde{\rho}_{\mu\chi}\tilde{\Gamma}^\chi_{\nu\omega} + \tilde{\rho}_{\chi\nu}\tilde{\Gamma}^\chi_{\mu\omega},$$
$$\left[\tilde{\rho}\tilde{R}_{\rho\sigma}\right] \triangleq \tilde{\rho}_{\mu\chi}\tilde{R}^\chi_{\nu\rho\sigma} + \tilde{\rho}_{\chi\nu}\tilde{R}^\chi_{\mu\rho\sigma}. \tag{132}$$

Then, for the same reason as Remark 18, based on Definition 37, we can strictly obtain

$$\tilde{f}_\rho \triangleq \tilde{p}_{\rho;\tau} = \tilde{m}_{\tau;\rho} - \tilde{p}_\sigma\tilde{\varepsilon}^\sigma_{\tau;\rho} + \left[\tilde{\rho}\tilde{R}_{\rho\sigma}\right]\tilde{\varepsilon}^\sigma_\tau. \tag{133}$$

In the mass-point model, $\tilde{m}_{\tau;\rho}$ and $\tilde{\varepsilon}^\sigma_{\tau;\rho}$ do not make sense, so Equation (133) turns into

$$\tilde{f}_\rho = \left[\tilde{\rho}\tilde{R}_{\rho\sigma}\right]\tilde{\varepsilon}^\sigma_\tau. \tag{134}$$

This is the affine connection representation of the force of interaction (e.g., the Lorentz force $\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ or $f_\rho = j^\sigma F_{\rho\sigma}$ of the electrodynamics).

Similar to Definition 16, let $\tilde{\mathscr{P}}(b, a)$ be the totality of paths on $\tilde{M}$ from point a to point b. And let $L \in \tilde{\mathscr{P}}(b, a)$ and parameter $\tilde{x}^\tau$ satisfy $\tau_a \triangleq \tilde{x}^\tau(a) < \tilde{x}^\tau(b) \triangleq \tau_b$. The affine connection representation of action in Minkowski coordinates can be defined as

$$\tilde{\mathfrak{s}}(L) \triangleq \int_L \tilde{D}\tilde{\rho} = \int_L \tilde{p}_\mu d\tilde{x}^\mu = \int_{\tau_a}^{\tau_b} \tilde{m}_\tau d\tilde{x}^\tau, \quad \tilde{s}(L) \triangleq \int_{\tau_a}^{\tau_b} \left(\gamma^\mu\tilde{\rho}_{;\mu} + \tilde{m}_\tau\right) d\tilde{x}^\tau. \tag{135}$$

There are more illustrations in Remark 41.

4.4. Affine Connection Representation of Dirac Equation

Discussion 39. Define Dirac algebras γ^μ and γ^α such that

$$\gamma^\mu = \tilde{C}^\mu_\alpha\gamma^\alpha, \quad \gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\tilde{\delta}^{\alpha\beta}, \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\tilde{G}^{\mu\nu}. \tag{136}$$

Suppose $(\tilde{M}, \tilde{\mathscr{G}})$ is orthogonal. According to Remark 35, $\tilde{G}^{\tau\tau} = 1$. Due to Discussion 38, in a gradient direction of $\tilde{\rho} \triangleq \tilde{\rho}_{\omega\nu}$, we have

$$\begin{aligned} \tilde{\rho}_{;\mu}\tilde{\rho}^{;\mu} = \tilde{\rho}_{;\tau}\tilde{\rho}^{;\tau} &\Longleftrightarrow \tilde{G}^{\mu\nu}\tilde{\rho}_{;\mu}\tilde{\rho}_{;\nu} \\ = \tilde{m}_\tau^2 &\Longleftrightarrow \left(\gamma^\mu\tilde{\rho}_{;\mu}\right)\left(\gamma^\nu\tilde{\rho}_{;\nu}\right) + \left(\gamma^\nu\tilde{\rho}_{;\nu}\right)\left(\gamma^\mu\tilde{\rho}_{;\mu}\right) \\ = 2\tilde{m}_\tau^2 &\Longleftrightarrow \left(\gamma^\mu\tilde{\rho}_{;\mu}\right)\left(\gamma^\nu\tilde{\rho}_{;\nu}\right) \\ = \tilde{m}_\tau^2 &\Longleftrightarrow \left(\gamma^\mu\tilde{\rho}_{;\mu}\right)^2 = \tilde{m}_\tau^2. \end{aligned} \tag{137}$$

Without loss of generality, take $\gamma^\mu\tilde{\rho}_{;\mu} = \tilde{m}_\tau$, that is,

$$\gamma^\mu\tilde{\rho}_{\omega\nu;\mu} = \tilde{m}_{\omega\nu\tau}. \tag{138}$$

Next, denote

$$\left[g\tilde{\Gamma}_\mu\right]^{\omega\nu} \triangleq \sum_\sigma \tilde{G}^{\nu\nu'}\tilde{\Gamma}_{\nu'\sigma\mu} + \sum_\kappa \tilde{G}^{\omega\omega'}\tilde{\Gamma}_{\omega'\kappa\mu}, \quad \tilde{D}^{\omega\nu}_\mu \triangleq \partial_\mu - \left[g\tilde{\Gamma}_\mu\right]^{\omega\nu}. \tag{139}$$

From Equation (138), it is obtained that

$$\begin{aligned}\sum_{\omega,\nu}\gamma^\mu\tilde{\rho}_{\omega\nu;\mu}=\sum_{\omega,\nu}\tilde{m}_{\omega\nu\tau}&\Longleftrightarrow\sum_{\omega,\nu}\gamma^\mu\left(\partial_\mu\rho_{\omega\nu}-\tilde{\rho}_{\omega\chi}\widetilde{\Gamma}^\chi_{\nu\mu}-\tilde{\rho}_{\chi\nu}\widetilde{\Gamma}^\chi_{\omega\mu}\right)\\ =\sum_{\omega,\nu}\tilde{m}_{\omega\nu\tau}&\Longleftrightarrow\sum_{\omega,\nu}\gamma^\mu\left(\partial_\mu\rho_{\omega\nu}-\tilde{\rho}_{\omega\nu}\sum_\sigma\widetilde{\Gamma}^\nu_{\sigma\mu}-\tilde{\rho}_{\omega\nu}\sum_\kappa\widetilde{\Gamma}^\omega_{\kappa\mu}\right)\\ =\sum_{\omega,\nu}\tilde{m}_{\omega\nu\tau}&\Longleftrightarrow\sum_{\omega,\nu}\gamma^\mu\left(\partial_\mu\rho_{\omega\nu}-\tilde{\rho}_{\omega\nu}\left[g\widetilde{\Gamma}_\mu\right]^{\omega\nu}\right)\\ =\sum_{\omega,\nu}\tilde{m}_{\omega\nu\tau}&\Longleftrightarrow\sum_{\omega,\nu}\gamma^\mu\left(\partial_\mu-\left[g\widetilde{\Gamma}_\mu\right]^{\omega\nu}\right)\tilde{\rho}_{\omega\nu}=\sum_{\omega,\nu}\tilde{m}_{\omega\nu\tau},\end{aligned}\tag{140}$$

that is,

$$\sum_{\omega,\nu}\gamma^\mu\tilde{D}^{\omega\nu}_\mu\tilde{\rho}_{\omega\nu}=\sum_{\omega,\nu}\tilde{m}_{\omega\nu\tau},\quad \tilde{D}^{\omega\nu}_\mu\triangleq\partial_\mu-\left[g\widetilde{\Gamma}_\mu\right]^{\omega\nu}.\tag{141}$$

We speak of the real-valued Equations (138) and (141) as affine Dirac equations.

Discussion 40. Next, we construct a kind of complex-valued representation of affine Dirac equation. The restriction of the charge $\tilde{\rho}_{\omega\nu}$ to $(\tilde{U},\tilde{x}^\mu)$ is a function $\tilde{\rho}_{\omega\nu}(\tilde{x}^\mu)$ with respect to the coordinates $(\tilde{x}^\mu)\triangleq(\tilde{x}^0,\tilde{x}^1,\tilde{x}^2,\tilde{x}^3)$. Let

$$\tilde{\mathrm{P}}_{\omega\nu}\left(\tilde{x}^0\right)\triangleq\int_{\left(\tilde{x}^1,\tilde{x}^2,\tilde{x}^3\right)}\tilde{\rho}_{\omega\nu}(\tilde{x}^\mu)d^3\tilde{x}.\tag{142}$$

Suppose a function $f_{\omega\nu}=f_{\omega\nu}(\tilde{x}^\mu)$ on $(\tilde{U},\tilde{x}^\mu)$ satisfies that

$$\tilde{\rho}_{\omega\nu}=(f_{\omega\nu})^2\tilde{\mathrm{P}}_{\omega\nu},\ \int_{\left(\tilde{x}^1,\tilde{x}^2,\tilde{x}^3\right)}(f_{\omega\nu})^2d^3\tilde{x}=1,\ \xi^\mu_\tau\frac{\partial f_{\omega\nu}}{\partial\tilde{x}^\mu}=0,\ \gamma^\mu\frac{\partial f_{\omega\nu}}{\partial\tilde{x}^\mu}=0.\tag{143}$$

We define $\psi_{\omega\nu}$ and $\tilde{\mathbb{M}}_{\omega\nu\tau}$ in the following way.

$$\begin{aligned}\tilde{y}_{\omega\nu}&\triangleq\int_L d\tilde{\rho}_{\omega\nu}=\int_L\frac{d\tilde{\rho}_{\omega\nu}}{d\tilde{x}^\tau}d\tilde{x}^\tau\\&=\int_L\left(\frac{d\left(f^2_{\omega\nu}\right)}{d\tilde{x}^\tau}\tilde{\mathrm{P}}_{\omega\nu}+f^2_{\omega\nu}\frac{d\tilde{\mathrm{P}}_{\omega\nu}}{d\tilde{x}^\tau}\right)d\tilde{x}^\tau\\&=f^2_{\omega\nu}\int_L\frac{d\tilde{\mathrm{P}}_{\omega\nu}}{d\tilde{x}^\tau}\mathrm{d}\tilde{x}^\tau\triangleq f^2_{\omega\nu}\tilde{Y}_{\omega\nu},\end{aligned}$$

$$\begin{aligned}\psi_{\omega\nu}&\triangleq f_{\omega\nu}e^{i\tilde{Y}_{\omega\nu}},\quad \tilde{m}_{\omega\nu\tau}\triangleq\tilde{\rho}_{\omega\nu;\tau}\\&=\left(f^2_{\omega\nu}\right)_{,\tau}\tilde{\mathrm{P}}_{\omega\nu}+f^2_{\omega\nu}\tilde{\mathrm{P}}_{\omega\nu;\tau}\\&=f^2_{\omega\nu}\tilde{\mathrm{P}}_{\omega\nu;\tau}\triangleq f^2_{\omega\nu}\tilde{\mathbb{M}}_{\omega\nu\tau}.\end{aligned}\tag{144}$$

In the QFT propagator, we usually take S in the path integral $\int e^{iS}\mathscr{D}\psi$ of a fermion in the form of

$$-\int\left(i\bar{\psi}\gamma^\mu D_\mu\psi-\bar{\psi}\tilde{\mathbb{M}}_\tau\psi\right)d^4\tilde{x},\tag{145}$$

where S and $d^4\tilde{x}$ are both covariant. We believe that the external spatial integral $\int_{(\tilde{x}^1,\tilde{x}^2,\tilde{x}^3)}d^3\tilde{x}$ is not an essential part for evolution, so for the sake of simplicity, we do not take into account the external spatial part $\int_{(\tilde{x}^1,\tilde{x}^2,\tilde{x}^3)}d^3\tilde{x}$ but only consider the evolution part $\int_L d\tilde{x}^0$. Meanwhile, in order to remain the covariance, $\int_L d\tilde{x}^0$ has to be replaced by $\int_L d\tilde{x}^\tau$. Thus, in affine connection representation of gauge fields, we shall consider an action in the form of

$$-\int_L\left(i\bar{\psi}\gamma^\mu D_\mu\psi-\bar{\psi}\tilde{\mathbb{M}}_\tau\psi\right)d\tilde{x}^\tau.\tag{146}$$

Concretely speaking, denote

$$\tilde{D}_{\omega\nu\mu}\triangleq\frac{\partial}{\partial\tilde{x}^\mu}-i\left[\tilde{\mathrm{P}}\widetilde{\Gamma}_\mu\right]_{\omega\nu},\quad\left[\tilde{\mathrm{P}}\widetilde{\Gamma}_\mu\right]_{\omega\nu}\triangleq\sum_\sigma\tilde{\mathrm{P}}_{\omega\nu}\widetilde{\Gamma}^\nu_{\sigma\mu}+\sum_\kappa\tilde{\mathrm{P}}_{\omega\nu}\widetilde{\Gamma}^\omega_{\kappa\mu}.\tag{147}$$

From Equation (135), we have

$$\tilde{s}_{\omega\nu}(L)\triangleq\int_L\left(\gamma^\mu\tilde{\rho}_{\omega\nu;\mu}+\tilde{m}_{\omega\nu\tau}\right)d\tilde{x}^\tau.\tag{148}$$

And from Equation (140), we know $\sum_{\omega,\nu}\gamma^\mu\tilde{\rho}_{\omega\nu;\mu}=\sum_{\omega,\nu}\gamma^\mu\tilde{D}^{\omega\nu}_\mu\tilde{\rho}_{\omega\nu}$. Then, it is obtained that

$$\begin{aligned}\tilde{s}_{\tilde{\rho}}(L)&\triangleq\sum_{\omega,\nu}\tilde{s}_{\omega\nu}(L)=\int_L\sum_{\omega,\nu}\left(\gamma^\mu\tilde{\rho}_{\omega\nu;\mu}+\tilde{m}_{\omega\nu\tau}\right)d\tilde{x}^\tau\\&=\int_L\sum_{\omega,\nu}\left(\gamma^\mu\tilde{D}^{\omega\nu}_\mu\tilde{\rho}_{\omega\nu}+\tilde{m}_{\omega\nu\tau}\right)d\tilde{x}^\tau\\&=\int_L\sum_{\omega,\nu}\left(\gamma^\mu\left(\partial_\mu\tilde{\rho}_{\omega\nu}-\left[g\widetilde{\Gamma}_\mu\right]^{\omega\nu}\tilde{\rho}_{\omega\nu}\right)+\tilde{m}_{\omega\nu\tau}\right)d\tilde{x}^\tau\\&=\int_L\sum_{\omega,\nu}\left(\gamma^\mu\left(\partial_\mu\tilde{y}_{\omega\nu}-\left[g\widetilde{\Gamma}_\mu\right]^{\omega\nu}\tilde{\rho}_{\omega\nu}\right)+\tilde{m}_{\omega\nu\tau}\right)d\tilde{x}^\tau\\&=\int_L\sum_{\omega,\nu}\left(\gamma^\mu\left(\partial_\mu(f^2_{\omega\nu}\tilde{Y}_{\omega\nu})-\left[g\widetilde{\Gamma}_\mu\right]^{\omega\nu}f^2_{\omega\nu}\tilde{\mathrm{P}}_{\omega\nu}\right)+f^2_{\omega\nu}\tilde{\mathbb{M}}_{\omega\nu\tau}\right)d\tilde{x}^\tau\\&=\int_L\sum_{\omega,\nu}\left(\gamma^\mu\left(\partial_\mu\tilde{Y}_{\omega\nu}-\left[\tilde{\mathrm{P}}\widetilde{\Gamma}_\mu\right]_{\omega\nu}\right)f^2_{\omega\nu}+f^2_{\omega\nu}\tilde{\mathbb{M}}_{\omega\nu\tau}\right)d\tilde{x}^\tau\\&=\int_L\sum_{\omega,\nu}\Big(f_{\omega\nu}e^{-i\tilde{Y}_{\omega\nu}}\gamma^\mu\left(f_{\omega\nu}e^{i\tilde{Y}_{\omega\nu}}\partial_\mu\tilde{Y}_{\omega\nu}-\left[\tilde{\mathrm{P}}\widetilde{\Gamma}_\mu\right]_{\omega\nu}f_{\omega\nu}e^{i\tilde{Y}_{\omega\nu}}\right)\\&\quad+f_{\omega\nu}e^{-i\tilde{Y}_{\omega\nu}}\tilde{\mathbb{M}}_{\omega\nu\tau}f_{\omega\nu}e^{i\tilde{Y}_{\omega\nu}}\Big)d\tilde{x}^\tau\\&=\int_L\sum_{\omega,\nu}\Big(-\bar{\psi}_{\omega\nu}i\gamma^\mu\left(e^{i\tilde{Y}_{\omega\nu}}\partial_\mu f_{\omega\nu}+f_{\omega\nu}e^{i\tilde{Y}_{\omega\nu}}i\partial_\mu\tilde{Y}_{\omega\nu}-i\left[\tilde{\mathrm{P}}\widetilde{\Gamma}_\mu\right]_{\omega\nu}\psi_{\omega\nu}\right)\\&\quad+\bar{\psi}_{\omega\nu}\tilde{\mathbb{M}}_{\omega\nu\tau}\psi_{\omega\nu}\Big)d\tilde{x}^\tau\\&=\int_L\sum_{\omega,\nu}\Big(-\bar{\psi}_{\omega\nu}i\gamma^\mu\left(\partial_\mu\left(f_{\omega\nu}e^{i\tilde{Y}_{\omega\nu}}\right)-i\left[\tilde{\mathrm{P}}\widetilde{\Gamma}_\mu\right]_{\omega\nu}\psi_{\omega\nu}\right)\\&\quad+\bar{\psi}_{\omega\nu}\tilde{\mathbb{M}}_{\omega\nu\tau}\psi_{\omega\nu}\Big)d\tilde{x}^\tau\\&=\int_L\sum_{\omega,\nu}\left(-\bar{\psi}_{\omega\nu}i\gamma^\mu\left(\partial_\mu-i\left[\tilde{\mathrm{P}}\widetilde{\Gamma}_\mu\right]_{\omega\nu}\right)\psi_{\omega\nu}+\bar{\psi}_{\omega\nu}\tilde{\mathbb{M}}_{\omega\nu\tau}\psi_{\omega\nu}\right)d\tilde{x}^\tau\\&=\int_L\sum_{\omega,\nu}\left(-\bar{\psi}_{\omega\nu}i\gamma^\mu\tilde{D}_{\omega\nu\mu}\psi_{\omega\nu}+\bar{\psi}_{\omega\nu}\tilde{\mathbb{M}}_{\omega\nu\tau}\psi_{\omega\nu}\right)d\tilde{x}^\tau\\&=-\int_L\sum_{\omega,\nu}\bar{\psi}_{\omega\nu}\left(i\gamma^\mu\tilde{D}_{\omega\nu\mu}-\tilde{\mathbb{M}}_{\omega\nu\tau}\right)\psi_{\omega\nu}d\tilde{x}^\tau.\end{aligned}\tag{149}$$

Thus, we have obtained a complex-valued representation of gradient direction of $\tilde{\rho}_{\omega\nu}$.

Remark 41. From the above discussion, we know in the gradient direction of $\rho_{\omega\nu}$ that

$$-\sum_{\omega,\nu}\bar{\psi}_{\omega\nu} i\gamma^{\mu}\tilde{D}_{\omega\nu\mu}\psi_{\omega\nu} d\tilde{x}^{\tau} = \sum_{\omega,\nu}\tilde{D}\tilde{\rho}_{\omega\nu}. \tag{150}$$

This shows that $s(L)$ and $\tilde{s}(L)$ in Definition 16 and Remark 17 are indeed applicable for constructing propagator by $e^{is(L)}$ and $e^{i\tilde{s}(L)}$ in affine connection representation of gauge fields. Therefore, the idea in Discussion 34 is reasonable.

4.5. From Classical Spacetime back to Full-Dimensional Space

Discussion 42. Now there is a problem. $(\tilde{M},\tilde{\mathcal{F}})$ and $(\tilde{M},\tilde{\mathcal{G}})$ cannot totally reflect the geometric properties of internal space of $(M,\mathcal{F})$ and $(M,\mathcal{G})$. Concretely speaking, in the previous section, we discuss the affine Dirac equation $\gamma^{\mu}\tilde{\rho}_{\omega\nu;\mu} = \tilde{m}_{\omega\nu\tau}$ on $(\tilde{M},\tilde{\mathcal{G}})$. Similar to Section 3.5, we have the affine Yang-Mills equation $\tilde{K}^{\mu}_{\nu\rho\sigma}{}^{:\rho} = \tilde{\rho}^{\mu}_{\nu}\gamma_{\sigma}$ on $(\tilde{M},\tilde{\mathcal{F}})$. Suppose there is no gravitational field, then the remaining nonvanishing equations are just only

$$\gamma^{\mu}\tilde{\rho}_{00;\mu} = \tilde{m}_{00\tau}, \quad \tilde{K}^{0}_{0\rho\sigma}{}^{:\rho} = \tilde{\rho}^{0}_{0}\gamma_{\sigma}. \tag{151}$$

There are multiple internal charges

$$\rho_{mn}(m, n = 4, 5, \cdots, \mathfrak{D}), \tag{152}$$

on $(M,\mathcal{F})$. We intend to use these ρ_{mn} to describe leptons and hadrons. However, via encapsulation of classical spacetime, $(\tilde{M},\tilde{\mathcal{F}})$ remains only one internal charge $\tilde{\rho}_{00}$, and it falls short. It is impossible for the only one real-valued field function $\tilde{\rho}_{00}$ to describe so many leptons and hadrons.

On the premise of not abandoning the $(1+3)$-dimensional spacetime, if we want to describe gauge fields, there is a method that to use some noncoordinate abstract degrees of freedom on the phase of $e^{iT_a\theta^a}$ of a complex-valued field function ψ. This way is effective, but not natural. It is not satisfactory for a theory to adopt a coordinate representation for external space but a noncoordinate representation for internal space.

A logically more natural way is required to abandon the framework of $(1+3)$-dimensional spacetime $(\tilde{M},\tilde{\mathcal{F}})$ and $(\tilde{M},\tilde{\mathcal{G}})$. We should put internal space and external space together to describe their unified geometry with the same spatial frame. On $(M,\mathcal{F})$ and $(M,\mathcal{G})$, there are enough real-valued field functions ρ_{mn} to describe leptons and hadrons and enough internal components $[mnP]$ of affine connection to describe gauge potentials.

Therefore, only on the full-dimensional $(M,\mathcal{F})$ and $(M,\mathcal{G})$ can total advantages of affine connection representation of gauge fields be brought into full play and thereby show complete details of geometric properties of gauge field. So we are going to stop the discussions about the classical spacetime $\tilde{M}$, but to focus on the full-dimensional manifold M.

Discussion 43. On M, due to $\Gamma_{MNP} = 1/2([MNP] + \{MNP\})$, $[MNP] = \delta_{AD}B^{D}_{M}(\partial B^{A}_{N}/\partial x^{P} + \binom{A}{BP}B^{B}_{N})$, and $G_{MN} = \delta_{AB}B^{A}_{M}B^{B}_{N}$, we know that gauge field and gravitational field can both be described by spatial frames B^{A}_{M} and C^{M}_{A} in a reference-system. Reference-system is the common origination of gauge field and gravitational field. The invariance under reference-system transformation is the common origination of gauge covariance and general covariance.

We adopt the components $[mnP]$ of $[MNP]$ with $m, n \in \{4, 5, \cdots, \mathfrak{D}\}$ to describe the gauge potentials of typical gauge fields such as electromagnetic, weak, and strong interaction fields and adopt the components ρ_{mn} of ρ_{MN} with $m, n \in \{4, 5, \cdots, \mathfrak{D}\}$ to describe the charges of leptons and hadrons. The physical meanings of the other components of ρ_{MN} and $[MNP]$ are not clear at present; maybe they could be used to describe dark matters and their interactions.

On orthogonal $(M,\mathcal{G})$ and $(M,\mathcal{F})$, there are full-dimensional field equations, i.e., affine Dirac equation and affine Yang-Mills equation

$$\begin{aligned} \gamma^{P}\rho_{MN;P} &= \rho_{MN;0}, \\ K^{M}_{NPQ}{}^{:P} &= \rho^{M}_{N}\gamma_{Q}, \end{aligned} \tag{153}$$

which reflect the on-shell evolution directions $\nabla\rho$ and ∇t, respectively. Their quantum evolutions are described by the propagators in Definition 32 or Discussion 34.

Discussion 44. On an orthogonal $(M,\mathcal{G})$, Equation (149) presents as a full-dimensional action

$$\begin{aligned} s_{\rho}(L) &= \int_{L}\sum_{M,N}\left(\gamma^{P}\rho_{MN;P} + \varepsilon^{P}_{0}\rho_{MN;P}\right)dx^{0} \\ &= -i\int_{L}\sum_{M,N}\bar{\psi}_{MN}\left(\gamma^{P}D_{MNP} + \varepsilon^{P}_{0}D_{MNP}\right)\psi_{MN}dx^{0}. \end{aligned} \tag{154}$$

If and only if $L_k : g \longrightarrow g'$ is an orthogonal transformation, L_k sends $s_{\rho}(L)$ to

$$\begin{aligned} s'_{\rho}(L) &= \int_{L}\sum_{M,N}\left(\gamma^{P'}\rho_{MN;P'} + \varepsilon^{P'}_{0'}\rho_{MN;P'}\right)dx^{0'} \\ &= -i\int_{L}\sum_{M,N}\overline{\psi'}_{MN}\left(\gamma^{P'}D'_{MNP'} + \varepsilon^{P'}_{0'}D'_{MNP'}\right)\psi'_{MN}dx^{0'}, \end{aligned} \tag{155}$$

where ρ_{MN} is determined by the reference-system $\mathfrak{f}\circ f$ but not $\mathfrak{g}\circ g$, so ρ_{MN} does not vary with the transformation $L_k : g \longrightarrow g'$. We see that in affine connection representation of gauge fields, the gauge transformations $\psi \mapsto \psi'$ and $D \mapsto D'$ essentially boil down to the reference-system transformation L_k.

Remark 45. For a general $(M, \mathscr{G})$, $\mathscr{G}$ is not necessarily orthogonal, so the corresponding action should be described by

$$s_{MN}(L) = \int_L \left(B_0^0 \gamma^P \rho_{MN;P} + \varepsilon_0^P \rho_{MN;P}\right) dx^0. \tag{156}$$

In this general case, Definition 16 and the method in Discussion 34 are also available and effective, where we take

$$s_{MN}(L) = \int_L D\rho_{MN}. \tag{157}$$

Remark 46. We see that the real-valued representation of action is more concise than the complex-valued representation of action. Hence, it is more convenient to adopt real-valued representations for field function, field equation, and action.

In the following sections, we are going to use $[MNP]$ to show the affine connection representations of electromagnetic, weak, and strong interaction fields and to adopt the real-valued representation $\rho_{MN;P}$ to discuss the interactions between gauge fields and elementary particles. They are based on the following definition.

Definition 47. Let $M = P \times N$, $r \triangleq dimP = 3$ and $\mathfrak{D} \triangleq dimM = 5 \text{ or } 6 \text{ or } 8$. Consider $\mathscr{F} = \mathfrak{f} \circ f$ and $\mathscr{G} = \mathfrak{g} \circ g$ that are defined by Equation (33), that is, $\forall p \in M$,

$$\left(U, \alpha^{A'}\right) \xrightarrow{\mathfrak{f}(p)} \left(U, \xi^A\right) \xrightarrow{f(p)} \left(U, x^M\right) \xleftarrow{g(p)} \left(U, \zeta^A\right) \xleftarrow{\mathfrak{g}(p)} \left(U, \beta^{A'}\right), \tag{158}$$

and furthermore, let

$$\begin{aligned} f(p)&: \xi^a = \xi^a(x^m), \xi^s = \delta_i^s x^i, \\ \mathfrak{f}(p)&: \alpha^{a'} = \alpha^{a'}\left(\xi^a\right), \alpha^{s'} = \delta_s^{s'} \xi^s, \\ g(p)&: \zeta^a = \zeta^a(x^m), \zeta^s = \delta_i^s x^i, \\ \mathfrak{g}(p)&: \beta^{a'} = \beta^{a'}\left(\zeta^a\right), \beta^{s'} = \delta_s^{s'} \zeta^s, \end{aligned} \tag{159}$$

$(s', s, i = 1, 2, 3\,; a', a, m, n = 4, 5, \cdots, \mathfrak{D})$ and both of $\mathscr{F}$ and $\mathscr{G}$ satisfy

$$(i) G_{mn} = \text{const}, (ii) \text{ when } m \neq n, G_{mn} = 0. \tag{160}$$

In the above extremely simplified case, we use $\mathscr{F}$ and $\mathscr{G}$ to show electromagnetic, weak, and strong interactions without gravitation.

5. Affine Connection Representation of the Gauge Field of Weak-Electromagnetic Interaction

Definition 48. Suppose $(M, \mathscr{F})$ and $(M, \mathscr{G})$ conform to Definition 47. Let $\mathfrak{D} = r + 2 = 5$ and both of $\mathscr{F}$ and $\mathscr{G}$ satisfy

$$G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}. \tag{161}$$

Thus, $\mathscr{F}$ and $\mathscr{G}$ can describe weak and electromagnetic interactions.

Proposition 49. *Let the holonomic connection of $(M, \mathscr{F})$ be Γ_{NP}^M and Γ_{MNP}. And let the coefficients of curvature tensor of $(M, \mathscr{F})$ be K_{NPQ}^M and K_{MNPQ}. Denote*

$$\begin{cases} B_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}\mathfrak{D}P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right), \\ A_P^3 \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}\mathfrak{D}P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right), \end{cases} \begin{cases} B_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K_{\mathfrak{D}\mathfrak{D}PQ} + K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}\right), \\ F_{PQ}^3 \triangleq \dfrac{1}{\sqrt{2}}\left(K_{\mathfrak{D}\mathfrak{D}PQ} - K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}\right), \end{cases}$$

$$\begin{cases} A_P^1 \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right), \\ A_P^2 \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right), \end{cases} \begin{cases} F_{PQ}^1 \triangleq \dfrac{1}{\sqrt{2}}\left(K_{(\mathfrak{D}-1)\mathfrak{D}PQ} + K_{\mathfrak{D}(\mathfrak{D}-1)PQ}\right), \\ F_{PQ}^2 \triangleq \dfrac{1}{\sqrt{2}}\left(K_{(\mathfrak{D}-1)\mathfrak{D}PQ} - K_{\mathfrak{D}(\mathfrak{D}-1)PQ}\right). \end{cases} \tag{162}$$

And denote $g \triangleq \sqrt{(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)})^2 + (G^{\mathfrak{D}\mathfrak{D}})^2}$. *Thus, the following equations hold spontaneously.*

$$\begin{aligned} B_{PQ} &= \frac{\partial B_Q}{\partial x^P} - \frac{\partial B_P}{\partial x^Q}, \\ F^3_{PQ} &= \frac{\partial A^3_Q}{\partial x^P} - \frac{\partial A^3_P}{\partial x^Q} + g\left(A^1_P A^2_Q - A^2_P A^1_Q\right), \\ F^1_{PQ} &= \frac{\partial A^1_Q}{\partial x^P} - \frac{\partial A^1_P}{\partial x^Q} + g\left(A^2_P A^3_Q - A^3_P A^2_Q\right), \\ F^2_{PQ} &= \frac{\partial A^2_Q}{\partial x^P} - \frac{\partial A^2_P}{\partial x^Q} + g\left(A^1_P A^3_Q - A^3_P A^1_Q\right). \end{aligned} \tag{163}$$

Proof. Due to Equation (159), it is obtained that the semi-metric of $(M, \mathfrak{f})$ satisfies

$$\left(B_{\mathfrak{f}}\right)^{s'}_{a} = 0, \quad \left(C_{\mathfrak{f}}\right)^{a}_{s'} = 0, \quad \left(B_{\mathfrak{f}}\right)^{a'}_{s} = 0,$$

$$\left(C_{\mathfrak{f}}\right)^{s}_{a'} = 0, \quad \left(B_{\mathfrak{f}}\right)^{s'}_{s} = \delta^{s'}_{s}, \quad \left(C_{\mathfrak{f}}\right)^{s}_{s'} = \delta^{s}_{s'}. \tag{164}$$

Then, compute $\binom{A}{BC}_{\mathfrak{f}} \triangleq 1/2(C_{\mathfrak{f}})^{A}_{A'}(\partial(B_{\mathfrak{f}})^{A'}_{B}/\partial\xi^C + \partial(B_{\mathfrak{f}})^{A'}_{C}/\partial\xi^B)$, and we obtain

$$\binom{s}{BC}_{\mathfrak{f}} = 0, \quad \binom{a}{tu}_{\mathfrak{f}} = 0, \quad \binom{a}{bC}_{\mathfrak{f}} \neq 0; \quad s, t, u = 1, 2, 3; \quad a, b = 4, 5, \cdots, \mathfrak{D}; \quad A, B, C = 1, 2, \cdots, \mathfrak{D}. \tag{165}$$

It is obtained from Equation (159) again that the semi-metric of (M, f) satisfies

$$B^s_m = 0, \quad C^m_s = 0, \quad B^a_i = 0, \quad C^i_a = 0, \quad B^s_i = \delta^s_i, \quad C^i_s = \delta^i_s. \tag{166}$$

Let $s', t', i, j, k = 1, 2, 3; a', b', m, n, p = 4, 5, \cdots, \mathfrak{D}$. Compute the metric of $(M, \mathcal{F})$, and we obtain

$$\begin{cases} G_{ij} = \delta_{s't'} B^{s'}_i B^{t'}_j + \delta_{a'b'} B^{a'}_i B^{b'}_j = \delta_{s't'} \delta^{s'}_i \delta^{t'}_j = \delta_{ij}, \\ G_{\text{in}} = \delta_{s't'} B^{s'}_i B^{t'}_n + \delta_{a'b'} B^{a'}_i B^{b'}_n = 0, \\ G_{mj} = \delta_{s't'} B^{s'}_m B^{t'}_j + \delta_{a'b'} B^{a'}_m B^{b'}_j = 0, \\ G_{mn} = B^{\mathfrak{D}-1}_m B^{\mathfrak{D}-1}_n + B^{\mathfrak{D}}_m B^{\mathfrak{D}}_n = \text{const}, \end{cases} \begin{cases} G^{ij} = \delta^{s't'} C^i_{s'} C^j_{t'} = \delta^{s't'} \delta^i_{s'} \delta^j_{t'} = \delta^{ij}, \\ G^{\text{in}} = \delta^{s't'} C^i_{s'} C^n_{t'} = 0, \\ G^{mj} = \delta^{s't'} C^m_{s'} C^j_{t'} = 0, \\ G^{mn} = C^m_{\mathfrak{D}-1} C^n_{\mathfrak{D}-1} + C^m_{\mathfrak{D}} C^n_{\mathfrak{D}} = \text{const}. \end{cases} \tag{167}$$

Compute the holonomic connection of $\mathcal{F}$ according to $\Gamma^M_{NP} \triangleq 1/2([{}^{M}_{NP}] + \{{}^{M}_{NP}\}) = 1/2(C^M_A \partial B^A_N/\partial x^P + C^M_A \binom{A}{BP}_{\mathfrak{f}} B^B_N)$, and it is obtained that

$$\begin{cases} \Gamma^i_{NP} = 0, \\ \Gamma^m_{jk} = 0, \\ \Gamma^m_{nP} = \dfrac{1}{2}\left(C^m_a \dfrac{\partial B^a_n}{\partial x^P} + C^m_a \binom{a}{bP}_{\mathfrak{f}} B^b_n\right), \\ \Gamma^m_{Np} = \dfrac{1}{2}\left(C^m_a \dfrac{\partial B^a_N}{\partial x^p} + C^m_a \binom{a}{Bp}_{\mathfrak{f}} B^B_N\right), \end{cases} \begin{cases} \Gamma_{iNP} = G_{iM'} \Gamma^{M'}_{NP} = G_{ii'} \Gamma^{i'}_{NP} = 0, \\ \Gamma_{mjk} = G_{mM'} \Gamma^{M'}_{jk} = G_{mm'} \Gamma^{m'}_{jk} = 0, \\ \Gamma_{mnP} = \dfrac{1}{2} \delta_{ab} B^b_m \left(\dfrac{\partial B^a_n}{\partial x^P} + \binom{a}{bP}_{\mathfrak{f}} B^b_n\right), \\ \Gamma_{mNp} = \dfrac{1}{2} \delta_{ab} B^b_m \left(\dfrac{\partial B^a_N}{\partial x^p} + \binom{a}{Bp}_{\mathfrak{f}} B^B_N\right). \end{cases} \tag{168}$$

Compute the coefficients of curvature of $\mathcal{F}$, that is,

$$K^m_{nPQ} \triangleq \frac{\partial \Gamma^m_{nQ}}{\partial x^P} - \frac{\partial \Gamma^m_{nP}}{\partial x^Q} + \Gamma^m_{HP} \Gamma^H_{nQ} - \Gamma^H_{nP} \Gamma^m_{HQ}, \quad K_{mnPQ} \triangleq G_{mM'} K^{M'}_{nPQ} = G_{mm'} K^{m'}_{nPQ}, \tag{169}$$

and then, we obtain

$$
\begin{aligned}
K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ} &= \frac{\partial \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q}}{\partial x^P} - \frac{\partial \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}}{\partial x^Q} + G^{\mathfrak{D}\mathfrak{D}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P}\Gamma_{\mathfrak{D}(\mathfrak{D}-1)Q} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\Gamma_{(\mathfrak{D}-1)\mathfrak{D}Q}\right),\\
K_{\mathfrak{D}(\mathfrak{D}-1)PQ} &= \frac{\partial \Gamma_{\mathfrak{D}(\mathfrak{D}-1)Q}}{\partial x^P} - \frac{\partial \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}}{\partial x^Q} + G^{\mathfrak{D}\mathfrak{D}}\left(\Gamma_{\mathfrak{D}\mathfrak{D}P}\Gamma_{\mathfrak{D}(\mathfrak{D}-1)Q} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\Gamma_{\mathfrak{D}\mathfrak{D}Q}\right) + G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\left(\Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\Gamma_{\mathfrak{D}(\mathfrak{D}-1)Q}\right),\\
K_{(\mathfrak{D}-1)\mathfrak{D}PQ} &= \frac{\partial \Gamma_{(\mathfrak{D}-1)\mathfrak{D}Q}}{\partial x^P} - \frac{\partial \Gamma_{(\mathfrak{D}-1)\mathfrak{D}P}}{\partial x^Q} + G^{\mathfrak{D}\mathfrak{D}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P}\Gamma_{\mathfrak{D}\mathfrak{D}Q} - \Gamma_{\mathfrak{D}\mathfrak{D}P}\Gamma_{(\mathfrak{D}-1)\mathfrak{D}Q}\right) + G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\Gamma_{(\mathfrak{D}-1)\mathfrak{D}Q} - \Gamma_{(\mathfrak{D}-1)\mathfrak{D}P}\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q}\right),\\
K_{\mathfrak{D}\mathfrak{D}PQ} &= \frac{\partial \Gamma_{\mathfrak{D}\mathfrak{D}Q}}{\partial x^P} - \frac{\partial \Gamma_{\mathfrak{D}\mathfrak{D}P}}{\partial x^Q} + G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\left(\Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\Gamma_{(\mathfrak{D}-1)\mathfrak{D}Q} - \Gamma_{(\mathfrak{D}-1)\mathfrak{D}P}\Gamma_{\mathfrak{D}(\mathfrak{D}-1)Q}\right).
\end{aligned}
\tag{170}
$$

Hence,

$$
\begin{aligned}
B_{PQ} &\triangleq \frac{1}{\sqrt{2}}\left(K_{\mathfrak{D}\mathfrak{D}PQ} + K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}\right) = \frac{1}{\sqrt{2}}\frac{\partial\left(\Gamma_{\mathfrak{D}\mathfrak{D}Q} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q}\right)}{\partial x^P} - \frac{1}{\sqrt{2}}\frac{\partial\left(\Gamma_{\mathfrak{D}\mathfrak{D}P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right)}{\partial x^Q} = \frac{\partial B_Q}{\partial x^P} - \frac{\partial B_P}{\partial x^Q}.\\
F^3_{PQ} &\triangleq \frac{1}{\sqrt{2}}\left(K_{\mathfrak{D}\mathfrak{D}PQ} - K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}\right) = \frac{1}{\sqrt{2}}\left(\frac{\partial \Gamma_{\mathfrak{D}\mathfrak{D}Q}}{\partial x^P} - \frac{\partial \Gamma_{\mathfrak{D}\mathfrak{D}P}}{\partial x^Q} + G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\left(\Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\Gamma_{(\mathfrak{D}-1)\mathfrak{D}Q} - \Gamma_{(\mathfrak{D}-1)\mathfrak{D}P}\Gamma_{\mathfrak{D}(\mathfrak{D}-1)Q}\right)\right)\\
&\quad - \frac{1}{\sqrt{2}}\left(\frac{\partial \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q}}{\partial x^P} - \frac{\partial \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}}{\partial x^Q} + G^{\mathfrak{D}\mathfrak{D}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P}\Gamma_{\mathfrak{D}(\mathfrak{D}-1)Q} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\Gamma_{(\mathfrak{D}-1)\mathfrak{D}Q}\right)\right) = \frac{\partial A^3_Q}{\partial x^P} - \frac{\partial A^3_P}{\partial x^Q}\\
&\quad + g\left(\Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\Gamma_{(\mathfrak{D}-1)\mathfrak{D}Q} - \Gamma_{(\mathfrak{D}-1)\mathfrak{D}P}\Gamma_{\mathfrak{D}(\mathfrak{D}-1)Q}\right) = \frac{\partial A^3_Q}{\partial x^P} - \frac{\partial A^3_P}{\partial x^Q} + g\left(A^1_P A^2_Q - A^2_P A^1_Q\right).
\end{aligned}
\tag{171}
$$

Then, F^1_{PQ} and F^2_{PQ} can also be computed similarly. □

Remark 50. Comparing the above conclusion and $U(1)\times SU(2)$ principal bundle theory, we know this proposition shows that the reference-system $\mathcal{F}$ indeed can describe weak and electromagnetic field.

The following proposition shows an advantage of affine connection representation, that is, affine connection representation spontaneously implies the chiral asymmetry of neutrinos, but $U(1)\times SU(2)$ principal bundle connection representation cannot imply it spontaneously.

Definition 51. According to Definition 11, let the charges of the above reference-system $\mathcal{F}$ be ρ_{mn}, where $m, n \in \{\mathfrak{D}-1, \mathfrak{D}\} = \{4, 5\}$. Then, $l \triangleq (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}, \rho_{\mathfrak{D}\mathfrak{D}})^T$ is said to be an electric charged lepton, and $\nu \triangleq (\rho_{\mathfrak{D}(\mathfrak{D}-1)}, \rho_{(\mathfrak{D}-1)\mathfrak{D}})^T$ is said to be a neutrino. l and ν are collectively denoted by L. Thus, $1/\sqrt{2}(1,1)L$ is said to be a left-handed lepton, and $1/\sqrt{2}(1,-1)L$ is said to be a right-handed lepton, denoted by

$$
\begin{cases}
l_L \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}\right),\\
l_R \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}\right),
\end{cases}
\quad
\begin{cases}
\nu_L \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)\mathfrak{D}}\right),\\
\nu_R \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-1)} - \rho_{(\mathfrak{D}-1)\mathfrak{D}}\right).
\end{cases}
\tag{172}
$$

Denote $(\Gamma_{\mathcal{G}})_{MNP}$ by Γ_{MNP} concisely. Then, we define on $(M, \mathcal{G})$ that

$$
\begin{cases}
Z_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P}\right),\\
A_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} - \Gamma_{\mathfrak{D}\mathfrak{D}P}\right),
\end{cases}
\quad
\begin{cases}
W^1_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right),\\
W^2_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right),
\end{cases}
\tag{173}
$$

and say A_P is (affine) electromagnetic potential, while Z_P, W^1_P, and W^2_P are (affine) weak gauge potentials.

Proposition 52. *If $(M, \mathcal{G})$ satisfies the symmetry condition $\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} = \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}$, then the geometric properties l and ν of $\mathcal{F}$ satisfy the following conclusions on $(M, \mathcal{G})$,*

$$\begin{cases} l_{L;P} = \partial_P l_L - g l_L Z_P - g l_R A_P - g \nu_L W_P^1, \\ l_{R;P} = \partial_P l_R - g l_R Z_P - g l_L A_P, \\ \nu_{L;P} = \partial_P \nu_L - g \nu_L Z_P - g l_L W_P^1, \\ \nu_{R;P} = \partial_P \nu_R - g \nu_R Z_P. \end{cases} \tag{174}$$

Proof. Let $H \in \{1, 2, 3, 4, 5\}$, $h \in \{4, 5\}$. It follows from Equation (168) that

$$\rho_{mn;P} = \partial_P \rho_{mn} - \rho_{Hn} \Gamma^H_{mP} - \rho_{mH} \Gamma^H_{nP} = \partial_P \rho_{mn} - \rho_{hn} \Gamma^h_{mP} - \rho_{mh} \Gamma^h_{nP}. \tag{175}$$

□

Then, Equations (172) and (173) lead to Equation (174).

Remark 53. From the above proposition, we see that some constraint conditions make the general linear group $GL(2, \mathbb{R})$ broken to a smaller group S, i.e.,

$$GL(2, \mathbb{R}) \xrightarrow{G_{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G_{\mathfrak{D}\mathfrak{D}}, \quad \Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} = \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}} S, \tag{176}$$

so that the chiral asymmetry of leptons arises in Equation (174) spontaneously.

Remark 54. Proposition 52 shows that

(1) In affine connection representation of gauge fields, the coupling constant g is possessed of a geometric meaning that it is in fact the metric of internal space. But it does not have such a clear geometric meaning in $U(1) \times SU(2)$ principal bundle connection representation

(2) At the most fundamental level, the coupling constant of Z_P and that of A_P are equal, i.e.,

$$g_Z = g_A = g. \tag{177}$$

Suppose there is a kind of medium. Z boson and photon move in it. Suppose Z field has interaction with the medium, but electromagnetic field A has no interaction with the medium. Thus, we have coupling constants

$$\tilde{g}_Z \neq g_A = g, \tag{178}$$

in the medium, and the Weinberg angle arises.

It is quite reasonable to consider a Higgs boson as a zero-spin pair of neutrinos, because in the Lagrangian, Higgs boson only couples with Z field and W field but does not couple with electromagnetic field and gluon field. If so, Higgs boson would lose its fundamentality and it would not have enough importance in a theory at the most fundamental level.

(3) The mixing of three generations of leptons does not appear in Proposition 52, but it can spontaneously arise in Proposition 63 due to the affine connection representation of the gauge field that is given by Definition 59

6. Affine Connection Representation of the Gauge Field of Strong Interaction

Definition 55. Suppose $(M, \mathscr{F})$ and $(M, \mathscr{G})$ conform to Definition 47. Let $\mathfrak{D} = r + 3 = 6$ and both of $\mathscr{F}$ and $\mathscr{G}$ satisfy

$$G^{(\mathfrak{D}-2)(\mathfrak{D}-2)} = G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}. \tag{179}$$

Thus, $\mathscr{F}$ and $\mathscr{G}$ can describe strong interaction.

Definition 56. According to Definition 11, let the charges of $\mathscr{F}$ be ρ_{mn}, where $m, n = 4, 5, \cdots, \mathfrak{D}$. Define

$$\begin{cases} d_1 \triangleq \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}, \quad \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right)^T, \\ d_2 \triangleq \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}, \quad \rho_{\mathfrak{D}\mathfrak{D}}\right)^T, \\ d_3 \triangleq \left(\rho_{\mathfrak{D}\mathfrak{D}}, \quad \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right)^T, \end{cases} \begin{cases} u_1 \triangleq \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)}, \quad \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right)^T, \\ u_2 \triangleq \left(\rho_{(\mathfrak{D}-1)\mathfrak{D}}, \quad \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right)^T, \\ u_3 \triangleq \left(\rho_{\mathfrak{D}(\mathfrak{D}-2)}, \quad \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right)^T. \end{cases} \tag{180}$$

We say d_1 and u_1 are red color charges, d_2 and u_2 are blue color charges, and d_3 and u_3 are green color charges. Then, d_1, d_2, and d_3 are said to be down-type color charges, and u_1, u_2, and u_3 are said to be up-type color charges. Their left-handed and right-handed charges are

$$\begin{cases} d_{1L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right), \\ d_{2L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}\right), \\ d_{3L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}\mathfrak{D}} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right), \end{cases} \begin{cases} d_{1R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right), \\ d_{2R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}\right), \\ d_{3R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}\mathfrak{D}} - \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right), \end{cases}$$
$$\begin{cases} u_{1L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right), \\ u_{2L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right), \\ u_{3L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right), \end{cases} \begin{cases} u_{1R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right), \\ u_{2R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} - \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right), \\ u_{3R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right). \end{cases} \quad (181)$$

On $(M, \mathcal{G})$, we denote

$$g_s \triangleq \sqrt{\left(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right)^2 + \left(G^{\mathfrak{D}\mathfrak{D}}\right)^2} = \sqrt{\left(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right)^2 + \left(G^{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right)^2} = \sqrt{\left(G^{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right)^2 + \left(G^{\mathfrak{D}\mathfrak{D}}\right)^2}.$$
$$\begin{cases} U_P^1 \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right), \\ V_P^1 \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right), \end{cases} \begin{cases} X_P^{23} \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}\right), \\ Y_P^{23} \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}\right), \end{cases}$$
$$\begin{cases} U_P^2 \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P}\right), \\ V_P^2 \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} - \Gamma_{\mathfrak{D}\mathfrak{D}P}\right), \end{cases} \begin{cases} X_P^{31} \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right), \\ Y_P^{31} \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right), \end{cases} \quad (182)$$
$$\begin{cases} U_P^3 \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}\mathfrak{D}P} + \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}\right), \\ V_P^3 \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}\mathfrak{D}P} - \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}\right), \end{cases} \begin{cases} X_P^{12} \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}\right), \\ Y_P^{12} \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} - \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}\right). \end{cases}$$

We notice that there are just only three independent ones in U_P^1, U_P^2, U_P^3, V_P^1, V_P^2, and V_P^3. Without loss of generality, let

$$\begin{cases} R_P \triangleq a_R U_P^1 + b_R U_P^2 + c_R U_P^3, \\ S_P \triangleq a_S U_P^1 + b_S U_P^2 + c_S U_P^3, \\ T_P \triangleq a_T U_P^1 + b_T U_P^2 + c_T U_P^3, \end{cases} \begin{cases} U_P^1 \triangleq \alpha_R R_P + \alpha_S S_P + \alpha_T T_P, \\ U_P^2 \triangleq \beta_R R_P + \beta_S S_P + \beta_T T_P, \\ U_P^3 \triangleq \gamma_R R_P + \gamma_S S_P + \gamma_T T_P, \end{cases} \quad (183)$$

where the coefficients matrix is nonsingular. Thus, it is not hard to find the following proposition true.

Proposition 57. *Let $\lambda_a (a = 1, 2, \cdots, 8)$ be the Gell-Mann matrices and $T_a \triangleq 1/2\lambda_a$ the generators of $SU(3)$ group. When $(M, \mathcal{G})$ satisfies the symmetry condition $\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P} = 0$, denote*

$$A_P \triangleq \frac{1}{2}\begin{pmatrix} A_P^{11} A_P^{12} A_P^{13} \\ A_P^{21} A_P^{22} A_P^{23} \\ A_P^{31} A_P^{32} A_P^{33} \end{pmatrix}, \quad (184)$$

where

$$A_P^{11} \triangleq S_P + \frac{1}{\sqrt{6}} T_P, A_P^{12} \triangleq X_P^{12} - iY_P^{12}, A_P^{13} \triangleq X_P^{31} - iY_P^{31},$$
$$A_P^{21} \triangleq X_P^{12} + iY_P^{12}, A_P^{22} \triangleq -S_P + \frac{1}{\sqrt{6}} T_P, A_P^{23} \triangleq X_P^{23} - iY_P^{23},$$
$$A_P^{31} \triangleq X_P^{31} + iY_P^{31}, A_P^{32} \triangleq X_P^{23} + iY_P^{23}, A_P^{33} \triangleq -\frac{2}{\sqrt{6}} T_P. \quad (185)$$

Thus, $A_P = T_a A_P^a$ *if and only if*

$$\begin{aligned} &A_P^1 = X_P^{12},\ A_P^2 = Y_P^{12},\ A_P^3 = S_P,\ A_P^4 = X_P^{31},\\ &A_P^5 = Y_P^{31},\ A_P^6 = X_P^{23},\ A_P^7 = Y_P^{23},\ A_P^8 = T_P. \end{aligned} \tag{186}$$

Remark 58. On the one hand, the above proposition shows that Definition 55 is an affine connection representation of strong interaction field. It does not define the gauge potentials as abstractly as that in principal $SU(3)$-bundle theory but endows gauge potentials with concrete geometric constructions.

On the other hand, the above proposition implies that if we take appropriate symmetry conditions, the algebraic properties of $SU(3)$ group can be described by the transformation group $GL(3,\mathbb{R})$ of internal space of $\mathscr{G}$. In other words, the exponential map

$$\exp : GL(3,\mathbb{R}) \longrightarrow U(3),\ [B_m^a] \mapsto e^{iT_a^m B_m^a} \tag{187}$$

defines a homomorphism, and $SU(3)$ is a subgroup of $U(3)$. Therefore, Definition 55 is compatible with $SU(3)$ theory.

7. Affine Connection Representation of the Unified Gauge Field

Definition 59. Suppose $(M,\mathscr{F})$ and $(M,\mathscr{G})$ conform to Definition 47. Let $\mathfrak{D} = r + 5 = 8$ and both of $\mathscr{F}$ and $\mathscr{G}$ satisfy

$$G^{(\mathfrak{D}-4)(\mathfrak{D}-4)} = G^{(\mathfrak{D}-3)(\mathfrak{D}-3)},\ G^{(\mathfrak{D}-2)(\mathfrak{D}-2)} = G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}. \tag{188}$$

Thus, $\mathscr{F}$ and $\mathscr{G}$ can describe the unified field of electromagnetic, weak, and strong interactions.

Definition 60. According to Definition 11, let the charges of $\mathscr{F}$ be ρ_{mn}, where $m, n = 4, 5, \cdots, \mathfrak{D}$. Define

$$\begin{cases} l \triangleq \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)}, \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\right)^T, \\ d_1 \triangleq \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}, \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right)^T, \\ d_2 \triangleq \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}, \rho_{\mathfrak{D}\mathfrak{D}}\right)^T, \\ d_3 \triangleq \left(\rho_{\mathfrak{D}\mathfrak{D}}, \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right)^T, \end{cases} \begin{cases} \nu \triangleq \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)}, \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}\right)^T, \\ u_1 \triangleq \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)}, \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right)^T, \\ u_2 \triangleq \left(\rho_{(\mathfrak{D}-1)\mathfrak{D}}, \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right)^T, \\ u_3 \triangleq \left(\rho_{\mathfrak{D}(\mathfrak{D}-2)}, \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right)^T. \end{cases} \tag{189}$$

And denote

$$\begin{aligned} &\begin{cases} l_L \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\right), \\ l_R \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} - \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\right), \end{cases} \begin{cases} \nu_L \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}\right), \\ \nu_R \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} - \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}\right), \end{cases} \\ &\begin{cases} d_{1L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right), \\ d_{2L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}\right), \\ d_{3L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}\mathfrak{D}} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right), \end{cases} \begin{cases} d_{1R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right), \\ d_{2R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}\right), \\ d_{3R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}\mathfrak{D}} - \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right), \end{cases} \\ &\begin{cases} u_{1L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right), \\ u_{2L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right), \\ u_{3L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right), \end{cases} \begin{cases} u_{1R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right), \\ u_{2R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} - \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right), \\ u_{3R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right). \end{cases} \end{aligned} \tag{190}$$

On $(M, \mathscr{G})$, we denote

$$\begin{cases} g \triangleq \sqrt{\left(G^{(\mathfrak{D}-4)(\mathfrak{D}-4)}\right)^2 + \left(G^{(\mathfrak{D}-3)(\mathfrak{D}-3)}\right)^2}, \\ g_s \triangleq \sqrt{\left(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right)^2 + \left(G^{\mathfrak{D}\mathfrak{D}}\right)^2} = \sqrt{\left(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right)^2 + \left(G^{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right)^2} \\ \quad = \sqrt{\left(G^{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right)^2 + \left(G^{\mathfrak{D}\mathfrak{D}}\right)^2}, \end{cases}$$
$$\begin{cases} Z_P \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-4)P} + \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-3)P}\right), \\ A_P \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-4)P} - \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-3)P}\right), \end{cases} \begin{cases} W_P^1 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-3)P} + \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-4)P}\right), \\ W_P^2 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-3)P} - \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-4)P}\right), \end{cases}$$
$$\begin{cases} U_P^1 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right), \\ V_P^1 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right), \end{cases} \begin{cases} X_P^{23} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}\right), \\ Y_P^{23} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}\right), \end{cases} \quad (191)$$
$$\begin{cases} U_P^2 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P}\right), \\ V_P^2 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} - \Gamma_{\mathfrak{D}\mathfrak{D}P}\right), \end{cases} \begin{cases} X_P^{31} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right), \\ Y_P^{31} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right), \end{cases}$$
$$\begin{cases} U_P^3 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}\mathfrak{D}P} + \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}\right), \\ V_P^3 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}\mathfrak{D}P} - \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}\right), \end{cases} \begin{cases} X_P^{12} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}\right), \\ Y_P^{12} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} - \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}\right). \end{cases}$$

Discussion 61. We know from Section 2.3 that the gauge frame matrix $[B_m^a] \in GL(5, \mathbb{R}), (a, m = 4, 5, \cdots, 8)$; therefore, when B_m^a are without any constraints, we can obtain a $GL(5, \mathbb{R})$ gauge theory. In consideration of the fact that the exponential map

$$\exp : GL(5, \mathbb{R}) \longrightarrow U(5), [B_m^a] \mapsto e^{iT_a^m B_m^a} \quad (192)$$

defines a homomorphism and $U(1) \times SU(2) \times SU(3)$ is a subgroup of $U(5)$. So there must exist some constraint conditions of B_m^a to make $GL(5, \mathbb{R})$ reduce to $U(1) \times SU(2) \times SU(3)$, i.e.,

$$GL(5, \mathbb{R}) \xrightarrow{\text{constraint conditions of } B_m^a} U(1) \times SU(2) \times SU(3). \quad (193)$$

More generally, suppose we do not know what the symmetry that can exactly describe "the real world" is, we just denote it by S; then, the map

$$GL(5, \mathbb{R}) \xrightarrow{\text{constraint conditions of } B_m^a} S \quad (194)$$

makes us be able to turn the problem of seeking for S into the problem of seeking for a set of constraint conditions of B_m^a. "To describe S" and "to describe the constraint conditions of B_m^a" are equivalent to each other.

Because gauge potentials Γ_{mnP} and particle fields ρ_{mn} are both constructed from the gauge frame field B_m^a, clearly here, it is more flexible and convenient "to describe the constraint conditions of B_m^a" than "to describe S."

Next, we have no idea what the best constraint conditions look like, but we can try to define a set of constraint conditions to see what can be obtained.

Definition 62. Similar to Remark 53, we define the constraint conditions as follows.

(1) 1st basic conditions

$$\begin{cases} G^{(\mathfrak{D}-4)(\mathfrak{D}-4)} = G^{(\mathfrak{D}-3)(\mathfrak{D}-3)}, \\ G^{(\mathfrak{D}-2)(\mathfrak{D}-2)} = G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}. \end{cases} \tag{195}$$

(2) 2nd basic conditions

$$\begin{cases} \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-4)P} = \Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-3)P}, \\ \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P} = 0. \end{cases} \tag{196}$$

(3) 1st conditions of PMNS mixing of leptons

$$\begin{cases} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-4)P} = c^{\mathfrak{D}-2}_{\mathfrak{D}-3}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P}, \\ \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-4)P} = c^{\mathfrak{D}-1}_{\mathfrak{D}-3}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P}, \\ \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-4)P} = c^{\mathfrak{D}}_{\mathfrak{D}-3}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P}, \end{cases} \begin{cases} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-3)P} = c^{\mathfrak{D}-2}_{\mathfrak{D}-4}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P}, \\ \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-3)P} = c^{\mathfrak{D}-1}_{\mathfrak{D}-4}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P}, \\ \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-3)P} = c^{\mathfrak{D}}_{\mathfrak{D}-4}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P}, \end{cases} \begin{cases} c^{\mathfrak{D}-2}_{\mathfrak{D}-3} = c^{\mathfrak{D}-2}_{\mathfrak{D}-4}, \\ c^{\mathfrak{D}-1}_{\mathfrak{D}-3} = c^{\mathfrak{D}-1}_{\mathfrak{D}-4}, \\ c^{\mathfrak{D}}_{\mathfrak{D}-3} = c^{\mathfrak{D}}_{\mathfrak{D}-4}. \end{cases} \tag{197}$$

(4) 2nd conditions of PMNS mixing of leptons

$$\begin{cases} \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} = \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}, \\ \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}, \\ \rho_{\mathfrak{D}(\mathfrak{D}-3)} = \rho_{\mathfrak{D}(\mathfrak{D}-4)}, \end{cases} \begin{cases} \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}, \\ \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}, \\ \rho_{(\mathfrak{D}-3)\mathfrak{D}} = \rho_{(\mathfrak{D}-4)\mathfrak{D}}. \end{cases} \tag{198}$$

(5) 1st conditions of CKM mixing of quarks

$$\begin{cases} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-2)P} = c^{\mathfrak{D}-4}_{\mathfrak{D}-2}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P}, \\ \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-1)P} = c^{\mathfrak{D}-4}_{\mathfrak{D}-1}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P}, \\ \Gamma^{\mathfrak{D}-3}_{\mathfrak{D}P} = c^{\mathfrak{D}-4}_{\mathfrak{D}}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P}, \end{cases} \begin{cases} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-2)P} = c^{\mathfrak{D}-3}_{\mathfrak{D}-2}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P}, \\ \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-1)P} = c^{\mathfrak{D}-3}_{\mathfrak{D}-1}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P}, \\ \Gamma^{\mathfrak{D}-4}_{\mathfrak{D}P} = c^{\mathfrak{D}-3}_{\mathfrak{D}}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P}, \end{cases} \begin{cases} c^{\mathfrak{D}-4}_{\mathfrak{D}-2} = c^{\mathfrak{D}-4}_{\mathfrak{D}-1} = c^{\mathfrak{D}-4}_{\mathfrak{D}}, \\ c^{\mathfrak{D}-3}_{\mathfrak{D}-2} = c^{\mathfrak{D}-3}_{\mathfrak{D}-1} = c^{\mathfrak{D}-3}_{\mathfrak{D}}. \end{cases} \tag{199}$$

(6) 2nd conditions of CKM mixing of quarks

$$\begin{cases} \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} = \rho_{\mathfrak{D}(\mathfrak{D}-3)}, \\ \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} = \rho_{\mathfrak{D}(\mathfrak{D}-4)}, \end{cases} \begin{cases} \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-3)\mathfrak{D}}, \\ \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-4)\mathfrak{D}}, \end{cases} \tag{200}$$

where c^m_n are constants.

Proposition 63. *When $(M, \mathscr{F})$ and $(M, \mathscr{G})$ satisfy the symmetry conditions (1), (2), (3), and (4) of Definition 62, denote*

$$\begin{aligned} l' \triangleq \Bigg(& \rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} + \frac{c^{\mathfrak{D}-2}_{\mathfrak{D}-4}}{2}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}\right) \\ & + \frac{c^{\mathfrak{D}-1}_{\mathfrak{D}-4}}{2}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}\right) + \frac{c^{\mathfrak{D}}_{\mathfrak{D}-4}}{2}\left(\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}}\right), \\ & \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)} + \frac{c^{\mathfrak{D}-2}_{\mathfrak{D}-3}}{2}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}\right) \\ & + \frac{c^{\mathfrak{D}-1}_{\mathfrak{D}-3}}{2}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}\right) + \frac{c^{\mathfrak{D}}_{\mathfrak{D}-3}}{2}\left(\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}}\right)\Bigg)^T \\ \nu' \triangleq \Bigg(& \rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} + \frac{c^{\mathfrak{D}-2}_{\mathfrak{D}-3}}{2}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}\right) \\ & + \frac{c^{\mathfrak{D}-1}_{\mathfrak{D}-3}}{2}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}\right) + \frac{c^{\mathfrak{D}}_{\mathfrak{D}-3}}{2}\left(\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}}\right), \\ & \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)} + \frac{c^{\mathfrak{D}-2}_{\mathfrak{D}-4}}{2}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}\right) \\ & + \frac{c^{\mathfrak{D}-1}_{\mathfrak{D}-4}}{2}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}\right) + \frac{c^{\mathfrak{D}}_{\mathfrak{D}-4}}{2}\left(\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}}\right)\Bigg)^T \end{aligned} \tag{201}$$

Then, the geometric properties l and ν of $\mathscr{F}$ satisfy the following conclusions on $(M, \mathscr{G})$.

$$\begin{cases} l_{L;P} = \partial_P l_L - g l_L Z_P - g l_R A_P - g \nu'_L W^1_P, \\ l_{R;P} = \partial_P l_R - g l_R Z_P - g l_L A_P, \\ \nu_{L;P} = \partial_P \nu_L - g \nu_L Z_P - g l'_L W^1_P, \\ \nu_{R;P} = \partial_P \nu_R - g \nu_R Z_P. \end{cases} \tag{202}$$

Proof. First, we compute the covariant differential of ρ_{mn} of $\mathscr{F}$.

$$\begin{aligned} \rho_{mn;P} = {} & \partial_P \rho_{mn} - \rho_{Hn}\Gamma^H_{mP} - \rho_{mH}\Gamma^H_{nP} = \partial_P \rho_{mn} - \rho_{(\mathfrak{D}-4)n}\Gamma^{\mathfrak{D}-4}_{mP} \\ & - \rho_{(\mathfrak{D}-3)n}\Gamma^{\mathfrak{D}-3}_{mP} - \rho_{(\mathfrak{D}-2)n}\Gamma^{\mathfrak{D}-2}_{mP} - \rho_{(\mathfrak{D}-1)n}\Gamma^{\mathfrak{D}-1}_{mP} \\ & - \rho_{\mathfrak{D}n}\Gamma^{\mathfrak{D}}_{mP} - \rho_{m(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-4}_{nP} - \rho_{m(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-3}_{nP} \\ & - \rho_{m(\mathfrak{D}-2)}\Gamma^{\mathfrak{D}-2}_{nP} - \rho_{m(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{nP} - \rho_{m\mathfrak{D}}\Gamma^{\mathfrak{D}}_{nP}. \end{aligned} \tag{203}$$

According to Definitions 60 and 62, by calculation, we obtain that

$$
\begin{gathered}
l_{L;P}=\partial_P l_L-gl_LZ_P-gl_RA_P-g\nu_LW_P^1\\
-\frac{1}{2}\left[c_{\mathfrak{D}-4}^{\mathfrak{D}-2}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}+\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}\right)+c_{\mathfrak{D}-3}^{\mathfrak{D}-2}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}+\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}\right)\right]\frac{g}{\sqrt{2}}W_P^1\\
-\frac{1}{2}\left[c_{\mathfrak{D}-4}^{\mathfrak{D}-1}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}+\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}\right)+c_{\mathfrak{D}-3}^{\mathfrak{D}-1}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}+\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}\right)\right]\frac{g}{\sqrt{2}}W_P^1\\
-\frac{1}{2}\left[c_{\mathfrak{D}-4}^{\mathfrak{D}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-3)}+\rho_{(\mathfrak{D}-3)\mathfrak{D}}\right)+c_{\mathfrak{D}-3}^{\mathfrak{D}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-4)}+\rho_{(\mathfrak{D}-4)\mathfrak{D}}\right)\right]\frac{g}{\sqrt{2}}W_P^1,\\
l_{R;P}=\partial_P l_R-gl_RZ_P-gl_LA_P,\\
\nu_{L;P}=\partial_P\nu_L-g\nu_LZ_P-gl_LW_P^1\\
-\frac{1}{2}\left[c_{\mathfrak{D}-4}^{\mathfrak{D}-2}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}+\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}\right)+c_{\mathfrak{D}-3}^{\mathfrak{D}-2}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}+\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}\right)\right]\frac{g}{\sqrt{2}}W_P^1\\
-\frac{1}{2}\left[c_{\mathfrak{D}-4}^{\mathfrak{D}-1}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}+\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}\right)+c_{\mathfrak{D}-3}^{\mathfrak{D}-1}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}+\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\right)\right]\frac{g}{\sqrt{2}}W_P^1\\
-\frac{1}{2}\left[c_{\mathfrak{D}-4}^{\mathfrak{D}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-4)}+\rho_{(\mathfrak{D}-4)\mathfrak{D}}\right)+c_{\mathfrak{D}-3}^{\mathfrak{D}}\left(\rho_{(\mathfrak{D}-3)\mathfrak{D}}+\rho_{\mathfrak{D}(\mathfrak{D}-3)}\right)\right]\frac{g}{\sqrt{2}}W_P^1,\\
\nu_{R;P}=\partial_P\nu_R-g\nu_RZ_P.
\end{gathered}
\tag{204}
$$

Then, according to definitions of l' and ν', we obtain that

$$
\begin{aligned}
l'_L={}&l_L+\frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-2}}{2\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}+\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}\right)+\frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-1}}{2\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}+\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}\right)\\
&+\frac{c_{\mathfrak{D}-4}^{\mathfrak{D}}}{2\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-4)}+\rho_{(\mathfrak{D}-4)\mathfrak{D}}\right)+\frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-2}}{2\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}+\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}\right)\\
&+\frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-1}}{2\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}+\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}\right)+\frac{c_{\mathfrak{D}-3}^{\mathfrak{D}}}{2\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-3)}+\rho_{(\mathfrak{D}-3)\mathfrak{D}}\right),\\
\nu'_L={}&\nu_L+\frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-2}}{2\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}+\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}\right)+\frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-1}}{2\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}+\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}\right)\\
&+\frac{c_{\mathfrak{D}-3}^{\mathfrak{D}}}{2\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-4)}+\rho_{(\mathfrak{D}-4)\mathfrak{D}}\right)+\frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-2}}{2\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}+\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}\right)\\
&+\frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-1}}{2\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}+\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}\right)+\frac{c_{\mathfrak{D}-4}^{\mathfrak{D}}}{2\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-3)}+\rho_{(\mathfrak{D}-3)\mathfrak{D}}\right).
\end{aligned}
\tag{205}
$$

Substitute them into the previous equations, and we obtain that

$$
\begin{gathered}
\begin{cases}
l_{L;P}=\partial_P l_L-gl_LZ_P-gl_RA_P-g\nu'_LW_P^1,\\
l_{R;P}=\partial_P l_R-gl_RZ_P-gl_L\mathrm{A}_P,
\end{cases}\\
\begin{cases}
\nu_{L;P}=\partial_P\nu_L-g\nu_LZ_P-gl'_LW_P^1,\\
\nu_{R;P}=\partial_P\nu_R-g\nu_RZ_P.
\end{cases}
\end{gathered}
\tag{206}
$$

□

Remark 64. The above proposition shows the geometric origin of PMNS mixing of weak interaction. In affine connection representation of gauge fields, PMNS mixing arises as a geometric property on manifold.

In conventional physics, e, μ, and τ have just only ontological differences, but they have no difference in mathematical connotation. By contrast, Proposition 63 tells us that leptons of three generations should be constructed by different linear combinations of $\{\rho_{pq},\rho_{qp}\}_{p=4,5;q=6,7,8}$. Thus, e, μ, and τ may have concrete and distinguishable mathematical connotations. For example, let a_μ, b_μ, $a_{\mu_n}^{\ m}$, $b_{\mu_n}^{\ m}$, a_τ, b_τ, $a_{\tau n}^{\ m}$, and $b_{\tau n}^{\ m}$ be constants; then, we might suppose that

$$
\begin{gathered}
\begin{cases}
e\triangleq l=\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)},\rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\right)^T,\\
\nu_e\triangleq\nu=\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)},\rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}\right)^T,
\end{cases}\\
\begin{cases}
\mu\triangleq a_\mu e+\frac{1}{2}\left(a_{\mu\mathfrak{D}-4}^{\ \mathfrak{D}-2}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}+a_{\mu\mathfrak{D}-4}^{\ \mathfrak{D}-1}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}+a_{\mu\mathfrak{D}-4}^{\ \mathfrak{D}}\rho_{\mathfrak{D}(\mathfrak{D}-4)},\right.\\
\qquad\left.a_{\mu\mathfrak{D}-3}^{\ \mathfrak{D}-2}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}+a_{\mu\mathfrak{D}-3}^{\ \mathfrak{D}-1}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}+a_{\mu\mathfrak{D}-3}^{\ \mathfrak{D}}\rho_{\mathfrak{D}(\mathfrak{D}-3)}\right)^T,\\
\nu_\mu\triangleq b_\mu\nu_e+\frac{1}{2}\left(b_{\mu\mathfrak{D}-3}^{\ \mathfrak{D}-2}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}+b_{\mu\mathfrak{D}-3}^{\ \mathfrak{D}-1}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}+b_{\mu\mathfrak{D}-3}^{\ \mathfrak{D}}\rho_{\mathfrak{D}(\mathfrak{D}-4)},\right.\\
\qquad\left.b_{\mu\mathfrak{D}-4}^{\ \mathfrak{D}-2}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}+b_{\mu\mathfrak{D}-4}^{\ \mathfrak{D}-1}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}+b_{\mu\mathfrak{D}-4}^{\ \mathfrak{D}}\rho_{\mathfrak{D}(\mathfrak{D}-3)}\right)^T,
\end{cases}\\
\begin{cases}
\tau\triangleq a_\tau\mu+\frac{1}{2}\left(a_{\tau\mathfrak{D}-4}^{\ \mathfrak{D}-2}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}+a_{\tau\mathfrak{D}-4}^{\ \mathfrak{D}-1}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}+a_{\tau\mathfrak{D}-4}^{\ \mathfrak{D}}\rho_{(\mathfrak{D}-4)\mathfrak{D}},\right.\\
\qquad\left.a_{\tau\mathfrak{D}-3}^{\ \mathfrak{D}-2}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}+a_{\tau\mathfrak{D}-3}^{\ \mathfrak{D}-1}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}+a_{\tau\mathfrak{D}-3}^{\ \mathfrak{D}}\rho_{(\mathfrak{D}-3)\mathfrak{D}}\right)^T,\\
\nu_\tau\triangleq b_\tau\nu_\mu+\frac{1}{2}\left(b_{\tau\mathfrak{D}-3}^{\ \mathfrak{D}-2}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}+b_{\tau\mathfrak{D}-3}^{\ \mathfrak{D}-1}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}+b_{\tau\mathfrak{D}-3}^{\ \mathfrak{D}}\rho_{(\mathfrak{D}-4)\mathfrak{D}},\right.\\
\qquad\left.b_{\tau\mathfrak{D}-4}^{\ \mathfrak{D}-2}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}+b_{\tau\mathfrak{D}-4}^{\ \mathfrak{D}-1}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}+b_{\tau\mathfrak{D}-4}^{\ \mathfrak{D}}\rho_{(\mathfrak{D}-3)\mathfrak{D}}\right)^T.
\end{cases}
\end{gathered}
\tag{207}
$$

Proposition 65. *When $(M,\mathscr{F})$ and $(M,\mathscr{G})$ satisfy the symmetry conditions (1), (2), (5), and (6) of Definition 62, denote*

$$
\begin{aligned}
d'_{1L}\triangleq{}&\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-1}^{\mathfrak{D}-3}\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}+\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}\right)+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-2}^{\mathfrak{D}-3}\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}+\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}\right)\\
&+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-1}^{\mathfrak{D}-4}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}+\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}\right)+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-2}^{\mathfrak{D}-4}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}+\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\right),\\
d'_{2L}\triangleq{}&\frac{1}{2\sqrt{2}}c_{\mathfrak{D}}^{\mathfrak{D}-3}\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}+\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}\right)+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-1}^{\mathfrak{D}-3}\left(\rho_{(\mathfrak{D}-4)\mathfrak{D}}+\rho_{\mathfrak{D}(\mathfrak{D}-4)}\right)\\
&+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}}^{\mathfrak{D}-4}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}+\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\right)+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-1}^{\mathfrak{D}-4}\left(\rho_{(\mathfrak{D}-3)\mathfrak{D}}+\rho_{\mathfrak{D}(\mathfrak{D}-3)}\right),\\
d'_{3L}\triangleq{}&\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-2}^{\mathfrak{D}-3}\left(\rho_{(\mathfrak{D}-4)\mathfrak{D}}+\rho_{\mathfrak{D}(\mathfrak{D}-4)}\right)+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}}^{\mathfrak{D}-3}\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}+\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}\right)\\
&+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-2}^{\mathfrak{D}-4}\left(\rho_{(\mathfrak{D}-3)\mathfrak{D}}+\rho_{\mathfrak{D}(\mathfrak{D}-3)}\right)+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}}^{\mathfrak{D}-4}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}+\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}\right),\\
u'_{1L}\triangleq{}&\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-2}^{\mathfrak{D}-3}\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}+\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}\right)+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-2}^{\mathfrak{D}-4}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}+\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}\right)\\
&+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-1}^{\mathfrak{D}-3}\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}+\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}\right)+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-1}^{\mathfrak{D}-4}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}+\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\right),\\
u'_{2L}\triangleq{}&\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-1}^{\mathfrak{D}-3}\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}+\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}\right)+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-1}^{\mathfrak{D}-4}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}+\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\right)\\
&+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}}^{\mathfrak{D}-3}\left(\rho_{(\mathfrak{D}-4)\mathfrak{D}}+\rho_{\mathfrak{D}(\mathfrak{D}-4)}\right)+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}}^{\mathfrak{D}-4}\left(\rho_{(\mathfrak{D}-3)\mathfrak{D}}+\rho_{\mathfrak{D}(\mathfrak{D}-3)}\right),\\
u'_{3L}\triangleq{}&\frac{1}{2\sqrt{2}}c_{\mathfrak{D}}^{\mathfrak{D}-3}\left(\rho_{(\mathfrak{D}-4)\mathfrak{D}}+\rho_{\mathfrak{D}(\mathfrak{D}-4)}\right)+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}}^{\mathfrak{D}-4}\left(\rho_{(\mathfrak{D}-3)\mathfrak{D}}+\rho_{\mathfrak{D}(\mathfrak{D}-3)}\right)\\
&+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-2}^{\mathfrak{D}-3}\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}+\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}\right)+\frac{1}{2\sqrt{2}}c_{\mathfrak{D}-2}^{\mathfrak{D}-4}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}+\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}\right).
\end{aligned}
\tag{208}
$$

Then, the geometric properties d_1, d_2, d_3, u_1, u_2, u_3 of $\mathscr{F}$ satisfy the following conclusions on $(M,\mathscr{G})$.

$$d_{1L;P} = \partial_P d_{1L} - g_s d_{1L} U_P^1 + g_s d_{2L} V_P^1 - g_s d_{3L} V_P^1 - g_s u_{1L} X_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} + \frac{g_s}{2} u_{2L} Y_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12} - g u'_{1L} W_P^1,$$

$$d_{2L;P} = \partial_P d_{2L} - g_s d_{2L} U_P^2 + g_s d_{3L} V_P^2 - g_s d_{1L} V_P^2 - g_s u_{2L} X_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} + \frac{g_s}{2} u_{3L} Y_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23} - g u'_{2L} W_P^1,$$

$$d_{3L;P} = \partial_P d_{3L} - g_s d_{3L} U_P^3 + g_s d_{1L} V_P^3 - g_s d_{2L} V_P^3 - g_s u_{3L} X_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} + \frac{g_s}{2} u_{1L} Y_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31} - g u'_{3L} W_P^1,$$

$$d_{1R;P} = \partial_P d_{1R} - g_s d_{1L} V_P^1 + g_s d_{2L} U_P^1 - g_s d_{3L} U_P^1 + g_s u_{1L} Y_P^{23} + \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12},$$

$$d_{2R;P} = \partial_P d_{2R} - g_s d_{2L} V_P^2 + g_s d_{3L} U_P^2 - g_s d_{1L} U_P^2 + g_s u_{2L} Y_P^{31} + \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23},$$

$$d_{3R;P} = \partial_P d_{3R} - g_s d_{3L} V_P^3 + g_s d_{1L} U_P^3 - g_s d_{2L} U_P^3 + g_s u_{3L} Y_P^{12} + \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31},$$

$$u_{1L;P} = \partial_P u_{1L} - g_s u_{1L} U_P^1 - \frac{g_s}{2} u_{2L} X_P^{12} - \frac{g_s}{2} u_{2L} Y_P^{12} - \frac{g_s}{2} u_{3L} X_P^{31} + \frac{g_s}{2} u_{3L} Y_P^{31} - g_s d_{1L} X_P^{23} + g_s d_{2L} Y_P^{23} - g_s d_{3L} Y_P^{23} - g d'_{1L} W_P^1,$$

$$u_{2L;P} = \partial_P u_{2L} - g_s u_{2L} U_P^2 - \frac{g_s}{2} u_{3L} X_P^{23} - \frac{g_s}{2} u_{3L} Y_P^{23} - \frac{g_s}{2} u_{1L} X_P^{12} + \frac{g_s}{2} u_{1L} Y_P^{12} - g_s d_{2L} X_P^{31} + g_s d_{3L} Y_P^{31} - g_s d_{1L} Y_P^{31} - g d'_{2L} W_P^1,$$

$$u_{3L;P} = \partial_P u_{3L} - g_s u_{3L} U_P^3 - \frac{g_s}{2} u_{1L} X_P^{31} - \frac{g_s}{2} u_{1L} Y_P^{31} - \frac{g_s}{2} u_{2L} X_P^{23} + \frac{g_s}{2} u_{2L} Y_P^{23} - g_s d_{3L} X_P^{12} + g_s d_{1L} Y_P^{12} - g_s d_{2L} Y_P^{12} - g d'_{3L} W_P^1,$$

$$u_{1R;P} = \partial_P u_{1R} - g_s u_{1R} U_P^1 + \frac{g_s}{2} u_{2R} X_P^{12} + \frac{g_s}{2} u_{2R} Y_P^{12} + \frac{g_s}{2} u_{3R} X_P^{31} - \frac{g_s}{2} u_{3R} Y_P^{31},$$

$$u_{2R;P} = \partial_P u_{2R} - g_s u_{2R} U_P^2 + \frac{g_s}{2} u_{3R} X_P^{23} + \frac{g_s}{2} u_{3R} Y_P^{23} + \frac{g_s}{2} u_{1R} X_P^{12} - \frac{g_s}{2} u_{1R} Y_P^{12},$$

$$u_{3R;P} = \partial_P u_{3R} - g_s u_{3R} U_P^3 + \frac{g_s}{2} u_{1R} X_P^{31} + \frac{g_s}{2} u_{1R} Y_P^{31} + \frac{g_s}{2} u_{2R} X_P^{23} - \frac{g_s}{2} u_{2R} Y_P^{23}. \quad (209)$$

Proof. Substitute Definition 60 into ρ_{mn} and consider Definition 62, then compute them, and then substitute $d'_{1L}, d'_{2L}, d'_{3L}, u'_{1L}, u'_{2L}, u'_{3L}$ into them, and we finally obtain the results. □

Remark 66. The above proposition shows a geometric origin of CKM mixing. We see that, in affine connection representation of gauge fields, $d'_{1L}, d'_{2L}, d'_{3L}, u'_{1L}, u'_{2L}, u'_{3L}$ arise as geometric properties on manifold. Detailed equations of CKM mixing can be obtained on an additional condition such as

$$\begin{aligned}
\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} &= a^{23}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + a^{24}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + a^{32}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + a^{42}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)},\\
\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} &= a^{13}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + a^{14}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + a^{31}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + a^{41}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)},\\
\rho_{\mathfrak{D}\mathfrak{D}} &= a^{03}\rho_{\mathfrak{D}(\mathfrak{D}-3)} + a^{04}\rho_{\mathfrak{D}(\mathfrak{D}-4)} + a^{30}\rho_{(\mathfrak{D}-3)\mathfrak{D}} + a^{40}\rho_{(\mathfrak{D}-4)\mathfrak{D}},\\
\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} &= b^{23}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + b^{13}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + b^{24}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + b^{14}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)},\\
\rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} &= b^{32}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + b^{31}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + b^{42}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + b^{41}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)},\\
\rho_{(\mathfrak{D}-2)\mathfrak{D}} &= b^{23}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + b^{03}\rho_{\mathfrak{D}(\mathfrak{D}-3)} + b^{24}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + b^{04}\rho_{\mathfrak{D}(\mathfrak{D}-4)},\\
\rho_{(\mathfrak{D}-1)\mathfrak{D}} &= b^{13}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + b^{03}\rho_{\mathfrak{D}(\mathfrak{D}-3)} + b^{14}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + b^{04}\rho_{\mathfrak{D}(\mathfrak{D}-4)},\\
\rho_{\mathfrak{D}(\mathfrak{D}-1)} &= b^{31}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + b^{30}\rho_{(\mathfrak{D}-3)\mathfrak{D}} + b^{41}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + b^{40}\rho_{(\mathfrak{D}-4)\mathfrak{D}}.
\end{aligned} \quad (210)$$

Definition 67. A particle is not an existence at the place of an individual point, and its concept is defined on the entire manifold. Concretely speaking, if the reference-system $\mathscr{F}$ satisfies

$$\begin{aligned}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} &= \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} = \rho_{\mathfrak{D}\mathfrak{D}} = \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} \\ &= \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-1)\mathfrak{D}} = \rho_{\mathfrak{D}(\mathfrak{D}-1)} \\ &= \rho_{\mathfrak{D}(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-2)\mathfrak{D}} = 0, \\ \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} &= \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} = \Gamma_{\mathfrak{D}\mathfrak{D}P} = \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} \\ &= \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P} = \Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} = \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P} \\ &= \Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} = \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P} = 0,\end{aligned} \tag{211}$$

we say $\mathscr{F}$ is a *lepton*; otherwise, $\mathscr{F}$ is a *hadron*.

Suppose $\mathscr{F}$ is a hadron. For d_1, d_2, d_3, u_1, u_2, u_3, if $\mathscr{F}$ satisfies that five of them are zero and the other one is nonzero, we say $\mathscr{F}$ is an *individual quark*.

Proposition 68. *There does not exist an individual quark. In other words, if any five ones of $d_1, d_2, d_3, u_1, u_2, u_3$ are zero, then $d_1 = d_2 = d_3 = u_1 = u_2 = u_3 = 0$.*

For an individual down-type quark, the above proposition is evidently true. Without loss of generality, let $u_1 = u_2 = u_3 = 0$ and $d_1 = d_2 = 0$; thus, $\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} = \rho_{\mathfrak{D}\mathfrak{D}} = 0$; hence, we must have $d_3 = 0$.

For an individual up-type quark, this paper has not made progress on the proof yet. Nevertheless, Proposition 68 provides the color confinement with a new geometric interpretation, which is significant in itself. It involves a natural geometric constraint of the curvatures among different dimensions.

8. Conclusions

(1) An affine connection representation of gauge fields is established in this paper. It has the following main points of view

(i) The holonomic connection Equation (6) contains more geometric information than Levi-Civita connection. It can uniformly describe gauge field and gravitational field

(ii) Time is the total spatial metric with respect to all dimensions of internal coordinate space and external coordinate space

(iii) Energy is the total momentum with respect to all dimensions of internal coordinate space and external coordinate space

(iv) On-shell evolution is described by gradient direction

(v) Quantum theory is a geometric theory of distribution of gradient directions. It has a geometric meaning discussed in Section 3.9

(2) In the affine connection representation of gauge fields, some physical objects are incorporated into the same geometric framework

(i) Gauge field and gravitational field can both be represented by affine connection. They have a unified coordinate description. Some parts of Γ^M_{NP} describe gauge fields such as electromagnetic, weak, and strong interaction fields. The other parts of Γ^M_{NP} describe gravitational field

(ii) Gauge field and elementary particle field are both geometric entities constructed from semimetric. The components ρ_{mn} of ρ_{MN} with $m, n \in \{4, 5, \cdots, \mathfrak{D}\}$ describe leptons and quarks, and the other components of ρ_{MN} may describe particle fields of dark matters

(iii) Physical evolutions of gauge field and elementary particle field have a unified geometric description. Their on-shell evolution and quantum evolution both present as geometric properties about gradient direction

(iv) CPT inversion can be geometrically interpreted as a joint transformation of full inversion of coordinates and full inversion of metrics

(v) Rest-mass is the total momentum with respect to internal space. It originates from geometric property of internal space. Energy, momentum, and mass have no essential difference in geometric sense

(vi) Quantum theory and gravitational theory have a unified geometric interpretation and the same view of time and space. They both reflect intrinsic geometric properties of manifold

(vii) The origination of coupling constants of interactions can be interpreted geometrically

(viii) Chiral asymmetry, PMNS mixing, and CKM mixing arise as geometric properties on manifold

(ix) There exists a geometric interpretation to the color confinement of quarks

In the affine connection representation, we can get better interpretations to these physical properties. Therefore, to represent gauge fields by affine connection is probably a necessary step towards the ultimate theory of physics.

Data Availability

No data were used to support this study.

Disclosure

Preprints have previously been published [43–47].

Conflicts of Interest

The author declares no conflicts of interest.

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