Gravitational-wave response of parametric amplifiers driven by radiation-induced dispersion force modulation

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The dispersion force between two perfectly conducting parallel plane surfaces, referred to as the Casimir force, depends not only on the optical properties of the interacting slabs but also on those of any additional material in the gap between the boundaries. We show by the effective medium analogy that a gravitational wave traveling through a Casimir cavity induces a time-dependent force possibly detectable by a differential opto-mechanical parametric amplification strategy in an asymmetrical torsion oscillator.

Keywords: Casimir force experimentation; micro-/nano-electromechanical systems; gravitational wave detectors; parametric amplifiers; gravitational wave interactions.

1. Introduction

Whether the effective medium formulation should be considered only as a powerful mathematical analogy or as a manifestation of an underlying physical reality is an enduring issue.[1] As often remarked by Whittaker,[2,3] the idea that gravitation might be explained in electrodynamical terms goes back at least to Fitzgerald, who speculated that "[g]ravity is probably due to a change of structure of the ether [a] produced by the presence of matter ..." (Ref. 5, p. 313). After Sakharov,[7] the project evolved into "emergent gravity"[8] – inconclusive but "a strong minority opinion".[1]

Since the early formulations by Gordon[9] and by Mandelstam and Tamm,[10–12] the analogy between metrics describing gravitational fields and the dielectric properties of optical media has been based on arbitrary choices of purely mathematical convenience. For instance, the formalism introduced by Landau and Lifshitz[13] in the widely cited problem in *The Classical Theory of Fields*,[14] published in 1941 (at the end of § 90 in modern Russian, Italian, and English editions), differs from that by Plebanski,[15–17] as the former "... gives a global view of electromagnetic fields with respect to a local static observer" whereas the latter presents "the viewpoint of an observer at infinity..."[18] (see also Ref. 19, p. 1048). On the other hand, "... while the analogy that leads to the refractive index tensor is not itself coordinate invariant, the physical observables that result at the end of any specific calculation are coordinate invariants".[20] Such a compelling conclusion naturally raises the

[a]Although Whittaker consistently conveys this "prophetic adumbration"[3] by Fitzgerald including the words "of the ether" also reproduced herein (see also Ref. 4, p. 301), such words are absent in both above references, in the 1902 edition of the *Scientific Writings*,[5] and indeed in the original text of the 1894 Nature article[6] reproduced in the *Writings*. It is unclear at this time whether Whittaker interpolated the text or was relying on different secondary sources.

issue of the relationship between physical phenomena observed in either domain. “Indeed, it is often not clear where the applicability of an analogue model starts and where it ends. So to what extent may the results obtained in one system be applied to the other? And is the analogue model useful in both senses. . . ?”[21]

For instance, the *dynamical* Casimir effect, due to a “sudden” index of refraction change, first emerged from consideration of the Hawking radiation and, via the principle of equivalence, from the Unruh-Davies effect.[22,23] Motivated by the same analogy, the present author extended to all orders the expression for the ‘self-force’ on a classical dipole in a static homogeneous gravitational field, written by Fermi[24] as a student at Pisa in 1921, confirming that, unlike later claims, such a result is independent of dipole orientation.[25] Consequently, it was shown that an ‘upward’ force – exactly equal to the *negative* gravitational mass equivalent of the van der Waals energy of two spherically symmetrical atoms[26] – is within range of BEC interferometry experiments.[27] Such a non-central self-force component of the distorted dipole-dipole field (Ref. 25, Figs. 1-5) provides a suggestive generalization to curved spacetime of the force between two *real* macroscopic slabs calculated by de Boer as a pairwise sum over all atoms.[28] This leads, within the limitations of the additive approximation (Ref. 29, Sec. 7.6), to the “tiny push in the upwards direction”[30] predicted by Green function techniques in *ideal* Casimir cavities in a weak gravitational field. The intuition that, since the “vacuum is probably a special kind of optical medium”,[31] the analogy should recover this result was recently confirmed by considering not the slabs but the gap quantum vacuum zero-point-energy.[32]

The idea of treating randomly perturbed spacetime as an inhomogeneous effective medium led Zipoy[33] and Winterberg[34] to the astronomical problem of possible stellar scintillation[35–37] due a stochastic gravitational wave background of cosmological and astrophysical origin (see Ref. 38 and Ref. 39, these Proceedings, and references therein). This analogy has been very recently further extended to the scattering of electromagnetic radiation in exact gravitational wave solutions.[18,40]

2. Gravitational wave-induced Casimir force modulation

The original proposal by the present author belongs within the above developmental pattern in that it predicts a novel general relativistic effect on the Casimir force between two plates separated by a gap of width s (see Ref. 38 and references therein).[b] A weak gravitational wave $h^{\rm TT}_{+,\times} \ll 1$ of wavelength $\lambda_{\rm GW} = 2\pi c/\omega_{\rm GW} \gg s$, traveling along the z-axis, introduces in the gap an effective medium[34] with index of refraction $n(\theta, \phi) = \sqrt{\epsilon\mu} \simeq 1 + \frac{1}{2}(h^{\rm TT}_{+} \cos 2\phi + h^{\rm TT}_{\times} \sin 2\phi) \sin^2 \theta$, where $(\theta,\ \phi)$ is the optical ray propagation direction, ϵ and μ are the dielectric constant and the magnetic permeability, respectively, and, as typical in the analogy, $\epsilon = \mu$. For an

[b]Notice that the magnitude of this effect, first given in Ref. 38, was estimated by assuming $\mu = 0$ and by using a well-known result by Dzyaloshinskii, Lifshitz, and Pitaevskii (Eq. (4.21) of Ref. 42). This underestimates the correct result by a factor of $\frac{3}{4}$ as first shown in Ref. 41.

order of magnitude estimate,[41] we treat this anisotropic problem with $h_\times^{\rm TT}=0$ by introducing an average appropriately weighted over all virtual photon directions, $\overline{h}_+^{\rm TT}$. The effective magneto-dielectric gap medium refraction index is, following Winterberg,[34] $\overline{n}_0^2 \simeq 1+\overline{h}_+^{\rm TT}$ and the Casimir force (Ref. 43, Eq. (90)) becomes:

$$F_{\rm Cas}(s) = P_{\rm Cas}(s)A_{\rm plate} = -\frac{\hbar c\pi^2 A_{\rm plate}}{240\, s^4}\sqrt{\frac{\mu_{3,0}}{\epsilon_{3,0}}}\left(\frac{2}{3}+\frac{1}{3}\frac{1}{\epsilon_{3,0}\,\mu_{3,0}}\right) = -\frac{\hbar c\pi^2 A_{\rm plate}}{240\, s^4}\left(\frac{2}{3}+\frac{1}{3}\frac{1}{n_0^2}\right) = -\frac{\hbar c\pi^2 A_{\rm plate}}{240\, s^4}\left(1-\frac{1}{3}\overline{h}_+^{\rm TT}\cos\omega_{\rm GW}t\right). \quad (1)$$

where $T=0$ K, $P_{\rm Cas}(s)$ is the Casimir pressure, $A_{\rm plate}$ is the facing area of the two parallel plates and fringing effects are neglected. The motivation[41] is provided by the fact that the time dependent component of this force with, for instance, $A_{\rm plate} = 1\ {\rm cm}^2$, $s = 10$ nm, and $\overline{h}_+^{\rm TT} = 10^{-20}$, is $\Delta F_{\rm Cas,0} = \frac{1}{720}(\hbar c\pi^2/s^4)A_{\rm plate}\overline{h}_+^{\rm TT} \simeq 4.33\times 10^4$ yN, well above the smallest forces ever measured ($\sim 10^2$–10^0 yN).[44,45]

3. Differential Casimir force detection strategy

Although the effect appears detectable, the signal manifests itself as a relative variation of order $\overline{h}_+^{\rm TT}$ of a much larger static Casimir force. Loosely inspired by recent results,[46] we propose to significantly increase the dynamic signal-to-static Casimir force response ratio by considering the *differential* Casimir torque due to two surfaces at different distances, $s_{0,1-2}$, from the sensing paddles of a torsion oscillator.

Let us generalize the typical treatment of the perturbation on *one* paddle of a torsion oscillator[47–51] to consider *two* planes separated from the facing paddles by unequal gaps (Fig. 1). Assuming for simplicity the center of mass of each paddle of length L and width W to be at a distance $L/2$ from the axis (Fig. 1a), the gap widths appearing at Eq. (1) become $s_{1-2}(t) = s_{0,1-2} \mp (L/2)\sin\theta$ so that, to order $O[(\theta)^1, (\overline{h}_+^{\rm TT})^1]$, the torque on each paddle is $\tau_{\rm Cas,1-2} = \pm[|F_{\rm Cas}(s_{0,1-2})|(L/2) + \hbar c\pi^2 A_{\rm plate}L^2\theta/(240\, s_{0,1-2}^5)]$, where the top (bottom) sign refers to paddle 1 (2), and corrections due to small deviations from plane parallelism can be neglected.[52] In the absence of gravitational waves ($\overline{h}_+^{\rm TT}=0$), the position of equilibrium, $\theta_{\rm eq}$, is determined by the static condition $\tau_{\rm Cas,1}+\tau_{\rm Cas,2}-\kappa_{\rm rot}\theta_{\rm eq}=0$, where $\kappa_{\rm rot}$ is the torsion constant, and, obviously, $|\theta_{\rm eq}| < |\theta_{\rm eq,max}| \simeq s_{0,2}/L$. The equation of motion is $I_{\rm osc}\ddot{\theta}(t) + \Gamma_{\rm rot}\dot{\theta} + \kappa_{\rm rot}\theta(t) = \tau_{\rm GW}(t)$, where $I_{\rm osc}$ is the moment of inertia, $\Gamma_{\rm rot}$ is the rotational friction coefficient, and $\tau_{\rm GW}(t)$ is the gravitational wave-induced Casimir torque. By redefining $\theta(t)-\theta_{\rm eq} \to \theta(t)$, we find:

$$\ddot{\theta}(t) + \frac{1}{\tau_{\rm d}}\dot{\theta}(t) + \omega_{0,\rm pert}^2\,\theta(t) = \frac{\hbar c\pi^2 A_{\rm plate}}{720}\left(\frac{1}{s_{0,1}^4}-\frac{1}{s_{0,2}^4}\right)\frac{L}{2I_{\rm osc}}\overline{h}_+^{\rm TT}\cos\omega_{\rm GW}t\,, \quad (2)$$

where $\omega_{0,\rm pert}^2 = \{\kappa_{\rm rot} - (\hbar c\pi^2 A_{\rm plate}L^2/240)[(1/s_{0,2}^5)+(1/s_{0,1}^5)]\}/I_{\rm osc} \to \omega_0^2 \equiv \kappa_{\rm rot}/I_{\rm osc}$ as $s_{0,1-2}\to\infty$ is the resonant frequency, perturbed by the Casimir force

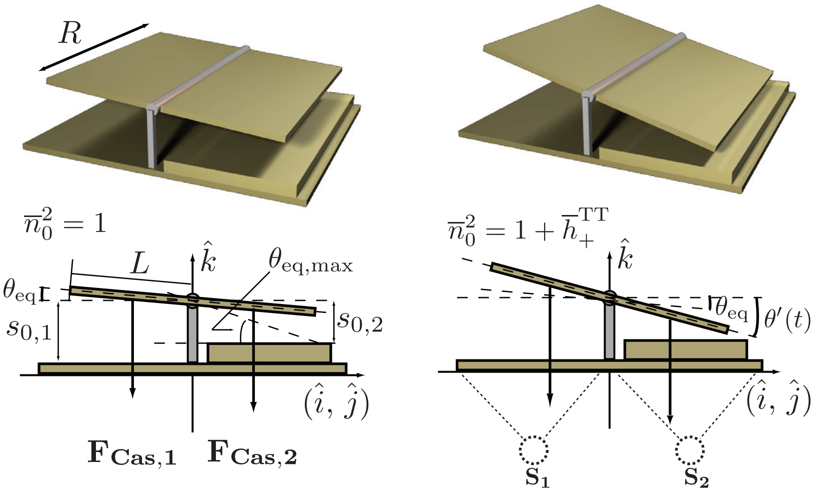

Fig. 1. Asymmetrical torsion oscillator for differential gravitational wave detection (not to scale). (a) The system at equilibrium ($\overline{h}_+^{\rm TT} = 0$). (b) Response to a gravitational wave traveling along the $\hat{\mathbf{k}}$ axis. The independent sources $S_{1,2}$ enable dispersion force time-modulation by back-illuminating *fixed* semiconducting boundaries at a frequency appropriate to yield parametric amplification.

gradients[48–51] and symmetrical under $s_{0,2} \leftrightarrow s_{0,1}$ permutation, and $1/\tau_{\rm d} = \Gamma_{\rm rot}/I_{\rm osc}$ is the decay time. The stationary *resonance* response ($\omega_{0,\rm pert} = \omega_{\rm GW}$) is:

$$\theta_{\rm res} = \frac{\hbar c \pi^2 A_{\rm plate}}{720} \left(\frac{1}{s_{0,1}^4} - \frac{1}{s_{0,2}^4} \right) \frac{L}{2 I_{\rm osc}} \frac{\overline{h}_+^{\rm TT}}{\omega_{\rm GW}^2} Q \to \tfrac{1}{3}\, \theta_{\rm eq}\, \overline{h}_+^{\rm TT} Q\,, \tag{3}$$

where $Q = \omega_0 \tau_{\rm d}$ is the quality factor and the rightmost limit applies asymptotically as $s_{0,1-2} \to \infty$. As a specific example, let us consider the following values: $L = 2$ cm, $W = 0.5$ cm, paddle thickness, $D = 0.1$ cm, material density, $\rho = 5 \times 10^3$ kg/m^3, $\overline{h}_+^{\rm TT} = 10^{-20}$, $\omega_0 = 10^2$ s^{-1}, and $Q = 3 \times 10^{11}$. Therefore, the mass of the two symmetrical paddles and their moment of inertia with respect to the axis of rotation are $2M_{\rm osc} = 2(\rho LWD) = 10^{-3}$ kg and $I_{\rm osc} = 2M_{\rm osc}L^2/3 = 8.3 \times 10^{-9}$ kg m^2, respectively, so that, from the expression for the natural frequency, $\kappa_{\rm rot} = 8.3 \times 10^{-5}$ N m. Choosing $s_{0,1} = 1.0$ μm and $s_{0,2} = 0.9$ μm, the system is in equilibrium for $\theta_{\rm eq} = -2.28 \times 10^{-6}$ rad $\ll \theta_{\rm eq,max} = 3.6 \times 10^{-4}$ rad and consistent with a negligible relative Casimir force non-parallelism error[52] $\simeq 5 \times 10^{-4}$. The angular response at the shifted resonance frequency ($\omega_{0,\rm pert} \simeq 94.6$ s^{-1}) is therefore $\theta_{\rm res} = 2.18 \times 10^{-15}$ rad, or a sensitivity $\theta_{\rm res}/\overline{h}_+^{\rm TT} \simeq 2 \times 10^5$ and a harmonic oscillator (HO) rotational energy $E_{\rm rot,res}/\hbar\omega_0 \simeq n_{\rm HO} + \frac{1}{2}$ with $n_{\rm HO} \gtrsim 1$. The angular Brownian fluctuation root-mean-square[53,54] is $<\theta^2>^{1/2}(T) = \sqrt{k_B T/\kappa_{\rm rot}} \simeq 1.82 \times 10^{-10}$ rad for $T = 100$ mK, or $\theta_{\rm res}/\theta_{\rm rms} \simeq 10^{-5}$ in this particular example. Notice that, asymptotically, $|\theta_{\rm res}/\theta_{\rm eq}| \to \frac{1}{3}\overline{h}_+^{\rm TT} Q \gg \frac{1}{3}\,\overline{h}_+^{\rm TT}$, that is, an improvement equal to Q.

4. Radiation-modulated Casimir force-driven parametric amplifier

The classical strategy of parametric amplification, instinctively learned by children as they grow up to reach wider oscillations of the playground swing,[55] is based on the periodic modulation of quantities that determine the free oscillator natural angular frequency ω_0, such as its stiffness.[50] The fundamental theory of this phenomenon shows that such a modulation must occur within finite frequency intervals centered around specific frequencies equal to $2\omega_0/n$, where $n \geq 1$ are integers, and that, in the presence of friction, a minimum required threshold of the natural frequency modulation magnitude exists for parametric resonance to ensue.[56] In the earliest reported application to mechanical oscillation amplification, the natural frequency was perturbed by introducing an electrostatic force gradient due to a *fixed* electrode biased with respect to a facing vibrating cantilever.[47] By periodically modulating the voltage while holding the system just below the parametric resonance threshold, vibrations caused by driving the cantilever with an external piezoelectric bimorph were amplified with demonstrated gains $\approx 10^2$. Parametric amplification cannot enhance the signal-to-thermal noise ratio, as Brownian vibrations are magnified as well, but it can moderate such factors as sensor and back-action induced noise.[47,51]

In the case of oscillators in regimes in which the Casimir force plays a dominant role, experimentation on standard, non-parametrically driven oscillators has already confirmed that sensor response can be highly non-linear.[49,57] Following the original proposal by the author,[38] the design of a Casimir-force driven torsional parametric amplifier has been presented,[51] theoretically establishing the feasibility to attain extremely high displacement gain factors ($G \gtrsim 10^3$) and reiterating the initial suggestion[38] to extend that approach to gravitational wave detection.[51] Although this concept closely follows the earlier electrostatic approach,[47] a fundamental difference exists. In the original system, the perturbing electrode is *fixed* and the electrostatic force gradient changes periodically because the potential difference is time-dependent. In the recent proposal, instead, "there is no such tunable parameter"[51] and the Casimir force gradient is modulated by *moving* a microsphere so as to periodically change its distance from the torsion oscillator. Here, we restore the design of Rugar and Grütter, so that the pump surface remains fixed,[41] whereas the Casimir force gradients are modulated via back-illumination (Fig. 1b and Ref. 41, Figs. 1-2) by employing the demonstrated dependence of dispersion forces in semiconductors on irradiation.[58] This approach – the dispersion force equivalent of electrostatic pumps – simplifies device design, enhances performance, reduces vibration noise, and paves the way for molecular-scale implementations.[59]

Let us show this strategy is feasible in Casimir force experimentation with silicon-gold cavities[38,60,61] by generalizing the oscillator frequency in Eq. (2) as $\omega_{0,\mathrm{pert}} \simeq \omega_0\{1 - \frac{1}{2}(A_{\mathrm{plate}}/\kappa_{\mathrm{rot}})(L/2)^2[P'_{\mathrm{Lif}}(s_{0,1}) + P'_{\mathrm{Lif}}(s_{0,2})]\}$, where $P'_{\mathrm{Lif}} = \partial P_{\mathrm{Lif}}/\partial s_0$, $\partial/\partial\theta \simeq (L/2)\partial/\partial s$, and $P_{\mathrm{Lif}} < 0$ is the Lifshitz pressure between real materials[42]. For ideal conductors, $P_{\mathrm{Lif}} \to P_{\mathrm{Cas}}$ and we recover our result for $\omega_{0,\mathrm{pert}}$.

Following a similar algorithm as Inui,[62] we compute the dispersion force by employing, in the Lifshitz theory integral (Ref. 42, Eq. (4.14) with $\epsilon_3 = 1$ and $T = 0$ K), the tabulated complex dielectric function data (Ref. 63, Tab. II) for unilluminated crystal silicon analytically best-fitted by a slightly modified Aoki and Adachi model[64] and by setting the parameter,[65] $\epsilon_{1,\infty} \equiv 1$, to ensure a proper convergence for $\omega_I \to \infty$ ($\omega_C = \omega_R + i\omega_I$). Under illumination, the silicon dielectric function displays an additional term dependent on incident power (Ref. 58, Fig. 6 and Ref. 66, Fig. 7a). Finally, the paddle optical properties are described by the dielectric function for gold as previously shown.[67,68] Since the charge carrier relaxation times are much shorter than the period of oscillation ($\tau_h = \tau_e \approx 0.4 \times 10^{-3}$ s $\ll 2\pi/\omega_0$), the system is approximately described by *quasi*-equilibrium states.[58,60] By estimating the required first derivatives with respect to the gap width, $P'_{\mathrm{Lif}}(s)$, by means of a centered finite difference approximation accurate to $O[(\Delta s)^2]$, the shifted natural frequency, for the same oscillator analyzed above, is revised to $\omega_{0,\mathrm{pert}} = 97.52\,\mathrm{s}^{-1}$ for unilluminated c:Si-Au. For Ar laser ($\omega_{\mathrm{Ar}} = 3.66\times10^{15}\,\mathrm{s}^{-1}$) illumination at an effective flux, $I_{\mathrm{Ar}} \simeq 24$ W/cm^2, typical of reported experiments,[60] we find $\omega_{0,\mathrm{pert}} = 96.85$ s^{-1}. This radiation-driven frequency shift time-modulation ($\Delta\omega_{0,\mathrm{pert}}/\omega_{0,\mathrm{pert}} \sim 1$ %) is consistent with values considered in mechanically driven pumps.[51] Furthermore, the parametric resonance condition with slight friction (Ref. 56, §27), written as $1/Q < \Delta\omega_{0,\mathrm{pert}}/\omega_{0,\mathrm{pert}}$, is satisfied. Hence a response as large as $\theta_{\mathrm{res,pump\ on}} = G\,\theta_{\mathrm{res,pump\ off}} \approx (5\times10^3)\,(2\times10^{-15})\,\mathrm{rad} = 10^{-11}$ rad, or a paddle linear center-of-mass displacement $z_{\mathrm{CM}} = \theta_{\mathrm{res,pump\ on}}L/2 \sim 10^{-13}$ m, can be attained by pumping the system below the self-sustained regime threshold. Finally, this response can be obtained even for lower values of Q provided that a higher gain G be chosen by operating arbitrarily close the same threshold.[47,51]

5. Conclusions

In the early phase of study of this novel approach, we have presented intriguing quantitative motivations to pursue possible detection schemes of Casimir force modulation by gravitational waves. The next step will be to remove the effective isotropic index of refraction approximation to obtain expressions for the effect of gravitational waves on Casimir forces between real boundaries with irradiation.

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References

1. C. Barceló, S. Liberati and M. Visser, *Living Rev. Relat* **8**, 12 (2005).
2. E. T. Whittaker, *Science* **LXVI**, 223 (1927).
3. E. T. Whittaker, *Nature* **121**, 1022 (1928).
4. E. T. Whittaker, *The Observatory* **L**, 301 (1927).
5. G. F. Fitzgerald, *The Scientific Writings of the Late George Francis Fitzgerald* (Hodges, Figgis & Co., Dublin and London, 1902).
6. G. F. Fitzgerald, *Nature* **49**, 381 (1894).
7. A. D. Sakharov, *Gen. Relat. Grav.* **32**, 365 (2000).
8. M. Visser, C. Barceló and S. Liberati, *Gen. Rel. Grav.* **34**, 1719 (2002).
9. W. Gordon, *Ann. Phys. (Berlin)* **72**, 421 (1923).
10. B. I. E. Tamm, *Zh. Russk. Fiz. -Khim. Obshch. O-va Chast. Fizika (J. Russ. Phys. Chem. Soc.)* **56**, 248 (1924).
11. B. I. E. Tamm, *Zh. Russk. Fiz. -Khim. Obshch. O-va Chast. Fizika (J. Russ. Phys. Chem. Soc.)* **57**, 209 (1925).
12. L. Mandelstam and I. Tamm, *Math. Ann.* **95**, 154 (1926).
13. K. Hall, 'Think less about foundations': A Short Course on the Course of Theoretical Physics of Landau and Lifshitz, in *Pedagogy and the Practice of Science*, ed. D. Kaiser (MIT Press, Cambridge, 2005) 253–286.
14. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, third edn. (Pergamon Press, Oxford, 1971).
15. J. Plebanski, *Phys. Rev.* **118**, 1396 (1960).
16. F. de Felice, *Gen. Rel. Grav.* **2**, 347 (1971).
17. A. M. Volkov, A. A. Izmest'ev and G. V. Skrotskiĭ, *Sov. Phys. JETP* **32**, 686 (1971).
18. D. Bini, A. Geralico and M. Haney, *Gen. Relat. Gravit.* **46**, 1644 (2014).
19. W. Schleich and M. O. Scully, General relativity and modern optics, in *New Trends in Atomic Physics, Les Houches 1982, Session XXXVI*, eds. G. Grynberg and R. Stora (Elsevier Science Publishers, 1982).
20. P. Boonserm, *et al.*, *Class. Quantum Grav.* **22**, 1905 (2005).
21. D. Faccio, *Contemp. Phys.* **53**, 97 (2012).
22. E. Yablonovitch, *Phys. Rev. Lett.* **62**, 1742 (1989).
23. C. M. Wilson, *et al.*, *Nature* **479**, 376 (2011).
24. E. Fermi, *Il Nuovo Cimento* **22**, 176 (1921).
25. F. Pinto, *Phys. Rev. D* **73**, p. 104020 (2006).
26. F. London, *Z. Phys. Chem. B* **11**, 222 (1930).
27. F. Pinto, *Int. J. Mod. Phys. D* **15**, 2235 (2006).
28. J. H. de Boer, *Trans. Faraday Soc.* **32**, 10 (1936).
29. P. W. Milonni, *The Quantum Vacuum* (Academic Press, San Diego, 1994).
30. G. Bimonte, *et al.*, *Phys. Rev. D* **74**, 085011 (2006).
31. X.-h. Ye and Q. Lin, *J. Opt. A: Pure Appl. Opt.* **10**, 075001 (2008).
32. B. Nazari and M. Nouri-Zonoz, *Phys. Rev. D* **85**, 044060 (2012).
33. D. M. Zipoy, *Phys. Rev.* **142**, 825 (1966).
34. F. Winterberg, *Il Nuovo Cimento B* **LIII B**, 264 (1968).
35. P. G. Bergmann, *Phys. Rev.* **70**, 486 (1946).
36. S. Chandrasekhar, *Mon. Not. R. astron. Soc.* **112**, 475 (1952).
37. H. von Scheffler, *Astron. Nachrichten* **284**, 21 (1957).
38. F. Pinto, *Int. J. Mod. Phys. D* **23**, 1442001 (2014).

39. F. Pinto, If detected, would hypothetical gravitational Casimir effects prove gravity quantization? (these *Proceedings*).
40. D. Bini, *et al. EPL* **102**, 20006 (2013).
41. F. Pinto, Gravitational-wave detection by dispersion force modulation in nanoscale parametric amplifiers, in *Proc. XIV International Conference on Topics in Astroparticle and Underground Physics (TAUP 2015)*, Turin, Italy, 2016).
42. I. Dzyaloshinskii, E. M. Lifshitz and L. P. Pitaevskii, *Adv. Phys.* **10**, 165 (1961).
43. C. Raabe and D.-G. Welsch, *Phys. Rev. A* **71**, 013814 (2005).
44. M. J. Biercuk, *et al.*, *Nature Nanotech.* **5**, 646 (2010).
45. S. Knünz, *et al.*, *Phys. Rev. Lett.* **105**, 013004 (2010).
46. G. Bimonte, *Phys. Rev. Lett.* **113**, 240405 (2014).
47. D. Rugar and P. Grutter, *Phys. Rev. Lett.* **67**, 699 (1991).
48. H. B. Chan, *et al.*, *Science* **291**, 1941 (2001).
49. H. Chan, *et al.*, *Phys. Rev. Lett.* **87**, 21 (2001).
50. H. B. Chan and C. Stambaugh, in *Applications of Nonlinear Dynamics*, (Springer, Berlin, 2009), 15.
51. M. Imboden, *et al.*, *J. Appl. Phys* **116**, 134504 (2014).
52. M. Bordag, G. L. Klimchitskaya and V. M. Mostepanenko, *Int. J. Mod. Phys. A* **10**, 2661 (1995).
53. G. Ising, *Philos. Mag.* **1**, 827 (1926).
54. S. K. Lamoreaux, *Phys. Rev. Lett.* **78**, 5 (1997).
55. F. Pinto, *Phys. Teach* **31**, 336 (1993).
56. L. D. Landau and E. M. Lifshitz, *Mechanics*, (Pergamon Press, Oxford, 1969).
57. F. M. Serry, D. Walliser and G. Maclay, *J Microelectromech. Sys.* **4**, 193 (1995).
58. W. Arnold, S. Hunklinger and K. Dransfeld, *Phys. Rev. B* **21**, 1713 (1980).
59. F. Pinto, *J. Appl. Phys.* **114**, 024310 (2013).
60. F. Chen, *et al.*, *Opt. Express*, **15**, 4823 (2007).
61. F. Pinto, *J. Phys. A* **41**, 164033 (2008).
62. N. Inui, *J. Phys. Soc. Japan*, **73**, 332 (2004).
63. D. E. Aspnes and A. A. Studna, *Phys. Rev. B* **27**, 985 (1983).
64. T. Aoki and S. Adachi, *J. Appl. Phys.*, **69**, 1574 (1991).
65. S. Adachi, *Phys. Rev. B*, **38**, 12966 (1988).
66. F. Chen, *et al.*, *Phys. Rev. B*, **76**, 035338 (2007).
67. A. O. Caride, *Phys. Rev. A*, **71**, 042901 (2005).
68. J. Laurent, *Phys. Rev. A*, **85**, 035426 (2012).