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RENORMALIZATION GROUP APPROACH FOR 2+1D U(1) LATTICE GAUGE THEORY IN FINITE TEMPERATURE LIMIT

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Renormalization group equations describing critical behaviour in the vicinity of the deconfinement phase transition in $3d$ $U(1)$ lattice gauge theory at finite temperature are obtained. These equations are used to check the validity of the Svetitsky-Yaffe conjecture regarding the critical behaviour of lattice $U(1)$ model.

1 Introduction

At high temperatures the partition function of the $3d$ $U(1)$ lattice gauge theory (LGT) coincides in the leading order of the high-temperature expansion with that of $2d$ XY model [1]. The latter is known to possess the Beresinsky-Kosterlitz-Thouless (BKT) phase tranisition of the infinite order. Moreover, at finite temperatures the $3d$ $U(1)$ LGT is invariant under global $U(1)$ rotations acting in a given time-slice of the three-dimensional lattice. These observations led Svetitsky and Yaffe to conjecture that the finite-temperature phase transition in $3d$ $U(1)$ model might belong to the BKT universality class [2]. That means that near the critical point the correlation function between the Polyakov loops should decrease like

$$\Gamma(R) \asymp \frac{1}{R^{\eta(T)}} , \tag{1}$$

for $\beta \geq \beta_c$, and

$$\Gamma(R) \asymp \exp\left[-\frac{R}{\xi(t)}\right], \ t = \frac{\beta_c}{\beta} - 1, \ \xi(t) \sim e^{bt^{-\nu}} , \tag{2}$$

for $\beta < \beta_c$. The critical indices $\eta(T_c)$ and ν for the XY model are known: $\eta(T_c) = 1/4, \ \nu = 1/2$. Thus, if the Svetitsky-Yaffe conjecture holds for $3d$ U(1) LGT, its critical indices should coincide with those of the XY model.

There seems to be no rigorous analytical proof that those indices do coincide. Recent numerical simulations of Ref.[3] indicate that this is the case, at least at strong coupling region ($\beta_s = 0$). The question remains open if β_s is non-vanishing. Here we attempt to study this problem analytically by constructing renormalization group equations perturbatively in β_s/β_t.

2 Definition of the model

As usually we define compact gauge field variables $\omega_l \in [0; 2\pi]$ on links of an anisotropic $3d$ lattice $\Lambda = L^2 \times N_t$. The anisotropic couplings are defined by

$$\beta_t = \frac{1}{g^2 a_t} , \quad \beta_s = \frac{\xi}{g^2 a_s} = \beta_t \, \xi^2 , \tag{3}$$

where a_s, a_t are lattice spacings in the temporal and spatial directions respectively, $\xi = \frac{a_t}{a_s}$, g^2 is a continnum coupling constant. The finite temperature limit is constructed as

$$\xi \to 0, \qquad N_t, L \to 0, \qquad a_t N_t = \beta , \tag{4}$$

where β is inverse temperature.

Now we can write down the partition function for the $3d$ $U(1)$ LGT as

$$Z = \int_0^{2\pi} \prod_l \frac{d\omega_l}{2\pi} \ \exp\left[\beta_s \sum_{p_s} \cos\omega(p_s) + \beta_t \sum_{p_t} \cos\omega(p_t)\right] . \tag{5}$$

Here, p_s and p_t are space- and time-oriented plaquettes, $w(p)$ is a plaquette angle defined in a standard way.

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3 An effective $2d$ sine-Gordon model

First, we perform duality transformations which are well-known in the context of the isotropic model. For the anisotropic model we find the dual representation for the partition function of the form

$$Z = \sum_{r(x)=-\infty}^{\infty} \prod_x \prod_{n=0}^{2} I_{r(x)-r(x+e_n)}(\beta_n) \ , \tag{6}$$

where $I_r(x)$ is the modified Bessel function.

Let us now redefine the variables in (6) in the following way

$$r(x,t) = r(x) + \sum_{\tau=1}^{t} \omega(x,\tau) \quad , \quad (\omega(x,\tau) = r(x,\tau) - r(x,\tau-1)) \ . \tag{7}$$

Here and below x contains only spatial coordinates. Temporal coordinate dependence is written explicitly.

Replacing the Bessel functions with their asymptotics at large β_t we obtain after some algebraic manipulations and integration over $\omega(x,\tau)$ the following representation for the partition function

$$Z = \sum_{r(x)=-\infty}^{\infty} Q(r(x)) \exp\left[-\frac{1}{2\beta_t}\sum_{x,n}(r(x)-r(x+n))^2\right] \ , \tag{8}$$

where notations have been used

$$Q(r(x)) = \prod_{K_n=0}^{L-1} \int_{-\infty}^{\infty} \prod_{n,t} dS_n(K,t) \exp\left\{-\frac{1}{2} S_n(K,t) A_{n,n'}(K;t,t') S_{n'}(-K,t') - i h_n(K) \sum_{t=2}^{N_t} S_n(K,t)\right\} \ ,$$

$$A_{n,n'}(K;t,t') = \beta_t \delta_{n,n'} \delta_{t,t'} + \beta_s b_1(\beta_s N_t) \nu(t,t') \left(1-\exp[iK_n]\right)\left(1-\exp[-iK_{n'}]\right) \ ,$$

$$\nu(t,t') = \min(t,t') - 1 - \frac{1}{N_t}(t-1)(t'-1) \ ,$$

$$b_1(x) = \frac{I_1(x)}{I_0(x)}, \ h_n(K) = \frac{1}{L}\sum_x (r(x)-r(x+e_n)) \exp[iKx] \ .$$

Integration over the $S_n(K,t)$ variables leads to the result which we present in the form

$$Z = \sum_{r(x)=-\infty}^{\infty} \exp\left[-\frac{1}{2} r(x) G(x,x')^{-1} r(x')\right] \ . \tag{9}$$

Here we have introduced the Green function $G(x,x')^{-1}$ given by

$$G(x,x')^{-1} = \sum_{n,n'} \left[B_{n,n'}(x,x') - B_{n,n'}(x-e_n,x') - B_{n,n'}(x,x'-e_{n'} + B_{n,n'}(x-e_n,x'-e_{n'}))\right] \ ,$$

$$B_{n,n'}(x,x') = \frac{1}{\beta_t}\delta_{n,n'}\delta_{x,x'} + \frac{1}{L^2}\sum_K \exp[iK(x-x')] \sum_{t,t'=2}^{N_t} A^{-1}_{n,n'}(K;t,t') \ . \tag{10}$$

Expanding the matrix $A^{-1}_{n,n'}$ up to the second order in $\frac{\beta_s}{\beta_t}$ we find

$$G(x,x')^{-1} = 2g^2\beta\left(\Delta - \frac{1}{6}\frac{\beta^3}{g^2 a_s^2}\Delta^2 + \frac{1}{120}\frac{\beta^6}{g^4 a_s^4}\Delta^3\right) \ . \tag{11}$$

with Δ being a two-dimensional Laplace operator.

To compute the partition function in (9) we use the Poisson summation formula. The result is a two-dimensional vortex model with the action

$$S_{vor} \ = \ \sum_{x,x'} m_x G(x,x') m'_x \ . \tag{12}$$

As is seen from (11) at large temperatures the interaction betwen vortices is of Coulomb type, i.e. logarithmic, up to corrections. That means in the thermodynamical limit only neutral configurations of vortices contribute to the partition function and, therefore one can apply the methods developed for the XY model to compute sums over vortex configurations. Adding a chemical potential term for the vortices we come finally to

$$Z = \int \prod_x d\phi(x) \exp\left[\frac{1}{2}\sum_{x,x'} \phi(x) G(x,x')^{-1} \phi(x') - 2y \sum_x \cos\left(2\pi\phi(x)\right)\right] . \tag{13}$$

The model (13) is of the sine-Gordon type. It follows from (11) that the leading order of the high-temperature expansion coincides exactly with the conventional sine-Gordon model. Corrections produce long-range interactions. This model is starting point for performing renormalization group transformations.

4 The renormalization group equations

To obtain the renormalization group equations we take the continuum limit of the model (13). How to perform the RG transformations for the conventional sine-Gordon model is well-known, see for instance [4]. We follow this strategy and introduce a momentum cutoff λ. Then, integrating out the high-momentum part of the field we get the statistical sum for momentum cutoff $\lambda' = \lambda - d\lambda$. This allows to construct the renormalization group equations in a quite standard way since new partition function turns out to coincide with the original one up to renormalization of all coupling constants and chemical potential. Introducing variables $\beta_{eff} = \frac{1}{2g^2\beta}$, $y = y_0 \exp[-\pi^2\beta_{eff}/2]$ and $\gamma = \frac{1}{6}\frac{\beta^3}{g^2}$ we obtain the renormalization group equation in the form

$$\begin{cases} dy &= -y\left(\pi\beta_{eff}(1+\gamma) - 2\right)\dfrac{da}{a} , \\ d\beta_{eff} &= -2\pi^4\alpha_2 y^2 {\beta_{eff}}^3 (1+\gamma)\dfrac{da}{a} , \\ d\gamma &= 2\gamma\dfrac{da}{a} + \gamma\dfrac{d\beta_{eff}}{\beta_{eff}} . \end{cases} \tag{14}$$

It is convenient to redefine variables according to $X = \pi\beta_{eff}(1+\gamma) - 2, Y = 4\pi\sqrt{\alpha_2}$. Let us recall that the values $X = Y = 0$ are fixed points of the XY model and γ is small by construction. This allows us to rewrite the system of equations near small values of X,Y and γ as

$$\begin{cases} dY &= -YX\dfrac{da}{a} , \\ dX &= -Y^2\dfrac{da}{a} + 4\gamma\dfrac{da}{a} , \\ d\gamma &= 2\gamma\dfrac{da}{a} . \end{cases} \tag{15}$$

This system of RG equations is the same as one for the XY model up to the terms containing γ. Perturbative in γ analysis shows that the renormalization flow of the XY model is not affected when γ is sufficiently small. When γ is growing we run out of the region of the validity of our equations.

5 Summary

RG equations calculated here coincides with the RG equations for the XY model up to a small term which does not affect the universality class. Thus, RG calculations support the Svetitsky-Yaffe conjecture, at least in the region of bare coupling constants where our approximations hold.

Hopefully this result can be extended by going to higher order terms of equations and by solving them to calculate exact values of the critical indices. It is also interesting and important to compute the critical indices from numerical simulations of the $3d$ $U(1)$ LGT. Such computations are now in progress.

References

[1] N. Parga, Phys. Lett. B **107**, 442 (1981)
[2] B. Svetitsky, L. Yaffe, Nucl. Phys. B **210**, 423 (1982)
[3] O. Borisenko, M. Gravina, A. Papa, J.Stat.Mech. (2008) 2008:P08009
[4] J. Kogut, Rev. Mod. Phys. **51**, 659 (1979)