DISS. ETH No. 20713

Superstring Theory on $AdS_3 \times S^3$ and the PSL(2|2) WZW Model

A dissertation submitted to

ETH Zürich

for the degree of

Doctor of Sciences

presented by

Sebastian Gerigk

Diplom-Physiker, Ruprecht-Karls-Universität Heidelberg

Date of birth
18 December, 1983

citizen of
Germany

accepted on the recommendation of

Prof. Dr. Matthias R. Gaberdiel, examiner
Prof. Dr. Niklas Beisert, co-examiner

2012

Abstract

This thesis is concerned with the formulation of superstring theory within the WZW model on PSL(2|2). This approach naturally yields a formulation of string theory on $AdS_3 \times S^3$ that is manifest space-time supersymmetric. The primary goal of the present work is to investigate how physical string states can be identified within the algebraic framework set by the PSL(2|2) WZW model.

As an important preparation, known results on the representation theory of $\mathfrak{psl}(2|2)$ are extended to include representations relevant in the context of string theory, namely those that are infinite-dimensional with respect to the $\mathfrak{sl}(2)$ subalgebra describing the space-time Anti-de-Sitter part. Then, motivated by recent insights into the structure of logarithmic conformal field theories, our understanding of these representations is used to give a proposal for the full space of states of the logarithmic conformal field theory underlying the PSL(2|2) WZW model. We furthermore present an appropriately generalised cohomological characterisation of the subsector of massless physical string states on this full space of states. Both the spectrum of massless states that is independent of the choice of the compactification manifold as well as the massless spectrum specific to compactifications on the four-torus T^4 are confirmed to agree with the supergravity answer.

Motivated by this success, the massive string spectrum in $AdS_3 \times S^3$ backgrounds is investigated. In particular, the physical state constraints of the hybrid formulation of string theory are applied to appropriately chosen vertex operators in order to extract information how massive physical string states can be identified in the PSL(2|2) WZW model. It is shown that the appropriate characterisation of physical string states at the first mass level is a natural generalisation of the description at the massless level. On these grounds, we propose that this naturally extended algebraic characterisation of physical string states holds at every mass level and confirm the proposal at the first two mass levels.

Zusammenfassung

Diese Arbeit beschäftigt sich mit der Darstellung von Superstringtheorie in $AdS_3 \times S^3$ im Rahmen des PSL(2|2)-WZW-Modells. Eine derartige Herangehensweise führt auf natürliche Weise zu einer Formulierung dieser Stringtheorie, die manifest supersymmetrisch in der Raumzeit ist. Das primäre Ziel der vorliegenden Arbeit ist die Beantwortung der Frage, wie die physikalischen Zustände des Strings durch die algebraischen Mittel des PSL(2|2)-WZW-Modells identifiziert werden können.

Als wichtige Vorbereitung werden zunächst bekannte Resultate zur Darstellungstheorie von $\mathfrak{psl}(2|2)$ in einer Weise erweitert, dass auch die für Stringtheorie relevanten Darstellungen behandelt werden können. Diese Darstellungen sind unendlich-dimensional bezüglich der $\mathfrak{sl}(2)$ Lie Unteralgebra, die den Anti-de-Sitter-Anteil der Raumzeit beschreibt. Mit Kenntnis der Eigenschaften dieser Darstellungen und motiviert durch kürzlich errungene Erkenntnisse zur Struktur logarithmischer konformer Feldtheorien ist es möglich, einen Zustandsraum für jene logarithmische konforme Feldtheorie zu konstruieren, die dem PSL(2|2)-WZW-Modell unterliegt. Des Weiteren wird eine verallgemeinerte kohomologische Charakterisierung der physikalischen masselosen Zustände des Strings auf diesem Zustandsraum eingeführt. In dieser Beschreibung wird sowohl das Spektrum bestimmt, das unabhängig von der genauen Wahl der Kompaktifizierung ist, wie auch das Spektrum im Falle von Kompaktifizierungen auf dem vier-dimensionalen Torus T^4. Es stellt sich heraus, dass beide mit dem jeweiligen Supergravitationsspektrum übereinstimmen.

Nach der erfolgreichen Beschreibung des masselosen Untersektors widmet sich die Arbeit dann der Untersuchung des massiven Stringspektrums in $AdS_3 \times S^3$ Gravitationshintergründen. Im Detail werden die Bedingungen an physikalische Zustände im Rahmen des sogenannten Hybridformalismus auf sinnvoll ausgewählte Vertexoperatoren angewandt. Auf diesem Wege können wichtige Erkenntnisse gewonnen werden, in welcher Weise massive Zustände im Rahmen des PSL(2|2)-WZW-Modells zu identifizieren sind. Es ergibt sich, dass die Identifizierung der leichtesten massiven Zustände eine natürliche Erweiterung der Beschreibung masseloser Zustände darstellt. Dies begründet die Vermutung, dass in der Tat das gesamte Spektrum durch eine derartige natürliche Erweiterung beschrieben werden kann. Diese Vermutung kann durch Vergleich des resultierenden Spektrums mit dem Spektrum des RNS-Strings auf erster und zweiter Massenstufe bestätigt werden.

Sebastian Gerigk

Superstring Theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$ and the $\mathrm{PSL}(2|2)$ WZW Model

"On my way came up with the answers,
I scratched my head, and the answers were gone."

Dave Matthews

Contents

Chapter 1

Introduction

1.1 String Theory and Supersymmetry

Even though quantum field theory, in particular Yang-Mills theory, has a long record of success in describing particle physics at today's accessible energies (see *e.g.* [5, 46] for a very recent experimental justification of the standard model), various reasons exist why one should expect a more fundamental guiding principle underlying the standard model of particle physics [154]. First of all, it does not include gravity, and even if one tries to treat gravitation in a quantum field theoretic way, a nonrenormalisable theory is obtained. But a fundamental theory is expected to describe gravity as well as the other fundamental forces, so that it is possible to investigate physical systems in which both gravitational and quantum aspects become important, *e.g.* near the horizon of black holes. Another noteworthy point is that the standard model has several free parameters which hopefully should be fixed by some underlying scheme. Furthermore, typical quantum field theoretic treatments of particle physics introduce a set of different energy scales like the electric-weak symmetry breaking scale, possibly the GUT scale and the Planck scale, which differ by many orders in their magnitudes.

Among several approaches to overcome these difficulties and formulate a theory that not only includes gravity but is also valid at all energy scales, string theory is often considered to be the most promising candidate. Roughly speaking, on the classical level it describes the dynamics of a one-dimensional object, the string, moving in space-time. Although originating from an attempt to effectively describe hadrons in particle physics [174], it was soon realised that the theoretical formulation quite naturally gives rise to a spin two particle that allows for an interpretation as graviton [164, 193] and hence string theory might serve as a theory of quantum gravity. In fact, demanding that the symmetries of the classical theory stay intact after quantisation requires the space-time to satisfy Einstein's equations of general relativity to lowest order. In this sense, general relativity is incorporated in string theory as the low energy effective theory. However, there are drawbacks to string theory in this purely bosonic formulation. First of all, it turns out that the spectrum contains a tachyonic state which renders the vacuum of the theory unstable. Furthermore, it does not give rise to any fermionic states, thus seemingly excluding it to be an UV completion of the well established standard model of particle physics. Fortunately, there exists a quite natural solution to these problems as we will now explain.

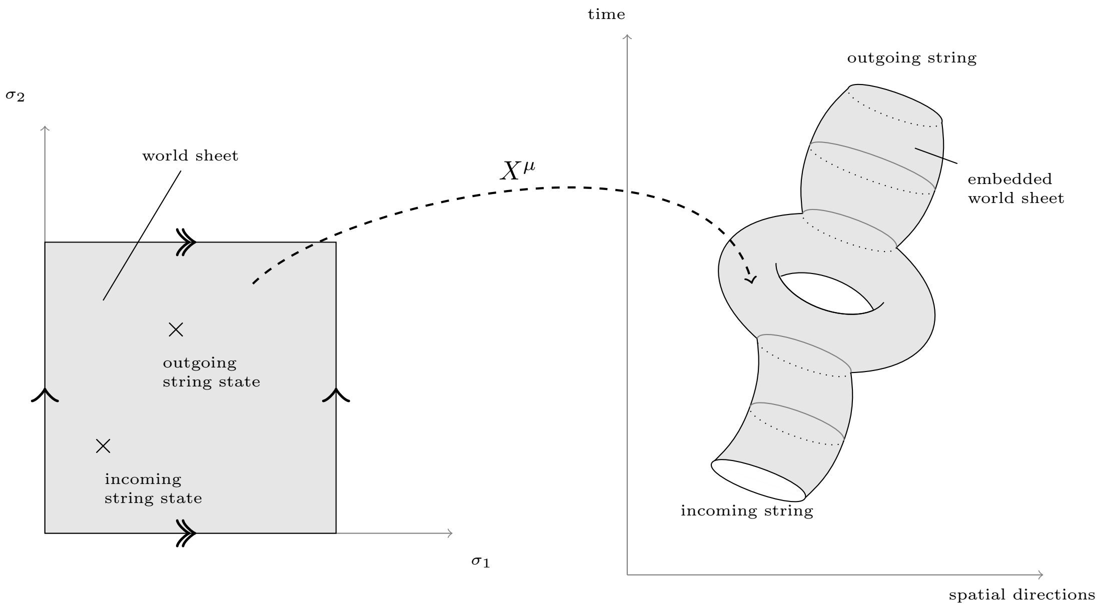


Figure 1.1: *A possible embedding of the string world sheet (shaded area) into the space-time. X^μ indicates the embedding function. In this example, the world sheet has the topology of a torus (the sides of the rectangle with the same number of arrowheads are supposed to be identified). The crosses on that torus indicate the insertion points of the incoming and outgoing string states from the point of view of the field theory living on the world sheet.*

The path the string takes in the space-time defines a two-dimensional surface which in analogy to a particle moving in Minkowski space is referred to as the world sheet (see Fig. 1.1). The embedding function of the world sheet into the space-time contains all dynamics of the string and thus string theory is basically given by a field theory in two dimensions with the fields taking values in the space-time manifold. The solution to the inconsistencies encountered before is then not to consider bosonic string theory but rather a supersymmetric version of the two-dimensional field theory on the world sheet called Ramond-Neveu-Schwarz or RNS string theory, for short [146, 147, 160]. This is achieved by adding fermionic fields to the action that serve as supersymmetric partners of the embedding fields. As a matter of fact, the inclusion of fermions to the world sheet theory also adds space-time fermions to the theory and the spectrum of the new theory can be truncated in such a way that the tachyonic state is removed from the physical spectrum. This truncation is an important element of RNS superstring theory which is called the GSO projection [100, 101]. Not only does it eliminate the tachyon from the spectrum but also renders the physical spectrum space-time supersymmetric. It is due to these findings that string theory is nevertheless considered to be a reasonable candidate to describe real world physics.

Although space-time supersymmetry can be obtained in the RNS formulation of string theory, it is not manifest as a symmetry of the classical action but only occurs after imposing the GSO projection. Nevertheless, a string theory with manifest space-time supersymmetry is desirable as supersymmetry typically imposes powerful constraints on physical quantities, thus hopefully simplifying calculations. A straightforward way to achieve this is by not adding a supersymmetric structure to the world sheet but to the space-time manifold instead. If the space-time manifold is flat Minkowski space, a geometric way of doing so is known from supersymmetric field theories, which can be formulated as "usual" field theories defined on superspace [183], a generalised version of Minkowski space that also includes fermionic, *i.e.* Grassmann-valued directions. Hence, in order to obtain a manifest space-time supersymmetric formulation of string theory, one could just consider a string moving in superspace rather than Minkowski space. A classical string action can be formulated in that setting which gives rise to the so called Green-Schwarz- or GS-string [109]. Even though space-time supersymmetric by construction, it is not known how to covariantly quantise the theory. It can, however, be quantised in light cone gauge and be shown to give the same spectrum as the RNS string in a flat gravitational background.

Let us summarise that there are two equivalent formulations of string theory so far. On the one hand, there is the RNS string that can be nicely treated with well-known techniques of conformal field theory, but lacks manifest space-time supersymmetry. On the other hand, the GS string yields a manifest space-time supersymmetric formulation, but it is difficult to quantise in a covariant manner.

In order to overcome these difficulties and combine the advantages of the RNS and GS description, yet another formulation of string theory was proposed that now goes under the name of *pure spinor string theory* [20,21]. Its world sheet matter content resembles the one of the GS string in that it describes an embedding of the world sheet into superspace. However, there are additional world sheet fields, most importantly a field transforming as Majorana-Weyl spinor in ten dimensions. This field is subject to the *pure spinor constraint*[1], where this framework obtained its name from. The pure spinor description comes together with a BRST operator, which is proportional to the pure spinor field, and whose cohomology defines the physical subspectrum. Although being only a well motivated proposal originally, its equivalence to the other string theories in a flat ten-dimensional gravitational background has been shown by now [23–26]. Furthermore, physical quantities have been calculated [22,27] and checked to agree with the results of the other formulations. Unfortunately, there are certain drawbacks to the pure spinor string. For us the most important is that compactifications to lower dimensions are yet not well understood [12] (see [44, 45] for recent developments in that direction). But as will be explained in detail later in this chapter, string theory has to be compactified in order to make contact to our four-dimensional physics. It has been attempted to bypass this problem by discussing the possibility of noncritical pure spinor superstrings [1, 106, 148, 192], *i.e.* pure spinor string descriptions that are *a priori* formulated in lower dimensions. However, apart from the analysis in [94], it is not clear whether these noncritical formulations describe the same string as the RNS or GS formulation compactified to lower dimensions.

[1]A spinor λ^a is a pure spinor if it satisfies $\lambda^a \gamma^m_{ab} \lambda^b = 0$, where γ^m are the Pauli matrices in ten dimensions.

1.2 The AdS/CFT Correspondence

In recent years, string theory in Anti-de-Sitter gravitational backgrounds has attracted a lot of interest, the reason being the discovery of the correspondence of these string theories to superconformal theories on the boundary [139] (for a comprehensive review the reader is referred to [2]). One can argue for such a correspondence by the following construction: starting with string theory in a flat Minkowski background, one adds some parrallel D3-branes at the origin. D-branes are solitonic states in string theory which, from a geometric perspective, describe hypersurfaces in the space-time on which the end points of open strings are restricted to lie [53, 135]. They also serve as sources of RR flux [156], but the most important property for us is that at low energies the open strings living on the D-branes become gauge bosons. In fact, if we have a stack of N D3-branes, the low energy effective action yields an $\mathcal{N} = 4$ U(N) Super-Yang-Mills theory which is located on the hypersurface defined by the D-branes [189]. Apart from the gauge theory on the brane, there are also closed string states living in the bulk, which give rise to a supergravity theory at low energies.

Following [2], we may consider this setup from another perspective. Since the D-branes are massive objects and charge flux in the normal space-time directions, they influence the space-time geometry. In fact, the space-time becomes $\mathrm{AdS}_5 \times \mathrm{S}^5$ in the near horizon limit [116]. When massive states are integrated out, two classes of states survive: massless states propagating in the bulk and any kind of string excitations located in the AdS_5 throat. The former gives rise to a supergravity theory at low energies, which we have already encountered before from the point of view of a gauge theory living on the stack of D3-branes. As both descriptions are based on the same model, the remaining elements should be identified, *i.e.* string theory (of type IIB) on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is dual to $\mathcal{N} = 4$ U(N) SYM theory [139].

The above construction serves as good motivation of the validity of the correspondence but heavily depends on elements of string theory, particularly on D-branes which are non-perturbative stringy objects. However, it is believed that the correspondence holds in a more general setting in that any quantum theory of gravity in an AdS background is dual to some CFT on the boundary [190]. Roughly speaking, the correlation functions of the dual CFT living on the boundary should coincide with the partition sum of the bulk theory of quantum gravity subject to certain boundary conditions [111, 190]. In fact, it has been recently attempted to constrain possible quantum gravities using this correspondence [191].

There is a lower dimensional analogue of this correspondence. It has been known for quite some time that the asymptotic symmetries of three-dimensional Anti-de-Sitter space AdS_3 are appropriately described by a Virasoro algebra whose central charge depends on the radius of the space as well as the gravitational constant [40]. Since the Virasoro algebra describes the symmetries underlying conformal field theories (cf. section 3.1), this strongly suggests that a duality of gravity theories on AdS_3 and two-dimensional conformal field theories exists. In particular, the AdS/CFT correspondence becomes notably accessible in this case basically due to two reasons: first of all, two-dimensional conformal field theories contain infinite many conserved currents and are thus fairly restrictive. Secondly, gravity in three dimensions does not contain any propagating degrees of freedom and hence the analysis on the gravity side is simplified. For example, a noteworthy success is the calculation of the entropy of the BTZ black hole [6] using the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence (see *e.g.* [132]). When the theory of quantum gravity in question is chosen to be string theory, one can perform a construction similar to the one before using D1- and D5-branes [59] which yields a correspondence of superconformal field theories at the boundary and string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$. In this sense, although interesting in its own right, the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence can be seen as a toy model for the larger, more complicated $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence.

1.3 String Compactifications to $\mathrm{AdS}_3 \times \mathrm{S}^3$

Anomaly cancelation in superstring theory requires the space-time to be ten-dimensional and these string theories are then said to be critical. In order to make contact to our four-dimensional physics, one is forced to dimensionally reduce the theory such that an effective four-dimensional theory is obtained. In more geometric terms, one considers critical string theories but takes six of the ten dimensions to be compactified, *i.e.* one assumes that the space-time takes the form $\mathbb{R}^{3,1} \times M$, where M is some six-dimensional, so called internal manifold. The low energy dynamics of the effective four-dimensional theory constrain the geometry of M, *e.g.* the existence of $\mathcal{N} = 1$ supersymmetry in the low energy effective theory requires the compactification manifold to be a Calabi-Yau manifold [41, 184].

However, since in this work we are primarily interested in contributing to the understanding of string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$, we will not consider compactifications to four but rather to six dimensions. More concretely, we will assume that the space-time is of the form $\mathrm{AdS}_3 \times \mathrm{S}^3 \times M$, where M is now a four-dimensional compactification manifold. As for the case of compactifications down to four dimensions, unbroken supersymmetry in six dimensions requires M to be Calabi-Yau. Apart from the four-torus T^4, there is only one such smooth space up to diffeomorphisms known as the K3 surface. These are the most common compactifications to six-dimensional space-time considered in the literature. When speaking of string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$, we will always implicitly mean critical string theory compactified appropriately.

String theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$ has been an active topic of research in the recent years. Not only does it play a crucial role in understanding the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, but it also serves as a traceable example of a string theory that is not defined in flat Minkowski space-time. The main reason for this is that AdS_3 allows one to introduce a consistent group structure on the manifold. In fact, as a differentiable manifold, it is isomorphic to the special linear group $\mathrm{SL}(2, \mathbb{R})$. From the world sheet perspective, string theory on AdS_3 can then be seen as a two-dimensional field theory whose fields take values in $\mathrm{SL}(2, \mathbb{R})$

[10, 114, 153] (of course, since we are considering critical string theory, one has to take care of the remaining dimensions as well, *e.g.* by compactification). More generally, it can be argued that string theory on group manifolds is closely connected to a well understood construction in conformal field theory, the Wess-Zumino-Witten models [92, 186]. Using conformal field theory techniques, many important results concerning string theory on AdS_3 have been obtained, *e.g.* it has been shown that the physical spectrum does not contain states of negative norm [65, 120], a crucial requirement for the consistency of string theory. For the bosonic string moving on AdS_3, this result has amongst other important insights been extended to all unitary $\mathrm{SL}(2,\mathbb{R})$ representations in [140–142].

As for string theory in flat backgrounds, one expects that string calculations become simpler if one could make space-time supersymmetry manifest. Starting from the RNS formulation for strings moving on $\mathrm{AdS}_3 \times \mathrm{S}^3$, it is possible to define such a manifest space-time supersymmetric framework, which now commonly goes under the name of the hybrid formulation [17–19, 30] (see also [125] for clarifications in the case of compactifications to four dimensions). It is discussed in detail in chapter 5 and the basic idea is to redefine the RNS world sheet fields in such a way that they behave like fields that embed the world sheet in six-dimensional superspace, therefore yielding a Green-Schwarz string like description. The name hybrid string originates from the fact that only the six uncompactified dimensions, *i.e.* eventually the degrees of freedom describing the $\mathrm{AdS}_3 \times \mathrm{S}^3$ part, have a Green-Schwarz like description while the compactification manifold is still modeled by RNS variables.

In flat space-time, the Green Schwarz formulation of the superstring was based on the idea of lifting space-time to its superspace version rather then the world sheet, hence introducing a manifest supersymmetric structure to the target space of the embedding fields. For curved space-times, in particular AdS_3, one could wish to take a similar path and might wonder what the correct lift to "superspace" is. In general, this is an involved problem if we want this supersymmetric structure to be globally defined. Fortunately, for some space-time geometries including $\mathrm{AdS}_3 \times \mathrm{S}^3$, there exists an algebraic way of determining the supersymmetric version. We have indicated before that $\mathrm{AdS}_3 \times \mathrm{S}^3$ is diffeomorphic to (a real form of) the Lie group $\mathrm{SL}(2) \times \mathrm{SL}(2)$. Locally, the Lie group is appropriately described by the associated Lie algebra. It is possible to add anticommuting generators to the Lie algebra, which can be thought of as the supercharges. However, the extension of the Lie bracket to also include these generators is subject to severe consistency constraints [121, 122]. The resulting algebraic structure is called Lie superalgebra and it can be lifted to a so called Lie supergroup by the exponential map as it is known from the theory of purely bosonic Lie groups. The Lie supergroup, interpreted as a Grassmannian manifold, *i.e.* a manifold with both bosonic and fermionic directions, can then be seen as a supersymmetrised version of the space one started with. In the case of $\mathrm{AdS}_3 \times \mathrm{S}^3$, the correct Lie supergroup to consider is called $\mathrm{PSL}(2|2)$. It is then not unreasonable to expect that superstrings moving on $\mathrm{AdS}_3 \times \mathrm{S}^3$ can be described by a nonlinear σ-model on $\mathrm{PSL}(2|2)$, possibly a Wess-Zumino-Witten model. In fact, one can argue that the physical spectrum of supergravity on $\mathrm{AdS}_3 \times \mathrm{S}^3$ can be arranged in representations of $\mathrm{PSL}(2|2)$ [54], indicating an underlying connection. It has further been shown that there is a whole family of nonlinear σ-models on PSL(2|2) that define conformal field theories [33] and one has to ask how to interpret these moduli from a string theoretic perspective.

We have already said that in string theory, $\mathrm{AdS}_3 \times \mathrm{S}^3$ can be obtained as the gravitational background by using a system of D1- and D5-branes. However, instead of D5-branes, we might also have used so called NS5-branes that can be thought of as the magnetic analog of the fundamental string [143]. While the D5-brane sources RR flux, the NS5-brane sources NSNS flux, under which the fundamental string is electrically charged. Hence, from a string theory perspective, given an $\mathrm{AdS}_3 \times \mathrm{S}^3$ geometry, there is a two-dimensional moduli space of vacuum configurations, parameterised by the number of D5- and NS5-branes we used. At each point in the moduli space, it turns out that the corresponding hybrid formulation gives rise to a nonlinear σ-model whose target space is the supergroup PSL(2|2) [30]. Although string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$ with RR flux has been analysed on a classical level [144, 150, 152, 159], a particular interesting point in the moduli space is the one where there is NSNS flux only, since this configuration is accessible in perturbative string theory. In fact, at this point in moduli space the string theory can be formulated as a WZW model using RNS variables [56], which in turn can be used to clarify the role of string theory in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence [99, 133, 134]. Using this approach, many detailed checks of the correspondence have been performed, *e.g.* three-point functions have been compared [42, 52, 55, 84, 98, 149, 175]. From the perspective of the hybrid formulation, this point is special because the nonlinear σ-model on PSL(2|2) becomes a Wess-Zumino-Witten model [30], which implies that the symmetry currents become holomorphic and powerful tools of conformal field theory can be used. So the moduli of nonlinear σ-models encountered before correspond to different vacuum configurations with NSNS and RR flux.

1.4 Conformal Field Theories with Supergroup Target Spaces

In the last decade, conformal field theories whose matter fields take values in some supergroup attracted a lot of interest. We already motivated how these models might appear in the context of string theory as some kind of Green-Schwarz string analogue in curved space-times that can be equipped with a group structure. However, they possess a much broader range of application in modern day physics. First of all, there are interesting from a fundamental point of view since Wess-Zumino-Witten models with supergroup target generically give rise to explicit examples of so called logarithmic conformal field theories [161,162,167]. In a loose sense, logarithmic conformal field theories can be thought of as rational conformal field theories but with the possibility of the two-point-functions having a logarithmic dependence on the distance of the inserted operators. In general, this yields a degeneration of conformal vacuum states [79,82,83,86]. Furthermore, they appear in the context of statistical mechanical models. For example, the Wess-Zumino-Witten model on the Lie supergroup called SU(2|1) is closely connected to a supersymmetrised version of the Ising model [31, 34, 163].

Due to their correspondence to string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$, especially nonlinear σ-models on PSL(2|2) were intensely explored in the literature recently [3, 4, 104, 105, 178]. Although noteworthy successes have been achieved in understanding string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$ in a space-time supersymmetric manner using the hybrid formulation, *e.g.* the vertex operators associated to massless string states have been identified and shown to coincide with the supergravity spectrum [61, 62], the analyses so far were mostly based on the point particle limit of string theory, where vertex operators become superfields in the target space. Furthermore, even in that case, the logarithmic nature of the conformal field theories associated with these models has not yet been included properly. But to gain insights into the complete $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, a full string theoretic understanding of these issues in backgrounds that contain not only NSNS flux but also RR flux is of major importance. The commonly accepted strategy to obtain this can be described as follows: one starts by investigating the PSL(2|2) WZW model, which is the most accessible point in moduli space, and then uses nonrenormalisation theorems specific to Lie supergroups [33] to marginally deform the theory to include RR flux as well [105].

It is the subject of the present work to contribute to the first step in this program, in particular to deepen the understanding of the connection between string theory in a pure NSNS flux background on the one hand and the PSL(2|2) Wess-Zumino-Witten model on the other. More concretely, by using the mathematical theory of representations and modules as well as applying tools of conformal field theory, an algebraic identification of the physical string spectrum within the full space of states of the logarithmic conformal field theory underlying the PSL(2|2) WZW model is obtained.

1.5 Overview

This work is organised as follows. We start off quite technically by reviewing the mathematics of Lie superalgebras and their representations in chapter 2. When discussing representations of Lie superalgebras, we usually take the modern point of view on representations in terms of module theory, which for illustration purposes is discussed in the context of Lie algebras in appendix A. This is closely connected to the notion of the BGG category, which loosely speaking is the set of all accessible Lie algebra representations, finite- and infinite-dimensional. Important notions introduced in this chapter are projective covers, which play an important role in the description of the massless string spectrum within the PSL(2|2) WZW model due to the logarithmic nature of the associated current algebra [105, 167]. For the use in later chapters, the projective covers of $\mathfrak{psl}(2|2)$ and the homomorphisms between them are constructed explicitly.

We turn back to the physics in chapter 3. Basic concepts of conformal field theory as well as string theory are recalled, including Wess-Zumino-Witten models on Lie groups and the BRST quantisation of string theory. The reader familiar with these subjects is invited to skip this part.

Chapter 4 discusses important basics and developments on the interplay of supersymmetry and string theory. In particular, superconformal algebras in two dimensions are discussed with particular emphasis on their significance as a world sheet symmetry in string theory. Special attention is given to the question how supersymmetry can be introduced in WZW models. As has been explained above, there are basically two ways this can be achieved: either one chooses to make the world sheet CFT supersymmetric by introducing fermionic superpartners of the currents, or one substitutes the Lie group target space of the WZW model by an appropriately chosen Lie supergroup, thus making the WZW model manifestly target space supersymmetric. Both possibilities are discussed in section 4.3 in general terms as well as applied to the case of string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$, which is described by a WZW model on $\mathrm{SL}(2) \times \mathrm{SU}(2)$.

In chapter 5, the hybrid formulation is discussed in quite some detail. In the first part of this chapter, it is argued that string theory is connected to the PSL(2|2) WZW model if an appropiate $\mathcal{N} = 2$ superconformal structure is added to the world sheet. The form of the associated superconformal algebra, which is crucial in order to identify the physical string states, is given explicitly in terms of normal ordered products of the $\widehat{\mathfrak{psl}}(2|2)_k$ current algebra and the so-called $\rho\sigma$-ghosts. Secondly, the physical state conditions of string theory quantised in the BRST framework are reformulated using the superconformal algebra. A first step is done towards an algebraic characterisation of physical massless string states in the PSL(2|2) WZW model by evaluating these physical state constraints on vertex operators in the hybrid formulation. We find that there are two distinguished kinds of states; those that depend on the choice of the compactification manifold and those that do not. The latter are thus called compactification-independent while the former are the compactification-dependent states. It is argued that the compactification-independent massless string spectrum is described by the cohomology of an appropriately chosen BRST operator [30,62]. For toroidal compactifications, the compactification-dependent states are shown to be described by the so-called socle of an indecomposable representation.

Motivated by recent results in logarithmic conformal field theory, we propose a detailed description of the spectrum of the logarithmic conformal field theory associated to the PSL(2|2) WZW model in chapter 6. Using the results in chapter 2, in particular those in section 2.2 on the representations of $\mathfrak{psl}(2|2)$ and the corresponding projective covers, we explain how to construct the full space of states of that logarithmic conformal field, following recent ideas of [86,87]. We then explain how the BRST operator on massless string states that are independent of the compactification can be formulated in our language, and study its cohomology. It is shown that this BRST cohomology reproduces precisely the physical spectrum of $\mathcal{N} = 2$ supergravity in six dimensions [54, 57]. Finally, a similar analysis is performed for compactification-dependent states for toroidal compactifications, which again agrees with supergravity result. The results obtained in this chapter are based on the publication [80].

After having discussed the massless case, we generalise our analysis to the massive spectrum that is independent of the compactification in chapter 7. First, we restrict ourselves to states at the first mass level. Following the same arguments as in the massless case, the physical state constraints in the hybrid formulation are evaluated on appropriately chosen vertex operators. An algebraic characterisation of the physical string states within the PSL(2|2) WZW model is obtained, which nicely resembles the one found in the massless case. The resulting spectrum agrees with the RNS spectrum if one takes into account contributions from the twisted $\mathcal{N} = 4$ superconformal field theory describing the internal manifold. We close this chapter by conjecturing a generalisation to all mass levels and collect evidence for it by comparing the implied spectrum to the RNS spectrum at the second mass level. The results of this chapter were published in [93].

Appendix A gives an overview of the theory of Lie algebras and their representations in the context of the BGG category. Although not strictly necessary for this thesis, it is instructive to see how notions like projective covers, duality and BGG reciprocity show up in the well known context of semisimple Lie algebras. Exemplarily, the ideas introduced in this appendix are applied to representations of $\mathfrak{sl}(2)$. In appendix B we present an important formal construction of projective representations of Lie superalgebras that is often used in the literature.

Appendix C summarises our choice of a basis for the Lie superalgebra $\mathfrak{psl}(2|2)$, technical details on OPE calculations are presented in appendix D and appendix E contains the spectrum of compactification-independent RNS string states at the first few mass levels.

Chapter 2

Lie Superalgebras and Their Representations

2.1 Lie Superalgebras

The importance of Lie algebras in physics originates form the fact that they describe continuous symmetries of physical theories and thus serve as underlying principles that constrain the moduli spaces of theories and make the analysis thereof more accessible. At least with the introduction of supersymmetry as a possible symmetry in high energy particle physics [103, 179, 182], it became apparent that one should extent the mathematical theory of Lie algebras to include generators with *symmetric* Lie bracket or *anticommutators* as well. These new generators are referred to as fermionic while the generators with an antisymmetric Lie bracket are called bosonic. Motivated by its supersymmetric origin the resulting algebraic structure was named a *Lie superalgebra.* A Lie superalgebra always contains a Lie algebra as a subalgebra, called the bosonic (Lie) subalgebra, by restricting to the bosonic generators. In fact, extending some Lie algebra, say $\mathfrak{g}^{(0)}$, to a Lie superalgebra $\mathfrak{g}$ that has $\mathfrak{g}^{(0)}$ as its bosonic subalgebra can be thought of as introducing a supersymmetric structure to Lie group that corresponds to $\mathfrak{g}^{(0)}$ by applying the exponential map. We now give a constructive definition of Lie superalgebras that closely resembles this physical intuition. Let us also mention that a concise overview can be found in [68].

A short comment on the notation: In the following and the remainder of this work, we will mostly denote commutators as well as anticommutators by square brackets, $[\,\cdot\,,\,\cdot\,]$. The grading of the Lie bracket should be clear from the context. However, when we wish to emphasize the grading, we will add the corresponding subscript to the Lie bracket, *i.e.* $[\,\cdot\,,\,\cdot\,]_-$ denotes the commutator while $[\,\cdot\,,\,\cdot\,]_+$ denotes the anticommutator.

2.1.1 Generalities

Let us assume we are given some semisimple Lie algebra $\mathfrak{g}^{(0)}$. As explained in the introductory remarks, we add a set of fermionic generators that span a vector space $\mathfrak{g}^{(1)}$ and consider the vector space $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ where $\mathfrak{g}^{(0)}$ inherits the original Lie algebraic structure. The Lie bracket $[\,\cdot\,,\,\cdot\,]$ on $\mathfrak{g}^{(0)}$ is now extended to cover the whole vector space $\mathfrak{g}$ by first letting $\mathfrak{g}^{(1)}$ transform in some representation ρ of $\mathfrak{g}^{(0)}$ under the adjoint action of $\mathfrak{g}^{(0)}$. So, if $\{t^a|a=1,\ldots,n\}$ is a basis of $\mathfrak{g}^{(0)}$ and $\{s^\mu|\mu=1,\ldots,m\}$ is a basis of $\mathfrak{g}^{(1)}$, the extended Lie bracket reads

$$[t^a, s^\mu] = \rho(t^a)^\mu{}_\nu s^\nu\,, \qquad \rho \in \mathbf{Rep}\,\mathfrak{g}^{(0)}\,. \tag{2.1}$$

Even though not strictly necessary, one usually requires ρ to be completely reducible and Lie superalgebras with that property are called *classical*. Since Lie superalgebras that are not classical are fairly complicated to deal with, it will always be tacitly assumed that the considered Lie superalgebra is classical. The crucial step is the introduction of a Lie bracket of two fermionic generators consistent with the structure we obtained so far. The conditions we impose on the Lie bracket are rather restrictive; it should be symmetric and should map to the Lie algebra $\mathfrak{g}^{(0)}$ we started with. Since $\mathfrak{g}^{(1)}$ transforms in some representation ρ, we are hence looking for a symmetric bilinear mapping $\rho\times\rho\to\mathbf{adj}$, where $\mathbf{adj}$ refers to the adjoint representation of $\mathfrak{g}^{(0)}$. Furthermore, our intuition from the theory of Lie algebras suggests that there should be an analogue of the Jacobi identity. In the context of Lie algebras, the Jacobi identity can be understood as an *invariance* condition on the Lie bracket seen as an antisymmetric map $\mathbf{adj}\times\mathbf{adj}\to\mathbf{adj}$,

$$\begin{aligned}\Big[[t^a,t^b],t^c\Big] + \Big[t^b,[t^a,t^c]\Big] &= \Big[\mathrm{ad}(t^a)t^b,t^c\Big] + \Big[t^b,\mathrm{ad}(t^a)t^c\Big]\\ &= \mathrm{ad}(t^a)[t^b,t^c] = \Big[t^a,[t^b,t^c]\Big]\,.\end{aligned} \tag{2.2}$$

Thus it seems reasonable to require the symmetric bilinear map $\rho\times\rho\to\mathbf{adj}$ that ultimately defines the anticommutator of fermionic generators to be invariant as well,

$$\begin{aligned}\Big[[t^a,s^\mu],s^\nu\Big] + \Big[s^\mu,[t^a,s^\nu]\Big] &= [\rho(t^a)^\mu{}_\kappa s^\kappa, s^\nu] + [s^\mu,\rho(t^a)^\nu{}_\kappa s^\kappa]\\ &= \mathrm{ad}(t^a)[s^\mu,s^\nu] = \Big[t^a,[s^\mu,s^\nu]\Big]\,.\end{aligned} \tag{2.3}$$

This yields the generalised Jacobi identity for Lie superalgebras. In fact, it turns out that for generic choices of the representation ρ such a bilinear map does not exist. In those cases in which it exists, the constraints are so restrictive that it is mostly unique.

We summarise that a Lie superalgebra $\mathfrak{g}$ is completely defined by the following data: a Lie algebra $\mathfrak{g}^{(0)}$ that serves as the Lie subalgebra of $\mathfrak{g}$, a $\mathfrak{g}^{(0)}$-representation ρ realised on the vectorspace $\mathfrak{g}^{(1)}$ and a symmetric bilinear invariant map $\rho\times\rho\to\mathbf{adj}$ that lifts to the anticommutator of fermionic generators. Like Lie algebras, Lie superalgebras can be classified [122] (see section 2.1.2). By construction, $\mathfrak{g}$ allows for a grading that will be denoted $|\cdot|$ defined by

$$|x| = \begin{cases} 0 & \text{if } x\in\mathfrak{g}^{(0)}\,,\\ 1 & \text{if } x\in\mathfrak{g}^{(1)}\,.\end{cases} \tag{2.4}$$

As an example, let us consider the simplest nontrivial Lie superalgebra, which is $\mathfrak{sl}(2|1)$. Its bosonic subalgebra is $\mathfrak{g}^{(0)}\simeq\mathfrak{sl}(2)\oplus\mathfrak{u}(1)$. With respect to the adjoint action of $\mathfrak{g}^{(0)}$, the fermionic generators arrange themselves in the representation $\mathbf{2}_{-1}\oplus\mathbf{2}_{+1}$.

The Lie superalgebra $\mathfrak{g}$ can be lifted to its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ defined as a quotient of its tensor algebra,

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/\langle x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y] \rangle \,. \tag{2.5}$$

It is also possible to prove an analogue of the Poincare-Birkhoff-Witt theorem for Lie superalgebras [50] along the same lines as for semisimple Lie algebras. Loosely speaking, it makes sure that once we specify an arbitrary order of the basis generators of $\mathfrak{g}$ and impose that the tensor product of two fermionic generators vanishes, the accordingly ordered tensor products generate a basis of $\mathcal{U}(\mathfrak{g})$. The importance of the theorem is due to the fact that it gives us an explicit form of the basis of $\mathcal{U}(\mathfrak{g})$.

Let us try to generalise important concepts in the theory of Lie algebras to Lie superalgebras in order to eventually define a Killing form on Lie superalgebras. The adjoint representation $\mathrm{ad}\,(\,\cdot\,)$ is defined as in the Lie algebra case, namely it associates with each element of a Lie superalgebra, say $\mathfrak{g}$, a linear map from $\mathfrak{g}$ to itself:

$$\mathrm{ad}\,(x) \equiv [x,\, \cdot\,] \,, \qquad \forall\, x \in \mathfrak{g} \,. \tag{2.6}$$

Let us emphasize that $[\,\cdot\,,\,\cdot\,]$ denotes the graded Lie bracket rather than the purely antisymmetric commutator. We can define a graded Lie bracket among the linear maps $\mathrm{ad}\,(x)$, $x \in \mathfrak{g}$, using the composition of linear maps,

$$[\mathrm{ad}\,(x)\,, \mathrm{ad}\,(y)] = \mathrm{ad}\,(x) \circ \mathrm{ad}\,(y) - (-)^{|x||y|} \mathrm{ad}\,(y) \circ \mathrm{ad}\,(x) \;. \tag{2.7}$$

Then $\mathrm{ad}\,(\,\cdot\,)$ becomes a homomorphism of Lie superalgebras, *i.e.*

$$\mathrm{ad}\,([x, y]) = [\mathrm{ad}\,(x)\,, \mathrm{ad}\,(y)] \,. \tag{2.8}$$

As for Lie algebras, this is in fact the defining property of Lie superalgebra representations (hence in retrospect justifying the name adjoint representation), which will be in detail discussed later.

The next ingredient we will need is a Lie superalgebra analogue of the trace. Recall from linear algebra that the trace (over some vector space V) is *invariant* in the sense that $\mathrm{tr}_V([A, B]_-) = \mathrm{tr}_V(A \circ B - B \circ A) = 0$, where A and B are linear maps of V to itself. Thus we may expect that the analogue in the Lie superalgebra case satisfies a similar invariance condition using the graded Lie bracket instead of the commutator. Indeed, a so called *supertrace* (over $\mathfrak{g}$)[1] can be defined satisfying the condition

$$\mathrm{str}_{\mathfrak{g}}([\mathrm{ad}\,(x)\,, \mathrm{ad}\,(y)]) = \mathrm{str}_{\mathfrak{g}}(\mathrm{ad}\,(x) \circ \mathrm{ad}\,(y)) - (-)^{|x||y|} \mathrm{str}_{\mathfrak{g}}(\mathrm{ad}\,(y) \circ \mathrm{ad}\,(x)) = 0 \,. \tag{2.9}$$

Using the linear map $\gamma(x) \equiv (-1)^{|x|} x$, $x \in \mathfrak{g}$, we can express the supertrace over $\mathfrak{g}$ of a linear map A from $\mathfrak{g}$ to itself in terms of the ordinary trace over the vector subspaces $\mathfrak{g}^{(0)}$

[1] In this context the Lie superalgebra $\mathfrak{g}$ should be treated as a $\mathbb{Z}_2$-graded vector space, $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$.

and $\mathfrak{g}^{(1)}$ as

$$\mathrm{str}_{\mathfrak{g}}(A) = \mathrm{tr}_{\mathfrak{g}}(\gamma \circ A) = \mathrm{tr}_{\mathfrak{g}^{(0)}}(A) - \mathrm{tr}_{\mathfrak{g}^{(1)}}(A)\,. \tag{2.10}$$

In principle, it is possible to define a supertrace over any $\mathbb{Z}_2$-graded vector space, not only $\mathfrak{g}$, and we will reconsider this issue later in section 2.1.3 when discussing general representations of Lie superalgebras.

In analogy with the ordinary Lie algebra case, we can define a Killing form on $\mathfrak{g}$ using the supertrace over the adjoint representation,

$$\kappa(x,y) \equiv \mathrm{str}_{\mathfrak{g}}\left(\mathrm{ad}\,(x) \circ \mathrm{ad}\,(y)\right) = \mathrm{tr}_{\mathfrak{g}}\left(\gamma \circ \mathrm{ad}\,(x) \circ \mathrm{ad}\,(y)\right)\,. \tag{2.11}$$

In the Lie algebra case, the Cartan criterion states that the Killing form is always nondegenerate if and only if the Lie algebra is semisimple. This is not true for Lie superalgebras. For example, for the Lie superalgebra which is of particular interest in the present work, $\mathfrak{psl}(2|2)$, the Killing form vanishes identically or, phrased differently, the dual Coxeter number is equal to zero. However, it can be shown [122, 165] that if $\mathfrak{g}$ is simple, any invariant bilinear form is either zero or nondegenerate and any two invariant forms are proportional to each other. The question whether a bilinear form like that exists at all is more subtle in the general case. At this point, we just state that it exists in all cases of interest to us, particularly for $\mathfrak{psl}(2|2)$.

By construction via the supertrace, the Killing form is invariant and supersymmetric, $\kappa(x,y) = (-1)^{|x||y|}\kappa(y,x)$. There is another quite handy property of the Killing form that should be mentioned. It is *consistent* (with the $\mathbb{Z}_2$-grading), which means that

$$\kappa(x,y) = 0 \quad \text{if } x \in \mathfrak{g}^{(0)}\,,\; y \in \mathfrak{g}^{(1)}\,. \tag{2.12}$$

This is actually quite easy to see if one writes

$$\mathrm{str}_{\mathfrak{g}}(\mathrm{ad}\,(x) \circ \mathrm{ad}\,(y)) = \mathrm{tr}_{\mathfrak{g}^{(0)}}(\mathrm{ad}\,(x) \circ \mathrm{ad}\,(y)) - \mathrm{tr}_{\mathfrak{g}^{(1)}}(\mathrm{ad}\,(x) \circ \mathrm{ad}\,(y)) \tag{2.13}$$

since by our constructive definition of Lie superalgebras it is clear that

$$\mathrm{ad}\,(x)\,\mathrm{ad}\,(y) : \mathfrak{g}^{(0)} \to \mathfrak{g}^{(1)} \qquad \text{and} \qquad \mathrm{ad}\,(x)\,\mathrm{ad}\,(y) : \mathfrak{g}^{(1)} \to \mathfrak{g}^{(0)} \tag{2.14}$$

if $x \in \mathfrak{g}^{(0)}$ and $y \in \mathfrak{g}^{(1)}$. Hence both traces vanish identically. Looking ahead, the consistency of the Killing form in the above sense will be of great importance in the context of Wess-Zumino-Witten models on Lie supergroups in section 4.3.2.

It is possible to define roots pretty much the same way as it is done for Lie algebras (see appendix A.1 for a review). In fact, for basic Lie superalgebras, *i.e.* those that allow for an invariant bilinear form (see section 2.1.2), the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ coincides with the CSA of the bosonic subalgebra $\mathfrak{g}^{(0)}$ and hence it is unique up to conjugation. However, one should now distinguish two sets of roots: those that correspond to roots of the bosonic subalgebra $\mathfrak{g}^{(0)}$ and those that correspond to fermionic generators. The earlier roots are usually said to be *even* while the latter are *odd* roots. Hence the root system decomposes as $\Delta = \Delta^{(0)} \cup \Delta^{(1)}$ into the set of even and odd roots, respectively. The roots of basic Lie superalgebras share many of the properties of Lie algebra roots. Their properties include amongst others (see [122] for a complete list)

1. if α is a root then $-\alpha$ is a root as well,
2. except for $\mathfrak{psl}(2|2)$, the root spaces $\mathfrak{g}_\alpha$ are one-dimensional,
3. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ if and only if $\alpha, \beta, \alpha + \beta \in \Delta$,
4. $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\alpha + \beta \neq 0$, where $(\cdot, \cdot)$ is the invariant form on $\mathfrak{g}$, and
5. if $\alpha \neq 0$ then $n\alpha \in \Delta$ for $n \neq \pm 1$, if and only if α is an odd root of nonzero length, *i.e.* $\alpha \in \Delta^{(1)}$ and $(\alpha, \alpha) \neq 0$. In this case, $n = \pm 2$.

We used that the invariant form restricted to $\mathfrak{h}$ induces a bilinear form on its dual $\mathfrak{h}^*$ by $(\alpha, \beta) = (h_\alpha, h_\beta)$, where h_α is defined by $(h_\alpha, \cdot) = \alpha(\cdot)$. Property 5 is specific to Lie superalgebras and, in contrast to the Lie algebra case, allows for roots that are multiples of other roots. From property 2 it is already suggestive that $\mathfrak{psl}(2|2)$ is special among basic Lie superalgebras. In fact, this can be observed in quite several contexts, *e.g.* when considering automorphisms of Lie superalgebras. The root system of $\mathfrak{psl}(2|2)$ is explicitly constructed in section 2.2.1. Instead of reviewing roots of Lie superalgebras here in detail, we rather stress one important difference: two Borel algebras may not be conjugate to each other in contrast to the case of Lie algebras and hence there are several non-equivalent choices of positive roots Δ^+. For example, we will see in section 2.2.1 that there are three non-equivalent possible positive root systems for $\mathfrak{psl}(2|2)$. Each of these gives rise to a different simple root system $\Delta_{i,0}$, $i \in 1, \ldots, n$, where simple roots are defined to be those positive roots that cannot be written as a sum of positive roots. In order to ensure that there is one-to-one correspondence between conjugacy classes of simple root systems and basic Lie superalgebras, we have to pick one simple root system that is special in some way. The usual choice is to consider the *distinguished* simple root system Δ_0, which is defined to be the up to conjugacy unique simple root system that contains the simple root system $\Delta_0^{(0)}$ of the bosonic subalgebra $\mathfrak{g}^{(0)}$ as a subset. Equivalently, it is the simple root system that has the maximal number of even roots.

It is possible to define Dynkin diagrams for Lie superalgebras as well, where one has to define different nodes for even and odd roots. Even roots are represented by a white node. It turns out useful to distinguish two set of fermionic roots with different properties. First, there are odd roots α of nonzero length with respect to the invariant form, $(\alpha, \alpha) \neq 0$. In that case, $2\,\alpha$ is a root as well. These roots are commonly represented by a black node and finally, the odd roots of zero length are represented by a gray node. Given the Dynkin diagram for a distinguished simple root system, the Dynkin diagram of the bosonic subalgebra can be obtained by simply removing all nodes associated to odd roots and the lines connecting them to other nodes.

2.1.1.1 Lie supergroups

We conclude this section by presenting the analog of the exponential map for Lie superalgebras which yields the notion of a Lie supergroup [14, 121, 122]. Let us start by looking at some Lie superalgebra $\mathfrak{g}$ of dimensions $n+m$ and superdimension $n-m$ and introduce real coordinates x_a, $a = 1, \ldots, n$ and Grassmann-valued coordinates θ_μ, $\mu = 1, \ldots, m$. Since $\theta_\mu\theta_\nu = -\theta_\nu\theta_\mu$, we see that elements $\theta_\mu s^\mu$ and $\theta_\nu s^\nu$ satisfy commutation relations rather than anticommutation relations in the sense that

$$[\theta_\mu s^\mu, \theta_\nu s^\nu]_- = \theta_\mu\theta_\nu [s^\mu, s^\nu]_+ , \tag{2.15}$$

Recall that the $\pm$ subscripts indicate whether we mean the commutator or anticommutator, $[x,y]_\pm \equiv xy \pm yx$. Hence the direct sum of vector spaces $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$, where $\mathfrak{g}^{(1)}$ should now be seen as a vector space over the Grassmann numbers, carries a Lie algebra structure rather than the structure of a Lie superalgebra. This means it is possible to act with the exponential map

$$g(x,\theta) \equiv \exp\left(x_a t^a + \theta_\mu s^\mu\right) . \tag{2.16}$$

The set of $g(x,\theta)$ has group-like structure and is called a *Lie supergroup*. From our definition of $g(x,\theta)$, we see that, like a Lie group is a differentiable manifold, a Lie supergroup is a Grassmannian manifold. From a physicist's perspective, this manifold might be thought of as a possibly curved analog of superspace. Note that results like the Campbell-Baker-Hausdorff-Dynkin formula hold for Lie supergroups as well.

2.1.2 Classification

In this section, an overview of the classification of Lie algebras is given. Since this is a rather vast subject, we refer the interested reader to the literature [121, 122, 165]. In the following, the Lie superalgebra $\mathfrak{g}$ is always assumed to be simple.

Up to now we thought of a Lie superalgebra as a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ with a (generalised) Lie bracket $[\cdot,\cdot]$ that respects that $\mathbb{Z}_2$-grading. However, this is a pretty general point of view and it should be possible to further categorise these Lie superalgebra.

As has already been indicated, the first restriction one usually imposes is that the $\mathfrak{g}^{(0)}$-representation the elements of $\mathfrak{g}^{(1)}$ transform in is completely reducible. These are the *classical Lie superalgebras* and those are at the time of writing the ones that are mainly considered in the literature. However, for completeness, let us state that the Lie superalgebras that are not classical are referred to as Cartan type Lie superalgebras [166].

Looking at classical Lie superalgebras only, we can actually take two paths of classifying them further. Since we already discarded not completely reducible $\mathfrak{g}^{(0)}$-representations for $\mathfrak{g}^{(1)}$, we may ask if $\mathfrak{g}^{(1)}$ is reducible as a $\mathfrak{g}^{(0)}$-representation. If this holds, the Lie superalgebra is said to be of type I. Then it can be shown that $\mathfrak{g}^{(1)}$ decomposes in two direct summands $\mathfrak{g}^{(1)} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{+1}$, where each of the summands yields a $\mathfrak{g}^{(0)}$-representation

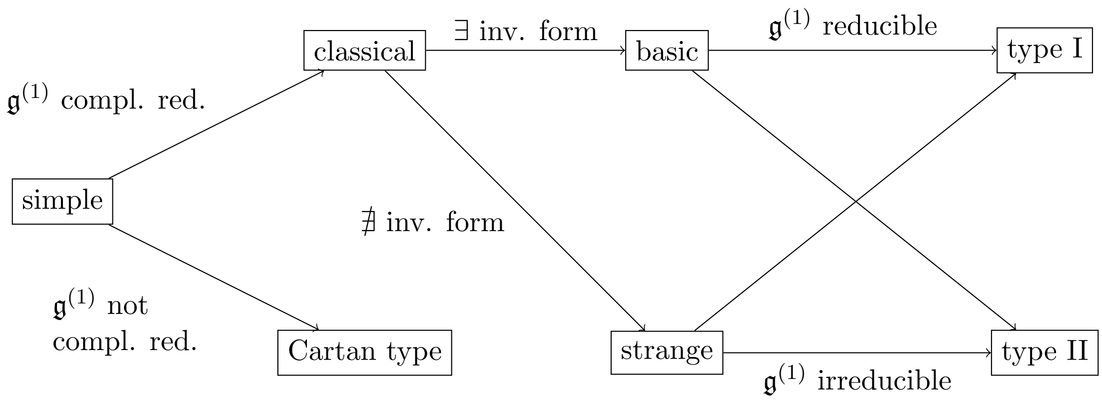


Figure 2.1: *Diagrammatic presentation of the classification of simple Lie superalgebras.*

on its own and $[\mathfrak{g}_{+1}, \mathfrak{g}_{+1}] = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0$. Note that this defines a $\mathbb{Z}$-grading of $\mathfrak{g}$ in the sense that

$$\text{type I:} \quad \mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} , \quad \mathfrak{g}^{(1)} = \mathfrak{g}_{+1} \oplus \mathfrak{g}_{-1} , \quad \text{such that} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} , \tag{2.17}$$

where $\mathfrak{g}_i = 0$ for $|i| > 2$. If $\mathfrak{g}^{(1)}$ is irreducible as a $\mathfrak{g}^{(0)}$-representation then it is called a Lie superalgebra of type II. In this case, a more complicated grading of vector spaces holds,

$$\text{type II:} \quad \mathfrak{g}^{(0)} = \mathfrak{g}_0 \oplus \mathfrak{g}_{+2} \oplus \mathfrak{g}_{-2} , \quad \mathfrak{g}^{(1)} = \mathfrak{g}_{+1} \oplus \mathfrak{g}_{-1} , \quad \text{such that} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} . \tag{2.18}$$

In contrast to type I Lie superalgebras, $\mathfrak{g}_{+1}$ and $\mathfrak{g}_{-1}$ in (2.18) are not $\mathfrak{g}^{(0)}$-representations individually. However, they both form representations of the Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}^{(0)}$.

Another path would have been to ask if for a given Lie superalgebra a non-degenerated invariant bilinear form exists. If it exists, the Lie superalgebra is called *basic* and *strange* otherwise. As a matter of fact, most Lie superalgebras are basic, the only strange exceptions are two series of Lie superalgebras denoted $P(n)$ and $Q(n)$. Of course, all Lie superalgebras for which the Killing form is not identically equal to zero are basic since then the Killing form serves as a non-degenerated invariant bilinear form. In fact, the only basic Lie superalgebras of type I for which the Killing form is vanishing are those that are based on the Lie algebra $\mathfrak{g}^{(0)} = \mathfrak{sl}(n) \oplus \mathfrak{sl}(n)$, which are denoted $\mathfrak{psl}(n|n)$.

2.1.3 Representations

We now turn to the question how to define representations of Lie superalgebra. Since in the remainder of this work we are only interested in Lie superalgebras of type I, particularly $\mathfrak{psl}(2|2)$, we restrict our discussion to these and circumvent the problems that arise for type II Lie superalgebras.

We start by introducing one explicit Lie superalgebra that is particularly accessible and generalises the concept of matrix representations of Lie algebras. Let V be a finite-dimensional $\mathbb{Z}_2$-graded vector space, $V = V^{(0)} \oplus V^{(1)}$, and consider the associated graduation of the space of endomorphisms, *i.e.* linear maps from V to itself, given by

$$\mathrm{End}_i(V) \equiv \{A \in \mathrm{End}(V) \mid AV^{(s)} \subset V^{(s+i)} \; \forall\, s \in \mathbb{Z}_2\} \,. \tag{2.19}$$

the space of endomorphisms becomes a Lie superalgebra if we define a Lie bracket as

$$[A, B] \equiv AB - (-1)^{|A||B|} BA \,. \tag{2.20}$$

In view of its bosonic analog, the resulting Lie superalgebra is often called $\mathfrak{gl}(m|n)$ where m and n are the dimensions of $V^{(0)}$ and $V^{(1)}$, respectively. By definition, the bosonic subalgebra is $\mathfrak{g}^{(0)} = \mathrm{End}_0(V)$. Being a finite-dimensional vector space, V is naturally isomorphic to $\mathbb{R}^m \oplus \mathbb{R}^n$ and therefore, given a choice of basis, $\mathfrak{gl}(m|n)$ has a matrix representation,

$$\mathbf{A} \equiv \left(\begin{array}{c|c} A_{m,m} & B_{m,n} \\ \hline C_{n,m} & D_{n,n} \end{array} \right) \,, \tag{2.21}$$

where the subscripts indicate the dimension of the matrix, *e.g.* $C_{n,m}$ is a $n \times m$ matrix. Note that the adjoint action of $\mathrm{End}_0(V)$ acts on $B_{m,n}$ with $A_{m,m}$ from the left and with $-D_{n,n}$ from the right,

$$\left[\left(\begin{array}{c|c} A_{m,m} & 0 \\ \hline 0 & D_{n,n} \end{array} \right), \left(\begin{array}{c|c} 0 & B_{m,n} \\ \hline 0 & 0 \end{array} \right) \right] = \left(\begin{array}{c|c} 0 & A_{m,m}B_{m,n} - B_{m,n}D_{n,n} \\ \hline 0 & 0 \end{array} \right) \,, \tag{2.22}$$

and vice versa for $C_{n,m}$. Hence the matrices $A_{m,m}$ and $D_{n,n}$ give representations of bosonic subalgebra $\mathrm{End}_0(V)$ acting on the vector space $\mathrm{End}_1(V)$, as it should be by the general constructive definition of a Lie superalgebra.

Following our intuition gained form Lie algebras, we now define (finite-dimensional) representations of Lie superalgebras. In particular, *a (linear) representations of Lie superalgebras* is a homomorphism π of Lie superalgebras from $\mathfrak{g}$ into $\mathfrak{gl}(m|n)$.

At this point it makes sense to reconsider the supertrace. In fact, $\mathfrak{gl}(m|n)$ allows for a quite natural definition of the supertrace,

$$\mathrm{str}\, \mathbf{A} \equiv \mathrm{tr}\, A_{m,m} - \mathrm{tr}\, D_{n,n} \,, \tag{2.23}$$

which can be checked to satisfy the invariance condition, $\mathrm{str}\,[\mathbf{A}, \mathbf{B}] = 0$, due to the invariance of the trace with respect to the usual commutator, $\mathrm{tr}\,[A, B]_- = 0$. Hence, given a representation π of a Lie superalgebra $\mathfrak{g}$, we can pull back the supertrace on $\mathfrak{gl}(m|n)$ to $\mathfrak{g}$,

$$\mathrm{str}_{\,\pi}(x) \equiv \mathrm{str}\, [\pi(x)] \,, \qquad x \in \mathfrak{g} \,. \tag{2.24}$$

This generalises the concept of the supertrace as the trace over the adjoint representation, $\mathrm{str}_{\,\mathfrak{g}}(\mathrm{ad}\,(x))$.

Even though the above approach to representations of Lie superalgebras is quite intuitive, it only works for finite-dimensional representations. However, when we wish to talk about infinite-dimensional representations, the above approach does not readily apply. As in our discussions of Lie algebras in appendix A, it is then more sensible to think about representations more abstractly in terms of $\mathcal{U}(\mathfrak{g})$-modules.

Having a notion of the universal algebra at hand, the definition of representations can actually be phrased analogously to the one for Lie algebras. By definition, Lie superalgebras of type I allow for a grading decomposition of the form $\mathfrak{g} \simeq \mathfrak{g}_{-1} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}_{+1}$. By the Poincaré-Birkhoff-Witt theorem for Lie superalgebras, we know that we can write

$$\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}_{-1})\mathcal{U}(\mathfrak{g}^{(0)})\mathcal{U}(\mathfrak{g}_{+1}) . \tag{2.25}$$

Now suppose we are given a highest weight $\mathcal{U}(\mathfrak{g}^{(0)})$-module $\mathcal{V}(\Lambda)$ of highest weight Λ. When we define $\mathfrak{b} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}_{+1}$, we can lift $\mathcal{V}(\Lambda)$ to become a $\mathcal{U}(\mathfrak{b})$-module by letting elements in $\mathfrak{g}_{+1}$ act trivially on $\mathcal{V}(\Lambda)$, *i.e.* $\mathfrak{g}_{+1}\mathcal{V}(\Lambda) \equiv 0$. A representation of $\mathfrak{g}$ can now be obtained from $\mathcal{V}(\Lambda)$, which should be viewed as at $\mathcal{U}(\mathfrak{b})$-module, by letting the generators of $\mathfrak{g}_{-1}$ act freely on $\mathcal{V}(\Lambda)$ modulo commutation relations. More formally, an $\mathcal{U}(\mathfrak{g})$-module $\mathcal{K}(\Lambda)$ is obtained from $\mathcal{V}(\Lambda)$ by

$$\mathcal{K}(\Lambda) \equiv \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}\mathcal{V}(\Lambda) \equiv \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathcal{V}(\Lambda) \simeq \mathcal{U}(\mathfrak{g}_{-1}) \otimes \mathcal{V}(\Lambda) . \tag{2.26}$$

The representation $\mathcal{K}(\Lambda)$ is called the *Kac module* of highest weight Λ [123]. It can be considered as the most elementary representation in the theory of Lie superalgebras. For completeness, let us state that in full generality, we could have started with any $\mathcal{U}(\mathfrak{b})$-module instead of $\mathcal{V}(\Lambda)$, say $\mathcal{M}$. Then the $\mathcal{U}(\mathfrak{g})$-module $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}\mathcal{M}$ is said to be *induced* from the $\mathcal{U}(\mathfrak{b})$-module $\mathcal{M}$. In this sense, the Kac module $\mathcal{K}(\Lambda)$ is induced from $\mathcal{V}(\Lambda)$ treated as an $\mathcal{U}(\mathfrak{b})$-module.

If $\mathcal{V}(\Lambda)$ is finite-dimensional, so is the Kac module $\mathcal{K}(\Lambda)$ because $\mathcal{U}(\mathfrak{g}_{-1})$ is also finite-dimensional. But $\mathcal{K}(\Lambda)$ might be reducible. In fact, it can be shown [123] that

$$\mathcal{K}(\Lambda) \text{ is irreducible} \qquad \Leftrightarrow \qquad (\Lambda + \rho, \alpha) \neq 0 \quad \forall\, \alpha \in \Delta_1^+ , \tag{2.27}$$

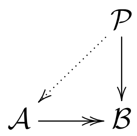
where ρ is constructed as follows. Let ρ_0 denote half the sum of all even positive roots, and ρ_1 half the sum of all odd positive roots,

$$\rho_0 = \tfrac{1}{2} \sum_{\alpha\in\Delta^{(0),+}} \alpha\,, \qquad \rho_1 = \tfrac{1}{2} \sum_{\alpha\in\Delta^{(1),+}} \alpha\,. \tag{2.28}$$

Then $\rho \equiv \rho_1 - \rho_2$ is the *Weyl vector*. A Kac module that is irreducible and its associated highest weight are called *typical*. If it is reducible, it is indecomposable by construction. It is then possible to identify a maximal submodule within $\mathcal{K}(\Lambda)$ and to take the quotient with respect to it in order to obtain an irreducible representation denoted $\mathcal{L}(\Lambda)$. The highest weight Λ for which $\mathcal{K}(\Lambda)$ is reducible as well as the Kac module $\mathcal{K}(\Lambda)$ itself is referred to as being *atypical*. It is quite common to call all modules atypical that either contain an atypical Kac module as a submodule or can be understood as a quotient module of an atypical Kac module, *e.g.* $\mathcal{L}(\Lambda)$ is said to be an atypical irreducible $\mathcal{U}(\mathfrak{g})$-module.

Projectivity

Projectivity in the category of Lie superalgebra representations can be defined the same way as in the case of semisimple Lie algebras (see appendix A.3 for a concise discussion of projectivity in that context). The property of a module, say $\mathcal{P}$, to be *projective* means that for any surjective homomorphism $\mathcal{A} \twoheadrightarrow \mathcal{B}$ and any homomorphism $\pi : \mathcal{P} \to \mathcal{B}$, there exists a homomorphism $\mathcal{P} \to \mathcal{A}$ such that the diagram



(2.29)

commutes. A representation $\mathcal{P}$ is the *projective cover* of $\mathcal{B}$ if it is projective, and if there exists a surjective homomorphism $\pi : \mathcal{P} \to \mathcal{B}$ such that no proper subrepresentation of $\mathcal{P}$ is mapped onto $\mathcal{B}$ by π.[2] In the present work, particularly chapter 6, the situation for atypical representations are of special interest, *i.e.* $\mathcal{B} = \mathcal{L}(\Lambda)$ — for the typical case, where $\mathcal{K}(\Lambda) = \mathcal{L}(\Lambda)$, the projective cover is simply $\mathcal{P}(\Lambda) = \mathcal{L}(\Lambda)$. Any representation $\mathcal{M}$ with head $\mathcal{L}(\Lambda)$ can be mapped onto $\mathcal{L}(\Lambda)$, and the projectivity property for $\mathcal{P}(\Lambda)$ then implies that for any such $\mathcal{M}$ we have a surjection $\mathcal{P}(\Lambda) \twoheadrightarrow \mathcal{M}$. Thus the projective cover $\mathcal{P}(\Lambda)$ is characterised by the property that any representation $\mathcal{M}$ 'headed' by $\mathcal{L}(\Lambda)$ can be obtained by taking a suitable quotient of $\mathcal{P}(\Lambda)$ with respect to a subrepresentation. Hence the projective cover $\mathcal{P}(\Lambda)$ of the irreducible representation $\mathcal{L}(\Lambda)$ is in some sense the largest indecomposable $\mathfrak{g}$-representation that has $\mathcal{L}(\Lambda)$ as its head. Note that this last condition depends on which category of representations we consider.

A quite general construction of projective Lie superalgebra representations, which is often used in the mathematical literature, is presented in appendix B.

2.2 The Lie Superalgebra $\mathfrak{psl}(2|2)$

Having discussed generalities about Lie superalgebra and their representations, we will now look at the specific example of $\mathfrak{psl}(2|2)$. This superalgebra will accompany us throughout the whole work, its physical relevance being due to the fact that it can be thought of as the tangent space of the superspace version of $\mathrm{AdS}_3 \times \mathrm{S}^3$.

2.2.1 The Lie superalgebra

In order to introduce the Lie superalgebra we will almost exclusively deal with in this thesis, let us start with the Lie superalgebra $\mathfrak{gl}(2|2)$. As we have said in section 2.1.3, this space is represented by 4×4 matrices and hence allows for a natural definition of the supertrace. Since the supertrace is invariant, the matrices with vanishing supertrace close among themselves under the action of the (graded) Lie bracket and thus define a

[2]A surjective homomorphism π with this property is sometimes also called *essential*. For more details on the use of projective modules and covers in representation theory see [117].

subalgebra,

$$\mathfrak{sl}(2|2) \equiv \{A \in \mathfrak{gl}(2|2) \mid \operatorname{str} A = 0\}\,. \tag{2.30}$$

The name of this subalgebra is again borrowed form the similar notation in the case of Lie algebras. However, this Lie superalgebra is not yet simple as it contains an ideal spanned by the identity $\mathbb{1}$. This is specific to Lie algebras of type $\mathfrak{sl}(n|n)$. To get a simple Lie algebra we have to take the quotient with respect to this ideal. This yields the Lie superalgebra $\mathfrak{psl}(2|2)$,

$$\mathfrak{psl}(2|2) \equiv \mathfrak{sl}(2|2)\,/\,\langle\mathbb{1}\rangle\,. \tag{2.31}$$

Why should physicists care about this, at this point, fairly abstract algebraic construct? To see this, let us investigate what the bosonic subalgebra reads. By construction it is the bosonic subalgebra of $\mathfrak{gl}(2|2)$, which is $\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)$ subject to the constraints that the supertrace as well as the trace vanishes. If $\mathbf{A}$ is a matrix in $\mathfrak{gl}(2|2)$ which lies in the bosonic subalgebra, *i.e.* it is of the form

$$\mathbf{A} = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)\,, \qquad A, B \in \mathfrak{gl}(2)\,, \tag{2.32}$$

the constraint imply

$$\operatorname{str}\mathbf{A} = \operatorname{tr} A - \operatorname{tr} B = 0\,, \quad \operatorname{tr}\mathbf{A} = \operatorname{tr} A + \operatorname{tr} B = 0 \qquad \Rightarrow \qquad \operatorname{tr} A = \operatorname{tr} B = 0\,. \tag{2.33}$$

So the bosonic subalgebra is $\mathfrak{g}^{(0)} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ which lifts, after choosing an appropriate real form of the Lie algebra, to the Lie group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$. This Lie group is in turn isomorphic to $\mathrm{AdS}_3 \times \mathrm{S}^3$ as differentiable manifolds. So the Lie superalgebra $\mathfrak{psl}(2|2)$, or rather the corresponding supergroup $\mathrm{PSL}(2|2)$, can be understood as introducing a supersymmetric structure to $\mathrm{AdS}_3 \times \mathrm{S}^3$. In other words, $\mathrm{PSL}(2|2)$ is in some sense the superspace analog of $\mathrm{AdS}_3 \times \mathrm{S}^3$.

Coming back to the algebraic properties of $\mathfrak{g} = \mathfrak{psl}(2|2)$, we extract right from the definition that it has six even and eight odd generators, so its dimension is 14 while its superdimension equals -2. In fact, $\mathfrak{g}^{(1)}$ decomposes into two representations that are conjugate to each other (the two "off-diagonal" matrices). Both transform in the $(\mathbf{2},\mathbf{2})$ of $\mathfrak{g}^{(0)}$ as the two-dimensional representation of $\mathfrak{sl}(2)$ is self-conjugate.

There are typically two choices for the generators found in the literature. The one that is more convenient from an algebraic perspective could be referred to as the Chevalley-Serre basis. Here, one starts from the Chevalley-Serre generators for each of the two $\mathfrak{sl}(2)$ that constitute the bosonic subalgebra. These are denoted $J^0, J^\pm$ and $K^0, K^\pm$, respectively. The Cartan subalgebra $\mathfrak{h}$ is spanned by J^0 and K^0. The fermionic generators, denoted by $S_\gamma^{\alpha\beta}$ with $\alpha, \beta, \gamma = \pm$, are chosen such that they are eigenvectors of the adjoint action of $\mathfrak{h}$,

$$\left[J^0, S_\gamma^{\alpha\beta}\right] = \alpha S_\gamma^{\alpha\beta}\,, \qquad \left[K^0, S_\gamma^{\alpha\beta}\right] = \beta S_\gamma^{\alpha\beta}\,. \tag{2.34}$$

Hence they correspond to the fermionic roots of $\mathfrak{psl}(2|2)$. The index γ gives the grading of the generator,

$$S_{\gamma}^{\alpha\beta} \in \mathfrak{g}_{\gamma}\,, \tag{2.35}$$

where the $\mathfrak{g}_{\pm 1}$ are defined as in (2.17). The second possibility is to make use of the isomorphism $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \simeq \mathfrak{so}(4)$. The generators of $\mathfrak{so}(4)$ are usually written as $K^{ab} = -K^{ba}$ with $a, b, c = 1, \ldots, 4$. The two sets of fermionic generators transform in the vector representation **4** under the adjoint action of $\mathfrak{g}^{(0)}$ and are denoted S_{γ}^{a}. Again, the subscript γ tells us the grading of the fermionic generator. Although both choices are equivalent in that they describe the same (complex) Lie superalgebra, it turns out that it is often more convenient to write expressions in this $\mathfrak{so}(4)$-basis. The complete set of commutation relations of $\mathfrak{psl}(2|2)$ in the two basis choices and how they can be converted into each other is in detail described in appendix C.

As has been indicated before, a drawback in the analysis of $\mathfrak{psl}(2|2)$ is the fact that the Killing form vanishes identically. However, there does exists an invariant form which in the $\mathfrak{so}(4)$-basis reads [105]

$$\left(K^{ab}, K^{cd}\right) = -\epsilon^{abcd} \qquad \left(S_{\alpha}^{a}, S_{\beta}^{b}\right) = -\epsilon_{\alpha\beta}\delta^{ab}\,, \qquad \left(K^{ab}, S_{\gamma}^{c}\right) = 0\,. \tag{2.36}$$

Note that it is not only invariant but also supersymmetric and consistent in the sense of eq. (2.12). Furthermore, by our general discussion of simple Lie superalgebras, it is unique up to scalar multiplication. In particular the property of being consistent will be extremely important when constructing affine models based on $\widehat{\mathfrak{psl}}(2|2)_k$ in section 4.3.2. According to the classification of Lie superalgebras, the properties we observed so far make $\mathfrak{psl}(2|2)$ a classical basic Lie superalgebra of type I. However, $\mathfrak{psl}(2|2)$ is special among all the basic Lie superalgebras since it is the only one for which the odd roots have multiplicity higher than one. This is mainly due to the fact that the **2** of $\mathfrak{sl}(2)$, which the fermionic generators transform in, is self-conjugate.

It is actually very instructive to look at the roots and the simple roots in some detail, not only to understand $\mathfrak{psl}(2|2)$ but also to gain an impression how to work with roots in the context of Lie superalgebras. The CSA $\mathfrak{h}$ of $\mathfrak{g} = \mathfrak{psl}(2|2)$ is the span of J^0 and K^0. In order to lift the multiplicity of odd roots, we extend $\mathfrak{h}$ by an generator Y such that

$$\left[Y, \mathfrak{g}^{(0)}\right] = 0 \qquad \text{and} \qquad [Y, x] = \pm x \;\; \forall\, x \in \mathfrak{g}_{\pm 1}\,. \tag{2.37}$$

In other words, the adjoint action of Y is nothing but the $\mathbb{Z}$-grading of Lie superalgebras of type I. Let us denote the extended CSA by $\tilde{\mathfrak{h}}$ and introduce the standard basis $\{\epsilon, \delta, \gamma\}$ on the dual space $\tilde{\mathfrak{h}}^*$ by setting

$$\epsilon\left(J^0\right) = \delta\left(K^0\right) = \gamma\left(Y\right) = 1 \tag{2.38}$$

and letting all other actions of the dual basis elements on generators of $\tilde{\mathfrak{h}}^*$ vanish. The roots of $\mathfrak{psl}(2|2)$ are then given by table 2.1.

even roots $\Delta^{(0)}$	$\pm\epsilon$, $\pm\delta$
odd roots $\Delta^{(1)}$	$\pm\frac{1}{2}(\epsilon+\delta)+\gamma$, $\pm\frac{1}{2}(\epsilon-\delta)+\gamma$, $\pm\frac{1}{2}(\epsilon+\delta)-\gamma$, $\pm\frac{1}{2}(\epsilon-\delta)-\gamma$

Table 2.1: *The roots of* $\mathfrak{psl}(2|2)$ *after extending the CSA to include the* $\mathbb{Z}$*-grading.*

Note that $\Delta^{(0)}$ is just the root system of the bosonic subalgebra $\mathfrak{g}^{(0)}$. Picking a set of positive roots corresponds to choosing a Borel algebra of $\mathfrak{g}$. As it has been explained in section 2.1.1, different choices of Borel algebras of Lie superalgebras are not necessarily conjugate to each other in contrast to the case of Lie algebras. For the bosonic subalgebra we always pick ϵ and δ to be positive, which defines the unique (up to conjugation) postive even root system $\Delta^{(0),+}$. However, there are three non-equivalent possibilities to pick the positive odd root system denoted by $\Delta_i^{(1),+} \subset \Delta^{(1)}$, $1 \leq i \leq 3$. They are given explicitly in table 2.2.

$\Delta_1^{(1),+}$	$\pm\frac{1}{2}(\epsilon+\delta)+\gamma$, $\pm\frac{1}{2}(\epsilon-\delta)+\gamma$
$\Delta_2^{(1),+}$	$\frac{1}{2}(\epsilon+\delta)\pm\gamma$, $\frac{1}{2}(\epsilon-\delta)\pm\gamma$
$\Delta_3^{(1),+}$	$\frac{1}{2}(\epsilon+\delta)\pm\gamma$, $\pm\frac{1}{2}(\epsilon-\delta)-\gamma$

Table 2.2: *The three possible choices of positive odd root systems* $\Delta_i^{(1),+}$, $i = 1, 2, 3$.

The complete positive root systems Δ_i^+ are then obtained by combining each of these possible choices with the positive even root system, $\Delta_i^+ \equiv \Delta^{(0),+} \cup \Delta_i^{(1),+}$. After the projection onto the original CSA $\mathfrak{h}$ as been performed, they can be depicted as in Fig. 2.2. Note that some odd roots have the same image under this projection, and thus some of the positive root systems in Fig. 2.2 contain roots with multiplicity two.

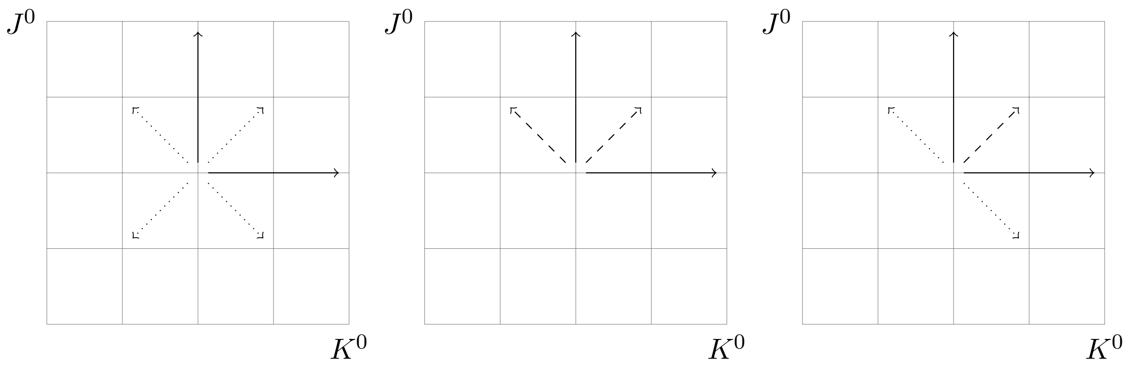


Figure 2.2: *The different possible positive root systems* Δ_i^+ *for* $i = 1, 2, 3$ *of* $\mathfrak{psl}(2|2)$*. Odd roots with simple multiplicity are drawn dotted while odd roots with multiplicity two are drawn dashed.*

Given a positive root system, we can ask for the set of simple roots. Clearly, the different positive root systems give rise to distinct simple root systems and it is interesting to see that they differ by the number of even roots (and, as a consequence, by the number of odd roots). The simple root systems $\Delta_{i,0} \subset \Delta_i^+$ corresponding to the respective choices of positive odd roots as well as their Dynkin diagrams are listed in table 2.3 and depicted in Fig. 2.3. Recall that by definition the distinguished simple root system is the one which contains the simple root system of the bosonic subalgebra $\mathfrak{g}^{(0)}$ as a subset. It is not difficult to see that in the case of $\mathfrak{psl}(2|2)$, the distinguished simple root system is $\Delta_{1,0}$ in table 2.3, which from now on we will simply denote by Δ_0. Note that, not surprisingly, we recover the Dynkin diagram of the bosonic subgroup if we cut out the odd root.

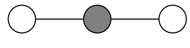
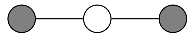
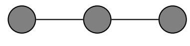
$\Delta_{1,0}$	$\epsilon,\, \delta,\, -\frac{1}{2}(\epsilon+\delta)+\gamma$	
$\Delta_{2,0}$	$\delta,\, \frac{1}{2}(\epsilon-\delta)\pm\gamma$	
$\Delta_{3,0}$	$\frac{1}{2}(\epsilon+\delta)+\gamma,\, \pm\frac{1}{2}(\epsilon-\delta)-\gamma$	

Table 2.3: *The simple root systems $\Delta_{i,0}$ and the corresponding Dynkin diagrams for each of the three choices of positive odd root systems $\Delta_i^{(1),+}$ in table 2.2. As introduced by Kac [122], even roots are associated to white nodes and gray nodes denote odd simple roots of zero length. $\Delta_{1,0}$ is the distinguished simple root system also denoted by Δ_0.*

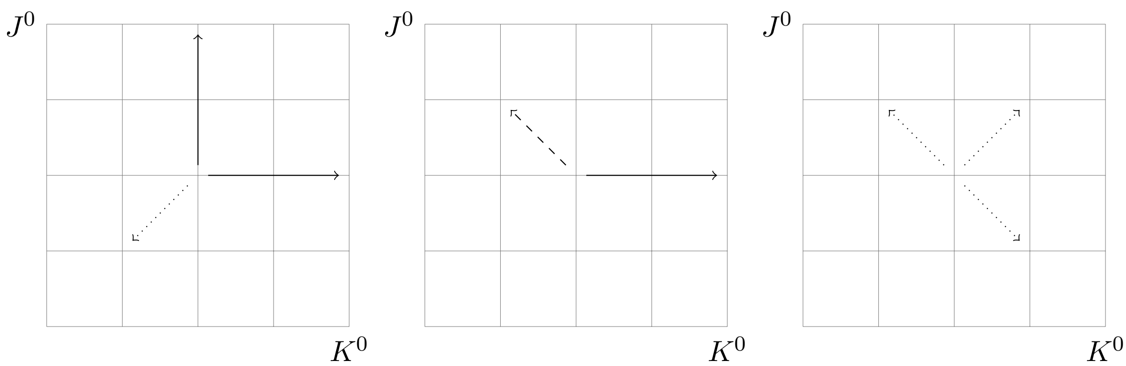


Figure 2.3: *The different possible positive simple systems $\Delta_{i,0}$ for $i = 1, 2, 3$ of $\mathfrak{psl}(2|2)$.*

Having an invariant form at hand, it is straightforward to construct the quadratic Casimir C_2 (cf. appendix C). It is an interesting property of $\mathfrak{psl}(2|2)$ that C_2 does not generate the whole center of the Lie superalgebra $Z(\mathfrak{psl}(2|2))$. In fact, there is an element of order six, denoted W_6, in the universal enveloping algebra $\mathcal{U}(\mathfrak{psl}(2|2))$ that does commute with all elements in $\mathfrak{psl}(2|2)$ (naturally embedded into the universal enveloping algebra). Both together generate the center $Z(\mathfrak{psl}(2|2)) = \langle C_2, W_6\rangle$ [33, 170]. Although we have constructed W_6 during this work, its length forbids to give its explicit form here. Looking ahead, in the affine model, *i.e.* the WZW-model on PSL(2|2), the operator W_6 is associated to a conformal field $W_6(z)$ pretty much the same way as C_2 is substituted by the Sugawara tensor $T(z)$. Both together yield an extension of the conformal symmetry called a $\mathcal{W}(2,6)$-symmetry [35, 38, 39, 194] (cf. section 4.3.2).

2.2.2 Kac modules and irreducible representations

For comparison to string theory on $AdS_3 \times S^3$ we will mainly be interested in representations whose decomposition which respect to the bosonic subalgebra $\mathfrak{g}^{(0)} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ leads to infinite-dimensional discrete series representations with respect to the first $\mathfrak{sl}(2)$ (that describes isometries on AdS_3), and finite-dimensional representations with respect to the second $\mathfrak{sl}(2)$ (that describes isometries to S^3). As in [105] we shall label them by a doublet of half-integers (j_1, j_2) where $j_1 \leq -\frac{1}{2}$ and $j_2 \geq 0$. The cyclic state of the corresponding representation is then characterised by

$$\begin{aligned} J^0 |j_1, j_2\rangle &= j_1 |j_1, j_2\rangle \ , \qquad K^0 |j_1, j_2\rangle = j_2 |j_1, j_2\rangle \ , \\ J^+ |j_1, j_2\rangle &= K^+ |j_1, j_2\rangle = (K^-)^{(2j_2+1)} |j_1, j_2\rangle = 0 \ . \end{aligned} \tag{2.39}$$

Here $J^0, J^\pm$ are the generators of the first $\mathfrak{sl}(2)$ with commutation relations

$$[J^0, J^\pm] = \pm J^\pm \ , \qquad [J^+, J^-] = 2J^0 \ , \tag{2.40}$$

while $K^0, K^\pm$ are the generators of the second $\mathfrak{sl}(2)$ that satisfy identical commutation relations. We denote the corresponding highest weight representation of $\mathfrak{g}^{(0)} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ by $\mathcal{V}(j_1, j_2)$.

Recall from (2.26) that for each representation $\mathcal{V}(j_1, j_2)$ of $\mathfrak{g}^{(0)}$ we can induce a representation of the full Lie superalgebra, the Kac module $\mathcal{K}(j_1, j_2)$, by taking all the modes in $\mathfrak{g}_{+1}$ to act trivially on all states in $\mathcal{V}(j_1, j_2)$, $\mathfrak{g}_{+1}\mathcal{V}(j_1, j_2) = 0$, and by taking the modes in $\mathfrak{g}_{-1}$ to be the fermionic creation operators. The dual construction, where $\mathfrak{g}_{+1}$ are taken to be the fermionic creation operators while $\mathfrak{g}_{-1}$ are annihilation operators, defines the dual Kac module $\mathcal{K}^\vee(j_1, j_2)$. The Kac module as well as the dual Kac module can be in turn decomposed in representations of the bosonic subalgebra, which is illustrated in Fig. 2.4. There we defined the weights

$$\lambda^\alpha_\beta = (j_1 + \tfrac{\alpha}{2}, j_2 + \tfrac{\beta}{2}) \, , \tag{2.41}$$

so *e.g.* $\lambda^+_- = (j_1 + \frac{1}{2}, j_2 - \frac{1}{2})$ and $\lambda^{++} = (j_1 + 1, j_2)$. In fact, this set of weights will be encountered several times throughout this work.

The grading ρ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ induces a grading on the Kac module, where we take all states in $\mathcal{V}(j_1, j_2)$ to have the same grade, say $g \in \mathbb{Z}$. If we want to stress this grade assignment, we shall sometimes write $\mathcal{K}_g(j_1, j_2)$. The states involving one fermionic generator from $\mathfrak{g}_{-1}$ applied to the states in $\mathcal{V}(j_1, j_2)$ then have grade $g - 1$, *etc.*

The characters of Kac modules have been discussed in [105]. However, in this work, we will be only interested in the highest weight states with respect to the bosonic subalgebra

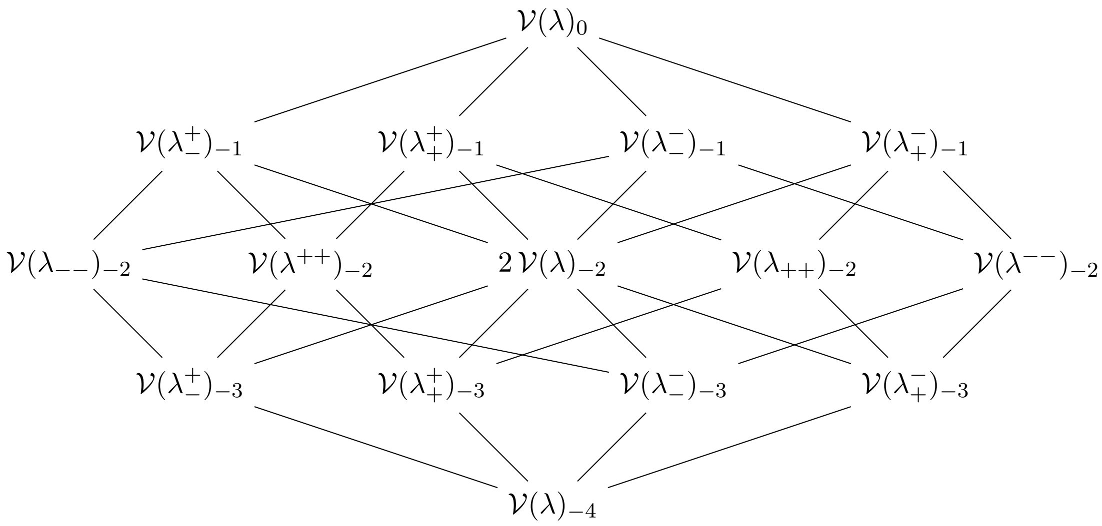


Figure 2.4: *Decomposition of the Kac-module $\mathcal{K}(\lambda)$ into $\mathfrak{g}^{(0)}$-representations. The lines indicate the action of the fermionic generators $\mathfrak{g}^{(1)}$.*

$\mathfrak{g}^{(0)} \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Let us define the branching function[3] of a Kac module $\mathcal{K}(\lambda)$ as

$$K_\lambda(x,y) = \mathrm{Tr}^{(0)}_{\mathcal{K}(\lambda)} \left(x^{J^0} y^{K^0} \right) , \tag{2.42}$$

where the trace $\mathrm{Tr}^{(0)}$ is taken over highest weight states with respect to $\mathfrak{g}^{(0)}$ only. Using Fig. 2.4, it is straightforward to evaluate the trace and we obtain

$$\begin{aligned} K_\lambda(x,y) &= x^{j_1} y^{j_2} \Big(2\,x^{\frac{1}{2}} y^{\frac{1}{2}} + 2\,x^{\frac{1}{2}} y^{-\frac{1}{2}} + 2\,x^{-\frac{1}{2}} y^{\frac{1}{2}} + 2\,x^{-\frac{1}{2}} y^{-\frac{1}{2}} + x + x^{-1} + y + y^{-1} + 4 \Big) \\ &= x^{j_1-1} y^{j_2-1} \left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right)^2 \left(x^{\frac{1}{2}} y^{\frac{1}{2}} + 1 \right)^2 . \end{aligned} \tag{2.43}$$

Note that

$$x^{l_1} y^{l_2} K_\lambda(x,y) = K_{\lambda'}(x,y) \qquad \text{with} \quad \lambda' = (j_1 + l_1, j_2 + l_2) . \tag{2.44}$$

We can use the criterion (2.27) to find the values of j_1 and j_2 for which this is the case. Due to the symmetry of the distinguished positive odd roots of $\mathfrak{psl}(2|2)$, we clearly have $\rho_2 = 0$ and thus the Weyl vector is $\rho = \frac{1}{2}(\epsilon + \delta)$. So using to (2.27) with $\Lambda = j_1\epsilon + j_2\delta$, we find that $\mathcal{K}(j_1, j_2)$ is atypical if and only if

$$\big((j_1 + \tfrac{1}{2})\epsilon + (j_2 + \tfrac{1}{2})\delta, \tfrac{1}{2}(\epsilon + \delta) \big) = 0 \qquad \Rightarrow j_1 - j_2 = 0 \tag{2.45}$$

$$\text{or} \quad \big((j_1 + \tfrac{1}{2})\epsilon + (j_2 + \tfrac{1}{2})\delta, \tfrac{1}{2}(\epsilon - \delta) \big) = 0 \qquad \Rightarrow j_1 + j_2 + 1 = 0 . \tag{2.46}$$

We conclude that, in the case at hand, *i.e.* for $j_1 < 0$ and $j_2 \geq 0$, the Kac module $\mathcal{K}(j_1, j_2)$ is atypical if and only if [105]

$$j_1 + j_2 + 1 = 0 . \tag{2.47}$$

[3]Technically, one would rather refer to the function $K_\lambda(x,y)$ as the generating function of branching rules for the decomposition of $\mathcal{K}(\lambda)$ into $\mathfrak{g}^{(0)}$-representations.

This condition is equivalent to the condition that the quadratic Casimir C_2 vanishes on the Kac module. We shall denote atypical Kac modules by a single index, $\mathcal{K}(j) \equiv \mathcal{K}(-j-1, j)$. The corresponding irreducible representation $\mathcal{L}(j) \equiv \mathcal{L}(-j-1, j)$ is then the quotient of $\mathcal{K}(j)$, where we devide out the largest proper subrepresentation M_1 of $\mathcal{K}(j)$

$$\mathcal{L}(j) = \mathcal{K}(j)/M_1 \ . \tag{2.48}$$

Atypical representations will play an important role in chapter 6, since these are the only representations that matter for the massless string states. The structure of the corresponding irreducible representations (with respect to the action of the bosonic subalgebra $\mathfrak{g}^{(0)}$) is described in Fig. 2.5.

$$\mathcal{L}_0(j \geq \tfrac{1}{2}): \quad \begin{array}{ccc} & \mathcal{V}_0(-j-1, j) & \\ \swarrow & & \searrow \\ \mathcal{V}_{-1}(-j-\tfrac{3}{2}, j-\tfrac{1}{2}) & & \mathcal{V}_{-1}(-j-\tfrac{1}{2}, j+\tfrac{1}{2}) \\ \searrow & & \swarrow \\ & \mathcal{V}_{-2}(-j-1, j) & \end{array} \qquad \mathcal{L}_0(0): \quad \begin{array}{cc} \mathcal{V}_0(-1,0) & \\ & \searrow \\ & \mathcal{V}_{-1}(-\tfrac{1}{2}, \tfrac{1}{2}) \\ & \swarrow \\ \mathcal{V}_{-2}(-1,0) & \end{array}$$

Figure 2.5: *The decompostion of atypical irreducible* $\mathfrak{g}$*-representations into* $\mathfrak{g}^{(0)}$*-components.*

The atypical Kac modules are reducible but not completely reducible. In order to describe their structure it is useful to introduce their composition series. This keeps track of how the various subrepresentations sit inside one another. More precisely, we first identify the largest proper subrepresentation M_1 of $\mathcal{K}(j)$, so that $\mathcal{L}(j) = \mathcal{K}(j)/M_1$ is irreducible; we call the irreducible representation $\mathcal{L}(j)$ the *head* of $\mathcal{K}(j)$. Then we repeat the same analysis with M_1 in place of $\mathcal{K}(j)$, *i.e.* we identify the largest subrepresentation M_2 of M_1 such that M_1/M_2 is a direct sum of irreducible representations. The composition series is then simply the sequence

$$\mathcal{L}(j) = \mathcal{K}(j)/M_1 \to M_1/M_2 \to M_2/M_3 \to \cdots \to M_{n-1}/M_n \ . \tag{2.49}$$

We shall write these composition series vertically, with the head of $\mathcal{K}(j)$ appearing in the first line, M_1/M_2 in the second, *etc.* The representation that appears in the last line of the composition series will be called the *socle*. It is the intersection of all (essential)[4] submodules. The composition series for the atypical Kac modules are shown in Fig. 2.6. Note that for the case of the atypical Kac modules $\mathcal{K}(j)$, both the head and the socle are isomorphic to the irreducible representation $\mathcal{L}(j)$. Finally, the composition series of the dual Kac module $\mathcal{K}^\vee(j)$ only differs by inverting the grading.[5]

We should stress that the Kac module (or dual Kac module) for $j = 0$ is special in the sense that the trivial one-dimensional representation $\mathbf{1} = (0, 0)$ appears in its composition

[4] A submodule U is essential if $U \cap V = 0$ implies $V = 0$ for all submodules V. In the cases of interest to us, this will always be the case.

[5] Note that the irreducible representations are self-dual, *i.e.* $\mathcal{L}_g^\vee(j) = \mathcal{L}_{g-2}(j)$.

series. It is important to note that this irreducible representation has grade -2, even though compared to the structure of the Kac module for the other values of j, one could have guessed that it has grade -1. The operator of grade zero that maps $\mathbf{1}_{-2}$ to $\mathcal{L}_{-2}(0)$ is simply J^-.

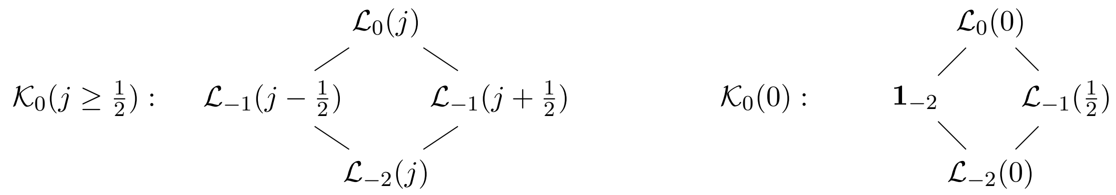


Figure 2.6: *Compostion series of Kac modules. The representation* $\mathbf{1}$ *appearing in* $\mathcal{K}(0)$ *is the trivial, i.e. the one-dimensional, representation of* $\mathfrak{psl}(2|2)$.

2.2.3 Projective covers

For the construction of the space of states of the underlying conformal field theory the projective covers defined in the general context in section 2.1.3 play an important role. As indicated there, the notion of a projective cover depends on the category considered, which we therefore need to specify: we will only work with representations that are completely decomposable under the action of $\mathfrak{g}^{(0)}$. This condition excludes, in particular, the Kac module $\mathcal{K}(0)$, since the arrow between $\mathbf{1}_{g-2}$ and $\mathcal{L}_{g-2}(0)$ is induced by J^-.

The projective cover of an irreducible $\mathcal{L}(j)$ can be constructed by using a generalised BGG duality [95, 195], which basically states that the multiplicity of the Kac module $\mathcal{K}(j')$ in the Kac composition series[6] of $\mathcal{P}(j)$ equals the multiplicity of the irreducible representation $\mathcal{L}(j')$ in the composition series of $\mathcal{K}(j)$. However, two complications arise. First, the generalised BGG duality only holds in situations where the multiplicities with which $\mathcal{L}(j)$ appears in $\mathcal{K}(j)$ is trivial. This problem was solved in [178, 195] by lifting $\mathfrak{psl}(2|2)$ to $\mathfrak{gl}(2|2)$, thereby making g an additional quantum number. Then the two copies of $\mathcal{L}(j)$ in $\mathcal{K}(j)$ can be distinguished. Additionally, the generalised BGG duality has only be shown for finite-dimensional modules so far. In this work, however, we shall assume that it also holds in the infinite-dimensional case, at least as long as j is sufficiently large ($j \geq 1$). This assumption will, *a posterori*, be confirmed by the fact that our analysis leads to sensible results. On the other hand, for $j \leq \frac{1}{2}$, we cannot directly apply BGG duality since $\mathcal{K}(0)$ is not part of our category. The projective covers for $j \leq \frac{1}{2}$ will be constructed in Section. 2.2.3.2, using directly the universal property of projective covers described above.

Applying the BGG duality to the projective covers of $\mathcal{P}(j)$ with $j \geq 1$, and observing that $\mathfrak{g}_{-1}$ generates the states within a Kac module (so that the arrows between different

[6] For the Kac composition series we successively look for submodules such that M_j/M_{j+1} is a direct sum of Kac modules (rather than a direct sum of irreducible modules).

Kac modules must come from $\mathfrak{g}_{+1}$), we obtain from Fig. 2.6 (compare [105])

$$\mathcal{P}_g(j): \qquad \mathcal{K}_g(0) \to \mathcal{K}_{g+1}(j-\tfrac{1}{2}) \oplus \mathcal{K}_{g+1}(j+\tfrac{1}{2}) \to \mathcal{K}_{g+2}(j)\,, \quad j \geq 1\,, \tag{2.50}$$

where g denotes again the $\mathbb{Z}$-grading introduced before, with the head of $\mathcal{P}_g(j)$ having grade g. In terms of the decomposition into irreducibe representations we then find (again using Fig. 2.6) the structure described in Fig. 2.7. Note that the projective cover $\mathcal{P}(j)$ covers both the Kac module $\mathcal{K}(j)$, as well as the dual Kac module $\mathcal{K}^\vee(j)$, since both of them are headed by the irreducible representation $\mathcal{L}(j)$.

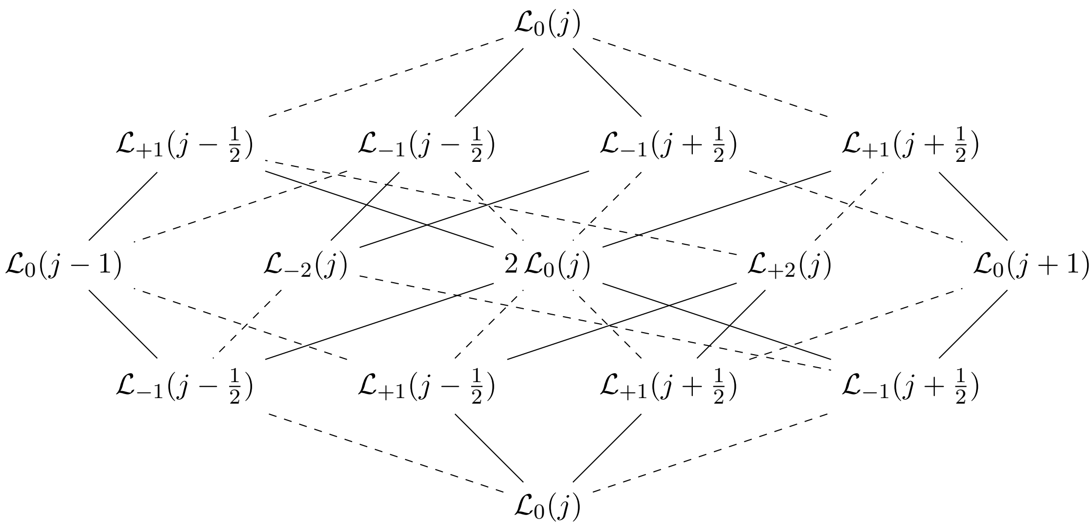


Figure 2.7: *The projective cover $\mathcal{P}_0(j)$ for $j \geq 1$ in terms of irreducible components. Solid lines correspond to mappings decreasing the grading by* 1, *while dashed lines increase it by* 1. *Note that the $\mathbb{Z}$-grading lifts almost the entire degeneracy except for the middle component $\mathcal{L}(j)$ with multiplicity* 2.

2.2.3.1 Homomorphisms

Before we come to discuss the projective covers for small j, let us briefly describe the various homormorphisms between different projective covers. In some sense the 'basic' homomorphisms (from which all other homomorphisms can be constructed by composition) are the homomorphisms (with $\sigma = \pm 1$)

$$s_\sigma^\pm : \mathcal{P}(j) \to \mathcal{P}(j+\tfrac{\sigma}{2})\,, \tag{2.51}$$

where the superscript $\pm$ indicates to which of the two irreducible representations $\mathcal{L}(j+\frac{\sigma}{2})$ the head of $\mathcal{P}(j)$ is mapped to, see Fig. 2.8 for an illustration of the map s_{+1}^+. We shall denote the image of this map by $\mathcal{M}_\sigma^\pm(j)$,

$$\mathcal{M}_\sigma^\pm(j) \equiv s_\sigma^\pm\big(\mathcal{P}(j)\big)\,. \tag{2.52}$$

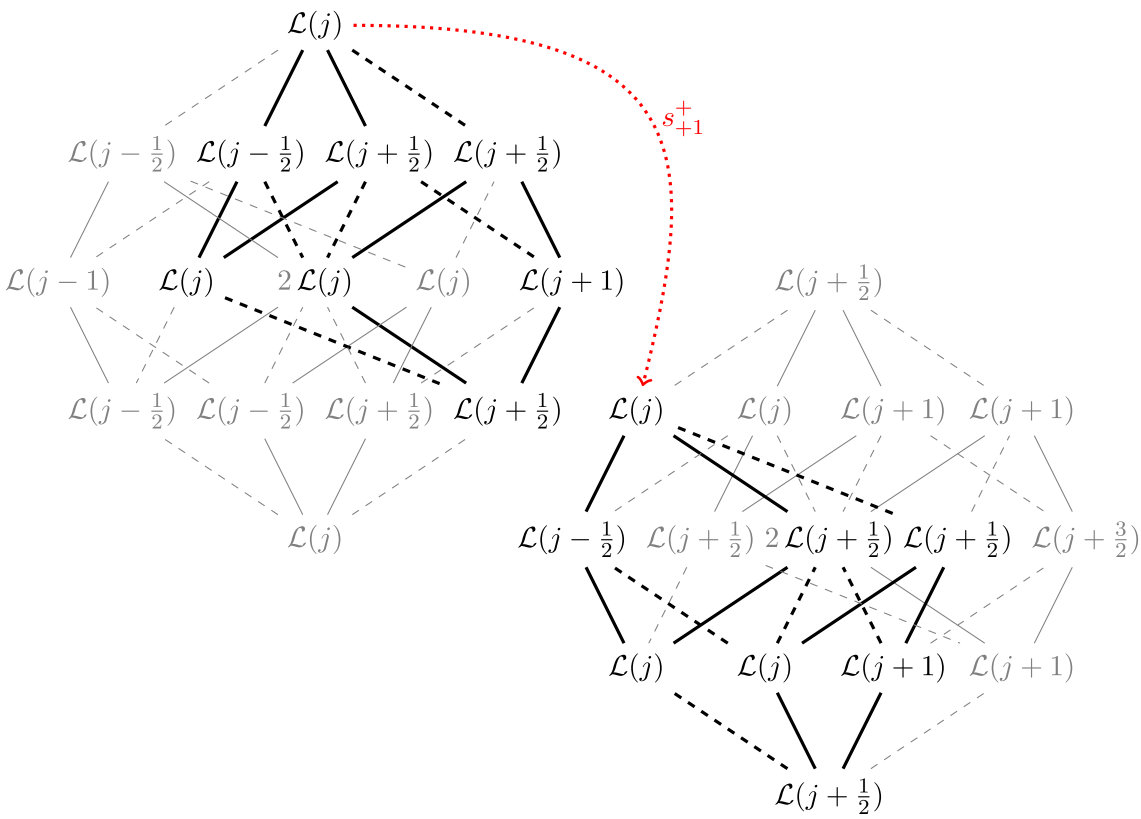


Figure 2.8: *Illustration of the maps* $s^\pm_\sigma : \mathcal{P}(j) \longrightarrow \mathcal{P}(j+\frac{\sigma}{2})$ *using the example of* s^+_{+1}.

Note that it follows from Fig. 2.8 that the kernel of $s^\pm_\sigma$ is isomorphic to $\mathcal{M}^\pm_\sigma(j-\frac{\sigma}{2})$. Thus we have the exact sequence

$$0 \longrightarrow \mathcal{M}^\pm_\sigma(j-\tfrac{\sigma}{2}) \stackrel{\iota}{\longrightarrow} \mathcal{P}(j) \stackrel{s^\pm_\sigma}{\longrightarrow} \mathcal{M}^\pm_\sigma(j) \longrightarrow 0 \ , \tag{2.53}$$

where ι denotes the inclusion $\mathcal{M}^\pm_\sigma(j-\frac{\sigma}{2}) \hookrightarrow \mathcal{P}(j)$.

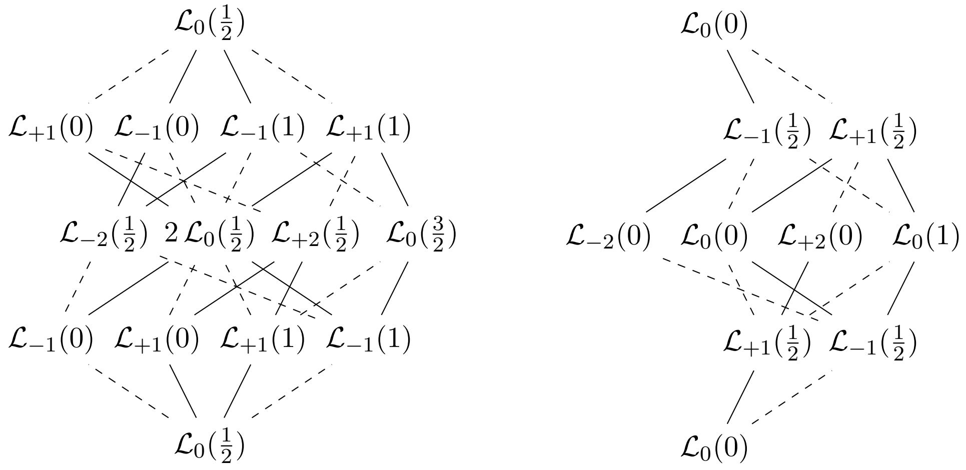
2.2.3.2 The projective covers for $j \leq \frac{1}{2}$

The cases of $\mathcal{P}(j)$ with $j = 0, \frac{1}{2}$ need to be discussed separately, since then BGG duality would give rise to a Kac composition for $\mathcal{P}(j)$ that contains $\mathcal{K}(0)$; however, as we have explained before, $\mathcal{K}(0)$ is not completely reducible with respect to $\mathfrak{g}^{(0)}$, and hence should not arise in our category. We therefore have to work from first principles, and construct $\mathcal{P}(j)$ by the property that any representation with head $\mathcal{L}(j)$ has to be covered by $\mathcal{P}(j)$.[7]

Our strategy to do so is as follows. Since we have already constructed $\mathcal{P}(1)$, we know that the subrepresentations of $\mathcal{P}(1)$ are part of our category. In particular, this is the case

[7]Note that the projective covers for $j = 0$ and $j = \frac{1}{2}$ that were suggested in section 2.4.2 of [105] do not seem to be consistent with these constraints: for their choices of projective covers it is not possible to cover both subrepresentations generated from $\mathcal{L}_{\pm 1}(0)$ at the first level of $\mathcal{P}(\frac{1}{2})$ by $\mathcal{P}(0)$. Indeed, $\mathcal{P}(\frac{1}{2})$ predicts that there is a map from each $\mathcal{L}_{\pm 1}(0)$ to the trivial representation in the middle line of $\mathcal{P}(\frac{1}{2})$, but according to their $\mathcal{P}(0)$, there is only one arrow from $\mathcal{L}(0)$ to the trivial representation at the first level, and this arrow cannot cover both maps in $\mathcal{P}(\frac{1}{2})$.

for the two subrepresentations whose head is $\mathcal{L}(\frac{1}{2})$ at the first level (and that we shall call $\mathcal{M}^{\pm}_{+1}(\frac{1}{2})$ by analogy to the above). The condition that both of them have to be covered by $\mathcal{P}(\frac{1}{2})$ puts then strong constraints on the structure of $\mathcal{P}(\frac{1}{2})$. Assuming in addition that the projective covers are all self-dual then also fixes the lower part of the $\mathcal{P}(\frac{1}{2})$, and we arrive at the representation depicted in Fig. 2.9(a). Note that this just differs from the naive extrapolation of Fig. 2.7 by the fact that the left most irreducible component in the middle line is missing.



(a) The projective cover $\mathcal{P}(\frac{1}{2})$ of $\mathcal{L}(\frac{1}{2})$. (b) The projective cover $\mathcal{P}(0)$ of $\mathcal{L}(0)$.

Figure 2.9: *The projective covers $\mathcal{P}(\frac{1}{2})$ and $\mathcal{P}(0)$.*

The same strategy can be applied to determine the projective cover $\mathcal{P}(0)$ of $\mathcal{L}(0)$. Now $\mathcal{P}(\frac{1}{2})$ contains the two subrepresentations generated by $\mathcal{L}(0)$ in the second line, and $\mathcal{P}(0)$ has to cover both of them. Again, assuming self-duality then leads to the projective cover depicted in Fig. 2.9(b). There is one more subtlety however: in $\mathcal{P}(0)$ it is consistent to have only one copy of $\mathcal{L}(0)$ at grade zero in the middle line. In order to understand why this is so, let us review the reason for the multiplicity of 2 of the corresponding $\mathcal{L}(j)$ representation for $j \geq \frac{1}{2}$. Let us denote the maps leading to and from the relevant $\mathcal{L}(j)$ representation in $\mathcal{P}(j)$ (with $j \geq 1$) by $\phi^{\pm}_{\pm 1}$ and $\bar{\phi}^{\pm}_{\pm 1}$, see Fig. 2.10. It now follows from the fact that $\mathcal{P}(j+\frac{1}{2})$ covers the subrepresentations generated by $\mathcal{L}(j+\frac{1}{2})$ that

$$\bar{\phi}^{-}_{-1} \circ \phi^{-}_{-1} = 0 \qquad \text{and} \qquad \bar{\phi}^{+}_{-1} \circ \phi^{+}_{-1} = 0\ , \tag{2.54}$$

since $\mathcal{P}(j+\frac{1}{2})$ does not contain the representation $\mathcal{L}(j-\frac{1}{2})$ at grade ± 2. The same argument applied to the two subrepresentations generated by $\mathcal{L}(j+\frac{1}{2})$ leads to

$$\bar{\phi}^{-}_{+1} \circ \phi^{-}_{+1} = 0 \qquad \text{and} \qquad \bar{\phi}^{+}_{+1} \circ \phi^{+}_{+1} = 0\ . \tag{2.55}$$

Now suppose that there was only one $\mathcal{L}(j)$ component at grade zero in the middle line of $\mathcal{P}(j)$. Since this one $\mathcal{L}(j)$ representation is in the image of all four $\phi^{\pm}_{\sigma}$, it would follow from the above that it would be annihilated by all four $\bar{\phi}^{\pm}_{\sigma}$. Thus the actual $\mathcal{P}(j)$ would not have any of the four lines represented by $\bar{\phi}^{\pm}_{\sigma}$, and as a consequence would not be

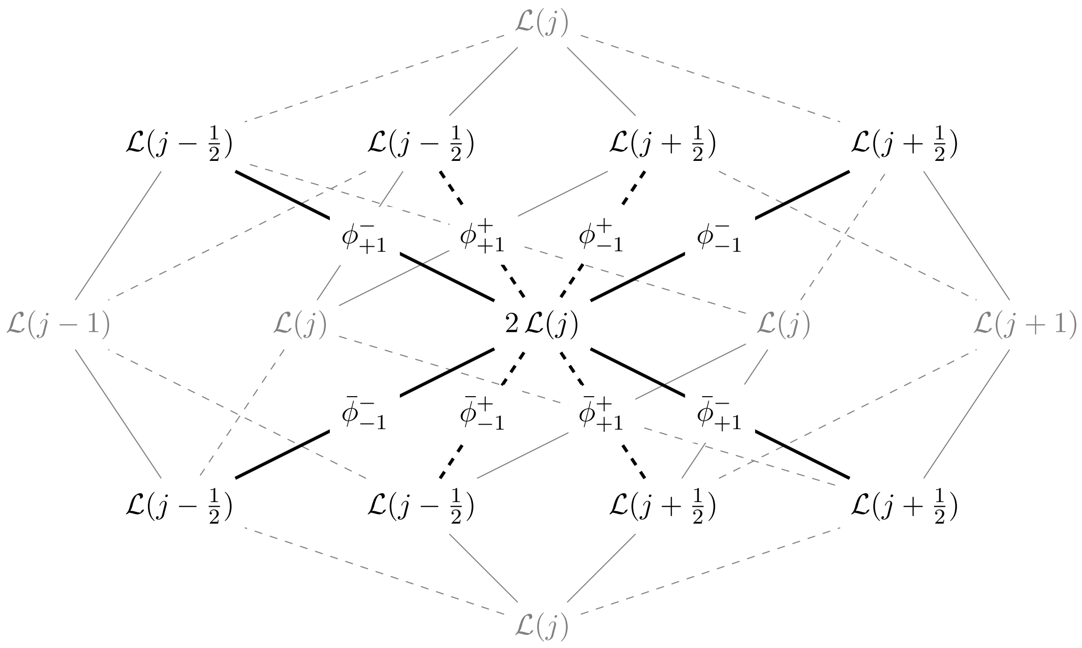


Figure 2.10: *The maps $\phi^{\pm}_{\pm 1}$ and $\bar\phi^{\pm}_{\pm 1}$ in $\mathcal{P}(j)$ with $j \geq 1$.*

self-dual. On the other hand, if the multiplicity is 2, there is no contradiction — and indeed multiplicity 2 is what the BGG duality suggests.

It is clear from Fig. 2.9(a) that the situation for $\mathcal{P}(\frac{1}{2})$ is essentially identical, but for $j = 0$ things are different since we do not have the analogues of $\phi^{\pm}_{+1}$ and $\bar\phi^{\pm}_{-1}$ any longer, see Fig. 2.9(b). Thus the constraints (2.54) and (2.55) are automatically satisfied, and do not imply that the multiplicity of $\mathcal{L}(0)$ at grade zero in the middle line of $\mathcal{P}(0)$ must be bigger than one.

By construction it is now also clear how to extend the definition of $s^{\pm}_{\sigma}$ in (2.51) to $j = \frac{1}{2}$ and $j = 0$ (where for $j = 0$ obviously only $\sigma = +1$ is allowed). Similarly we extend the definition of $\mathcal{M}^{\pm}_{\sigma}(j)$ as in (2.52).

2.3 The Affine Lie Superalgebra $\widehat{\mathfrak{psl}}(2|2)_k$

The affine version of $\mathfrak{psl}(2|2)$ at affine level k is denoted by $\widehat{\mathfrak{g}} = \widehat{\mathfrak{psl}}(2|2)_k$ and its commutation relations are listed in Appendix C. The zero modes of $\widehat{\mathfrak{g}}$ define a $\mathfrak{psl}(2|2)$ subalgebra within $\widehat{\mathfrak{g}}$, commonly referred to as the horizontal subalgebra. In abuse of notation, the horizontal subalgebra will also be called $\mathfrak{g}$. It should be clear from the context whether it refers to the Lie superalgebra $\mathfrak{psl}(2|2)$ or the horizontal subalgebra of $\widehat{\mathfrak{psl}}(2|2)_k$.

Affine Lie superalgebras allow for a decomposition into eigenspaces with respect to the adjoint action of the Virasoro zero-mode L_0 which is obtained from the Sugawara construction. It is the so-called level decomposition, which should not to be confused with the affine level k of the affine Lie superalgebra $\widehat{\mathfrak{g}}$,

$$\widehat{\mathfrak{g}} \quad \simeq \quad \bigoplus_{n\in\mathbb{Z}} \widehat{\mathfrak{g}}_n \,, \tag{2.56}$$

where $\widehat{\mathfrak{g}}_n$ is the vector space spanned by all modes of mode number n. Each direct summand transforms in the adjoint representation $\mathcal{L}_1(\frac{1}{2})$ under the adjoint action of the horizontal subalgebra $\widehat{\mathfrak{g}}_0 = \mathfrak{g}$.

Representations of $\widehat{\mathfrak{g}}$ are most easily generated in a similar way as Kac modules are generated in the case of Lie superalgebras. We start with a set of states that transform as a Kac module $\mathcal{K}(\lambda)$ under the action of the horizontal subalgebra $\mathfrak{g}$. These states are called the affine ground states and $\mathcal{K}(\lambda)$ is the affine ground state representation. In order to expand the set of affine ground states to a representation of the full affine algebra $\widehat{\mathfrak{g}}$, we let all positive modes annihilate the states in $\mathcal{K}(\lambda)$, $\widehat{\mathfrak{g}}_n\,\mathcal{K}(\lambda) = 0$ for $n \geq 1$. The negative modes act freely on the affine ground states modulo commutation relations. The resulting representation is called an affine Kac module of weight λ, $\widehat{\mathcal{K}}(\lambda)$. It contains additional singular vectors if the weight $\lambda = (j_1, j_2)$ satisfies $j_1 - j_2 \in k\,\mathbb{Z}$ or $j_1 + j_2 + 1 \in k\,\mathbb{Z}$ [104]. As we have explained, for applications to string theory, we are interested in the case where $j_1 \leq -\frac{1}{2}$ and $j_2 \in \frac{1}{2}\,\mathbb{N}$. Furthermore, consistency of string theory requires the spin values to be bounded by $-\frac{k}{2} - 1 < j_1$ [65] and $j_2 \leq \frac{k}{2}$. So for $k \geq 3$, the only remaining condition for the appearance of singular vectors is $j_1 + j_2 + 1 = 0$, which means that already the ground state representation contains singular vectors. It has been argued that in this case all affine submodules are generated from singular vectors in the affine ground state representation [104]. Since for massive string states we will see that $j_1 + j_2 + 1 \neq 0$, all affine Kac modules that we will encounter in chapter 7 are irreducible. Therefore we will restrict to these in the remainder of this section.

Like the affine Lie superalgebra $\widehat{\mathfrak{g}}$ itself, affine Kac modules $\widehat{\mathcal{K}}(\lambda)$ allow for a level decomposition as well,

$$\widehat{\mathcal{K}}(\lambda) \quad \simeq \quad \bigoplus_{n\in\mathbb{N}} \widehat{\mathcal{K}}^{(n)}(\lambda) \,, \tag{2.57}$$

where $\widehat{\mathcal{K}}^{(n)}(\lambda)$ is the L_0-eigenspace of eigenvalue $\frac{1}{2k}C_2(\lambda) + n$. Note that $\frac{1}{2k}C_2(\lambda)$ is the L_0-eigenvalue of the ground state representation. Because of $[L_0, \mathfrak{g}] = 0$, each direct summand yields a representation of the horizontal subalgebra $\mathfrak{g}$. Clearly, $\widehat{\mathcal{K}}^{(0)}(\lambda) \simeq \mathcal{K}(\lambda)$ as $\mathfrak{g}$-representations. Since $\widehat{\mathfrak{g}}_1$ transforms in the adjoint representation, the first level of the Kac-modules decomposes under the action of $\mathfrak{g}$ as

$$\begin{aligned}
\widehat{\mathcal{K}}_0^{(1)}(\lambda)\Big|_{\mathfrak{g}} \quad &\simeq \quad \mathcal{L}_1(\tfrac{1}{2}) \otimes \mathcal{K}_0(\lambda) \\
&\simeq \quad 2\,\mathcal{K}_0(\lambda) \oplus \mathcal{K}_0(\lambda^{++}) \oplus \mathcal{K}_0(\lambda^{--}) \oplus \mathcal{K}_0(\lambda_{++}) \oplus \mathcal{K}_0(\lambda_{--}) \\
&\qquad \oplus \bigoplus_{\alpha,\beta=\pm} \left(\mathcal{K}_{-1}(\lambda^\alpha_\beta) \oplus \mathcal{K}_{+1}(\lambda^\alpha_\beta)\right) \,.
\end{aligned} \tag{2.58}$$

Here we have added subscripts to the modules keeping track of the grading for later use. The above result on the tensor product of an finite-dimensional with an infinite-dimensional representation of $\mathfrak{g}$ agrees with the expectation one might have gained from the analysis of tensor products of finite-dimensional representations [104]. Further evidence for the decomposition in (2.58) can be obtained by considering the characters of the representations on both sides of the equation. Let $\mathrm{ch}_{\mathcal{L}(\frac{1}{2})}(x,y)$ and $\mathrm{ch}_{\mathcal{K}(\lambda)}(x,y)$ denote the characters of $\mathcal{L}(\frac{1}{2})$ and $\mathcal{K}(\lambda)$, respectively:

$$\mathrm{ch}_{\mathcal{L}(\frac{1}{2})}(x,y) = \sum_{(\mu_1,\mu_2)\in\Lambda\left(\mathcal{L}(\frac{1}{2})\right)} x^{\mu_1} y^{\mu_2}\,, \tag{2.59}$$

$$\mathrm{ch}_{\mathcal{K}(\lambda)}(x,y) = \underbrace{\frac{x^{j_1}}{1-x^{-1}}\frac{\left(y^{-j_2}-y^{j_2+1}\right)}{1-y}}_{=\mathrm{ch}_{\mathcal{V}(\lambda)}(x,y)} \left(x^{\frac{1}{2}}+y^{\frac{1}{2}}+x^{-\frac{1}{2}}+y^{-\frac{1}{2}}\right)^2\,, \tag{2.60}$$

where $\Lambda\left(\mathcal{L}(\frac{1}{2})\right)$ denotes the set of weights of $\mathcal{L}(\frac{1}{2})$ including multiplicities, *i.e.* the eigenvalues of the $\mathfrak{psl}(2|2)$ generators in appendix C under the adjoint action of J^0 and K^0. These characters satisfy the relation

$$\mathrm{ch}_{\mathcal{L}(\frac{1}{2})}\mathrm{ch}_{\mathcal{K}(\lambda)} = \sum_{\mu\in\Lambda\left(\mathcal{L}(\frac{1}{2})\right)} \mathrm{ch}_{\mathcal{K}(\lambda+\mu)}\,. \tag{2.61}$$

which implies a decomposition of the form (2.58) assuming that $\mathcal{L}(\frac{1}{2})\otimes\mathcal{K}(\lambda)$ is fully reducible. This will be the case if the quadratic Casimir $C_2(\lambda)$ is a non-zero integer, since then j_1 is generically not a half-integer, and therefore $j_1+j_2+n\neq 0$ for all $n\in\mathbb{Z}$. Then all weights appearing in the decomposition (2.58) are typical, which in turn implies full reducibility. We also constructed the cyclic states of any Kac module in the direct summand explicitly.[8]

The $\mathfrak{g}$-representations appearing at the second level can be found by noting that these states are generated from the affine ground states by either acting with bilinears of elements in $\widehat{\mathfrak{g}}_1$ or a single element of $\widehat{\mathfrak{g}}_2$, where the latter is again transforming in the adjoint representation $\mathcal{L}(\frac{1}{2})$ of $\mathfrak{g}$. Bilinears of elements in $\widehat{\mathfrak{g}}_1$ transform as the symmetric part[9] of the tensor product representation $\mathcal{L}(\frac{1}{2})\otimes\mathcal{L}(\frac{1}{2})$, which is [104]

$$\mathrm{Sym}\Big(\mathcal{L}(\tfrac{1}{2})\otimes\mathcal{L}(\tfrac{1}{2})\Big) \quad\simeq\quad \mathcal{K}(0,1)\oplus\mathcal{K}(1,0)\oplus\mathbf{1}\,, \tag{2.62}$$

where $\mathbf{1}$ denotes the trivial representation of $\mathfrak{g}$, associated to the trace. Hence the decomposition of the states at the second level into $\mathfrak{g}$-representations is given by

$$\widehat{\mathcal{K}}^{(2)}(\lambda)\Big|_{\mathfrak{g}} \simeq \Big[\Big(\mathcal{K}(0,1)\oplus\mathcal{K}(1,0)\Big)\otimes\mathcal{K}(\lambda)\Big]\oplus\mathcal{K}(\lambda)\oplus\Big[\mathcal{L}(\tfrac{1}{2})\otimes\mathcal{K}(\lambda)\Big]\,. \tag{2.63}$$

[8] In order to find the cyclic states in $\widehat{\mathcal{K}}^{(1)}(\lambda)$, we have decomposed $\widehat{\mathcal{K}}^{(1)}(\lambda)$ into weight spaces with respect to the zero modes J_0^0 and K_0^0. In contrast to the full space $\widehat{\mathcal{K}}^{(1)}(\lambda)$, these weight spaces are finite-dimensional and hence it is possible to write down an explicit basis for each of them. It is then just a matter of linear algebra to evaluate the cyclic state conditions $J_0^+\psi = K_0^+\psi = S_{+,0}^{\alpha\beta}\psi = 0$ on each of these weight spaces.

[9] Since the representations contain fermionic as well as bosonic states, the symmetric part of the tensor product should be understood as the antisymmetric combination whenever both entries are fermionic.

The tensor products on the right hand side can, of course, be evaluated if necessary. For example, the first direct summand decomposes as

$$\begin{aligned}\Big(\mathcal{K}(0,1)\oplus\mathcal{K}(1,0)\Big)\otimes\mathcal{K}(\lambda)\simeq\ & 12\,\mathcal{K}(\lambda)\oplus\mathcal{K}(\lambda^{4+})\oplus\mathcal{K}(\lambda^{4-})\oplus\mathcal{K}(\lambda_{4+})\oplus\mathcal{K}(\lambda_{4-})\\ &\oplus\ 6\Big(\mathcal{K}(\lambda^{++})\oplus\mathcal{K}(\lambda^{--})\oplus\mathcal{K}(\lambda_{++})\oplus\mathcal{K}(\lambda_{--})\Big)\\ &\oplus\ 2\left(\bigoplus_{\substack{|\alpha|+|\beta|=4\\ |\alpha|,|\beta|\geq 1}}\mathcal{K}(\lambda^{\alpha}_{\beta})\right)\oplus\ 8\left(\bigoplus_{\alpha,\beta=\pm}\mathcal{K}(\lambda^{\alpha}_{\beta})\right)\ .\end{aligned}\tag{2.64}$$

The decomposition of the remaining tensor product in (2.63) has already been given in (2.58). In a similar fashion, the decomposition at higher level can be determined.

Chapter 3

Conformal Field Theory and BRST Quantisation

In this chapter, we will review some aspects of superstring theory and conformal field theory that will become important in the later chapters. It is not intended to present a complete treatment of this vast subject and we rather refer to the standard literature [107, 108, 154, 155]. However, aspects relevant for later discussions will be discussed in detail and notation will be set.

3.1 Conformal Field Theory

We start by reviewing conformal field theory and thereby introducing important tools that will be helpful in the following. Needless to say that in this work, we usually have in mind its application in the context of string theory. Therefore, we specify our discussions to two-dimensional conformal field theories living on the complex plane $\mathbb{C}$.

The literature on conformal field theory is vast. Apart from rather comprehensive treatments [58], there also exist reviews on conformal field theory that aim especially at applications in the context of statistical mechanics [43, 97] or string theory [36]. In fact, most recent textbooks on string theory contain more or less thorough introductions to conformal field theory [128,154,155,177]. Approaches that shed more light on the algebraic and axiomatic structure of conformal field theories can be found in [78, 81, 102, 168].

When considering conformal field theories in this thesis, we will usually work in two-dimensional Euclidean space and choose complex coordinates defined by

$$z = \tfrac{1}{\sqrt{2}}(x + iy)\,, \qquad \bar{z} = \tfrac{1}{\sqrt{2}}(x - iy)\,, \tag{3.1}$$

where x, y are Euclidean coordinates in $\mathbb{R}^2 \simeq \mathbb{C}$. It makes sense to think of z and $\bar{z}$ as being independent and set $z^* = \bar{z}$ if necessary, where $\cdot^*$ denotes complex conjugation. The metric in the new coordinates reads

$$g_{z\bar{z}} = g_{\bar{z}z} = 1\,, \qquad g_{\bar{z}\bar{z}} = g_{zz} = 0\,. \tag{3.2}$$

The partial derivatives take the form

$$\partial \equiv \partial_z = \tfrac{1}{\sqrt{2}}(\partial_x - i\partial_y)\,, \qquad \bar\partial \equiv \partial_{\bar z} = \tfrac{1}{\sqrt{2}}(\partial_x + i\partial_y) \tag{3.3}$$

such that $\partial z = \bar\partial\bar z = 1$ and $\partial\bar z = \bar\partial z = 0$. Sometimes the partial derivative with respect to z is called the holomorphic derivative and the one with respect to $\bar z$ is called the antiholomorphic derivative.

3.1.1 Conformal transformations

In general, conformal transformations are maps $M \to M$, where M is a differentiable manifold, that locally preserve the metric $g_{\mu\nu}$ up to a scalar factor $\Omega(x) \neq 0$, sometimes called the conformal factor,

$$g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(x) = \Omega(x) g_{\mu\nu}(x)\,, \quad x \in M\,. \tag{3.4}$$

In particular, conformal transformations are angle-preserving and they form a group. If we choose M to be flat space, possibly with mixed signature (m,n), the conformal group consists of the Poincaré group, *i.e.* rotations $SO(m,n)$ and translations, as well as the scaling operation $x^\mu \to \lambda x^\mu$, $\lambda \neq 0$, and the special conformal transformations. The latter are given by

$$x^\mu \mapsto \frac{x^\mu - b^\mu x^2}{1 - 2\,b\cdot x + b^2 x^2}\,, \quad b \in \mathbb{R}^{(m,n)}\,. \tag{3.5}$$

If we define the inversion i on $\mathbb{R}^{(m,n)}\backslash\{0\}$ to be a local scaling of the form $i\colon\ x^\mu \mapsto x^\mu/x^2$, the special conformal transformation is the composition $i \circ t_{-b} \circ i$, where t_{-b} is a translation $x^\mu \mapsto x^\mu - b^\mu$.

In two Euclidean dimensions, the conformal group as described above takes a nice form, as it coincides with the Möbius group given by the transformations

$$z \mapsto \frac{az+b}{cz+d}\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2\,. \tag{3.6}$$

These are the conformal transformations that are globally defined in that they describe invertible mappings from the complex plane to itself. However, locally one finds an infinite-dimensional space of generators of conformal transformations. This phenomenon is specific to two dimensions. Explicitly, infinitesimal local conformal transformations are

$$z \mapsto z' = z + \epsilon f(z)\,, \qquad \bar z \mapsto \bar z' = \bar z + \epsilon h(\bar z) \tag{3.7}$$

with a meromorphic function $f(z)$ and an antimeromorphic function $h(\bar z)$. Indeed, using the metric in (3.2), it is easily checked that

$$g_{z'z'} = g_{\bar z'\bar z'} = 0\,, \qquad g_{z'\bar z'} = g_{\bar z' z'} = 1 - \epsilon(f'(z) + h'(\bar z)) \equiv \Omega(z,\bar z)\,. \tag{3.8}$$

Following [58], in order to uncover the Lie algebraic structure associated with local conformal transformations, let us now consider a complex function $\phi : \mathbb{C} \to \mathbb{C}$. Under local conformal transformations, it behaves as

$$\phi'(z', \bar{z}') = \phi(z, \bar{z}) = \phi(z', \bar{z}') - \epsilon f(z)\partial'\phi(z', \bar{z}') - \epsilon h(\bar{z})\bar{\partial}'\phi(z', \bar{z}') \,. \tag{3.9}$$

Performing a Laurent expansion of f and g, the change in the scalar field under local conformal transformations has the form

$$\epsilon\delta\phi = \sum_n \epsilon \left(c_n l_n \phi(z, \bar{z}) + \bar{c}_n \bar{l}_n \phi(z, \bar{z}) \right) \,. \tag{3.10}$$

The c_n can be interpreted as the parameters of the infinitesimal conformal transformations whose generators are

$$l_n = -z^{n+1}\partial \,, \quad \bar{l}_n = -\bar{z}^{n+1}\bar{\partial} \,. \tag{3.11}$$

By construction, this gives us a representation of a Lie algebra, the conformal algebra, and it is straightforward to calculate the corresponding commutation relations,

$$[l_n, l_m] = (n - m)\, l_{m+n} \,, \quad \left[\bar{l}_n, \bar{l}_m\right] = (n - m)\, \bar{l}_{m+n} \,, \qquad \left[l_n, \bar{l}_m\right] = 0 \,. \tag{3.12}$$

The conformal algebra therefore consists of two copies of a simple but infinite-dimensional Lie algebra called Witt algebra. It is instructive to note that it contains an $\mathfrak{sl}(2)$ subalgebra spanned by $\{l_n \,|\, n = 0, \pm 1\}$. This is exactly the Lie algebra that corresponds to global conformal transformations in (3.6). The extension to a larger symmetry algebra in the local case is special to two dimensions and demanding invariance under these symmetries turns out to put strong restrictions on quantum field theories.

3.1.2 Conformal Ward identities and the operator product expansion

After getting an impression of what conformal transformations are, we are now in a position to ask what one means by a field theory, and ultimately a quantum field theory, to be invariant under such transformations [13]. Let us assume for the moment that our theory has an action S. Under infinitesimal transformations the action functional changes as

$$\delta S = \int d^2x \, \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = \int d^2x \, T^{\mu\nu} \delta g_{\mu\nu} \,, \tag{3.13}$$

possibly times a constant subject to convention. By the definition of conformal transformations, $\delta g_{\mu\nu}$ is proportional to the original metric $g_{\mu\nu}$ and hence the variation of S under conformal transformations is proportional to the trace of the energy momentum tensor. We conclude that a theory is conformally invariant at the classical level if the trace of the energy momentum tensor vanishes,

$$\delta S = 0 \quad \text{under conformal transformations} \quad \Leftrightarrow \quad T^{\mu}{}_{\mu} = 0 \,. \tag{3.14}$$

In complex coordinates $z, \bar{z}$, the traceless condition translates to the vanishing of the off-diagonal components,

$$T_{z\bar{z}} = T_{\bar{z}z} = 0\,. \tag{3.15}$$

Since the energy momentum tensor is the Noether current associated to translations, it is conserved. With the vanishing of the off-diagonal components, we find that the remaining components are holomorphic and antiholomorphic, respectively,

$$\bar{\partial}T(z) = \partial\bar{T}(\bar{z}) = 0\,, \tag{3.16}$$

where we introduced the simplified notation $T(z) \equiv T_{zz}(z,\bar{z})$ and $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(z,\bar{z})$ to emphasize that T_{zz} and $T_{\bar{z}\bar{z}}$ are respectively independent of $\bar{z}$ and z.

Let us now go one step further and consider conformally invariant theories at the quantum level. The dynamics of a quantum field theory is determined by a set of fields and the correlation functions between them. Again, for the sake of the argument, let us assume that we have a classical action S that is invariant under some symmetry, which we will eventually take to be the conformal symmetry. Then the correlation functions are given by the path integral including insertions of the fields at the points $x_1, x_2, \dots, x_n$,

$$\langle \Phi_1(x_1)\Phi_2(x_2)\dots\Phi_n(x_n)\rangle = \int \prod_{i=1}^{n} [d\Phi_i]\, \Phi_1(x_1)\Phi_2(x_2)\dots\Phi_n(x_n)e^{-S[\Phi]}\,. \tag{3.17}$$

If we further assume that the path integral measure is invariant under that symmetry as well, we immediately see that the correlation function inherits the same behavior under global symmetry transformations as the product of the classical fields. However, under local symmetries, the classical action changes as $S[\Phi] + \frac{1}{2\pi}\int d^2x\, j^\mu \partial_\mu \epsilon$, where j^μ is the *classically* conserved current. In addition, the fields might transform as $\Phi_i(x) \to \Phi'_i(x)$, so at the insertion points we have

$$\Phi'_i(x_i) - \Phi(x_i) = \epsilon(x)\,(\delta\Phi_i)(x)\,\delta(x-x_i)\,. \tag{3.18}$$

We may now ask how the complete correlation function changes. Plugging in our variations into (3.17), we obtain at first order in ϵ,

$$\begin{aligned}\frac{1}{2\pi}\partial_\mu\langle j^\mu(x)\Phi_1(x_1)\Phi_2(x_2)\dots\Phi_n(x_n)\rangle \\ = -\sum_{i=1}^{n}\delta(x-x_i)\langle\Phi_1(x_1)\Phi_2(x_2)\dots(\delta\Phi_i)(x)\dots\Phi_n(x_n)\rangle\,.\end{aligned} \tag{3.19}$$

This is the Ward identity associated with the classical symmetry generated by j^μ. So far, this holds for quantum field theories in arbitrary dimensions. Let us now focus on the case of a two-dimensional field theory and impose that j is not only classically conserved but also holomorphic. After integrating both sides of the Ward identity over some subset of the complex plane $X \subset \mathbb{C}$ such that only $x_i \in X$, and using Stokes' theorem, we obtain in complex coordinates

$$\frac{1}{2\pi i}\oint_{\partial X} dz\,\langle j(z)\Phi_1(z_1,\bar{z}_1)\ldots\rangle = -\langle(\delta\Phi_1)(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\ldots\rangle\,. \tag{3.20}$$

Since this holds independently of the field insertions at points away from $(z_1, \bar{z}_1)$, it is common to write

$$\frac{1}{2\pi i}\oint_{\partial X} dz\, j(z)\Phi_1(z_1,\bar{z}_1) = -(\delta\Phi_1)(z_1,\bar{z}_1)\,, \tag{3.21}$$

which has to be understood as an operator equation in the sense that the equality holds in any correlation function. In fact, this might be seen as a part of a more general concept in quantum field theories, the one of *operator product expansion* or OPE for short. Loosely speaking, the OPE basically tells us what happens when two field insertions approach each other within correlation functions in that the two approaching fields can be substituted by a sum of single fields located at one of these points,

$$\langle\Phi_i(z,\bar{z})\Phi_j(w,\bar{w})\ldots\rangle = \sum_k \mathcal{C}_{ij}^k(z-w,\bar{z}-\bar{w})\langle\Phi_k(w,\bar{w})\ldots\rangle\,, \tag{3.22}$$

where the fields Φ_i might also be derivatives of some primary fields. The operator product expansion is commonly written without stressing that it should be interpreted as being a part of a correlation function,

$$\Phi_i(z,\bar{z})\Phi_j(w,\bar{w}) = \sum_k \mathcal{C}_{ij}^k(z-w,\bar{z}-\bar{w})\Phi_k(w,\bar{w})\,. \tag{3.23}$$

Due to the existence of powerful tools in complex analysis, the concept of operator product expansions is particularly important in two dimensions and therefore is one of the major elements of two-dimensional conformal field theories. Since the field insertions in correlation functions are taken to be radial-ordered, it should not make a difference in which order we write down the fields in the OPE,

$$\Phi_i(z,\bar{z})\Phi_j(w,\bar{w}) = \epsilon_{ij}\Phi_j(w,\bar{w})\Phi_i(z,\bar{z})\,. \tag{3.24}$$

Here we introduced the possibility of the fields being Grassmann-valued by adding a factor ϵ_{ij} which takes the value -1 if both fields of the OPE are Grassmann-valued and 1 otherwise. Going back to our discussion of symmetries at quantum level, we note that (3.21) actually allows us to make a statement about the OPE of j with Φ_1. The contour integral just extracts the residue of the OPE, that is the first order pole. Hence we know something about the structure of the OPE,

$$j(z)\Phi_1(w,\bar{w}) = \ldots + \frac{-\delta\Phi_1(w,\bar{w})}{z-w} + \ldots\,. \tag{3.25}$$

After this general discussion of symmetries, we can finally investigate how the conformal symmetry shows up at the quantum level. The analysis will be restricted to the holomorphic current $T(z)$ since the analysis for the antiholomorphic component is similar.

We know that $z^nT(z)$ is a classically conserved current for any n. For $n=0$, we find the current associated with translation invariance. Under infinitesimal translations any field behaves as $\delta\Phi(z,\bar{z}) = -\epsilon\partial\Phi(z,\bar{z})$. By our argumentation above, this fixes the first order pole of the OPE of $T(z)$ with $\Phi(w,\bar{w})$,

$$T(z)\Phi_1(w,\bar{w}) = \ldots + \frac{\partial\Phi_1(w,\bar{w})}{z-w} + \ldots\,. \tag{3.26}$$

The rotations and dilatations are more interesting since they correspond to the current $zT(z)$. Due to the additional factor of z multiplying $T(z)$, actually the second order pole of the OPE is picked out by the contour integral in (3.21) plus some contributions from the first order pole. Since we already know the first order pole and because the field $\Phi(w,\bar{w})$ transforms under scaling of the complex coordinate as $(\delta\Phi)(w,\bar{w}) = -h\,\Phi(w,\bar{w}) - w\partial\Phi(w,\bar{w})$, $h\in\mathbb{R}$, we obtain the OPE

$$T(z)\Phi_1(w,\bar{w}) = \ldots + \frac{h\,\Phi_1(w,\bar{w})}{(z-w)^2} + \frac{\partial\Phi_1(w,\bar{w})}{z-w} + (\text{terms nonsingular for } z\to w)\,. \tag{3.27}$$

In principle we could continue this way and determine the poles order by order. However, the family of fields for which all higher poles vanish is especially important and these field are called *Virasoro primary* fields. Summarising, $\Phi(w,\bar{w})$ is Virasoro primary of conformal weight h if its OPE with the energy momentum tensor is

$$T(z)\Phi(w,\bar{w}) \sim \frac{h\,\Phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\Phi(w,\bar{w})}{z-w}\,. \tag{3.28}$$

Here we wrote $\sim$ instead of an equality sign to indicate that we are ignoring nonsingular terms of the OPE.

One may wonder whether the energy momentum tensor itself is a primary field. In order to answer this question, we will argue for the most general form of the OPE of $T(z)$ with itself. First, we note that $T(z)$ has conformal weight two. According to (3.27), this already fixes the poles of order one and two. Furthermore, it can be shown that unitarity of a conformal field theory requires all fields of the theory to have positive conformal weight (except for the vacuum which has $h=0$) [36]. Then, since z^{-1} has conformal weight one, the highest order pole in the OPE is maximally quartic, which in general can be multiplied by a constant. So the TT-OPE reads

$$T(z)T(w) \quad \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,. \tag{3.29}$$

No pole of third order appears because it would not be consistent with the symmetry $T(z)T(w) = T(w)T(z)$. The only possibility for a pole of third order to be consistent with this symmetry would be that it is proportional to the derivative of a field of conformal weight zero. But the only such field is the vacuum $\Omega(z)$, which is translation invariant and thus $\partial\Omega(z) = 0$ holds. The parameter c is called the *central charge* and is possibly the most important parameter in the characterisation of a conformal field theory.

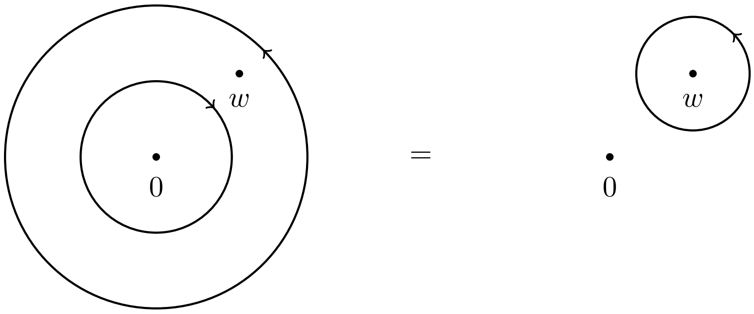


Figure 3.1: *Illustration of the change of contour in (3.32).*

Let us close this section with some final but crucial remarks on the operator product expansion and its connections to Laurent modes. Suppose that we are given a meromorphic field, say $\Psi(z)$. Then one can perform a Laurent expansion,

$$\Psi(z) = \sum_{n\in\mathbb{Z}} \Psi_n z^{-n-h_\Psi} \tag{3.30}$$

where h_Ψ denotes the conformal weight of Ψ. The modes can be extracted from $\Psi(z)$ by an appropriate contour integral,

$$\Psi_n = \oint dz\, z^{n+h_\Psi-1}\Psi(z)\,, \tag{3.31}$$

where we suppressed a factor of $(2\pi i)^{-1}$ as will be often done in the following. One might just consider the definition of the contour integral to contain such a factor. What happens if we insert a mode like this in the correlation function? In particular, let us look at the correlation function

$$\begin{aligned}\langle[\Psi_n,\Phi(w,\bar{w})]\ldots\rangle &= \langle\Psi_n\Phi(w,\bar{w})\ldots\rangle - \langle\Phi(w,\bar{w})\Psi_n\ldots\rangle \\ &= \oint_{C^>} dz\, z^{n+h_\Psi-1}\langle\Psi(z)\Phi(w,\bar{w})\ldots\rangle - \oint_{C^<} dz\, z^{n+h_\Psi-1}\langle dz\,\Psi(z)\Phi(w,\bar{w})\ldots\rangle\,,\end{aligned}$$

where $C^>$ is a contour around $z=0$ with $|z|>|w|$ and similar the contour $C^<$ is a contour around $z=0$ with $|z|<|w|$. The contour has to be chosen like this in order for the mode insertion to be on the left and right side of $\Phi(w,\bar{w})$, respectively, since the correlation function is radial-ordered by definition. By a change of contour (cf. Fig. 3.1), this can be written as

$$\langle[\Psi_n,\Phi(w,\bar{w})]\ldots\rangle = \oint_w dz\, z^{n+h_\Psi-1}\langle\Psi(z)\Phi(w,\bar{w})\ldots\rangle\,. \tag{3.32}$$

The contour is now taken around w and it is assumed that no other insertion points than w are surrounded by the contour. Hence, in an operator sense, the following relationship is obtained:

$$\mathrm{ad}\,(\Psi_n)\,\Phi(w,\bar{w}) \equiv [\Psi_n,\Phi(w,\bar{w})] = \oint_w dz\, z^{n+h_\Psi-1}\Psi(z)\Phi(w,\bar{w})\,. \tag{3.33}$$

Note that this basically defines for every mode Ψ_n a map from the space of fields to itself, which has been denoted $\mathrm{ad}\,(\Psi_n)$. If the reader feels uncomfortable inserting a Laurent mode in the correlation function, he can just treat this as the definition of the adjoint action of the Laurent mode Ψ_n applied to the field $\Phi(w,\bar{w})$. If Φ is itself meromorphic, it can be expanded in a Laurent series as well and so does the new field $\mathrm{ad}\,(\Psi_n)\,\Phi(w)$. By linearity of $\mathrm{ad}\,(\Psi_n)$, we find

$$\begin{aligned}[\Psi_n,\Phi_m] = \mathrm{ad}\,(\Psi_n)\,\Phi_m &= \oint_0 dw\, w^{m+h_\Phi-1}\mathrm{ad}\,(\Psi_n)\,\Phi(w) \\ &= \oint_0 dw\, w^{m+h_\Phi-1}\oint_w dz\, z^{n+h_\Psi-1}\Psi(z)\Phi(w)\,. \end{aligned} \tag{3.34}$$

This way, if all fields are meromorphic, we can extract a Lie algebraic structure from the OPEs.

As a quite important example, we can ask what kind of Lie algebraic structure the Laurent modes of the energy momentum tensor describe. The Laurent modes are typically denoted L_n. Since both fields are meromorphic, we can use the formalism developed in (3.34) to extract this structure. A straightforward calculation yields

$$[L_m,L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}\,. \tag{3.35}$$

This Lie algebra is referred to as *Virasoro algebra.* Note that for $c=0$, we recover the Witt algebra (3.12) that describes infinitesimal conformal transformations on the complex plane, so the Virasoro algebra can be considered as a central extension thereof. Like the Witt algebra, the Virasoro algebra also contains an $\mathfrak{sl}(2)$ subalgebra spanned by $\{L_0, L_{\pm1}\}$. That means that the Lie algebra of global conformal transformations is not centrally extended at the quantum level and stays unchanged.

3.1.3 Vertex operator algebras and the operator-state correspondence

At this point, it makes sense to introduce another concept familiar from quantum field theory. The fields of a quantum field theory are in general assumed to be in one-to-one correspondence with states in some space $\mathcal{H}$, possibly a Hilbert space, in that fields applied to the vacuum (at infinite past) yield the corresponding state. In this spirit, a conformal field in the complex plane, say $\Phi(z,\bar{z})$, evaluated at the origin, generates the associated state Φ from the vacuum Ω,

$$\Phi = \lim_{z,\bar{z}\to 0}\Phi(z,\bar{z})\Omega\,. \tag{3.36}$$

This is a quite general and important concept in two-dimensional conformal field theory called *operator-state correspondence.* When we now restrict ourselves to consider meromorphic conformal field theories, which are the most relevant in the following chapters, it is indeed often useful to think of them as a space of states $\mathcal{H}$ and a map $V(\cdot,z)$ from $\mathcal{H}$ to operator-valued fields on $\mathbb{C}$ such that $V(\Phi,z)\equiv\Phi(z)$ is the conformal field associated with $\Phi\in\mathcal{H}$. The field $V(\Phi,z)$ is also called the *vertex operator* of Φ, a notation that originates from string theory. By the operator-state correspondence, the operator product expansion (3.23) induces an algebraic structure on $\mathcal{H}$, which is called a vertex algebra. If we further assume the existence of a state ω that corresponds to a conformal Noether current $T(z)$ with an OPE as in (3.29), the algebra is called a *vertex operator algebra* [88, 90, 124], sometimes abbreviated by VOA. Let us mention that we implicitly also assumed that $\mathcal{H}$ contains a single vacuum state Ω that is invariant under global conformal transformations and hence corresponds to the identity via the operator-state correspondence, $\Omega(z) = V(\Omega, z) = \mathbb{1}$. For the sake of a concise but clear notation, we stress that $\Phi(z)$ always denotes a conformal field, but when we omit the argument and just write Φ, we mean the corresponding state.

Performing a Laurent expansion of $\Phi(z)$ and using (3.36), it is not difficult to convince oneself that

$$\Phi = \Phi_{-h_\Phi}\Omega \quad \text{and} \quad \Phi_{-h_\Phi+n}\Omega = 0 \quad \text{for all } n \geq 1\,. \tag{3.37}$$

The latter requirement is necessary in order for the limit in (3.36) to be well-defined. The vacuum itself should be invariant under automorphisms of the complex plane, *i.e.* under global conformal transformations. Hence $L_n\Omega = 0$ for $n = 0, \pm 1$. It is possible to show [78, 102] that the operator product expansion can be written as

$$\Psi(z)\Phi(w) = \sum_{n \leq h_\Phi} \{\Psi_n\Phi\}(w)(z-w)^{-n-h_\Psi}\,, \tag{3.38}$$

where $\{\Psi_n\Phi\}(z)$ denotes the conformal field associated to the state $\Psi_n\Phi$. In VOA notation, $\{\Psi_n\Phi\}(z) \equiv V(\Psi_n\Phi, z)$. Thus from the point of view of vertex operator algebras, the OPEs just encode the action of the various modes of Ψ on the vertex operator $\Phi(w)$. However, it is often necessary to know the modes of the vertex operator $\{\Psi_n\Phi\}(w)$ as well. Luckily, this has been worked out [37]. After introducing a shorthand notation $\Phi_{(n)} = \Phi_{n-h_\Phi+1}$, the result reads

$$\{\Psi_{(-n)}\Phi\}_{(-m)} = \sum_{l\geq 0} \binom{m+l-1}{l} \left[\Psi_{(-n-l)}\Phi_{(-m+l)} + \epsilon_{\Psi\Phi}(-1)^{n+1}\Phi_{(-m-n-l)}\Psi_{(l)}\right]\,, \tag{3.39}$$

where $\epsilon_{\Psi\Phi} = -1$ if both Ψ and Φ are fermionic and $\epsilon_{\Psi\Phi} = 1$ otherwise.

An important application of this lemma appears in the context of normal-ordered products. The *normal-ordered product*, denoted as $(\cdot)$, of two meromorphic fields is defined as the nonsingular part of the OPE,

$$(\Psi(z)\Phi(w)) \equiv \Psi(z)\Phi(w) - (\text{terms singular in the limit } z \to w)\,. \tag{3.40}$$

After taking the normal ordered product, clearly the limit $z \to w$ is well-defined. We write

$$(\Psi\Phi)(w) \equiv \lim_{z\to w} (\Psi(z)\Phi(w))\,. \tag{3.41}$$

Using (3.38), we immediately see that $(\Psi\Phi)(w) = \{\Psi_{-h_\Psi}\Phi\}(w)$. What state in $\mathcal{H}$ does this correspond to? This is where formula (3.39) comes in. Since $(\Psi\Phi)(w)$ must have conformal weight $h_\Phi + h_\Psi$, the state associated with $(\Psi\Phi)(w)$ is

$$(\Psi\Phi)_{-h_\Phi - h_\Psi}\Omega = \{\Psi_{-h_\Psi}\Phi\}_{-h_\Phi - h_\Psi}\Omega = \Psi_{-h_\Psi}\Phi_{-h_\Phi}\Omega\,. \tag{3.42}$$

So the normal-ordered product of two vertex operators corresponds to the multi-particle state with two excitations of the respective corresponding states.

With our discussion in 3.1.3 in mind, we can apply the techniques introduced in this section to give an interpretation of Virasoro primary states in terms of representation theory. Let $\Phi(w,\bar{w})$ be a conformal field, not necessarily meromorphic. Using (3.38) and the defintion of Virasoro primaries in terms of its OPE with the (meromorphic) energy momentum tensor $T(z)$ in (3.28), we conclude that the fields $\{L_n\Phi\}(w,\bar{w})$ vanish for all $n \geq 1$, which is equivalent to saying that L_n annihilates Φ. Furthermore, $L_0\Phi = h\Phi$. Hence, by the operator-state correspondence, a Virasoro primary field is associated with a highest weight state of conformal weight h with respect to the Virasoro algebra in $\mathcal{H}$,

$$\Phi(w,\bar{w}) \quad \text{is Virasoro primary} \quad \Leftrightarrow \quad L_0 = h\Phi\,, \quad L_n\Phi = 0 \quad \forall\, n \geq 1 \tag{3.43}$$

Due to this correspondence, we will refer to these states as Virasoro primaries as well. Virasoro primary states may serve as generating states of a Verma module over the Virasoro algebra, which we will call a Virasoro module for short. In fact, one can expect that the space of states $\mathcal{H}$ decomposes into representations of the Virasoro algebra.

Of course, there is much more to say about conformal field theory and the interested reader is referred to the literature in the beginning of this section and references therein. For the present purposes, we stop at this point and present a popular example of a conformal field theory.

3.2 Wess-Zumino-Witten Models

An important source of two-dimensional non-linear σ-models are Wess-Zumino-Witten (WZW) models [92]. The advantage of WZW models in contrast to generic non-linear σ-models is that the target manifold is equipped with a group structure which can be used to highly simplify calculations.

3.2.1 The classical theory

Let us start with an embedding $g(z,\bar{z})$ of some Riemannian manifold W without boundaries into the Lie group G. The derivative $\partial_\mu g(z,\bar{z})$ yields a vector in the tangent space $T_{g(z,\bar{z})}G$ at the point $g(z,\bar{z})$. It can be pulled back to the tangent space at the identity by applying the group inverse $g^{-1}(z,\bar{z})$ either from the left or from the right, which yields two currents

$$\mathcal{J}_\mu(z,\bar{z}) = g^{-1}(z,\bar{z})\partial_\mu g(z,\bar{z})\,, \qquad \mathcal{J}'_\mu(z,\bar{z}) = \partial_\mu g(z,\bar{z})g^{-1}(z,\bar{z}) \tag{3.44}$$

Since the tangent space at the identity is nothing but the corresponding Lie algebra $\mathfrak{g}$, both currents are $\mathfrak{g}$-valued. The associated $\mathfrak{g}$-valued one-forms are in complex coordinates given by

$$\mathcal{J} = \mathcal{J}_z dz + \mathcal{J}_{\bar{z}} d\bar{z}\,, \qquad \mathcal{J}' = \mathcal{J}'_z dz + \mathcal{J}'_{\bar{z}} d\bar{z}\,. \tag{3.45}$$

From a geometric point of view the current $\mathcal{J}$ can be considered as the pullback of the Maurer-Cartan form onto W. We are now in the position of writing down a kinetic term S_{kin} for the non-linear σ-model that respects the Lie group symmetry:

$$S_{\text{kin}}[g] = \gamma \int_W d^2z\, \text{Tr}\left(g^{-1}\partial g g^{-1}\bar{\partial} g\right) = \gamma \int_W \text{Tr}\left(\mathcal{J} \wedge \star\mathcal{J}\right) = \gamma \int_W \text{Tr}\left(\mathcal{J}' \wedge \star\mathcal{J}'\right)\,, \tag{3.46}$$

where Tr refers to the trace with respect to some unitary representation of $\mathfrak{g}$ such that the trace is real. The overall constant $\gamma \in \mathbb{R}$ is yet to be determined. By using the cyclicity of the trace and that

$$\bar{\partial}(gg^{-1}) = (\bar{\partial}g)g^{-1} + g^{-1}\bar{\partial}g = 0 \qquad \Rightarrow \qquad \bar{\partial}g^{-1} = -g^{-1}(\bar{\partial}g)g^{-1} \tag{3.47}$$

we can write the kinetic term in a more compact form,

$$S_{\text{kin}}[g] = \gamma \int d^2z\, \text{Tr}\left(\partial g \bar{\partial} g^{-1}\right)\,. \tag{3.48}$$

Varying this action we obtain

$$\delta S_{\text{kin}}[g] = \int d^2z\, \text{Tr}\left[\delta g\left(g^{-1}\partial\bar{\partial}g g^{-1} - \partial\bar{\partial}g^{-1}\right)\right] \tag{3.49}$$

and hence the equation of motion becomes

$$g^{-1}\partial\bar{\partial}g - (\partial\bar{\partial}g^{-1})g = \partial(g^{-1}\bar{\partial}g) + \bar{\partial}(g^{-1}\partial g) = 0\,. \tag{3.50}$$

In terms of the differential form $\mathcal{J}$ the equation of motion reads

$$\bar{\partial}\mathcal{J}_z + \partial\mathcal{J}_{\bar{z}} = 0 \qquad \Leftrightarrow \qquad d\star\mathcal{J} = 0\,. \tag{3.51}$$

We would like the field theory we are considering to factorise into a holomorphic and an antiholomorphic part, thus extending the Lie group symmetry to a full $G \otimes G$-symmetry locally. The symmetry can be thought of as the left and the right action of G on the embedding field $g(z)$. This means that the two terms in (3.51) have to vanish separately, which is equivalent to additionally demanding that $\mathcal{J}$ is closed, $d\mathcal{J} = 0$. However, this is generically not the case because $\mathcal{J}$ is subject to the Maurer-Cartan equation [113],

$$d\mathcal{J} + \mathcal{J} \wedge \mathcal{J} = 0\,, \tag{3.52}$$

and hence $d\mathcal{J} \neq 0$ for non-Abelian groups G. In [186] it has been shown that this problem can be overcome by adding an additional topological term to the action, now commonly called the *Wess-Zumino-term*,

$$kS_B^{\rm WZ}[\tilde{g}] = \frac{-ik}{24\pi}\int_B \mathrm{Tr}\left(\widetilde{\mathcal{J}}\wedge\widetilde{\mathcal{J}}\wedge\widetilde{\mathcal{J}}\right) . \tag{3.53}$$

Here B is a three-dimensional manifold with $\partial B = W$, $\widetilde{\mathcal{J}} = \tilde{g}^{-1}\partial_\mu \tilde{g}\, dx^\mu$, where $\tilde{g}$ is the extension of g to B, *i.e.* $\tilde{g}|_{\partial B} = g$, and the x^μ are local coordinates on B. The free parameter k, called the *level*, multiplying $S_B^{WZ}[\tilde{g}]$ should be restricted to be an integer as we will argue now. For the sake of the argument, let us assume that $W \simeq \mathrm{S}^2$ and thus $B \simeq \mathrm{D}^3$, the three-dimensional disc, and suppose that we are given two extensions $\tilde{g}_1$ and $\tilde{g}_2$, both mapping B to G. Since we are interested in the dynamics of g only, the physics should be independent of which extension $\tilde{g}_i$ we are considering. This is the case if they contribute equally to the path integral measure, *i.e.* if

$$e^{ikS_B^{WZ}[\tilde{g}_1]} = e^{ikS_B^{WZ}[\tilde{g}_2]} \quad \Rightarrow e^{ik(S_B^{WZ}[\tilde{g}_1]-S_B^{WZ}[\tilde{g}_2])} = 1\,. \tag{3.54}$$

Now consider two copies of B, labelled by subscripts 1 and 2, and let $\tilde{g}_i$ map B_i to G. Demanding the B_i to have opposite orientation, we can glue them together by identifying their boundaries $\partial B_1 = \partial B_2$. The resulting space is $B_1 - B_2 \simeq \mathrm{S}^3$. Since $\tilde{g}_i|_{\partial B_i} = g$ for both $i = 1$ and $i = 2$, they induce a continuous map $\tilde{g}$ from the three-sphere S^3 to the Lie group G by demanding $\tilde{g}|_{B_i} = \tilde{g}_i$. Hence, we can write

$$S_B^{WZ}[\tilde{g}_1] - S_B^{WZ}[\tilde{g}_2] = S_{\mathrm{S}^3}^{WZ}[\tilde{g}] \tag{3.55}$$

It can be shown that the latter expression yields the winding number of the map $\tilde{g}$ multiplied by 2π, hence $S_{\mathrm{S}^3}^{WZ}[\tilde{g}] = 2\pi\nu$ with $\nu \in \mathbb{Z}$ [89, 180]. Loosely speaking, $S_{\mathrm{S}^3}^{WZ}[\tilde{g}]$ counts the number of times the image of $\tilde{g}$ wraps the nontrivial three-cycle in G. So (3.54) only holds if $k \in \mathbb{Z}$ as well. Thus the full action reads

$$S^{\rm WZW} = S_{\rm kin} + kS^{\rm WZ}\,, \quad k \in \mathbb{Z}\,, \tag{3.56}$$

where we simplified the notation by dropping the arguments as well as the subscript in the WZ term. This action has two free parameters, γ and k. Demanding invariance of the whole action under the variation $g \to g + \delta g$ yields the equation of motion

$$\gamma\, d\star\mathcal{J} + \frac{k}{16\pi} d\mathcal{J} = 0\,. \tag{3.57}$$

So, if we choose $\gamma = \frac{k}{16\pi}$, we find that the equation of motion becomes $\partial\mathcal{J}_{\bar{z}} = 0$. The nonlinear σ-model defined by the action

$$S^{\rm WZW} = \frac{k}{16\pi}\int_W \mathrm{Tr}\,(\mathcal{J}\wedge\star\mathcal{J}) + \frac{k}{24\pi}\int_B \mathrm{Tr}\left(\widetilde{\mathcal{J}}\wedge\widetilde{\mathcal{J}}\wedge\widetilde{\mathcal{J}}\right) \tag{3.58}$$

is called the *Wess-Zumino-Witten (WZW) model.* Of course, the Maurer-Cartan equation (3.52) still holds, hence $\bar{\partial}\mathcal{J}_z$ cannot be set to zero as well. So is there no conserved holomorphic current? Well, we have neglected up to now the one-form $\mathcal{J}'$. In fact, $\mathcal{J}_{\bar{z}}$

being antiholomorphic implies that $\mathcal{J}'_z$ is holomorphic because

$$\partial\mathcal{J}_{\bar{z}} = \partial(g^{-1}\bar{\partial}g) = -g^{-1}\partial g g^{-1}\bar{\partial}g + g^{-1}\partial\bar{\partial}g = g^{-1}\bar{\partial}(\partial g g^{-1})g = g^{-1}\bar{\partial}\mathcal{J}'_z g\,. \tag{3.59}$$

We observe that $\partial\mathcal{J}_{\bar{z}}$ and $\bar{\partial}\mathcal{J}'_z$ are connected to each other by an inner automorphism of $\mathfrak{g}$. Thus $\mathcal{J}'_z$ is a conserved holomorphic current whenever $\mathcal{J}_{\bar{z}}$ is conserved. From now on we drop the index of the current and simply refer to $\mathcal{J}_{\bar{z}}$ and $\mathcal{J}'_z$ as the holomorphic and antiholomorphic current, respectively, denoted by

$$\mathcal{J}(z) \equiv -k\mathcal{J}'_z(z)\,, \qquad \bar{\mathcal{J}}(\bar{z}) \equiv -k\mathcal{J}_{\bar{z}}(\bar{z})\,. \tag{3.60}$$

Note that we also rescaled the currents for later convenience.

3.2.2 The current algebra and the Sugawara construction

We have defined two conserved $\mathfrak{g}$-valued currents, one of which is holomorphic and one antiholomorphic. After choosing a basis $\{t^a | a = 1, \ldots, \dim\mathfrak{g}\}$ for the Lie algebra $\mathfrak{g}$, the currents can be expanded in that basis

$$\mathcal{J}(z) = \sum_{a=1}^{\dim\mathfrak{g}} \mathcal{J}^a(z)\kappa_{ab}t^b\,, \qquad \bar{\mathcal{J}}(\bar{z}) = \sum_{a=1}^{\dim\mathfrak{g}} \bar{\mathcal{J}}^a(\bar{z})\kappa_{ab}t^b\,, \tag{3.61}$$

where $\kappa^{ab} = (t^a, t^b)$ are the matrix elements of the Killing form and κ_{ab} are the matrix elements of its inverse, which exists since $\mathfrak{g}$ is semisimple by assumption. The currents generate locally the left and right action of G on the Lie group embedding function $g(z,\bar{z})$ [58]. Since the argument is similar for $\mathcal{J}(z)$ and $\bar{\mathcal{J}}(\bar{z})$, let us look at the holomorphic current only. In its infinitesimal form, the transformation of g generated by $\mathcal{J}^a(z)$ reads

$$g(z,\bar{z}) \to g'(z,\bar{z}) = (1 + \epsilon_a(z)t^a)g(z,\bar{z})\,. \tag{3.62}$$

Hence, the holomorphic current itself changes as

$$\begin{aligned}\delta\mathcal{J} &= -k\delta(\partial g)g^{-1} = -k\partial(\epsilon_a t^a g)g^{-1} + k(\partial g)g^{-1}\epsilon_a t^a \\ &= -k\partial\epsilon_a t^a + \epsilon_a t^a\mathcal{J} - \mathcal{J}\epsilon_a t^a = -k\partial\epsilon_a t^a + \epsilon_a[t^a, \mathcal{J}]\,,\end{aligned} \tag{3.63}$$

where we suppressed the coordinate dependence for readability. Expanding $\mathcal{J}$ in the chosen Lie algebra basis, we obtain

$$\delta\mathcal{J}^a = -k\partial\epsilon_b\kappa^{ab} + i\epsilon_b f^{ab}{}_c\mathcal{J}^c\,, \tag{3.64}$$

where $f^{ab}{}_c$ are the structure constants of $\mathfrak{g}$, $[t^a, t^b] = if^{ab}{}_c t^c$, and Lie algebra indices are raised and lowered by the Killing metric. We may now use the Ward identity (3.21) to conclude that the associated current algebra reads

$$\mathcal{J}^a(z)\mathcal{J}^b(w) \sim \frac{k\kappa^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c \mathcal{J}^c(w)}{z-w}\,, \tag{3.65}$$

$$\bar{\mathcal{J}}^a(\bar z)\bar{\mathcal{J}}^b(\bar w) \sim \frac{k\kappa^{ab}}{(\bar z-\bar w)^2} + \frac{i f^{ab}{}_c \bar{\mathcal{J}}^c(\bar w)}{\bar z-\bar w}\,, \tag{3.66}$$

$$\mathcal{J}^a(z)\bar{\mathcal{J}}^b(w) \sim 0\,. \tag{3.67}$$

Here we also included the result one obtains for the antiholomorphic currents. Let us now concentrate on the holomorphic current $\mathcal{J}(z)$ again. Performing a Laurent mode expansion, $\mathcal{J}^a(z) = \sum_n \mathcal{J}^a_n z^{n-1}$, we see that the commutator of the modes is

$$\begin{aligned}\left[\mathcal{J}^a_m, \mathcal{J}^b_n\right] &= \oint_0 dw\, w^m \oint_w dz\, z^n \mathcal{J}^a(z)\mathcal{J}^b(w) \\ &= i f^{ab}{}_c \mathcal{J}^c_{m+n} + k m \delta_{m+n}\,.\end{aligned} \tag{3.68}$$

Therefore the current algebra is equivalent to the commutation relation of the affine Lie algebra $\widehat{\mathfrak{g}}$.

In order to uncover the conformal symmetry of the WZW model, we have to find the conserved current associated to it. Indeed, given the current algebra, an energy momentum tensor can be constructed [130]. A reasonable guess would be to postulate that the energy momentum tensor structurally agrees with the classical one but with the product of the classical currents substituted with the normal ordered product,

$$T(z) = \alpha\, \kappa_{bc}\Big(\mathcal{J}^b\mathcal{J}^c\Big)(z)\,, \tag{3.69}$$

with a constant α possibly subject to quantum corrections to be determined. In order to fix the constant, we demand that $\mathcal{J}$ is a Virasoro primary field of conformal weight one. The singular part of the OPE of $\mathcal{J}^a(z)$ with $T(w)$ is

$$\begin{aligned}\mathcal{J}^a(z)\kappa_{bc}\Big(\mathcal{J}^b\mathcal{J}^c\Big)(w) \sim \kappa_{bc}\oint_w \frac{dx}{x-w}\bigg(&\frac{k\kappa^{ab}\mathcal{J}^c(w)}{(z-x)^2} + \frac{i f^{ab}{}_d}{z-x}\bigg[\frac{k\kappa^{dc}}{(x-w)^2} + \frac{i f^{dc}{}_e \mathcal{J}^e(w)}{x-w} \\ &+ \Big(\mathcal{J}^d\mathcal{J}^c\Big)(w)\bigg] + \frac{k\kappa^{ac}\mathcal{J}^b(w)}{(z-w)^2} + \frac{i f^{ac}{}_d\big(\mathcal{J}^b\mathcal{J}^d\big)(w)}{z-w}\bigg) \\ = \frac{2\,k\mathcal{J}^a(w) - \kappa_{bc}f^{ab}{}_d f^{dc}{}_e \mathcal{J}^e(w)}{(z-x)^2} &+ \frac{i f^a{}_{cd}\left[\big(\mathcal{J}^d\mathcal{J}^c\big)(w) + \big(\mathcal{J}^c\mathcal{J}^d\big)(w)\right]}{z-w}\end{aligned} \tag{3.70}$$

The first order pole vanishes because it is a contraction of an antisymmetric with a symmetric tensor. Making use of the Lie algebraic relation

$$-\kappa_{bd}\Big(i f^{ad}{}_c\Big)\Big(i f^{cb}{}_a\Big) = \kappa_{bd} f^{ad}{}_c f^{cb}{}_a = -2h^\vee \dim\mathfrak{g}\,, \tag{3.71}$$

where $h^\vee$ is the dual Coxeter number, *i.e.* the quadratic Casimir element of the adjoint representation divided by two, the OPE simplifies to

$$\mathcal{J}^a(z)T(w) \sim 2\alpha\,(k+h^\vee)\frac{\mathcal{J}^a(z)}{(z-w)^2}\,. \tag{3.72}$$

Therefore we see that in order for $\mathcal{J}^a(z)$ to be a (Virasoro primary) field of conformal weight one, we have to set $\alpha = (2k+2h^\vee)^{-1}$. Hence,

$$T(z) = \frac{1}{2(k+h^\vee)}\kappa_{bc}\Big(\mathcal{J}^b\mathcal{J}^c\Big)\,. \tag{3.73}$$

However, we still have to check that $T(z)$ generates a Virasoro algebra to uncover the claimed conformal symmetry. Indeed, using that we fixed α such that $\mathcal{J}^a(z)$ is primary and therefore its OPE with $T(z)$ becomes rather simple, one can show along the same lines that [58]

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{3.74}$$

with central charge $c = \frac{k\dim\mathfrak{g}}{(k+h^\vee)}$. The energy momentum tensor we have constructed out of the currents is often referred to as *Sugawara tensor.* We conclude that the WZW model yields indeed a conformal field theory with additional $G\otimes G$ symmetry. The Sugawara tensor is a holomorphic field and thus one can perform a Laurent expansion as usual for the energy momentum tensor of a conformal field theory,

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\,. \tag{3.75}$$

However, using (3.39), we can give a explicit expression of the Virasoro modes in terms of affine modes,

$$L_n = \frac{1}{2\,(k+h^\vee)}\sum_{m\in\mathbb{Z}}\kappa_{ab}\,:\!\mathcal{J}^a_{n-m}\mathcal{J}^b_m\!:\,, \tag{3.76}$$

where $:\cdot:$ denotes creation-annihilation ordering, *i.e.* the modes are arranged according to their mode number in increasing order. For example, $:\!\mathcal{J}_1\mathcal{J}_{-1}\!:= \mathcal{J}_{-1}\mathcal{J}_1$.

3.2.3 Primary states

Up to now, we have only discussed the symmetries of the WZW model and their respective currents. However, in general there are more fields contributing to the spectrum of the theory. Since the symmetry currents of the WZW model in (3.65) - (3.67) are either holomorphic or antiholomorphic, we locally have a $\widehat{\mathfrak{g}}_k \otimes \widehat{\mathfrak{g}}_k$ symmetry underlying the WZW model. Thus the space of states should take the form

$$\mathcal{H} = \bigoplus_{ij} m_{ij}\,\mathcal{H}_i\otimes\bar{\mathcal{H}}_j\,, \tag{3.77}$$

where $\mathcal{H}_i$ and $\bar{\mathcal{H}}_j$ are unitary representations of $\widehat{\mathfrak{g}}_k$, respectively. The matrix m_{ij} encodes the multiplicities and should be chosen such that the theory is modular invariant [58].

For now, we will consider the holomorphic sector only. Let $\{t^a|a=1,\dots,n\}$ denote the generators of the Lie algebra $\mathfrak{g}$. A state Φ_ρ in the holomorphic sector of the WZW model on G is defined to be primary if

$$\mathcal{J}^a_n\Phi_\rho = 0\,, \qquad \mathcal{J}^a_0\Phi = \rho(t^a)\Phi_\rho\,, \tag{3.78}$$

for $n \geq 1$ and any representation ρ of $\mathfrak{g}$. By the operator-state correspondence of a conformal field theory, this can be equivalently written as

$$\mathcal{J}^a(z)\Phi_\rho(w) \quad \sim \quad \frac{\rho(t^a)\Phi_\rho(w)}{z-w}\,, \tag{3.79}$$

where $\Phi_\rho(z)$ is the vertex operator associated to Φ_ρ. The definition of primary states implies that they transform in representations of $\mathfrak{g}$ with respect to the action of the zero modes. A $\widehat{\mathfrak{g}}$-module is generated from such a primary $\mathfrak{g}$-representation by the action of the negative modes. When we decompose the affine algebra as a vector space into negative, positive and zero-modes, $\widehat{\mathfrak{g}} = \mathfrak{g}^< \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^>$, and choose a highest weight representation $\mathcal{V}(\lambda)$ of $\mathfrak{g}^0$, we can write the associated $\widehat{\mathfrak{g}}$-module $\mathcal{M}_{\mathcal{V}(\lambda)}$ as

$$\mathcal{M}_{\mathcal{V}(\lambda)} = \mathcal{U}(\widehat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{g}^0\oplus\mathfrak{g}^>)} \mathcal{V}(\lambda)\,, \tag{3.80}$$

where we have taken $\mathcal{V}(\lambda)$ to be annihilated by all positive modes, $\mathfrak{g}^>\mathcal{V}(\lambda) = 0$.

When we think of the current algebra (3.65) as an affine Lie algebra, we note that the modes of the Sugawara tensor are actually elements of the universal enveloping algebra $\mathcal{U}(\widehat{\mathfrak{g}})$ of the affine Lie algebra $\widehat{\mathfrak{g}}$. So due to (3.80), the $\widehat{\mathfrak{g}}$-module $\mathcal{M}$ can be decomposed into Virasoro modules. In particular, basically by construction of the Sugawara tensor, the primary states are also Virasoro primaries. This is clear from the expression of the Virasoro modes (3.76). Let us also note that the action of L_0 on primary states becomes quite simple,

$$L_0\Phi_{\mathcal{V}(\lambda)} = \frac{1}{2\,(k+h^\vee)}C_2(\lambda)\Phi_{\mathcal{V}(\lambda)}\,, \tag{3.81}$$

where $C_2(\lambda)$ is the eigenvalue of the quadratic Casimir evaluated on the $\mathfrak{g}$-representation $\mathcal{V}(\lambda)$. Hence the conformal weight of $\Phi_{\mathcal{V}(\lambda)}$ is $\frac{1}{2\,(k+h^\vee)}C_2(\lambda)$.

3.2.4 The SU(2) WZW model

As an example, let us consider the SU(2) WZW model. From the point of view of nonlinear σ-models, this corresponds to strings moving on S^3, which is isomorphic to SU(2) as differentiable manifolds. It is possibly the best understood non-Abelian WZW model and hence serves as a good example of the power of an additional affine symmetry in a conformal field theory. The reason that makes this WZW model particularly easy to handle is the fact that SU(2) is compact and hence the unitary representations are finite-dimensional. We will loosely follow the discussion in [36] in this section.

In an appropriate basis of $\mathfrak{su}(2)$, the current algebra takes the form

$$J^0(z)J^\pm(w) \sim \frac{\pm J^\pm(w)}{z-w}\,, \qquad J^+(z)J^-(w) \sim \frac{k}{(z-w)^2} + \frac{2\,J^0(w)}{z-w}\,,$$
$$J^0(z)J^0(w) \sim \frac{\frac{k}{2}}{(z-w)^2}\,, \tag{3.82}$$

and thus the corresponding commutation relations of $\widehat{\mathfrak{su}}(2)_k$ are

$$\left[J^0_m, J^\pm_n\right] = \pm J^\pm_{m+n}\,, \quad \left[J^+_m, J^-_n\right] = 2\,J^0_{m+n} + k\,m\,\delta_{m+n}\,, \quad \left[J^0_m, J^0_n\right] = \tfrac{k}{2} m \delta_{m+n}\,. \tag{3.83}$$

The central extensions as given above are consistent with the Jacobi identity. The Killing form and its inverse are given by

$$\kappa^{+-} = \kappa_{+-} = 1\,, \quad \kappa^{00} = \tfrac{1}{2}\,, \quad \kappa_{00} = 2 \tag{3.84}$$

with all other matrix elements vanishing. Since the quadratic Casimir of a spin j representation of $\mathfrak{su}(2)$ is given by $2j(j+1)$ and the adjoint representation is the spin 1 representation, we conclude that the dual Coxeter number is $h^\vee = 2$. Alternatively, one could have used (3.71) to determine $h^\vee$, where one has to be careful about the appearances of imaginary units. Now it is straightforward using (3.73) to give the energy momentum tensor,

$$T(z) = \frac{1}{2(k+2)}\left[2\left(J^0J^0\right)(z) + \left(J^+J^-\right)(z) + \left(J^-J^+\right)(z)\right]\,. \tag{3.85}$$

With the affine algebra $\mathfrak{su}(2)$ in (3.83) at hand, let us discuss allowed representations. According to our definition of primary states, we are looking for states

$$J^0_n\Phi_j = J^\pm_n\Phi_j = 0 \quad \text{for } n \geq 0\,. \tag{3.86}$$

With respect to the zero modes, the primary states form a representation of $\mathfrak{su}(2)$. In the end, we are interested in unitary representation and it is known that all unitary representations of $\mathfrak{su}(2)$ are finite-dimensional. And since all finite-dimensional representations are highest weight representations, we can further restrict our search for primary states to $\mathfrak{su}(2)$ highest weight states by demanding

$$J^0_0\Phi_j = j\Phi_j\,, \qquad J^+_0\Phi_j = 0 \tag{3.87}$$

with $j \in \frac{1}{2}\mathbb{N}$. The whole zero mode representation generated from Φ_j is then primary.

Apart form the $\mathfrak{su}(2)$ spanned by the zero modes, there are further $\mathfrak{su}(2)$ subalgebras given by the family

$$I^+ = J^+_{-n}\,, \quad I^- = J^-_n\,, \quad I^0 = J^0_0 - \frac{nk}{2}\,, \tag{3.88}$$

with $n \geq 1$. Any representation has to be unitary with respect to any $\mathfrak{su}(2)$ embedded in $\widehat{\mathfrak{su}}(2)_k$ and hence the eigenvalues of J^0_0 have to be half-integers. Since all states in the affine module generated from Φ_j have half-integer eigenvalues with respect to J^0_0, we conclude that unitarity requires k to be an integer in agreement with the geometric analysis in section 3.2.1. Obviously, $I^-\Phi_j = 0$ and thus Φ_j generates a lowest weight representation with respect to the $\mathfrak{su}(2)$ spanned by $\{I^\pm, I^0\}$. In order for this representation to be unitary, its I_0^0-eigenvalue has to be nonpositive. We conclude that

$$j - \frac{nk}{2} \leq 0 \quad \text{for all} \quad n \geq 1 \qquad \Rightarrow \qquad j \leq \frac{k}{2}\,. \tag{3.89}$$

Therefore admissible representations for the SU(2) WZW model at level $k \in \mathbb{N}$ are affine $\widehat{\mathfrak{su}}(2)_k$ highest weight modules of highest weight $0 \leq j \leq \frac{k}{2}$. The conformal weights of the primary states are

$$h(\Phi_j) = \frac{j(j+1)}{k(k+2)} \geq 0\,. \tag{3.90}$$

Having identified the set of admissible $\widehat{\mathfrak{su}}_k(2)$ representations, we have to combine the holomorphic with the antiholomorphic sector in order to obtain the full WZW spectrum. This requires the matrix of multiplicities m_{ij} in (3.77) to be chosen such that the full theory is modular invariant. A natural and consistent choice is the *diagonal* theory where $m_{ij} = \delta_{ij}$. However, there also exist nondiagonal choices which are consistent with modular invariance [58].

Comments on the SL(2, ℝ) WZW model

The SL(2, ℝ) WZW model is a close relative of the SU(2) WZW model. It is of particular interest as it also plays an important role in string theory on AdS_3 due to the diffeomorphism $AdS_3 \simeq SL(2,\mathbb{R})$. Hence the SL(2, ℝ) WZW model typically appears in analyses of string theory on this space (cf. *e.g.* [65, 99, 134, 140–142]). However, it is noncompact, which makes the analysis more complicated even though the current algebra is similar.

Representations of $\widehat{\mathfrak{sl}}(2)_k$ are generated from representations of $\mathfrak{sl}(2)$ as has been explained in the general context. The spectrum of unitary $\mathfrak{sl}(2)$ representations is known [60]. Apart from the trivial representation, there are four types of unitary representations denoted $\mathcal{D}_j^\pm$, $\mathcal{C}_{1/2+is}^\alpha$ and $\mathcal{E}_j^\alpha$ with $j > 0$ and $s \in \mathbb{R}$. It was argued in [140] that the latter does not contribute to the spectrum of the WZW model. The structure of the argument is quite instructive. The limit $k \to \infty$ corresponds to large string tension and hence the theory reduces to quantum mechanics on AdS_3. Thus the Hilbert space should agree with the space of square-integrable functions $\mathcal{L}^2(AdS_3)$, which decomposes with respect to the global $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry into $\mathcal{D}_j^\pm \otimes \mathcal{D}_j^\pm$ with $j > 1/2$ and $\mathcal{C}_{1/2+is}^\alpha \otimes \mathcal{C}_{1/2+is}^\alpha$. So we discard $\mathcal{E}_j^\alpha$ and only discuss the remaining representations. The representations $\mathcal{D}_j^\pm$ are called discrete series representations and are basically irreducible Verma modules with highest weight $-j$ or lowest weight j. The representations $\mathcal{C}_{1/2+is}^\alpha$, $s \in \mathbb{R}$, are called continuous series representations and are special since the are neither highest nor lowest weight representations [173]. The present work we will be mainly concerned with the discrete series representations, in particular those that are highest weight representations, *i.e.* $\mathcal{D}_j^-$.

As has been indicated, the SL$(2,\mathbb{R})$ WZW model can be used to define a consistent theory of strings moving on $\mathrm{AdS}_3 \times M$, where M is some additional manifold necessary to make the string theory critical [92, 140]. In that case physical string states should arrange in representations of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. However, string theory induces more constraints on the type of representations that are allowed. In particular, in order to avoid ghost states, *i.e.* physical states of negative norm, it is necessary to bound the weight of primary fields by $j \leq \frac{k}{2}$ [65].

3.3 BRST Quantisation of String Theory

In this section we review the BRST quantisation procedure in string theory to fix diffeomorphism invariance and Weyl rescaling. The gauge fixed string obtained by this procedure serves as the starting point for a definition of the hybrid string in chapter 5.

3.3.1 The bc system

One conformal field theory which will be of great importance in the following is the so called bc system. It contains two holomorphic primaries: a field $b(z)$ of conformal weight λ and a field $c(z)$ of conformal weight $1-\lambda$. The action in conformal gauge reads

$$S^{\mathrm{bc}} = \int d^2 z \, b \bar{\partial} c \,. \tag{3.91}$$

Their OPEs read

$$b(z)c(w) \sim \frac{\varepsilon}{z-w}\,, \qquad c(z)b(w) \sim \frac{1}{z-w}\,, \tag{3.92}$$

where we left the statistics of b and c undetermined. For fermionic statistics one must choose $\varepsilon = 1$ while $\varepsilon = -1$ gives b and c bosonic statistics. Demanding that b and c are primary, it is easy to find their infinitesimal behaviour under conformal transformations $\epsilon(z)$:

$$\begin{aligned} \delta b &= -\epsilon \partial b - \lambda (\partial \epsilon) b \,, \\ \delta c &= -\epsilon \partial c - (1-\lambda)(\partial \epsilon) c \,. \end{aligned}$$

By using Noether's trick, we can then determine the energy momentum tensor of this system:

$$T^{bc} = (\partial b) c - \lambda \partial (bc) \,. \tag{3.93}$$

In order to check the correct conformal weights of b and c, note that depending on the chosen statistics, the normal ordering statisfies $(bc)(z) = -\varepsilon\,(cb)(z)$. Using the OPEs in (3.92), it is straightforward to determine the OPE of the energy momentum tensor with itself,

$$T^{bc}(z) T^{bc}(w) = \frac{-\varepsilon\,[6\lambda(\lambda-1)+1]}{(z-w)^4} + \frac{2T^{bc}(w)}{(z-w)^2} + \frac{\partial T^{bc}(w)}{z-w}\,, \tag{3.94}$$

from which one easily extracts the central charge of the bc system,

$$c_{bc} = -\varepsilon\,[12\lambda(\lambda-1)+2]\ . \tag{3.95}$$

The anticommuting bc system, *i.e.* $\varepsilon = 1$, is actually equivalent to the conformal field theory of a free boson with background charge. To see this, note that the OPE of the normal ordered product (bc) with itself,

$$(bc)(z)(bc)(w) \sim \frac{1}{(z-w)^2} + \frac{(bc)+(cb)}{z-w} = \frac{1}{(z-w)^2}\,, \tag{3.96}$$

mimics the OPE of an Abelian current $i\partial H(z)$, where $H(z)$ is a boson satisfying the OPE $H(z)H(w) \sim -\ln(z-w)$. This suggests that one can identify $(bc) = i\partial H$. But the conformal field theory is not complete without giving an expression for the appropriate energy momentum tensor. In general, it is not simply the energy momentum tensor of the free boson $-\frac{1}{2}\left(\partial H^2\right)$ since then the OPE of the energy momentum tensor with the current $i\partial H$, and hence equivalently with (bc), would not have a pole of third order. But this is in contrast to the observation in the original theory that the OPE is given by

$$T^{bc}(z)(bc)(w) \sim \frac{2\lambda-1}{(z-w)^3} + \frac{(bc)}{(z-w)^2} + \frac{\partial(bc)}{z-w}\,. \tag{3.97}$$

In particular, the current (bc) is not a Virasoro primary. However, we can introduce a so called screening charge Λ to the energy momentum tensor to take care of that third order pole,

$$T^{\Lambda}(z) = -\frac{1}{2}(\partial H^2) + \Lambda\partial^2 H\,. \tag{3.98}$$

Using that $\partial^2 H(z)\partial H(w) \sim 2\,(z-w)^{-3}$, we find the correct screening charge for the bc system,

$$\Lambda = \frac{i}{2}(1-2\lambda)\,. \tag{3.99}$$

Note that, if we chose to bosonise (cb) instead of (bc), the screening charge would differ by a sign. Indeed, the ghost current in bosonic string theory is usually defined as $-(bc)$ [154]. One could have asked why this argument does not work for commuting bc systems, *i.e.* $\varepsilon = -1$. A counterexample will be presented in section 3.3.3.

As b and c are meromorphic functions, there exist Laurent expansions

$$b(z) = \sum_n b_n z^{-n-\lambda}\,, \qquad c(z) = \sum_n c_n z^{-n+\lambda-1}\,. \tag{3.100}$$

Depending on the chosen statistics, *i.e.* the value of ε, we obtain the following (anti)commutators of the modes

$$[b_m, c_n] = \oint_{w=0} dw\, w^{n-\lambda} \oint_{z=w} dz\, z^{m+\lambda-1} b(z)c(w) = \varepsilon\delta_{m+n}\,, \tag{3.101}$$

$$[c_m, b_n] = \oint_{w=0} dw\, w^{n+\lambda-1} \oint_{z=w} dz\, z^{m-\lambda} c(z)b(w) = \delta_{m+n}\,. \tag{3.102}$$

Thus we see, as expected, that the modes anticommute if $\varepsilon = 1$ and commute if $\varepsilon = -1$.

Let us now consider the radial ordered product $b(z)c(w)$, *i.e.* let $|z| > |w|$. By inserting the mode expansions from above, we obtain the following equality:

$$\begin{aligned} b(z)c(w) &= \sum_{m,n} b_m c_n z^{-m-\lambda} w^{-n+\lambda-1} \\ &= \sum_{m<n} b_m c_n z^{-m-\lambda} w^{-n+\lambda-1} + \sum_{m\geq n} (-\varepsilon c_n b_m + \varepsilon \delta_{m,n}) z^{-m-\lambda} w^{-n+\lambda-1} \\ &= \sum_{m,n} {:}b_m c_n{:}\, z^{-m-\lambda} w^{-n+\lambda-1} + \varepsilon \sum_{n\geq 0} z^{-n-(\lambda+\varrho)} w^{n+(\lambda+\varrho)-1} \\ &= {:}b(z)c(w){:} + \varepsilon \frac{w^{(\lambda+\varrho)-1}}{z^{\lambda+\varrho}} \sum_{n\geq 0} \left(\frac{w}{z}\right)^n = {:}b(z)c(w){:} + \varepsilon \left(\frac{w}{z}\right)^{(\lambda+\varrho)-1} \frac{1}{z-w}\,, \end{aligned}$$

where $\varrho = 0$ for an expansion in integer modes and $\varrho = \frac{1}{2}$ for an expansion in half integer modes. Here ${:}\cdot{:}$ denotes creation-annihilation-ordering, which for the zero modes is defined as ${:}b_0 c_0{:} = -\varepsilon c_0 b_0$. We want this ordering to be consistent with the normal ordering defined by subtracting singular terms of the OPE. We find that

$$(b(z)c(w)) = {:}b(z)c(w){:} + \varepsilon \frac{\left(\frac{w}{z}\right)^{\lambda+\varrho-1} - 1}{z-w}\,. \tag{3.103}$$

By definition, $(b(z)c(w))$ has no singular terms when taking the limit $z \to w$. In particular,

$$\lim_{z\to w} \frac{\left(\frac{w}{z}\right)^{\lambda+\varrho-1} - 1}{z-w} = \frac{1-\varrho-\lambda}{w} \tag{3.104}$$

and thus we find for the bc-number current $j^{\text{bc}} = -\lim_{z\to w}(b(z)c(w))$ the following charge:

$$N^{\text{bc}} = \frac{1}{2\pi i} \oint dw\, j^{\text{bc}}(w) = -\sum_{n\neq 0} {:}b_{-n} c_n{:} + \delta_\varrho c_0 b_0 + \varepsilon(\lambda + \varrho - 1)\,. \tag{3.105}$$

Hence the normal ordering constant is $a^\lambda = \varepsilon(\lambda + \varrho - 1)$.

Conformal Fermions

The arguably most important bc-system is the one with $\lambda = \frac{1}{2}$ and $\varepsilon = 1$. In this case, both fields $b(z)$ and $c(z)$ have conformal weight $\frac{1}{2}$ and they obey fermionic statistics. In fact, this is just the theory of (complex) fermions in two dimensions commonly denoted as $\bar{\Psi}(z) = b(z)$ and $\Psi(z) = c(z)$. The energy momentum tensor is

$$T^{\bar{\Psi}\Psi} = \big((\partial\bar{\Psi})\Psi\big) - \tfrac{1}{2}\big(\partial(\bar{\Psi}\Psi)\big) = \tfrac{1}{2}\big((\partial\bar{\Psi})\Psi\big) - \tfrac{1}{2}\big(\bar{\Psi}\partial\Psi\big)\,. \tag{3.106}$$

We may also express this conformal field theory in terms of real fermions by writing $\Psi(z) = \frac{1}{\sqrt{2}}(\psi^0 + i\psi^1)$ and $\bar{\Psi}(z) = \frac{1}{\sqrt{2}}(\psi^0 - i\psi^1)$. In that notation the energy momentum tensor becomes

$$T^{\psi\psi} = T^{\bar{\Psi}\Psi} = -\tfrac{1}{2}\left(\psi^0\partial\psi^0 + \psi^1\partial\psi^1\right) , \tag{3.107}$$

which is just the well-known energy momentum tensor of two real fermions in two dimensions with Euclidean metric. Note that since the complex fermions have $\lambda = \frac{1}{2}$, there is no need to introduce a screening charge when bosonised. Hence the complex fermion is equivalent to one real boson, as is also suggested by the central charge of the associated Virasoro algebra, which is one in both cases.

3.3.2 Covariant gauge fixing and physical string states

Let us now review one important place where *bc*-systems appear in the context of string theory, namely the gauge-fixing of local diffeomorphism invariance on the world sheet. Treating the invariance under diffeomorphisms as a gauge symmetry of the action, fixing it requires us to perform the Faddeev-Popov procedure and hence including appropriate functional determinants in the path integral [74,119,126]. Following the general Faddeev-Popov procedure, these determinants can be written in such a way that they add $(2,-1)$ *bc*-systems to the classical action. The exposition below follows [72].

The Polyakov action of the bosonic string is given by

$$S = \int d^2x\, \sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\,, \tag{3.108}$$

which is invariant under local diffeomorphisms and Weyl rescaling of the world sheet metric $h^{\alpha\beta}$. At the quantum level, one rather considers the generating functional or partition sum [157],

$$Z = \int \frac{[dX][dh]}{V} e^{-S}\,, \tag{3.109}$$

where V is the volume of the gauge orbit of equivalent metrics under diffeomorphisms and Weyl rescalings. If we wish to fix diffeomorphism invariance,

$$\delta h_{zz} = \nabla_z\zeta_z\,, \quad \delta h_{\bar{z}\bar{z}} = \nabla_{\bar{z}}\zeta_{\bar{z}}\,, \quad \delta h_{z\bar{z}} = \nabla_{\bar{z}}\zeta_z + \nabla_z\zeta_{\bar{z}}\,, \tag{3.110}$$

in such a way that the worldsheet metric coincides with the conformal metric $\hat{h}$, which satisfies

$$\hat{h}_{zz} = \hat{h}_{\bar{z}\bar{z}} = 0\,, \tag{3.111}$$

we need to insert delta distributions $\delta(h_{zz})$ and $\delta(h_{\bar{z}\bar{z}})$ into the path integral, but Z should stay unchanged. This describes exactly the setting of the Faddeev-Popov method. For better readability, let us adopt the notation $\zeta \equiv \zeta_z$ and $\bar{\zeta} \equiv \zeta_{\bar{z}}$. The crucial observation in this procedure applied to our case is that (see *e.g.* [181] for a proof)

$$\int [d\zeta]\delta(h_{zz} - \hat{h}^\zeta_{zz}) \det\left[\frac{\delta\hat{h}^\zeta_{zz}}{\delta\zeta}\right] = \int [d\bar{\zeta}]\delta(h_{\bar{z}\bar{z}} - \hat{h}^{\bar{\zeta}}_{\bar{z}\bar{z}}) \det\left[\frac{\delta\hat{h}^{\bar{\zeta}}_{\bar{z}\bar{z}}}{\delta\bar{\zeta}}\right] = 1\,, \tag{3.112}$$

where $\hat{h}^\zeta_{zz}$ and $\hat{h}^{\bar{\zeta}}_{\bar{z}\bar{z}}$ are the gauge transformed metric components, *i.e.* $\hat{h}^\zeta_{zz} = \hat{h}_{zz} + \nabla_z\zeta$

and $\hat{h}^{\bar\zeta}_{\bar z\bar z} = \hat{h}_{\bar z\bar z} + \nabla_{\bar z}\bar\zeta$, and det[·] denotes the functional determinant. So we can just insert unity represented by the integral in (3.112) into the generating functional

$$\begin{aligned} Z &= \int \frac{[dX][d\zeta][d\bar\zeta][dh]}{V}\delta(h_{zz} - \hat{h}^{\zeta}_{zz})\delta(h - \hat{h}^{\bar\zeta}_{\bar z\bar z}) \det\left[\frac{\delta\hat{h}^{\zeta}_{zz}}{\delta\zeta}\right] \det\left[\frac{\delta\hat{h}^{\bar\zeta}_{\bar z\bar z}}{\delta\bar\zeta}\right] e^{-S[X,h]} \\ &= \int \frac{[dX][d\zeta][d\bar\zeta][dh_{z\bar z}]}{V} \det\left[\frac{\delta\hat{h}^{\zeta}_{zz}}{\delta\zeta}\right] \det\left[\frac{\delta\hat{h}^{\bar\zeta}_{\bar z\bar z}}{\delta\bar\zeta}\right] e^{-S[X,\hat{h}^{\zeta}_{zz},\hat{h}^{\bar\zeta}_{\bar z\bar z},h_{z\bar z}]} . \end{aligned} \tag{3.113}$$

The generating functional can be further simplified by replacing the integration variables X and $h_{z\bar z}$ by $X^{\zeta,\bar\zeta}$ and $h^{\zeta,\bar\zeta}_{z\bar z}$, respectively, and assuming the gauge invariance of the measure $[dX^{\zeta,\bar\zeta}][dh^{\zeta,\bar\zeta}_{z\bar z}] = [dX][dh_{z\bar z}]$. Using the gauge invariance of the classical action,

$$S[X^{\zeta,\bar\zeta}, \hat{h}^{\zeta}_{zz}, \hat{h}^{\bar\zeta}_{\bar z\bar z}, h^{\zeta,\bar\zeta}_{z\bar z}] = S[X, \hat{h}_{zz}, \hat{h}_{\bar z\bar z}, h_{z\bar z}] = S[X, 0, 0, h_{z\bar z}] , \tag{3.114}$$

the generating functional becomes

$$Z = \int \frac{[dX][d\zeta][d\bar\zeta][dh_{z\bar z}]}{V} \det\left[\frac{\delta\hat{h}^{\zeta}_{zz}}{\delta\zeta}\right] \det\left[\frac{\delta\hat{h}^{\bar\zeta}_{\bar z\bar z}}{\delta\bar\zeta}\right] e^{-S[X,h_{z\bar z}]} . \tag{3.115}$$

Using (3.110) we can evaluate the functional derivatives. The functional determinants become

$$\det\left[\frac{\delta\hat{h}^{\zeta}_{zz}}{\delta\zeta}\right] = \det[\nabla_z] , \qquad \det\left[\frac{\delta\hat{h}^{\bar\zeta}_{\bar z\bar z}}{\delta\zeta}\right] = \det[\nabla_{\bar z}] . \tag{3.116}$$

Here, it is important to keep in mind that in this case ∇_z denotes the covariant derivative that maps rank one tensors to rank two tensors, while $\nabla_{\bar z}$ maps tensors of rank -1 to tensors of rank -2.[1] After inserting these expressions for the functional determinant, we have no dependence on the gauge parameters ζ and $\bar\zeta$ any more. Therefore we can evaluate the path integral over ζ and $\bar\zeta$ to give the volume of the gauge orbit of diffeomorphisms. The generating functional simplifies to

$$Z = \int \frac{[dX][dh_{z\bar z}]}{V'} \det[\nabla_z] \det[\nabla_{\bar z}]\, e^{-S[X,h_{z\bar z}]} , \tag{3.117}$$

where V' is now the volume of the gauge orbit of Weyl rescaling only. It is a useful fact that functional determinants have a representation in terms of path integrals. In fact, if $\mathcal{O}$ is a differential operator, the functional integral can be written as

$$\det\mathcal{O} = \int [db][dc] e^{-\int d^2z\, b\mathcal{O}c} , \tag{3.118}$$

[1]For the definition of the rank of an tensor in complex coordinates in two dimensions see [71]. Basically, a tensor t with n lower z-indices, $t_{zz...}$ or with n upper $\bar z$-indices, $t^{\bar z\bar z\cdots}$ has rank n. Similarly, a tensor t with n upper z-indices, $t^{zz\cdots}$ or with n lower $\bar z$-indices, $t_{\bar z\bar z...}$ has rank $-n$. The rank coincides with the difference of chiral conformal weights, $h - \bar h$.

where b and c are Grassmann-valued fields. So we can write the generating functional as

$$Z = \int \frac{[dX][db][dc][d\bar{b}][d\bar{c}][dh_{z\bar{z}}]}{V'} \, e^{-S'[X,h_{z\bar{z}},b,c,\bar{b},\bar{c}]}$$
$$\text{with} \quad S' = S + \int d^2z \, (b_{zz}\nabla_{\bar{z}}c^z + \bar{b}^{zz}\nabla_z \bar{c}_z) \, . \tag{3.119}$$

Since $\nabla_{\bar{z}}$ is a mapping from rank -1 tensors to tensors of rank -2, the c-ghost has to be a tensor of rank -1 as well, hence the notation c^z. In order for the overall rank to vanish, the b-ghost must be a rank two tensor. Similar arguments hold for ∇_z. Since we have imposed the conformal metric with $\hat{h}_{zz} = \hat{h}_{\bar{z}\bar{z}} = 0$, the only non-vanishing Christoffel symbols are Γ^z_{zz} and $\Gamma^{\bar{z}}_{\bar{z}\bar{z}}$, which in turn implies that $\nabla_{\bar{z}}c = \bar{\partial}c$. The covariant derivative $\nabla_z c_z$ reads in the conformal metric $\nabla_z c_z = h^{z\bar{z}}\partial h_{z\bar{z}}c_z$. Let us introduce the short-hand notation

$$b \equiv b_{zz} \, , \quad c \equiv c^z \, , \quad \bar{b} \equiv b^{zz}h^{z\bar{z}} \quad \text{and} \quad \bar{c} \equiv h_{z\bar{z}}c_z \, . \tag{3.120}$$

The equation of motions deduced from the gauge fixed action S' tell us that b and c are holomorphic fields while $\bar{b}$ and $\bar{c}$ are antiholomorphic. As the rank of a tensor is nothing but the difference of chiral weights, $h - \bar{h}$, we find that both c and $\bar{c}$ have conformal weight -1 while b and $\bar{b}$ have conformal weight 2. Hence they yield bc-systems with $\lambda = 2$, one being chiral, the other one being antichiral.

Note that we have not fixed the metric completely since $h_{z\bar{z}}$ is still a dynamical field, sometimes called the Liouville field. This implies that there is an equation of motion associated to it, which forces us to set

$$\frac{\delta S'}{\delta h_{z\bar{z}}} = 0 \quad \Rightarrow \quad T_{z\bar{z}} = 0 \, , \tag{3.121}$$

which means that the theory associated to the action S' is classically conformally invariant.

In the case of the superstring, the classical action does not only depend on the embedding fields X but also on their superpartners ψ. Indeed, to conserve supersymmetry after gauge fixing, we have to introduce superpartners of b and c as well, which are commonly denoted by β and γ. Since b and c are anticommuting, β and γ are commuting fields. Furthermore, their conformal weight will differ by $\frac{1}{2}$, so β has conformal weight $\frac{3}{2}$ and γ has conformal weight $-\frac{1}{2}$. Hence β and γ add a *commuting* bc-system with $\lambda = \frac{3}{2}$ to the action. It is also possible to obtain these systems by a gauge-fixing procedure as above and the interested reader is referred to [72]. Note that demanding the central charge to vanish in the covariant gauge-fixed theory yields directly the critical dimension of the string. According to (3.95) the conformal ghosts b, c with $\lambda = 2$ have a central charge of $c_{bc} = -26$, which is canceled by the central charge of 26 free bosons. Hence bosonic string theory is critical in a 26-dimensional space-time. If the fermionic superpartners are included, the superconformal ghosts β, γ also contribute to the total central charge with $c_{\beta\gamma} = 11$ and each supersymmetric free boson/fermion pair contributes another $\frac{3}{2}$ to the central charge. The critical dimension D thus satisfies

$$\tfrac{3}{2}D + c_{bc} + c_{\beta\gamma} = \tfrac{3}{2}D + 15 = 0\,. \tag{3.122}$$

So criticality of the superstring is obtained in ten dimensions.

As in Yang-Mills theory, the complete gauge-fixed action S' still has a residual symmetry that originates from the original gauge symmetry, in our case the local diffeomorphism invariance. It is called the BRST symmetry [11, 67]. Loosely speaking, even though we imposed some gauge-fixing condition, we are still free to choose that condition any way we want. The associated conserved current is [155]

$$j_{\mathrm{BRST}} = \gamma G_{\mathrm{matter}} + c(T_{\mathrm{matter}} - \tfrac{3}{2}\beta\partial\gamma - \tfrac{1}{2}\gamma\partial\beta - b\partial c) - b\gamma^2 + \partial(c\beta\gamma) + \partial^2 c \tag{3.123}$$

in the case of the superstring. The BRST current of the bosonic string is easily recovered by setting $\beta = \gamma = 0$, where usually another term $\frac{1}{2}\partial^2 c$ is added to the current in order to make sure that it transforms as a tensor. Here, T_{matter} is the energy momentum tensor of the original action S but with the world sheet metric fixed, $h = \hat{h}$. Since we are considering $\mathcal{N} = 1$ superstrings, we have a supercurrent as well, which has been denoted by G_{matter}.

Any matrix element between physical states should be independent of the choice of the gauge-fixing condition, which implies that physical states are invariant under the BRST symmetry. Hence the zero-mode of the BRST current, which generates the BRST symmetry, truncates the space of states $\mathcal{H}$ to the physical sector in that the BRST operator

$$Q = \oint dz\, j_{\mathrm{BRST}}(z) \tag{3.124}$$

annihilates physical states. It can be checked that the BRST operator is nilpotent of second order, $Q^2 = 0$, if we are considering the string in its critical dimension, *i.e.* $D = 26$ for the bosonic and $D = 10$ for the superstring. Due to the nilpotency, states in the image of Q are always physical. However, they have vanishing inner product with any physical state including themselves because

$$\langle\phi|\,(Q\,|\psi\rangle) = \Big(\langle\phi|\,Q^\dagger\Big)\,|\psi\rangle = 0\,. \tag{3.125}$$

We assumed that Q is hermitian, which seems reasonable because otherwise $Q^\dagger$ would generate another symmetry and there is no candidate for it [154]. States in the image of Q are sometimes referred to as *spurious* states. Since they do not contribute any nontrivial matrix elements between physical states, they should be discarded from the spectrum. Hence the physical subsector within the space of states $\mathcal{H}$ coincides with the Q-cohomology,

$$\mathcal{H}_{\mathrm{BRST}} = H_Q(\mathcal{H})\,. \tag{3.126}$$

However, to obtain the physical spectrum with correct multiplicities, note that states will transform in representations of the zero-modes of the bc ghosts and, in the R sector, in representations of β_0 and γ_0. In order to pick one of the states in these representations, one imposes the additional constraints

$$b_0\psi = \beta_0\psi = 0\,. \tag{3.127}$$

This is sometimes called the Siegel gauge. Since the ghost vacuum is taken to be annihilated by all positive ghost modes, the Siegel gauge can be rephrased in a more algebraic way using (3.105),

$$N^{bc}\psi = \psi\,, \quad N^{\beta\gamma}\psi = \begin{cases} -\frac{1}{2}\psi & \text{in the R sector}\,, \\ -\psi & \text{in the NS sector}\,. \end{cases} \tag{3.128}$$

By the commutation relations $[Q, b_n]_+ = L_n$ and $[Q, \beta_r]_- = G_r$, where L_n and G_r are the modes of T_{matter} and G_{matter}, respectively, this implies that physical states have conformal weight zero and, in the R sector, are annihilated by G_0. It can be shown that the subsector of physical states obtained by the BRST procedure does not contain any ghost states, *i.e.* states of negative norm [69, 70, 176].

3.3.3 Bosonising the superstring ghost fields

As we have just seen, one of the most important superconformal theories in superstring theory is the combination of the $(2, -1)$ and the $(\frac{3}{2}, -\frac{1}{2})$ bc system. This system is added to the original superstring action by the usual Faddeev-Popov procedure when gauge fixing the superdiffeomorphism invariance. More concretely, when fixing the world sheet metric to be conformal, the Faddeev-Popov procedure introduces the *conformal ghosts* b and c described by a $(2, -1)$ bc system. When going to the superconformal version of it, we need to introduce the supersymmetric partners of these fields as well, denoted by β and γ. The latter is described by a $(\frac{3}{2}, -\frac{1}{2})$ bc system. We will now discuss the bosonisation of both systems.

First we recall the OPEs of the fields:

$$\begin{aligned} b(z)c(w) &\sim \tfrac{1}{z-w}\,, \qquad & \beta(z)\gamma(w) &\sim -\tfrac{1}{z-w}\,, \\ c(z)b(w) &\sim \tfrac{1}{z-w}\,, \qquad & \gamma(z)\beta(w) &\sim \tfrac{1}{z-w}\,. \end{aligned}$$

Note the extra minus sign in the $\beta\gamma$ OPE since these are *commuting* fields. The OPEs of the number currents (bc) and $(\beta\gamma)$ follow directly from the OPEs above:

$$(bc)(z)(bc)(w) \sim \tfrac{1}{(z-w)^2} \quad \text{and} \quad (\beta\gamma)(z)(\beta\gamma)(w) \sim -\tfrac{1}{(z-w)^2}\,. \tag{3.129}$$

The conformal ghosts are easily bosonised by introducing a scalar boson σ with the OPE $\sigma(z)\sigma(w) \sim -\ln(z-w)$ such that $(bc) \equiv -i\partial\sigma$. The sign of the current is chosen such that the antighost b has ghost number -1 and c has ghost number $+1$. Then the conformal ghosts are given by

$$b = e^{-i\sigma} \quad \text{and} \quad c = e^{i\sigma}\,. \tag{3.130}$$

It is straightforward to verify that these exponentials satisfy the correct OPE.

The superconformal ghosts are tricky to bosonise [72, 129]. Due to the extra minus sign in the $\beta\gamma$ OPE, the OPE of the number current with itself gains a minus sign as well. Introducing a scalar boson ϕ with the usual OPE $\phi(z)\phi(w) \sim -\ln(z-w)$, the current is

$$(\beta\gamma) = \partial\phi \tag{3.131}$$

with no imaginary unit in front. Due to this missing unit, the would-be bosonised superconformal ghosts $e^{-\phi}$ and e^{ϕ} do not have the correct OPEs, *e.g.*

$$e^{-\phi}(z)e^{\phi}(w) \sim (z-w)\,, \qquad e^{\phi}(z)e^{\phi}(w) \sim \tfrac{1}{z-w}\,. \tag{3.132}$$

In order to resolve this problem, additional fields η and ξ are included, which have non-singular OPEs with ϕ. Then the superconformal ghosts are assumed to take the form

$$\beta = \left(\partial\xi e^{-\phi}\right) \quad \text{and} \quad \gamma = \left(\eta e^{\phi}\right). \tag{3.133}$$

Demanding that these normal ordered products satisfy the correct $\beta\gamma$ OPEs yields

$$\partial\xi(z)\eta(w) \sim -\tfrac{1}{(z-w)^2}\,, \qquad \eta(z)\partial\xi(w) \sim \tfrac{1}{(z-w)^2} \tag{3.134}$$

and therefore

$$\eta(z)\xi(w) \sim \tfrac{1}{z-w} \quad \text{and} \quad \xi(z)\eta(w) \sim \tfrac{1}{z-w}\,. \tag{3.135}$$

The third-order pole in the OPE

$$\begin{aligned} T^{\beta\gamma}(z)(\beta\gamma)(w) &= \left((\partial\beta\gamma) - \tfrac{3}{2}\partial(\beta\gamma)\right)(z)(\beta\gamma)(w) \\ &= \frac{-2}{(z-w)^3} + \mathcal{O}((z-w)^{-2}) \end{aligned} \tag{3.136}$$

fixes the screening charge of the energy momentum tensor T^{ϕ} of ϕ to be $\Lambda^{\phi} = -1$ such that the energy momentum tensor reads

$$T^{\phi} = -\tfrac{1}{2}(\partial\phi)^2 - \partial^2\phi\,, \tag{3.137}$$

and thus determines the conformal weights of the exponentials,

$$h\left(e^{n_\phi \phi}\right) = -\tfrac{1}{2}n_\phi^2 - n_\phi\,. \tag{3.138}$$

Using the known conformal weights of β and γ, one determines the conformal weights of η and ξ to be $h^{\eta} = 1$ and $h^{\xi} = 0$, respectively. This identifies the $\eta\xi$ system as a $(1,0)$ bc system.

For a complete bosonisation of the superconformal ghost system we have to bosonise the $\eta\xi$ system as well. But having integer conformal weights, this is easily done along the lines of the conformal ghosts b and c. Thus we introduce a scalar boson κ and write η and

ξ as

$$\eta = e^{i\kappa} \quad \text{and} \quad \xi = e^{-i\kappa}\,. \tag{3.139}$$

The known conformal weights of η and ξ determine the screening charge in its energy momentum tensor,

$$T^{\kappa} = -\tfrac{1}{2}(\partial\kappa)^2 - \tfrac{i}{2}\partial^2\kappa\,. \tag{3.140}$$

This way we can find the conformal weights of the corresponding exponentials:

$$h\left(e^{n_\kappa \kappa}\right) = -\tfrac{1}{2}n_\kappa^2 - \tfrac{1}{2}in_\kappa\,. \tag{3.141}$$

This completes the bosonisation process of the superconformal ghosts. The energy momentum tensor of the whole bosonised ghost system is just the sum of the individual ones, yielding

$$T^{\sigma\phi\kappa} = -\tfrac{1}{2}(\partial\sigma)^2 + \tfrac{3}{2}\partial^2(i\sigma) - \tfrac{1}{2}(\partial\phi)^2 - \partial^2\phi - \tfrac{1}{2}(\partial\kappa)^2 - \tfrac{1}{2}\partial^2(i\kappa)\,. \tag{3.142}$$

The central charge can easily be determined by recognising that for a general holomorphic scalar boson X with screening charge Λ the following OPE holds:

$$\begin{aligned} T^X(z)T^X(w) &= (-\tfrac{1}{2}(\partial X)^2 + \Lambda\partial^2 X)(z)(-\tfrac{1}{2}(\partial X)^2 + \Lambda\partial^2 X)(w) \\ &\sim \frac{1+12\Lambda^2}{2(z-w)^4} + \frac{2T^X(w)}{(z-w)^2} + \frac{\partial T^X(w)}{z-w}\,. \end{aligned} \tag{3.143}$$

Hence the the central charge is $c_X^\Lambda = 12\Lambda^2 + 1$. Since we know the screening charges of the bosonised ghosts, it is easy to see that the individual central charges are

$$c_\sigma = -26\,, \qquad c_\phi = 13\,, \qquad c_\kappa = -2\,. \tag{3.144}$$

We see that the central charge of the σ boson fits the central charge of the conformal *bc* ghost system as it should. Furthermore, $c_\phi + c_\kappa = 11$, which similarly fits the central charge of the superconformal $\beta\gamma$ ghost system. This provides a check that our bosonised conformal field theory is equivalent to the original one.

In the bosonised theory, the physical state conditions have to be reformulated in terms of the new bosons. An important observation is that physical states should be independent of the zero mode of ξ. The reason for this is that the β and γ ghosts just inherit a dependence on the derivative of ξ but not on ξ itself. In order for the bosonisation procedure to be invertible, we have to make sure that physical states are independent of that zero mode or, phrased differently, restrict ourselves to the kernel of η_0,

$$\eta_0\psi = 0\,. \tag{3.145}$$

This is the *small* Hilbert space in contrast to the *large* Hilbert space that also allows dependences on ξ_0. In fact, the BRST cohomology on the large Hilbert space is trivial. The $\beta\gamma$ ghost number in terms of bosonised ghosts follows directly from (3.131). It turns out to be useful to define the so called *ghost picture* or *picture-counting* operator,

$$\Xi \equiv \oint dz\ [(\eta\xi) - \partial\phi] = \oint dz\ [(\eta\xi) - (\beta\gamma)] = N^{\eta\xi} - N^{\beta\gamma}\,. \tag{3.146}$$

The intuitive definition of the ghost number would be to count both bc and $\beta\gamma$ ghosts, $N^{bc} + N^{\beta\gamma}$. However, it turns out that it is convenient to shift the ghost number such that it is independent of whether we are considering the R or the NS sector. Hence, let us define the ghost number to be

$$J_{\text{ghost}} \equiv N^{bc} + N^{\beta\gamma} + \Xi = N^{bc} + N^{\eta\xi}\,. \tag{3.147}$$

With the ghost picture and ghost number operator at hand, one could rephrase the physical state conditions on the small Hilbert space $\mathcal{H}$ as

$$\psi \in H_Q(\mathcal{H})\,, \quad J_{\text{ghost}}\psi = \psi\,, \quad \Xi\,\psi = \begin{cases} -\frac{1}{2}\psi & \text{in the R sector}\,, \\ -\psi & \text{in the NS-sector}\,. \end{cases} \tag{3.148}$$

Looking a little ahead, the ghost number current as defined above will later serve as the U(1) current of an $\mathcal{N} = 2$ superconformal algebra [17].

The way the physical state conditions in the bosonised theory have been derived here suggests that only in the -1 and $-\frac{1}{2}$ ghost picture one can obtain the correct string spectrum. These ghost pictures are referred to as the *canonical* ghost pictures. However, the correct spectrum can actually be obtained in any ghost picture because there exist isomorphisms, called the *picture changing* operators, that map the n ghost picture to the $n \pm 1$ ghost picture [138]. The nice property of the canonical ghost picture is that the target space supersymmetry generators take a particularly nice form, but they do carry ghost picture themselves. Hence restricting ourselves to one ghost picture breaks manifest target space supersymmetry. Later, we will fix the ghost picture in a way that circumvents this problem.

String Theory and Supersymmetry

4.1 Superconformal Algebras in Two Dimensions

In order to introduce a supersymmetric structure on the string world sheet, an understanding of the superconformal algebras in two dimensions is essential. Here we review these algebras and their properties as well as their relevance in the context of string theory.

4.1.1 The $\mathcal{N} = 1$ superconformal algebra

Bosonic string theory is unstable as its spectrum contains a space-time tachyon [107, 154]. Furthermore, the spectrum does not contain space-time fermions at all. One way to overcome these problems is adding fermionic fields on the world sheet [96, 146, 147, 160]. These free world sheet fermions $\psi^\mu(z)$ have conformal weight $\frac{1}{2}$. We recall the relevant OPEs

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{z-w}\,, \quad T(z)\psi^\mu(w) \sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi^\mu(w)}{z-w}\,. \tag{4.1}$$

Note that $T(z)$ here refers to the energy momentum tensor of the full theory including the fermions. The supersymmetric structure is then easily obtained by realising that one can construct an additional field of weight $\frac{3}{2}$ by taking the normal ordered product of the matter fields,

$$G(z) = \tfrac{1}{\sqrt{2}}(\psi_\mu \partial X^\mu)\,. \tag{4.2}$$

It is not difficult to check that $G(z)$ is primary,

$$T(z)G(w) \sim \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}\,. \tag{4.3}$$

Using the explicit realisation of $G(z)$ in (4.2), we can also determine the GG-OPE,

$$G(z)G(w) \sim \frac{\frac{D}{2}}{(z-w)^3} + \frac{T(w)}{z-w}\,, \tag{4.4}$$

At this point, we left the number of space-time dimensions D undetermined. Now the important thing to note is that $T(z)$ and $G(w)$ close among each other. Hence, they define an algebraic structure independent of their explicit realisation. In our derivation, we just discovered it in the context of string theory, but the structure itself is valid on its own. In fact, it may be considered more fundamental since the world sheet fields transform in representations of that algebra.

Leaving the string theory point of view and just considering the current algebra defined by $T(z)$ and $G(z)$, it seems that one could have two free parameters; the central charge c of the conformal algebra in the OPE of $T(z)$ with itself and a possibly different constant multiplying the third order pole of the GG-OPE. However, our string theoretic discussion suggests that they are connected because in that example, both are proportional to the number of space time dimensions D. In particular, expressed in terms of c, the third order pole in the GG-OPE should be $\frac{c}{3}$. Indeed, this relation between the two superficially free parameters does not only hold in the context of string theory but in general. To see this, one simply imposes the Jacobi identity on the algebra generated by $T(z)$ and $G(z)$.

The algebra generated by $T(z)$ and $G(w)$ is referred to as the $\mathcal{N}=1$ *superconformal algebra* and implies the existence of supersymmetry on the world sheet. The parameter $\mathcal{N}$ counts the number of supersymmetries and in the case at hand, $G(z)$ is the only supersymmetry. To summarise, we write down the algebra in terms of OPEs,

$$\begin{aligned} T(z)T(w) &\sim \frac{c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,, \\ T(z)G(w) &\sim \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}\,, \\ G(z)G(w) &\sim \frac{\frac{c}{3}}{(z-w)^3} + \frac{T(w)}{z-w}\,. \end{aligned} \tag{4.5}$$

as well as the set of implied commutation relations of the modes,

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}\,, \\ [L_m, G_r] &= \left(\tfrac{n}{2} - r\right) G_{m+r}\,, \\ [G_r, G_s] &= L_{r+s} + \frac{c}{3}(r^2 - \tfrac{1}{4})\delta_{r+s}\,. \end{aligned} \tag{4.6}$$

Clearly, $m, n \in \mathbb{Z}$, but we have not yet addressed what kind of moding the supercurrent possesses. It being a fermionic field, we can impose antiperiodic (Neveu-Schwarz or NS sector) or periodic (Ramond or R sector) boundary conditions[1] on $G(z)$. The mode expansions for each of these boundary conditions read

$$G(z) = \sum_{r\in\mathbb{Z}+\varrho} G_r z^{-r-\frac{3}{2}} \qquad \text{with} \qquad \begin{cases} \varrho = \frac{1}{2} & (\text{NS})\,, \\ \varrho = 0 & (\text{R})\,. \end{cases} \tag{4.7}$$

Recall that the Virasoro algebra contains a Lie subalgebra $\mathfrak{sl}(2)$ spanned by $L_{\pm1}$ and L_0. Looking at (4.6) with antiperiodic boundary conditions (*i.e.* the NS sector), we note that $L_{\pm1}$, L_0, $G_{\pm\frac{1}{2}}$ close among each other, hence generating a Lie superalgebra. The theory of Lie superalgebras and their representations has been the subject of chapter 2 and in

[1] Here we mean the boundary conditions that are imposed on the SCFT when defined on the cylinder. If mapped to the complex plane, a branch cut is introduced that interchanges the boundary conditions [155]. That is why $G(z)$ in (4.7) is actually periodic in the NS sector in the sense that $G(e^{2\pi i}z) = G(z)$ and vice versa for the R sector.

the classification of Lie superalgebras, the above Lie superalgebra goes under the name $\mathfrak{osp}(1|2)$.

We have seen that by introducing world sheet fermions, the conformal symmetry is naturally extended to a superconformal symmetry. However, even though supersymmetry on the world sheet is manifest as a symmetry of the action, it is not at all evident that we obtain space-time supersymmetry that way. In the RNS formulation of string theory, the fermionic zero modes in the Ramond sector generate an $\mathrm{SO}(1, D-1)$ Clifford algebra and the Ramond vacuum has to transform in a representation thereof. This gives rise to states that transform as fermions with respect to the space-time Poincaré group. But space-time supersymmetry is not obtained until one truncates the spectrum, which is called the Gliozzi-Sherk-Olive or GSO projection [100, 101]. From the perspective of the superconformal symmetry on the world sheet, it turns out that the emergence of spacetime supersymmetry is closely connected to a further extension of the world sheet $\mathcal{N} = 1$ superconformal algebra, which is the subject of the following section.

4.1.2 The $\mathcal{N} = 2$ superconformal algebra and target space supersymmetry

If an U(1) current $J(z)$ is added to the $\mathcal{N} = 1$ superconformal algebra such that the supercurrent $G(z)$ separates into two parts of opposite charge, $G(z) = G^+(z) + G^-(z)$, then it is enhanced to an $\mathcal{N} = 2$ superconformal algebra. The defining OPEs are (taken from [138])

$$\begin{aligned}
T(z)T(w) &\sim \frac{\frac{3}{2}\hat{c}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \,, \\
T(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} \,, \qquad & J(z)J(w) &\sim \frac{\hat{c}}{(z-w)^2} \,, \\
T(z)G^\pm(w) &\sim \frac{\frac{3}{2}G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(w)}{z-w} \,, \qquad & J(z)G^\pm(w) &\sim \pm\frac{G^\pm(w)}{z-w} \,,
\end{aligned} \tag{4.8}$$

$$G^+(z)G^-(w) \sim \frac{\hat{c}}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w) + \frac{1}{2}\partial J(w)}{z-w} \,.$$

Here we set $\hat{c} = \frac{1}{3}c$. Written in terms of modes, the corresponding affine algebra reads

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{\hat{c}}{4}m(m^2-1)\delta_{m+n} \,, \qquad & [L_m, J_n] &= -nJ_{m+n} \,, \\
[L_m, G_r^\pm] &= \left(\tfrac{m}{2} - r\right) G_{n+r}^\pm \,, & [J_m, J_n] &= \hat{c}\, m\, \delta_{m+n} \,, \\
[G_r^+, G_s^-] &= \frac{\hat{c}}{2}\left(r^2 - \tfrac{1}{4}\right)\delta_{r+s} + \tfrac{1}{2}(r-s)J_{r+s} + L_{r+s} \,, & [J_m, G_r^\pm] &= \pm G_{m+r}^\pm \,,
\end{aligned} \tag{4.9}$$

where $n, m \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \varrho$ with $\varrho = \frac{1}{2}$ (NS sector) or $\varrho = 0$ (R sector) as usual. As for the $\mathcal{N} = 1$ superconformal algebra, the $\mathcal{N} = 2$ algebra also contains a subset of modes that close among each other and thus generate a Lie superalgebra. These modes are L_0, $L_{\pm 1}$, J_0, $G^\pm_{\frac{1}{2}}$ and $G^\pm_{-\frac{1}{2}}$ and the Lie superalgebra they generate is called $\mathfrak{osp}(2|2)$ [107, 171]. Let us also note that in the literature, the anticommutation relation of the supercurrent modes sometimes takes a different form. However, the versions of anticommutation relations just differ by a rescaling of $G_n^\pm$.

The importance of the $\mathcal{N}=2$ superconformal algebra in string theory is due to its relation to target space supersymmetry. In fact, the extension of $\mathcal{N}=1$ to $\mathcal{N}=2$ supersymmetry on the world sheet, *i.e.* the existence of an U(1) current $J(z)$ such that the supercurrent decomposes as above, implies $\mathcal{N}=1$ target space supersymmetry [155, 185]. The converse holds as well. Given the target space supercharges, an U(1)-current with the appropriate properties can be constructed [8].

4.1.2.1 The chiral ring and chiral primaries

In an $\mathcal{N}=2$ superconformal theory (super)primary fields are defined by Virasoro primaries that are also primary with respect to the supercurrents. That means, $|h,q\rangle$ is (super)primary if

$$L_n\,|h,q\rangle = G_r^\pm\,|h,q\rangle = J_n\,|h,q\rangle = 0 \qquad \forall\, n,r>0\,, \tag{4.10}$$

where h and q are the conformal weight and the U(1)-charge of $|h,q\rangle$, respectively, *i.e.* $L_0\,|h,q\rangle = h\,|h,q\rangle$ and $J_0\,|h,q\rangle = q\,|h,q\rangle$. Of special interest is the subsector of *(anti)chiral primaries.* In the NS sector, they additionally to (4.10) satisfy the conditions

$$G^+_{-\frac{1}{2}}\,|h,q\rangle = 0 \quad \text{(chiral)}\,, \qquad G^-_{-\frac{1}{2}}\,|h,q\rangle = 0 \quad \text{(antichiral)}\,. \tag{4.11}$$

This subsector is notable since, as we will see soon, it naturally carries a ring structure. But first we note that in unitary theories we have the following property of primary states:

$$\begin{aligned} 0 \le \langle h,q|\, G^\mp_{\frac{1}{2}} G^\pm_{-\frac{1}{2}}\,|h,q\rangle &= \langle h,q|\left[G^\mp_{\frac{1}{2}}, G^\pm_{-\frac{1}{2}}\right]|h,q\rangle \\ &= \langle h,q|\,(L_0 \mp \tfrac{1}{2}J_0)\,|h,q\rangle = h \mp \tfrac{q}{2}\,. \end{aligned} \tag{4.12}$$

Since $h-\frac{q}{2}\ge 0$ and $h+\frac{q}{2}\ge 0$ hold simultaneously, this in particular implies that $h-\frac{|q|}{2}\ge 0$. Thus the conformal weight of primary states with U(1) charge q is always bounded from below, $h\ge\frac{|q|}{2}$. Equality in (4.12) is obtained for chiral states and antichiral states; in these cases we find that $h=\frac{q}{2}$ and $h=-\frac{q}{2}$, respectively. Since these are the only cases for which equality in (4.12) holds, we have found a simple characterisation of the chiral and antichiral subsector.

Using this simple characterisation, we next have a look at their OPEs. Suppose $\phi_a(z)$ and $\phi_b(z)$ are chiral primary fields. Their most general OPE reads

$$\phi_a(z)\phi_b(w) = \sum_c \sum_{n\ge 0} \frac{\partial^n \phi_c(w)}{(z-w)^{h_a+h_b-h_c-n}} \tag{4.13}$$

as it follows by conservation of the conformal weight. We stress that the field ϕ_c is not a chiral primary a priori. Using the conservation of the U(1)-charge q and eq. (4.12), we find that

$$h_a + h_b - h_c \leq \frac{q_a + q_b}{2} - \frac{|q_c|}{2} = \frac{q_c}{2} - \frac{|q_c|}{2} \leq 0\,. \tag{4.14}$$

Thus the OPE in eq. (4.13) does not contain any singular terms. Thus the limit $z \to w$ is well defined and we can define a product between the fields $\phi_a(z)$ and $\phi_b(z)$ by

$$(\phi_a \cdot \phi_b)(w) = \lim_{z \to w} \phi_a(z)\phi_b(w)\,. \tag{4.15}$$

This is just the normal ordered product for fields with nonsingular OPE. A few comments on this product are in order. First, terms with $h_a + h_b - h_c - n < 0$ vanish in the limit $z \to w$. Since $n \geq 0$ and eq. (4.14) holds, we see that only the terms with $n = 0$ and $h_c = \frac{q_c}{2}$ survive. This implies that the product as defined in eq. (4.15) of two chiral primaries again gives a chiral primary. Therefore the product in eq. (4.15) induces a ring structure on the subsector of chiral primaries, the so called *chiral ring*. This algebraic structure makes this subsector so important and accessible [136, 137] .

4.1.2.2 Spectral flow

There exists a continuous family of automorphisms of the affine algebra in (4.9) which is commonly called the spectral flow and denoted by SF_θ. The modes of the $\mathcal{N} = 2$ SCA are mapped according to the following rules [36]:

$$\mathrm{SF}_\theta\, L_n = L_n + \theta J_n + \frac{\theta^2}{2}\hat{c}\,\delta_n\,, \tag{4.16}$$

$$\mathrm{SF}_\theta\, J_n = J_n + \theta\,\hat{c}\,\delta_n\,, \tag{4.17}$$

$$\mathrm{SF}_\theta\, G^\pm_r = G^\pm_{r\pm\theta}\,. \tag{4.18}$$

Here $\theta \in \mathbb{R}$. It is easily checked that the spectral flowed modes satisfy the commutation relations in (4.9). In particular, choosing $\theta = \frac{1}{2}$, we see from (4.18) that half-integer moding of the supercurrents $G^\pm$ is replaced by integer moding and vice versa. In other words, the boundary conditions on the supercurrents are changed by spectral flow.

As we have said before, in the RNS formulation of string theory, space-time fermions arise in the R sector while space-time bosons come from the NS sector. The spectral flow, however, connects both sectors. In particular, as it is an automorphism of the $\mathcal{N} = 2$ superconformal algebra, there is an equal number of fermionic and bosonic space-time degrees of freedom. This shows again the deep connection between space-time supersymmetry and the extension of the $\mathcal{N} = 1$ to the $\mathcal{N} = 2$ superconformal current algebra on the world sheet [110].

4.1.2.3 Twisting the $\mathcal{N}=2$ superconformal algebra

We can redefine the energy momentum tensor of the $\mathcal{N}=2$ superconformal algebra by performing a *twist* of the algebra with parameter χ:

$$T^{\chi}(z) := T(z) + \chi\partial J(z)\,. \tag{4.19}$$

This essentially means that the conformal weight of fields is changed proportionally to its U(1) charge weighted by the parameter χ. The operator algebra in eq. (4.8) is changed to

$$\begin{aligned}
T^{\chi}(z)T^{\chi}(w) &\sim \frac{\left(\frac{3}{2}-6\chi^2\right)\hat{c}}{(z-w)^4} + \frac{2T^{\chi}(w)}{(z-w)^2} + \frac{\partial T^{\chi}(w)}{z-w}\,, \\
T^{\chi}(z)G^{\pm}(w) &\sim \frac{\left(\frac{3}{2}\mp\chi\right)G^{\pm}(w)}{(z-w)^2} + \frac{\partial G^{\pm}(w)}{z-w}\,, \\
T^{\chi}(z)J(w) &\sim \frac{-2\chi\hat{c}}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}\,, \\
G^{+}(z)G^{-}(w) &\sim \frac{\hat{c}}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T^{\chi}(w) + \frac{1-2\chi}{2}\partial J(w)}{z-w}\,,
\end{aligned} \tag{4.20}$$

and all other OPEs, which do not involve the energy momentum tensor, are left unchanged. From these OPEs it is clear that the case $\chi=\frac{1}{2}$ is special. First of all, we see that the quartic pole in the TT OPE vanishes. In other words, the energy momentum tensor is primary after the twist. Furthermore, the ∂J term in the first order pole in the G^+G^- OPE vanishes. This property will turn out to be useful later when we discuss the $\mathcal{N}=4$ superconformal algebra and the definition of physical states in that context.

However, we are paying a price for this. In fact, the TJ-OPE tells us that after twisting the algebra, the U(1) current J is not primary anymore. In particular, its third order pole is exactly given by $\hat{c}$, which corresponds to introducing a global U(1) background charge to the theory. From the gauge theory perspective, this is equivalent to twisting the U(1) gauge bundle which means a change of the bundle topology. Therefore this twist of the superconformal algebra by a parameter $\chi=\frac{1}{2}$ is referred to as *topological twist* [187, 188]. We will denote the topologically twisted energy momentum tensor by $T^{\mathrm{tw}}(z)$. In the following, when we refer to a twisted algebra, we always mean an algebra on which the topological twist $\chi=\frac{1}{2}$ has been performed. The twist with parameter $\chi=-\frac{1}{2}$ is the inverse topological twist.

As implied before, the topological twist changes the conformal weight of the fields. More concretely, the conformal weights of the superconformal currents all become positive integers as summarised in table 4.1. Thus we only have an integer mode expansions for all currents. In particular, note that $G^+(z)$ is a nilpotent current of order two and thus can be considered as a BRST current with BRST operator G_0^+ [145]. Indeed, we will see that the $\mathcal{N}=1$ gauge-fixed superstring can be considered as a twisted $\mathcal{N}=2$ $\hat{c}=2$ string [29].

conformal weight w.r.t.	$T^{(\mathrm{tw})}(z)$	$G^+(z)$	$G^-(z)$	$J(z)$
$T(z)$	2	$\frac{3}{2}$	$\frac{3}{2}$	1
$T^{\mathrm{tw}}(z)$	2	1	2	1

Table 4.1: *Change of conformal weights by topologically twisting the $\mathcal{N}=2$ superconformal algebra.*

4.1.3 The $\mathcal{N}=4$ superconformal algebra

The $\mathcal{N}=2$ superconformal algebra gives rise to an $\mathcal{N}=4$ superconformal algebra by lifting the Abelian current $J(z)$ to a non-Abelian $\mathfrak{su}(2)$ current algebra. Demanding that both G^+ and G^- are components of independent doublets under the zero mode algebra $\mathfrak{su}(2)$ then adds new fermionic currents as we will see below [29].

We will now construct the $\mathcal{N}=4$ superconformal algebra from eq. (4.8). First, we add two currents $J^{++}(z)$ and $J^{--}(z)$ of U(1) charge ± 2 such that the following OPEs hold:

$$J(z)J^{\pm\pm}(w) \sim \frac{\pm 2J^{\pm\pm}(w)}{z-w}\,, \qquad J^{++}(z)J^{--}(w) \sim \frac{\hat{k}}{(z-w)^2} + \frac{J(w)}{z-w}\,. \tag{4.21}$$

These currents lift $J(z)$ to an $\widehat{\mathfrak{su}}(2)_{\hat{k}}$ current algebra. The level $\hat{k}$ is determined by demanding the consistency of the operator algebra. In order to see this, note that the J-modes fulfill

$$\begin{gathered}[J_n, J_m] = n\,\hat{c}\,\delta_{n+m}\,, \qquad [J_n, J^{\pm\pm}_m] = \pm 2\, J^{\pm\pm}_{n+m}\,, \\ [J^{++}_n, J^{--}_m] = J_{m+n} + n\,\hat{k}\,\delta_{n+m}\,. \end{gathered} \tag{4.22}$$

Using the Jacobi identity, we find that

$$\begin{aligned} l\,\hat{c}\,\delta_{k+n+m} &= [J_l, [J^{++}_n, J^{--}_m]] \\ &= -[J^{--}_m, [J_l, J^{++}_n]] - [J^{++}_n, [J^{--}_m, J_l]] = 2\,\hat{k}\,l\,\delta_{k+n+m} \end{aligned}$$

and thus $\hat{k} = \hat{c}/2$. For now we simply presume that we are given such currents J^{++} and J^{--} satisfying the OPEs in (4.21) and postpone the discussion of their existence.

Having constructed the consistent $\widehat{\mathfrak{su}}(2)_{\hat{k}}$ current algebra, we need to specify the remaining OPEs. This is easily done by specifying the $\mathfrak{su}(2)$ representation $T(z)$ and $G^\pm(z)$ transform in. It is a natural choice to choose the smallest one possible. As $T(z)$ carries zero U(1) charge, we say that it transforms in the **1** of $\mathfrak{su}(2)$. $G^+(z)$ having charge $+1$ can be taken to be the highest weight state of a **2**, whose lower component is

$$\tilde{G}^-(z) \equiv [J^{--}_0, G^+(z)] = \frac{1}{2\pi i}\oint dw\, J^{--}(w)G^+(z)\,. \tag{4.23}$$

Similarly, we take $G^-(z)$ to be the lower component of a **2** with upper component

$$\tilde{G}^+(z) \equiv -[J^{++}_0, G^-(z)] = -\frac{1}{2\pi i}\oint dw\, J^{++}(w)G^-(z)\,. \tag{4.24}$$

This way we obtain four supercurrents. The remaining OPEs are now easily determined from the ones in eq. (4.8). For example, we find that

$$\begin{aligned} G^+(z)\tilde{G}^+(w) &= \frac{1}{2\pi i}\oint_w dx\, J^{++}(x)G^+(z)G^-(w) \\ &\sim \frac{1}{2\pi i}\oint_w dx\, J^{++}(x)\left[\frac{\hat{c}}{(z-w)^3}+\frac{J(w)}{(z-w)^2}+\frac{T(w)+\frac{1}{2}\partial J(w)}{z-w}\right] \\ &= \frac{2J^{++}(w)}{(z-w)^2}+\frac{\partial J^{++}(w)}{z-w}\,. \end{aligned}$$

The other OPEs are found along the same lines:

$$\begin{aligned} \tilde{G}^+(z)\tilde{G}^-(w) &\sim G^+(z)G^-(w)\,, \\ G^-(z)\tilde{G}^-(w) &\sim \frac{2J^{--}(w)}{(z-w)^2}+\frac{\partial J^{--}(w)}{z-w}\,. \end{aligned} \tag{4.25}$$

Having the complete set of OPEs, we can determine the associated infinite-dimensional Lie superalgebra generated by the modes using the standard procedure. Similar to the superconformal algebras with $\mathcal{N}=1$ and $\mathcal{N}=2$ supercurrents, there exist a finite subset of modes in the NS sector that close among each other. These are

$$L_0,\ L_{\pm 1},\ J_0,\ J_0^{\pm\pm},\ G^{\pm}_{\frac{1}{2}},\ G^{\pm}_{-\frac{1}{2}},\ \tilde{G}^{\pm}_{\frac{1}{2}}\quad\text{and}\quad \tilde{G}^{\pm}_{-\frac{1}{2}}\,. \tag{4.26}$$

They generate a Lie superalgebra that plays a prominent role in this thesis, namely $\mathfrak{psl}(2|2)$ [171].

Of course, the $\mathcal{N}=4$ superconformal algebra that we have just constructed may be twisted as well. However, it is important to note that with respect to the twisted energy momentum tensor, J^{++} and J^{--} have conformal weight zero and two, respectively, as it is easily seen from the OPE

$$\begin{aligned} T^{\text{tw}}(z)J^{\pm\pm}(w) &= \big(T(z)+\tfrac{1}{2}\partial J(z)\big)J^{\pm\pm}(w) \\ &\sim \frac{(1\mp 1)J^{\pm\pm}(w)}{(z-w)^2}+\frac{\partial J^{\pm\pm}(w)}{z-w}\,. \end{aligned} \tag{4.27}$$

Therefore one can deduce that $\tilde{G}^+$ and $\tilde{G}^-$ have conformal weight one and two with respect to the twisted energy momentum tensor. Note that the former zero mode of J^{++} becomes the $n=1$ mode J^{++}_1 after twisting and similarly the former zero mode of J^{--} becomes the $n=-1$ mode J^{--}_{-1}. Thus, after twisting, we have

$$\tilde{G}^-(z) = [J^{--}_{-1}, G^+(z)]\,, \tag{4.28}$$

$$\tilde{G}^+(z) = -[J^{++}_1, G^-(z)]\,. \tag{4.29}$$

4.1.4 An example: RNS string theory on T^4

As an example, let us consider RNS strings on T^4. This is an $\mathcal{N}=1$ SCFT with four bosonic embedding fields $X^j(z)$ and their fermionic partners $\psi^j(z)$. The former have conformal dimension one while the latter have conformal dimension $\frac{1}{2}$. For later convenience, we define their complex version by

$$\mathbf{X}^j(z) = \tfrac{1}{\sqrt{2}}\left(X^{2j} + iX^{2j+1}\right), \qquad \bar{\mathbf{X}}^j(z) = \tfrac{1}{\sqrt{2}}\left(X^{2j} - iX^{2j+1}\right), \tag{4.30}$$

$$\Psi^j_\pm(z) = \tfrac{1}{\sqrt{2}}\left(\psi^{2j} \pm i\psi^{2j+1}\right). \tag{4.31}$$

The coefficient in front of the linear combinations is chosen such that the OPEs are

$$\mathbf{X}^j(z)\bar{\mathbf{X}}^k(w) \sim -\frac{\delta^{jk}}{(z-w)^2}, \quad \Psi^j_\pm(z)\Psi^k_\mp(w) \sim \frac{\delta^{jk}}{z-w} \tag{4.32}$$

and all other OPEs are non-singular. We know explicit realisations of the energy momentum tensor T and the supercurrent G that satisfy the commutation relations of the $\mathcal{N}=1$, $\hat{c}=2$ superconformal algebra in terms of the matter fields,

$$T(z) = -\partial\mathbf{X}^j\partial\bar{\mathbf{X}}_j - \tfrac{1}{2}\Psi^j_+\partial(\Psi_-)_j - \tfrac{1}{2}\Psi^j_-\partial(\Psi_+)_j\,, \tag{4.33}$$

$$G(z) = \sqrt{2}\left(\partial\bar{\mathbf{X}}\Psi_+ + \partial\mathbf{X}\Psi_-\right). \tag{4.34}$$

This makes manifest that $(\partial\bar{\mathbf{X}}^j, \Psi^j_-)$ and $(\partial\mathbf{X}^j, \Psi^j_+)$ each give one $\mathcal{N}=1$ supermultiplet each in the sense that

$$\{G_{-\frac{1}{2}}\Psi^j_+\}(z) = \sqrt{2}\partial\mathbf{X}^j(z)\,, \quad \{G_{\frac{1}{2}}\mathbf{X}^j\}(z) = \sqrt{2}\Psi^j_+\,, \tag{4.35}$$

and similarly for $\partial\bar{\mathbf{X}}^j$ and Ψ^j_-.

It is well-known that there exists an U(1) current, the fermion number current, that is defined by

$$J(z) = (\Psi^j_+\Psi_{-,j})(z)\,. \tag{4.36}$$

With this definition, the complex fermionic fields $\Psi^j_\pm$ have fermion number ±1 while the bosonic fields are neutral. Furthermore, T is neutral as well but the supercurrent splits in two components, one with negative and one with positive fermion number,

$$G^+(z) = \sqrt{2}(\partial\bar{\mathbf{X}}^j\Psi_{+,j})(z)\,, \tag{4.37}$$

$$G^-(z) = \sqrt{2}(\partial\mathbf{X}^j\Psi_{-,j})(z)\,. \tag{4.38}$$

This new set of currents generate an $\mathcal{N}=2$, $\hat{c}=2$ superconformal algebra.

To further extend the algebra, we define two normal ordered products of fields as follows:

$$J^{++}(z) = i(\Psi^1_+\Psi^2_+)(z)\,, \qquad J^{--}(z) = i(\Psi^1_-\Psi^2_-)(z)\,. \tag{4.39}$$

It is not difficult to see that both of these fields are currents in the sense that they have conformal dimension one. Furthermore, one notes that one of them has fermion number +2 while the other has fermion number −2. Finally, the OPE between them reads

$$J^{++}(z)J^{--}(w) \sim \frac{1}{(z-w)^2} + \frac{J(z)}{z-w} . \tag{4.40}$$

Hence we lifted the fermion current to a full affine $\mathfrak{su}(2)$ algebra with level $k = 1$. In order to obtain an $\mathcal{N} = 4$ superconformal algebra, we define further fields to complete the representations G^+ and G^- sit in,

$$\tilde{G}^-(z) := \{J_0^{--}G^+\}(z) = -\sqrt{2}i\,\epsilon_{jk}(\partial\bar{\mathbf{X}}^j\Psi_-^k)(z)\,, \tag{4.41}$$

$$\tilde{G}^+(z) := \{J_0^{++}G^+\}(z) = \sqrt{2}i\,\epsilon_{jk}(\partial\mathbf{X}^j\Psi_+^k)(z)\,. \tag{4.42}$$

By construction, we have uncovered the $\mathcal{N} = 4$ $\hat{c} = 2$ superconformal algebra within string theory on T^4. The matter fields arrange themselves in representations of the Lie superalgebra $\mathfrak{psl}(2|2)$ spanned by the modes in (4.26) as depicted in Fig. 4.1. We see that the matter fields make up two small $\mathcal{N} = 4$ supermultiplets.

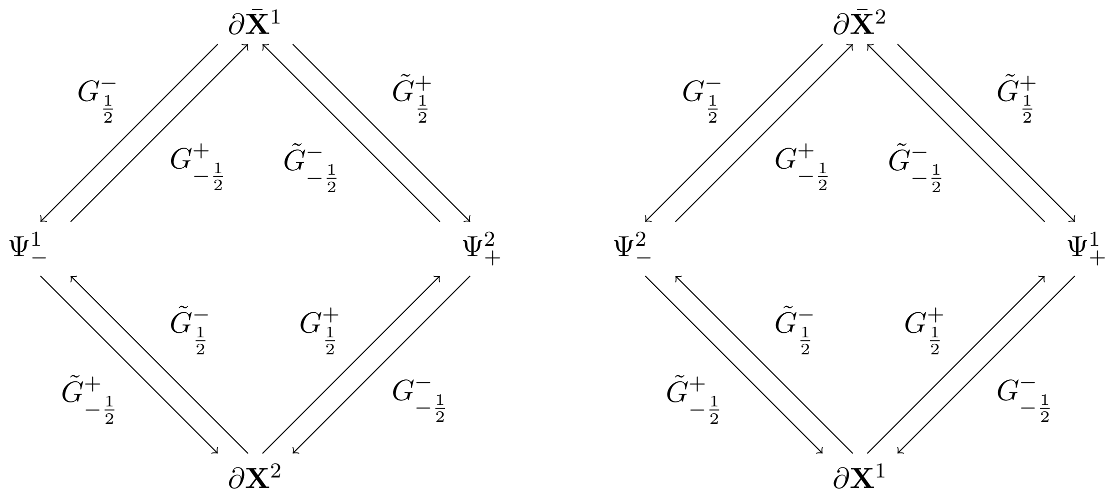


Figure 4.1: *The* $\mathcal{N} = 4$ *multiplets of string theory on* T^4

Finally, we perform a topological twist. As this just means to add background charge changing the conformal weight of the fields proportional to their charge, it implies that the fields Ψ_-^j become "currents" (h=1) while the Ψ_+^j have vanishing conformal weight now. Of course, according to table 4.1, the supercurrents have now integer conformal weight as well. Hence it makes sense to consider the action of the horizontal subalgebra only, that is, the action of the zero modes. The multiplet structure with respect to the zero modes is depicted in Fig. 4.2. This mimics the multiplet structure before twisting, but with the fields Ψ_+^j replaced with $\partial\Psi_+^j$. This is necessary as the zero mode action does not change the conformal weight, but the Ψ_+^j have vanishing conformal weight after twisting. The partial derivative, or equivalently the action of the Virasoro mode L_{-1}, takes care of this. With respect to the full $\mathcal{N} = 4$ zero mode action the Ψ_+^j are singlets, *i.e.* they are annihilated by all zero modes, except the fermion number. The twisted conformal field theory presented

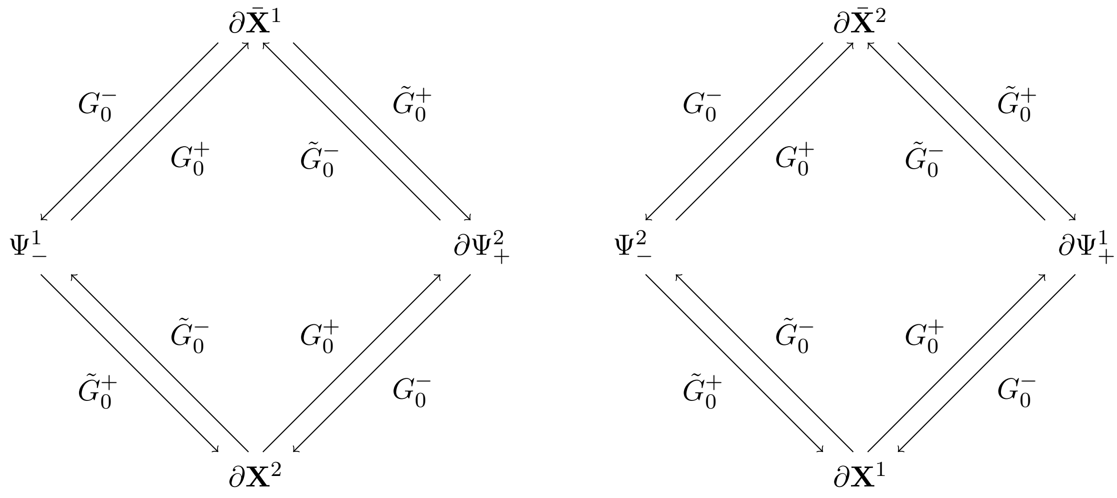


Figure 4.2: *The $\mathcal{N} = 4$ multiplets of string theory on T^4 after the topologically twist considering the zero-mode action only.*

here will play an important role later on when considering compactification-dependent physical states of string theory compactified on T^4.

4.2 Space-Time Fields

We have discussed the superstring in terms of its world sheet description, *i.e.* we have introduced supersymmetry on the world sheet by adding fermionic partners to the world sheet conformal field theory. Since actually we are interested in space-time fermions, as those are the ones appearing in effective low energy theories, one expects also the existence of fields in the spinor representation of the space-time Lorentz group $SO(9,1)$ (or rather its Lie algebra $\mathfrak{so}(9,1)$). In the following we will show how the world sheet fermions give rise to space-time Lorentz generators and representations thereof.

In the Ramond sector, the zero modes of the world sheet fermions act on the Ramond vacuum like the ten-dimensional gamma matrices because

$$[\psi_0^\mu, \psi_0^\nu]_+ = \eta^{\mu\nu} \,. \tag{4.43}$$

If one has a representation of the Clifford algebra, a representation of the corresponding Lorentz group comes basically for free. In general, the corresponding Lorentz generators are given by

$$\Sigma^{\mu\nu} = [\psi_0^\mu, \psi_0^\nu]_- \,. \tag{4.44}$$

Therefore we expect the Lorentz currents in the superstring world sheet theory also to be bilinears in the fermions.

Instead of the Lorentz group $SO(9,1)$ we will be discussing its Wick-rotated version

$SO(10)$. First, we pair the ten fermions into five currents. A common choice [47] is

$$i\partial H^a = \left(\psi^{2a}\psi^{2a+1}\right). \tag{4.45}$$

There is no need to introduce screening charges for the H^a as has been argued in section 3.3.1. Having defined bosons H^a, we can define vertex operators[2]

$$e^{i\alpha\cdot H}, \quad \text{where} \quad \alpha\cdot H = \sum_{a=1}^{5}\alpha_a H^a. \tag{4.46}$$

The conformal weights of these operators are easily found to be $h^\alpha = \frac{1}{2}\sum_a \alpha_a^2 = \frac{1}{2}\alpha^2$. The $SO(10)$ space-time symmetry corresponds to a Noether current on the world sheet, *i.e.* a world sheet field of conformal weight 1. Furthermore, since $e^{\pm iH}$ gives rise to the fermions and we are considering normal ordered products of them, we also recognise that $\alpha_a = 0, \pm 1$. Using eq. (4.46) then yields the conditions

$$\alpha^2 = 2, \quad \alpha_a \in \{-1, 0, 1\}. \tag{4.47}$$

But this is just a description of the D_5 root system, *i.e.* the root system of the Lie algebra $\mathfrak{so}(10)$. Indeed, recalling the OPEs

$$\begin{aligned} e^{i\alpha\cdot H}(z)e^{i\beta\cdot H}(w) &\sim (z-w)^{\alpha\cdot\beta}e^{i(\alpha+\beta)\cdot H}(z), \quad \text{if } \beta \neq -\alpha, \\ e^{i\alpha\cdot H}(z)e^{-i\alpha\cdot H}(w) &\sim \tfrac{1}{(z-w)^2} + \tfrac{\partial H}{z-w}, \end{aligned} \tag{4.48}$$

we see that the modes

$$E^\alpha_m = \oint dz\, z^m e^{i\alpha\cdot H}(z) \tag{4.49}$$

fulfill the commutation relations of the Kac-Moody algebra $\widehat{\mathfrak{so}}(10)_1$ at affine level one:

$$[E^\alpha_m, E^\beta_n] = \begin{cases} E^{\alpha+\beta}_{m+n} & \text{if } \alpha\cdot\beta = -1, \\ n\delta_{m+n} + \alpha\cdot\partial H & \text{if } \alpha\cdot\beta = -2, \\ 0 & \text{otherwise}. \end{cases} \tag{4.50}$$

The horizontal subalgebra, *i.e.* the subalgebra spanned by the zero modes, gives a realisation of the space-time Lorentz symmetry $\mathfrak{so}(10)$. It is then possible to define states transforming in any representation of the space-time symmetry $\mathfrak{so}(10)$. We just need to specify the highest weight state by giving a weight vector ω in the D_5 weight lattice Λ_{D_5}. By the above construction, this yields a vertex operator of the form

$$S_\omega(z) = e^{i\omega\cdot H}(z), \tag{4.51}$$

[2] In order to ensure the correct commutation relations of the vertex operators $e^{i\alpha\cdot H}$, cocycle factors should be included in the definition as well [131]. However, here and in the following, we assume that a consistent choice of cocycles exists and they are implicitly included in the definition of these vertex operators.

which, by the operator-state-correspondence, is associated with a state

$$|\omega\rangle = S_\omega(0)\Omega\,. \tag{4.52}$$

This is the highest weight state of the representation specified by ω. In order to obtain space-time fermions, we need states in the $\mathbf{16_s}$. Therefore we choose one of the following weight vectors:

$$\omega = \left(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\right) \in \tfrac{1}{2}\left(\mathbb{Z}^\times\right)^5\,, \tag{4.53}$$

where $\mathbb{Z}^\times = \{-1, +1\}$ is the group of units of $\mathbb{Z}$. The weight vectors with an even number of minus signs are the states of the $\mathbf{16_s}$, while those with an odd number of minus signs give rise to the conjugate representation $\mathbf{16_c}$. In the following, we will sometimes just write S^a, where a denotes the spinor index of the $\mathbf{16_s}$, and S_a in the case of the conjugate spinor representation $\mathbf{16_c}$.

However, we note that the fields $S^a(z)$ that correspond to the weight vector ω have conformal weight $\frac{5}{8}$. This can be healed by using the ghost fields of section 3.3.3. Recalling the conformal weight of $e^{n_\phi \phi}$ from eq. (3.138), we see that $h\left(e^{-\frac{1}{2}\phi}\right) = \frac{3}{8}$. So the actual vertex operator we need to consider is

$$e^{-\frac{1}{2}\phi} S_a(z)\,. \tag{4.54}$$

This fits with our earlier observation that physical Ramond states have ghost number $-\frac{1}{2}$. In fact, (4.54) can be interpreted as the target space supersymmetry generating current in the $-\frac{1}{2}$ ghost picture.

4.3 Introducing Supersymmetry in WZW Models

Like in flat space, there are basically two ways of extending WZW models on G to include supersymmetry. Either fermionic partners of the embedding fields in the non-linear σ-model action are added, which yields $\mathcal{N} = 1$ supersymmetry on the world sheet in string theory terms; or one tries to achieve target space supersymmetry by choosing G to be a Grassmannian manifold, the analogue of superspace in the flat case. In algebraic terms, one would rather say that G should be a Lie supergroup instead of a Lie group. Even though the first approach seems to be straightforward, there are some subtleties to be dealt with [99, 134].

4.3.1 Supersymmetric WZW models

The first objective in this section is to explain how to introduce world sheet supersymmetry to the game. This might be thought of as the WZW analogue of going from bosonic string theory to the RNS superstring. The resulting theory will be called a *supersymmetric* WZW model, which has to be distinguished from the WZW model on Lie supergroups that have target space supersymmetry rather than world sheet supersymmetry.

So suppose we are given some current algebra $\mathcal{J}^a$. In the same way as fermionic degrees of freedom are introduced in the transition from the bosonic string to the superstring, we can introduce fermionic partners of the currents $\mathcal{J}^a$. Because the supercurrent transforms in a singlet with respect to the zero mode subalgebra of the current algebra, the fermionic partners transform in the same representation as the currents, *i.e.* they transform in the adjoint representation as well. So let us denote the fermionic partner of $\mathcal{J}^a$ by ψ^a. Including the new fields, the OPEs become

$$\mathcal{J}^a(z)\mathcal{J}^b(w) \sim \frac{k\kappa^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c\mathcal{J}^c(w)}{z-w}\,, \tag{4.55}$$

$$\mathcal{J}^a(z)\psi^b(w) \sim \frac{if^{ab}{}_c\psi^c(w)}{z-w}\,, \tag{4.56}$$

$$\psi^a(z)\psi^b(w) \sim \frac{k\kappa^{ab}}{z-w}\,. \tag{4.57}$$

It is possible to redefine the bosonic currents in such a way that their OPEs with the fermions become nonsingular. From the OPE

$$f^a{}_{bc}\Big(\psi^b\psi^c\Big)(z)\psi^d(w) \sim -\frac{2\,kf^{ad}{}_c\psi^c(w)}{z-w} \tag{4.58}$$

it is evident that one can linearly combine $f^a{}_{bc}\big(\psi^b\psi^c\big)(z)$ with $\mathcal{J}^a$ in such a way that the fermions become free. Hence we define new currents

$$\tilde{\mathcal{J}}^a(z) \equiv \mathcal{J}^a(z) + \tfrac{i}{2k}f^a{}_{bc}\Big(\psi^b\psi^c\Big)(z)\,. \tag{4.59}$$

Using the Jacobi identity, one can check that these currents generate the same current algebra as the $\mathcal{J}^a$ except for a shift in the affine level due to the OPE between the bilinears in the fermionic fields,

$$\tilde{\mathcal{J}}^a(z)\tilde{\mathcal{J}}^b(w) \sim \frac{(k-h^\vee)\kappa^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c\tilde{\mathcal{J}}^c(w)}{z-w}\,, \tag{4.60}$$

where $h^\vee$ is the dual Coxeter number introduced in (3.71). Hence we see that the supersymmetrised $\hat{\mathfrak{g}}_k$ current algebra is isomorphic to a $\hat{\mathfrak{g}}_{k-h^\vee}$ current algebra plus free fermions. It is straightforward to write down the energy momentum tensor for the latter using the Sugawara construction,

$$T(z) = \frac{1}{2k}\kappa_{ab}\Big(\tilde{\mathcal{J}}^a\tilde{\mathcal{J}}^b\Big) - \frac{1}{2k}\kappa_{ab}\Big(\psi^a\partial\psi^b\Big)\,. \tag{4.61}$$

Of course, one can always trade the new currents $\tilde{\mathcal{J}}^a$ for the original ones $\mathcal{J}^a$ using definition (4.59) in order to obtain the energy momentum tensor of the supersymmetric WZW model. Using our result for the central charge of WZW models, we immediately get the central charge of the supersymmetric WZW model,

$$c_{\hat{\mathfrak{g}}} = \frac{(k-h^\vee)\dim\mathfrak{g}}{k} + \frac{\dim\mathfrak{g}}{2} = \frac{\frac{3}{2}k-h^\vee}{k}\dim\mathfrak{g}\,. \tag{4.62}$$

In order to uncover the world sheet supersymmetry, we have to find a current that exchanges the supersymmetric partners $\mathcal{J}^a$ and ψ^a. A suitable current is

$$G(z) = \frac{1}{k}\left[\kappa_{ab}\Big(\tilde{\mathcal{J}}^a\psi^b\Big) - \frac{if_{abc}}{6k}\Big(\psi^a\psi^b\psi^c\Big)\right] , \tag{4.63}$$

which can be checked to define an $\mathcal{N} = 1$ superconformal algebra together with $T(z)$.

For application to string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$, the supersymmetric version of the $\widehat{\mathfrak{sl}}(2)_k \oplus \widehat{\mathfrak{su}}(2)_{k'}$ current algebra is of particular interest [99]. Due to the automorphism of affine Lie algebras $\widehat{\mathfrak{sl}}(2)_k \simeq \widehat{\mathfrak{su}}(2)_{-k}$, this current algebra is the same as the $\widehat{\mathfrak{su}}(2)_{-k} \oplus \widehat{\mathfrak{su}}(2)_{k'}$ current algebra. The $\widehat{\mathfrak{su}}(2)_k$ current algebra has been discussed at length in section 3.2.4, where we also determined its Sugawara tensor, which we denote by $T_{(k)}(z)$. Hence the Sugawara tensor of the complete $\widehat{\mathfrak{sl}}(2)_k \oplus \widehat{\mathfrak{su}}(2)_{k'}$ current algebra can be written as $T_{(-k)}(z) + T_{(k')}(z)$ and the total central charge of the associated Virasoro algebra is just the sum of the central charges of the individual current algebras,

$$c_{(\widehat{\mathfrak{sl}}_k(2)\oplus\widehat{\mathfrak{su}}_{k'}(2))} = c_{\widehat{\mathfrak{su}}_{-k}(2)} + c_{\widehat{\mathfrak{su}}_{k'}(2)} = \frac{\frac{3}{2}k+2}{k} + \frac{\frac{3}{2}k'-2}{k'} . \tag{4.64}$$

Criticality of string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$ requires the central charge to be $c = 9$, which yields the relation $k = k'$ when imposed on (4.64). So when having string theory in mind, one has to make sure that the affine levels of the current algebras associated with the AdS_3 and the S^3 part coincide.

For later use, the important point to keep in mind from this discussion is that the RNS string on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}$ with NSNS flux only, where $\mathcal{M}$ is some compactification manifold, has an isomorphic description in terms of an $\mathrm{SL}(2) \times \mathrm{SU}(2)$ WZW model plus a conformal field theory of six free fermions plus some internal CFT on $\mathcal{M}$. The existence of such a description is crucial in order to reformulate RNS string theory in a manifest target space supersymmetric way in chapter 5.

4.3.2 WZW models on Lie supergroups

The discussion of WZW models and its symmetries has been limited to Lie groups so far. However, one might wonder whether WZW models can be defined on supergroups as well. And indeed, it is possible, but one needs to take care in order to get the signs straight. The arguments are basically the same as in the Lie group case in section 3.2.

The first problem in the Lie supergroup case is that there might be no Killing form. However, we assume that there exists at least some supersymmetric invariant bilinear form, which we will also denote by κ^{AB}, defined by the supertrace

$$\kappa^{AB} = \mathrm{str}\,\left(t^A t^B\right) , \tag{4.65}$$

where the t^A are the generators of $\mathfrak{g}$ in some representation such that κ^{AB} is not vanishing. We will assume that this bilinear form inherits all properties of the Killing form, *i.e.*

invariance, supersymmetry and consistency (cf. section 2.1.1). Here capital letters $A, B, \ldots$ describe indices of the bosonic as well as of the fermionic generators. We introduce a grading of indices by

$$|A| = \begin{cases} 0 & \text{if } t^A \in \mathfrak{g}^{(0)} \\ 1 & \text{if } t^A \in \mathfrak{g}^{(1)} \end{cases}, \qquad |A+B| \equiv (|A|+|B|) \bmod 2\,, \tag{4.66}$$

where the Lie superalgebra was decomposed in a bosonic and a fermionic part $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$. Supersymmetry of the bilinear form implies that $\kappa^{AB} = (-1)^{|A||B|}\kappa^{BA}$. We define the inverse κ_{AB} by

$$\kappa_{AB}\kappa^{BC} = (-1)^{|A|}\delta_A^C \quad \Rightarrow \quad \kappa_{BA}\kappa^{BC} = \kappa_{AB}\kappa^{CB} = \delta_A^C\,. \tag{4.67}$$

The latter equality holds due to the consistency of the invariant form. In the Lie superalgebra case, the structure constants are defined by

$$[t^A, t^B] = i f^{AB}{}_C t^C\,. \tag{4.68}$$

By definition, they are antisupersymmetric in the first two indices, $f^{AB}{}_C = -(-1)^{|A||B|} f^{BA}{}_C$. We further define

$$f^{ABC} \equiv f^{AB}{}_D \kappa^{DC} \quad \overset{(4.67)}{\Rightarrow} \quad f^{AB}{}_C = f^{ABD}\kappa_{CD}\,. \tag{4.69}$$

The structure constants with only upper indices are completely antisupersymmetric because

$$\begin{aligned} i f^{ABC} &= \mathrm{str}\,\left([t^A, t^B]t^C\right) = \mathrm{str}\,\left(t^A t^B t^C\right) - (-1)^{|A||B|}\mathrm{str}\,\left(t^B t^A t^C\right) \\ &= \mathrm{str}\,\left(t^C t^A t^B\right)(-1)^{|C||A+B|} - (-1)^{|A||B|}(-1)^{|B||A+C|}\mathrm{str}\,\left(t^A t^C t^B\right) \\ &= -(-1)^{|A||C|}\left[\mathrm{str}\,\left(t^A t^C t^B\right) - (-1)^{|B||C|}\mathrm{str}\,\left(t^C t^A t^B\right)\right] \\ &= -(-1)^{|A||C|}\mathrm{str}\,\left([t^A, t^C]t^B\right) = -(-1)^{|A||C|} i f^{ACB}\,. \end{aligned}$$

After these preliminary remarks on the appropriate supersymmetric generalisation of the Killing form and the structure constants, we now turn to the actual discussion of the WZW model. Let $g(z, \bar{z})$ take values in some Lie supergroup G with an associated Lie superalgebra $\mathfrak{g}$ that is simple. In the same way as a Lie group is a differentiable manifold, G is a Grassmannian manifold and hence, from a string theoretic perspective, one might think of the WZW model as a string moving in superspace, which then makes target space supersymmetry manifest.

The classical theory can be constructed using the same arguments as in section 3.2.1 only with the supertrace replacing the usual trace in the action. Instead of repeating the above discussion, we rather turn our attention immediately to the discussion on the quantum level, *i.e.* the current algebra.

The currents are defined as in the non-supersymmetric case. In particular, the holomorphic current is $\mathcal{J}(z) = -k\partial g g^{-1}$. As before, it generates the local left action of G. The variation of the current is

$$\begin{aligned}\delta J &= -k\partial\epsilon_A t^A + [\epsilon_A t^A, \mathcal{J}]_- \\ &= -k\partial\epsilon_A t^A + \epsilon_A \mathcal{J}^C \kappa_{CB}[t^A, t^B] \\ &= -k\partial\epsilon_A t^A + i\epsilon_A f^{AB}{}_D \kappa_{CB} J^C t^D \,,\end{aligned} \tag{4.70}$$

where ϵ_A is Grassmann-valued if and only if A is a fermionic index. We indicated one commutator with a subscript minus to emphasise that this is an actual commutator in contrast to the graded commutator $[\cdot,\cdot]$, which becomes an anticommutator if both entries are fermionic. Using the Ward identity (3.21), we obtain

$$\mathcal{J}^A \mathcal{J}^B = \frac{(-1)^{|B|}\kappa^{AB}}{(z-w)^2} + \frac{i(-1)^{|B|(1+|C|)} f^{AB}{}_C J^C}{z-w} \,. \tag{4.71}$$

This looks like a current algebra that corresponds to an affine Lie algebra except for some unusual signs. Let us have a closer look at those signs. First we note that everything works out if $|B| = 0$, *i.e* the bosonic part of the Lie algebra yields a current algebra as expected. If $|B| = 1$, we have to distinguish two cases. Let us first assume that $|A| = 0$. Then $|C| = 1$ because $[\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}] \subset \mathfrak{g}^{(1)}$. So the sign in front of the first order pole vanishes. Furthermore, the second order pole vanishes completely since $\kappa^{AB} = 0$ if A and B have opposite grading. We are left with the case $|A| = |B| = 1$. In that case, the OPE gets an overall minus sign. This minus sign can be dealt with in both type I and type II Lie superalgebras, but in view of the main subject of the rest of this work, we restrict to the type I case. Recall that in type I Lie superalgebras the fermionic part further decomposes as

$$\mathfrak{g}^{(1)} = \mathfrak{g}_{+1} \oplus \mathfrak{g}_{-1} \quad \text{such that} \quad [\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1}] = 0\,, \quad \left[\mathfrak{g}^{(0)}, \mathfrak{g}_{\pm 1}\right] \subset \mathfrak{g}_{\pm 1}\,. \tag{4.72}$$

In other words, the fermionic part $\mathfrak{g}^{(1)}$ is reducible as a $\mathfrak{g}^{(0)}$-representation. But this means that the OPE of two fermionic currents is only nonsingular if one of them corresponds to an element in $\mathfrak{g}_{+1}$ and the other one to an element in $\mathfrak{g}_{-1}$. Thus in order to take care of the overall minus sign, we should rescale the currents $\mathcal{J}^A \to -\mathcal{J}^A$ if $t^A \in \mathfrak{g}_{+1}$. To summarise, after the rescaling we recover the usual form of the current algebra

$$\mathcal{J}^A \mathcal{J}^B = \frac{\kappa^{AB}}{(z-w)^2} + \frac{i f^{AB}{}_C J^C}{z-w} \tag{4.73}$$

in the Lie superalgebra case. A Sugawara tensor can be defined by

$$T(z) \equiv \frac{1}{2(k+h^\vee)} \big(\mathcal{J}^A \mathcal{J}^B\big) \kappa_{BA} \,. \tag{4.74}$$

Note the unusual order of indices in κ_{BA}. The reason for this lies implicitly in the definition of the inverse in (4.67). As in the Lie group case, we wish the currents $\mathcal{J}^A(z)$ to be primary with respect to $T(z)$. A calculation along the lines of (3.70) yields

$$\mathcal{J}^A(z)\big(\mathcal{J}^B\mathcal{J}^C\big)(w)\kappa_{CB} = \frac{if^{ABC}\kappa_{CB}}{(z-w)^3} + \frac{2h^\vee \mathcal{J}^A(w)}{(z-w)^2} + \frac{k\left[\kappa^{AB}\kappa_{CB}\mathcal{J}^C(w) + (-1)^{|A||B|}\kappa^{AC}\kappa_{CB}J^B(w)\right]}{z-w}\,.$$

A few comments on that expression are in order. The pole of third order vanishes due to the symmetry properties of the bilinear form and the structure constants. The factor of $(-1)^{|A||B|}$ in the second term of the first order pole comes from the fact that the fermionic currents are Grassmann-valued and hence anticommuting. Furthermore, note that

$$(-1)^{|A||B|}\kappa^{AC}\kappa_{CB} = (-1)^{|A||C|}\kappa^{AC}\kappa_{CB} = (-1)^{|C||B|}\kappa^{AC}\kappa_{CB}\,. \tag{4.75}$$

Now it is clear why we chose the fairly unusual order of indices in the definition of $T(z)$. Only with this choice, the terms in the first order pole sum up to give the Kronecker delta and hence the OPE for primary fields is reproduced.

Along the same lines, it can be checked that $T(z)$ indeed generates conformal transformations. Since this calculation is a little tedious, let us here only derive the central charge, which is given by the fourth order pole of the TT OPE. Using that the $\mathcal{J}^A$ are Virasoro primary, we obtain

$$\begin{aligned} T(z)\big(\mathcal{J}^A\mathcal{J}^B\big)(w) &\sim \frac{1}{2(k+h^\vee)}\oint_w \frac{dx}{(x-w)^3}\left(\frac{k\kappa^{AB}}{(z-x)^2} + \frac{2k\kappa^{AB}}{(z-x)(x-w)}\right) + \mathcal{O}\Big((z-w)^{-3}\Big) \\ &= \frac{1}{2(k+h^\vee)}\frac{k\kappa^{AB}}{(z-w)^4} + \mathcal{O}\Big((z-w)^{-3}\Big) \end{aligned} \tag{4.76}$$

and so the central charge of a WZW-model based on a Lie supergroup with Lie superalgebra $\mathfrak{g}$ is

$$c = \frac{k\,\kappa^{AB}\kappa_{BA}}{k+h^\vee} = \frac{k\,(-1)^{|A|}\delta^A_A}{k+h^\vee} = \frac{k\,\mathrm{sdim}\,\mathfrak{g}}{k+h^\vee}\,. \tag{4.77}$$

It is instructive to emphasise the importance of the correct choice of the order of indices in the definition of the Sugawara tensor. The additional sign in the definition of the inverse in (4.67) was crucial in order to be able to define a current algebra in the first place. This minus sign required us to define the Sugawara tensor (4.74) with that unusual order of indices in κ such that the currents are Virasoro primary. In terms of the central charge we observe that due to our choices the central charge of the Virasoro algebra is proportional to the superdimension rather than to the dimension of $\mathfrak{g}$ as a vector space.

4.3.3 The PSL(2|2) WZW model

Of particular importance in this work is the PSL(2|2) WZW model. From a string theoretic point of view the interest in this model originates from the important work by Berkovits, Vafa and Witten [30], which is reviewed in detail in chapter 5. Loosely speaking, they succeeded to show that superstring theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$ with NSNS flux only can be appropriately described by a PSL(2|2) WZW model equipped with an additional superconformal structure on the world sheet, *i.e.* a set of fields living on the world sheet that generate a (twisted) superconformal algebra. It is suggestive that the first step in understanding superstring theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$ in a way that target space supersymmetry is manifest would be to ignore the superconformal structure and investigate the PSL(2|2) WZW model alone. A lot of progress has been made in this direction in the last decade, which is reviewed in the present section [33, 51, 104, 105].

As has already been indicated in the last section, the Lie superalgebra $\mathfrak{psl}(2|2)$ has the property that its Killing form and hence the dual Coxeter number vanishes, $h^\vee = 0$. This has particularly remarkable consequences. The first one is the form of the Sugawara tensor, which becomes

$$T(z) \equiv \frac{1}{2k}(\mathcal{J}^A \mathcal{J}^B)\kappa_{BA}\,. \tag{4.78}$$

This agrees with the classical form of the energy momentum tensor, *i.e.* the constant multiplying the normal ordered product is not subject to quantum corrections. Indeed, the vanishing of the dual Coxeter number implies that double contractions of two structure constants always vanish. This has the remarkable effect that the nonlinear σ-model on PSL(2|2) is conformally invariant *even without including the Wess-Zumino term* [33]. The model with $k = 0$, which corresponds just to the kinetic term of the WZW action, is often referred to as the *principal chiral model*. However, the currents are only holomorphic and antiholomorphic, respectively, if we consider the theory at the WZW point in moduli space and it is only at this point where the powerful tools of complex analysis and vertex operator algebra are directly applicable. Hence a common strategy in attacking the problem of understanding the whole moduli space of conformal field theories with $\widehat{\mathfrak{psl}}(2|2)_k$ symmetry is to first understand the theory at the WZW point and then deform this theory to other points in the moduli space, hoping that the vanishing of double contractions of structure constants might imply non-renormalisation theorems [105]. From a string theory perspective, the WZW point corresponds to the case of a pure NSNS flux background which can be treated in the RNS formulation as well. The deformation away from the WZW point is then equivalent to adding RR flux to the background.

In the present work, we will try to contribute to the first step of the above strategy, *i.e* understanding the PSL(2|2) WZW model and its connection to string theory with NSNS flux only. Hence let us concentrate on the WZW point. Another effect of the vanishing dual Coxeter number is that the central charge is independent of the level,

$$c = \frac{k\,\mathrm{sdim}\,\mathfrak{g}}{k} = \mathrm{sdim}\,\mathfrak{g}\,. \tag{4.79}$$

It actually coincides with the superdimension of $\mathfrak{psl}(2|2)$ which equals -2. Conformal field theories with $c = -2$ are known and are closely connected to the topic of logarithmic field theories [82, 86, 127]. Indeed, it has been argued that the PSL(2|2) WZW model gives rise to a logarithmic field theory [105] like many other WZW models on supergroups (see *e.g.* [163, 167]). The properties of logarithmic conformal field theories, namely the appearance of reducible but indecomposable representations, have a strong influence on the massless sector of the theory, which will be the topic of the next section.

4.3.4 The spectrum of the LCFT underlying the PSL(2|2) WZW model

Ultimatively, we want to describe the conformal field theory whose BRST cohomology describes the physical string states on $AdS_3 \times S^3$. For the case where we just have pure NSNS flux, this is the WZW model based on the supergoup PSL(2|2) [30]. Non-linear σ-models with supergroup targets lead to logarithmic conformal field theories [105, 163, 167]. We can therefore apply the general ideas of [86, 87] in order to construct their spectrum. This is best described as a certain quotient space of the tensor products of projective covers as we will explain now.

The spectrum of the WZW model based on the supergroup PSL(2|2) can be described in terms of representations of the affine Lie superalgebra based on $\mathfrak{psl}(2|2)$. As is familiar from the usual WZW models, affine representations are uniquely characterised by the representations of the zero modes that simply form a copy of $\mathfrak{psl}(2|2)$; these zero modes act on the Virasoro highest weight states. In order to describe the spectrum of the conformal field theory, we therefore only have to explain which combinations of representations of the zero modes appear for left- and right-movers. In fact, in this section we shall only study these massless 'ground states', and thus the affine generators will not make any appearance. The massive spectrum (for which the affine generators will play an important role) will be discussed in detail in chapter 7.

The structure of the ground states $\mathcal{H}^{(0)}$ should be determined by the harmonic analysis of the supergroup. This point of view suggests [158] that $\mathcal{H}^{(0)}$ is the quotient of $\hat{\mathcal{H}}$ by a subrepresentation $\mathcal{N}$

$$\mathcal{H}^{(0)} = \hat{\mathcal{H}}/\mathcal{N} \ , \qquad \text{where} \qquad \hat{\mathcal{H}} = \bigoplus_{(j_1,j_2)} \mathcal{P}(j_1,j_2) \otimes \overline{\mathcal{P}(j_1,j_2)} \ , \tag{4.80}$$

and the sum runs over all (allowed) irreducible representations $\mathcal{L}(j_1, j_2)$, with $\mathcal{P}(j_1, j_2)$ the corresponding projective cover. The relevant quotient should be such that, with respect to the left-moving action of $\mathfrak{psl}(2|2)$, we can write

$$\mathcal{H}^{(0)} = \bigoplus_{(j_1,j_2)} \mathcal{P}(j_1,j_2) \otimes \overline{\mathcal{L}(j_1,j_2)} \ , \tag{4.81}$$

and similarly with respect to the right-moving action. Furthermore, the analysis of a specific class of logarithmic conformal field theories in [86, 87] suggests, that the subrepresentation $\mathcal{N}$ has a general simple form that we shall explain below. This ansatz was obtained in [86] for the $(1, p)$ triplet models by studying the constraints the bulk spectrum has to obey in order to be compatible with the analogue of the identity boundary condition (that had been previously proposed). In [87] essentially the same ansatz was used in an example where a direct analogue of the identity boundary condition does not exist, and again the resulting bulk spectrum was found to satisfy a number of non-trivial consistency conditions, thus justifying the ansatz a posteriori. Given the close structural similarity between the projective covers of [87] and those of the atypical representations above, it seems very plausible that the ansatz of [87] will also lead to a sensible bulk spectrum in our context, and as we shall see this expectation is borne out by our results.

In the following we shall only consider the 'atypical' part of $\mathcal{H}^{(0)}$, since, using the mass-shell condition, these are the only representations that appear for the massless string states. Actually, it is only for these sectors that the submodules $\mathcal{N}$ are non-trivial (since for typical (j_1, j_2), the projective cover $\mathcal{P}(j_1, j_2)$ agrees with the irreducible representation $\mathcal{L}(j_1, j_2)$, and hence $\mathcal{N}$ has to be trivial).

Following [86, 87] we then propose that the subspace $\mathcal{N}$ by which we want to divide out $\hat{\mathcal{H}}$, is spanned by the subrepresentations

$$\mathcal{N}^{\pm}_{\sigma}(j) \quad = \quad \left(s^{\pm}_{\sigma} \otimes \overline{\mathrm{id}} - \mathrm{id} \otimes \overline{(s^{\pm}_{\sigma})^{\vee}}\right)\left(\mathcal{P}(j-\tfrac{\sigma}{2}) \otimes \overline{\mathcal{P}(j)}\right) \ , \tag{4.82}$$

where $s^{\pm}_{\sigma}$ was defined in section 2.2.3.1, and $j \geq \max\{0, \frac{\sigma}{2}\}$ with $\sigma = \pm 1$. It is easy to see from the definition of $s^{\pm}_{\sigma}$, see Fig. 2.8, that the dual homomorphism equals

$$(s^{\pm}_{\sigma})^{\vee} = s^{\mp}_{-\sigma} \ . \tag{4.83}$$

Together with (2.52), we can then write the two terms as

$$\begin{aligned} s^{\pm}_{\sigma} \otimes \overline{\mathrm{id}}\left(\mathcal{P}(j-\tfrac{\sigma}{2}) \otimes \overline{\mathcal{P}(j)}\right) &= \mathcal{M}^{\pm}_{\sigma}(j-\tfrac{\sigma}{2}) \otimes \overline{\mathcal{P}(j)} \subset \left(\mathcal{P}(j) \otimes \overline{\mathcal{P}(j)}\right) \\ \mathrm{id} \otimes \bar{s}^{\mp}_{-\sigma}\left(\mathcal{P}(j-\tfrac{\sigma}{2}) \otimes \overline{\mathcal{P}(j)}\right) &= \mathcal{P}(j-\tfrac{\sigma}{2}) \otimes \overline{\mathcal{M}^{\mp}_{-\sigma}(j)} \subset \left(\mathcal{P}(j-\tfrac{\sigma}{2}) \otimes \overline{\mathcal{P}(j-\tfrac{\sigma}{2})}\right) \ , \end{aligned} \tag{4.84}$$

and therefore the two subrepresentations in (4.82) are individual subrepresentations of different direct summands of $\hat{\mathcal{H}}$. Dividing out by $\mathcal{N}$ therefore identifies

$$\left(\mathcal{P}(j) \otimes \overline{\mathcal{P}(j)}\right) \supset \ \mathcal{M}^{\pm}_{\sigma}(j-\tfrac{\sigma}{2}) \otimes \overline{\mathcal{P}(j)} \ \sim \ \mathcal{P}(j-\tfrac{\sigma}{2}) \otimes \overline{\mathcal{M}^{\mp}_{-\sigma}(j)} \ \subset \left(\mathcal{P}(j-\tfrac{\sigma}{2}) \otimes \overline{\mathcal{P}(j-\tfrac{\sigma}{2})}\right) \ . \tag{4.85}$$

Note that this equivalence relation does not preserve the $\mathbb{Z}$-grading: for example, by considering the corresponding heads, we get the equivalence relation

$$\left(\mathcal{P}(j) \otimes \overline{\mathcal{P}(j)}\right) \supset \ \mathcal{L}_{\pm 1}(j-\tfrac{\sigma}{2}) \otimes \overline{\mathcal{L}_0(j)} \ \sim \ \mathcal{L}_0(j-\tfrac{\sigma}{2}) \otimes \overline{\mathcal{L}_{\mp 1}(j)} \ \subset \left(\mathcal{P}(j-\tfrac{\sigma}{2}) \otimes \overline{\mathcal{P}(j-\tfrac{\sigma}{2})}\right) \ . \tag{4.86}$$

We shall sometimes denote the corresponding equivalence classes by $[\,\cdot\,]$. It is not difficult to see that this equivalence relation leads to a description of $\mathcal{H}^{(0)}$ as in eq. (4.81). Indeed, iteratively applying the above equivalence relation we can choose the representative in such a way that the right-moving factor, say, is the head of the projective cover; this is sketched in Fig. 4.3.

Before concluding this subsection, let us briefly comment on possible generalisations of our ansatz to WZW models on other supergroups, for example those discussed in [158]. Let us label the irreducible representations by λ, and their projective covers by $\mathcal{P}(\lambda)$.

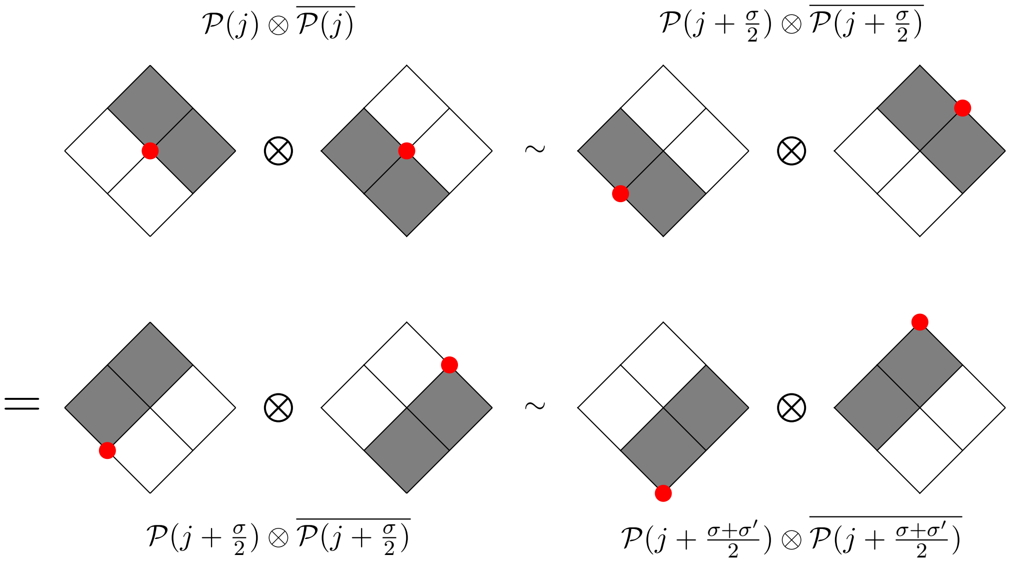


Figure 4.3: *Schematic presentation of the equivalence relation. Each big square represents a projective cover $\mathcal{P}$, and the shaded regions describe the subrepresentations $\mathcal{M}^{\pm}_{\sigma}$ of $\mathcal{P}$. The red dots mark exemplary equivalent irreducible components $\mathcal{L}\otimes\overline{\mathcal{L}}$ in $\mathcal{P}(j)\otimes\overline{\mathcal{P}(j)}$ and $\mathcal{P}(j+\frac{\sigma}{2})\otimes\overline{\mathcal{P}(j+\frac{\sigma}{2})}$, respectively. Note that by applying the equivalence relation, the right-moving irreducible is lifted by one level, while the left-moving one is lowered one level, until the right-moving irreducible is at the head of some projective cover.*

Thus the analogue of (4.80) is

$$\mathcal{H}^{(0)}=\hat{\mathcal{H}}/\mathcal{N}\ ,\qquad \text{where}\qquad \hat{\mathcal{H}}=\bigoplus_{\lambda}\mathcal{P}(\lambda)\otimes\overline{\mathcal{P}(\lambda)}\ . \tag{4.87}$$

In order to construct $\mathcal{N}$ it is again sufficient to concentrate on the atypical sectors since otherwise $\mathcal{P}(\lambda)=\mathcal{L}(\lambda)$ is irreducible and the intersection of $\mathcal{N}$ with $\mathcal{P}(\lambda)\otimes\overline{\mathcal{P}(\lambda)}$ must be trivial. If λ is atypical, on the other hand, $\mathcal{P}(\lambda)$ is only indecomposable, and it contains a maximal proper submodule that we denote by $\mathcal{M}(\lambda)$. Its head is in general a direct sum of irreducible representations $\mathcal{L}(\mu_i)$. Each direct summand generates a submodule $\mathcal{M}(\mu_i)$ of $\mathcal{P}(\lambda)$ which is covered by the projective cover $\mathcal{P}(\mu_i)$. Thus we have the homomorphisms $s_{\mu_i}:\mathcal{P}(\mu_i)\to\mathcal{P}(\lambda)$ via

$$s_{\mu_i}:\mathcal{P}(\mu_i)\twoheadrightarrow\mathcal{M}(\mu_i)\hookrightarrow\mathcal{P}(\lambda)\ . \tag{4.88}$$

The dual homomorphism are then of the form $s^{\vee}_{\mu_i}:\mathcal{P}^{\vee}(\lambda)\longrightarrow\mathcal{P}^{\vee}(\mu_i)$, where the dual representation $\mathcal{M}^{\vee}$ is obtained from $\mathcal{M}$ by exchanging the roles of $\mathfrak{g}_{+1}$ and $\mathfrak{g}_{-1}$. If we assume the projective covers to be self-dual, $\mathcal{P}^{\vee}(\mu)=\mathcal{P}(\mu)$, the dual homomorphisms are of the form

$$s^{\vee}_{\mu_i}:\mathcal{P}(\lambda)\longrightarrow\mathcal{P}(\mu_i)\ . \tag{4.89}$$

It is then again natural to define $\mathcal{N}$ as the vector space generated by

$$\mathcal{N}_{\mu_i}=\left(s_{\mu_i}\otimes\overline{\mathrm{id}}-\mathrm{id}\otimes\overline{s^{\vee}_{\mu_i}}\right)\left(\mathcal{P}(\mu_i)\otimes\overline{\mathcal{P}(\lambda)}\right)\ . \tag{4.90}$$

By the same arguments as above, the resulting quotient space $\mathcal{H}^{(0)}$ then has the desired form [158]

$$\mathcal{H}^{(0)} = \bigoplus_{\lambda} \mathcal{P}(\lambda) \otimes \overline{\mathcal{L}(\lambda)} \tag{4.91}$$

with respect to the left-action. Thus it seems natural that our ansatz for the bulk spectrum will also apply more generally to WZW models on basic type I supergroups, provided that the projective covers are all self-dual $\mathcal{P}^{\vee}(\lambda) \simeq \mathcal{P}(\lambda)$.

The Hybrid String

In this chapter, we will review the construction of the hybrid string. The hybrid formalism allows one to redefine the RNS worldsheet fields in such a way that $\mathcal{N}=1$ space-time supersymmetry in the noncompactified dimensions is made manifest. First applied to string compactifications to four dimensions [15], the idea was soon extended to be applicable to compactifications to six dimensions as well [29]. Shortly afterwards, it was shown that this concept can be used to discuss superstrings moving on an $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}^4$ gravitational background with RR-flux [30]. Furthermore, it was argued that in the case of vanishing RR-flux, superstrings on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}^4$ are appropriately described by an $PSL(2|2)$ WZW model.

5.1 The Gauge-Fixed Superstring as an $\mathcal{N}=4$ Topological String

Superstring theory on some target space is described by a nonlinear σ-model on the world sheet. This σ-model needs be quantised and a convenient and covariant method for doing so is given by the BRST procedure; the superdiffeomorphism invariance of the world sheet is fixed, thereby introducing two bc-systems whose generalities were discussed in section 3.3.1. Namely, these are a $(2,-1)$ bc-system, whose doublet of fields are consequently called b and c, and a $(\frac{3}{2},\frac{1}{2})$ bc-system with fields denoted by β and γ. A residual gauge invariance survives by choosing the gauge parameters to be proportional to the c-ghost. The current generating this residual gauge symmetry is the BRST-current j_{BRST}. The spectrum of the theory is then given by the cohomology of the zero mode of j_{BRST} and demanding that physical states have ghost number $+1$. In addition, the ghost-picture has to be fixed.

This is all standard string theory in the RNS formalism. The interesting fact is that, even though the superdiffeomorphism invariance has been fixed, one can identify a set of fields within the gauge-fixed symmetry that gives rise to an $\mathcal{N}=2$ superconformal algebra. By the construction in 4.1.3 this can be extended an $\mathcal{N}=4$ structure. We will see how the fields have to be chosen and which form the physical state conditions take.

First, let us give the set of fields satisfying the OPEs of the $\mathcal{N}=2$ superconformal algebra [16, 17, 28]:

$$T = T_{\mathcal{N}=1}\,, \qquad G^+ = j_{\mathrm{BRST}}\,, \qquad G^- = b\,, \qquad J = J_{\mathrm{ghost}} = cb + \eta\xi\,, \tag{5.1}$$

where the BRST current j_{BRST} is explicitly given by

$$j_{\mathrm{BRST}} = \gamma G_{\mathrm{matter}} + c(T_{\mathrm{matter}} - \tfrac{3}{2}\beta\partial\gamma - \tfrac{1}{2}\gamma\partial\beta - b\partial c) - \gamma^2 b + \partial^2 c + \partial(c\xi\eta)\,. \tag{5.2}$$

A few comments on the notation are in order. $T_{\mathcal{N}=1} = T_{\mathrm{matter}} + T_{\mathrm{ghost}}$ and $G = G_{\mathrm{matter}} + G_{\mathrm{ghost}}$ are the energy momentum tensor and the supercurrent of the gauge-fixed $\mathcal{N} = 1$ superstring theory, respectively, and J_{ghost} is the ghost number current. Note that the algebra as given above *does not* give the OPEs of an $\mathcal{N} = 2$ superconformal algebra but rather a twisted version of it, *e.g.* note that G^+ and G^- have conformal weight one and two, respectively, and by construction the TT-OPE has no anomalous term since the superconformal ghosts have been included. Of course, the algebra can easily be untwisted by adding a term $\frac{1}{2}\partial J$ to T. For a proof that the fields in (5.1) indeed generate a topologically twisted $\mathcal{N} = 2$ superconformal algebra the interested reader is referred to appendix D.1.

The physical state conditions in the BRST quantisation of the superstring can directly be rewritten in terms of the twisted $\mathcal{N} = 2$ superconformal generators. From the point of view of the RNS formulation, G^+ and J are nothing but the BRST current and ghost number current, respectively, and hence physical states have to satisfy

$$G_0^+\Phi = (J_0 - 1)\Phi = 0 \qquad \text{up to a BRST invariance} \qquad \Phi \sim \Phi + G_0^+\Lambda\,. \tag{5.3}$$

Furthermore, physical states are independent of the zero mode of ξ as it has been discussed in section 3.3.3. This is not yet covered by the conditions above. Recall that we have to restrict to the so called small Hilbert space which is given by the kernel of η_0. Thus we see that physical states of the gauge-fixed $\mathcal{N} = 1$ string in its critical dimension cannot be characterised using only the $\mathcal{N} = 2$ superconformal algebraic structure. But noting that the composite fields

$$J^{++} = (c\eta)\,, \qquad J^{--} = (b\xi) \tag{5.4}$$

together with the ghost current J generate an affine $\widehat{\mathfrak{su}}(2)$ algebra, we can lift the $\mathcal{N} = 2$ superconformal algebra to an $\mathcal{N} = 4$ superconformal algebra. The additional (twisted) super currents are given by [30]

$$\tilde{G}^+ = \eta\,, \tag{5.5}$$

$$\begin{aligned}\tilde{G}^- &= b(e^{\phi}G_{\mathrm{matter}} + \eta e^{2\phi}\partial b - c\partial\xi) \\ &\quad - \xi(T_{\mathrm{matter}} - \tfrac{3}{2}\beta\partial\gamma - \tfrac{1}{2}\gamma\partial\beta + 2b\partial c - c\partial b) + \partial^2\xi\,. \end{aligned} \tag{5.6}$$

Then restricting to the small Hilbert space can be written as the condition $\tilde{G}_0^+\Phi = \eta_0\Phi = 0$. Since the η_0-cohomology is trivial, we may also write the gauge parameter Λ as $\eta_0\Lambda'$. Doing so, the physical state conditions in the $\mathcal{N}=4$ context read

$$\boxed{G_0^+\Phi^+ = \tilde{G}_0^+\Phi^+ = (J_0-1)\Phi^+ = 0\,, \qquad \Phi^+ \sim \Phi^+ + G_0^+\tilde{G}_0^+\Lambda'} \tag{5.7}$$

We have added a superscript + to Φ in order to indicate that this state has J-charge one. These are exactly the conditions on physical states of the $\mathcal{N}=4$ topological string [29]. So we have seen that the gauge-fixed $\mathcal{N}=1$ superstring inherits the structure of an $\mathcal{N}=4$ topological string such that the tools of topological string theory can be used to gain results in critical $\mathcal{N}=1$ string theory.

The physical state conditions in (5.7) can be written in a more compact form as the $(G_0^+,\tilde{G}_0^+)$-cohomology of the kernel of J_0-1. By the $(G_0^+,\tilde{G}_0^+)$-cohomology we mean the intersection of the kernels of G_0^+ and $\tilde{G}_0^+$ modulo states in the image of the product $G_0^+\tilde{G}_0^+$. Therefore, we denote the spectrum of physical states of the $\mathcal{N}=4$ topological string, and hence of the $\mathcal{N}=1$ superstring, by

$$\boxed{\mathcal{H}^{\text{physical}} = H^0_{(G_0^+,\tilde{G}_0^+)}\Big(\ker(J_0-1)\Big)} \tag{5.8}$$

5.2 Redefinition to Superspace Variables

In the previous section we uncovered the structure of an $\mathcal{N}=4$ topological algebra within the gauge-fixed $\mathcal{N}=1$ string theory in its critical dimension. In this section, we will give a redefinition of the world sheet fields of the latter in Green-Schwarz-like variables such that target space supersymmetry in six dimensions is manifest. Rewriting the $\mathcal{N}=4$ superconformal generators in these variables then gives us a way to determine the physical spectrum in a manifestly target space supersymmetric manner.

5.2.1 Six-dimensional superspace embedding fields

Starting from the field content of the RNS world sheet theory, we will now define the world sheet fields of the hybrid string. Since the process of redefining the world sheet fields is technical and lengthy, the reader is advised to use fig. 5.1 to keep track of the various steps and fields.

In section 3.3.3 we discussed the physical state conditions in the RNS formulation. In particular, we argued that a copy of the physical spectrum exists for any integer multiple of the ghost picture and one should pick one picture to determine physical states. However, picking a ghost picture breaks space-time supersymmetry because the space-time supersymmetry currents carry ghost picture themselves (We will soon give an explicit form of the space-time supersymmetry currents). Hence space-time supersymmetry is only recovered after applying the necessary number of ghost picture raising and lowering operators in order to end up in the chosen ghost picture.

Let us assume the world sheet CFT separates into a bosonic current algebra $\mathcal{J}^a$, free fermions ψ^a and the ghost systems. The examples to keep in mind are RNS string theory on a flat Minkowski background where $\mathcal{J}^\mu = i\partial X^\mu$ and on an $\mathrm{AdS}_3 \times \mathrm{S}^3$ background [65, 99]. The fermions are supposed to be normalised as

$$\psi^a(z)\psi^b(w) \sim \frac{\eta^{ab}}{z-w} \,. \tag{5.9}$$

We saw above how to construct fields corresponding to states that transform in spinor representations of the space-time symmetry in section 4.2. In particular, we found fields S_ω transforming in the $\mathbf{16_s}$ and $\mathbf{16_c}$ representations of SO(10). We can break these spinors down to spinors in six dimensions by fixing the last two entries in ω to be *e.g.* $+\frac{1}{2}$, so ω takes the form $\omega = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}\right)$ where the number of minus signs depends on whether we are consider chiral or antichiral Weyl spinors. This way, we break the Majorana-Weyl representations of $SO(10)$ down to representation of $\mathrm{SO}(6) \simeq \mathrm{SU}(4)$ as follows:

$$\mathbf{16_s} \longrightarrow 2\ (\mathbf{4_s} \oplus \mathbf{4_c})\ , \qquad \mathbf{16_c} \longrightarrow 2\ (\mathbf{4_s} \oplus \mathbf{4_c})\ . \tag{5.10}$$

Demanding CPT invariance, the pseudo-real nature of $\mathbf{4_c}$ requires the six-dimensional supersymmetry generators to transform in a doublet of two $\mathbf{4_c}$'s [172]. Using (5.10), we can break down the ten-dimensional supersymmetry generators transforming in the $\mathbf{16_s}$ to six dimensions and identify the appropriate doublet. The six-dimensional supersymmetry generators in the $-\frac{1}{2}$ ghost picture read

$$q^\pm_\epsilon = \oint dz \left(e^{-\frac{1}{2}\phi + \frac{i}{2}\epsilon\cdot H \pm \frac{i}{2}H_C^{\mathrm{RNS}}}\right)(z)\,, \tag{5.11}$$

where $\epsilon = (\pm,\pm,\pm)$ with an even number of minus signs and, for flat compactifications, $H_C^{\mathrm{RNS}} = H^4 + H^5$. If we are considering nontrivial compactification manifolds $\mathcal{M}^4$, *i.e.* if $\mathcal{M}^4$ is not the four-torus T^4, it can be shown that $\mathcal{N}=1$ space-time supersymmetry in six dimensions requires the internal CFT to contain a boson H_C^{RNS} whose OPE is

$$H_C^{\mathrm{RNS}}(z)H_C^{\mathrm{RNS}}(w) \sim -2\ln(z-w) \tag{5.12}$$

such that the space-time supersymmetry generators still take the form in (5.11) [8]. This is closely connected to the extension of the $\mathcal{N}=1$ superconformal algebra on the world sheet in the RNS formulation to an extended superconformal algebra, which will be important later in this work when determining the massive RNS spectrum of states that are independent of the choice of $\mathcal{M}^4$.

But coming back to our present discussion, the generators in (5.11) have been defined in the $-\frac{1}{2}$ picture, but an equivalent expression can be found in any $n-\frac{1}{2}$ picture with $n \in \mathbb{Z}$ by applying picture raising and lowering operators. Since our intention is to make space-time supersymmetry manifest, we should fix the ghost picture in such a way that acting with a supersymmetry generator on any state followed by its hermitian conjugate[1]

[1] Here, by the hermitian conjugate of a supersymmetry generator q we mean the supersymmetry generator $q^\dagger$ such that the anticommutator is proportional to the translation operator, $[q_a^\dagger, q_b]_+ \propto \sigma^\mu_{ab}P_\mu$.

does not change ghost picture. Otherwise we would need to make use of the picture raising and lowering operators to recover the original state. Hence, we pick q_ϵ^- to be in the $-\frac{1}{2}$ picture, *i.e.* it takes the form in (5.11), while q_ϵ^+ lives in the $+\frac{1}{2}$ picture, $q_\epsilon^+ \to \hat{q}_\epsilon^+ = Zq_\epsilon^+$, where Z denotes the picture raising operator. The new rather complicated expression for q_ϵ^+ and a check that they indeed give rise to a supersymmetry algebra in six dimensions can be found in [30].

Having fixed the ghost picture of the space-time supersymmetry generators in a consistent way, we could make its action manifest by constructing fields that transform in appropriate representations under the supersymmetry algebra. One should have the superspace formulation in mind, where supersymmetry generates translations in the odd directions[2], $[q_\epsilon^\alpha, \theta_\beta^{\tilde{\epsilon}}]_+ = \delta_\beta^\alpha \delta_\epsilon^{\tilde{\epsilon}}$. Indeed, using the RNS fields, one can construct fields transforming like this as follows [17, 30]:

$$\theta_-^\epsilon(z) = \exp\left(\tfrac{1}{2}\phi + \tfrac{i}{2}\epsilon_i H^i + \tfrac{i}{2}H_C^{\mathrm{RNS}}\right) , \tag{5.13}$$

$$\theta_+^\epsilon(z) = \exp\left(-\tfrac{3}{2}\phi + \tfrac{i}{2}\epsilon_i H^i - \tfrac{i}{2}H_C^{\mathrm{RNS}}\right) , \tag{5.14}$$

where ϵ now has an odd number of minus signs[3]. In other words, with respect to the six-dimensional Poincaré group, it transforms in the $\mathbf{4_s}$. Here a subtlety shows up as θ_-^ϵ and θ_+^ϵ are not independent fields [17, 30], so not both can be assumed to be free fields. Following the literature, we will work with $\theta^\epsilon \equiv \theta_-^\epsilon$ and discard θ_+^ϵ. This breaks half of the supersymmetry charges, however, they can be recovered by using harmonic variables [18, 19, 94].

The conjugated momentum to θ^ϵ denoted by p_ϵ is defined such that the following OPE holds:

$$p_\epsilon(z)\theta^{\tilde{\epsilon}}(w) \sim \frac{\delta_\epsilon^{\tilde{\epsilon}}}{z-w} . \tag{5.15}$$

A explicit construction of the conjugated momentum of basic RNS fields can be given by

$$p_\epsilon(z) \equiv \exp\left(-\tfrac{1}{2}\phi - \tfrac{i}{2}\epsilon_i H^i - \tfrac{i}{2}H_C^{\mathrm{RNS}}\right) \tag{5.16}$$

because the OPE is easily seen to be

$$\begin{aligned} p_\epsilon(z)\theta^{\tilde{\epsilon}}(w) &= (z-w)^{\frac{1}{4}-\frac{1}{2}-\frac{\epsilon\cdot\tilde{\epsilon}}{4}}\left(\exp\left(-\tfrac{1}{2}\phi - \tfrac{i}{2}\epsilon_i H^i - \tfrac{i}{2}H_C^{\mathrm{RNS}}\right)\exp\left(\tfrac{1}{2}\phi + \tfrac{i}{2}\tilde{\epsilon}_i H^i + \tfrac{i}{2}H_C^{\mathrm{RNS}}\right)\right) \\ &\sim \begin{cases} \frac{1}{z-w} & \text{if}\, \epsilon\cdot\tilde{\epsilon} = 3 \\ 0 & \text{if}\, \epsilon\cdot\tilde{\epsilon} = -1 \end{cases} . \end{aligned} \tag{5.17}$$

Note that even though p_ϵ has a lower spinor index, its index ϵ has an odd number of minus signs due to the additional minus in the definition in (5.16). Thus, since $\epsilon, \tilde{\epsilon} = (\pm^3)$ with an odd number of minus signs, we know that $\epsilon \cdot \tilde{\epsilon} = 3$ implies that $\epsilon = \tilde{\epsilon}$ and in turn $\epsilon \cdot \tilde{\epsilon} = -1$ implies $\epsilon \neq \tilde{\epsilon}$. Using this equivalence, (5.15) follows and thus verifies the explicit

[2] The Kronecker delta of spinor indices ϵ should be read as $\delta_\epsilon^{\tilde{\epsilon}} = 1$ if $\tilde{\epsilon} = -\epsilon$ and zero otherwise.

[3] Usually, an upper ϵ index implies an odd number of minus signs while a lower ϵ index means that it contains an even number of minus signs. However, the conjugated momentum p_ϵ is an exception to that rule.

form of p_ϵ.

We have defined fermionic embedding coordinates θ^ϵ. However, our goal is to have a world sheet description of string theory that consists of two non-interacting CFTs - one being manifest target space supersymmetric in six dimensions and the other described by usual RNS fields in the remaining four dimensions. In order to achieve that both CFTs do not interact with each other we have to make sure that the OPEs between fields of the two CFTs are nonsingular. But due to the appearance of H_C^{RNS} in the fermionic embedding coordinates, this will not be the case whenever the RNS field in the compactified directions, say χ_C^{RNS}, carries non-vanishing U(1)-charge with respect to the current $i\partial H_C^{\mathrm{RNS}}$. We can take care of this singularity realising that θ^ϵ and p_ϵ both come with a contribution of the ghost field ϕ as well. The idea is to add a ϕ contribution to χ_C^{RNS} in such a way that the singular part of the OPE is canceled. This is done by redefining the fields of the internal CFT as

$$\chi_C^{\mathrm{RNS}} \longrightarrow \chi_C^{\mathrm{GS}} \equiv \left(e^{n(\phi+i\kappa)}\chi_C^{\mathrm{RNS}}\right) = \left(\gamma^n \chi_C^{\mathrm{RNS}}\right) . \tag{5.18}$$

This makes sure that the OPEs are nonsingular, where we assumed that the U(1)-charge of χ_C^{RNS} is n. The appearance of $i\kappa$ seems a little arbitrary at this point but must be included, so that there exists a linear combination of the fields ϕ, κ and H_C^{RNS} that has nonsingular OPEs with all other fields defined so far. This linear combination, called ρ, will be discussed in detail later. More abstractly, (5.18) can be written as a similarity transformation of the superconformal theory that describes the internal manifold M generated by $R = \oint dz\, (\phi + i\kappa) J_C^{\mathrm{RNS}}$, where $J_C^{\mathrm{RNS}} = i\partial H_C^{\mathrm{RNS}}$. This means that every field of the superconformal field theory describing M, say V_C^{RNS}, is transformed by the adjoint action of R:

$$V_C^{\mathrm{GS}} \equiv \mathrm{Ad}(R) V_C^{\mathrm{RNS}} = e^R V_C^{\mathrm{RNS}} e^{-R} = \sum_{n=0}^{\infty} \tfrac{1}{n!} \mathrm{ad}(R) V_C^{\mathrm{RNS}} , \tag{5.19}$$

where $\mathrm{ad}(R) V_C^{\mathrm{RNS}}(w) = [R, V_C^{\mathrm{RNS}}(w)]$. This defines the so called Green-Schwarz like variables denoted by a GS superscript. Let us consider what the similarity transformed fields look like for fields χ_C^{RNS} of U(1)-charge n with respect to J_C^{RNS}, *i.e.* for fields that satisfy the following OPE:

$$J_C^{\mathrm{RNS}}(z) \chi_C^{\mathrm{RNS}}(w) \sim \frac{n \chi_C^{\mathrm{RNS}}(w)}{z-w} . \tag{5.20}$$

Because

$$\mathrm{ad}\left(\oint dz \left((i\kappa + \phi) J_C^{\mathrm{RNS}}\right)(z)\right) \chi_C^{\mathrm{RNS}}(w) = n(i\kappa + \phi)\chi_C^{\mathrm{RNS}}(w) \tag{5.21}$$

the Green-Schwarz like field associated with $\chi_C^{\mathrm{RNS}}(w)$ reads

$$\chi_C^{\mathrm{GS}}(w) \equiv \sum_{m=0}^{\infty} \tfrac{1}{m!} \left[n(i\kappa + \phi)\right]^m \chi_C^{\mathrm{RNS}}(w) = e^{n(i\kappa+\phi)} \chi_C^{\mathrm{RNS}}(w) , \tag{5.22}$$

which reproduces (5.18) as required. In particular, the transformation (5.18) holds for the supercurrents $G_C^{\pm\mathrm{RNS}}$, which will be important later.

Let us stop a moment and look at what we have done so far. We took the fermions that are added to the world sheet current algebra and, assuming they are free fields, partially redefined them to obtain fermionic superspace embedding coordinates in six dimensions plus RNS fields describing the internal manifold. The latter have been similarity transformed such that they have a nonsingular OPE with the superspace embedding fields. But we did not yet take care about the bosonised superconformal ghosts ϕ, κ. After the redefinitions and similarity transformations performed so far, they will have singular OPEs with both the superspace embedding fields and the RNS fields in the four compactified dimensions. However, we might look for a combination of the bosons in our description which does have nonsingular OPEs with the remaining fields. Let us denote the general linear combination by

$$\rho(\{\alpha_i\}) = \alpha_1\phi + \alpha_2 i\kappa + \alpha_3 iH_C^{\mathrm{RNS}} \,. \tag{5.23}$$

The relevant OPEs of $\partial\rho(\{\alpha_i\})$ with the other fields are

$$\partial\rho(z)p_a(w) \sim \frac{(-\alpha_1 + 2\alpha_3)p_a(w)}{z-w} \,,$$

$$\partial\rho(z)\Big(e^{n(\phi+i\kappa)}\chi_C^{\mathrm{RNS}}\Big)(w) \sim \frac{n(-\alpha_1+\alpha_2+\alpha_3)\big(e^{n(\phi+i\kappa)}\chi_C^{\mathrm{RNS}}\big)(w)}{z-w} \,,$$

from which we conclude that they are nonsingular if and only if $\alpha_1 = 2\alpha_2 = 2\alpha_3$. Further demanding the standard normalisation of bosons, $\rho(z)\rho(w) \sim -\ln(z-w)$, fixes $\alpha_3 = 1$. This finally defines the unique free boson

$$\rho = -2\phi - i\kappa - iH_C^{\mathrm{RNS}} \tag{5.24}$$

which has nonsingular OPEs with the superspace embedding fields as well as the internal superconformal field theory. The contribution of ρ to the full energy momentum tensor is computed by determining the conformal weight of $e^{n\rho}$. For convenience, we recall the conformal weights of the linear combined bosons:

$$h\left(e^{n_\phi\phi}\right) = -\tfrac{1}{2}n_\phi^2 - n_\phi \,, \quad h\left(e^{n_\kappa\kappa}\right) = -\tfrac{1}{2}n_\kappa^2 - \tfrac{1}{2}in_\kappa \,, \quad h\left(e^{n_H H^C_{\mathrm{RNS}}}\right) = -n_H^2 \,. \tag{5.25}$$

Recall that H^C_{RNS} can be seen as a sum of two bosons and thus the conformal weight of $e^{n_H H^C_{\mathrm{RNS}}}$ is twice the usual conformal weight $-\frac{1}{2}n_H^2$ that is obtained when bosonising complex fermions. Thus

$$\begin{aligned} h\left(e^{n\rho}\right) &= -\tfrac{1}{2}(-2n)^2 - (-2n) - \tfrac{1}{2}(-in)^2 - \tfrac{1}{2}i(-in) - (-in)^2 \\ &= -\tfrac{1}{2}n^2 + \tfrac{3}{2}n \,. \end{aligned} \tag{5.26}$$

Having found the conformal weight of the exponentials of ρ, we can easily write down its energy momentum tensor:

$$T^\rho = -\tfrac{1}{2}(\partial\rho)^2 + \tfrac{3}{2}\partial^2\rho \,. \tag{5.27}$$

We are still not done since we have ignored the conformal ghosts b and c up to now.

In the hybrid formulation, the conformal ghosts are bosonised in the standard way,

$$b = e^{-i\sigma} \qquad \text{and} \qquad c = e^{i\sigma}\,, \tag{5.28}$$

hence adding yet another boson σ to the theory, whose OPE is nonsingular with any other field of the hybrid formulation by construction. The known conformal weights of both b and c can be used to determine the energy momentum tensor in the bosonised form, which is

$$T^{\sigma} = -\tfrac{1}{2}(\partial\sigma)^2 + \tfrac{3}{2}\partial^2(i\sigma)\,. \tag{5.29}$$

Furthermore, since the OPEs of ρ and σ with all other fields and among each other are nonsingular by construction, the energy momentum tensor of the complete $\rho\sigma$-system is just the sum of the individual energy momentum tensors,

$$T^{\rho\sigma} = -\tfrac{1}{2}\left((\partial\rho)^2 + (\partial\sigma)^2\right) + \tfrac{3}{2}\partial^2(\rho + i\sigma)\,. \tag{5.30}$$

This concludes the definition of fields involved in the hybrid formulation. If the 10-dimensional target-space $\mathcal{M}^{10}$ decomposes as $\mathcal{M}^6 \times \mathcal{M}^4$ in such a way that $\mathcal{M}^6$ is maximally supersymmetric, it consists of

1) a current algebra $\mathcal{J}^a$ describing the geometry of $\mathcal{M}^6$,

2) 4+4 anticommuting bosons p_ϵ and θ^ϵ that can be understood as the fermionic coordinates in the superspace version of $\mathcal{M}^6$,

3) a topologically twisted $\mathcal{N} = 2$ $\hat{c} = 2$ superconformal field theory on $\mathcal{M}^4$ and

4) two additional free bosons ρ and σ from now on referred to as the $\rho\sigma$-system.

The equivalence of the hybrid description and the RNS formulation of string theory in a flat Minkowski background is shown by complete bosonisation in section 5.2.3.

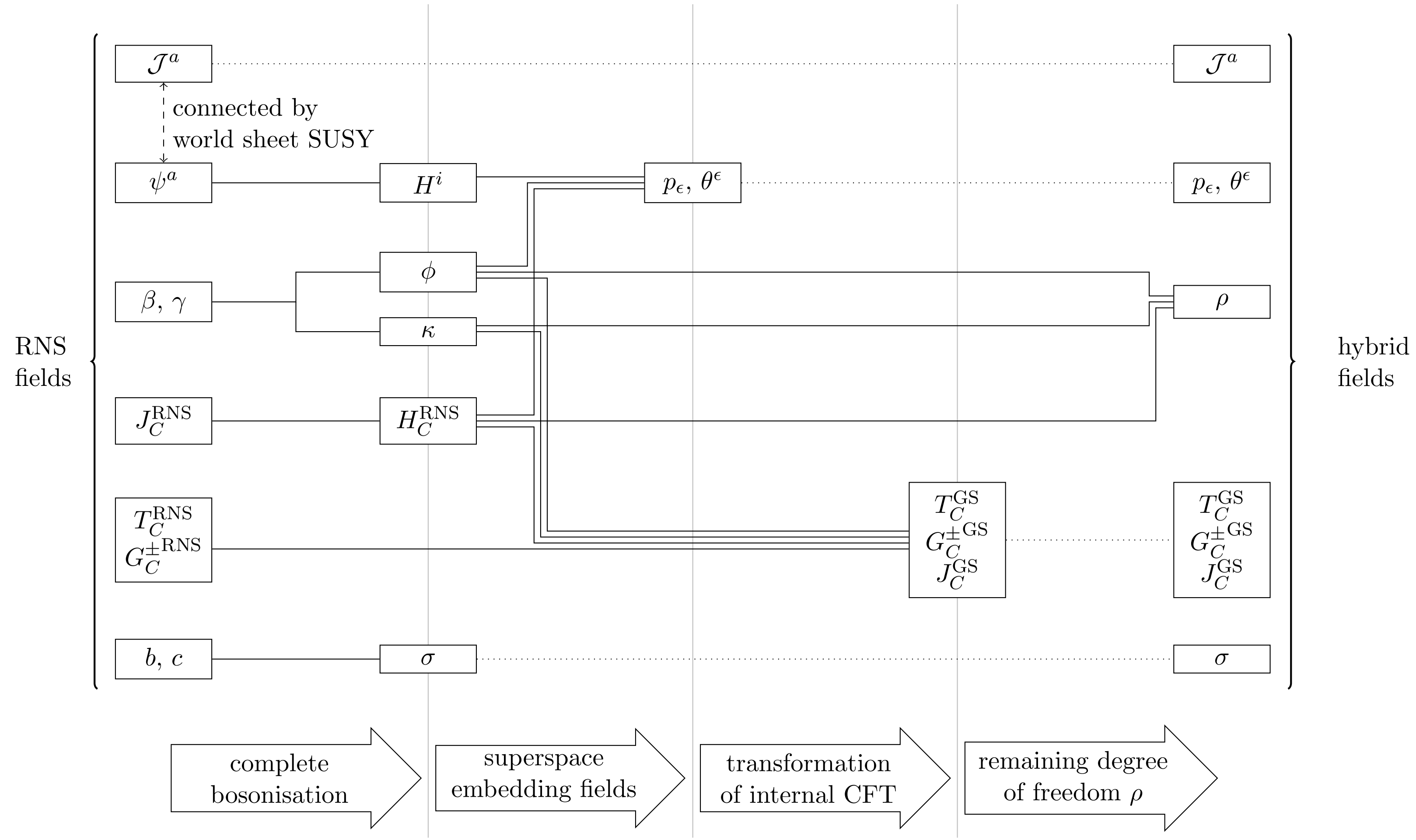


Figure 5.1: *Diagrammatic presentation of the various steps in going from the RNS formulation to the hybrid formulation. Solid black lines indicate which fields are used to define new fields. Note that we left out the last similarity transformation of the internal conformal field theory which just brings the superconformal algebra into a more convenient form.*

5.2.2 Superconformal constraints

Starting from the field content of the covariant gauge-fixed superstring, we have defined new fundamental world sheet fields such that space-time supersymmetry becomes manifest. However, there is more to string theory than just the matter content of conformal field theory on the world sheet. It also comes with a description of which string states should be considered to represent nonequivalent physical states. As we have seen, this description can be formulated using the $\mathcal{N}=4$ extension of the $\mathcal{N}=2$ superconformal algebra in (5.1). Thus in order to identify RNS string theory within the larger manifestly space-time supersymmetric theory, this algebra has to be formulated in terms of the new fundamental fields. Once that is done, we can extend it to its $\mathcal{N}=4$ version and then look for physical string states by using the cohomological characterisation in (5.8).

Let us start with the energy momentum tensor in the new fundamental fields. Having determined the energy momentum tensors of the ρ- and σ-systems above, we can use the fact that we are mostly dealing with free fields to obtain the whole energy momentum tensor in the hybrid formulation,

$$T = T_{\mathcal{J}} - p_\epsilon \partial\theta^\epsilon - \tfrac{1}{2}\left((\partial\rho)^2 + (\partial\sigma)^2\right) + \tfrac{3}{2}\partial^2(\rho + i\sigma) + T_C^{\mathrm{GS}}\,, \tag{5.31}$$

where $T_{\mathcal{J}}$ is the energy momentum tensor of the bosonic embedding coordinates and T_C^{GS} is the (similarity transformed) energy momentum tensor of the SCFT in the compactified space. As a side remark, we can extract a severe restriction from this energy momentum tensor on the possible current algebras generated by $\mathcal{J}^a$ and hence on possible geometric backgrounds of string theory. Since $T(z)$ as given in (5.31) is just a reformulation of critical string theory including the superconformal ghosts, the associated central charge is supposed to vanish. But calculating the central charge from (5.31) directly yields a central charge of $c_{\mathcal{J}} + c_{p\theta} + c_{\rho\sigma}$ in a hopefully obvious notation. Note that the superconformal field theory on $\mathcal{M}$ is topologically twisted, hence it does not contribute to the total central charge. The $\rho\sigma$ system contributes a central charge of two (cf. eq. (3.143)) while the central charge of the Grassmann-valued fields is -8. Thus the total central charge only vanishes if the central charge associated with $T_{\mathcal{J}}$ is $c_{\mathcal{J}} = 6$. Therefore allowed background choices are flat Minkowski space, where the central charge is the same as the number of dimensions, and $\mathfrak{sl}(2)_k \oplus \mathfrak{su}(2)_{k'} \simeq \mathrm{AdS}_3 \times \mathrm{S}^3$ provided that $k = k'$ (cf. section 4.3.1).

But let us come back to the question how to express the $\mathcal{N}=2$ superconformal algebra in terms of the manifest space-time supersymmetric fields. The G^--current is easily reformulated as it coincides with the b-ghost. Hence $G^- = e^{i\sigma}$. The U(1)-current is

$$J = cb + \eta\xi = i\partial\sigma + i\partial\kappa = \partial(\rho + i\sigma) + J_C^{\mathrm{GS}}\,, \tag{5.32}$$

where we used the definition of ρ in (5.24) and $J_C^{\mathrm{RNS}} = i\partial H_C^{\mathrm{RNS}}$ to show that in Green-Schwarz like variables one has

$$J_C^{\mathrm{GS}} = \mathrm{Ad}\left[(i\kappa + \phi)J_C^{\mathrm{RNS}}\right] J_C^{\mathrm{RNS}} = J_C^{\mathrm{RNS}} + 2\partial(\rho + i\kappa)\,. \tag{5.33}$$

It is important in this calculation to keep in mind that the OPE of H_C^{RNS} with itself has an additional factor of two (cf. (5.12)).

The hardest but arguably most important part is expressing G^+ in hybrid variables. For this, we recall the form of it in RNS variables from (5.1):

$$G^+ = (\gamma G_{\text{matter}}) + c(T_{\mathcal{J}} + T^{\beta\gamma} + \tfrac{1}{2}T^{bc}) - \big(\gamma^2 b\big) + \partial^2 c + \partial(c\xi\eta)\,, \tag{5.34}$$

where $T^{\beta\gamma}$ and T^{bc} are the energy-momentum-tensors of the respective ghost systems. Following [30], we first note that

$$-\big(\gamma^2 b\big) = -\big(e^{-2\rho - i\sigma}\, P\big)\,, \tag{5.35}$$

where P is the antisymmetrised normal ordered product of all four p_ϵ. To see this, we use that

$$P := \big(p_{(+,+,+)}p_{(+,-,-)}p_{(-,+,-)}p_{(-,-,+)}\big) = \Big(e^{-2\phi - 2iH_C^{\text{RNS}}}\Big) \tag{5.36}$$

to evaluate the normal ordered product

$$\big(e^{-2\rho - i\sigma}\, P\big) = \Big(e^{2(\phi + i\kappa) - i\sigma}\Big) = \big(\gamma^2 b\big)\,. \tag{5.37}$$

In order to rephrase the term $c(T_{\mathcal{J}} + T^{\beta\gamma} + \frac{1}{2}T^{bc})$ in hybrid variables, we note that $T_{\mathcal{J}} + T^{\beta\gamma}$ coincides with the full hybrid energy-momentum tensor without the σ-contributions since those come from T^{bc}. Furthermore, we have to realise that $\big(cT^{bc}\big) = (-2\, cb\partial c)$ as $c^2 = 0$ due to the anticommuting nature of c. Using this and (5.31) we can rewrite

$$\begin{aligned} c(T_{\mathcal{J}} + T^{\beta\gamma} + \tfrac{1}{2}T^{bc}) = e^{i\sigma}\Big(T_{\mathcal{J}} - p_\epsilon \partial\theta^\epsilon - \tfrac{1}{2}\left((\partial\rho)^2 - (\partial\sigma)^2\right) \\ + \tfrac{1}{2}\partial^2(3\rho + i\sigma) + T_C^{\text{GS}}\Big)\,. \end{aligned} \tag{5.38}$$

Finally, the last term $\partial^2 c + \partial(c\xi\eta)$ can be expressed as

$$\partial^2 c + \partial(c\xi\eta) = \partial^2(e^{i\sigma}) + \partial(e^{i\sigma}(\partial\rho + J_C^{\text{GS}}))\,. \tag{5.39}$$

The most interesting term is $(\gamma G_{\text{matter}})$ as it depends on the specific model considered. For example, in flat space the supercurrent G_{matter} in the RNS formulation is of the form $\psi_n J^n$, while there is an additional trilinear term in the fermionic fields if we are considering string theory on $\text{AdS}_3 \times \text{S}^3 \times \mathcal{M}$ (cf. eq. (4.63)). In order to analyse this term, let us separate G_{matter} in a six-dimensional hybrid part, say G^6, and a four-dimensional compactified part, $G_{\text{matter}} = G^6 + G_C^{\text{RNS}}$. For example, in flat Minkowski space, we have

$$G^6 = \sum_{\mu=0}^{5} (\mathcal{J}^\mu \psi_\mu) \quad \text{and} \quad G_C^{\text{RNS}} = \sum_{\mu=6}^{9} (\mathcal{J}^\mu \psi_\mu)\,. \tag{5.40}$$

The part of $(\gamma G_{\text{matter}})$ that corresponds to the internal supercurrent G_C^{RNS} can be written as

$$\left(\gamma G_C^{\rm RNS}\right) = \left(\gamma G_C^{+\,\rm RNS}\right) + \left(\gamma G_C^{-\,\rm RNS}\right) = G_C^{+\,\rm GS} + \left(e^{-2\rho} P G_C^{-\,\rm GS}\right). \tag{5.41}$$

The first term in the above equation follows directly from the definition of Green-Schwarz like variables, $G_C^{+\,\rm GS} = \gamma G_C^{+\,\rm RNS}$. The second term, on the other hand, is more involved. Using that $G_C^{-\,\rm GS} = \gamma^{-1} G_C^{+\,\rm RNS}$ and (5.36), one obtains

$$\left(e^{-2\rho} P G_C^{-\,\rm GS}\right) = \left(e^{2\phi+2i\kappa} G_C^{-\,\rm GS}\right) = \left(\gamma^2 G_C^{-\,\rm GS}\right) = \left(\gamma G_C^{-\,\rm RNS}\right). \tag{5.42}$$

Let us now turn to γG^6 and within that term to the universal part $G^6_{\sf flat} = \psi_n J^n$ that corresponds to the RNS supercurrent in flat space. In order to obtain how this may be expressed in hybrid variables, we first determine the OPE of two p_ϵ's:

$$p_\epsilon(z) p_{\tilde\epsilon}(w) = (z-w)^{\frac14+\frac14\epsilon\cdot\tilde\epsilon}\left(e^{-\frac12\phi-\frac{i}{2}\epsilon\cdot H-\frac{i}{2}H_{\rm C}^{\rm RNS}}(z) e^{-\frac12\phi-\frac{i}{2}\tilde\epsilon\cdot H-\frac{i}{2}H_{\rm C}^{\rm RNS}}(w)\right), \tag{5.43}$$

Let us assume that $\epsilon \neq \tilde\epsilon$. Then we extract from the OPE the normal ordered product

$$(p_\epsilon p_{\tilde\epsilon})(z) = \left(e^{-\phi-iH_{\rm C}^{\rm RNS}} e^{-\frac{i}{2}(\epsilon+\tilde\epsilon)\cdot H}\right)(z). \tag{5.44}$$

The nice fact about the normal ordered product above is that $\frac12(\epsilon+\tilde\epsilon)$ has one component equal to ±1 while all other components vanish. Thus the second exponential in (5.44) reproduces the RNS fermion fields in the six uncompactified dimensions times a factor $e^{-\phi-iH_{\rm C}^{\rm RNS}}$. We obtain

$$\frac12\left((p_\epsilon p_{\tilde\epsilon})\mathcal{J}^{\epsilon\tilde\epsilon}\right) = \left(e^{-\phi} e^{-iH_{\rm C}^{\rm RNS}} \psi_m \mathcal{J}^m\right). \tag{5.45}$$

Here we made use of the bispinor representation of vectors in six dimensions, so we can write the current $\mathcal{J}^n$ as $\mathcal{J}^{\epsilon\tilde\epsilon} = -\mathcal{J}^{\tilde\epsilon\epsilon}$. The latter is normalised such that the above relation holds. Apart from the factor including the ghost fields, the right-hand side coincides with the matter supercurrent in the RNS formulation. This additional factor can be taken care of by taking the normal ordered product with $e^{-\rho}$,

$$\frac{1}{2\sqrt{k}}\left(e^{-\rho}(p_\epsilon p_{\tilde\epsilon})\mathcal{J}^{\epsilon\tilde\epsilon}\right) = \frac{1}{\sqrt{k}}\left(e^{\phi+i\kappa}\psi_m\mathcal{J}^m\right) = \left(\gamma G^6_{\sf flat}\right) \tag{5.46}$$

as the OPE satisfies $e^{-\rho}(z)e^{-\phi-iH_{\rm C}^{\rm RNS}}(w) = \mathcal{O}(1)$. For flat target spaces, this would be it. However, we are ultimately interested in nontrivial target space geometries, in particular $\mathrm{AdS}_3 \times \mathrm{S}^3$. Here additional trilinears in the fermions in G^6 arise and we will now argue that they can be expressed in terms of p_ϵ as well. In order to do that, let us consider the case $\epsilon = \tilde\epsilon$ such that $\epsilon\cdot\tilde\epsilon = 3$. Then, according to (5.44), $p_\epsilon(z)p_\epsilon(w) = \mathcal{O}(z-w)$ and hence $\left(p_\epsilon^2\right)(z) = 0$. But by taking the derivative of the OPE with respect to w we can obtain a nontrivial normal ordered product, in particular

$$(p_\epsilon \partial p_\epsilon)(z) \;=\; -\left(e^{-\phi-iH_{\rm C}^{\rm RNS}} e^{-i\epsilon\cdot H}\right)(z) \tag{5.47}$$

This is a normal ordered product of three RNS fermion fields in the six uncompactified dimensions times factor $e^{-\phi}e^{-iH_{\mathrm{C}}^{\mathrm{RNS}}}$. It is convenient to distinguish the RNS fermion fields in the six uncompactified dimensions. Let ψ^r denote the superpartners of the $\mathfrak{sl}(2)$-currents and χ^r the superpartners of the $\mathfrak{su}(2)$ currents corresponding to AdS_3 and S^3, respectively. It is also assumed that we are working in Euclidean AdS_3 such that

$$\psi^r(z)\psi^s(w) \sim \frac{\delta^{rs}}{z-w}\,, \qquad \chi^r(z)\chi^s(w) \sim \frac{\delta^{rs}}{z-w}\,. \tag{5.48}$$

Then we can construct complex fermions by defining

$$\Psi_j^{\pm} = \tfrac{1}{\sqrt{2}}(\chi^j \pm i\psi^j)\,. \tag{5.49}$$

with the bosonisation scheme $i\partial H^r = (\Psi_r^+\Psi_r^-)$ and $\Psi_r^{\pm} = e^{\pm iH^r}$. Summing over all chiral spinor indices then yields

$$\begin{aligned}\sum_\epsilon (p_\epsilon \partial p_\epsilon) &= -\sqrt{2}\Big(e^{-\phi}e^{-iH_{\mathrm{C}}^{\mathrm{RNS}}}(i^3\psi^0\psi^1\psi^2 + \chi^0\chi^1\chi^2)\Big)\\ &= -\frac{\sqrt{2}}{6}\Big(e^{-\phi}e^{-iH_{\mathrm{C}}^{\mathrm{RNS}}}\epsilon_{rst}(-i\psi^r\psi^s\psi^t + \chi^r\chi^s\chi^t)\Big)\,.\end{aligned} \tag{5.50}$$

Hence, when the normal ordered product of conjugated momenta $\sum_\epsilon (p_\epsilon \partial p_\epsilon)$ is expressed in RNS variables, the structure constants of $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ pop up due to combinatorical reasons. In particular, in the case of RNS string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$, the supercurrent has an additional term that is proportional to a product of three fermions contracted with the structure constant. So when we express the supercurrent in terms of hybrid variables, we can take care of that trilinear term in the fermions by adding an appropriate multiple of $\sum_\epsilon (p_\epsilon \partial p_\epsilon)$. Hence, the term γG^6 may be expressed as

$$\big(\gamma G^6\big) = \frac{1}{2\sqrt{k}}\left(e^{-\rho}\left[(p_\epsilon p_{\tilde\epsilon})\mathcal{J}^{\epsilon\tilde\epsilon} + \nu\sum_\epsilon p_\epsilon \partial p_\epsilon\right]\right)\,, \tag{5.51}$$

where $\nu \in \mathbb{C}$ depends on the particular normalisation of the structure constants in the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ current algebra $\mathcal{J}^m$. The flat case corresponds to the choice $\nu = 0$. This finally concludes our discussion of the reformulation the $\mathcal{N} = 2$ constraints. In order to cover both cases simultaneously whenever possible, we will denote $\big(\gamma G^6\big) = (e^{-\rho}Q)$, where the field Q takes the general form above, namely

$$Q = \frac{1}{2\sqrt{k}}\left((p_\epsilon p_{\tilde\epsilon})\mathcal{J}^{\epsilon\tilde\epsilon} + \nu\sum_\epsilon p_\epsilon \partial p_\epsilon\right) \tag{5.52}$$

To summarise so far, we have found the $\mathcal{N}=2$ constraints in hybrid variables:

$$\begin{aligned}
T &= T_{\mathcal{J}} - p_\epsilon \partial\theta^\epsilon - \tfrac{1}{2}\left((\partial\rho)^2 + (\partial\sigma)^2\right) + \tfrac{3}{2}\partial^2(\rho + i\sigma) + T_C^{\rm GS}\,, && (5.53)\\
G^+ &= -e^{-2\rho - i\sigma}\,P + e^{-\rho}Q \\
&\quad + e^{i\sigma}\left(T_{\mathcal{J}} + p_\epsilon\partial\theta^\epsilon - \tfrac{1}{2}\left((\partial\rho)^2 - (\partial\sigma)^2\right) + \tfrac{1}{2}\partial^2(3\rho + i\sigma) + T_C^{\rm GS}\right) \\
&\quad + \partial^2(e^{i\sigma}) - \partial(e^{i\sigma}(\partial\rho + J_C^{\rm GS})) + G_C^{+\,\rm GS} + e^{-2\rho}\,P\,G_C^{-\,\rm GS} && (5.54)\\
G^- &= e^{-i\sigma}\,, && (5.55)\\
J &= \partial(\rho + i\sigma) + J_C^{\rm GS}\,, && (5.56)
\end{aligned}$$

where normal ordering has been supressed for readability. The last step will consist of yet another similarity transformation generated by $R = \oint dz\left(e^{i\sigma}G_C^{-\,\rm GS}\right)$ in order to put these constraints in a nicer form. Recall that a superconformal generator, say $V(w)$, is transformed by the adjoint action of R:

$$\mathrm{Ad}(R) = e^R V(w) e^{-R} = \sum_{n=0}^{\infty} \tfrac{1}{n!}\mathrm{ad}(R)V(w)\,, \tag{5.57}$$

where $\mathrm{ad}(R)V(w) = [R, V(w)] = \oint_w dz\left(e^{i\sigma}G_C^{-\,\rm GS}\right)(z)V(w)$. The normal ordered product $\left(e^{i\sigma}G_C^{-\,\rm GS}\right)$ is a conformal field neutral under J of conformal dimension one. Thus

$$\left(e^{i\sigma}G_C^{-\,\rm GS}\right)(z)J(w) \sim 0\,, \qquad \left(e^{i\sigma}G_C^{-\,\rm GS}\right)(z)T(w) \sim \frac{\left(e^{i\sigma}G_C^{-\,\rm GS}\right)}{(z-w)^2} \tag{5.58}$$

and hence $\mathrm{ad}(R)T = \mathrm{ad}(R)J = 0$. For the G^--constraint we obtain

$$\mathrm{ad}^n(R)G^- = \begin{cases} G_C^{-\,\rm GS} & \text{if}\, n = 1 \\ 0\,, & \text{if}\, n \geq 2\,, \end{cases} \tag{5.59}$$

so the similarity transformation adds a term $G_C^{-\,\rm GS}$ to the G^--current. But the similarity transformation has the most drastic effect on the G^+-current. We will now analyse the adjoint action of R on those summands on which it acts nontrivially. We obtain

$$\begin{array}{lll}
\mathrm{ad}(R) \;\; \left(e^{-2\rho - i\sigma}\,P\right) & = & \left(e^{-2\rho}\,P\right), \\
\mathrm{ad}(R) \;\; \left(e^{i\sigma}T_C^{\rm GS}\right) & = & \left(e^{2i\sigma}G_C^{-\,\rm GS}\right), \\
\mathrm{ad}(R) \;\; G_C^{+\,\rm GS} & = & -\partial^2 e^{i\sigma} + \partial\left(e^{i\sigma}J_C^{\rm GS}\right) - \left(e^{i\sigma}T_C^{\rm GS}\right), \\
\mathrm{ad}^2(R) \;\; G_C^{+\,\rm GS} & = & -2\left(e^{2i\sigma}G_C^{-\,\rm GS}\right)
\end{array} \tag{5.60}$$

After this similarity transformation we arrive at the $\mathcal{N}=2$ superconformal structure of the hybrid formulation:

$$T = T_{\mathcal{J}} - p_\epsilon \partial\theta^\epsilon - \tfrac{1}{2}\left((\partial\rho)^2 + (\partial\sigma)^2\right) + \tfrac{1}{2}\partial^2(\rho + i\sigma) + T_C^{\mathrm{GS}}\,, \tag{5.61}$$

$$G^+ = -e^{-2\rho - i\sigma}\, P + e^{-\rho} Q + e^{i\sigma}\mathcal{T} + G_C^{+\,\mathrm{GS}} \tag{5.62}$$

$$G^- = e^{-i\sigma} + G_C^{-\,\mathrm{GS}}\,, \tag{5.63}$$

$$J = \partial(\rho + i\sigma) + J_C^{\mathrm{GS}}\,, \tag{5.64}$$

where the fields appearing in the G^+-current are defined by

$$P = \left(p_{(+,+,+)} p_{(+,-,-)} p_{(-,+,-)} p_{(-,-,+)}\right)\,, \tag{5.65}$$

$$Q = \frac{1}{2\sqrt{k}}\left((p_\epsilon p_{\tilde{\epsilon}})\mathcal{J}^{\epsilon\tilde{\epsilon}} + \nu \sum_\epsilon p_\epsilon \partial p_\epsilon\right)\,, \tag{5.66}$$

$$\mathcal{T} = T_{\mathcal{J}} + (p_\epsilon \partial\theta^\epsilon) - \tfrac{1}{2}J^2 + \tfrac{1}{2}\partial J\,. \tag{5.67}$$

By applying the similarity transformation generated by R, we were able to decouple the generators of the $\mathcal{N} = 2$ world sheet superconformal symmetry into a six-dimensional target-space supersymmetric part and an additional part that consists of the generators of a four-dimensional $\mathcal{N} = 2$ superconformal field theory.

5.2.3 Equivalence to the RNS string in a flat background

At this point in our derivation we have two formulations of the superstring - the original RNS string and the hybrid string that has manifest supersymmetry in six dimensions. We want to show that both formulations are equivalent.

Ignoring the bosonic embedding coordinates and the conformal ghosts, which are the same in both formulations, we are left with two energy momentum tensors:

$$T^{\mathrm{RNS}} = \frac{1}{2}\sum_{a=1}^{5}\left(\partial(iH^a)\right)^2 - \tfrac{1}{2}(\partial\phi)^2 - \partial^2\phi + \tfrac{1}{2}\left(\partial(i\kappa)\right)^2 + \tfrac{1}{2}\partial^2(i\kappa)\,,$$

$$\begin{aligned} T^{\mathrm{Hybrid}} &= \frac{1}{2}\sum_{a=1}^{4}\left((\partial(i\lambda^a)\right)^2 + \frac{1}{2}\sum_{a=1}^{4}\partial^2(i\lambda^a) \\ &\quad + \sum_{a=1}^{2}\left[\frac{1}{2}\left(\partial(iH^a_{\mathrm{GS}})\right)^2 + \frac{1}{2}\partial^2(iH^a_{\mathrm{GS}})\right] - \frac{1}{2}(\partial\rho)^2 + \frac{3}{2}\partial^2\rho\,. \end{aligned}$$

Here we bosonised all fields. In particular, we bosonised all RNS worldsheet fermions by introducing a set of bosons H^a and the superspace embedding coordinates p_a and θ^a were bosonised in favour of four bosons λ^a. It is important to note that the complex fermions in the compactified directions, $\bar{\Psi}^I_{\mathrm{GS}}$ and Ψ^I_{GS}, have conformal weight 1 and 0 as the internal CFT is topologically twisted. Thus they are bosonised as an $(1, 0)$ bc-system, introducing a screening charge of $\frac{1}{2}$. Our goal is to show that both energy momentum tensors are identical by choosing an appropriate linear combination of the bosons in the

RNS formulation.

For convenience, we define two 7-tuples of fields,

$$(\mathcal{S}^{\text{hyb}})^T = \begin{pmatrix} i\lambda^1 & i\lambda^2 & i\lambda^3 & i\lambda^4 & iH^1_{\text{GS}} & iH^2_{\text{GS}} & \rho \end{pmatrix} , \tag{5.68}$$

$$(\mathcal{S}^{\text{RNS}})^T = \begin{pmatrix} iH^1 & iH^2 & iH^3 & iH^4 & iH^5 & i\kappa & \phi \end{pmatrix} . \tag{5.69}$$

The energy momentum tensors can then be written as

$$\begin{aligned} T^{\text{RNS}} &= \frac{1}{2}(\partial\mathcal{S}^{\text{RNS}})^T M \partial\mathcal{S}^{\text{RNS}} + (\Lambda^{\text{RNS}})^T \partial^2 \mathcal{S}^{\text{RNS}} , \\ T^{\text{hybrid}} &= \frac{1}{2}(\partial\mathcal{S}^{\text{hyb}})^T M \partial\mathcal{S}^{\text{hyb}} + (\Lambda^{\text{hyb}})^T \partial^2 \mathcal{S}^{\text{hyb}} , \end{aligned}$$

where $M = \text{diag}(1,1,1,1,1,1,-1)$ and the Λs are 7-tuples of the respective screening charges. The latter read

$$\left(\Lambda^{\text{hyb}}\right)^T = \begin{pmatrix} \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{3}{2} \end{pmatrix} , \tag{5.70}$$

$$\left(\Lambda^{\text{RNS}}\right)^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & -1 \end{pmatrix} . \tag{5.71}$$

Now assuming that there is a matrix A such that

$$\mathcal{S}^{\text{hyb}} = A\,\mathcal{S}^{\text{RNS}} , \tag{5.72}$$

the equality of both bosonised energy momentum tensor yields the following constraints on the matrix A:

$$A^T M A = M , \qquad A^T \Lambda^{\text{hyb}} = \Lambda^{\text{RNS}} . \tag{5.73}$$

The first condition in (5.73) tells us that $A \in SO(6,1)$ by definition of $SO(6,1)$. Then $A^T \in SO(6,1)$ as well. Therefore a sufficient condition for the existence of a matrix A is that Λ^{hyb} and Λ^{RNS} have the same norm squared,

$$\left(\Lambda^{\text{hyb}}\right)^T M \Lambda^{\text{hyb}} = \left(\Lambda^{\text{RNS}}\right)^T M \Lambda^{\text{RNS}} \tag{5.74}$$

and indeed it is easy to check that this is the case for the screening charge tuples in (5.70) and (5.71). This finishes the proof of the equivalence of the RNS and hybrid string in their bosonised forms.

5.2.4 The hybrid string on $\text{AdS}_3 \times \text{S}^3$ and the $\text{PSL}(2|2)$ WZW model

We have defined the hybrid string in six-dimensional backgrounds that are maximally supersymmetric, namely flat Minkowski space as well as $\text{AdS}_3 \times \text{S}^3$. From a differential geometric point of view, the latter is isomorphic to the Lie group $\text{SL}(2) \times \text{SU}(2)$. We have seen in section 2.2 that there is an algebraic construction of the supersymmetric analog of that group by extending its Lie algebra to a Lie superalgebra. Loosely speaking, this results in a superspace version of the target space and one might guess that the corresponding nonlinear σ-model describes strings moving on $AdS_3 \times S^3$ such that target space supersymmetry is manifest. However, at first sight it is not obvious that this nonlinear σ-model will describe the "same" string as is described by the RNS formulation. For example, there is more than one supergroup whose bosonic subgroup is isomorphic to $AdS_3 \times S^3$ as differential manifolds. So which one is the correct one to look at?

Above we have shown that the critical RNS string can be formulated in a way that the free world sheet fermions give rise to Grassmann-valued embedding coordinates and their canonical momenta, which we called the hybrid formulation of string theory. Thus at the end of the day, the hybrid string is a nonlinear σ-model whose target space is a Grassmannian manifold (plus an additional $\mathcal{N}=2$ supersymmetry structure on the world sheet). The crucial observation [30, 105, 158] is that the hybrid string can be understood as a free field realisation of the PSL(2|2) WZW model as long as there is NSNS flux only (cf. section 4.3.3)

Following [9, 30, 105], let us denote the currents of the SL(2) × SU(2) WZW model describing $AdS_3 \times S^3$ by $j^{ab}(z)$, where $a, b, c, \ldots = 1, \ldots, 4$ are $\mathfrak{so}(4)$-indices. Similar, we change the basis such that the Grassmann coordinates and their conjugated momenta read θ^a and p_a respectively. The OPEs of the currents in that basis are [105]

$$j^{ab}(z)j^{cd}(w) \sim -\frac{k\epsilon^{abcd} + 2\delta^{a[c}\delta^{d]b}}{(z-w)^2} + \frac{i(\delta^{ac}j^{bd}(w) - \delta^{ad}j^{bc}(w) - \delta^{bc}j^{ad}(w) + \delta^{bd}j^{ac}(w))}{z-w}, \tag{5.75}$$

$$p_a(z)\theta^b(w) \sim \frac{\delta_a^b}{z-w} \tag{5.76}$$

and all other OPEs vanishing. In particular, the Grassmann-valued coordinates and momenta have nonsingular OPEs with the WZW currents. From the hybrid perspective, this is not surprising since the derivation required us to start with a model for the RNS string in which the world sheet fermions are free. From these fields, we can construct PSL(2|2) WZW currents the following way:

$$K^{ab} \equiv j^{ab} - i\Big(\theta^{[a}p^{b]}\Big), \tag{5.77}$$

$$S_+^a \equiv k\partial\theta^a + \frac{i}{2}\epsilon_{abcd}\Big(\theta^b\Big[j^{cd} - i\Big(\theta^c p^d\Big)\Big]\Big), \tag{5.78}$$

$$S_-^a \equiv p^a. \tag{5.79}$$

Here the $\mathfrak{so}(4)$-indices are raised and lowered with the metric δ^{ab} and thus $p^a = p_a$. It can be checked that these currents give a $\mathfrak{psl}(2|2)$ current algebra.

Formulation of the $\mathcal{N}=2$ superconformal generators in terms of the WZW currents

An important aspect that distinguishes the pure PSL(2|2) WZW model from the hybrid string is the additional $\mathcal{N}=2$ supersymmetric structure (5.61) - (5.64) on the world sheet. In order to perform string theoretic analyses within the PSL(2|2) WZW model, it is necessary to rephrase this current algebra in terms of the WZW currents. Since apart from the $\rho\sigma$ ghosts the $\mathcal{N}=2$ currents only involve j^{ab} as well as the conjugated momenta p_a, the substitution of the WZW currents is straightforward. The only problematic term is the field Q since we have to determine the parameter ν in (5.52). In order to make contact to (5.50), we first have to choose a basis of the WZW currents j^{ab} in (5.75) such that the $\mathfrak{su}(2)$ structure constants are proportional to the Levi-Civita tensor in three dimensions, ϵ_{rst}. We achieve this by defining the currents

$$\mathcal{J}^1_{(1)} \equiv \frac{i}{\sqrt{2}}\left(j^{12}+j^{34}\right)\,, \quad \mathcal{J}^2_{(1)} \equiv \frac{i}{\sqrt{2}}\left(j^{14}+j^{23}\right)\,, \qquad \mathcal{J}^3_{(1)} \equiv \frac{i}{\sqrt{2}}\left(j^{24}-j^{13}\right)\,, \tag{5.80}$$

$$\mathcal{J}^1_{(2)} \equiv \frac{1}{\sqrt{2}}\left(j^{12}-j^{34}\right)\,, \quad \mathcal{J}^2_{(2)} \equiv \frac{1}{\sqrt{2}}\left(-j^{14}+j^{23}\right)\,, \quad \mathcal{J}^3_{(2)} \equiv \frac{1}{\sqrt{2}}\left(j^{24}+j^{13}\right)\,. \tag{5.81}$$

The normalisation of the currents was chosen such that the second order poles of their OPEs resemble the one in (4.55) with $\kappa^{ab}=\delta^{ab}$. For this choice of κ^{ab} the results in section 5.2.2, particularly the bosonisation scheme for the fermions, apply after performing an additional rescaling of the fermions $\psi \to \psi' = \frac{1}{\sqrt{k}}\psi$. The OPEs between the currents $\mathcal{J}^r_{(j)}$ read

$$\mathcal{J}^r_{(1)}(z)\mathcal{J}^s_{(1)}(w) \sim \frac{(k+2)\,\delta^{rs}}{(z-w)^2} - \frac{\sqrt{2}\epsilon^{rst}\delta_{tu}\mathcal{J}^u_{(1)}}{z-w}\,, \tag{5.82}$$

$$\mathcal{J}^r_{(2)}(z)\mathcal{J}^s_{(2)}(w) \sim \frac{(k-2)\,\delta^{rs}}{(z-w)^2} - \frac{i\sqrt{2}\epsilon^{rst}\delta_{tu}\mathcal{J}^u_{(2)}}{z-w}\,, \tag{5.83}$$

$$\mathcal{J}^r_{(1)}(z)\mathcal{J}^s_{(2)}(w) \sim 0\,, \tag{5.84}$$

where $r,s,t,u=1,2,3$. The currents $\mathcal{J}^r_{(1)}$ give an $\mathfrak{su}(2)$ current algebra at level $k+2$ describing the AdS$_3$ part while $\mathcal{J}^r_{(2)}$ generate an $\mathfrak{su}(2)$ current algebra at level $k-2$ associated with the sphere S^3. It is then possible to extract the $\mathfrak{su}(2)$ structure constants, $if^{(1)}_{rst} = -\sqrt{2}\epsilon_{rst}$ for the AdS$_3$ part and $if^{(2)}_{rst} = -i\sqrt{2}\epsilon_{rst}$ for the S^3 part. With these f_{rst} we can write the equality (5.50) as

$$\sum_{a=1}^{4}(p_a\partial p_a) = \frac{1}{6}\Big(e^{-\phi}e^{-iH^{\mathrm{RNS}}_{\mathrm{C}}}(f^{(1)}_{rst}\psi^r\psi^s\psi^t + f^{(2)}_{rst}\chi^r\chi^s\chi^t)\Big)\,. \tag{5.85}$$

The supercurrent in (4.63) can then be written as (recall that the fermions were rescaled according to $\psi \to \psi' = \frac{1}{\sqrt{k}}\psi$)

$$\begin{aligned}(\gamma G) &= \frac{1}{\sqrt{k}}\left[\Big(\gamma\,\psi_r\mathcal{J}^r_{(1)}\Big) + \Big(\gamma\,\chi_r\mathcal{J}^r_{(2)}\Big) - i\frac{f^{(1)}_{rst}}{6}\left(\gamma\,\psi^r\psi^s\psi^t\right) - i\frac{f^{(2)}_{rst}}{6}\left(\gamma\,\chi^r\chi^s\chi^t\right)\right] \\ &= \frac{1}{\sqrt{k}}\left[\Big(\gamma\,\psi_r\mathcal{J}^r_{(1)}\Big) + \Big(\gamma\,\chi_r\mathcal{J}^r_{(2)}\Big) - i\sum_{a=1}^{4}\left(e^{-\rho}p_a\partial p_a\right)\right] \\ &= \frac{1}{2\sqrt{k}}\left[\Big(e^{-\rho}(p_ap_b)j^{ab}\Big) - 2i\sum_{a=1}^{4}\left(e^{-\rho}p_a\partial p_a\right)\right]\end{aligned} \tag{5.86}$$

Comparing this expression to (5.51), one indeed finds that ν should be set to $-2i$ such that Q reads

$$Q = \frac{1}{2\sqrt{k}} \left[\Big((p_a p_b) j^{ab} \Big) - 2i (p_a \partial p_a) \right] , \tag{5.87}$$

where summation over repeated indices is understood. Before writing this in terms of the WZW currents, we note that

$$\sum_{\substack{1 \le b \le 4 \\ b \ne a}} (p_a p_b)(z) \Big(p^a \theta^b \Big)(w) = -\frac{3(p_a(z) p_a(w))}{z-w}$$

$$\Rightarrow \quad \Big((p_a p_b) \Big(p_a \theta^b \Big) \Big)(w) = 3 \left(p_a \partial p_a \right) . \tag{5.88}$$

Since $j^{ab} = K^{ab} + i \big(p^{[a} \theta^{b]} \big)$, the field Q reads in terms of the WZW currents

$$Q = \frac{1}{2\sqrt{k}} \left[\Big(\Big(S^a_- S^b_- \Big) K_{ab} \Big) + 4i \big(S^a_- \partial S^a_- \big) \right] . \tag{5.89}$$

The order in which the normal ordered product is evaluated on the triple of fields is crucial since the normal ordered product is neither commutative nor associative for non-Abelian current algebras. The factor of $4i$ multiplying $\big(S^a_- \partial S^a_- \big)$, or equivalently the choice $\nu = -2i$, can also be justified *a posterori* since only this choice will yield agreement of the physical hybrid string spectrum at the first mass level with the one obtained in the RNS formulation. For completeness, we also state that

$$P = \big(S^1_- S^2_- S^3_- S^4_- \big) \qquad \text{and} \qquad T_{\mathcal{J}} + p_\epsilon \partial \theta^\epsilon = T_{\mathrm{PSL}(2|2)} \, . \tag{5.90}$$

We now have expressions for all generators of the $\mathcal{N} = 2$ superconformal algebra in terms of the PSL(2|2) WZW currents and the $\rho\sigma$ ghosts.

5.3 Physical state conditions on the massless states

At this point we have two equivalent description of the superstring both on flat space-time as well as in an $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathcal{M}^4$ background: the usual RNS formulation and the hybrid formulation. The latter has the advantage of having manifest target space supersymmetry as the six uncompactified directions are nicely described by a WZW model on PSL(2|2). However, a drawback is the amount of additional structures required: a topological twisted superconformal field theory on $\mathcal{M}^4$, two ghost fields ρ and σ as well as a $\mathcal{N} = 2$ superconformal algebra on the worldsheet which is essential in the definition of the physical subsector.

The aim of this section is to obtain physical state conditions on the Hilbert space of the WZW model only. Our approach is to evaluate the physical state constraints on hybrid vertex operators until we are left with constraints formulated in terms of the WZW currents and normal ordered products thereof. We will first concentrate on the massless ground states.

Let us start by first writing down the form of the most general vertex operator that carries no affine excitations. It looks like

$$V = (\phi_{m,n}\, e^{m\rho+in\sigma}\psi_{\mathcal{M}})(z)\,. \tag{5.91}$$

Here $\phi_{m,n}$ is vertex operator of the supergroup WZW model. It is neutral under the U(1)-charge of the $\mathcal{N}=2$ superconformal algebra. $e^{m\rho+in\sigma}$ specifies the ρ and σ charge of V. Derivatives of ρ and σ are not allowed at this point because they would correspond to affine excitations. Similarly, the WZW vertex operators $\phi_{m,n}$ correspond to affine ground states of representations of $\widehat{\mathfrak{psl}}(2|2)_k$, thus excluding affine excitations in the WZW model. Finally, $\psi_{\mathcal{M}}$ is a vertex operator of the twisted superconformal field theory on the compactification manifold $\mathcal{M}$. It is restricted to be a groundstate of that SCFT as we are discussing the massless sector. This means that its conformal weight is zero. Since the SCFT on $\mathcal{M}$ is twisted, the $\mathcal{N}=4$ superconformal algebra on $\mathcal{M}$ contains fields of vanishing conformal weight apart from the vacuum Ω, namely the $\mathfrak{su}(2)$ current J^{++}. Depending on the choice of $\mathcal{M}$ there might be additional matter fields of conformal weight zero, *e.g.* for $\mathcal{M}=\mathrm{T}^4$ the complex fermions Ψ_+ are relevant as well.

We can restrict V even further. We have shown above that the energy-momentum tensors of the hybrid string and the RNS string are equivalent, at least in flat backgrounds. In the RNS formulation it is well-known that the Siegel gauge $b_0\psi=0$ together with the BRST-kernel condition implies that $T_0\psi=[Q_0,b_0]_+\psi=0$. Here we denoted the Virasoro zero-mode of the full hybrid formulation as T_0 in order to distinguish it from the Virasoro zero-mode of the pure PSL(2|2) WZW model that is denoted by L_0. Due to the equivalence, we can impose the same condition on states in the hybrid formulation,

$$[T_0,V]=0\,. \tag{5.92}$$

This condition holds *after* the other physical state conditions (5.7) have been imposed. But a solution to (5.92) can only exist if a sufficiently high power of T_0, say T_0^p for some $p\geq 1$, annihilates V *before* applying the constraints (5.7). Note that T_0 does not necessarily act diagonally due to the possible appearance of atypical representations of $\mathfrak{psl}(2|2)$ as discussed in section 2.2.2. Indeed, since we are interested in the massless groundstates at the moment, we demand that $\phi_{m,n}$ sits in such an atypical representation, which are characterised by $L_0^r\phi_{m,n}=0$ for some $r\geq 1$. We further note that $\mathrm{ad}^p(T_0)V=0$ implies that m and n have to be chosen such that the conformal weight of $e^{m\rho+in\sigma}$ vanishes. If V carries charge q with respect to the superconformal U(1)-current, we obtain the following constraints on possible values of m and n,

$$h\left(e^{m\rho+in\sigma}\right) = -\tfrac{1}{2}(m^2-n^2)+\tfrac{3}{2}(m-n) \;=\; h=0\,, \tag{5.93}$$

$$n-m \;=\; q\,. \tag{5.94}$$

These are solved by $m=n$ for neutral V and $m=\frac{1}{2}(3-q)$, $n=\frac{1}{2}(3+q)$ for charged $\rho\sigma$-vertex operators. Note that in the case of non-vanishing conformal weight, $h\neq 0$, we must have $m \neq n$, *i.e.* all U(1)-neutral $\rho\sigma$-vertex operators have zero conformal weight. For completeness, let us add that for general conformal weight h (and $q \neq 0$, of course), m and n are given by

$$(m,n) = \left(\frac{3q - 2h - q^2}{2q}, \frac{3q - 2h + q^2}{2q} \right) . \tag{5.95}$$

In the following, we will assume that $\phi_{m,n}$ are states in atypical Kac modules $\mathcal{K}(j)$. In that case L_0 is diagonalisable and annihilates the complete Kac module. Due to this assumption the derivation of appropriate constraints on physical string states below simplifies significantly and it can then be seen as a reformulation of the derivation of physical state constraints in [30]. In chapter 6, the resulting conditions will be generalised to atypical representations on which L_0 is not diagonalisable, namely projective covers, thus making contact to our previous discussion of the spectrum of the PSL(2|2) WZW model in section 4.3.4.

5.3.1 The compactification-independent spectrum

Independent of the choice of the internal manifold $\mathcal{M}$, in the hybrid formulation the string dynamics on $\mathcal{M}$ are always modeled by a superconformal field theory on the world sheet. So even without specifying the matter content of it, we know that there are at least two vertex operators that have vanishing conformal weight, J^{++} and the vacuum Ω. According to our ansatz for ground state vertex operators (5.91), this means that we may set $\psi_{\mathcal{M}} = \Omega$ or $\psi_{\mathcal{M}} = J^{++}$. This yields in principle two families of valid vertex operators

$$\begin{aligned} \left(\phi_{m,n} e^{m+in\sigma} \Omega\right) &= \left(\phi_{m,n} e^{m+in\sigma}\right) , \\ \left(\hat{\phi}_{m,n}\, e^{m+in\sigma}\, J^{++}\right) &= \left(\hat{\phi}_{m,n}\, e^{m\rho+in\sigma+iH_C^{\rm GS}}\right) . \end{aligned} \tag{5.96}$$

Recall that $\phi_{m,n}$ and $\hat{\phi}_{m,n}$ are vertex operators of the WZW model. In comparison to the ansatz (5.91) we have added a hat to one of the WZW vertex operator only to distinguish both families. However, we can use the condition that physical vertex operators must have unit U(1) charge (cf. eq. (5.8)) to single out one vertex operator in each family. By construction, the WZW vertex operators are U(1) neutral. Therefore, because Ω has vanishing U(1) charge and J^{++} has U(1) charge 2, the exponentials must carry U(1) charge $q = 1$ and $q = -1$, respectively. Then using (5.93) and (5.94), we can fix m and n and obtain two allowed *compactification-independent* vertex operators, namely

$$\left(\phi_{1,2} e^{1+2i\sigma}\right) \qquad \text{and} \qquad \left(\phi_{2,1}\, e^{2\rho+i\sigma+iH_C^{\rm GS}}\right) . \tag{5.97}$$

Here we dropped the hat from the second WZW vertex operator $\hat{\phi}_{2,1}$ since there remains no ambiguity.

The next step is to check for which vertex operators of the WZW model, $\phi_{2,1}$ and $\phi_{1,2}$, the above hybrid vertex operators are physical. First, the OPE

$$\begin{aligned}\tilde{G}^+(z)(\phi_{1,2}\, e^{\rho+i2\sigma})(w) &= e^{\rho+iH_C^{\rm GS}}(z)(\phi_{1,2}\, e^{\rho+i2\sigma})(w)\\ &= \sum_{d\geq 0}(z-w)^{-1+d}\left(\partial^d(e^{\rho+iH_C^{\rm GS}})\,\phi_{1,2}\, e^{\rho+i2\sigma}\right)\end{aligned} \tag{5.98}$$

implies that the first order pole only vanishes if $\phi_{1,2}=0$. Hence the first vertex operator in (5.97) contains no physical degrees of freedom and thus does not contribute to the on-shell spectrum. For the second vertex operator in (5.97), one checks that the $\tilde{G}_0^+=0$ condition is satisfied. Therefore, all that is left is to check is the G_0^+ condition on the second vertex operator in (5.97).

Since each term in G^+ comes with its own exponential of ρ- and σ-ghosts, we can consider them independently. We obtain the following set of OPEs:

$$\begin{aligned}&(e^{i\sigma}\mathcal{T})(z)\left(\phi_{2,1}\, e^{2\rho+i\sigma+iH_C^{\rm GS}}\right)(w)\\ &\qquad =\sum_l\sum_{d\geq 0}(z-w)^{1+d+l}\left(\partial^d e^{i\sigma}\,\{L_{-2-l}\phi_{2,1}\}e^{2\rho+i\sigma+iH_C^{\rm GS}}\right)(w)\,,\\ &(e^{-\rho}Q)(z)\left(\phi_{2,1}\, e^{2\rho+i\sigma+iH_C^{\rm GS}}\right)(w)\\ &\qquad =\sum_l\sum_{d\geq 0}(z-w)^{2+d+l}\left(\partial^d e^{-\rho}\,\{Q_{-3-l}\phi_{2,1}\}e^{2\rho+i\sigma+iH_C^{\rm GS}}\right)(w)\,,\\ &(e^{-2\rho-i\sigma}P)(z)\left(\phi_{2,1}\, e^{2\rho+i\sigma+iH_C^{\rm GS}}\right)(w)\\ &\qquad =\sum_l\sum_{d\geq 0}(z-w)^{3+d+l}\left(\partial^d e^{-2\rho-i\sigma}\,\{P_{-4-l}\phi_{2,1}\}e^{\rho+2i\sigma+iH_C^{\rm GS}}\right)(w)\,.\end{aligned}$$

Demanding that the first order pole to vanish implies that,

$$L_d\,\phi_{1,2}=Q_d\,\phi_{1,2}=P_d\,\phi_{1,2}=0 \qquad \forall\, d\geq 0\,. \tag{5.99}$$

Most of these conditions are satisfied by construction, *e.g.* $\phi_{1,2}$ was chosen to be annihilated by all positive affine modes and $L_0\,\phi_{1,2}=0$ by assumption. However, the conditions

$$Q_0\,\phi_{1,2}=P_0\,\phi_{1,2}=0 \tag{5.100}$$

are nontrivial. Note that neither the ghosts nor the fields of the topological twisted SCFT make any appearance in this physical state condition. It defines a subsector in the space of states in the PSL(2|2) WZW model only.

We still have to figure out which states are $\tilde{G}_0^+G_0^+$-trivial. To perform this analysis, it is helpful to realise that $\tilde{G}_0^+G_0^+$ is of U(1) charge two. This restricts the possible form of the preimage if we require the image to coincide with the second vertex operator in (5.97). In fact, the pre-image vertex operator should be of the form $(\psi_{2,1}e^{2\rho+i\sigma})$, where $\psi_{2,1}$ is again a vertex operator of the supergroup WZW model associated with an affine ground state (as for $\phi_{m,n}$, the subscripts of $\psi_{2,1}$ indicate the number of $\rho\sigma$ ghosts is the corresponding exponential). For the sake of readability, let us apply the different summands that $G^+(z)$ consists of individually. From the first summand we obtain

$$\begin{aligned} &(e^{-2\rho-i\sigma}P)(z)\left(\psi_{2,1}e^{2\rho+i\sigma}\right)(w) \\ &= \sum_l \sum_{d\geq 0} \quad (z-w)^{3+l+d}\left(\tfrac{1}{d!}\partial^d(e^{-2\rho-i\sigma})e^{2\rho+i\sigma}\{P_{-4-l}\psi_{2,1}\}\right) \\ &\qquad \xrightarrow{\text{1st order pole}} \sum_{d\geq 0}\left(\tfrac{1}{d!}\partial^d(e^{-2\rho-i\sigma})e^{2\rho+i\sigma}\{P_d\psi_{2,1}\}\right). \end{aligned} \tag{5.101}$$

The second summand contributes

$$\begin{aligned} &(e^{-\rho}Q)(z)\left(\psi_{2,1}e^{2\rho+i\sigma}\right)(w) \\ &= \sum_l \sum_{d\geq 0} \quad (z-w)^{2+l+d}\left(\tfrac{1}{d!}\partial^d(e^{-\rho})e^{3\rho+i\sigma}\{Q_{-l-3}\psi_{2,1}\}\right) \\ &\qquad \xrightarrow{\text{1st order pole}} \sum_{d\geq 0}\left(\tfrac{1}{d!}\partial^d(e^{-\rho})e^{2\rho+i\sigma}\{Q_d\psi_{2,1}\}\right) \end{aligned} \tag{5.102}$$

while the third summands gives us

$$\begin{aligned} &(e^{i\sigma}\mathcal{T})(z)\left(\psi_{2,1}e^{2\rho+i\sigma}\right)(w) \\ &= \sum_l \sum_{d\geq 0}(z-w)^{1+l+d}\left(\tfrac{1}{d!}\partial^d(e^{-\rho})e^{2\rho+i\sigma}\{L_{-l-2}\psi_{2,1}\}\right) \\ &\qquad \xrightarrow{\text{1st order pole}} \sum_{d\geq 0}\tfrac{1}{d!}\partial^d(e^{i\sigma})e^{2\rho+i\sigma}\{L_d\psi_{2,1}\}, \end{aligned} \tag{5.103}$$

where we used that

$$\left(e^{i\sigma}(-\tfrac{1}{2}J^2+\tfrac{1}{2}\partial J)\right)(z)e^{2\rho+i\sigma}(w)\sim 0\,. \tag{5.104}$$

Combining these results, the full first order pole reads

$$\begin{aligned} \left\{G_0^+\left(\psi_{2,1}e^{2\rho+i\sigma}\right)\right\}(z) = \sum_{d\geq 0}\Big[&-\tfrac{1}{d!}\partial^d(e^{-2\rho-i\sigma})e^{2\rho+i\sigma}\{P_d\psi_{2,1}\}+\tfrac{1}{d!}\partial^d(e^{-\rho})e^{2\rho+i\sigma}\{Q_d\psi_{2,1}\} \\ &+\tfrac{1}{d!}\partial^d(e^{i\sigma})e^{2\rho+i\sigma}\{L_d\psi_{2,1}\}\Big] \end{aligned} \tag{5.105}$$

Next the action of $\tilde{G}_0^+$ has to be evaluated on this first order pole. In order to simplify the calculation, we may use that $\psi_{2,1}$ is an affine groundstate and thus annihilated by positive modes. The pole then simplifies to

$$\left\{G_0^+\left(\psi_{2,1}e^{2\rho+i\sigma}\right)\right\}(z) = -\{P_0\psi_{2,1}\}+e^{\rho+i\sigma}\{Q_0\psi_{2,1}\}+e^{2\rho+2i\sigma}\{L_0\psi_{2,1}\}\,. \tag{5.106}$$

It is not difficult to determine the first order pole with $\tilde{G}^+ = e^{\rho + iH_C^{\rm GS}}$. One obtains

$$\left\{\tilde{G}_0^+\left\{G_0^+\left(\psi_{2,1}e^{2\rho+i\sigma}\right)\right\}\right\} = \left(e^{2\rho+\sigma+iH_C^{\rm GS}}\{Q_0\psi_{2,1}\}\right) + \left(\partial(\rho + iH_C^{\rm GS})e^{3\rho+2i\sigma}\{L_0\psi_{2,1}\}\right). \tag{5.107}$$

The second term vanishes by assumption since $\psi_{2,1}$ has vanishing quadratic Casimir. Hence, if we take $\psi_{2,1}$ to be a state of an atypical groundstate representation of an affine Verma module, *i.e.* all positive modes annihilate on $\psi_{2,1}$ and $L_0\psi_{2,1} = 0$, the trivial states read

$$\left\{\tilde{G}_0^+\left\{G_0^+\left(\psi_{2,1}e^{2\rho+i\sigma}\right)\right\}\right\} = \left(e^{2\rho+\sigma+iH_C^{\rm GS}}\{Q_0\psi_{2,1}\}\right) \tag{5.108}$$

As the consequence, states in the PSL(2|2) WZW model are physical if they satisfy (5.100) up to the gauge freedom

$$\phi_{2,1} \sim \phi_{2,1} + Q_0\,\psi_{2,1}\,. \tag{5.109}$$

Putting everything together, we can identify the physical massless string spectrum $\mathcal{H}_0^{\rm ind}$ that does not dependent on the choice of compactification within the $PSL(2|2)$ WZW model. It coincides with the union of the Q_0-cohomology evaluated on the ground state representation, which is an atypical Kac module $\mathcal{K}(j)$, with the kernel of P_0,

$$\boxed{\mathcal{H}_0^{\rm ind} \simeq H_{Q_0}\left(\mathcal{K}(j)\right) \cap \ker_{\mathcal{K}(j)} P_0\,.} \tag{5.110}$$

The representation theory of $\mathfrak{psl}(2|2)$ has been discussed in chapter 2. It will allow us to compute the Q_0-cohomology explicitly even on projective covers $\mathcal{P}(j)$ in chapter 6, yielding agreement with the supergravity answer. In fact, we will see there that it is sufficient to determine the Q_0-cohomology as it is completely annihilated by P_0.

5.3.2 The massless compactification-dependent states on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$

In the case of compactifications on T^4, we can give an explicit realisation of the $\mathcal{N} = 4$ topologically twisted superconformal field theory with $\hat{c} = 2$ and it has been discussed in detail in section 4.1.4. We have seen there that after the topological twist there were four fields of vanishing conformal weight, the vacuum field Ω, the $\mathfrak{su}(2)$-current J^{++} and two complex fermions Ψ_+^a, $a = 1, 2$. As has been discussed, the first two fields exist in every admissible topologically twisted $\mathcal{N} = 4$ SCFT. However, the additional fermion fields are specific to the compactification manifold chosen, *i.e.* to T^4. In order to make contact with the the hybrid field $H_C^{\rm GS} = H^3 + H^4$, we recall that the complex fermions can be bosonised as

$$\Psi_+^a = e^{iH^a}\,. \tag{5.111}$$

Hence, in addition to (5.97) there are two supplementary vertex operator that may contribute to the physical spectrum,

$$V_{\mathrm{T}^4} \equiv \sum_{m\in\mathbb{Z}}\left(\phi_m e^{m(\rho+i\sigma)}\Psi_+^a\right) = \sum_{m\in\mathbb{Z}}\left(\phi_m e^{m(\rho+i\sigma)+iH^a}\right)\,. \tag{5.112}$$

Note that the complex fermions have U(1)-charge one such that the exponential in ρ and σ must be neutral under the U(1) current, which in turn implies $m = n$ and that the exponential has vanishing conformal weight. This fits nicely with our assumption that the WZW vertex operators ϕ_m correspond to states in atypical Kac modules $\mathcal{K}(j)$ since then the whole vertex operator V_{T^4} has conformal weight zero. Because vertex operators like V_{T^4} depend on the choice of the compactification manifold $\mathcal{M} = \mathrm{T}^4$, they are called *compactification-dependent*.

We can now evaluate the physical state conditions $G_0^+ = \tilde{G}_0^+ = 0$. The latter gives rise to the OPE

$$\begin{aligned}\tilde{G}^+(z)\sum_{m\in\mathbb{Z}}\Big(\phi_m e^{m(\rho+i\sigma)+iH^a}\Big)(w) &= \sum_{m\in\mathbb{Z}} e^{\rho+iH_C^{\mathrm{GS}}}(z)\Big(\phi_m e^{m(\rho+i\sigma)+iH^a}\Big)(w)\\ &= \sum_{m\in\mathbb{Z}}(z-w)^{1-m}\Big(e^{\rho+iH_C^{\mathrm{GS}}}(z)\phi_m e^{m(\rho+i\sigma)+iH^a}(w)\Big)\,.\end{aligned} \tag{5.113}$$

The summands with $m \leq 1$ do not contribute to the singular part of the OPE. For $m \geq 2$, demanding the first order-pole to vanish yields $\phi_m = 0$. After imposing this result on the vertex operators in (5.112), the OPE with $G^+(z)$ reads

$$\begin{aligned}G^+(z)\sum_{m\leq 1}\Big(\phi_m e^{m(\rho+i\sigma)+iH^a}\Big)(w) = \sum_{\substack{m\leq 1\\ l\geq 0}}(z-w)^{m+l}\Bigg(\Big[e^{i\sigma}(z)\{L_{-2-l}\phi_m\}\\ + e^{-\rho}(z)\{Q_{-3-l}\phi_m\} - e^{-2\rho-i\sigma}(z)\{P_{-4-l}\phi_m\}\Big]e^{m(\rho+i\sigma)+iH^a}(w)\Bigg)\end{aligned} \tag{5.114}$$

such that the first order pole reads

$$\begin{aligned}\sum_{d\geq 0}\Bigg[&\Big((\partial^d e^{i\sigma})e^{\rho+i\sigma+iH^a}\{L_d\phi_1\}\Big)\\ &+\Big((\partial^d e^{i\sigma})e^{iH^a}\{L_{d-1}\phi_0\}\Big)+\Big((\partial^d e^{-\rho})e^{\rho+i\sigma+iH^a}\{Q_{d-1}\phi_1\}\Big)\\ &+\sum_{m\leq 1}\Big((\partial^d e^{i\sigma})e^{(m-2)(\rho+i\sigma)+iH^a}\{L_{m+d-3}\phi_{m-2}\}\Big)\\ &\qquad+\Big((\partial^d e^{-\rho})e^{(m-1)(\rho+i\sigma)+iH^a}\{Q_{m+d-3}\phi_{m-1}\}\Big)\\ &\qquad-\Big((\partial^d e^{-2\rho-i\sigma})e^{m(\rho+i\sigma)+iH^a}\{P_{m+d-3}\phi_m\}\Big)\Bigg]\,.\end{aligned}$$

This pole vanishes if the following conditions on ϕ are satisfied:

$$L_d\phi_1 = 0 \tag{5.115}$$

$$L_{-1}\phi_0 + Q_{-1}\phi_1 = 0\,, \tag{5.116}$$

$$L_d\phi_0 = Q_d\phi_1 = 0\,, \tag{5.117}$$

$$L_{m-3}\phi_{m-2} + Q_{m-3}\phi_{m-1} - P_{m-3}\phi_m = 0\,, \tag{5.118}$$

$$L_{m-2}\phi_{m-2} - P_{m-2}\phi_m = 0\,, \tag{5.119}$$

$$Q_{m-2}\phi_{m-1} - 2\,P_{m-2}\phi_m = 0\,, \tag{5.120}$$

$$Q_{m-1+d}\phi_{m-1} = P_{m-1+d}\phi_m = L_{m-1+d}\phi_{m-2} = 0 \tag{5.121}$$

for $d \geq 0$ and $m \leq 1$. We did not yet use our assumption that ϕ_m sits in the ground state representation of an affine representation, *i.e.* the ϕ_m are annihilated by the positive modes of the supergroup currents. In this case, we observe from (5.121) that for any $m \leq -3$

$$P_{-4}\phi_m = 0 \quad \Rightarrow \quad \phi_m = 0 \tag{5.122}$$

since the negative modes act almost freely[4] on ϕ_m. Let us try to make this a little more explicit. Suppose Φ is some state in the affine ground state representation, *i.e.* it is annihilated by all positive $\widehat{\mathfrak{psl}}(2|2)_k$ modes. Then $P_{-4}\Phi$ can schematically be written as

$$P_{-4}\Phi = \sum_{\substack{m+n+k+l=-4\\ m\leq n\leq k\leq l\leq 0}} w_{mnkl}\,\varepsilon_{abcd}\,S^a_{-,m}S^b_{-,n}S^c_{-,k}S^d_{-,l}\Phi \tag{5.123}$$

with some appropriately chosen coefficients $w_{mnkl} \in \mathbb{C}$. Now, any individual state contained in the above sum is linearly independent of the others, so they have to vanish independently. In particular, we can choose $m = n = k = l = -1$ such that $\varepsilon_{abcd}S^a_{-,-1}S^b_{-,-1}S^c_{-,-1}S^d_{-,-1}\Phi = 0$, which in turn implies that $\Phi = 0$ because the (-1)-modes act freely on Φ. Intuitively, from a more physical perspective, one may think of P_{-4} as the operator that generates the state associated with the field P from the vacuum as P has conformal weight four. But adding a particle to some state in the vacuum representation should not vanish. Hence $\Phi = 0$.

Therefore, at this point we already know that $\phi_m \neq 0$ only for $-2 \leq m \leq 1$. This observation can be imposed on (5.119) for $m = -2$ and along the same lines as before one obtains

$$P_{-4}\phi_{-2} = 0 \quad \Rightarrow \quad \phi_{-2} = 0\,. \tag{5.124}$$

Similarly, (5.118) yields

$$P_{-4}\phi_{-1} = 0 \quad \Rightarrow \quad \phi_{-1} = 0\,. \tag{5.125}$$

So only ϕ_0 and ϕ_1 are non-vanishing and might carry physical degrees of freedom. However,

[4]The action of the negative modes is called "almost free" since they are subject to an equivalence relation induced by the nontrivial commutation relations. The action of elements in $\mathcal{U}(\mathfrak{n}^-)$ instead is free.

they are still subject to the residual physical state conditions

$$L_0\phi_1 = L_0\phi_0 = Q_0\phi_1 = Q_0\phi_0 = P_0\phi_1 = P_0\phi_0 = 0\,, \tag{5.126}$$

and additionally

$$P_{-1}\phi_1 = 0\,, \tag{5.127}$$

$$P_{-3}\phi_0 = 0\,, \tag{5.128}$$

$$P_{-2}\phi_1 - Q_{-2}\phi_0 = 0\,, \tag{5.129}$$

$$L_{-1}\phi_0 + Q_{-1}\phi_1 = 0\,. \tag{5.130}$$

Now we make use of the fact that $\mathfrak{psl}(2|2)$ is a Lie superalgebra of type I, which implies that it has a decomposition of the form $\mathfrak{g} \simeq \mathfrak{g}_{-1} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}_{+1}$. Recall that the field P is actually defined to be the normal ordered product of all currents associated to $\mathfrak{g}_{-1}$ (not to be confused with the vector space of affine (-1)-modes, that is denoted by $\widehat{\mathfrak{g}}_{-1}$). Hence if applied to affine $\widehat{\mathfrak{psl}}(2|2)_k$ ground states, condition (5.128) tells us that every physical state must be annihilated by any element of $\mathfrak{g}_{-1}$. Again, this can be seen from an argument similar to our argument for (5.122). For any affine ground state, which we call Φ as before, $P_{-3}\Phi$ can be written as

$$P_{-3}\Phi = \sum_{\substack{m+n+l=-3 \\ m\le n\le l\le 0}} w_{mnl}\, \varepsilon_{abcd}\, S^a_{-,m} S^b_{-,n} S^c_{-,l} S^d_{-,0}\Phi \tag{5.131}$$

with some coefficients $w_{mnl} \in \mathbb{C}$. Due to the linear independence of the summands, they have to vanish independently. We can choose $m = n = l = -1$ such that

$$\varepsilon_{abcd} S^a_{-,-1} S^b_{-,-1} S^c_{-,-1} S^d_{-,0}\Phi = 0\,, \tag{5.132}$$

which in turn implies that $S^d_{-,0}\Phi = 0$ for any $d = 1, \ldots, 4$.

In view of our discussion in section 4.3.4, these constraints should be ultimately applied to projective covers $\mathcal{P}(j)$, which will be done in the next chapter. However, as a first instructive step, we may apply these constraints to (atypical) Kac-modules $\mathcal{K}(j)$. Then eq. (5.128) tells us that ϕ_0 coincides with the "lowest" $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$-representation $\mathcal{V}(j)$ in $\mathcal{K}(j)$. Furthermore, using similar arguments as before, eq. (5.127) demands that valid states ϕ_1 are annihilated by any combination of three fermionic generators in $\mathfrak{g}_{-1}$. This basically reduces the Kac modules, viewed as a vector space, to its (13-dim) "lower half" as depicted in figure 5.2. Then evaluating eq. (5.130) on this subspace and using ϕ_0 as determined above reduces the subspace to the submodule $\mathcal{L}(j)$, *i.e* the socle of $\mathcal{K}(j)$. This can be shown by performing a calculation using an explicit construction of the Kac module. Although cumbersome, a calculation of this sort is quite accessible due to the fact that Kac modules are induced modules by definition.

To summarise, if the hybrid physical state conditions for compactification-dependent massless string states in the case of compactifications on T^4 are applied to an atypical Kac

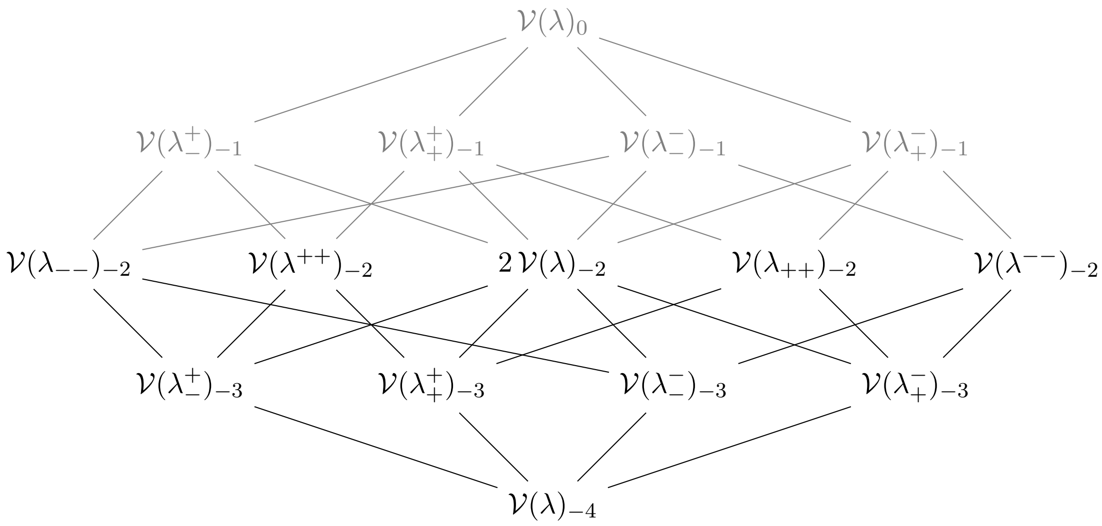


Figure 5.2: *The "lower half" of a Kac module as it is determined by (5.127).*

module $\mathcal{K}(j)$, they can be equivalently formulated from the point of view of the PSL(2|2) WZW model by saying that the physical states coincide with the socle of $\mathcal{K}(j)$.

$$\boxed{\mathcal{H}_0^{\mathrm{T}^4} \simeq \operatorname{soc}\mathcal{K}(j) \simeq \mathcal{L}(j)}\,. \tag{5.133}$$

We will see in chapter 6 how to generalise this result to cover projective covers as well.

Chapter 6

The Massless String Spectrum within the PSL(2|2) WZW Model

We have shown in the previous chapter that RNS string theory with pure NSNS flux has an equivalent description in terms of a supergroup WZW model, which in turn defines a logarithmic conformal field theory, or LCFT for short (see section 4.3.3). Let us remind the reader that, using ideas that had been developed before for the analysis of the logarithmic triplet models, we have made a detailed proposal for the spectrum of this LCFT in section 4.3.4. Indeed, by employing our detailed knowledge of the structure of the projective covers from section 2.2.3, we gave a natural proposal for how the left- and right-moving projective representations have to be coupled together, leading to a description of the full spectrum as the quotient space of the direct sum of tensor products of the projective representations. This fixed the spectrum of the underlying world sheet CFT, from which one can then obtain the string spectrum as a suitable BRST cohomology.

In this section, in order to check our proposal (4.80), we calculate the BRST cohomology for the massless string states. For this case the BRST cohomology was previously studied in terms of the vertex operators in [62]. We explain how the BRST operators of section 5.3 can be lifted to act on the projective covers (from which the LCFT spectrum can be obtained by quotienting). It is then straightforward to determine their common cohomology, and hence the massless physical string spectrum. We find that the resulting spectrum agrees precisely with the supergravity prediction of [54, 57], including the truncations that appear for small momenta.

6.1 The Compactification-Independent Spectrum

Since we have constructed the spectrum of the underlying LCFT in section 4.3.4 we need to define the BRST operator. For the massless sector, the cohomological description of the physical sector (5.8) can be simplified to (5.110), and we can identify the BRST operator with a suitable operator in the universal enveloping algebra of $\mathfrak{psl}(2|2)$. There is a subtlety about how this BRST operator can be lifted to the direct sum of projective covers, see section 6.1.1, but once this is achieved, it is straightforward to determine its cohomology. As has already been said, we find that the cohomology agrees precisely with the supergravity spectrum, see section 6.1.2. This generalises and refines the recent analysis of [178]; in particular, we explain in more detail how left- and right-moving degrees of freedom are coupled together, and we are able to obtain also the correct spectrum for small KK-momenta. (Naively extending the analysis of [178] to small momenta would not have correctly reproduced the expected result.)

6.1.1 The BRST-operator and its cohomology

In this chapter we are only interested in massless physical states. As we have argued in section 5.3, these appear as ground states of affine representations for which the ground state representation is atypical, and hence we can restrict ourselves to the corresponding projective covers in $\hat{\mathcal{H}}$ (see (4.80)). Because we are only interested in the ground states, we can ignore the affine excitations.

In section 5.3.1, we have further argued that the cohomological description of the string spectrum in (5.8) simplifies on ground state representations of vanishing quadratic Casimir and reduces to the cohomology (5.110) of the BRST operator $\mathrm{Q}_{\text{hybrid}} = K_{ab} S^a_- S^b_-$, as well as its right-moving analogue. Because the $\mathfrak{so}(4)$ indices are all contracted, $\mathrm{Q}_{\text{hybrid}}$ commutes with $\mathfrak{g}^{(0)}$, and it follows from a straightforward computation that it also commutes with $\mathfrak{g}_{-1}$. For the following it will be convenient to define more generally

$$\mathrm{Q}_\alpha = K_{ab} S^a_\alpha S^b_\alpha \ , \qquad \alpha = \pm \, , \tag{6.1}$$

with $\mathrm{Q}_- \equiv \mathrm{Q}_{\text{hybrid}}$. Note that Q_α has $\mathbb{Z}$-grading 2α.

From now on we shall work with the basis of generators of $\mathfrak{g}$ given in Appendix C, for which we have

$$\begin{aligned} \mathrm{Q}_\alpha = -i\big[& S^-_{1\alpha} S^+_{1\alpha} (J^0 + K^0) + S^-_{2\alpha} S^+_{2\alpha} (J^0 - K^0) \\ & + S^+_{2\alpha} S^-_{1\alpha} K^+ + S^-_{2\alpha} S^-_{1\alpha} J^+ + S^+_{1\alpha} S^-_{2\alpha} K^- + S^+_{1\alpha} S^+_{2\alpha} J^- \big] \ . \end{aligned} \tag{6.2}$$

Using the commutation relations of Appendix C, we find by a direct calculation

$$[S^\pm_{m\beta}, \mathrm{Q}_\gamma] = i\, \varepsilon_{\beta\gamma} S^\pm_{m\gamma}\, C_2 \qquad \mathrm{Q}^2_\alpha = S^4_\alpha\, C_2 \ , \tag{6.3}$$

where $S^4_\alpha = S^+_{2\alpha} S^-_{2\alpha} S^+_{1\alpha} S^-_{1\alpha}$, and C_2 is the quadratic Casimir of $\mathfrak{psl}(2|2)$. Thus if the quadratic Casimir vanishes on a given representation $\mathcal{R}$, $C_2(\mathcal{R}) = 0$, the operator Q_α is nilpotent and commutes with the full $\mathfrak{psl}(2|2)$ algebra on $\mathcal{R}$, *i.e.* it defines a nilpotent $\mathfrak{psl}(2|2)$-homomorphism from $\mathcal{R}$ to itself. In particular, the cohomology of Q_α on $\mathcal{R}$ then organises itself into representations of $\mathfrak{psl}(2|2)$.

An important class of representations on which the quadratic Casimir vanishes are the atypical Kac modules $\mathcal{K}(j)$ with $j \geq \frac{1}{2}$. For each $\mathcal{K}(j)$ there are two non-trivial homomorphisms $\mathcal{K}(j) \to \mathcal{K}(j)$: apart from the identity we have the homomorphism q_- that maps the head of $\mathcal{K}(j)$ to its socle and that has $\mathbb{Z}$-grading -2. Since the identity operator is not nilpotent, we conclude that the BRST operators $\mathrm{Q}_\pm$ must be equal to

$$\text{on } \mathcal{K}(j): \qquad \mathrm{Q}_+ = 0 \ , \qquad \mathrm{Q}_- = q_- \ . \tag{6.4}$$

Similarly, on the dual Kac module, $\mathcal{K}^\vee(j)$, the BRST operator Q_- is trivial, while Q_+ now agrees with the non-trivial homomorphism q_+ that maps the head of $\mathcal{K}^\vee(j)$ to its socle (which has now grade +2)

$$\text{on } \mathcal{K}^\vee(j): \qquad \mathrm{Q}_+ = q_+ \ , \qquad \mathrm{Q}_- = 0 \ . \tag{6.5}$$

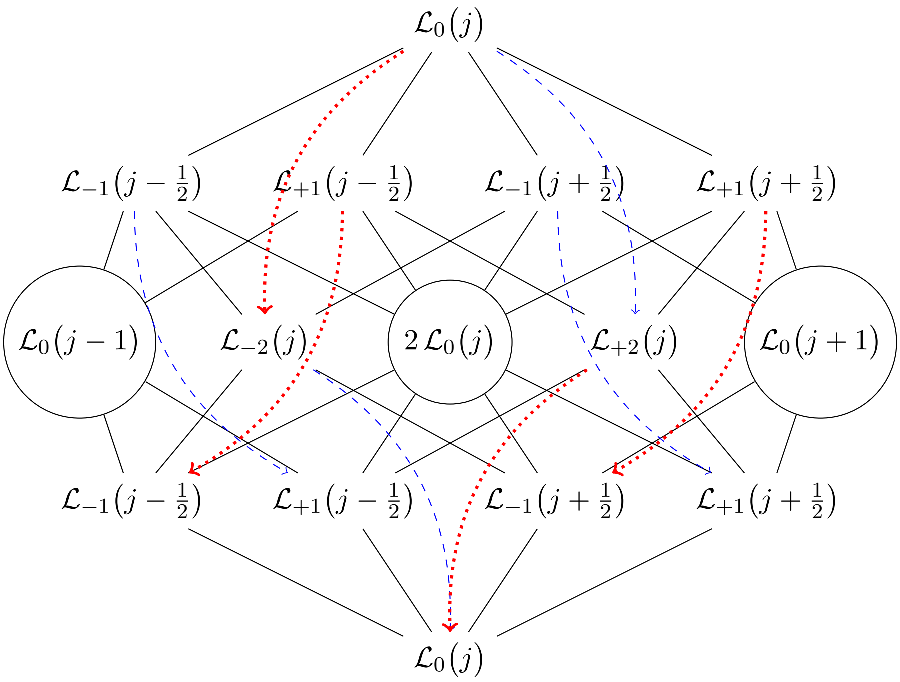


Figure 6.1: *The action of the BRST operators $\mathcal{Q}_+$ (blue, dashed arrows) and $\mathcal{Q}_-$ (red, dotted arrows) on the projective cover $\mathcal{P}(j)$ for $j \geq 1$. The irreducible representations that generate the common cohomology of $\mathcal{Q}_+$ and $\mathcal{Q}_-$ have been circled.*

Next we need to discuss the relation between Kac modules and the full CFT spectrum $\mathcal{H}^{(0)}$ proposed in (4.80). Using similar arguments as in section 4.3.4, it is not difficult to see that, as a vector space, $\mathcal{H}^{(0)}$ is isomorphic to

$$\mathcal{H}^{(0)} = \bigoplus_{(j_1,j_2)} \mathcal{K}(j_1,j_2) \otimes \overline{\mathcal{K}(j_1,j_2)} \ . \tag{6.6}$$

On the atypical representations (that correspond to the massless states) the BRST operators $\mathrm{Q}_\pm$ (defined as acting on the two Kac modules) are then indeed nilpotent. However, this definition of $\mathrm{Q}_\pm$ does not agree with the usual zero mode action on $\mathcal{H}^{(0)}$ since (6.6) is only true as a vector space, but not as a representation of the two superalgebra actions. (Indeed, with respect to the left-moving superalgebra, say, the correct action is given by (4.81)). In order to define the BRST operators on the full space of states it is therefore more convenient to lift $\mathrm{Q}_\pm$ to the projective covers. This requires a little bit of care as the operators $\mathrm{Q}_\pm$, as defined above, are not nilpotent on $\mathcal{P}(j)$. In fact, the quadratic Casimir does not vanish on $\mathcal{P}(j)$ since it maps, for example, the head of $\mathcal{P}(j)$ to $\mathcal{L}_0(j)$ in the middle line, see Fig. 2.7. However, the projectivity property guarantees that there exist nilpotent operators

$$\mathcal{Q}_\pm : \mathcal{P}(j) \to \mathcal{P}(j) \ , \qquad \mathcal{Q}_\pm^2 = 0 \, , \qquad [\mathfrak{psl}(2|2), \mathcal{Q}_\pm]\big|_{\mathcal{P}(j)} = 0 \ . \tag{6.7}$$

For example, for the case of Q_-, we apply (2.29) with $\mathcal{A} = \mathcal{P}(j)$ and $\mathcal{B} = \mathcal{K}(j)$, and thus conclude that there exists a homomorphism $\mathcal{Q}_- : \mathcal{P}(j) \to \mathcal{P}(j)$ such that

$$\begin{array}{ccc} & & \mathcal{P}(j) \\ & \overset{\mathcal{Q}_-}{\swarrow} & \big\downarrow {\scriptstyle \mathrm{Q}_- \circ \pi_{\mathcal{K}}} \\ \mathcal{P}(j) & \xrightarrow{\ \pi_{\mathcal{K}}\ } & \mathcal{K}(j) \end{array} \qquad\qquad \pi_{\mathcal{K}} \circ \mathcal{Q}_- = \mathrm{Q}_- \circ \pi_{\mathcal{K}} \, , \tag{6.8}$$

where $\pi_{\mathcal{K}}$ is the surjective homomorphism from $\mathcal{P}(j)$ to $\mathcal{K}(j)$. Furthermore, it follows from the structure of the projective cover, see Fig. 2.7 and Fig. 2.9(a), that there is only one homomorphism on $\mathcal{P}(j)$ of $\mathbb{Z}$-grading -2, namely the one that maps the head $\mathcal{L}_0(j)$ of $\mathcal{P}(j)$ to $\mathcal{L}_{-2}(j)$ in the middle line. Its square vanishes (for example, because there is no homomorphism of $\mathbb{Z}$-grading -4), and thus we conclude that $\mathcal{Q}_-$ is nilpotent. The argument for $\mathcal{Q}_+$ is analogous. The resulting action of $\mathcal{Q}_-$ and $\mathcal{Q}_+$ on $\mathcal{P}(j)$ with $j \geq 1$ is depicted in Fig. 6.1. For $j = \frac{1}{2}$, the analysis is essentially the same, the only difference being the absence of the left-most irreducible representation in the middle line.

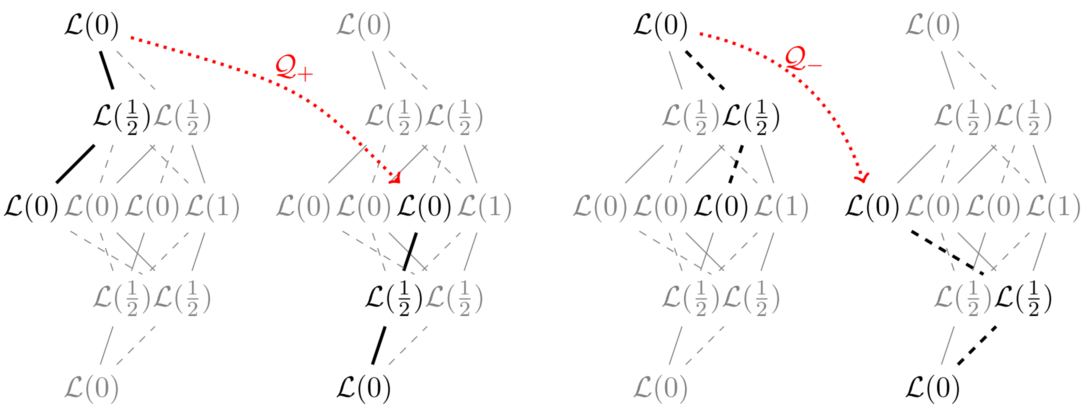
For $j = 0$ we can argue along similar lines, however with one small modification. Recall that for $j = 0$ the Kac module $\mathcal{K}(0)$, see Fig. 2.6, is not part of our category (and a similar statement applies to the dual Kac module $\mathcal{K}^\vee(0)$). However, our category *does* contain an analogue of the Kac module for $j = 0$, which we shall denote by $\hat{\mathcal{K}}(0)$. It is the quotient of the projective cover $\mathcal{P}(0)$ by the subrepresentation $\mathcal{M}^+_{-1}(\frac{1}{2})$ (see section 2.2.3.1 for a definition), and likewise for the dual Kac module; their diagrammatic form is given by

$$\hat{\mathcal{K}}(0) : \quad \begin{array}{ccc} \mathcal{L}(0) & & \\ & \diagdown & \\ & & \mathcal{L}(\tfrac{1}{2}) \\ & \diagup & \\ \mathcal{L}(0) & & \end{array} \qquad\qquad \hat{\mathcal{K}}^\vee(0) : \quad \begin{array}{ccc} \mathcal{L}(0) & & \\ & \ddots & \\ & & \mathcal{L}(\tfrac{1}{2}) \\ & \cdot^{\cdot^{\cdot}} & \\ \mathcal{L}(0) & & \end{array}$$

The quadratic Casimir vanishes on $\hat{\mathcal{K}}(0)$ and $\hat{\mathcal{K}}^\vee(0)$, and thus $\mathrm{Q}_\pm$ are nilpotent homomorphisms on $\hat{\mathcal{K}}(0)$ and $\hat{\mathcal{K}}^\vee(0)$. By the same arguments as above, we can then lift $\mathrm{Q}_\pm$ to nilpotent homomorphisms $\mathcal{Q}_\pm$ on $\mathcal{P}(0)$, and their structure is given in Fig. 6.2.

6.1.2 The physical spectrum

According to section 5.3.1, the (massless) physical states of the string theory are described by the common cohomology of $\mathcal{Q}_-$ and $\bar{\mathcal{Q}}_-$, where $\bar{\mathcal{Q}}_\pm$ are the corresponding



(a) The BRST operator Q_+ acting on the projective cover $\mathcal{P}(0)$.

(b) The BRST operator $\mathcal{Q}_-$ acting on the projective cover $\mathcal{P}(0)$.

Figure 6.2: *The action of the operators $\mathcal{Q}_\pm$ on $\mathcal{P}(0)$.*

right-moving BRST operators. Since $\mathcal{Q}_-$ and $\bar{\mathcal{Q}}_-$ commute with one another, the common cohomology simply consists of those states that are *simultaneously* annihilated by $\mathcal{Q}_-$ and $\bar{\mathcal{Q}}_-$, modulo states that are *either* in the image of $\mathcal{Q}_-$ or $\bar{\mathcal{Q}}_-$.

Given the explicit form of the various BRST operators, see Fig. 6.1 and Fig. 6.2, it is clear that on the actual space of states (4.80), we have the equivalences

$$\mathcal{Q}_\pm \otimes \bar{\mathrm{id}} \;\cong\; \mathrm{id} \otimes \bar{\mathcal{Q}}_\mp \ . \tag{6.9}$$

We may therefore equivalently characterise the (massless) physical string states as lying in the common BRST cohomology of $\mathcal{Q}_-$ and $\mathcal{Q}_+$. Note that since $\mathcal{Q}_-$ and $\bar{\mathcal{Q}}_-$ obviously commute, the same must be true for $\mathcal{Q}_-$ and $\mathcal{Q}_+$; this can be easily verified from their explicit action on the projective covers.

Since these two BRST operators now only act on the left-movers, we can work with the representatives as described in (4.81). From the description of the BRST operators, see in particular Fig. 6.1, we conclude that the common cohomology of $\mathcal{Q}_\pm$ equals for $j \geq 1$

$$H^0\left(\mathcal{P}(j)\right) \simeq \mathcal{L}(j-1) \oplus 2\,\mathcal{L}(j) \oplus \mathcal{L}(j+1) \ , \qquad j \geq 1 \ . \tag{6.10}$$

For $j = \frac{1}{2}$, the only difference is the absence of the left-most irreducible representation in the middle line, and we have instead

$$H^0\left(\mathcal{P}(\tfrac{1}{2})\right) \simeq 2\,\mathcal{L}(\tfrac{1}{2}) \oplus \mathcal{L}(\tfrac{3}{2}) \ , \tag{6.11}$$

while for $j = 0$ we get from Fig. 6.2

$$H^0\left(\mathcal{P}(0)\right) \simeq \mathcal{L}(0) \oplus \mathcal{L}(1) \ . \tag{6.12}$$

Here both $\mathcal{L}(0)$ and $\mathcal{L}(1)$ appear in the middle line of $\mathcal{P}(0)$, and $\mathcal{L}(0)$ is the middle of the three $\mathcal{L}(0)$'s.

The actual cohomology of interest is then simply the tensor product of these BRST cohomologies for the left-movers, with the irreducible head coming from the right-movers; thus we get altogether

$$\begin{aligned}\mathcal{H}_{\text{phys}} &= \Big[\Big(\mathcal{L}(0)\oplus\mathcal{L}(1)\Big)\otimes\overline{\mathcal{L}(0)}\Big] \;\oplus\; \Big[\Big(2\mathcal{L}(\tfrac{1}{2})\oplus\mathcal{L}(\tfrac{3}{2})\Big)\otimes\overline{\mathcal{L}(\tfrac{1}{2})}\Big] \\ &\quad\oplus \bigoplus_{j\geq 1}\Big[\Big(\mathcal{L}(j-1)\oplus 2\mathcal{L}(j)\oplus\mathcal{L}(j+1)\Big)\otimes\overline{\mathcal{L}(j)}\Big] \\ &= \Big(\mathcal{L}(0)\otimes\overline{\mathcal{L}(0)}\Big) \;\oplus\; \Big(\mathcal{L}(0)\otimes\overline{\mathcal{L}(1)}\Big) \;\oplus\; \Big(\mathcal{L}(1)\otimes\overline{\mathcal{L}(0)}\Big) \\ &\quad\oplus \bigoplus_{j\geq\frac{1}{2}}\Big[\Big(\mathcal{L}(j+1)\otimes\overline{\mathcal{L}(j)}\Big) \;\oplus\; 2\Big(\mathcal{L}(j)\otimes\overline{\mathcal{L}(j)}\Big) \;\oplus\; \Big(\mathcal{L}(j)\otimes\overline{\mathcal{L}(j+1)}\Big)\Big]\,. \end{aligned} \tag{6.13}$$

The spectrum for $j\geq 1$ fits directly the KK-spectrum of supergravity on $\mathrm{AdS}_3\times\mathrm{S}^3$ [54,57]. It therefore remains to check the low-lying states. In order to compare our results with [54,57], we decompose the physical spectrum with respect to the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ Lie algebra[1] corresponding to the bosonic Lie generators K^a and $\bar{K}^a$; the relevant representations are therefore labelled by $(j_2,\bar{\jmath}_2)$. For the first few values of $(j_2,\bar{\jmath}_2)$, the multiplicities are worked out in Tab. 6.1. The multiplicities of the last column reproduce precisely the results of [54], see eq. (6.2) of that paper with $n_T=1$.

$(j_2,\bar{\jmath}_2)$	$\mathfrak{psl}(2\|2)$-rep	# in $\mathfrak{psl}(2\|2)$-rep	# in $\mathcal{H}$	$\sum$
$(0,0)_{S^3}$	$\mathcal{L}(0)\otimes\overline{\mathcal{L}(0)}$	4	4	6
	$\mathcal{L}(\frac{1}{2})\otimes\overline{\mathcal{L}(\frac{1}{2})}$	1	2	
$(0,\frac{1}{2})_{S^3}$	$\mathcal{L}(0)\otimes\overline{\mathcal{L}(0)}$	2	2	8
	$\mathcal{L}(\frac{1}{2})\otimes\overline{\mathcal{L}(\frac{1}{2})}$	2	4	
	$\mathcal{L}(0)\otimes\overline{\mathcal{L}(1)}$	2	2	
$(\frac{1}{2},\frac{1}{2})_{S^3}$	$\mathcal{L}(0)\otimes\overline{\mathcal{L}(0)}$	1	1	13
	$\mathcal{L}(\frac{1}{2})\otimes\overline{\mathcal{L}(\frac{1}{2})}$	4	8	
	$\mathcal{L}(0)\otimes\overline{\mathcal{L}(1)}$	1	1	
	$\mathcal{L}(1)\otimes\overline{\mathcal{L}(0)}$	1	1	
	$\mathcal{L}(1)\otimes\overline{\mathcal{L}(1)}$	1	2	
$(0,1)_{S^3}$	$\mathcal{L}(0)\otimes\overline{\mathcal{L}(1)}$	4	4	7
	$\mathcal{L}(\frac{1}{2})\otimes\overline{\mathcal{L}(\frac{1}{2})}$	1	2	
	$\mathcal{L}(\frac{1}{2})\otimes\mathcal{L}(\frac{3}{2})$	1	1	

Table 6.1: *Decompostion of $\mathcal{H}_{\text{phys}}$ under $\mathfrak{so}(4)$. The first column denotes the $\mathfrak{so}(4)$ representations, the second enumerates the irreducible $\mathfrak{psl}(2|2)$ representations which contain the relevant $\mathfrak{so}(4)$ representation. The third column lists its multiplicity within the $\mathfrak{psl}(2|2)$ representation, and the fourth its overall multiplicity in $\mathcal{H}_{\text{phys}}$. Finally, the last column sums the multiplicities from the different $\mathfrak{psl}(2|2)$ representations.*

[1]These generators span the isometry group $\mathfrak{so}(4)=\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ of S^3.

6.2 The Compactification-Dependent Spectrum on T^4

In section 5.3.2 we have argued that massless compactification-dependent states, if the internal manifold is T^4, are of the form $\big(\phi\, e^{\rho+i\sigma}\Psi^a_+\big)(z)$ with $a = 1, 2$. In contrast to the physical state condition on compactification-independent states, which can be written from the point of view of $\mathfrak{psl}(2|2)$ representation theory as taking the cohomology of Q_0, the physical state conditions on compactification-dependent states (5.127) - (5.130) are a little more complicated to handle. We have already argued in section 5.3.2 that when these conditions are applied to atypical Kac modules $\mathcal{K}(j)$, only the socle of the Kac module survives, *i.e.* the minimal submodule within $\mathcal{K}(j)$. The socle of $\mathcal{K}(j)$ is isomorphic to the irreducible representation $\mathcal{L}(j)$.

But in view of our discussion of the spectrum of the PSL(2|2) WZW model in section 4.3.4, one should rather think of projective covers at the massless level than Kac modules. So the constraints (5.127) - (5.130) should rather be evaluated on projective covers $\mathcal{P}(j)$. However, the reason why these constraints could be evaluated on Kac modules before was that they are induced modules and thus comparatively easily accessible by direct computations. Unfortunately, this is not the case for projective covers, so we have to pick a different strategy: knowing that the physical states within Kac modules are those that lie in the socle of $\mathcal{K}(j)$, we will try to appropriately generalise these concepts to include projective covers.

6.2.1 Lifting the physical state constraints to projective covers

In order to find out which states satisfy the physical state constraints for compactification-dependent states on T^4, we have to reduce the projective cover to Kac modules. This would allow us to use our previous results from section 5.3.2 to draw conclusion for projective covers. Let us define two Borel algebras $\mathfrak{b}^\pm = \mathfrak{g}^{(0)} \oplus \mathfrak{g}_{\pm 1}$. Note that the operators in question, *i.e.* those appearing in (5.127) - (5.130), are all elements of $\mathcal{U}(\mathfrak{b}^-)$. Hence, the physical state constraints single out the same states independent of whether we treat the projective cover $\mathcal{P}(j)$ as a full $\mathcal{U}(\mathfrak{g})$-module or as an $\mathcal{U}(\mathfrak{b}^-)$-module. But as an $\mathcal{U}(\mathfrak{b}^-)$-module, the projective cover is not simple. In fact, it takes the following form:

$$\mathcal{P}(j)\Big|_{\mathcal{U}(\mathfrak{b}^-)} = 2\,\mathcal{K}(j)\Big|_{\mathcal{U}(\mathfrak{b}^-)} \oplus \mathcal{K}(j+\tfrac{1}{2})\Big|_{\mathcal{U}(\mathfrak{b}^-)} \oplus \mathcal{K}(j-\tfrac{1}{2})\Big|_{\mathcal{U}(\mathfrak{b}^-)}\,. \tag{6.14}$$

The physical state constraints are easily evaluated on the right hand side of (6.14) since it is just a direct sum of atypical Kac modules. Because only the socle of $\mathcal{K}(j)$ statisfies the physical state constraints, we obtain that the physical spectrum within $\mathcal{P}(j)$ is given by

$$\mathrm{Phys}\,\mathcal{P}(j)\Big|_{\mathcal{U}(\mathfrak{b}^-)} = 2\,\mathcal{L}(j)\Big|_{\mathcal{U}(\mathfrak{b}^-)} \oplus \mathcal{L}(j+\tfrac{1}{2})\Big|_{\mathcal{U}(\mathfrak{b}^-)} \oplus \mathcal{L}(j-\tfrac{1}{2})\Big|_{\mathcal{U}(\mathfrak{b}^-)}\,. \tag{6.15}$$

Here we denoted the subspace of $\mathcal{P}(j)$ that satisfy (5.127) - (5.130) by $\mathrm{Phys}\,\mathcal{P}(j)$ and used that according to section 5.3.2, $\mathrm{Phys}\,\mathcal{K}(j) = \mathrm{soc}\,\mathcal{K}(j) = \mathcal{L}(j)$. We can bring the right hand side of (6.15) in a nicer form by noting that the *dual* Kac module $\mathcal{K}^\vee(j)$ treated as an

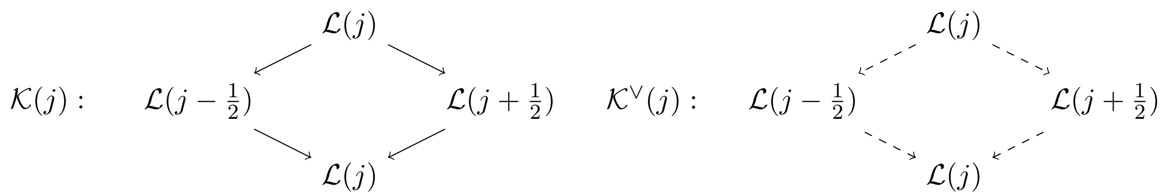


Figure 6.3: *The decomposition of the atypical Kac module $\mathcal{K}(j)$ and its dual $\mathcal{K}^\vee(j)$ into irreducible components. Solid arrows decrease the grading by 1 while the dashed one increase the grading by one. So loosely speaking, solid lines correspond to elements in $\mathcal{U}(\mathfrak{b}^-)$, and dashed line are associated with elements in $\mathcal{U}(\mathfrak{b}^+)$.*

$\mathcal{U}(\mathfrak{b}^-)$-module decomposes into a direct sum as well,

$$\mathcal{K}^\vee(j)\Big|_{\mathcal{U}(\mathfrak{b}^-)} = 2\,\mathcal{L}(j)\Big|_{\mathcal{U}(\mathfrak{b}^-)} \oplus \mathcal{L}(j+\tfrac{1}{2})\Big|_{\mathcal{U}(\mathfrak{b}^-)} \oplus \mathcal{L}(j-\tfrac{1}{2})\Big|_{\mathcal{U}(\mathfrak{b}^-)} . \tag{6.16}$$

This can be seen from the decomposition depicted in fig. 6.3. Hence we can write

$$\mathrm{Phys}\,\mathcal{P}(j)\Big|_{\mathcal{U}(\mathfrak{b}^-)} = \mathcal{K}^\vee(j)\Big|_{\mathcal{U}(\mathfrak{b}^-)} . \tag{6.17}$$

This strongly suggests that, if we consider the full $\mathcal{U}(\mathfrak{g})$-module structure, the physical states transform as a dual Kac module,

$$\mathrm{Phys}\,\mathcal{P}(j) = \mathcal{K}^\vee(j) . \tag{6.18}$$

In the following, we will assume that this holds. Fig. 6.4 shows where these states are situated within the projective cover $\mathcal{P}(j)$. Comparing fig. 6.4 to fig. 6.1, one sees that the submodule of $\mathcal{P}(j)$ consisting of compactification-dependent states on T^4 can actually be equivalently characterised as *the image of the BRST operators* $\mathcal{Q}_-$,

$$\mathrm{Phys}\,\mathcal{P}(j) = \mathrm{im}_{\mathcal{P}(j)}\mathcal{Q}_- . \tag{6.19}$$

This also holds for Kac modules in the sense that $\mathrm{Phys}\,\mathcal{K}(j) = \mathrm{im}_{\mathcal{K}(j)}\mathcal{Q}_-$. However, it is not in general true that for any module the compactification-dependent states on T^4 are given by the image of $\mathcal{Q}_-$, take *e.g.* the atypical irreducible representation $\mathcal{L}(j)$: The physical state constraints vanish on the full module, thus $\mathrm{Phys}\,\mathcal{L}(j) = \mathcal{L}(j)$, but $\mathcal{Q}_-(\mathcal{L}(j)) = 0$.

So far, we have only discussed the action of the physical state constraints on atypical projective covers. However, to get a hand on the full string spectrum, we have to take the right movers into account as well as the quotient in (4.87). The full space $\hat{\mathcal{H}}$ is simply a direct sum of products of left- and right-moving degrees of freedom and thus the physical state constraints can be evaluated on each factor independently. But these states are subject to the equivalence relation induced by taking the quotient with respect to the submodule $\mathcal{N}$ (cf. section 4.3.4). At this point in our analysis, the characterisation (6.19) of the physical compactification-dependent states in $\mathcal{P}(j)$ as the image of $\mathcal{Q}_-$ becomes

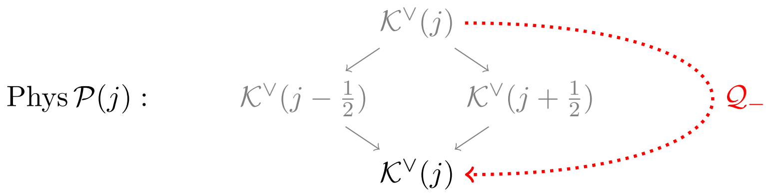


Figure 6.4: *The submodule of $\mathcal{P}(j)$ that satisfies the physical state constraints for compactification-dependent states on T^4. For comparison, we also indicated the action of $\mathcal{Q}_-$ showing that* $\text{Phys}\,\mathcal{P}(j) = \text{im}_{\mathcal{P}(j)}\mathcal{Q}_-$.

extremely useful. We have already argued in the previous section (cf. (6.9)) that on the actual space of states $\mathcal{H}^{(0)}$ we have the equivalences

$$\mathcal{Q}_\pm \otimes \bar{\text{id}} \;\cong\; \text{id} \otimes \bar{\mathcal{Q}}_\mp \,. \tag{6.20}$$

So working with the representatives (4.81), the submodule of compactification-dependent states on T^4 are given by the images of

$$\mathcal{Q}_- \otimes \overline{\text{id}} \qquad \text{for left-movers}\,, \tag{6.21}$$

$$\mathcal{Q}_+ \otimes \overline{\text{id}} \qquad \text{for right-movers}\,. \tag{6.22}$$

Thus, as in the case of compactification-independent states, the compactification-dependent states on T^4 can be characterised by the action of the BRST operators $\mathcal{Q}_\pm$ on the left-moving factor only.

6.3 The Full Massless Physical String Spectrum on $AdS_3 \times S^3 \times T^4$

At this point in our discussion, we have found two classes of physical states: those that are independent of the particular choice of compactification manifold and those specific to a torodial compactification. This allows us to give an expression for the full physical string spectrum on $AdS_3 \times S^3 \times T^4$ by combining these two classes. However, we have to be a little careful when gluing left- and right-moving degrees of freedom together. In fact, there are four possibilities how to combine them: either both left- and right-movers are compactification-independent, or one of them is compactification-dependent and the other one is not, or both are specific to compactifications on T^4. In this section, we will explain how the physical state conditions found in sections 6.1 and 6.2 should be applied to each of these possible combinations.

For simplicity, we denote compactification-independent states by ϕ^- and compactification-dependent states by ϕ^0_a, $a = 1, 2$. Each set of states is assumed to arrange in projective covers $\mathcal{P}(j)$. Putting left- and right-moving degrees of freedom together, we thus find a direct sum of nine $\mathcal{P}(j) \otimes \overline{\mathcal{P}(j)}$ summands associated to states of the form

$$\phi^- \otimes \overline{\phi^-}\,, \quad \phi^0_a \otimes \overline{\phi^-}\,, \quad \phi^- \otimes \overline{\phi^0_a}\,, \quad \phi^0_a \otimes \overline{\phi^0_b}\,. \tag{6.23}$$

with individually are subject to the equivalence relation (4.87) discussed before. Recall that this relation allows us to pick a gauge in which each one of the direct summands takes the form (4.91), which in the following we will assume. The physical state conditions on the full spectrum may differ on every summand depending on whether compactification-dependent states are involved. So in the chosen gauge, the spectrum can be written as

$$\bigoplus_j \Big(\Big[\mathcal{P}(j) \otimes \overline{\mathcal{L}(j)}\Big]_{\text{ii}} \oplus 2\, \Big[\mathcal{P}(j) \otimes \overline{\mathcal{L}(j)}\Big]_{\text{di}} \oplus 2\, \Big[\mathcal{P}(j) \otimes \overline{\mathcal{L}(j)}\Big]_{\text{id}} \oplus 4\, \Big[\mathcal{P}(j) \otimes \overline{\mathcal{L}(j)}\Big]_{\text{dd}} \Big)\,, \tag{6.24}$$

where the subscript on each summand indicates whether one is considering compactification-dependent (d) or compactification-independent (i) states with respect to the left- or right-movers, *e.g.* the direct summand $\Big[\mathcal{P}(j) \otimes \overline{\mathcal{L}(j)}\Big]_{\text{di}}$ means that the left-movers are taken to be compactification-dependent while the right-movers are compactification-independent. We will now consider each direct summand for given j seperately. The multiplicities of the direct summand are due to the fact that there are two distinguished complex fermions Ψ^a_+, $a = 1, 2$, of conformal weight zero on the four-torus T^4. So the direct summands with subscript di and id appear twice while the dd-summand has multiplicity $2^2 = 4$.

If both left- and right-moving states are compactification-independent, we already found that both the $\mathcal{Q}_-$- and $\mathcal{Q}_+$-cohomology has to be evaluated on the left-moving factor, which is the projective cover in our gauge. This leaves us with a physical spectrum of the form

$$\mathcal{H}^{\text{ii}}_{\text{phys}} = \bigoplus_j \left(\mathcal{L}(j-1) \oplus 2\mathcal{L}(j) \oplus \mathcal{L}(j+1)\right) \otimes \overline{\mathcal{L}(j)}\,. \tag{6.25}$$

Now let us consider states whose left-moving part is compactification-dependent, *i.e.* we ask for the physical states that sit within $\Big[\mathcal{P}(j) \otimes \overline{\mathcal{L}(j)}\Big]_{\text{di}}$ in (6.24). This direct summand corresponds to states of the form $\phi^0_a \otimes \overline{\phi^-}$. Since the right-moving degrees of freedom are compactification-independent, we have to evaluate the $\mathcal{Q}_-$-cohomology on the right-moving factor, which is equivalent taking the $\mathcal{Q}_+$-cohomology on the left-moving factor (cf. eq. (6.9)). Hence, in the gauge (4.87), we are interested in the cohomology

$$H_{\mathcal{Q}_+}(\mathcal{P}(j)) \simeq \mathcal{K}(j-\tfrac{1}{2}) \oplus \mathcal{K}(j+\tfrac{1}{2})\,. \tag{6.26}$$

On the resulting representation content one has to evaluate the compactification-dependent physical state conditions, *i.e.* the image of $\mathcal{Q}_-$ should be identified with the physical states. So the physical spectrum reads

$$\mathcal{Q}_- \left[H_{\mathcal{Q}_+}(\mathcal{P}(j))\right] \simeq \mathcal{Q}_- \left(\mathcal{K}(j-\tfrac{1}{2})\right) \oplus \mathcal{Q}_- \left(\mathcal{K}(j+\tfrac{1}{2})\right) \simeq \mathcal{L}(j-\tfrac{1}{2}) \oplus \mathcal{L}(j+\tfrac{1}{2})\,, \tag{6.27}$$

where we made use of $\text{im}_{\mathcal{K}(j)}\mathcal{Q}_- = \mathcal{L}(j)$, as has been argued before (see Fig. 6.3). A similar argument applies if we consider states that are compactification-independent with respect to the left-movers and compactification-dependent with respect to the right-movers. However, in this case we have to look for the image of $\mathcal{Q}_+$ applied to the $\mathcal{Q}_-$-cohomology. We obtain

$$\mathcal{Q}_+\left[H_{\mathcal{Q}_-}(\mathcal{P}(j))\right] \simeq \mathcal{Q}_+\left(\mathcal{K}^\vee(j-\tfrac{1}{2})\right) \oplus \mathcal{Q}_+\left(\mathcal{K}^\vee(j+\tfrac{1}{2})\right) \simeq \mathcal{L}(j-\tfrac{1}{2}) \oplus \mathcal{L}(j+\tfrac{1}{2})\,. \quad (6.28)$$

Due to the multiplicity of two for the di- and id-summand in (6.24) they add four physical multiplets $\left(\mathcal{L}(j-\frac{1}{2})+\mathcal{L}(j+\frac{1}{2})\right)\otimes\overline{\mathcal{L}(j)}$ to the physical spectrum. Therefore the physical spectrum originating from states where either the left- or the right-movers are chosen to be compactification-dependent is

$$\mathcal{H}^{\mathrm{id}}_{\mathrm{phys}} \oplus \mathcal{H}^{\mathrm{di}}_{\mathrm{phys}} = \bigoplus_j 4\left(\mathcal{L}(j-\tfrac{1}{2}) \oplus \mathcal{L}(j+\tfrac{1}{2})\right) \otimes \overline{\mathcal{L}(j)}\,. \quad (6.29)$$

The last case to consider are those states where the left- as well as the right-movers are compactification-dependent, *i.e.* the direct summand $\left[\mathcal{P}(j)\otimes\overline{\mathcal{L}(j)}\right]_{\mathrm{dd}}$ in (6.24). In this case, we have to look at the image of $\mathcal{Q}_- \circ \mathcal{Q}_+ = \mathcal{Q}_+ \circ \mathcal{Q}_-$. No cohomology has to be taken at all. It is not difficult to see that the image is identical to the socle of $\mathcal{P}(j)$,

$$\mathcal{Q}_-\left(\mathcal{Q}_+\mathcal{P}(j)\right) = \mathcal{Q}_-\,\mathcal{K}(j) = \mathcal{L}(j) = \operatorname{soc}\mathcal{P}(j)\,. \quad (6.30)$$

Again, as there are two relevant fermionic fields on the torus T^4, states of this form come with a multiplicity of four. So their contribution to the physical spectrum is

$$\mathcal{H}^{\mathrm{dd}}_{\mathrm{phys}} = \bigoplus_j 4\,\mathcal{L}(j)\otimes\overline{\mathcal{L}(j)} \quad (6.31)$$

Putting it all together we obtain the full massless physical spectrum for string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$,

$$\begin{aligned}\mathcal{H}_{\mathrm{phys}} &= \mathcal{H}^{\mathrm{ii}}_{\mathrm{phys}} \oplus \mathcal{H}^{\mathrm{id}}_{\mathrm{phys}} \oplus \mathcal{H}^{\mathrm{di}}_{\mathrm{phys}} \oplus \mathcal{H}^{\mathrm{dd}}_{\mathrm{phys}} \\ &= \bigoplus_j \Bigg[\Big(\underbrace{\mathcal{L}(j-1)\oplus 2\,\mathcal{L}(j)\oplus\mathcal{L}(j+1)}_{\text{compactification-independent}}\Big) \oplus 4\,\Big(\underbrace{\mathcal{L}(j-\tfrac{1}{2})\oplus\mathcal{L}(j)\oplus\mathcal{L}(j+\tfrac{1}{2})}_{\text{compactification-dependent}}\Big)\Bigg]\otimes\overline{\mathcal{L}(j)} \\ &= \bigoplus_j \left(\mathcal{L}(j-1)\oplus 4\,\mathcal{L}(j-\tfrac{1}{2})\oplus 6\,\mathcal{L}(j)\oplus 4\,\mathcal{L}(j+\tfrac{1}{2})\oplus\mathcal{L}(j+1)\right)\otimes\overline{\mathcal{L}(j)}\,,\end{aligned}$$

in agreement with the supergravity spectrum of type II string theory compactified on T^4 in [54].

Chapter 7

Massive Hybrid States

In this chapter, we give a description of the compactification-independent massive string spectrum within the PSL(2|2) WZW model. In the first section, we identify the vertex operators that correspond to physical string states at the first mass level in the hybrid formulation; the resulting string spectrum is then shown to agree with the RNS spectrum in section 7.2. Based on this result and those obtained in the previous chapters, we present a conjecture for an algebraic characterisation of the full string spectrum in section 7.3, which is then checked to hold at the second mass level. Apart from the results presented here, the reader should be aware of appendix D.2, where an examplary calculation of a relevant OPE is given, and appendix D.3, which lists OPEs of vertex operators of the $\rho\sigma$ system; the latter are of particular importance when evaluating the physical state constraints in section 7.1.

7.1 Compactification-Independent String States at the First Level

In this section, we will analyse the lightest massive string states in the hybrid formulation that are independent of the choice of the compactification manifold M. As for the massless case, our goal is to find a description that allows us to identify these physical compactification-independent massive string states within the PSL(2|2) WZW model. This would imply that it is not necessary to work with the complicated $\mathcal{N} = 2$ superconformal structure (5.61) - (5.64), which plays an important role in the hybrid formulation, in order to determine the physical string spectrum. Rather we only have to understand the algebraic structure of the affine Lie superalgebra $\widehat{\mathfrak{psl}}(2|2)$ and its representations.

Our strategy to achieve this is similar to the one applied in section 5.3.1: we start in the hybrid formulation and note that the hybrid vertex operators factorise into a vertex operator of the PSL(2|2) WZW model, and a vertex operator containing the ghost fields ρ and σ as well as fields of the $\mathcal{N} = 4$ superconformal algebra on the compactification manifold. Evaluating the physical state conditions (5.8) on these vertex operators result in conditions on the PSL(2|2) WZW vertex operator alone, thus reducing the hybrid description to an algebraic description in the context of the PSL(2|2) WZW model.

7.1.1 Evaluation of the hybrid physical states constraints

Our goal is to find an appropriate description of the massive string states within the PSL(2|2) WZW model. From the hybrid formulation we know that physical states have vanishing conformal weight, so the quadratic Casimir $C_2(\lambda)$ of the horizontal subalgebra of $\mathfrak{psl}(2|2)$ has to be negative on the ground state representation. We will be considering here states at the first level, whose ground states therefore satisfy

$$\frac{1}{2k} C_2(\lambda) = L_0 = -1\,. \tag{7.1}$$

Hence any affine descendant has conformal weight greater or equal to zero. Note that (7.1) defines a set of allowed weights for affine Kac-modules.

Recall that in the massless case (see section 5.3.1), the vertex operators of compactification-independent physical states were of the form

$$V_0^+ \equiv \left(\phi_{2,1}\, e^{2\rho+i\sigma} J_C^{++}\right) = \left(\phi_{2,1}\, e^{2\rho+i\sigma+iH}\right), \tag{7.2}$$

where $\phi_{2,1}$ is a vertex operator of the target space supersymmetric theory in six dimensions, *i.e.* of the PSL(2|2) WZW model. The guiding principle in arguing for this ansatz is that it must have vanishing conformal weight, unit $U(1)$ charge and it must be massless from the point of view of the six-dimensional theory, *i.e.* $L_0^{\mathrm{WZW}}\phi = 0$. The reader might wonder why V_0^+ should be considered compactification-independent even though it apparently involves excitations on the internal manifold M, namely the field J_C^{++}. The reason for this is that, independent of the choice of M, there is always a topologically twisted $\mathcal{N} = 4$ superconformal field theory associated with it. Hence the field J_C^{++} exists independently of the choice of the internal manifold. Following this philosophy, we generalise the vertex operator in (7.2) to include the first affine excitations of the WZW currents and of the scalar bosons ρ, σ and $H \equiv H_C^{\mathrm{RNS}}$. Thus our ansatz to describe compactification-independent states on the first massive level is

$$V^+ \equiv \left(\phi_{1,0} e^{\rho+iH}\right) + \Big((\phi_{2,1} + \partial\rho\, \phi_{2,1}^{\rho} + i\partial\sigma\, \phi_{2,1}^{\sigma} + i\partial H\, \phi_{2,1}^{H}) e^{2\rho+i\sigma+iH}\Big)\,. \tag{7.3}$$

Here $\phi_{2,1}$ is a vertex operator of the PSL(2|2) WZW model of zero conformal weight while $\phi_{1,0}$, $\phi_{2,1}^{\rho}$, $\phi_{2,1}^{\sigma}$ and $\phi_{2,1}^{H}$ are associated to affine ground states of the PSL(2|2) WZW model of conformal weight -1.[1] It is easily checked that V^+ has $U(1)$-charge one and vanishing conformal weight, as required by the physical state conditions in the hybrid formulation.

We will now evaluate the physical state conditions (5.8) on V^+. Both $\tilde{G}^+$ and G^+ have conformal weight $h = 1$, so the action of their zero-modes on V^+ is given by the first order pole of the respective OPE. Since the first order pole of any OPE, say $\phi(z)\psi(w)$, coincides with the residue of that OPE at the singular locus, we will denote it by $\mathrm{Res}\,\phi(z)\psi(w)$.

[1] Following section 5.3.1, the subscripts of the WZW vertex operators, say $\phi_{m,n}$, associate it with the corresponding exponential in the $\rho\sigma$ ghosts, $e^{m\rho+in\sigma+iH}$, *e.g.* the subscripts of $\phi_{2,1}$ imply that the vertex operator is part of the normal ordered product involving the exponential $e^{2\rho+i\sigma+iH}$ in the ansatz (7.3).

The $\tilde{G}_0^+$ condition

We first check what constraints on the ϕ's are imposed by the $\tilde{G}_0^+ = 0$ condition. Recalling that $\tilde{G}^+ = e^{\rho+iH}$ we can determine the residue of the OPE with each summand. The evaluation of the residues, although straightforward, is in some cases cumbersome. The reader interested in the technical details of the calculation is referred to appendix D.2, where the rather involved calculation of the residue in (7.11) is presented in detail. The relevant residues for the $\tilde{G}_0^+$ condition are

$$\mathrm{Res}\, e^{\rho+iH}(z)\,\big(\phi_{1,0}e^{\rho+iH}\big)(w) = 0\,, \tag{7.4}$$

$$\mathrm{Res}\, e^{\rho+iH}(z)\,\big(\phi_{2,1}e^{2\rho+i\sigma+iH}\big)(w) = 0\,, \tag{7.5}$$

$$\mathrm{Res}\, e^{\rho+iH}(z)\,\big(\phi^{\sigma}_{2,1}i\partial\sigma e^{2\rho+i\sigma+iH}\big)(w) = 0\,, \tag{7.6}$$

$$\mathrm{Res}\, e^{\rho+iH}(z)\,\Big(\phi^{\rho}_{2,1}\partial\rho e^{2\rho+i\sigma+iH}\Big)(w) = -\Big(\phi^{\rho}_{2,1}e^{3\rho+i\sigma+2iH}\Big)(w)\,, \tag{7.7}$$

$$\mathrm{Res}\, e^{\rho+iH}(z)\,\Big(\phi^{\rho}_{2,1}\partial\rho e^{2\rho+i\sigma+iH}\Big)(w) = 2\big(\phi^{H}_{2,1}e^{3\rho+i\sigma+2iH}\big)(w)\,. \tag{7.8}$$

Hence we conclude that physical states have to satisfy

$$\phi^{\rho}_{2,1} = 2\,\phi^{H}_{2,1} \tag{7.9}$$

in order for the full residue to vanish. In the following, this relation is imposed on V^+.

The G_0^+ condition

We now turn to the second kernel condition $G_0^+ = 0$. The calculation can be simplified by noting that the terms proportional to some exponential $e^{m\rho+in\sigma+iH}$ in the first order pole of the OPE have to vanish independently. Hence we will consider them separately:

Terms proportional to $e^{2\rho+2i\sigma+iH}$:

Note that normal ordered products proportional to $e^{2\rho+2i\sigma+iH}$ only appear in the OPE of $\big(e^{i\sigma}\mathcal{T}\big)$ with summands of V^+ proportional to $e^{2\rho+i\sigma+iH}$. Thus the residue of this OPE has to vanish separately. One obtains

$$\begin{aligned}&\mathrm{Res}\big(e^{i\sigma}\mathcal{T}\big)(z)\big(\phi_{2,1}e^{2\rho+i\sigma+iH}\big)(w)\\ &\qquad= \big(i\partial\sigma\{L_1\phi_{2,1}\}e^{2\rho+2i\sigma+iH}\big)(w)\,,\end{aligned} \tag{7.10}$$

$$\begin{aligned}&\mathrm{Res}\big(e^{i\sigma}\mathcal{T}\big)(z)\big(i\partial\sigma\phi^{\sigma}_{2,1}e^{2\rho+i\sigma+iH}\big)(w)\\ &\qquad= -\big(\big(\{L_{-1}\phi^{\sigma}_{2,1}\} + 2\partial(\rho+i\sigma)\phi^{\sigma}_{2,1}\big)\,e^{2\rho+2i\sigma+iH}\big)(w)\,,\end{aligned} \tag{7.11}$$

$$\begin{aligned}&\mathrm{Res}\big(e^{i\sigma}\mathcal{T}\big)(z)\big(\partial(2\rho+iH)\phi^{H}_{2,1}e^{2\rho+i\sigma+iH}\big)(w)\\ &\qquad= -\big(\partial(2\rho+iH)\phi^{H}_{2,1}e^{2\rho+2i\sigma+iH}\big)(w)\,.\end{aligned} \tag{7.12}$$

The full residue therefore vanishes if

$$\phi^{\sigma}_{2,1} = \phi^{H}_{2,1} = 0 \qquad \text{and} \tag{7.13}$$

$$L_1\phi_{2,1} = 0\,. \tag{7.14}$$

Condition (7.13) together with (7.9) tells us that all ghost excitations at the first affine level are unphysical. As a consequence, we only have to look at the space of states of the WZW model. The second condition further restricts the physical sector to include only Virasoro primaries up to the first level. Of course, the field $\phi_{1,0}$ has not yet been restricted in any way except for being an affine ground state; it is therefore a Virasoro primary by construction. Hence we reduced physical vertex operators to be of the form

$$V^{+} = \big(\phi_{1,0}e^{\rho+iH}\big) + \big(\phi_{2,1}e^{2\rho+i\sigma+iH}\big) \tag{7.15}$$

with Virasoro primaries $\phi_{1,0}$ and $\phi_{2,1}$.

Terms proportional to $e^{\rho+i\sigma+iH}$:

The next step is to look at those terms of the residue proportional to $e^{\rho+i\sigma+iH}$. These may come from the OPE of $\big(e^{i\sigma}\mathcal{T}\big)$ with terms in $\big(\phi_{1,0}e^{\rho+iH}\big)$ and from the OPE of $(e^{-\rho}Q)$ with $\big(\phi_{2,1}e^{2\rho+i\sigma+iH}\big)$. Their residues read

$$\begin{aligned}
&\mathrm{Res}\big(e^{i\sigma}\mathcal{T}\big)(z)\big(\phi_{1,0}e^{\rho+iH}\big)(w)\\
&\qquad = \big((\{L_{-1}\phi_{1,0}\} + i\partial\sigma\{L_0\phi_{1,0}\} + \partial(\rho+i\sigma)\phi_{1,0})\, e^{\rho+i\sigma+iH}\big)(w)\\
&\qquad = \big((\{L_{-1}\phi_{1,0}\} + \partial\rho\phi_{1,0})\, e^{\rho+i\sigma+iH}\big)(w)\,,
\end{aligned} \tag{7.16}$$

$$\begin{aligned}
&\mathrm{Res}\big(e^{-\rho}Q\big)(z)\big(\phi_{2,1}e^{2\rho+i\sigma+iH}\big)(w)\\
&\qquad = \big((\{Q_0\phi_{2,1}\} - \partial\rho\{Q_1\phi_{2,1}\})\, e^{\rho+i\sigma+iH}\big)(w)\,,
\end{aligned} \tag{7.17}$$

where we have used that $L_0\phi_{1,0} = -\phi_{1,0}$. Again, we demand the full residue to vanish. We find that $\phi_{1,0}$ is determined by $\phi_{2,1}$,

$$\phi_{1,0} = Q_1\phi_{2,1}\,, \tag{7.18}$$

and that $\phi_{2,1}$ is subject to the constraint,

$$Q_0\phi_{2,1} + L_{-1}\phi_{1,0} = (Q_0 + L_{-1}Q_1)\phi_{2,1} = 0\,. \tag{7.19}$$

Note that (7.18) implies that $\phi_{1,0}$ does not carry physical degrees of freedom even though it is non-vanishing. Thus it is enough to know all the physical degrees of freedom contained in $\phi_{2,1}$. Apart from the condition that it has to be a Virasoro primary with respect to the WZW model, it also lies in the kernel of $Q_0 - L_{-1}Q_1$.

Terms proportional to e^{iH}*:*

Finally, we take a look at terms of the residue uncharged with respect to ρ as well as σ. In other words, we consider terms proportional to e^{iH} originating from the OPE of $(e^{-\rho}Q)$ with $\big(\phi_{1,0}e^{\rho+iH}\big)$ and the OPE of $\big(e^{-2\rho-i\sigma}P\big)$ with $\big(\phi_{2,1}e^{2\rho+i\sigma+iH}\big)$,

$$\begin{aligned}\mathrm{Res}\big(e^{-\rho}Q\big)(z)\big(\phi_{1,0}e^{\rho+iH}\big)(w)&\\ &= \big((\{Q_{-1}\phi_{1,0}\} - \partial\rho\{Q_0\phi_{1,0}\})\, e^{iH}\big)(w)\,, \end{aligned} \tag{7.20}$$

$$\begin{aligned}\mathrm{Res}\big(e^{-2\rho-i\sigma}P\big)(z)\big(\phi_{2,1}e^{2\rho+i\sigma+iH}\big)(w)&\\ &= \big((\{P_0\phi_{2,1}\} - \partial(2\rho+i\sigma)\{P_1\phi_{2,1}\})\, e^{iH}\big)(w)\,. \end{aligned} \tag{7.21}$$

From those and the identification in (7.18) we see that $\phi_{2,1}$ is also subject to the constraints

$$(P_0 - Q_{-1}Q_1)\phi_{2,1} = Q_0Q_1\phi_{2,1} = P_1\phi_{2,1} = 0\,. \tag{7.22}$$

Terms proportional to $e^{-\rho-i\sigma+iH}$*:*

There is still one OPE left to consider, namely the OPE of $\big(e^{-2\rho-i\sigma}P\big)$ with $\big(\phi_{1,0}e^{\rho+iH}\big)$. Its residue is

$$\begin{aligned}\mathrm{Res}\big(e^{-2\rho-i\sigma}P\big)(z)\big(\phi_{1,0}e^{\rho+iH}\big)(w)&\\ &= \big((\{P_{-1}\phi_{1,0}\} - \partial(2\rho+i\sigma)\{P_0\phi_{1,0}\})\, e^{-\rho-i\sigma+iH}\big)(w)\,. \end{aligned} \tag{7.23}$$

So we must demand

$$P_{-1}Q_1\phi_{2,1} = P_0Q_1\phi_{2,1} = 0\,. \tag{7.24}$$

in order to exploit the kernel condition completely.

Let us summarise our results so far. We have seen that the compactification-independent physical spectrum of the hybrid string can be identified within the PSL(2|2) WZW model alone, at least up to the first level. The physical hybrid string states within the full WZW-spectrum are subject to the following kernel conditions

$$L_1\phi = (Q_0 + L_{-1}Q_1)\phi = (P_0 - Q_{-1}Q_1)\phi = Q_0Q_1\phi = P_{-1}Q_1\phi = P_0Q_1\phi = P_1\phi = 0\,. \tag{7.25}$$

The interpretation of the first constraint is simply that physical hybrid string states have to be Virasoro primary as already said above. We will see later that apart from the restriction to be Virasoro primaries, the only significant condition to impose is the second one in (7.25), namely $(Q_0 + L_{-1}Q_1)\phi = 0$. All other kernel conditions are then automatically satisfied, at least at the first level.

Gauge degrees of freedom

Having found a set of kernel conditions on physical states, eq. (7.25), we may now investigate which of these are gauge trivial. In particular, we are interested in the gauge freedoms of the WZW vertex operator $\phi_{2,1}$ as it is the only one carrying physical degrees of freedom.

Recall that gauge trivial states in the hybrid formulation are of the form $\tilde{G}_0^+ G_0^+ \Lambda^-$. In order for that vertex operator to be of the same form as $\phi_{2,1} e^{2\rho+i\sigma+iH}$, a sensible ansatz for Λ^- is

$$\Lambda^- = \Lambda_{1,0} e^{\rho} + \Lambda_{2,1} e^{2\rho+i\sigma}\,, \tag{7.26}$$

where $\Lambda_{1,0}$ and $\Lambda_{2,1}$ are vertex operators associated to WZW states that lie in an affine Kac module of lowest conformal weight -1. In particular, $\Lambda_{1,0}$ has conformal weight -1, *i.e.* it corresponds to an affine ground state, and $\Lambda_{2,1}$ has vanishing conformal weight so that Λ^- has conformal weight zero as well. As before, the terms in $\tilde{G}_0^+ G_0^+ \Lambda^-$ can again be distinguished by the exponential in the $\rho\sigma$-ghosts, which can be considered individually.

Terms proportional to $e^{3\rho+2i\sigma+iH}$:

First, we look at terms proportional to $e^{3\rho+2i\sigma+iH}$. They only arise from the OPE of $\big(e^{i\sigma}\mathcal{T}\big)$ with $\big(\Lambda_{2,1}e^{2\rho+i\sigma}\big)$. From eq. (7.10) we can immediately extract that

$$\begin{aligned}&\mathrm{Res}\big(e^{i\sigma}\mathcal{T}\big)(z)\big(\Lambda_{2,1}e^{2\rho+i\sigma}\big)(w)\\ &\qquad\qquad = \big(i\partial\sigma\{L_1\Lambda_{2,1}\}e^{2\rho+2i\sigma+iH}\big)(w)\,.\end{aligned} \tag{7.27}$$

Applying $\tilde{G}_0^-$ to this first order pole, we obtain the required terms proportional to $e^{3\rho+2i\sigma+iH}$. Since such terms do not appear in V^+, we have to require that these terms vanish. So the gauge parameter $\Lambda_{2,1}$ has to be annihilated by L_1. In other words, it has to be a Virasoro primary.

Terms proportional to $e^{2\rho+i\sigma+iH}$:

Next, we take a look at the terms proportional to $e^{2\rho+i\sigma+iH}$. These may be regarded as the most important ones because they describe the gauge freedom of $\phi_{2,1}$. As before, they arise from the OPE of $\big(e^{i\sigma}\mathcal{T}\big)$ with terms in $\big(\Lambda_{1,0}e^{\rho+iH}\big)$ and from the OPE of $(e^{-\rho}Q)$ with $\big(\Lambda_{2,1}e^{2\rho+i\sigma+iH}\big)$. From eq. (7.16) and (7.17) we know that

$$\begin{aligned}&\mathrm{Res}\big(e^{i\sigma}\mathcal{T}\big)(z)(\Lambda_{1,0}e^{\rho})(w) + \big(e^{-\rho}Q\big)(z)\big(\Lambda_{2,1}e^{2\rho+i\sigma}\big)(w)\\ &\qquad = \big((\{L_{-1}\Lambda_{1,0}\} + \{Q_0\Lambda_{2,1}\} + (\Lambda_{1,0} - \{Q_1\Lambda_{2,1}\})\partial\rho)\, e^{\rho+i\sigma}\big)(w)\,.\end{aligned} \tag{7.28}$$

We have to demand that $\Lambda_{1,0} = Q_1\Lambda_{2,1}$ since otherwise $\tilde{G}_0^+$ applied to the first order pole above yields a vertex operator involving ρ-ghost excitations. Substituting $\Lambda_{1,0}$, we then obtain a gauge freedom of $\phi_{2,1}$,

$$\phi_{2,1} \quad \sim \quad \phi_{2,1} + (Q_0 + L_{-1}Q_1)\Lambda_{2,1}\,. \tag{7.29}$$

Note that the operator that acts on the gauge parameter $\Lambda_{2,1}$ is exactly the deformation of Q_0 that appears in the kernel conditions (7.25) suggesting a cohomological description.

Terms proportional to $e^{\rho+iH}$:

The terms proportional to $e^{\rho+iH}$ in $\tilde{G}_0^+ G_0^+ \Lambda^-$ arise from the OPE of $(e^{-\rho}Q)$ with $(\Lambda_{1,0}e^{\rho})$ and the OPE of $\big(e^{-2\rho-i\sigma}P\big)$ with $\big(\Lambda_{2,1}e^{2\rho+i\sigma}\big)$. Using (7.20) and (7.21), we obtain

$$\begin{aligned}&\operatorname{Res}\big(e^{-\rho}Q\big)(z)(\Lambda_{1,0}e^{\rho})(w) - \big(e^{-2\rho-i\sigma}P\big)(z)\big(\Lambda_{2,1}e^{2\rho+i\sigma}\big)(w)\\ &\quad = (\{Q_{-1}\Lambda_{1,0}\} - \{P_0\Lambda_{2,1}\} + \partial(2\rho+i\sigma)\{P_1\Lambda_{2,1}\} - \partial\rho\{Q_0\Lambda_{1,0}\})(w) \equiv \mathcal{F}(w)\,. \end{aligned} \tag{7.30}$$

We temporarily denote this first order term by $\mathcal{F}(w)$. Note that it involves no exponential in the $\rho\sigma$ ghosts at all. Hence when we determine the first order pole with $\tilde{G}^+ = e^{\rho+iH}$, the first two terms vanish. In the end, it simplifies to

$$\operatorname{Res} e^{\rho+iH}(z)\mathcal{F}(w) = \big((2\{P_1\Lambda_{2,1}\} - \{Q_0\Lambda_{1,0}\})e^{\rho+iH}\big)\,. \tag{7.31}$$

Therefore, the gauge parameter $\Lambda_{2,1}$ induces a change of $\phi_{1,0}$ as

$$\phi_{1,0} \sim \phi_{1,0} + (2P_1 - Q_0Q_1)\Lambda_{2,1}\,. \tag{7.32}$$

But $\phi_{1,0}$ carries no independent physical degrees of freedom since it descends from $\phi_{2,1}$ by applying Q_1. Now using that $L_1\Lambda_{2,1} = 0$ because of (7.27) and $\Lambda_{2,1} \in \widehat{\mathcal{K}}^{(1)}(\lambda)$ with λ chosen such that $L_0\Lambda_{2,1} = 0$, one can show that

$$(2P_1 - Q_0Q_1)\Lambda_{2,1} = Q_1(Q_0 + L_{-1}Q_1)\Lambda_{2,1} \tag{7.33}$$

and hence the change in $\phi_{1,0}$ in (7.32) descends from the gauge freedom of $\phi_{2,1}$ as one would have hoped. We conclude that physical states ϕ are only well defined up to a gauge freedom,

$$\phi \quad \sim \quad \phi + (Q_0 + L_{-1}Q_1)\Lambda \qquad \text{with} \quad L_1\Lambda = 0\,. \tag{7.34}$$

Thus we have arrived at a classification of the physical spectrum at the first level in terms of the algebraic structure of $\mathfrak{psl}(2|2)$.

7.1.2 The hybrid vertex operator Q and its properties[2]

Before we proceed to further evaluate the physical state conditions found in the previous section, let us emphasize the particular importance of the hybrid vertex operator Q in the formulation of these conditions. This should come as no big surprise as it already

[2]The reader willing to take some algebraic properties of Q for granted may skip this section.

served as the BRST operator for the massless spectrum (see sections 5.3.1 and 6.1). However, as soon as one leaves the affine ground states, the situation becomes more involved as *e.g.* the order of normal ordered product in the definition of Q becomes crucial. In this section, we want to address these subtleties and present some properties of the modes Q_n that will be particularly useful in the following.

The field Q as defined[3] in (5.89),

$$Q \equiv \frac{1}{2\sqrt{k}}\left[\left(K_{ab}\left(S_-^a S_-^b\right)\right) + 4i\left(S_-^a \partial S_-^a\right)\right] , \tag{7.35}$$

can be easily written in terms of modes using general results on the mode expansion of the normal ordered product of two vertex operators [37,85]. Specifically, if $\psi(z)$ and $\chi(w)$ are two vertex operators, the n-th mode of the product equals

$$(\psi\chi)_n = \sum_{L\geq 0}\left(\psi_{-h_\psi - L}\chi_{n+h_\psi+L} + \varepsilon\,\chi_{n+h_\psi-1-L}\psi_{-h_\psi+1+L}\right) , \tag{7.36}$$

where h_ψ is the conformal weight of ψ. Here $\varepsilon = -1$ if both ψ and χ are fermionic and $\varepsilon = +1$ otherwise. With this result it is straightforward to write down a mode expression for the n-th mode of Q. For the individual summands of Q we get

$$\begin{aligned}(K^{ab}(S_-^a S_-^b))_n &= \sum_{m,l\geq 0}\Big[K^{ab}_{-m-1}S^a_{-,-l-1}S^b_{-,l+m+n+2} + K^{ab}_{-m-1}S^a_{-,m+n+1-l}S^b_{-,l} \\ &\qquad + S^a_{-,-l-1}S^b_{-,n-m+l+1}K^{ab}_m + S^a_{-,n-m-l}S^b_{-,l}K^{ab}_m\Big] \\ &= \sum_{m,l\in\mathbb{Z}} K^{ab}_{-m}S^a_{-,-l}S^b_{-,n+m+l}\,,\end{aligned} \tag{7.37}$$

where $\sum_{l\in\mathbb{Z}}[K^{ab}_m, S^a_{-l}S^b_{n+l}] = 0$ was used, and

$$\begin{aligned}(S^a\partial S^a)_n &= \sum_{m\geq 0}\left((-n-m-2)S^a_{-,-m-1}S^a_{-,n+m+1} + (n-m+1)S^a_{-,n-m}S^a_{-,m}\right) \\ &= \sum_{m\in\mathbb{Z}}(m-n-1)\,S^a_{-,m}S^a_{-,n-m} = \sum_{m\in\mathbb{Z}}(-m-1)\,S^a_{-,n-m}S^a_{-,m}\,.\end{aligned} \tag{7.38}$$

An important property of Q is its commutation relation with operators in $\mathfrak{g}_{+1}$,

$$[Q_0, S^a_{+,0}] = -i\sqrt{k}\left(S^a_- T^{\mathrm{WZW}}\right)_0 , \tag{7.39}$$

where $\alpha, \beta = \pm 1$. This relation can be proven by noting that

$$\left[S^a_{+,0}, Q(w)\right] = \left\{S^a_{+,0}\, Q_{-3}\Omega\right\}(w)\,, \tag{7.40}$$

where Ω is the conformal vacuum annihilated by all non-negative modes of the WZW-

[3]Recall that whenever $\mathfrak{so}(4)$-indices $a, b, c, \ldots$ appear twice within a term, summation is understood.

currents. The state associated to the field Q by the operator-state-correspondence is

$$Q_{-3}\Omega = \frac{1}{2\sqrt{k}}\left[\left(K^{ab}\left(S^a_- S^b_-\right)\right)_{-3}\Omega + 4i(S^a\partial S^a)_{-3}\Omega\right]$$
$$= \frac{1}{2\sqrt{k}}\left(K^{ab}_{-1}S^a_{-1}S^b_{-1} + 4iS^a_{-1}S^a_{-2}\right)\Omega\,. \tag{7.41}$$

It is straightforward to evaluate the commutators of $S^c_{+,0}$ with Q_{-3}. First, we consider the commutator

$$\left[S^c_{+,0}, K^{ab}_{-1}S^a_{-,-1}S^b_{-,-1}\right] = 2i\,S^c_{-,-1}S^a_{+,-1}S^a_{-,-1} - \tfrac{i}{2}\varepsilon^{cbef}S^a_{-,-1}[K^{ab}_{-1}, K^{ef}_{-1}]_+ + \Xi^c_{-3}\,. \tag{7.42}$$

The auxiliary operator Ξ^c_{-3} collects all residual terms that involve elementary commutators of the modes of the WZW currents and its explicit form will be discussed later. The anticommutator $[K^{ab}_{-1}, K^{ef}_{-1}]_+$ can be considered as a tensor T^{abef} with the symmetry properties

$$T^{abef} = -T^{baef} = -T^{abfe} = T^{efab}\,. \tag{7.43}$$

For such tensor, the following identity holds

$$\varepsilon^{cabe}T^{adbe} = -\tfrac{1}{4}\varepsilon^{abef}T^{abef}\delta^d_c\,. \tag{7.44}$$

So the commutator (7.42) simplifies to

$$\left[S^c_{+,0}, K^{ab}_{-1}S^a_{-,-1}S^b_{-,-1}\right] = -i\,S^c_{-,-1}\left(\tfrac{1}{4}\varepsilon^{abef}K^{ab}_{-1}, K^{ef}_{-1} - 2\,S^a_{+,-1}S^a_{-,-1}\right) + \Xi^c_{-3}\,. \tag{7.45}$$

The term in brackets should be recognised as the state that corresponds to the Sugawara energy momentum tensor $L^{\mathrm{WZW}}_{-2}\Omega$, multiplied by $2k$. So in order for (7.39) to be valid, one would hope that the the residual term Ξ^c_{-3} is canceled by the commutator of $S^c_{+,0}$ with the remaining term in $Q_{-3}\Omega$. Evaluating Ξ^c_{-3}, we obtain

$$\Xi^c_{-3} = -\tfrac{i}{2}\varepsilon^{cbef}S^a_{-1}\left[K^{ab}_{-1}, K^{ef}_{-1}\right] - 2i\left\{S^a_{+,-1}, S^c_{-,-1}\right\}S^a_{-,-1}$$
$$- i\,\varepsilon^{cbef}\left[K^{ab}_{-1}, S^a_{-,-1}\right]K^{ef}_{-1} + \tfrac{i}{2}\varepsilon^{caef}K^{ab}_{1}\left[K^{ef}_{-1}, S^b_{-,-1}\right]$$
$$= -2\,\varepsilon^{caef}S^a_{-,-1}K^{ef}_{-2} + 2\,\varepsilon^{caef}K^{ef}_{-1}S^a_{-,-2}\,, \tag{7.46}$$

which is indeed canceled by the commutator

$$\left[S^c_{+,0}, 4iS^a_{-1}S^a_{-2}\right] = -2\,\varepsilon^{caef}K^{ef}_{-1}S^a_{-,-2} + 2\,\varepsilon^{caef}S^a_{-,-1}K^{ef}_{-2} = -\Xi^c_{-3}\,. \tag{7.47}$$

Therefore acting with $S^e_{+,0}$ on $Q_{-3}\Omega$ yields

$$S^e_{+,0}Q_{-3}\Omega = \frac{i}{2\sqrt{k}}S^e_{-,-1}\left(-\tfrac{1}{4}\varepsilon_{abcd}K^{ab}_{-1}K^{cd}_{-1} + S^a_{+,-1}S^a_{-,-1} - S^a_{-,-1}S^a_{+,-1}\right)\Omega$$
$$= i\sqrt{k}\,S^e_{-,-1}L_{-2}\Omega = i\sqrt{k}\left(S^e_- T^{\mathrm{WZW}}\right)_{-3}\Omega\,, \tag{7.48}$$

and we conclude that

$$\big[S^a_{+,0}, Q(w)\big] = i\sqrt{k}\big(S^a_- T^{\mathrm{WZW}}\big)(w)\,, \tag{7.49}$$

which in particular implies the commutation relation (7.39).

The right-hand side of (7.39) can be expanded in modes by using (7.36),

$$\big(S^a_- T^{\mathrm{WZW}}\big)_0 = \sum_{n\in\mathbb{Z}} : L_{-n} S^a_{-,n} : \,. \tag{7.50}$$

Recall that $:\cdot:$ refers to creation-annihilation-ordering. Note that there is no ordering ambiguity for the zero-modes since L_0 commutes with $S^{\alpha\beta}_{-,0}$. From (7.50) we see that the commutation relation (7.39) implies the particularly interesting fact that when acting on the subspace of Virasoro primary states of conformal weight zero, Q_0 commutes with $\mathfrak{g}_{+1}$ up to terms that are Virasoro descendants.

From the explicit realisations of the normal ordered products (7.37) and (7.38) in terms of modes, we can draw further conclusions. From (7.37) and the commutation relations $[L_{n'}, \mathcal{J}_n] = -n\mathcal{J}_{n+n'}$, where $\mathcal{J}$ may denote any current of the PSL(2|2) WZW model, one obtains

$$\Big[L_{n'}, \Big(K_{ab}\Big(S^a_- S^b_-\Big)\Big)_n\Big] = (2n'-n)\Big(K_{ab}\Big(S^a_- S^b_-\Big)\Big)_{n+n'}\,. \tag{7.51}$$

Therefore $\big(K_{ab}\big(S^a_- S^b_-\big)\big)$ is a Virasoro primary field of conformal weight 3. The additional term (7.38) of Q spoils this property. Indeed, we find

$$\Big[L_{n'}, \Big(S^a_- \partial S^b_-\Big)_n\Big] = \sum_{m\in\mathbb{Z}} \left(\frac{n'(1-n'+2m)}{m+1} - n\right)(-m-1) S^a_{-,n+n'-m} S^a_{-,m}\,. \tag{7.52}$$

However, if restricted to $\widehat{\mathcal{K}}^{(1)}(\lambda)$, *i.e.* states at the first affine level, and choosing $n=0$ and $n'=1$, the above commutator simplifies to

$$\Big[L_1, \Big(S^a_- \partial S^b_-\Big)_0\Big]\Big|_{\widehat{\mathcal{K}}^{(1)}(\lambda)} = \sum_{m=0}^{1}(-2m)\, S^a_{-,1-m} S^a_{-,m} = -2S^a_{-,0}S^a_{-,1} = 2\Big(S^a_- \partial S^b_-\Big)_1\,. \tag{7.53}$$

Combining this with (7.51), we obtain

$$[L_1, Q_0]\Big|_{\widehat{\mathcal{K}}^{(1)}(\lambda)} = 2Q_1\,. \tag{7.54}$$

This commutation relation will be important in (7.62).

7.1.3 Cohomological description of physical string states

So far we have worked in the hybrid formulation and reduced the physical state conditions to algebraic requirements on states in the WZW factor. We will now have a closer look at the algebraic constraints (7.25) and (7.34). Recall that WZW vertex operators $\phi_{2,1}(z)$ and $\Lambda_{2,1}(z)$ correspond to states of the WZW model at the first level of vanishing conformal weight, hence from a representation theoretic point of view, we are looking for $\phi, \Lambda \in \widehat{\mathcal{K}}^{(1)}(\lambda)$ with λ chosen such that $L_0\phi = L_0\Lambda = 0$. This is exactly the case if $C_2(\lambda) = -2k$. In the following we will assume that λ has been chosen in this manner.

Let us first consider some of the kernel conditions (7.25) in detail. The condition $L_1\phi = 0$ implies that physical string states are Virasoro primaries. We can define a projection $\Pi^{(1)}$ onto the kernel of L_1, by

$$\Pi^{(1)} = \mathbb{1} + \tfrac{1}{2}L_{-1}L_1 \,. \tag{7.55}$$

Note that we are working at the first level with the ground states having conformal weight -1. Furthermore, any affine ground state $|\lambda\rangle \in \widehat{\mathcal{K}}^{(1)}(\lambda)$ satisfies $\Pi^{(1)}L_{-1}\,|\lambda\rangle = 0$. This implies that the first level decomposes as $\widehat{\mathcal{K}}^{(1)}(\lambda) = \ker L_1 \oplus \operatorname{im} L_{-1}$. Thus restricting $\widehat{\mathcal{K}}^{(1)}(\lambda)$ to the $\mathfrak{g}$-submodule of Virasoro primaries is, morally speaking, equivalent to identifying the submodule of Virasoro descendants and removing it from the decomposition in (2.58). Note that L_{-1} applied to the ground state representation $\widehat{\mathcal{K}}^{(0)}(\lambda)\big|_{\mathfrak{g}} \simeq \mathcal{K}(\lambda)$ yields a copy of it at the first level,

$$\operatorname{im} L_{-1} \simeq \mathcal{K}(\lambda) \quad \subset \quad \widehat{\mathcal{K}}^{(1)}(\lambda)\,. \tag{7.56}$$

Therefore the subspace of Virasoro primaries at the first level decomposes into $\mathfrak{g}$-representations as (recall that the subscript denotes the grading of the cyclic state of the respective Kac-module)

$$\begin{aligned} \ker_{\widehat{\mathcal{K}}^{(1)}(\lambda)} L_1\Big|_{\mathfrak{g}} \simeq\; & \mathcal{K}_0(\lambda) \oplus \mathcal{K}_0(\lambda^{++}) \oplus \mathcal{K}_0(\lambda^{--}) \oplus \mathcal{K}_0(\lambda_{++}) \oplus \mathcal{K}_0(\lambda_{--}) \\ & \oplus \bigoplus_{g=\pm 1} \Big(\mathcal{K}_g(\lambda^+_+) \oplus \mathcal{K}_g(\lambda^+_-) \oplus \mathcal{K}_g(\lambda^-_+) \oplus \mathcal{K}_g(\lambda^-_-)\Big)\,. \end{aligned} \tag{7.57}$$

The next condition of interest is the deformation of Q_0 in (7.25), namely $Q_0 + L_{-1}Q_1$. In order to get some insight into the meaning of this operator we can proceed as follows. Having the projection (7.55) onto $\ker L_1$ at hand, we can project Q_0 onto $\ker L_1$ simply by multiplying $\Pi^{(1)}$ on both sides. The projected operator is denoted as $Q_0^\Pi \equiv \Pi^{(1)}Q_0\Pi^{(1)}$. Since the Virasoro modes L_n commute with the horizontal subalgebra of $\widehat{\mathfrak{psl}}(2|2)_k$, the projection $\Pi^{(1)}$ commutes with it as well. Thus we can use the commutation relation (see section 7.1.2 for a proof)

$$\Big[Q_0, S^{\alpha\beta}_{+,0}\Big] = -i\sqrt{k}\Big(S^{\alpha\beta}T^{\mathrm{WZW}}\Big)_0 = -i\sqrt{k}\sum_{n\in\mathbb{Z}} {:}S^{\alpha\beta}_{-,-n}L_n{:} \tag{7.58}$$

to calculate the commutation relation

$$\begin{aligned} \Big[Q_0^\Pi, S^{\alpha\beta}_{+,0}\Big] &= \Big[\Pi^{(1)}Q_0\Pi^{(1)}, S^{\alpha\beta}_{+,0}\Big] = \Pi^{(1)}\,\Big[Q_0, S^{\alpha\beta}_{+,0}\Big]\,\Pi^{(1)} \\ &= -i\sqrt{k}\,(S^{\alpha\beta}_{-,0}L_0^\Pi + \Pi^{(1)}L_{-1}S^{\alpha\beta}_{-,1}\Pi^{(1)} + \Pi^{(1)}S^{\alpha\beta}_{-,-1}L_1\Pi^{(1)})\,, \end{aligned} \tag{7.59}$$

where $L_n^\Pi \equiv \Pi^{(1)}L_n\Pi^{(1)}$ is a shorthand notation for the projected Virasoro modes. Because $\Pi^{(1)}L_{-1} = 0$ when applied to affine ground states in $\widehat{\mathcal{K}}^{(0)}(\lambda)$ and $L_1\Pi^{(1)} = 0$ on $\widehat{\mathcal{K}}^{(1)}(\lambda)$, the last two terms in (7.59) vanish. Hence

$$\left[Q_0^\Pi, S_{+,0}^{\alpha\beta}\right] = -i\sqrt{k}\, S_{-,0}^{\alpha\beta} L_0^\Pi\,. \tag{7.60}$$

It is easy to check that the zero mode L_0 commutes with $\Pi^{(1)}$ using the commutation relations of Virasoro modes. Since this implies that $L_0^\Pi = \Pi^{(1)} L_0$, the above commutator vanishes when applied to the kernel of L_0, *i.e.* to states of zero conformal weight. Because λ is chosen such that $\widehat{\mathcal{K}}^{(1)}(\lambda)$ is annihilated by L_0, *i.e.* $C_2(\lambda) = -2k$, we obtain

$$\left[Q_0^\Pi, S_{+,0}^{\alpha\beta}\right] = 0 \qquad \text{on } \widehat{\mathcal{K}}^{(1)}(\lambda)\,. \tag{7.61}$$

Since both Q_0 and $\Pi^{(1)}$ commute with the $\mathfrak{g}^{(0)} \oplus \mathfrak{g}_{-1}$ as well, we conclude that Q_0^Π commutes with the full horizontal algebra $\mathfrak{g}$ of $\widehat{\mathfrak{g}}$ and hence induces a $\mathfrak{g}$-homomorphism on $\widehat{\mathcal{K}}^{(1)}(\lambda)$.

As soon as we have imposed the physical state condition $L_1\phi = 0$, $\Pi^{(1)}$ obviously acts on the remaining states like the identity. The action of Q_0^Π can be evaluated on $\phi \in \ker L_1$,

$$Q_0^\Pi \phi = \Pi^{(1)} Q_0 \Pi^{(1)}\, \phi = (\mathbb{1} + \tfrac{1}{2} L_{-1} L_1) Q_0 \phi = (Q_0 + L_{-1} Q_1)\phi\,, \tag{7.62}$$

where we used (7.54). But the last expression in (7.62) coincides exactly with the operator in (7.25). Thus we have obtained a nice algebraic interpretation of one of the operators appearing in (7.25). Namely, it is simply Q_0 appropriately corrected such that it induces a $\mathfrak{g}$-homomorphism on $\ker L_1 \subset \widehat{\mathcal{K}}^{(1)}(\lambda)$.

Before we continue analysing the kernel conditions, a discussion of the gauge degrees of freedom (7.34) is in order. The gauge parameter Λ has to be Virasoro primary as well and hence according to our discussion before, the gauge freedom can be equivalently written as

$$\phi \quad \sim \quad \phi + Q_0^\Pi \Lambda \qquad \text{with} \quad L_1 \Lambda = 0\,. \tag{7.63}$$

In other words, physical states are only defined up to states in the image of the $\mathfrak{g}$-homomorphism induced by Q_0^Π. Clearly, this is only well-defined if $(Q_0^\Pi)^2 = 0$. Indeed, note that $(Q_0^\Pi)^2$ is a $\mathfrak{g}$-homomorphism of grading -4. But the decomposition in (2.58) implies that no nontrivial $\mathfrak{g}$-homomorphism mapping $\widehat{\mathcal{K}}^{(1)}(\lambda) \to \widehat{\mathcal{K}}^{(1)}(\lambda)$ of grading -4 exists. Hence the $\mathfrak{g}$-homomorphism induced by Q_0^Π is nilpotent and the Q_0^Π-cohomology is well-defined on the submodule of Virasoro primaries.

The spectrum of Virasoro primaries has been given in (7.57). Since Q_0^Π is an operator of grading -2, we can immediately state that all Kac-modules of zero grading will contribute to the Q_0^Π-cohomology. Unfortunately, just by considering the grading of Q_0^Π, we cannot make any statement on the remaining Kac-modules in (7.57) because the $\mathfrak{g}$-homomorphism

$$Q_0^\Pi : \qquad \mathcal{K}_{+1}(\lambda_\beta^\alpha) \longrightarrow \mathcal{K}_{-1}(\lambda_\beta^\alpha)\,, \qquad \alpha, \beta = \pm\,, \tag{7.64}$$

might be nontrivial. Using the explicit realisation of Q_0 in (7.37) and (7.38), one can show that the induced homomorphism from $\mathcal{K}_{+1}(\lambda_\beta^\alpha)$ to $\mathcal{K}_{-1}(\lambda_\beta^\alpha)$ is indeed nontrivial for any combination of α and β. In this sense, the action of Q_0^Π is maximal. Hence we conclude that

$$H_{Q_0^\Pi}(\ker L_1)\Big|_{\mathfrak{g}} \simeq \mathcal{K}_0(\lambda)\oplus\mathcal{K}_0(\lambda^{++})\oplus\mathcal{K}_0(\lambda^{--})\oplus\mathcal{K}_0(\lambda_{++})\oplus\mathcal{K}_0(\lambda_{--})\,. \tag{7.65}$$

We now turn our attention to the other conditions in (7.25). We want to show that they are all automatically satisfied once ϕ is taken to be an element of the Q_0^Π-cohomology (7.65). We begin by showing $(P_0 - Q_{-1}Q_1)\phi = 0$. As for Q_0, the zero-mode P_0 commutes with $\mathfrak{g}_{-1}\oplus\mathfrak{g}^{(0)}$, but a priori does not commute with $\mathfrak{g}_{+1}$. However, taking into account the correction term $-Q_{-1}Q_1$, we find that at the first level the commutation relation

$$\begin{aligned}&[P_0 - Q_{-1}Q_1, S^{\alpha\beta}_{+,0}]\\ &\qquad = i\sqrt{k}\left(S^{\alpha\beta}_{-,0}(Q_0 + Q_{-1}L_1 + L_{-1}Q_1) + (Q_{-1}S^{\alpha\beta}_{-,1} + S^{\alpha\beta}_{-,-1}Q_1)L_0\right)\end{aligned} \tag{7.66}$$

holds. When acting on states that are annihilated by both L_0 and L_1, the above commutation relations simplify to

$$[P_0 - Q_{-1}Q_1, S^{\alpha\beta}_{+,0}]\Big|_{\ker L_1\cap\ker L_0} = i\sqrt{k}\,S^{\alpha\beta}_{-,0}(Q_0 + L_{-1}Q_1) = i\sqrt{k}\,S^{\alpha\beta}_{-,0}\,Q_0^\Pi\Big|_{\ker L_1}\,, \tag{7.67}$$

where we have made use of (7.62). So after restricting to the subspace of Virasoro primaries within $\widehat{\mathcal{K}}^{(1)}(\lambda)$ and imposing the physical state condition $Q_0^\Pi\phi = 0$, the deformed operator $P_0 - Q_{-1}Q_1$ induces a $\mathfrak{g}$-homomorphism of grading -4 on the resulting subspace. But since the Q_0^Π-kernel only involves Kac-modules of grading 0 and -1, it is clear that $P_0 - Q_{-1}Q_1$ annihilates all states in that kernel. Thus the $(P_0 - Q_{-1}Q_1)$-kernel condition is trivially satisfied once we reduced the physical subspace to be a part of the Q_0^Π-kernel. Hence the $(P_0 - Q_{-1}Q_1)$-kernel condition may be discarded.

The two operators $P_{-1}Q_1$ and P_0Q_1 commute with the bosonic subalgebra $\mathfrak{g}^{(0)}$ of $\mathfrak{g}$ and hence induce homomorphisms of $\mathfrak{g}^{(0)}$-representations. However, these operators have grading -6 and hence they induce trivial homomorphisms because the Q_0^Π-kernel as well as the affine ground states $\widehat{\mathcal{K}}^{(0)}(\lambda)$ only involve $\mathfrak{g}^{(0)}$-representations that have gradings between 0 and -5.

The remaining operators in (7.25), Q_0Q_1 and P_1, cannot be deduced to be trivial simply by an analysis of their gradings. However, using their explicit realisation in terms of modes, one finds that they indeed vanish on the direct summands of $\widehat{\mathcal{K}}^{(1)}(\lambda)$ given in (7.65) that are neither in the kernel nor in the image of Q_0^Π.

Thus we have shown that physical string states at the first level in the PSL(2|2) WZW model can be described by the Q_0^Π-cohomology evaluated on the subspace of Virasoro primaries of conformal weight zero,

$$\mathcal{H}^{(1),\mathrm{PSL}}_{\mathrm{phys}} \simeq H_{Q_0^\Pi}\left(\ker_{\widehat{\mathcal{K}}^{(1)}(\lambda)} L_1\right) \tag{7.68}$$

with the weight λ chosen such that $L_0 = 0$ is satisfied. Note that this is the same description of physical states as in the case of the massless sector. There the physical sector was given by the Q_0-cohomology (see eq. (5.110)). But Q_0^{Π} reduces to Q_0 on affine ground states as they are all Virasoro primaries. Hence (7.68) can be considered as the natural generalisation of the description of the massless sector.

7.2 Comparison with the RNS String Spectrum

In the previous section we have succeed to give a description of the physical states of the hybrid formulation within the PSL(2|2) WZW model. Next we want to show that our result is in agreement with the spectrum one obtains for the RNS string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$ [65, 99]. In fact, we will be considering the full string spectrum not restricted to the first massive level. This will allow us to deduce that the massive compactification-independent physical spectrum fits into representations of $\mathfrak{g}$ at all mass levels.

Let us assume that the NS vacuum has $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ quantum numbers $\lambda = (j_1, j_2)$. The generating function of physical $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ highest weight states of the RNS-string [77] on $\mathrm{AdS}_3 \times \mathrm{S}^3$ in the NS and R sector are, respectively,

$$
\begin{aligned}
&F^{\mathrm{NS}}(x,y|q) \\
&\quad = x^{j_1} y^{j_2} q^{\frac{1}{2k}C_2(\lambda)-\frac{1}{2}} \prod_{n\geq 1} \frac{(1+xq^{n-\frac{1}{2}})(1+x^{-1}q^{n-\frac{1}{2}})(1+yq^{n-\frac{1}{2}})(1+y^{-1}q^{n-\frac{1}{2}})}{(1-xq^n)(1-x^{-1}q^n)(1-yq^n)(1-y^{-1}q^n)}\,, \quad (7.69)\\
&F^{\mathrm{R}}(x,y|q) \\
&\quad = x^{j_1+\frac{1}{2}} y^{j_2+\frac{1}{2}} q^{\frac{1}{2k}C_2(\lambda)-\frac{1}{4}} \prod_{n\geq 1} \frac{(1+xq^n)(1+x^{-1}q^{n-1})(1+yq^n)(1+y^{-1}q^{n-1})}{(1-xq^n)(1-x^{-1}q^n)(1-yq^n)(1-y^{-1}q^n)}\,. \quad (7.70)
\end{aligned}
$$

In order to apply the GSO-projection later, we also have to determine the generating functions with the insertion of $(-1)^F$; they read

$$
\begin{aligned}
&F^{\widetilde{\mathrm{NS}}}(x,y|q) \\
&\quad = x^{j_1} y^{j_2} q^{\frac{1}{2k}C_2(\lambda)-\frac{1}{2}} \prod_{n\geq 1} \frac{(1-xq^{n-\frac{1}{2}})(1-x^{-1}q^{n-\frac{1}{2}})(1-yq^{n-\frac{1}{2}})(1-y^{-1}q^{n-\frac{1}{2}})}{(1-xq^n)(1-x^{-1}q^n)(1-yq^n)(1-y^{-1}q^n)}\,, \quad (7.71)\\
&F^{\widetilde{\mathrm{R}}}(x,y|q) \\
&\quad = x^{j_1+\frac{1}{2}} y^{j_2+\frac{1}{2}} q^{\frac{1}{2k}C_2(\lambda)-\frac{1}{4}} \prod_{n\geq 1} \frac{(1-xq^n)(1-x^{-1}q^{n-1})(1-yq^n)(1-y^{-1}q^{n-1})}{(1-xq^n)(1-x^{-1}q^n)(1-yq^n)(1-y^{-1}q^n)} \\
&\quad = x^{j_1+\frac{1}{2}} y^{j_2+\frac{1}{2}} (1-x^{-1})(1-y^{-1}) q^{\frac{1}{2k}C_2(\lambda)-\frac{1}{4}}\,. \quad (7.72)
\end{aligned}
$$

We are interested in the spectrum of compactification-independent states, *i.e.* the subsector of physical states that are always present independent of the choice of M. However, the choice of M is restricted to manifolds that yield target space supersymmetry in six dimensions. The existence of supersymmetry in a six-dimensional spacetime requires the $\mathcal{N}=1$ superconformal symmetry of the non-linear σ-model with target space M to be extended to an $\mathcal{N} = 4$ superconformal symmetry [7, 169]. Hence the fields generating the $\mathcal{N} = 4$ superconformal algebra are always present and should be considered as compactification-independent, even though they correspond to excitations on the compactification manifold. The character for some representation $\mathcal{D}$ of the $\mathcal{N} = 4$ superconformal algebra is defined as

$$\chi^{\mathcal{D}}_{\mathcal{N}=4}(z|q) = \mathrm{Tr}_{\mathcal{D}}\left(q^{L_0} z^{\mathcal{J}_0}\right) , \tag{7.73}$$

where $\mathcal{J}_0$ is the $U(1)$-charge of the superconformal algebra. In [63], the characters have been determined for large classes of representations. For the compactification-independent spectrum only the vacuum representations in the NS and R sector are of interest. Their respective characters are

$$\chi^{\mathrm{R}}_{\mathcal{N}=4}(z|q) = q^{\frac{1}{4}} \frac{i\vartheta_{10}^2(z|q)}{\vartheta_{11}(z^2|q)\eta^3(q)} \sum_{m\in\mathbb{Z}} \left(\frac{z^{4m+1}}{(1+z^{-1}q^{-m})^2} - \frac{z^{-4m-1}}{(1+zq^{-m})^2} \right) q^{2m^2+m} , \tag{7.74}$$

$$\chi^{\mathrm{NS}}_{\mathcal{N}=4}(z|q) = q^{\frac{1}{4}} \frac{i\vartheta_{00}^2(z|q)}{\vartheta_{11}(z^2|q)\eta^3(q)} \sum_{m\in\mathbb{Z}} \left(\frac{z^{4m+1}}{(1+zq^{m+\frac{1}{2}})^2} - \frac{z^{-4m-1}}{(1+z^{-1}q^{m+\frac{1}{2}})^2} \right) q^{2m^2+m} . \tag{7.75}$$

The first factor containing theta functions[4] is the character of the corresponding Verma module while the infinite sum encodes the singular (and redundant) vectors within that Verma module. Note that the characters of the two sectors are connected by spectral flow,

$$\chi^{\mathrm{R}}_{\mathcal{N}=4}(z|q) = z q^{\frac{1}{4}} \chi^{\mathrm{NS}}_{\mathcal{N}=4}(z q^{\frac{1}{2}}|q) . \tag{7.76}$$

Since the supercurrents have odd $U(1)$-charge and the bosonic currents of the $\mathcal{N} = 4$ superconformal algebra have even $U(1)$-charge, the insertion of $(-1)^F$ is easily implemented by substituting z by $-z$. The partition sum of compactification-independent physical states in each sector is then given by the GSO-projected product of the $\mathrm{AdS}_3 \times \mathrm{S}^3$ string partition sum with the respective $\mathcal{N} = 4$ superconformal vacuum character. Thus the full compactification-independent generating function including both the NS- and the R-sector is given by

$$\begin{aligned} Z^{\mathrm{indep}}(x,y|q) = \; & \frac{1}{2}\Big(F^{\mathrm{NS}}(x,y|q)\,\chi^{\mathrm{NS}}_{\mathcal{N}=4}(z|q) - F^{\widetilde{\mathrm{NS}}}(x,y|q)\chi^{\mathrm{NS}}_{\mathcal{N}=4}(-z|q)\Big)\Big|_{z=0} \\ & + \frac{1}{2}\Big(F^{\mathrm{R}}(x,y|q)\,\chi^{\mathrm{R}}_{\mathcal{N}=4}(z|q) - F^{\widetilde{\mathrm{R}}}(x,y|q)\chi^{\mathrm{R}}_{\mathcal{N}=4}(-z|q)\Big)\Big|_{z=0} . \end{aligned} \tag{7.77}$$

[4]Our conventions for the theta functions are

$$\begin{aligned} \vartheta_{00}(z|q) &= \prod_{m\geq 1} (1-q^m)(1+zq^{m-\frac{1}{2}})(1+z^{-1}q^{m-\frac{1}{2}}) , \\ \vartheta_{10}(z|q) &= q^{\frac{1}{8}}(z^{\frac{1}{2}}+z^{-\frac{1}{2}}) \prod_{m\geq 1} (1-q^m)(1+zq^m)(1+z^{-1}q^m) , \\ \vartheta_{11}(z|q) &= iq^{\frac{1}{8}}(z^{\frac{1}{2}}-z^{-\frac{1}{2}}) \prod_{m\geq 1} (1-q^m)(1-zq^m)(1-z^{-1}q^m) . \end{aligned}$$

They satisfy the relations $\vartheta_{10}(z|q) = q^{\frac{1}{8}} z^{\frac{1}{2}} \vartheta_{00}(zq^{\frac{1}{2}}|q)$ and $\vartheta_{11}(z^2|q) = -z^2 q^{\frac{1}{2}} \vartheta_{11}(z^2 q|q)$.

Expanding it in powers of q, we obtain

$$Z^{\text{indep}}(x,y|q) = \sum_{n\in\mathbb{N}} \mathcal{A}_n(x,y) q^{n+\frac{1}{2k}C_2(\lambda)} \,. \tag{7.78}$$

The coefficient functions $\mathcal{A}_n(x,y)$ can be understood as the generating functions of the physical states at the n-th mass level that are highest weight states with respect to $\mathfrak{g}^{(0)}$; for the massless level, *i.e.* $n=0$, the coefficient function is for example

$$\mathcal{A}_0(x,y) = x^{j_1-1}y^{j_2-1}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)^2 (xy+1)\,. \tag{7.79}$$

It turns out that the coefficient function $\mathcal{A}_n(x,y)$ for $n\geq 1$ seems to factorise into a factor $K_\lambda(x,y)$ and a residual polynomial factor. This factorisation property of the coefficient functions has been checked explicitly up to mass level $n=6$. Hence we can write

$$\mathcal{A}_n(x,y) = K_\lambda(x,y)\sum_{r,s=0}^{4n}(A_n)_{rs}\, x^{\frac{r}{2}-n}y^{\frac{s}{2}-n}\,, \tag{7.80}$$

where the A_n are $(4n+1)\times(4n+1)$ matrices. Since the function $x^{l_1}y^{l_2}K_\lambda(x,y)$ is just the branching function for $\mathcal{K}(\lambda+(l_1,l_2))$, we see that the massive states of the RNS string on $\text{AdS}_3\times\text{S}^3$ can be arranged into a direct sum of Kac modules with respect to $\mathfrak{g}$. Then the matrices A_n have the interpretation of encoding the multiplicities of Kac modules at the various mass levels. The explicit form of the matrices A_n with $n=1,\ldots,6$ can be found in appendix E. For the case $n=1$ that is of primary interest to us, the relevant coefficient function is

$$\begin{aligned}\mathcal{A}_1(x,y) &= K_\lambda(x,y)(x^{-1}+x+y^{-1}+y+1)\,.\\ &= K_\lambda(x,y)+K_{\lambda^{++}}(x,y)+K_{\lambda^{--}}(x,y)+K_{\lambda_{++}}(x,y)+K_{\lambda_{--}}(x,y)\,.\end{aligned} \tag{7.81}$$

The associated matrix A_1 can be easily extracted from this expression and it is found to be

$$A_1 = \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&0&0\\ 1&0&1&0&1\\ 0&0&0&0&0\\ 0&0&1&0&0\end{pmatrix}\,. \tag{7.82}$$

Hence the physical compactification-independent RNS string states on $\text{AdS}_3\times\text{S}^3$ can be uniquely arranged in the following direct sum of $\mathfrak{psl}(2|2)$ representations:

$$\mathcal{H}^{(1),\text{RNS}}_{\text{phys}} \simeq \mathcal{K}(\lambda)\oplus\mathcal{K}(\lambda^{++})\oplus\mathcal{K}(\lambda^{--})\oplus\mathcal{K}(\lambda_{++})\oplus\mathcal{K}(\lambda_{--})\,. \tag{7.83}$$

Eq. (7.83) should now be compared with our result obtained in the hybrid formulation in (7.68). Indeed, the spectra on first level agree nicely,

$$\mathcal{H}^{(1),\text{PSL}}_{\text{phys}} \quad = \quad \mathcal{H}^{(1),\text{RNS}}_{\text{phys}} \, . \tag{7.84}$$

This is a good consistency check of our analysis.

7.3 A Conjecture and its Confirmation at the Second Affine Level

With our previous results in mind, it seems natural to expect that the cohomological characterisation of physical string states in the PSL(2|2) WZW model generalises to all mass levels. In particular, at level n, we conjecture that the space of physical states is given by

$$\mathcal{H}^{(n)}_{\text{phys}} \quad \simeq \quad H_{Q_0^\Pi}\left(\text{Vir}\,\widehat{\mathcal{K}}^{(n)}(\lambda)\right) \, , \tag{7.85}$$

where λ is chosen such that $C_2(\lambda) = -n$, and $\text{Vir}\,\widehat{\mathcal{K}}^{(n)}(\lambda)$ denotes the $\mathfrak{g}$-submodule of Virasoro primaries within $\widehat{\mathcal{K}}^{(n)}(\lambda)$, *i.e.*

$$\text{Vir}\,\widehat{\mathcal{K}}^{(n)}(\lambda) \quad \equiv \quad \left\{\phi \in \widehat{\mathcal{K}}^{(n)}(\lambda) \,\middle|\, L_m\phi = 0 \quad \forall m \geq 1\right\} . \tag{7.86}$$

Furthermore Π is the projection onto the subspace of Virasoro primaries. We assume that any state can be uniquely decomposed into a Virasoro primary and a Virasoro descendant. In other words, our assumption is that $\widehat{\mathcal{K}}(\lambda)$ can be written as a direct sum of the space of Virasoro primaries and the space of Virasoro descendants. This is true since any Verma module with respect to the $c = -2$ Virasoro algebra of highest weight $h \leq -1$ is irreducible [66]. Hence the space of Virasoro descendants does not contain Virasoro primaries since they would generate a subrepresentation. In the rest of this section, we confirm the conjectured characterisation of physical compactification-independent string states in the PSL(2|2) WZW model at the second level.

We start by defining the modified branching function of $\widehat{\mathcal{K}}(\lambda)$, which is

$$\chi_{\widehat{\mathcal{K}}_0(\lambda)}(x, y, u|q) = \text{Tr}^{(0)}_{\widehat{\mathcal{K}}_0(\lambda)}\left(u^{\varrho} x^{J^0} y^{K^0} q^{L_0 - \frac{1}{2k}C_2(\lambda)}\right) , \tag{7.87}$$

where the trace $\text{Tr}^{(0)}$ is only taken over highest weight states with respect to the bosonic subalgebra $\mathfrak{g}^{(0)}$ of the horizontal subalgebra $\mathfrak{g}$. We have furthermore introduced a chemical potential u that keeps track of the grading of these states. $C_2(\lambda)$ is the value of the quadratic Casimir evaluated on the ground state representation $\mathcal{K}(\lambda)$; by subtracting $\frac{1}{2k}C_2(\lambda)$ from the L_0-eigenvalue the branching function has no poles in q. Evaluating the trace yields

$$\chi_{\widehat{\mathcal{K}}_0(\lambda)}(x, y, u|q) = x^{j_1} y^{j_2} \prod_{n\geq 1} \frac{\prod_{\alpha,\beta=\pm 1}(1 + u x^{\frac{\alpha}{2}} y^{\frac{\beta}{2}} q^n)(1 + u^{-1} x^{\frac{\alpha}{2}} y^{\frac{\beta}{2}} q^{n-1})}{(1 - x^{-1}q^n)(1 - xq^n)(1 - y^{-1}q^n)(1 - yq^n)(1 - q^n)^2} \, . \tag{7.88}$$

Loosely speaking, the factors in the numerator correspond to the fermionic generators of $\mathfrak{g}$ while the factors in the denominator are associated to the bosonic ones. The first step on the way to determine the claimed space of physical states is then to identify the subspace of Virasoro primaries. By our assumption that $\widehat{\mathcal{K}}(\lambda)$ decomposes into Virasoro primaries and descendants, it is sufficient to eliminate the Virasoro Verma module generated by the affine ground states. This is most easily achieved by multiplying the character $\chi_{\widehat{\mathcal{K}}(\lambda)}$ with $q^{-\frac{1}{24}}\eta(q)$, the inverse of character of the Virasoro Verma module . The spectrum of Kac modules that are Virasoro primary can be extracted form the resulting character by expanding it as

$$\begin{aligned}\chi_{\mathrm{Vir}\,\widehat{\mathcal{K}}(\lambda)}(x,y,u|q) &= q^{-\frac{1}{24}}\eta(q)\,\chi_{\widehat{\mathcal{K}}(\lambda)}(x,y,u|q)\\ &= K_\lambda(x,y)\sum_{\substack{g\in\mathbb{Z}\\ n\in\mathbb{N}}}\sum_{r,s=0}^{4n}\left(D_g^n\right)_{rs}\,x^{\frac{r}{2}-n}y^{\frac{s}{2}-n}u^g q^n\,. \end{aligned}\tag{7.89}$$

In analogy to the matrices C_n defined below eq. (7.79), the matrices D_g^n encode the multiplicities of the various Kac-modules of grading g at level n that are Virasoro primaries. Clearly, $D_0^0 = 1$ and $D_g^0 = 0$ for all $g \neq 0$. For $n = 1$ we get

$$D_0^1 = \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&0&0\\ 1&0&1&0&1\\ 0&0&0&0&0\\ 0&0&1&0&0 \end{pmatrix}, \qquad D_{\pm1}^1 = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&1&0\\ 0&0&0&0&0\\ 0&1&0&1&0\\ 0&0&0&0&0 \end{pmatrix}, \tag{7.90}$$

which agrees with (7.57). We have seen that all Kac modules of grading $+1$, which are counted by D_{+1}^1, are mapped by Q_0^Π to Kac modules of grading -1, which are counted by D_{-1}^1. Since $D_{-1}^1 = D_{+1}^1$, none of them survive in the cohomology. Thus D_0^1 encodes the physical string spectrum at first level in agreement with (7.65) and (7.68).

Expanding $\chi_{\mathrm{Vir}\,\widehat{\mathcal{K}}(\lambda)}$ to second order in q, we obtain the following set of matrices:

$$D_0^2 = \begin{pmatrix} 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&2&0&4&0&2&0&0\\ 0&0&0&0&0&0&0&0&0\\ 1&0&4&0&8&0&4&0&1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&2&0&4&0&2&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0 \end{pmatrix},\ D_{\pm1}^2 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&1&0&4&0&4&0&1&0\\ 0&0&0&0&0&0&0&0&0\\ 0&1&0&4&0&4&0&1&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$D^2_{\pm 2} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&2&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{pmatrix} .$$

We will need the projection operator onto the Virasoro primaries up to second level. It is realised in terms of Virasoro modes as

$$\Pi^{(2)} = \mathbb{1} + \tfrac{1}{2} L_{-1} L_1 + \tfrac{1}{10} L_{-1}^2 L_1^2 + \tfrac{1}{15} L_{-2} L_2 + \tfrac{1}{30} \left(L_{-1}^2 L_2 + L_{-2} L_1^2 \right) . \tag{7.91}$$

Note that

$$\left(\Pi^{(2)}\right)^2 = \Pi^{(2)} \quad + \quad (\text{terms annihilating states in } \widehat{\mathcal{K}}^{(n)}(\lambda) \text{ for } n = 0, 1, 2) , \tag{7.92}$$

and that $\left(\mathbb{1} - \Pi^{(2)}\right)$ clearly maps to Virasoro descendants. Furthermore, eq. (7.37) and (7.38) give an explicit realisation of Q_0 as an infinite sum of products of $\widehat{\mathfrak{g}}$-modes. When applied to states at the second level, this infinite sum truncates to a finite number of terms. Thus, using (7.91), we can work with an expression for Q_0^Π that has only finitely many terms. After identifying the cyclic states of the Kac modules at the second level $\widehat{\mathcal{K}}^{(2)}(\lambda)$ of $\widehat{\mathcal{K}}(\lambda)$, it is possible to check with this realisation of Q_0^Π that

- the homomorphism induced by Q_0^Π satisfies $\left(Q_0^\Pi\right)^2 = 0$ and hence the Q_0^Π-cohomology is well defined,
- no Kac module of grading +2 lies in the kernel of Q_0^Π, *i.e.* the Kac modules in D^2_{+2} do not contribute to the cohomology and neither does their image in D^2_0,
- every Kac module of grading −2 lies in the image of Q_0^Π, *i.e.* the Kac modules in D^2_{-2} do not contribute to the cohomology and neither does their preimage in D^2_0.

The Kac modules of odd grading are less intuitive as the action of Q_0^Π is not maximal in the sense that not every Kac module in D^2_{+1} is mapped to one in D^2_{-1}. In fact, by a brute force calculation using the explicit realisations of the Kac modules as well as the operators Q_0 and Π, one finds that four Kac-modules in D^2_{+1} are in the kernel of Q_0^Π. As a consequence, the same set of Kac modules does not lie in the image of Q_0^Π in D^2_{-1}. In our matrix notation the states that survive in cohomology can be encoded as

$$\widetilde{D}^2_{\pm 1} \equiv \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

and hence the spectrum of Kac module surviving in cohomology is encoded in the matrix

$$D^2_0 - D^2_{-2} - D^2_{+2} + \widetilde{D}^2_{-1} + \widetilde{D}^2_{+1} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&2&0&2&0&2&0&0\\ 0&0&0&2&0&2&0&0&0\\ 0&0&2&0&4&0&2&0&0\\ 0&0&0&2&0&2&0&0&0\\ 0&0&2&0&2&0&2&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}. \tag{7.93}$$

This matrix is the same as the matrix A_2 in appendix E, which tells us the multiplicities and $\mathfrak{g}$-quantum numbers of physical Kac modules in the RNS formulation. We therefore conclude that our conjecture holds at the second level.

Chapter 8

Conclusions and Outlook

In the last decades, string theory has been an active and fruitful topic of research and has influenced a variety of areas: among other things it offers resolutions of short-comings of the standard model in particle physics and addresses the challenge of formulating a consistent theory of quantum gravity. It also has had a notable impact in abstract mathematics, most impressively in the context of the monstrous moonshine conjecture [48] and in algebraic geometry by inspiring the concept of mirror symmetry of Calabi-Yau manifolds (see [115] and references therein). In recent years, however, a particular class of string theories has made its way into the literature; these are string theories in which the string lives in an Anti-de-Sitter gravitational background.

The interest in string theories on Anti-de-Sitter space-times is twofold: from the point of view of elementary particle physics, string theory is considered to be a promising candidate for a consistent UV completion of the standard model, which also naturally incorporates gravity. As such it is a theory of quantum gravity. However, actually quantising string theory in nontrivial geometric backgrounds, *i.e.* curved space-times, is a difficult exercise. In this regard, Anti-de-Sitter backgrounds are particularly compelling because, due to their large amount of symmetry, they provide string backgrounds in which string theory can be quantised. Thus, even though our universe most certainly is not Anti-de-Sitter[1], it serves as a class of toy models from which we can gain insights into the quantisation of the string.

From a more conceptual point of view, these theories are important due to the role they play in the context of the AdS/CFT correspondence, which conjectures that string theories, or quantum gravities in general, on Anti-de-Sitter spaces are equivalent to conformal field theories living on the boundary of that Anti-de-Sitter space. Since it thus connects quantum field theories to theories of gravitation, it might be considered as a quite fundamental concept in physics, even entering application-oriented areas such as quantum chromodynamics [64] and condensed matter physics [112]. Therefore, it is desirable to gain deeper insights into the correspondence, collect more evidence for its consistency and eventually, possibly in a simplified setting, give a proof of it. But this certainly requires a well-established understanding of quantum strings living on Anti-de-Sitter gravitational backgrounds.

This thesis succeeded in making progress in understanding string theory on Anti-de-

[1] Recent measurements [151] suggest a small but positive value of the cosmological constant while Anti-de-Sitter space-times are solutions to Einstein's equations with negative cosmological constant.

Sitter spaces as it was concerned with the question how string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3$ can be identified within the larger context of the PSL(2|2) WZW model. The close connection of these conformal field theories has been first noticed in [30], where a reformulation of RNS string theory, now commonly called hybrid string theory, was shown to give rise to nonlinear σ-models on PSL(2|2). The WZW point in moduli space of nonlinear σ-models is associated with string theory in a background with pure NSNS flux.

In more detail, we started with the discussion of representation theory of Lie superalgebras, in particular the representation theory of $\mathfrak{psl}(2|2)$. In section 2.2.3 we were able to determine the composition series of projective covers in the case of small $\mathfrak{sl}(2)$ eigenvalues (in an appropriately chosen category), thus extending previous results on projective covers of $\mathfrak{psl}(2|2)$ representations [95, 104, 105, 195].

Then, after reviewing important concepts in conformal field theory and string theory as well as the derivation of the hybrid string, we have given a detailed description of the PSL(2|2) WZW model that underlies the hybrid formulation of $\mathrm{AdS}_3 \times \mathrm{S}^3$ for pure NSNS flux in section 4.3.3. Following recent insights into the structure of logarithmic conformal field theories [86, 87, 105, 163, 167, 178] one expects that the space of states has the structure of a quotient space of a direct sum of tensor products of projective covers. We have worked out the details of this proposal: in particular, we have given a fairly explicit description of all the relevant projective covers and explained in detail how the quotient space can be defined.

In chapter 6, we discussed how the physical spectrum of massless string states can be identified within the full space of states of the logarithmic conformal field theory underlying the PSL(2|2) WZW model. For the subspectrum of massless states that is independent of the choice of the internal manifold, we argued that the known characterisation originating from the hybrid formulation in terms of BRST operators [30, 62] must be appropriately adjusted in order to be applicable to the logarithmic CFT. We have described the structure of the resulting BRST operators in detail and determined their common cohomology. The resulting massless compactification-independent string states reproduce precisely the supergravity prediction of [54,57], including the truncation at small KK momenta. We then specified our analysis to toroidial compactifications in section 6.2. Using the physical state constraints for compactification-dependent states as they arose in the hybrid formulation in section 5.3.2, we identified the physical massless string states specific to compactifications on T^4 within the full space of states of the logarithmic CFT. We observed that they coincide with the image of the BRST operators whose common cohomologies describe the compactification-independent massless spectrum. This yields an intriguingly simple characterisation of the compactification-dependent spectrum which again is consistent with the supergravity analysis of [54, 57].

We have also succeeded to evaluate the physical state constraints of the hybrid formulation at the first mass level in chapter 7. We have found a surprisingly accessible and elegant description of the physical compactification-independent string spectrum in the context of the PSL(2|2) WZW model. In particular, we showed that the physical string states can be identified within the PSL(2|2) WZW model by taking an appropriate cohomology of the full WZW spectrum at the first affine level. This cohomological description of the physical states reduces to the one known for the supergravity spectrum when applied to the affine ground states. Hence it yields an appropriate generalisation of the description of compactification-independent physical string states in the massless sector. This strongly suggests that the description of physical states we found can be further extended to describe also the physical states at any mass level. A possible generalisation to all mass levels was conjectured and we checked that it agrees at the second level with the on-shell RNS string spectrum.

Having a simple algebraic characterisation of physical string states within the PSL(2|2) WZW model at hand, it would be desirable to calculate correlation functions between these states in the PSL(2|2) WZW model and compare them to the known correlation functions [42, 84, 175] in the RNS formulation. This may give further insights into the connections of string theory on $AdS_3 \times S^3$ to the PSL(2|2) WZW model.

Within the program of understanding the string theory side of the AdS_3/CFT_2 correspondence, the natural next step would be to investigate how our description of physical string states changes if the PSL(2|2) WZW model is marginally deformed such that RR flux is included to the string background. Even though no RNS description of string theory exists in that case, from the point of view of the hybrid formulation this is equivalent to leaving the WZW point in moduli space. In general, calculations away from the WZW point are difficult to perform due to the fact that the currents are not holomorphic anymore, which forbids the application of complex analytic tools. But since there exists a non-renormalisation theorem specific to the PSL(2|2) WZW model [33], one may expect that it is possible to keep track of how the description of physical states changes along any path in the moduli space that starts at the WZW point. In particular, one should be able to gain a deeper understanding of the non-renormalisation theorem of [55] in a target space supersymmetric setting and possibly embed it into a larger scheme that also includes massive states. Although this seems like a challenging task, success in that respect would be a major step forward in mastering string theory in RR backgrounds and thus working out the AdS_3/CFT_2 correspondence in much greater detail.

Acknowledgements

I am grateful to my advisor Matthias R. Gaberdiel for continuous encouragement, instructive discussions and guidance through this thesis. His deep understanding of theoretical physics combined with his ability to explain even the most complicated issues in an accessible way often renewed my fascination for the elegance of the mathematical description of nature. His motivating words during the tough periods of this thesis were highly appreciated and kept me on the right path.

Special thanks go to Niklas Beisert for his acceptance to be the co-examiner of this thesis.

I also would like to thank the present and former members of the string theory group at ETH for helpful and often entertaining discussions: C. Candu, M. de Leeuw, S. Hohenegger, K. Jin, M. Kelm, I. Kirsch, D. Persson, M. Rosso, C. Schmidt-Colinet, B. Schwab, P. Suchanek, C. Vollenweider, R. Volpato and S. Wood.

Many thanks go to V. Beaud, S. Bieri, D. Egli, M. Fraas, P. Grech, A. Lisibach, M. Porta, K. Schnelli, C. Stark, M. Walter and G. Zhou for creating such a cooperative and stimulating working atmosphere and to the secretaries for providing helpful guidance through the jungle of bureaucracy.

Furthermore, I thank my good friends J. Dahlhaus, J. Döveling, L. M. & H. Haas, N. Malchus, P. Nathaus, L. Püttmann and T. Pokorny for moral support and always lifting my spirits.

I am grateful to my parents Helga and Bernd, and to my brother Fabian for their constant unconditional support and for giving me the opportunity to pursue my path in life. Moreover, I appreciate the encouragement from the rest of my family, particularly Ann-Kristin, Karolin, Angelika and Werner. I am grateful to Julia for all her understanding, patience and support during the work on this thesis.

This research was supported by the Swiss National Science Foundation.

Appendix A

Lie Algebras and the BGG Category $\mathscr{O}$

In this appendix we mainly set notation and introduce concepts of representation theory of Lie algebras which are similarly obtained in the Lie superalgebra case.

A.1 A Very Short Review on Lie Algebras

Let us start by recalling basic facts on Lie algebras. Lie algebras play an important role in theoretical physics since infinitesimally small symmetry transformations usually give rise to such Lie algebraic structures (see *e.g.* [49, 73, 91]). As a consequence, physical entities (quantum mechanical states, fields *etc.*) transform in representations thereof. So understanding Lie algebras and eventually their extension to Lie superalgebras turns out useful in analysing symmetric theories since they put strong restrictions on the physical entities. This section is mainly based on the standard literature [76, 118].

Formally, a Lie algebra $\mathfrak{g}$ is a vector space equipped with an antisymmetric bilinear map

$$[\cdot,\cdot]\colon\ \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}\,. \tag{A.1}$$

By definition, this bilinear form, called Lie bracket, is also required to satisfy the Jacobi identity,

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]\,. \tag{A.2}$$

A quite intuitive realisation of a Lie algebra is given by the space of linear maps from some finite-dimensional complex vector space V to itself, denoted $\mathrm{End}(V)$ for endomorphisms. As V is finite-dimensional, it is isomorphic to some $\mathbb{R}^n$ and $\mathrm{End}(V)\sim gl(n)$. It becomes a Lie algebra if we define $[x,y]\equiv xy-yx$ for all matrices $x,y\in gl(n)$. If we mean the associated Lie algebra rather then the algebra of matrices, we write $\mathfrak{gl}(n)$. It is suggestive that any finite-dimensional Lie algebra can be represented as matrices if we just take n large enough. Indeed, a linear (finite-dimensional) *representation* of $\mathfrak{g}$ is defined as a homomorphism of Lie algebras $\pi:\mathfrak{g}\to\mathfrak{gl}(n)$.

It is actually quite easy to see that any finite-dimensional Lie algebra has at least one finite-dimensional representation in the above sense. By the defintion of a Lie algebra, $\mathfrak{g}$ is a vector space. Furthermore, the Lie bracket defines a map from $\mathfrak{g}$ to $\mathrm{End}(\mathfrak{g})$ by mapping any $x \in \mathfrak{g}$ to $\pi(x) \equiv [x, \cdot]$. Due to the Jacobi identity, it can be shown to be a homomorphism of Lie algebras, $[\pi(x), \pi(y)] \equiv \pi([x, y])$. This is a very special representation which exists for any Lie algebra called the *adjoint representation* and denoted $\mathrm{ad}\,(\cdot)$. Since everything is finite-dimensional, we can identify $\mathfrak{g} \simeq \mathbb{R}^{\dim \mathfrak{g}}$ and $\mathrm{End}(\mathfrak{g}) \simeq \mathbb{R}^{\dim \mathfrak{g} \times \dim \mathfrak{g}}$ if we wish to write everything in terms of matrices. For later purposes, let us emphasize that the notation of the adjoint representation is not restricted to finite-dimensional Lie algebras. However, in that case the adjoint representation will be infinite-dimensional as well. Infinite-dimensional representations of finite-dimensional Lie algebras will be discussed in sect. A.2.

We call a representation π of $\mathfrak{g}$ *reducible* if $\pi(\mathfrak{g})$ there exists a proper subspace W of V that is invariant under the action of $\pi(\mathfrak{g})$. Loosely speaking, starting from an element in W, we will never recover the whole vector space V by just acting with $\pi(\mathfrak{g})$. This implies that the restriction of π to W is a representation of $\mathfrak{g}$ itself, called a *subrepresentation.* If π has no subrepresentation, it is said to be *irreducible.* Furthermore, if π can be written as a direct sum of irreducible representations, it is called *completely* or *fully reducible.*

By construction, the adjoint representation is intrinsically connected to the structure of the Lie algebra itself. A Lie algebra is said to be *simple* if the adjoint representation is irreducible. If it is reducible, the subspace $I \subset \mathfrak{g}$ such that $\mathrm{ad}\,(\cdot)\,|_I$ is a subrepresentation is called an *ideal* of $\mathfrak{g}$. So a Lie algebra being simple is equivalent to saying that it has no other ideal than zero. Simple Lie algebras are particular nice since they might be thought of as containing no smaller "building blocks".

From now on, we restrict ourselves to simple Lie algebras. Given any Lie algebra $\mathfrak{g}$, we can look for a subset $\mathfrak{h}$ of linearly independent elements or *generators*[1] such that $\mathrm{ad}\,(\mathfrak{h})\,\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] = 0$. If $\mathfrak{h}$ is maximal, then it is called the Cartan subalgebra (CSA). The elements of $\mathfrak{h}$ in the adjoint representation, $\mathrm{ad}\,(x)$ for $x \in \mathfrak{h}$, can be simultaneously diagonalised by choosing an appropriate basis of $\mathfrak{g}$. So it is sensible to look for eigenvectors e_α of the adjoint action,

$$\mathrm{ad}\,(h)\, e_\alpha = \alpha(h) e_\alpha \,, \qquad h \in \mathfrak{h}\,. \tag{A.3}$$

The linear functional $\alpha \in \mathfrak{h}^*$ maps any h to the associated eigenvalue of e_α under the adjoint action of h and is called a *root.* The *root system* denoted Δ is the set of all roots that do not equal zero. Since the set of eigenvectors e_α together with the generators of the CSA yield a basis of $\mathfrak{g}$, we obtain the *Cartan decomposition* of the Lie algebra,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \,, \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{ad}\,(h)\, x = \alpha(h) x\}\,. \tag{A.4}$$

Roots have the following properties (see *e.g.* [76]):

1. It follows from the Jacobi identity that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

[1] There is different definitions of generators in the literature. In some cases, generators refers to a set of elements of a Lie algebra such that the whole Lie algebra can be obtained by evaluating successively the Lie bracket thereof and then considering the linear span. However, in the present work, generators will refer to basis elements of $\mathfrak{g}$.

2. It can also be shown that all so-called root spaces are one-dimensional, *i.e.* $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$.

3. If $\alpha \in \Delta$, then $n\alpha \in \Delta$ only for $n = \pm 1$.

Point 3 suggests that we can separate the roots in two sets of postive and negative roots by requiring that if α is a positive root then $-\alpha$ is negative. Furthermore, for consistency, we should also demand that postivity is conserved under addition, *i.e.* if α and β are positive roots and $\alpha + \beta \in \Delta$ then $\alpha + \beta$ is a positive root as well. The set of positive (resp. negative) roots is denoted by Δ^+ (resp. Δ^-). Another important notion is that of a *simple* root which is a positive (negative) root that cannot be written as the sum of two positive (negative) roots. The set of simple roots is denoted by Δ_0^+ (Δ_0^-). For later use, we further define

$$\mathfrak{n}^\pm \equiv \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha \tag{A.5}$$

such that the Cartan decomposition becomes

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \,. \tag{A.6}$$

The Lie subalgebra $\mathfrak{b}^+ \equiv \mathfrak{h} \oplus \mathfrak{n}^+$ is called the Borel algebra. Of course, the positive root system is not unique and therefore, the Borel subalgebra is not uniquely defined. However, it can be shown [76] that any two Borel subalgebras, say $\mathfrak{b}$ and $\mathfrak{b}'$, are conjugated to each other, *i.e.* there exists an inner automorphism of $\mathfrak{g}$ such that $\mathfrak{b}$ is mapped to $\mathfrak{b}'$ (the same holds for the CSA). Looking ahead, this is not true for Lie superalgebras.

If we take any representation ρ of $\mathfrak{g}$ instead of the adjoint one, the associated vector space V can be decomposed in eigenspaces under the action of $\rho(\mathfrak{h})$,

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda \,, \qquad V_\lambda = \{ v \in V \mid \rho(h)v = \lambda(h)v \} \,, \tag{A.7}$$

where $\lambda \in \mathfrak{h}^*$. The linear functionals λ are called *weights*, V_λ are the associated weight spaces and Λ is the set of all weights. In that sense, the roots are just the weights of the adjoint representation.

Apart from the Lie bracket, there are additional algebraic operations defined on $\mathfrak{g}$. This becomes clear, if one considers the Lie algebra $gl(n)$. Since elements thereof are just $n \times n$ matrices, *e.g.* the trace is well-defined. So given a representation ρ, the notion of the trace can be pulled back to $\mathfrak{g}$,

$$\mathrm{tr}_\rho(x) \equiv \mathrm{tr}\, (\rho(x)) \ , \quad x \in \mathfrak{g} \,. \tag{A.8}$$

The pullback of the trace has the valuable property that it vanishes on the commutator,

$$\mathrm{tr}_\rho([x,y]) = \mathrm{tr}\, (\rho([x,y])) = \mathrm{tr}\, ([\rho(x),\rho(y)]) = \mathrm{tr}\, (\rho(x)\rho(y)) - \mathrm{tr}\, (\rho(y)\rho(x)) = 0 \,, \tag{A.9}$$

where we used that ρ is a homomorphism of Lie algebras, and thus preserves the Lie bracket, and the cyclicality of the trace. Therefore $\mathrm{tr}_\rho(\cdot)$ defines an invariant[2] linear functional on $\mathfrak{g}$. We can further construct a symmetric bilinear form using the trace in the same manner since $\mathrm{tr}\,(xy)$ surely defines a symmetric bilinear form on $gl(n)$. Hence $\mathrm{tr}\,(\rho(x)\rho(y))$ defines a symmetric bilinear form on $\mathfrak{g}$ that can be checked to be invariant by a simple calculation using again that ρ preserves the Lie bracket. At first sight, this seems to give a whole family of invariant symmetric bilinear forms. However, it can be shown [76] that if $\mathfrak{g}$ is simple any two invariant symmetric bilinear forms are proportional to each other. Hence up to a multiplying constant, the form is unique. In fact, there is a natural choice for ρ which is the adjoint representation of $\mathfrak{g}$ since it exists for any Lie algebra. Therefore the *Killing form* is defined as

$$(x, y) \equiv \mathrm{tr}\ (\mathrm{ad}\,(x)\,\mathrm{ad}\,(y))\ , \quad x, y \in \mathfrak{g} \tag{A.10}$$

which can be shown to be non-degenerate on semisimple Lie algebras (Cartan criterion for semisimplicity). Its restriction to the CSA yields a symmetric bilinear form on $\mathfrak{h}$ which in turn induces a bilinear form on the dual space $\mathfrak{h}^*$ and hence on the root space.

Up to now we only discussed finite-dimensional representations. If we wish to include infinite-dimensional representations as well, it makes sense to reexamine the notion of a representation. Note that the definition of a representation as a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathfrak{gl}(n)$ is specific to finite-dimensional representations. However, let us try to extract the important aspects of this definition that eventually might allow for a generalisation.

Note that in making $gl(n)$ a Lie algebra it was crucial that $gl(n)$ has the structure of an associative algebra since the Lie bracket was defined in terms of the algebra multiplication, $[x, y] = xy - yx$. In fact, any associative algebra, not necessarily finite-dimensional, can be made into a Lie algebra this way. For example, given a vector space V we can look at its tensor algebra $T(V)$ and define the Lie bracket to be $[x, y] = x \otimes y - y \otimes x$. However, we cannot simply substitute $gl(n)$ in our defintion of a representation by some associative algebra because we also have to make sure that there exists a homomorphism of Lie algebras mapping $\mathfrak{g}$ to that associative algebra. Since $\mathfrak{g}$ is a vector space, let us consider the tensor algebra $T(\mathfrak{g})$ with a Lie bracket as defined before. $T(\mathfrak{g})$ has a natural grading by "counting the number of tensor products", $T_n(\mathfrak{g}) = \bigotimes_{i=1}^n \mathfrak{g}$. Furthermore, there is a natural vector space homomorphism ν from $\mathfrak{g}$ to $T(\mathfrak{g})$ by identifying $\mathfrak{g} \simeq T_1(\mathfrak{g})$, but it is not a homomorphism of Lie algebras because

$$[\nu(\mathfrak{g}), \nu(\mathfrak{g})] \ \subset\ T_2(\mathfrak{g}) \quad \text{while} \quad \nu([x, y]) \in T_1(\mathfrak{g})\,. \tag{A.11}$$

[2]In general, assume that we are given $n + 1$ representations of $\mathfrak{g}$, say ρ_i, with associated vector spaces V_i. A multilinear map

$$f\colon\ V_1 \times \ldots \times V_n \to V_{n+1}$$

is said to be *invariant* if

$$\sum_{i=1}^{n} f(v_1, \ldots, \rho_i(x)v_i, \ldots, v_n) = \rho_{n+1}(x) f(v_1, \ldots, v_n) \quad \forall\, v_i \in V_i\,,\ x \in \mathfrak{g}\,.$$

However, this problem can be dealt with by simply identifying elements of the form $\nu(x) \otimes \nu(y) - \nu(y) \otimes \nu(x)$ in $T_2(\mathfrak{g})$ with the elements of the form $\nu([x,y])$ in $T_1(\mathfrak{g})$. The resulting algebra is

$$\mathcal{U}(\mathfrak{g}) \equiv T(\mathfrak{g}) \ / \ \langle [x,y] - x \otimes y + y \otimes x \rangle \tag{A.12}$$

and it is called the *universal enveloping algebra*. In the following, the tensor product will be suppressed such that $xy \equiv x \otimes y$.

Having found a generalisation of $gl(n)$, it should be asked how the vector space V has to be appropriately generalised. In fact, the vector space V just served as something that $gl(n)$ could act on and in that sense should be rather seen as a $gl(n)$-module. Hence after substituting $gl(n)$ by the universal enveloping algebra, we should think about representations as $\mathcal{U}(\mathfrak{g})$-modules. This may be the most general meaning one could give to the notion of a representation of a Lie algebra and in the following we will use the notions of representation theory and module theory interchangably. For example, irreducible representations will be identified with simple $\mathcal{U}(\mathfrak{g})$-module.

A.2 The BGG-Category $\mathscr{O}$

The category of representations, *i.e.* the category of (left) $\mathcal{U}(\mathfrak{g})$-modules $\mathcal{U}(\mathfrak{g})$-**Mod**, is yet not well enough understood to extract interesting structures. However, there is a certain subcategory called the *Bernstein-Gelfand-Gelfand-category* or *BGG-category* for short. Its objects are $\mathcal{U}(\mathfrak{g})$-modules M that satisfy the following three axioms [117]:

($\mathscr{O}$1) M is finitely generated.

($\mathscr{O}$2) M is semisimple with respect to the CSA of $\mathfrak{g}$, *i.e.* it decomposes into weight spaces: $M = \bigoplus_{\lambda \in \mathfrak{g}^*} M_\lambda$.

($\mathscr{O}$3) M is locally $\mathcal{U}(\mathfrak{n}^+)$-finite, which means that the subspace $\mathcal{U}(\mathfrak{n}^+)\, v$ is finite-dimensional for every $v \in M$.

Although this definition is fairly abstract, all conditions are natural as they apply to the most common representations and building blocks thereof. For example, *highest weight representations* are objects in $\mathscr{O}$. These are by definition $\mathcal{U}(\mathfrak{g})$-modules M that are generated from a *maximal vector* $v^+ \in M$, *i.e.* a vector satisfying $\mathcal{U}(\mathfrak{n}^+) \cdot v^+ = 0$. Since M is generated by v^+, any element of M, say v, in M can be written as $v = g \cdot v^+$ for some $g \in \mathcal{U}(\mathfrak{g})$. Abstractly, we can thus write $M = \mathcal{U}(\mathfrak{g}) \cdot v^+$. If λ is the weight of v^+, it is called the *highest weight* of M.

A very important construction is the so called Verma module $\mathcal{V}$. It is constructed as follows: Take some weight $\lambda \in \mathfrak{h}^*$ and let the vector v_λ satisfy $hv_\lambda = \lambda(h)v_\lambda$ for every $h \in \mathfrak{h}$. Then v_λ can be interpreted as a left $\mathcal{U}(\mathfrak{h})$-module. It is easily lifted to a $\mathcal{U}(\mathfrak{b}^+)$-module by letting v_λ be annihilated by every element of $\mathfrak{n}^+$. We can further lift v_λ to a left $\mathcal{U}(\mathfrak{g})$-module by simply letting elements in $\mathcal{U}(\mathfrak{n}^-)$ act freely on v_λ. Because the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ has naturally the structure of an $\mathcal{U}(\mathfrak{g})$-bimodule, this can be written more formally as taking the tensor product of $\mathcal{U}(\mathfrak{b}^+)$-modules,

$$\mathcal{V}(\lambda) \equiv \mathrm{Ind}_{\mathfrak{b}^+}^{\mathfrak{g}} v_\lambda := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}^+)} v_\lambda \,. \tag{A.13}$$

This defines the Verma module of weight λ. Formally, $\mathrm{Ind}_{\mathfrak{b}^+}^{\mathfrak{g}} - \equiv \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}^+)} -$ is a functor from $\mathcal{U}(\mathfrak{b}^+)$-**Mod** to $\mathcal{U}(\mathfrak{g})$-**Mod**. Note that $v^+ \equiv \mathbb{1} \otimes v_\lambda \in \mathcal{V}(\lambda)$ is an nontrivial element of $\mathcal{V}(\lambda)$ annihilated by $\mathfrak{n}^+$ and is thus maximal. Furthermore, it generates $\mathcal{V}(\lambda)$. We conclude that every Verma module is a highest weight module and hence an object in $\mathscr{O}$.

In general, the Verma module is not irreducible as further maximal vectors of weight $\mu < \lambda$ might appear within $\mathcal{V}(\lambda)$. For example, if $n = \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}^+$, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ is *the coroot of* α, then $(e_{-\alpha})^{n+1} v^+$ is a maximal vector of weight $\mu = \lambda - (n+1)\alpha$. Therefore, there is a monomorphism $\mathcal{V}(\mu) \to \mathcal{V}(\lambda)$. The two weights μ and λ are called *linked*. An irreducible representation, *i.e.* a simple module, can be obtained from the Verma module $\mathcal{V}(\lambda)$ by identifying the maximal proper submodule, say M, and taking the quotient

$$L(\lambda) \equiv \mathcal{V}(\lambda)/M \,. \tag{A.14}$$

Note that $L(\lambda)$ is simple by construction.

A further important concept is the *block decomposition* of $\mathscr{O}$ that we will now explain. In order to do this, we need to introduce the *center* of a semisimple Lie algebra, denoted by $Z(\mathfrak{g})$, which consists of all elements in $\mathcal{U}(\mathfrak{g})$ that commute with $\mathfrak{g}$:

$$Z(\mathfrak{g}) = \{ z \in \mathcal{U}(\mathfrak{g}) \,|\, [z, \mathfrak{g}] = 0 \} \,. \tag{A.15}$$

It is clear that the center forms a subalgebra of $\mathcal{U}(\mathfrak{g})$. An important element of $Z(\mathfrak{g})$ is given by the quadratic Casimir C_2 which is constructed as follows; let $(\cdot, \cdot)$ denote the Killing metric of $\mathfrak{g}$ and furthermore take t^i, $i = 1, \ldots, n$ to be a basis of $\mathfrak{g}$. The coefficients of the Killing metric are $\kappa^{ij} = (t^i, t^j)$, with κ_{ij} being its inverse. The quadratic Casimir can then be written in an explicit form as

$$C_2 = \kappa_{ij} t^i t^j \,. \tag{A.16}$$

This implies that $Z(\mathfrak{g})$ is always nontrivial since it at least contains C_2. However, usually more elements will generate $Z(\mathfrak{g})$.

The action of $Z(\mathfrak{g})$ on highest weight modules is particular interesting. Let M be a highest weight module with maximal vector v^+. Using the definition of the center we obtain for every $h \in \mathfrak{h}$

$$h \cdot (z \cdot v^+) = z \cdot (h \cdot v^+) = \lambda(h) z \cdot v^+ \,, \quad z \in Z(\mathfrak{g}) \,. \tag{A.17}$$

Because the highest weight space is one-dimensional, we conclude that $z \cdot v^+ = \chi_\lambda(z)v^+$ with $\chi_\lambda(z) \in \mathbb{C}$. It is the *central character* associated to λ. Note that since

$$z \cdot M = z \cdot (\mathcal{U}(\mathfrak{g}) \cdot v^+) = \mathcal{U}(\mathfrak{g})(z \cdot v^+) = \chi_\lambda(z)M \tag{A.18}$$

the element z acts diagonally on M with the same scalar and thus χ_λ is specific to the highest weight module M with highest weight λ.

For general modules M the situation is more complicated. However, it can be shown that every module in $\mathcal{O}$ allows for a filtration

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M \tag{A.19}$$

such that M_{i+1}/M_i is a highest weight module [117]. Therefore we conclude that an element $z \in Z(\mathfrak{g})$ acts as a scalar (within each quotient) plus a nilpotent part (which maps *e.g.* M_{i+1} to the submodule M_i divided out). Thus, given some central character $\chi(z)$, we can define an independent submodule of M, *i.e.* it appears as a direct summand of M, by defining

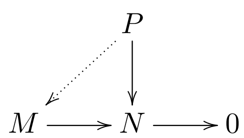
$$M\downarrow_\chi = \{m \in M \,|\, (z - \chi(z))^{n(z)}m = 0 \quad \text{for some positive integer } n(z)\}\,. \tag{A.20}$$

For fixed χ the full subcategory of $\mathcal{O}$ consisting of objects $M\downarrow_\chi$ as above is called a *block* $\mathcal{O}_\chi$. This generalises the Casimir decomposition of finite-dimensional representations of semisimple Lie algebras into a direct sum of irreducible representations. Of particular importance is the *principal block* $\mathcal{O}_0$ for the trivial central character $\chi = 0$. The BGG category decomposes as

$$\mathcal{O} = \sum_\chi \mathcal{O}_\chi\,. \tag{A.21}$$

A.3 Projective Modules and Covers

We will now introduce the categoric concept of projective objects in the category theoretic framework and then specialise it to the case of $\mathcal{O}$. Suppose one is given an epimorphism mapping an object onto another $M \to N \to 0$ and a homomorphism $P \to N$. The object P is called *projective* if there exist a homomorphism $P \to M$ such that the diagram

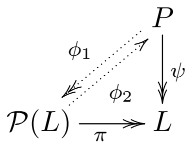


commutes. One of many equivalent definitions is that the functor $\mathrm{Hom}(P, -)$ to the category of Abelian groups **Ab** is exact. A category is said to *have enough projectives* if for each object N there exists a projective object P and a homomorphism $\pi : P \to N$.

Let us now specifiy to the BGG-category $\mathcal{O}$, so our objects are now given by $\mathcal{U}(\mathfrak{g})$-

modules obeying axioms ($\mathscr{O}1$) – ($\mathscr{O}3$). The projective objects are referred to as projective modules and epimorphisms coincide with surjections. Indeed, it can be shown that $\mathscr{O}$ has enough projectives. In fact, for every $M \in \mathscr{O}$ there exist a projective module $\mathcal{P}(M)$ with an *essential* homomorphism $\pi : \mathcal{P}(M) \to M$.[3] A homomorphism being essential means that no proper submodule of $\mathcal{P}(M)$ is mapped onto M. The tuple $(\mathcal{P}(M), \pi)$ is called the *projective cover of* M.

It is important to recognise that every projective module in $\mathscr{O}$ can be written as a direct sum of projective covers. Let P be a projective module. Then there exists a epimorphism $P \to L$ with L being simple. As has been discussed, there exists a projective cover of L, so we arrive at the following diagram.



Since both P and $\mathcal{P}(L)$ are projective there exist homomorphisms $\phi_1 : P \to \mathcal{P}(L)$ and $\phi_2 : \mathcal{P}(L) \to P$ such that

$$\pi \circ \phi_1 = \psi \,, \tag{A.22}$$

$$\psi \circ \phi_2 = \pi \,. \tag{A.23}$$

Since π is essential, we conclude that ϕ_1 is surjective, because otherwise $\phi_1(P)$ gives a proper submodule mapped by π onto L. Furthermore, inserting (A.23) in (A.22) tells us that $\pi \circ (\phi_1 \circ \phi_2) = \pi$. In particular,

$$\pi\left((\phi_1 \circ \phi_2)\,\mathcal{P}(L)\,\right) = \pi(\,\mathcal{P}(L)\,) = L \,. \tag{A.24}$$

Since π is essential, we conclude that $\phi_1 \circ \phi_2(\,\mathcal{P}(L)\,) = \mathcal{P}(L)$ and thus $\phi_1 \circ \phi_2 = \mathrm{id}_{\mathcal{P}(L)}$. Therefore we recover the universal property of the direct sum. So $\mathcal{P}(L)$ is a direct summand of P. Note that this proof only requires the existence of a projective covers for the module L and is independent of the special structures of $\mathscr{O}$. Furthermore, it tells us that if the projective cover of L exists, it is unique.

A.4 Duality

Given a finite-dimensional $\mathcal{U}(\mathfrak{g})$-module M, we can define a standard action of $\mathcal{U}(\mathfrak{g})$ on the dual vector space M^* by

$$(g \cdot f)(v) = -f(g \cdot v) \qquad \forall\, v \in M \,, \tag{A.25}$$

[3] In more general terms, for every artinian module category the fact that enough projective modules exists implies the existence of a projective cover for every module.

where $g \in \mathfrak{g}$ and $f \in M^*$. This definition still lifts M^* to a $\mathcal{U}(\mathfrak{g})$-module if M is infinite-dimensional but it in general it will not be an object in $\mathscr{O}$. Furthermore, the action as defined in (A.25) is inconvenient in several aspects, *e.g.* it maps simple highest weight modules of weight λ to lowest weight modules with weight $-\lambda$.

The latter inconvenience is best dealt with by introducing a different action of $\mathfrak{g}$ on M^*. In order to do this, we have to introduce the so-called *transpose map* $\tau : \mathfrak{g} \to \mathfrak{g}$. Given a root decomposition of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$, τ acts on the Cartan subalgebra $\mathfrak{h}$ as the identity and interchanges the root spaces, *i.e.* $\mathfrak{g}_\alpha$ gets mapped to $\mathfrak{g}_{-\alpha}$ and vice versa. Clearly, τ is an involution. It lifts to an involutory automorphism on $\mathcal{U}(\mathfrak{g})$ by defining $\tau(xy) = \tau(x)\tau(y)$, $x, y \in \mathfrak{g}$. An action of $\mathfrak{g}$ on M^* is then defined by

$$(g \cdot f)(v) = f(\tau(g) \cdot v) \qquad \forall\, v \in M\,. \tag{A.26}$$

We will call to this action as the *dual action* while the action in (A.25) will be referred to as the *conjugated action*. It can easily be checked that the dual action lifts M^* to a $\mathcal{U}(\mathfrak{g})$-module (a representation of $\mathfrak{g}$).

Before analysing the dual action let us attack the other problem that M^* might not be an object in $\mathscr{O}$ anymore. In the case of infinite-dimensional modules, M^* might not satisfy the defining axioms of $\mathscr{O}$. This can be healed by taking the weight space decomposition of M, $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, and taking the dual vector space of each M_λ individually. These are by the BGG-axioms finite-dimensional, so $(M_\lambda)^*$ has the same dimension. Given an element f in $(M_\lambda)^*$ it can naturally be identified with an element in M^* by setting $f(v) = 0$ for all $v \in M_\mu$ whenever $\mu \neq \lambda$. Therefore $(M_\lambda)^* \subset M^*$ as vector spaces. On the other hand, given a element f of weight λ in M^* with respect to the dual action (A.26), it vanishes on M_μ whenever $\mu \neq \lambda$ because

$$(h \cdot f)(v_\mu) = \lambda(h) f(v_\mu) = f(\tau(h) \cdot v_\mu) = f(h \cdot v_\mu) = \mu(h) f(v_\mu)\,, \qquad \forall\, h \in \mathfrak{h} \tag{A.27}$$

So we conclude that $(M^*)_\lambda$ consists of forms vanishing on all M_μ, $\mu \neq \lambda$, as well. Thus we have $(M^*)_\lambda = (M_\lambda)^* =: M^*_\lambda$. We define the *dual module* $M^\vee \subset M^*$ as

$$M^\vee = \oplus_{\lambda \in \mathfrak{h}^*} M^*_\lambda\,. \tag{A.28}$$

It is closed under the dual action of $\mathfrak{g}$ and thus it indeed defines a module. The submodule $M^\vee$ of M^* can be checked to be an object in $\mathscr{O}$ [117] and in $-^\vee$ defines a contravariant functor: Given a short exact sequence of $\mathcal{U}(\mathfrak{g})$-modules

$$0 \longrightarrow L \xrightarrow{\phi} M \xrightarrow{\psi} N \longrightarrow 0 \tag{A.29}$$

the sequence

$$0 \longrightarrow N^\vee \xrightarrow{\psi^\vee} M^\vee \xrightarrow{\phi^\vee} L^\vee \longrightarrow 0 \tag{A.30}$$

is exact as well. The dual homomorphism $\phi^\vee$ is given by the standard action $(\phi^\vee(f))(v) = f(\phi(v))$ for all $v \in L$. Note that if $M^\vee$ is simple, so is M, because if M is not simple, there exist a exact sequence like (A.29) with L being a proper submodule. Hence, $N^\vee = (M/L)^\vee$ is a proper submodule of $M^\vee$ and $M^\vee$ is not simple. Furthermore, $(M^\vee)^\vee$ is naturally isomorphic to M as it consists of reflexive vector spaces. Thus M is simple if and only if $M^\vee$ is simple. Furthermore, by construction of $M^\vee$, we have that M and $M^\vee$ have the same set of weights counting multiplicities. In other the words, the formal character of M and $M^\vee$ is the same $\operatorname{char} M = \operatorname{char} M^\vee$. So if M is simple, they have the same highest weight and thus $M = M^\vee$. All simple modules are *self-dual*.[4]

An Example

To illustrate how duality acts, let us consider as an example $\mathfrak{g} = \mathfrak{sl}(2)$. The simple root is simply $\alpha = 2$ and the weight lattice is $\mathbb{Z}$. Verma modules of highest weight $\lambda \leq 1$ are simple. According to the above argument it follows that $\mathcal{V}(\lambda) = \mathcal{V}(\lambda)^\vee$ in those cases. But what if $\lambda \geq 0$? Then $\mathcal{V}(\lambda)$ is reducible but not indecomposable. In particular, it contains $\mathcal{V}(-\lambda-2)$ as a simple submodule. Let v_μ denote the element in $\mathcal{V}(\lambda)$ of weight μ. Clearly, $\mu \in \lambda - 2\,\mathbb{N}$. As we have argued, to each v_μ we may associate a form $f_\mu \in \mathcal{V}(\lambda)^*_\mu \subset \mathcal{V}(\lambda)^\vee$ such that $f_\mu(v_\nu) = \delta_{\mu\nu}$. The question is what is the structure of the module whose elements are f_μ? Let us look for highest weight vectors:

$$(J^+ \cdot f_\mu)(v) = f_\mu(J^- v) = 0\,, \qquad \forall\, v \in \mathcal{V}_\lambda\,. \tag{A.31}$$

By the definition of the Verma module every element of $\mathcal{V}_\lambda$ can be represented in the form $J^- w$ for some $w \in \mathcal{V}_\lambda$ *except for the generating vector* v_λ. So in order that $J^+ \cdot f_\mu$ vanishes on every vector we must have $\mu = \lambda$. f_λ is the unique highest weight vector in $\mathcal{V}(\lambda)^\vee$. Similarly, we can look for lowest weight vectors $J^- \cdot f_\mu = 0$. Analysing this the same way as before yields that there is only one such element $f_{-\lambda}$. So we identified a simple submodule $L(\lambda)$ in $\mathcal{V}(\lambda)^\vee$. Furthermore, note that

$$(J^+ \cdot f_{-\lambda-2})(v_\mu) = f_{-\lambda-2}(J^- v_\mu) = f_{-\lambda-2}(v_{\mu-2}) = \delta_{-\lambda,\mu} \tag{A.32}$$

and therefore $J^+ \cdot f_{-\lambda-2} = f_{-\lambda}$. So we have the following structure

$$\mathcal{V}(\lambda)^\vee : \qquad L(\lambda) \xleftarrow{\;J^+\;} \mathcal{V}(-\lambda-2)\,, \tag{A.33}$$

while the original structure was given by

$$\mathcal{V}(\lambda) : \qquad L(\lambda) \xrightarrow{\;J^-\;} \mathcal{V}(-\lambda-2)\,. \tag{A.34}$$

The simple components are unchanged but the maps are inverted.

In order to emphasize the difference of duality, $-^\vee$, to conjugation, $-^*$, let us have a short look how $\mathcal{V}(\lambda)^*$ looks like. Firstly, it is easily checked that $\mathcal{V}(\lambda)^*$ is a simple *lowest* weight module of lowest weight $-\lambda$ whenever $\lambda \leq -1$. For $\lambda \geq 0$, an analogous analysis

[4] Note that this is not the case if we would have taken the conjugated action instead of the dual one, even though it lifts the vector space $M^\vee$ to a $\mathcal{U}(\mathfrak{g})$ module as well.

as before can be performed to obtain the modular structure of $\mathcal{V}(\lambda)^*$:

$$\mathcal{V}(\lambda)^* : \qquad L(\lambda) \simeq L(\lambda)^* \xleftarrow{J^-} \mathcal{V}(-\lambda-2)^* \,. \tag{A.35}$$

In the case of $\mathfrak{g} = \mathfrak{sl}(2)$, the irreducible representations $L(\lambda)$ are self-conjugated, which was used above. However, for other Lie algebras this is generically not the case.

Before concluding this section let us also emphasize that for the definition of $-^\vee$ only a root decomposition $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in\Delta} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ was necessary. Such a decomposition exists for Lie superalgebras of type I, which are the subject of chapter 2, as well. So $-^\vee$ is also well-defined for these Lie superalgebras and the above results apply. In particular, simple modules are self-dual.

A.5 BGG Reciprocity

Several aspects of representation theory of Lie algebras from the point of view of module theory have been covered so far and illustrated in the case of $\mathfrak{sl}(2)$. An important point was the introduction of projective covers that in a loose sense may be thought of as the "maximal completion" of a module. But how to find the projective cover in the general case? This is indeed a tricky business.

But let us postpone the question of how to find the projective cover a little, and first think about what kind of module is the most accessible for explicit calculations. The Verma module $\mathcal{V}(\lambda)$ seems to be quite a good choice as it is an induced module, which means that we can start with some vector of weight λ and obtain any other element of the module by applying Lie algebra generators associated to negative roots. By the PBW theorem, after choosing an ordering of the negative roots, we even get a unique presentation of each element. However, a drawback is that the Verma module by construction is infinite-dimensional and in many cases contains nontrivial submodules.

The simplest way to get rid of these submodules is to identify the maximal proper submodule in $\mathcal{V}(\lambda)$, say $M(\lambda)$, that contains any other submodule of $\mathcal{V}(\lambda)$ as a submodule. In other words, it is the union of all proper submodules of $\mathcal{V}(\lambda)$. By construction, the quotient $L(\lambda) \equiv \mathcal{V}(\lambda)/M(\lambda)$ is simple (or, in terms of usual representation theory, irreducible). The simple modules $L(\lambda)$ are of particular importance in the context of the BGG category $\mathscr{O}$ as they have some kind of universal character in the following sense. Any module M in $\mathscr{O}$ possesses a filtration in submodules,

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_{n-1} \subset M_n = M \,, \tag{A.36}$$

such that the quotients M_{i+1}/M_i are isomorphic to $L(\mu)$ for some weight μ. Although the filtration above is not unique, its length n is and so is the set of simple modules that result as quotients of form M_{i+1}/M_i [117]. Hence, the multiplicity of the simple module $L(\mu)$ in M is well-defined without specifying the filtration. Commonly the multiplicity of $L(\mu)$ in M is denoted by $[M\colon L(\mu)]$. Clearly, by construction, $[\mathcal{V}(\lambda)\colon L(\lambda)] = 1$. For completeness, let us add that the natural number n is called the Jordan-Hölder length or simply the length of the composition.

In principle, a Jordan-Hölder composition, as we will call the filtration of the form (A.36), is quite easily constructed for Verma modules $\mathcal{V}(\lambda)$ since every element is in general known explicitly in a unique presentation, assumed an ordering for $\mathcal{U}(\mathfrak{n}^-)$ has been fixed. One just has to look for highest weight states in $\mathcal{V}(\lambda)$ and identify the submodules this way.

But what about modules that are not Verma modules, in particular those that are "bigger" than Verma modules like *e.g.* projective covers? Of course, they also possess a Jordan-Hölder composition but this is usually not found that easily. In order to gain better access to these kinds of modules, we have to introduce yet another kind of filtration to the game, the so called *standard filtration*. A standard filtration is a filtration of the form (A.36) but such that the quotients are *Verma modules* $\mathcal{V}(\mu)$ of some weight μ rather than simple modules. Therefore the standard filtration is also sometimes referred to as Verma flag. In contrast to the Jordan-Hölder composition, not every module in the BGG category $\mathscr{O}$ does possess a standard filtration; an obvious counterexample is the simple module $L(\lambda)$ if the associated Verma module $\mathcal{V}(\lambda)$ is not simple. However, as for Jordan-Hölder compositions, the length of the standard filtration and the multiplicity with which the Verma module $\mathcal{V}(\mu)$ appears is unique, if the standard filtration exists. In order to distinguish this multiplicity from the Jordan-Hölder multiplicity, it is commonly denoted by round brackets rather than square brackets, *i.e.* the multiplicity of $\mathcal{V}(\mu)$ in the standard filtration of some module M is denoted by $(M:\mathcal{V}(\lambda))$.

After these remarks, the path to gain an impression what the projective cover $\mathcal{P}(\lambda)$ of a Verma module $\mathcal{V}(\lambda)$, and thus of the simple modules $\mathcal{L}(\lambda)$, looks like follows two steps. First, it can be shown [117] that even though not every module possesses a standard filtration, every *projective* modules does. Hence in particular the projective cover $\mathcal{P}(\lambda)$ has a standard filtration. The second step is the impressive result of [32], which is nowadays referred to as *BGG reciprocity* or sometimes *BGG duality*. It states that the two kinds of multiplicities are connected by

$$(\mathcal{P}(\lambda)\colon \mathcal{V}(\mu)) = [\mathcal{V}(\mu) : L(\lambda)]\ . \tag{A.37}$$

As we have argued, the right hand side of this equation is in principle quite accessible. Hence, we can determine the multiplicities of the Verma modules appearing in the standard filtration of $\mathcal{P}(\lambda)$ and gain a handle on its structure.

Let us again consider our favorite example of $\mathfrak{sl}(2)$. There are three nontrivial multiplicities in the Jordan-Hölder composition,

$$[\mathcal{V}(\lambda) : L(\lambda)] = [\mathcal{V}(\lambda) : L(-\lambda-2)] = [\mathcal{V}(-\lambda-2) : L(-\lambda-2)] = 1\,, \tag{A.38}$$

where $\lambda \geq 0$. Applying the BGG reciprocity (A.37), the standard filtration of $\mathcal{P}(\lambda)$ only has one non-vanishing multiplicity,

$$(\mathcal{P}(\lambda)\colon \mathcal{V}(\lambda)) = 1\,, \tag{A.39}$$

which implies that the projective cover of $\mathcal{P}(\lambda)$ of $\mathcal{V}(\lambda)$, and hence $L(\lambda)$, is exactly the Verma module, $\mathcal{P}(\lambda) = \mathcal{V}(\lambda)$ if the weight λ is positive. For the linked weight $-\lambda-2$, the Verma module is itself simple, $\mathcal{V}(-\lambda-2) = L(-\lambda-2)$. The BGG reciprocity with (A.38) gives us two non-vanishing multiplicities in the standard filtration of $\mathcal{P}(-\lambda-2)$,

$$(\mathcal{P}(-\lambda-2)\colon \mathcal{V}(\lambda)) = (\mathcal{P}(-\lambda-2)\colon \mathcal{V}(-\lambda-2)) = 1\,. \tag{A.40}$$

It follows that we must have $\mathcal{P}(-\lambda-2)/\mathcal{V}(\lambda) = \mathcal{V}(-\lambda-2)$. The standard filtration is therefore

$$0 \subset \mathcal{V}(\lambda) \subset \mathcal{P}(-\lambda-2)\,. \tag{A.41}$$

This allows us to construct the projective cover which is schematically illustrated in figure A.1. Note that the projective cover is not a highest weight module.

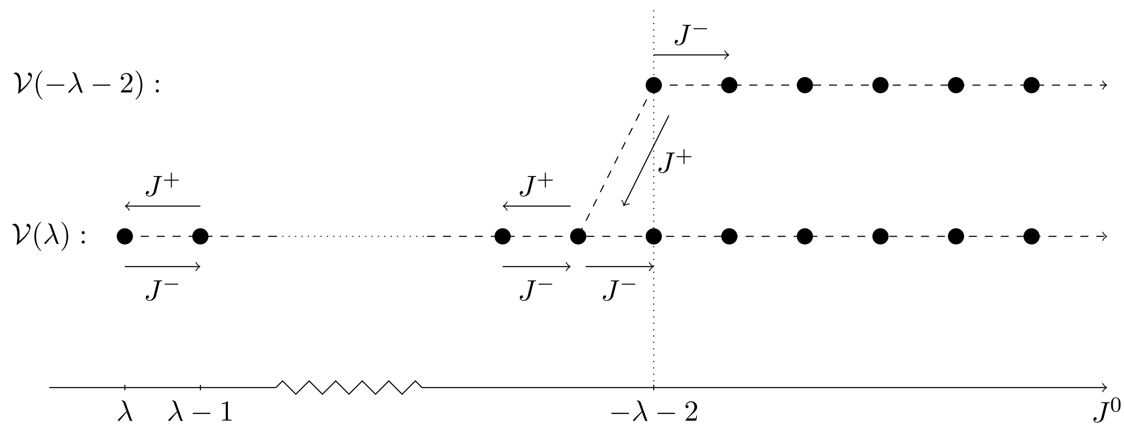


Figure A.1: *Illustration of the projective cover $\mathcal{P}(-\lambda-2)$ for positive weights λ of the simple module $L(-\lambda-2) = \mathcal{V}(-\lambda-2)$ for $\mathfrak{g} = \mathfrak{sl}(2)$. Each dot corresponds to an element of the module. They are arranged according to their weights. Note that the weight spaces $\mathcal{P}(-\lambda-2)_\mu$ for $\mu \leq -\lambda-2$ are two-dimensional.*

It is instructive to write down the Jordan-Hölder composition of $\mathcal{P}(-\lambda-2)$ and compare it to the standard filtration (A.41). It is given by

$$0 = L(-\lambda-2) = \mathcal{V}(-\lambda-2) \subset \mathcal{V}(\lambda) \subset \mathcal{P}(-\lambda-2)\,. \tag{A.42}$$

Note that the Jordan-Hölder length is three while the length of the standard filtration is two. The multiplicities in the Jordan-Hölder composition of the projective cover are

$$[\mathcal{P}(-\lambda-2) : L(-\lambda-2)] = 2\,, \qquad [\mathcal{P}(-\lambda-2) : L(\lambda)] = 1\,. \tag{A.43}$$

This concludes our discussion of Lie algebras and their representations in the terms of module theory. We hope that the reader could gain a good impression of important concepts in the well-known setting of Lie algebras. They will repeatingly appear later in the context of Lie superalgebra. Let us close by saying that we just scratched the surface of the broad subject of the BGG category. The reader interested in more details is referred to [117].

A Construction of Projective Lie Superalgebra Representations

Unfortunately, at the time of writing, the literature on Lie superalgebra representations is mainly concerned with finite-dimensional representations since even in that case, the representation theory of Lie superalgebras is quite involved. Since our main focus will be on infinite-dimensional representations, we will later adopt a working definition of a sensible subcategory of $\mathcal{U}(\mathfrak{g})$-modules in section 2.2.2. However, in the present section, a construction of projective $\mathcal{U}(\mathfrak{g})$-modules from projective $\mathcal{U}(\mathfrak{g}^{(0)})$-modules is presented which is independent of the dimension of the modules involved [195]. This is the most common construction of projective $\mathcal{U}(\mathfrak{g})$-modules and a central element in the analysis of finite-dimensional Lie superalgebra representations. Note that this construction does not imply that there are enough projectives in the sense that any $\mathcal{U}(\mathfrak{g})$-modules has a projective cover. Furthermore, the projective modules obtained this way are typically not indecomposable. Let us state that this construction of projective modules will actually not be important in the remainder of this work, however, it serves as an instructive example to study the property of being projective and the construction of $\mathcal{U}(\mathfrak{g})$-modules by using the functor of induction.

First, let us start by noticing an important fact on induced Lie superalgebra representation. Let $\mathcal{M}$ be a $\mathcal{U}(\mathfrak{g})$-module and $\mathcal{A}$ a $\mathcal{U}(\mathfrak{g}^{(0)})$-module. Then each element in $\mathrm{Hom}_{\mathfrak{g}^{(0)}}(\mathcal{A},\mathcal{M})$ is associated to an element in $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{(0)}}\mathcal{A},\mathcal{M})$ and vice versa. We will denote the induced homomorphism by $-^\sharp$. Depending on which kind of homomorphism ϕ we have, the induced homomorphism $\phi^\sharp$ is given by

$$\begin{aligned} &\phi\in\mathrm{Hom}_{\mathfrak{g}^{(0)}}(\mathcal{A},\mathcal{M})\Rightarrow\phi^\sharp\in\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{(0)}}\mathcal{A},\mathcal{M}\right): \quad \phi^\sharp\,(x\otimes v)\equiv x\phi(v)\,,\\ &\phi\in\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{(0)}}\mathcal{A},\mathcal{M}\right)\Rightarrow\phi^\sharp\in\mathrm{Hom}_{\mathfrak{g}^{(0)}}(\mathcal{A},\mathcal{M}): \quad \phi^\sharp(v)\equiv\phi(1\otimes v)\,. \end{aligned} \tag{B.1}$$

It is not difficult to see that $-^\sharp$ is an involution, *i.e.* it squares to the identify. If $\phi\in\mathrm{Hom}_{\mathfrak{g}^{(0)}}(\mathcal{A},\mathcal{M})$, we have

$$(\phi^\sharp)^\sharp(v)=\phi^\sharp(1\otimes v)=1\,\phi(v)=\phi(v)\,. \tag{B.2}$$

Similarly, if we have $\phi \in \mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{(0)}}\mathcal{A},\mathcal{M}\right)$, we obtain

$$(\phi^\sharp)^\sharp(x\otimes v) = x\phi^\sharp(v) = x\phi(1\otimes v) = \phi(x\otimes v)\,, \tag{B.3}$$

where in the last step we used that ϕ is a homomorphism of $\mathcal{U}(\mathfrak{g})$-modules. Therefore $-^\sharp$ is an isomorphism and we find [195]

$$\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{(0)}}\mathcal{A},\mathcal{M}\right) \quad\cong\quad \mathrm{Hom}_{\mathfrak{g}^{(0)}}(\mathcal{A},\mathcal{M})\,. \tag{B.4}$$

This essential fact on induced Lie superalgebra representations will turn out to be important to prove that $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{(0)}}\mathcal{A}$ is projective whenever $\mathcal{A}$ is. So let $\mathcal{A}$ be a projective $\mathcal{U}(\mathfrak{g}^{(0)})$-module and suppose one is given two $\mathcal{U}(\mathfrak{g})$-modules $\mathcal{M}$ and $\mathcal{N}$ such that the following diagram holds:

$$\begin{array}{ccccc} & & \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{(0)}}\mathcal{A} & & \\ & & \big\downarrow f & & \\ \mathcal{M} & \xrightarrow[g]{} & \mathcal{N} & \longrightarrow & 0 \end{array} \quad .$$

In other words, we are given an epimorphism $\mathcal{M}\to\mathcal{N}$. We need to show that there exists an homomorphism $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{(0)}}\mathcal{A}\to\mathcal{M}$ such that the above diagram is commutative. In order to do this, we use the isomorphism $-^\sharp$. Using that $\mathcal{A}$ is projective, we obtain the following commutative diagram of $\mathcal{U}(\mathfrak{g}^{(0)})$-modules:

$$\begin{array}{ccccc} & & \mathcal{A} & & \\ & \overset{h}{\swarrow} & \big\downarrow f^\sharp & & \\ \mathcal{M} & \xrightarrow[g]{} & \mathcal{N} & \longrightarrow & 0 \end{array} \quad .$$

Thus this implies the existence of a homomorphism $h^\sharp : \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{(0)}}\mathcal{A}\to\mathcal{M}$ of $\mathcal{U}(\mathfrak{g})$-modules:

$$\begin{array}{ccccc} & & \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{(0)}}\mathcal{A} & & \\ & \overset{h^\sharp}{\swarrow} & \big\downarrow f & & \\ \mathcal{M} & \xrightarrow[g]{} & \mathcal{N} & \longrightarrow & 0 \end{array} \quad .$$

All that is left to prove is the commutativity of the above diagram. Let $x\otimes v\in \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{(0)}}\mathcal{A}$. It is not difficult to see that

$$g\circ h^\sharp(x\otimes v) = g(x\cdot h(v)) = x\cdot(g\circ h)(v) = x\cdot f^\sharp(v) = f(x\otimes v)\,, \tag{B.5}$$

where we used that $\mathcal{A}$ is projective and $-^\sharp$ is an involution. Thus we see that $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}^{(0)}}\mathcal{A}$ is projective as an $\mathcal{U}(\mathfrak{g})$-module.

Bases and Commutator Relations

The Lie superalgebra $\mathfrak{g} = \mathfrak{psl}(2|2)$ can be decomposed as

$$\mathfrak{g} = \mathfrak{g}_{+1} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}_{-1}\,, \tag{C.1}$$

where $\mathfrak{g}^{(0)}$ is the bosonic subalgebra and $\mathfrak{g}^{(1)} = \mathfrak{g}_{+1} \oplus \mathfrak{g}_{-1}$ gives the fermionic generators. The bosonic generators are denoted by K^{ab} with $\mathfrak{so}(4)$-indices a, b and the fermionic generators are denoted by $S^a_\alpha \in \mathfrak{g}_\alpha$ where $\alpha = \pm$. Hence the index α corresponds to the $\mathbb{Z}$-grading ϱ as explained in sect. 2.2.1. For later use, we also define $\varepsilon_{\alpha\beta}$ as

$$\varepsilon_{+-} = -\varepsilon_{-+} = 1\,, \qquad \varepsilon_{++} = \varepsilon_{--} = 0\,. \tag{C.2}$$

In the basis used in [62,178], the commutation relations of the affine version $\widehat{\mathfrak{psl}}(2|2)_k$ read

$$\begin{aligned}
[K^{ab}_m, K^{cd}_n] &= i\left(\delta^{ac}K^{bd}_{m+n} - \delta^{bc}K^{ad}_{m+n} - \delta^{ad}K^{bc}_{m+n} + \delta^{bd}K^{ac}_{m+n}\right) + nk\delta_{n+m}\varepsilon^{abcd}\\
[K^{ab}_m, S^c_{\gamma,n}] &= i\left(\delta^{ac}S^b_{\gamma,m+n} - \delta^{bc}S^a_{\gamma,m+n}\right)\\
[S^a_{\alpha,m}, S^b_{\beta,n}] &= \tfrac{i}{2}\,\varepsilon_{\alpha\beta}\,\varepsilon^{abcd}K^{cd}_{m+n} + nk\varepsilon_{\alpha\beta}\delta^{ab}\delta_{m+n}\ ,
\end{aligned}$$

where indices are raised and lowered with the invariant $\mathfrak{so}(4)$-metric δ^{ab}, so upper and lower indices do not need to be distinguished. We will mainly just write upper indices for readability and impose the summation convention if the same index appears twice as upper (lower) index as well. An appropriate basis change can be made by defining [105]

$$\begin{aligned}
J^0_n &= \tfrac{1}{2}\left(K^{12}_n + K^{34}_n\right) & K^0_n &= \tfrac{1}{2}\left(K^{12}_n - K^{34}_n\right)\\
J^\pm_n &= \tfrac{1}{2}\left(K^{14}_n + K^{23}_n \pm iK^{24}_n \mp iK^{13}_n\right) & K^\pm_n &= \tfrac{1}{2}\left(-K^{14}_n + K^{23}_n \mp iK^{24}_n \mp iK^{13}_n\right)\\
S^{\pm\pm}_{\alpha,n} &= S^1_{\alpha,n} \pm iS^2_{\alpha,n} & S^{\pm\mp}_{\alpha,n} &= S^3_{\alpha,n} \pm iS^4_{\alpha,n}\ ,
\end{aligned}$$

for which the commutation relations are explicitly given by

$$[J^0_m, J^0_n] = -\frac{k}{2}\, m\, \delta_{m+n}\,, \qquad [K^0_m, K^0_n] = \frac{k}{2}\, m\, \delta_{m+n}\,,$$

$$[J^0_m, J^\pm_n] = \pm\, J^\pm_{n+m}\,, \qquad [K^0_m, K^\pm_n] = \pm\, K^\pm_{n+m}\,,$$

$$[J^+_m, J^-_n] = 2J^0_{n+m} - mk\,\delta_{m+n}\,, \qquad [K^+_m, K^-_n] = 2K^0_{n+m} + mk\,\delta_{m+n}\,,$$

$$[J^0_m, S^{\pm\pm}_{\alpha,n}] = \pm\frac{1}{2}\, S^{\pm\pm}_{\alpha,m+n}\,, \qquad [J^0_m, S^{\pm\mp}_{\alpha,n}] = \pm\frac{1}{2}\, S^{\pm\mp}_{\alpha,m+n}\,,$$

$$[K^0_m, S^{\pm\pm}_{\alpha,n}] = \pm\frac{1}{2}\, S^{\pm\pm}_{\alpha,m+n}\,, \qquad [K^0_m, S^{\pm\mp}_{\alpha,n}] = \mp\frac{1}{2}\, S^{\pm\mp}_{\alpha,m+n}\,,$$

$$[J^\pm_m, S^{\mp\mp}_{\alpha,n}] = \pm S^{\pm\mp}_{\alpha,m+n}\,, \qquad [J^\pm_m, S^{\mp\pm}_{\alpha,n}] = \mp S^{\pm\pm}_{\alpha,m+n}\,,$$

$$[K^\pm_m, S^{\mp\mp}_{\alpha,n}] = \pm S^{\mp\pm}_{\alpha,m+n}\,, \qquad [K^\pm_m, S^{\pm\mp}_{\alpha,n}] = \mp S^{\pm\pm}_{\alpha,m+n}\,,$$

$$[S^{\pm\pm}_{\alpha,m}, S^{\pm\mp}_{\beta,n}] = \mp 2\epsilon_{\alpha\beta}\, J^\pm_{n+m}\,, \qquad [S^{\pm\pm}_{\alpha,m}, S^{\mp}_{2\beta,n}] = \pm 2\epsilon_{\alpha\beta}\, K^\pm_{n+m}\,,$$

$$[S^{++}_{\alpha,m}, S^{--}_{\beta,n}] = 2\epsilon_{\alpha\beta}\big(J^0_{n+m} - K^0_{n+m}\big) - 2\, mk\, \epsilon_{\alpha\beta}\, \delta_{m+n}\,,$$

$$[S^{+-}_{\alpha,m}, S^{-+}_{\beta,n}] = 2\epsilon_{\alpha\beta}\big(J^0_{n+m} + K^0_{n+m}\big) - 2\, mk\, \epsilon_{\alpha\beta}\, \delta_{m+n}\,.$$

Of course, the commutation relations of $\mathfrak{psl}(2|2)$ can be extracted from the affine commutation relations by looking at the zero modes only, *i.e.* the horizontal subalgebra. The quadratic Casimir operator of $\mathfrak{psl}(2|2)$ is

$$C_2 = C_2^{\rm bos} + C_2^{\rm fer} \tag{C.3}$$

with

$$C_2^{\rm bos} = -2(J^0)^2 - (J^+J^- + J^-J^+) + 2(K^0)^2 + (K^+K^- + K^-K^+) \tag{C.4}$$

$$C_2^{\rm fer} = \frac{\epsilon^{\alpha\beta}}{2}\sum_{m=1}^{2}\Big(S^+_{m\alpha}S^-_{m\beta} + S^-_{m\alpha}S^+_{m\beta}\Big) = \sum_{m=1}^{2}\big(S^+_{m-}S^-_{m+} + S^-_{m-}S^+_{m+}\big)\ . \tag{C.5}$$

The operator $C_2^{\rm fer}$ is the only bilinear in the fermionic generators that commutes with the bosonic subalgebra $\mathfrak{g}^{(0)}$. Evaluated on a irreducible representation of highest weight $\lambda = (j_1, j_2)$, *e.g.* the Kac module $\mathcal{K}(\lambda)$, it takes the value

$$C_2(\lambda) \equiv -2\, j_1(j_1+1) + 2\, j_2(j_2+1)\,. \tag{C.6}$$

Various OPE Calculations

In this appendix, we give some additional technical details to the calculations performed in this thesis.

D.1 The $\mathcal{N}=2$ Superconformal Structure of the Gauge-fixed Superstring

We argued in section 5.1 that the fields of the $\mathcal{N}=1$ RNS string after gauge-fixing superdiffeomorphism invariance give rise to a topologically twisted $\mathcal{N}=2$ superconformal algebra. More concretely, it was claimed that the fields in (5.1) generate this algebra. This has first been noticed in [16] and it is of major importance in the deduction of the hybrid string [30]. For most OPEs between the fields given in (5.1) it is fairly clear that they fulfill the OPEs in (4.20) with $\chi=\frac{1}{2}$, *e.g.* since j_{BRST} and b are Virasoro primary fields of conformal weight 1 and 2, respectively, it follows that

$$T(z)G^{\pm}(w)\sim\frac{\left(\frac{3}{2}\mp\frac{1}{2}\right)G^{\pm}(w)}{(z-w)^2}+\frac{\partial G^{\pm}(w)}{z-w}\,, \tag{D.1}$$

where $T=T_{\mathcal{N}=1}$, $G^+=j_{\mathrm{BRST}}$ and $G^-=b$, in agreement with (4.20). Here we wrote $T(z)$ instead of $T^{\mathrm{tw}}(z)$ in order not to clutter the notation. It should be understood that we are considering the topologically twisted algebra. It is also not difficult to see that j_{BRST} and b have ghost number ±1, hence

$$J(z)G^{\pm}(w)\sim\frac{\pm G^{\pm}(w)}{z-w}\,. \tag{D.2}$$

The OPE of $T_{\mathcal{N}=1}$ with J_{ghost} can be determined using the energy momentum tensors in (3.93), where we have to keep in mind that b and c give rise to a $\lambda=2$ bc system while η

and ξ define a bc system with $\lambda = 1$. We find

$$\begin{aligned} T_{\mathcal{N}=1}(z)J_{\text{ghost}}(w) &= (-\partial bc - 2\, b\partial c - \eta\partial\xi)(z)(cb + \eta\xi)(w) \\ &\sim \frac{-3+1}{(z-w)^3} + \frac{-2(b(z)c(w)) - (c(z)b(w)) + (\eta(z)\xi(w))}{(z-w)^2} \\ &\qquad\qquad + \frac{(\partial bc) + 2(\partial cb) - (\partial\xi\eta)}{z-w} \\ &= \frac{-2}{(z-w)^3} + \frac{J_{\text{ghost}}(w)}{(z-w)^2} + \frac{\partial J_{\text{ghost}}(w)}{z-w}\,, \end{aligned} \tag{D.3}$$

where we ignored the term T_{matter} and the energy momentum tensor associated with the boson ϕ in $T_{\mathcal{N}=1}$ since they have nonsingular OPEs with the fields appearing in J_{ghost}. The OPE in (D.3) fits with the corresponding $\mathcal{N} = 2$ OPE in (4.20) for $\chi = \frac{1}{2}$ if the central charge is taken to be $\hat{c} = 2$.

The most important OPE completing the $\mathcal{N} = 2$ structure is the one between G^+ and G^-. Since G^+ is a complicated normal ordered product of fields while G^- is simply the b-ghost, it is actually more convenient to check the G^-G^+-OPE rather than the G^+G^--OPE. Using (4.20) with $\chi = \frac{1}{2}$ it is straightforward to see that

$$\begin{aligned} G^-(z)G^+(w) &\sim \frac{\hat{c}}{(z-w)^3} - \frac{J(z)}{(z-w)^2} + \frac{T(z)}{z-w} \\ &= \frac{\hat{c}}{(z-w)^3} - \frac{J(w)}{(z-w)^2} + \frac{T(w) - \partial J(w)}{z-w}\,. \end{aligned} \tag{D.4}$$

Recall that the supercurrents possess fermionic statistics despite their conformal weights being integers. Imposing (5.1), this OPE reads in RNS variables

$$\begin{aligned} b(z)j_{\text{BRST}}(w) &\sim \frac{2}{(z-w)^3} + \frac{-(cb)(w) + (\xi\eta)(w)}{(z-w)^2} \\ &\quad + \frac{T_{\text{matter}}(w) - \frac{3}{2}(\beta\partial\gamma)(w) - \frac{1}{2}(\gamma\partial\beta)(w) - (b\partial c)(w) + \partial(\xi\eta)(w)}{z-w} \\ &= \frac{2}{(z-w)^3} + \frac{-J(w)}{(z-w)^2} + \frac{\overbrace{T_{\text{matter}}(w) + T^{\beta\gamma}(w) + T^{bc}(w)}^{=T_{\mathcal{N}=1}} + \partial\overbrace{(bc + \eta\xi)(w)}^{=-J(w)}}{z-w} \\ &= \frac{2}{(z-w)^3} - \frac{J(w)}{(z-w)^2} + \frac{T(w) - \partial J(w)}{z-w}\,, \end{aligned} \tag{D.5}$$

where we made use of (3.93), in particular that $-(b\partial c) = T^{bc} + \partial(bc)$. This result agrees with the OPE in (D.4). Hence, we showed that the fields in (5.1) indeed generate a topologically twisted $\mathcal{N} = 2$, $\hat{c} = 2$ superconformal algebra as claimed.

D.2 Residues in the Hybrid Formulation

Although the determination of the first order poles in section 7.1.1 is mostly straightforward, in some cases the analysis is somewhat more involved. In this appendix, we explain the calculation of the residue for one representative case, namely (7.11), in order to present the technical tools used to extract the residue and to give an impression of the basic philosophy of our calculation.

Recall that the OPE in (7.11) is

$$\big(e^{i\sigma}\mathcal{T}\big)(z)\big(i\partial\sigma\phi^{\sigma}_{2,1}e^{2\rho+i\sigma+iH}\big)(w)\,. \tag{D.6}$$

Since the $\rho\sigma$-deformed energy momentum tensor $\mathcal{T}(z)$ is a sum of vertex operators, we can consider each summand individually. First, let us check the OPE

$$\big(e^{i\sigma}T^{\mathrm{WZW}}\big)(z)\big(i\partial\sigma\phi^{\sigma}_{2,1}e^{2\rho+i\sigma+iH}\big)(w)\,. \tag{D.7}$$

Using the elementary OPEs

$$T^{\mathrm{WZW}}(z)\phi^{\sigma}_{2,1}(w)=\sum_{l\in\mathbb{Z}}\{L_{-2-l}\phi^{\sigma}_{2,1}\}(z-w)^{l} \tag{D.8}$$

as well as

$$\begin{aligned}\big(e^{i\sigma}\big)(z)\big(i\partial\sigma e^{2\rho+i\sigma+iH}\big)(w)=&\sum_{d\geq 0}(z-w)^{d+1}\frac{1}{d!}\Big(i\partial\sigma(\partial^{d}e^{i\sigma})e^{2\rho+i\sigma+iH}\Big)\\&-\sum_{d\geq 0}(z-w)^{d}\frac{1}{d!}\Big((\partial^{d}e^{i\sigma})e^{2\rho+i\sigma+iH}\Big)\,,\end{aligned} \tag{D.9}$$

we deduce the following OPE

$$\begin{aligned}&\big(e^{i\sigma}T^{\mathrm{WZW}}\big)(z)\big(i\partial\sigma\phi^{\sigma}_{2,1}e^{2\rho+i\sigma+iH}\big)(w)\\&\quad=\sum_{l\in\mathbb{Z},d\geq 0}\Bigg((z-w)^{l+d+1}\frac{1}{d!}\Big(\{L_{-2-l}\phi^{\sigma}_{2,1}\}i\partial\sigma(\partial^{d}e^{i\sigma})e^{2\rho+i\sigma+iH}\Big)\\&\qquad-(z-w)^{l+d}\frac{1}{d!}\Big(\{L_{-2-l}\phi^{\sigma}_{2,1}\}(\partial^{d}e^{i\sigma})e^{2\rho+i\sigma+iH}\Big)\Bigg)\,.\end{aligned} \tag{D.10}$$

In this argument it is important to realise that the $\rho\sigma$-ghosts are free and thus have non-singular OPEs with vertex operators of the PSL(2|2)-WZW-model. The first order pole can be determined by extracting the summand with $l+d+1=-1$ and $l+d=-1$, respectively. This yields

$$\begin{aligned}\operatorname{Res}\left(e^{i\sigma}T^{\mathrm{WZW}}\right)(z)\left(i\partial\sigma\phi^{\sigma}_{2,1}e^{2\rho+i\sigma+iH}\right)(w)\\ =\sum_{d\geq 0}\frac{1}{d!}\Big(\{L_d\phi^{\sigma}_{2,1}\}i\partial\sigma(\partial^d e^{i\sigma})e^{2\rho+i\sigma+iH}\Big)\\ -\frac{1}{d!}\Big(\{L_{d-1}\phi^{\sigma}_{2,1}\}(\partial^d e^{i\sigma})e^{2\rho+i\sigma+iH}\Big)\Big).\end{aligned} \tag{D.11}$$

We can now use the properties of the vertex operator $\phi^{\sigma}_{2,1}$ to truncate the summation. Note that by construction of the ansatz V^{+}, the vertex operator $\phi^{\sigma}_{2,1}$ is an affine ground state and hence is annihilated by all positive modes. Thus the residue simplifies to

$$\begin{aligned}\operatorname{Res}\left(e^{i\sigma}T^{\mathrm{WZW}}\right)(z)\left(i\partial\sigma\phi^{\sigma}_{2,1}e^{2\rho+i\sigma+iH}\right)(w)\\ =-\left(\{L_{-1}\phi^{\sigma}_{2,1}\}e^{2\rho+2i\sigma+iH}\right).\end{aligned} \tag{D.12}$$

Next, we look at the deformation term. Useful elementary OPEs are

$$\begin{aligned}\left(e^{i\sigma}(\partial\rho)^2\right)(z)\left(i\partial\sigma e^{2\rho+i\sigma+iH}\right)(w)\sim\tfrac{-4}{(z-w)^2}\left(e^{2\rho+2i\sigma+iH}\right)(w)\\ +\tfrac{4}{z-w}\left(\partial\rho e^{2\rho+2i\sigma+iH}\right)(w)\,,\end{aligned} \tag{D.13}$$

$$\left(e^{i\sigma}\partial\rho\, i\partial\sigma\right)(z)\left(i\partial\sigma e^{2\rho+i\sigma+iH}\right)(w)\sim 0\,, \tag{D.14}$$

$$\begin{aligned}\left(e^{i\sigma}(i\partial\sigma)^2\right)(z)\left(i\partial\sigma e^{2\rho+i\sigma+iH}\right)(w)\sim\tfrac{1}{(z-w)^2}\left(e^{2\rho+2i\sigma+iH}\right)(w)\\ +\tfrac{2}{z-w}\left(i\partial\sigma e^{2\rho+2i\sigma+iH}\right)(w)\,,\end{aligned} \tag{D.15}$$

$$\left(e^{i\sigma}\partial^2\rho\right)(z)\left(i\partial\sigma e^{2\rho+i\sigma+iH}\right)(w)\sim\tfrac{-2}{(z-w)^2}\left(e^{2\rho+2i\sigma+iH}\right)(w) \tag{D.16}$$

$$\begin{aligned}\left(e^{i\sigma}\, i\partial^2\sigma\right)(z)\left(i\partial\sigma e^{2\rho+i\sigma+iH}\right)(w)\sim\tfrac{-1}{(z-w)^2}\left(e^{2\rho+2i\sigma+iH}\right)(w)\\ -\tfrac{2}{z-w}\left(i\partial\sigma e^{2\rho+2i\sigma+iH}\right)(w)\,.\end{aligned} \tag{D.17}$$

Taking the appropriate sum, we obtain

$$\begin{aligned}\Big(e^{i\sigma}\Big(-\tfrac{1}{2}(\partial\rho+i\partial\sigma)^2+\tfrac{1}{2}\partial^2(\rho+i\sigma)\Big)\Big)(z)\left(i\partial\sigma e^{2\rho+i\sigma+iH}\right)(w)\\ \sim-\tfrac{2}{z-w}\left(\partial(\rho+i\sigma)e^{2\rho+2i\sigma+iH}\right).\end{aligned} \tag{D.18}$$

The residue is now apparent. Combining eq. (D.12) and eq. (D.18), we obtain the full residue of the OPE of interest,

$$\begin{aligned}\operatorname{Res}\left(e^{i\sigma}\mathcal{T}\right)(z)\left(i\partial\sigma\phi^{\sigma}_{2,1}e^{2\rho+i\sigma+iH}\right)(w)\\ =-\left(\{L_{-1}\phi^{\sigma}_{2,1}\}e^{2\rho+2i\sigma+iH}\right)-2\left(\phi^{\sigma}_{2,1}\partial(\rho+i\sigma)e^{2\rho+2i\sigma+iH}\right).\end{aligned} \tag{D.19}$$

This is the residue given in (7.11). The remaining residues can be determined in a similar way.

D.3 OPEs of the $\rho\sigma$ System

In this section we list some OPEs that play an important role when evaluating the physical state constraints in section 7.1.1. We use the fields defined in table D.1 in order to avoid a convoluted notation. Whenever we write the field $\mathcal{Y}^{(m,n)}$, the OPE holds for both cases $\mathcal{Y}^{(m,n)} = Y^{(m,n)}$ and $\mathcal{Y}^{(m,n)} = \tilde{Y}^{(m,n)}$ (cf. table D.1). It is also helpful to keep in mind the Laurent expansions of the normal ordered products

$$\left(Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\right) = \sum_{j\geq 0}(z-w)^j\left(\mathcal{Y}^{(a+c,b+d)}P_j^{(a,b)}\right)(w)\,,$$

$$\left(\tilde{Y}^{(a,b)}(z)\tilde{Y}^{(c,d)}(w)\right) = \sum_{j\geq 0}(z-w)^j\left(\tilde{Y}^{(a+c,b+d)}\tilde{P}_j^{(a,b)}\right)(w)\,.$$

They can be used to bring the OPEs below in the general form (3.23). With this preliminary remarks in mind, we now give the list of helpful OPEs. They have been partially confirmed using [75].

$$Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w) = (z-w)^{bd-ac}\left(Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\right)$$

$$\tilde{Y}^{(a,b)}(z)\tilde{Y}^{(c,d)}(w) = (z-w)^{bd-ac+2}\left(\tilde{Y}^{(a,b)}(z)\tilde{Y}^{(c,d)}(w)\right)$$

$$\begin{aligned}\left(Y^{(a,b)}\right)(z)\left(\partial\rho\mathcal{Y}^{(c,d)}\right)(w) &= \oint_w \frac{dx}{x-w}\Bigg[\frac{-aY^{(a,b)}(z)}{x-z}\mathcal{Y}^{(c,d)}(w) \\ &\qquad + \partial\rho(x)(z-w)^{bb'-aa'}\left(Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\right)\Bigg] \\ &= a(z-w)^{bb'-aa'-1}\left(Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\right) \\ &\qquad + (z-w)^{bb'-aa'}\left(Y^{(a,b)}(z)\partial\rho(w)\mathcal{Y}^{(c,d)}(w)\right)\end{aligned}$$

$$\begin{aligned}\left(Y^{(a,b)}\right)(z)\left(i\partial\sigma\mathcal{Y}^{(c,d)}\right)(w) &= \oint_w \frac{dx}{x-w}\Bigg[\frac{bY^{(a,b)}(z)}{x-z}\mathcal{Y}^{(c,d)}(w) \\ &\qquad + i\partial\sigma(x)(z-w)^{bb'-aa'}\left(Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\right)\Bigg] \\ &= -b(z-w)^{bb'-aa'-1}\left(Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\right) \\ &\qquad + (z-w)^{bb'-aa'}\left(Y^{(a,b)}(z)i\partial\sigma(w)\mathcal{Y}^{(c,d)}(w)\right)\end{aligned}$$

$$\begin{aligned}\Big(Y^{(a,b)}J^2\Big)(z)\Big(\partial\rho\mathcal{Y}^{(c,d)}\Big)(w) &= (z-w)^{bd-ac-3}(2+a(c-d))(c-d)\Big(Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\\ &+(z-w)^{bd-ac-2}\Big(-2(1+a(c-d))\Big(J(z)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\\ &\qquad+(c-d)^2\Big(\partial\rho(w)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\Big)\\ &+(z-w)^{bd-ac-1}\Big[a\Big(J^2(z)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\\ &\qquad-2(c-d)\Big(J(z)\partial\rho(w)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\Big]\\ &+(z-w)^{bd-ac}\Big(J^2(z)\partial\rho(w)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\end{aligned}$$

$$\begin{aligned}\Big(Y^{(a,b)}J^2\Big)(z)\Big(i\partial\sigma Y^{(c,d)}\Big)(w) &= (z-w)^{bd-ac-3}(-2-b(c-d))(c-d)\Big(Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\\ &+(z-w)^{bd-ac-2}\Big[2(1+b(c-d))\Big(J(z)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\\ &\qquad+(c-d)^2\Big(i\partial\sigma(w)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\Big]\\ &+(z-w)^{bd-ac-1}\Big[-b\Big(J^2(z)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\\ &\qquad-2(c-d)\Big(J(z)i\partial\sigma(w)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\Big]\\ &+(z-w)^{bd-ac}\Big(J^2(z)i\partial\sigma(w)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\end{aligned}$$

$$\begin{aligned}\Big(Y^{(a,b)}\partial J\Big)(z)\Big(\partial\rho Y^{(c,d)}\Big)(w) &= (z-w)^{bd-ac-3}\big(2+a(c-d)\big)\Big(Y^{(a,b)}(z)Y^{(c,d)}(w)\Big)\\ &+(z-w)^{bd-ac-2}(c-d)\Big(\partial\rho(w)Y^{(a,b)}(z)Y^{(c,d)}(w)\Big)\\ &+(z-w)^{bd-ac-1}a\Big(\partial J(z)Y^{(a,b)}(z)Y^{(c,d)}(w)\Big)\\ &+(z-w)^{bd-ac}\Big(\partial J(z)\partial\rho(w)Y^{(a,b)}(z)Y^{(c,d)}(w)\Big)\end{aligned}$$

$$\begin{aligned}\Big(Y^{(a,b)}\partial J\Big)(z)\Big(i\partial\sigma Y^{(c,d)}\Big)(w) &= (z-w)^{bd-ac-3}\big(-2-b(c-d)\big)\Big(Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\\ &+(z-w)^{bd-ac-2}(c-d)\Big(i\partial\sigma(w)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\\ &-(z-w)^{bd-ac-1}b\Big(\partial J(z)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\\ &+(z-w)^{bd-ac}\Big(\partial J(z)i\partial\sigma(w)Y^{(a,b)}(z)\mathcal{Y}^{(c,d)}(w)\Big)\end{aligned}$$

These elementary OPEs are the building blocks for the OPEs in section 7.1.1. In particular, let us define $\mathcal{X} \equiv -\frac{1}{2}J^2 + \frac{1}{2}\partial J$, so that the deformed energy momentum tensor reads $\mathcal{T} = T^{\mathrm{WZW}} + \mathcal{X}$. Then we can deduce the following OPEs necessary for evaluating the $G^+ = 0$ constraint in the hybrid formulation.

$Y^{(m,n)}(z)$	$e^{m\rho+in\sigma}(z)$
$\tilde{Y}^{(m,n)}(z)$	$e^{m\rho+in\sigma+iH}(z)$
$J(z)$	$\partial(\rho+i\sigma)(z)$
$P_j^{(m,n)}$	$\frac{1}{j!}(\partial(m\rho+in\sigma)+\partial)^{j-1}\partial(m\rho+in\sigma)$
$\tilde{P}_j^{(m,n)}$	$\frac{1}{j!}(\partial(m\rho+in\sigma+iH)+\partial)^{j-1}\partial(m\rho+in\sigma+iH)$

Table D.1: *Short hand notations for the OPEs listed in appendix D.3. Here $j \geq 1$ and $P_0^{(m,n)}$ and $\tilde{P}_0^{(m,n)}$ are defined to be 1.*

$$\begin{aligned}\left(Y^{(0,1)}\mathcal{X}\right)(z)\mathcal{Y}^{(c,d)}(w) = &(z-w)^{d-2}\left[-\tfrac{1}{2}(c-d-1)(c-d)\right]\left(Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\\ &+(z-w)^{d-1}(c-d)\left(J(z)Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\\ &+(z-w)^{d}\left[\tfrac{1}{2}\left(J^2(z)Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\right.\\ &\qquad\left.-\tfrac{1}{2}\left(\partial J(z)Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\right]\end{aligned}$$

$$\begin{aligned}\left(Y^{(0,1)}\mathcal{X}\right)(z)\left(\partial\rho\mathcal{Y}^{(c,d)}\right)(w) = &(z-w)^{d-3}(d-c+1)\left(Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\\ &+(z-w)^{d-2}\left[-\tfrac{1}{2}(c-d-1)(c-d)\right]\left(\partial\rho(w)Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\\ &+(z-w)^{d-2}\left(J(z)Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\\ &+(z-w)^{d-1}(c-d)\left(J(z)\partial\rho(w)Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\\ &+(z-w)^{d}\left(\mathcal{X}(z)\partial\rho(w)Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\end{aligned}$$

$$\begin{aligned}\left(Y^{(0,1)}\mathcal{X}\right)(z)\left(i\partial\sigma\mathcal{Y}^{(c,d)}\right)(w) = &(z-w)^{d-3}\left[\tfrac{1}{2}(c-d-1)(2+c-d)\right]\left(Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\\ &+(z-w)^{d-2}\left[-\tfrac{1}{2}(c-d-1)(c-d)\right]\left(i\partial\sigma(w)Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\\ &+(z-w)^{d-2}(d-c-1)\left(J(z)Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\\ &+(z-w)^{d-1}\left(\mathcal{X}(z)Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\\ &+(z-w)^{d-1}(c-d)\left(J(z)(i\partial\sigma)(w)Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\\ &+(z-w)^{d}\left(\mathcal{X}(z)(i\partial\sigma)(w)Y^{(0,1)}(z)\mathcal{Y}^{(c,d)}(w)\right)\end{aligned}$$

Note that for $c = 2$ and $d = 1$ the singular part of the last OPE becomes

$$\left(Y^{(0,1)}\mathcal{X}\right)(z)\left(i\partial\sigma\tilde{Y}^{(2,1)}\right)(w) \sim -2\,(z-w)^{-1}\left(J(z)Y^{(0,1)}(z)\tilde{Y}^{(c,d)}(w)\right) \tag{D.20}$$

reproducing our result in (D.18).

Some Coefficient Functions $\mathcal{A}_n(x, y)$

In this appendix we write the q-expansion of $Z^{\text{indep}}(x, y|q)$ as

$$Z^{\text{indep}}(x,y|q) = \sum_{n\in\mathbb{N}} \mathcal{A}_n(x,y) q^n = \mathcal{A}_0(x,y) + \sum_{n\in\mathbb{N}\neq 0} K_\lambda(x,y) R(x,y) q^n \,. \tag{E.1}$$

The functions $R(x, y)$ are polynomial functions of the form

$$R(x,y) = \sum_{r,s=0}^{4n} (A_n)_{rs}\, x^{\frac{r}{2}-n} y^{\frac{s}{2}-n} \,, \tag{E.2}$$

where the A_n are $(4n+1)\times(4n+1)$ matrices. These matrices tell us the multiplicities of the various Kac-modules at the different mass levels. The first matrices are given by

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} , \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 4 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$A_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 4 & 0 & 8 & 0 & 4 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 10 & 0 & 10 & 0 & 4 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 8 & 0 & 13 & 0 & 8 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 4 & 0 & 10 & 0 & 10 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 4 & 0 & 8 & 0 & 4 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$A_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 5 & 0 & 10 & 0 & 5 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 16 & 0 & 16 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 5 & 0 & 20 & 0 & 30 & 0 & 20 & 0 & 5 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 16 & 0 & 36 & 0 & 36 & 0 & 16 & 0 & 4 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 10 & 0 & 30 & 0 & 44 & 0 & 30 & 0 & 10 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 4 & 0 & 16 & 0 & 36 & 0 & 36 & 0 & 16 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 5 & 0 & 20 & 0 & 30 & 0 & 20 & 0 & 5 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 16 & 0 & 16 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 5 & 0 & 10 & 0 & 5 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$A_5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 5 & 0 & 10 & 0 & 5 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 18 & 0 & 18 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 24 & 0 & 39 & 0 & 24 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 28 & 0 & 60 & 0 & 60 & 0 & 28 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 5 & 0 & 24 & 0 & 68 & 0 & 101 & 0 & 68 & 0 & 24 & 0 & 5 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 18 & 0 & 60 & 0 & 120 & 0 & 120 & 0 & 60 & 0 & 18 & 0 & 4 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 10 & 0 & 39 & 0 & 101 & 0 & 144 & 0 & 101 & 0 & 39 & 0 & 10 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 4 & 0 & 18 & 0 & 60 & 0 & 120 & 0 & 120 & 0 & 60 & 0 & 18 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 5 & 0 & 24 & 0 & 68 & 0 & 101 & 0 & 68 & 0 & 24 & 0 & 5 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 28 & 0 & 60 & 0 & 60 & 0 & 28 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 24 & 0 & 39 & 0 & 24 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 18 & 0 & 18 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 5 & 0 & 10 & 0 & 5 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$A_6 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 5 & 0 & 10 & 0 & 5 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 18 & 0 & 18 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 25 & 0 & 41 & 0 & 25 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 30 & 0 & 68 & 0 & 68 & 0 & 30 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 30 & 0 & 92 & 0 & 134 & 0 & 92 & 0 & 30 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 30 & 0 & 98 & 0 & 200 & 0 & 200 & 0 & 98 & 0 & 30 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 5 & 0 & 25 & 0 & 92 & 0 & 230 & 0 & 317 & 0 & 230 & 0 & 92 & 0 & 25 & 0 & 5 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 18 & 0 & 68 & 0 & 200 & 0 & 368 & 0 & 368 & 0 & 200 & 0 & 68 & 0 & 18 & 0 & 4 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 10 & 0 & 41 & 0 & 134 & 0 & 317 & 0 & 429 & 0 & 317 & 0 & 134 & 0 & 41 & 0 & 10 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 4 & 0 & 18 & 0 & 68 & 0 & 200 & 0 & 368 & 0 & 368 & 0 & 200 & 0 & 68 & 0 & 18 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 5 & 0 & 25 & 0 & 92 & 0 & 230 & 0 & 317 & 0 & 230 & 0 & 92 & 0 & 25 & 0 & 5 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 30 & 0 & 98 & 0 & 200 & 0 & 200 & 0 & 98 & 0 & 30 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 30 & 0 & 92 & 0 & 134 & 0 & 92 & 0 & 30 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 30 & 0 & 68 & 0 & 68 & 0 & 30 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 25 & 0 & 41 & 0 & 25 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 18 & 0 & 18 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 5 & 0 & 10 & 0 & 5 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Bibliography

[1] I. Adam, P. A. Grassi, L. Mazzucato, Y. Oz, and S. Yankielowicz, *Non-Critical Pure Spinor Superstrings*, JHEP **03** (2007), 091 [arXiv:hep-th/0605118].

[2] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Large N Field Theories, String Theory and Gravity*, Phys.Rept. **323** (2000), 183 [arXiv:hep-th/9905111].

[3] S. K. Ashok, R. Benichou, and J. Troost, *Asymptotic Symmetries of String Theory on* $\mathrm{AdS}_3 \times \mathrm{S}^3$ *with Ramond-Ramond Fluxes*, JHEP **0910** (2009), 051 [arXiv:0907.1242].

[4] ———, *Conformal Current Algebra in Two Dimensions*, JHEP **0906** (2009), 017 [arXiv:0903.4277].

[5] ATLAS Collaboration, *Combined Search for the Standard Model Higgs Boson in pp Collisions at* $\sqrt{s} = 7$ *TeV with the ATLAS Detector*, Phys.Rev. **D86** (2012), 032003 [arXiv:1207.0319].

[6] M. Banados, C. Teitelboim, and J. Zanelli, *The Black Hole in Three-Dimensional Space-Time*, Phys.Rev.Lett. **69** (1992), 1849 [arXiv:hep-th/9204099].

[7] T. Banks and L. J. Dixon, *Constraints on String Vacua with Space-Time Supersymmetry*, Nucl.Phys. **B307** (1988), 93.

[8] T. Banks, L. J. Dixon, D. Friedan, and E. J. Martinec, *Phenomenology and Conformal Field Theory or Can String Theory Predict the Weak Mixing Angle?*, Nucl.Phys. **B299** (1988), 613.

[9] I. Bars, *Free Fields and New Cosets of Current Algebras*, Phys.Lett. **B255** (1991), 353.

[10] I. Bars and D. Nemeschansky, *String Propagation in Backgrounds with Curved Space-Time*, Nucl.Phys. **B348** (1991), 89.

[11] I. Batalin and G. Vilkovisky, *Relativistic S Matrix of Dynamical Systems with Boson and Fermion Constraints*, Phys.Lett. **B69** (1977), 309.

[12] O. A. Bedoya and N. Berkovits, *GGI Lectures on the Pure Spinor Formalism of the Superstring*, Oct 2009 [arXiv:0910.2254].

[13] A. Belavin, A. M. Polyakov, and A. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, Nucl.Phys. **B241** (1984), 333.

[14] F. Berezin and V. Tolstoi, *The Group With Grassmann Structure* UOSP$(1, 2)$, Commun.Math.Phys. **78** (1981), 409.

[15] N. Berkovits, *Covariant Quantization of the Green-Schwarz Superstring in a Calabi-Yau Background*, Nucl. Phys. **B431** (1994), 258 [arXiv:hep-th/9404162].

[16] ———, *The Ten-Dimensional Green-Schwarz Superstring is a Twisted Neveu-Schwarz-Ramond String*, Nucl.Phys. **B420** (1994), 332 [arXiv:hep-th/9308129].

[17] ———, *A New Description of the Superstring*, Rio de Janeiro 1995: Particles and Fields, 1996, p. 390.

[18] ———, *Quantization of the Superstring in Ramond-Ramond Backgrounds*, Class. Quant. Grav. **17** (2000), 971 [arXiv:hep-th/9910251].

[19] ———, *Quantization of the Type II Superstring in a Curved Six-Dimensional Background*, Nucl. Phys. **B565** (2000), 333 [arXiv:hep-th/9908041].

[20] ———, *Super-Poincare Covariant Quantization of the Superstring*, JHEP **04** (2000), 018 [arXiv:hep-th/0001035].

[21] ———, *ICTP Lectures on Covariant Quantization of the Superstring*, Sep 2002 [arXiv:hep-th/0209059].

[22] ———, *Multiloop Amplitudes and Vanishing Theorems Using the Pure Spinor Formalism for the Superstring*, JHEP **09** (2004), 047 [arXiv:hep-th/0406055].

[23] ———, *Pure Spinor Formalism as an $\mathcal{N} = 2$ Topological String*, JHEP **10** (2005), 089 [arXiv:hep-th/0509120].

[24] ———, *Explaining the Pure Spinor Formalism for the Superstring*, JHEP **01** (2008), 065 [arXiv:0712.0324].

[25] N. Berkovits and C. R. Mafra, *Equivalence of Two-Loop Superstring Amplitudes in the Pure Spinor and RNS Formalisms*, Phys.Rev.Lett. **96** (2006), 011602 [arXiv:hep-th/0509234].

[26] N. Berkovits and D. Z. Marchioro, *Relating the Green-Schwarz and Pure Spinor Formalisms for the Superstring*, JHEP **0501** (2005), 018 [arXiv:hep-th/0412198].

[27] N. Berkovits and N. Nekrasov, *Multiloop Superstring Amplitudes from Non-Minimal Pure Spinor Formalism*, JHEP **0612** (2006), 029 [arXiv:hep-th/0609012].

[28] N. Berkovits and C. Vafa, *On the Uniqueness of String Theory*, Mod.Phys.Lett. **A9** (1994), 653 [arXiv:hep-th/9310170].

[29] ———, $\mathcal{N} = 4$ *Topological Strings*, Nucl.Phys. **B433** (1995), 123 [arXiv:hep-th/9407190].

[30] N. Berkovits, C. Vafa, and E. Witten, *Conformal Field Theory of AdS Background with Ramond-Ramond Flux*, JHEP **9903** (1999), 18 [arXiv:hep-th/9902098].

[31] D. Bernard, *(Perturbed) Conformal Field Theory Applied to 2-D Disordered Systems: An Introduction*, Cargese 1995: Low-dimensional applications of quantum field theory, 1995, p. 19.

[32] J. H. Bernstein, I. M. Gel'fand, and S. I. Gel'fand, *Category of* $\mathfrak{g}$*-Modules*, Funkts. Anal. Prilozh. **10** (1976), no. 2, 1.

[33] M. Bershadsky, S. Zhukov, and A. Vaintrob, PSL$(n|n)$ *Sigma Model as a Conformal Field Theory*, Nucl. Phys. **B559** (1999), 205 [arXiv:hep-th/9902180].

[34] M. Bhaseen, J. Caux, I. Kogan, and A. Tsvelik, *Disordered Dirac Fermions: The Marriage of Three Different Approaches*, Nucl.Phys. **B618** (2001), 465 [arXiv:cond-mat/0012240].

[35] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel, et al., $\mathcal{W}$*-Algebras with Two and Three Generators*, Nucl.Phys. **B361** (1991), 255.

[36] R. Blumenhagen and E. Plauschinn, *Introduction to Conformal Field Theory*, Lecture Notes Physics, vol. 779, Springer Verlag, 2009.

[37] R. E. Borcherds, *Vertex Algebras, Kac-Moody Algebras, and the Monster*, Proc.Nat.Acad.Sci. **83** (1986), 3068.

[38] P. Bouwknegt and K. Schoutens, $\mathcal{W}$ *Symmetry in Conformal Field Theory*, Phys.Rept. **223** (1993), 183 [arXiv:hep-th/9210010].

[39] ———, $\mathcal{W}$ *Symmetry*, Adv.Ser.Math.Phys. **22** (1995), 1.

[40] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, Commun.Math.Phys. **104** (1986), 207.

[41] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, *Vacuum Configurations for Superstrings*, Nucl.Phys. **B258** (1985), 46.

[42] C. A. Cardona and C. A. Nunez, *Three-Point Functions in Superstring Theory on* $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$, JHEP **06** (2009), 009 [arXiv:0903.2001].

[43] J. Cardy, *Conformal Field Theory and Statistical Mechanics*, Jul 2008 [arXiv:0807.3472].

[44] O. Chandia, I. Linch, William D., and B. C. Vallilo, *Compactification of the Heterotic Pure Spinor Superstring I*, JHEP **0910** (2009), 060 [arXiv:0907.2247].

[45] O. Chandia, W. D. Linch, and B. Carlini Vallilo, *Compactification of the Heterotic Pure Spinor Superstring II*, JHEP **1110** (2011), 098 [arXiv:1108.3555].

[46] CMS Collaboration, *Search for a Fermiophobic Higgs Boson in pp Collisions at* $\sqrt{s} = 7$ *TeV*, (2012) [arXiv:1207.1130v1].

[47] S. R. Coleman, *The Quantum Sine-Gordon Equation as the Massive Thirring Model*, Phys.Rev. **D11** (1975), 2088.

[48] J. Conway and S. Norton, *Monstrous Moonshine*, Bull.London Math.Soc. **11** (1979), no. 308.

[49] J. F. Cornwell, *Group Theory in Physics*, Acad. Press, San Diego ; London [u.a.], 1997.

[50] L. Corwin, Y. Ne'eman, and S. Sternberg, *Graded Lie Algebras in Mathematics and Physics (Bose-Fermi Symmetry)*, Rev.Mod.Phys. **47** (1975), 573.

[51] T. Creutzig, *Branes in Supergroups*, (2009) [arXiv:0908.1816], Ph.D. Thesis.

[52] A. Dabholkar and A. Pakman, *Exact Chiral Ring of* AdS(3)/CFT(2), Adv. Theor. Math. Phys. **13** (2009), 409 [arXiv:hep-th/0703022].

[53] J. Dai, R. Leigh, and J. Polchinski, *New Connections Between String Theories*, Mod.Phys.Lett. **A4** (1989), 2073.

[54] J. de Boer, *Six-Dimensional Supergravity on* $\mathrm{S}^3 \times \mathrm{AdS}_3$ *and 2d Conformal Field Theory*, Nucl. Phys. **B548** (1999), 139 [arXiv:hep-th/9806104].

[55] J. de Boer, J. Manschot, K. Papadodimas, and E. Verlinde, *The Chiral Ring of* $\mathrm{AdS}_3/\mathrm{CFT}_2$ *and the Attractor Mechanism*, JHEP **03** (2009), 030 [arXiv:0809.0507].

[56] J. de Boer, H. Ooguri, H. Robins, and J. Tannenhauser, *String Theory on* AdS_3, JHEP **9812** (1998), 026 [arXiv:hep-th/9812046].

[57] S. Deger, A. Kaya, E. Sezgin, and P. Sundell, *Spectrum of* $D = 6$, $N = 4b$ *Supergravity on* $\mathrm{AdS}_3 \times \mathrm{S}^3$, Nucl. Phys. **B536** (1998), 110 [arXiv:hep-th/9804166].

[58] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*, Springer New York, USA:, 1997.

[59] R. Dijkgraaf, J. M. Maldacena, G. W. Moore, and E. P. Verlinde, *A Black Hole Farey Tail*, (2000) [arXiv:hep-th/0005003].

[60] L. J. Dixon, M. E. Peskin, and J. D. Lykken, $\mathcal{N} = 2$ *Superconformal Symmetry and SO(2,1) Current Algebra*, Nucl.Phys. **B325** (1989), 329.

[61] L. Dolan, *Vertex Operators for* AdS_3 *with Ramond Background*, J.Math.Phys. **42** (2001), 2978 [arXiv:hep-th/0010219].

[62] L. Dolan and E. Witten, *Vertex Operators for* AdS(3) *Background with Ramond-Ramond Flux*, JHEP **11** (1999), 003 [arXiv:hep-th/9910205].

[63] T. Eguchi and A. Taormina, *Character Formulas For The* $\mathcal{N} = 4$ *Superconformal Algebra*, Phys.Lett. **B200** (1988), 315.

[64] J. Erlich, *Recent Results in AdS/QCD*, PoS CONFINEMENT8, 2008, p. 032.

[65] J. M. Evans, M. R. Gaberdiel, and M. J. Perry, *The No Ghost Theorem for* AdS(3) *and the Stringy Exclusion Principle*, Nucl.Phys. **B535** (1998), 152 [arXiv:hep-th/9806024].

[66] B. L. Feigin and D. B. Fuchs, *Verma Modules over the Virasoro Algebra*, Functional Analysis and Its Applications **17** (1984), no. 3, 241.

[67] E. Fradkin and G. Vilkovisky, *Conformal Off Mass Shell Extension and Elimination of Conformal Anomalies in Quantum Gravity*, Phys.Lett. **B73** (1978), 209.

[68] L. Frappat, P. Sorba, and A. Sciarrino, *Dictionary on Lie Superalgebras*, (1996) [arXiv:hep-th/9607161].

[69] M. Freeman and D. I. Olive, *BRS Cohomology in String Theory and the No Ghost Theorem*, Phys.Lett. **B175** (1986), 151.

[70] I. Frenkel, H. Garland, and G. Zuckerman, *Semiinfinite Cohomology and String Theory*, Proc.Nat.Acad.Sci. **83** (1986), 8442.

[71] D. Friedan, *Introduction to Polyakov's String Theory*, Proc. of Conference: C82-08-02 (Les Houches Sum.School 1982:0839), 1982.

[72] D. Friedan, E. J. Martinec, and S. H. Shenker, *Conformal Invariance, Supersymmetry and String Theory*, Nucl. Phys. **B271** (1986), 93.

[73] J. Fuchs and C. Schweigert, *Symmetries, Lie Algebras and Representations*, 1. publ. ed., Cambridge University Press, Cambridge [u.a.], 1997.

[74] K. Fujikawa, *On the Path Integral of Relativistic Strings*, Phys.Rev. **D25** (1982), 2584.

[75] A. Fujitsu, *ope.math: Operator Product Expansions in Free Field Realizations of Conformal Field Theory*, Comput.Phys.Commun. **79** (1994), 78.

[76] W. Fulton and J. Harris, *Representation Theory*, 9. print. ed., Springer Science + Business Media, New York, NY, 2004.

[77] M. R. Gaberdiel, *unpublished notes.*

[78] ———, *An Introduction to Conformal Field Theory*, Rept.Prog.Phys. **63** (2000), 607 [arXiv:hep-th/9910156].

[79] ———, *An Algebraic Approach to Logarithmic Conformal Field Theory*, Int.J.Mod.Phys. **A18** (2003), 4593 [arXiv:hep-th/0111260].

[80] M. R. Gaberdiel and S. Gerigk, *The Massless String Spectrum on $AdS_3 \times S^3$ from the Supergroup*, JHEP **10** (2011), 045 [arXiv:1107.2660].

[81] M. R. Gaberdiel and P. Goddard, *Axiomatic Conformal Field Theory*, Commun.Math.Phys. **209** (2000), 549 [arXiv:hep-th/9810019].

[82] M. R. Gaberdiel and H. G. Kausch, *A Rational Logarithmic Conformal Field Theory*, Phys.Lett. **B386** (1996), 131 [arXiv:hep-th/9606050].

[83] ———, *A Local Logarithmic Conformal Field Theory*, Nucl.Phys. **B538** (1999), 631 [arXiv:hep-th/9807091].

[84] M. R. Gaberdiel and I. Kirsch, *Worldsheet Correlators in* AdS(3)/CFT(2), JHEP **04** (2007), 050 [arXiv:hep-th/0703001].

[85] M. R. Gaberdiel and A. Neitzke, *Rationality, Quasirationality and Finite $\mathcal{W}$-Algebras*, Commun.Math.Phys. **238** (2003), 305 [arXiv:hep-th/0009235].

[86] M. R. Gaberdiel and I. Runkel, *From Boundary to Bulk in Logarithmic CFT*, (2007) [arXiv:0707.0388].

[87] M. R. Gaberdiel, I. Runkel, and S. Wood, *Fusion Rules and Boundary Conditions in the $c = 0$ Triplet Model*, J. Phys. **A42** (2009), 325403 [arXiv:0905.0916].

[88] T. Gannon, *Moonshine Beyond the Monster*, Cambridge Univ. Press, Cambridge [u.a.], 2006.

[89] K. Gawedzki, *Conformal Field Theory: A Case Study*, Apr 1999 [arXiv:hep-th/9904145].

[90] R. W. Gebert, *Introduction to Vertex Algebras, Borcherds Algebras, and the Monster Lie Algebra*, Int.J.Mod.Phys. **A8** (1993), 5441 [arXiv:hep-th/9308151].

[91] H. Georgi, *Lie algebras in Particle Physics*, 2. ed., Perseus Books, Reading, Mass., 1999.

[92] D. Gepner and E. Witten, *String Theory on Group Manifolds*, Nucl.Phys. **B278** (1986), 493.

[93] S. Gerigk, *String States on* $\mathrm{AdS}_3 \times \mathrm{S}^3$ *from the Supergroup*, Aug 2012 [arXiv:1208.0345].

[94] S. Gerigk and I. Kirsch, *On the Relation between Hybrid and Pure Spinor String Theory*, JHEP **03** (2010), 106 [arXiv:0912.2347].

[95] J. Germoni, *Indecomposable representations of special linear lie superalgebras*, Journal of Algebra **209** (1998), no. 2, 367.

[96] J.-L. Gervais and B. Sakita, *Field Theory Interpretation of Supergauges in Dual Models*, Nucl.Phys. **B34** (1971), 632.

[97] P. H. Ginsparg, *Applied Conformal Field Theory*, (1988) [arXiv:hep-th/9108028].

[98] G. Giribet, A. Pakman, and L. Rastelli, *Spectral Flow in* AdS(3)/CFT(2), JHEP **06** (2008), 013 [arXiv:0712.3046].

[99] A. Giveon, D. Kutasov, and N. Seiberg, *Comments on String Theory on* AdS(3), Adv. Theor. Math. Phys. **2** (1998), 733 [arXiv:hep-th/9806194].

[100] F. Gliozzi, J. Scherk, and D. I. Olive, *Supergravity and the Spinor Dual Model*, Phys.Lett. **B65** (1976), 282.

[101] ———, *Supersymmetry, Supergravity Theories and the Dual Spinor Model*, Nucl.Phys. **B122** (1977), 253.

[102] P. Goddard, *Meromorphic Conformal Field Theory*, Infinite dimensional Lie Algebras and Lie Groups (V. G. Kac, ed.), World Scientific, Singapore, New Jersey, Hong Kong, 1989, p. 556.

[103] Y. Golfand and E. Likhtman, *Extension of the Algebra of Poincare Group Generators and Violation of p Invariance*, JETP Lett. **13** (1971), 323.

[104] G. Gotz, T. Quella, and V. Schomerus, *Tensor Products of* $\mathfrak{psl}(2|2)$ *Representations*, (2005) [arXiv:hep-th/0506072].

[105] ———, *The WZNW Model on* $\mathrm{PSU}(1,1|2)$, JHEP **03** (2007), 003 [arXiv:hep-th/0610070].

[106] P. A. Grassi and N. Wyllard, *Lower-Dimensional Pure Spinor Superstrings*, JHEP **12** (2005), 007 [arXiv:hep-th/0509140].

[107] M. B. Green, J. Schwarz, and E. Witten, *Superstring Theory. Vol. 1: Introduction*, Cambridge University Press, 1987.

[108] ———, *Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies and Phenomenology*, Cambridge University Press, 1987.

[109] M. B. Green and J. H. Schwarz, *Covariant Description of Superstrings*, Phys.Lett. **B136** (1984), 367.

[110] B. R. Greene, *String Theory on Calabi-Yau Manifolds*, Jun 1996 [arXiv:hep-th/9702155], Lectures given at TASI96.

[111] S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge Theory Correlators from Noncritical String Theory*, Phys.Lett. **B428** (1998), 105 [arXiv:hep-th/9802109].

[112] S. A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, Class.Quant.Grav. **26** (2009), 224002 [arXiv:0903.3246].

[113] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[114] M. Henningson, S. Hwang, P. Roberts, and B. Sundborg, *Modular Invariance of* SU(1, 1) *Strings*, Phys.Lett. **B267** (1991), 350.

[115] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror Symmetry*, American Mathematical Society, Providence, RI, 2003.

[116] G. T. Horowitz and A. Strominger, *Black Strings and p-Branes*, Nucl.Phys. **B360** (1991), 197.

[117] J. E. Humphreys, *Representations of Semisimple Lie Algebras in the BGG Category $\mathscr{O}$*, Graduate Studies in Mathematics, vol. 94, American Mathematical Society, 2008.

[118] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, 7. ed., Springer, New York ; Heidelberg [u.a.], 1997.

[119] S. Hwang, *Covariant Quantization of the String in Dimensions $D \leq 26$ Using a BRS Formulation*, Phys.Rev. **D28** (1983), 2614.

[120] ———, *No Ghost Theorem for* SU(1, 1) *String Theories*, Nucl.Phys. **B354** (1991), 100.

[121] V. G. Kac, *A Sketch of Lie Superalgebra Theory*, Commun.Math.Phys. **53** (1977), 31.

[122] ———, *Lie Superalgebras*, Advances in Mathematics **26** (1977), no. 1, 8.

[123] ———, *Representations of Classical Lie Superalgebras*, Lecture Notes in Mathematics, vol. 676, Springer Verlag, 1978.

[124] ———, *Vertex Algebras for Beginners*, 2. ed., American Mathematical Society, Providence, RI, 2001.

[125] J. Kappeli, S. Theisen, and P. Vanhove, *Hybrid Formalism and Topological Amplitudes*, (2006) [arXiv:hep-th/0607021].

[126] M. Kato and K. Ogawa, *Covariant Quantization of String Based on BRS Invariance*, Nucl.Phys. **B212** (1983), 443.

[127] H. Kausch, *Extended Conformal Algebras Generated by a Multiplet of Primary Fields*, Phys.Lett. **B259** (1991), 448.

[128] E. Kiritsis, *String Theory in a Nutshell*, Princeton Univ. Press, Princeton, 2007.

[129] V. G. Knizhnik, *Covariant Fermionic Vertex in Superstrings*, Phys. Lett. **B160** (1985), 403.

[130] V. Knizhnik and A. Zamolodchikov, *Current Algebra and Wess-Zumino Model in Two Dimensions*, Nucl.Phys. **B247** (1984), 83.

[131] V. A. Kostelecky, O. Lechtenfeld, W. Lerche, S. Samuel, and S. Watamura, *Conformal Techniques, Bosonization and Tree Level String Amplitudes*, Nucl.Phys. **B288** (1987), 173.

[132] P. Kraus, *Lectures on Black Holes and the* AdS(3)/CFT(2) *Correspondence*, Lect.Notes Phys. **755** (2008), 193 [arXiv:hep-th/0609074].

[133] D. Kutasov, F. Larsen, and R. G. Leigh, *String Theory in Magnetic Monopole Backgrounds*, Nucl. Phys. **B550** (1999), 183 [arXiv:hep-th/9812027].

[134] D. Kutasov and N. Seiberg, *More Comments on String Theory on* AdS(3), JHEP **04** (1999), 008 [arXiv:hep-th/9903219].

[135] R. Leigh, *Dirac-Born-Infeld Action from Dirichlet Sigma Model*, Mod.Phys.Lett. **A4** (1989), 2767.

[136] W. Lerche, C. Vafa, and N. P. Warner, *Addendum to 'Chiral Rings in* $\mathcal{N} = 2$ *Superconformal Theories'*, (1989).

[137] ———, *Chiral Rings in* $\mathcal{N} = 2$ *Superconformal Theories*, Nucl.Phys. **B324** (1989), 427.

[138] D. Lüst and S. Theisen, *Lectures on string theory*, Lecture Notes in Physics, vol. 346, Springer Verlag, 1989.

[139] J. M. Maldacena, *The large N limit of Superconformal Field Theories and Supergravity*, Adv. Theor. Math. Phys. **2** (1998), 231 [arXiv:hep-th/9711200].

[140] J. M. Maldacena and H. Ooguri, *Strings in* AdS(3) *and* SL(2, R) *WZW Model. I*, J. Math. Phys. **42** (2001), 2929 [arXiv:hep-th/0001053].

[141] ———, *Strings in* AdS(3) *and the* SL(2, R) *WZW Model. III: Correlation Functions*, Phys. Rev. **D65** (2002), 106006 [arXiv:hep-th/0111180].

[142] J. M. Maldacena, H. Ooguri, and J. Son, *Strings in* AdS(3) *and the* SL(2, R) *WZW Model. II: Euclidean Black Hole*, J. Math. Phys. **42** (2001), 2961 [arXiv:hep-th/0005183].

[143] J. M. Maldacena and A. Strominger, AdS(3) *Black Holes and a Stringy Exclusion Principle*, JHEP **9812** (1998), 005 [arXiv:hep-th/9804085].

[144] R. R. Metsaev and A. A. Tseytlin, *Type IIB Superstring action in* AdS(5) × S(5) *Background*, Nucl. Phys. **B533** (1998), 109 [arXiv:hep-th/9805028].

[145] A. Neitzke and C. Vafa, *Topological Strings and Their Physical Applications*, Oct 2004 [arXiv:hep-th/0410178].

[146] A. Neveu and J. Schwarz, *Factorizable Dual Model of Pions*, Nucl.Phys. **B31** (1971), 86.

[147] A. Neveu, J. Schwarz, and C. B. Thorn, *Reformulation of the Dual Pion Model*, Phys.Lett. **B35** (1971), 529.

[148] Y. Oz, *The Pure Spinor Formulation of Superstrings*, Class. Quant. Grav. **25** (2008), 214001 [arXiv:0910.1195].

[149] A. Pakman and A. Sever, *Exact* $\mathcal{N} = 4$ *correlators of* AdS(3)/CFT(2), Phys. Lett. **B652** (2007), 60 [arXiv:0704.3040].

[150] J. Park and S.-J. Rey, *Green-Schwarz Superstring on* $\mathrm{AdS}_3 \times \mathrm{S}^3$, JHEP **9901** (1999), 001 [arXiv:hep-th/9812062].

[151] S. Perlmutter et al., *Measurements of* Ω *and* Λ *from 42 high redshift supernovae*, Astrophys.J. **517** (1999), 565–586 [arXiv:astro-ph/9812133].

[152] I. Pesando, *The GS Type IIB Superstring Action on* $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$, JHEP **9902** (1999), 007 [arXiv:hep-th/9809145].

[153] P. Petropoulos, *Comments on* $SU(1,1)$ *String Theory*, Phys.Lett. **B236** (1990), 151.

[154] J. Polchinski, *String Theory. Vol. 1: An Introduction to the Bosonic String*, Cambridge University Press, 1998.

[155] ——, *String Theory. Vol. 2: Superstring Theory and Beyond*, Cambridge University Press, 1998.

[156] J. Polchinski, *Dirichlet Branes and Ramond-Ramond charges*, Phys.Rev.Lett. **75** (1995), 4724 [arXiv:hep-th/9510017].

[157] A. M. Polyakov, *Quantum Geometry of Bosonic Strings*, Phys.Lett. **B103** (1981), 207.

[158] T. Quella and V. Schomerus, *Free Fermion Resolution of Supergroup WZNW Models*, JHEP **09** (2007), 085 [arXiv:0706.0744].

[159] J. Rahmfeld and A. Rajaraman, *The GS String Action on* $\mathrm{AdS}_3 \times \mathrm{S}^3$ *with Ramond-Ramond Charge*, Phys. Rev. **D60** (1999), 064014 [arXiv:hep-th/9809164].

[160] P. Ramond, *Dual Theory for Free Fermions*, Phys.Rev. **D3** (1971), 2415.

[161] N. Read and H. Saleur, *Exact Spectra of Conformal Supersymmetric Nonlinear Sigma Models in Two Dimensions*, Nucl. Phys. **B613** (2001), 409 [arXiv:hep-th/0106124].

[162] L. Rozansky and H. Saleur, *Quantum Field Theory for the Multivariable Alexander- Conway Polynomial*, Nucl. Phys. **B376** (1992), 461.

[163] H. Saleur and V. Schomerus, *On the* SU(2|1) *WZNW Model and its Statistical Mechanics Applications*, Nucl. Phys. **B775** (2007), 312 [arXiv:hep-th/0611147].

[164] J. Scherk and J. H. Schwarz, *Dual Models for Nonhadrons*, Nucl.Phys. **B81** (1974), 118.

[165] M. Scheunert, *The Theory of Lie Superalgebras: an Introduction*, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 1979.

[166] M. Scheunert, W. Nahm, and V. Rittenberg, *Classification of All Simple Graded Lie Algebras Whose Lie Algebra Is Reductive. 2. Construction of the Exceptional Algebras*, J.Math.Phys. **17** (1976), 1640.

[167] V. Schomerus and H. Saleur, *The* GL(1|1) *WZW Model: From Supergeometry to Logarithmic CFT*, Nucl. Phys. **B734** (2006), 221 [arXiv:hep-th/0510032].

[168] M. Schottenloher, *A Mathematical Introduction to Conformal Field Theory*, 2. ed. ed., Springer, Berlin ; Heidelberg, 2008.

[169] N. Seiberg, *Observations on the Moduli Space of Superconformal Field Theories*, Nucl.Phys. **B303** (1988), 286.

[170] A. Sergeev, *The Invariant Polynomials on Simple Lie Superalgebras*, Oct 1998 [arXiv:math/9810111].

[171] A. Sevrin, W. Troost, and A. Van Proeyen, *Superconformal Algebras in Two Dimensions with* $\mathcal{N} = 4$, Phys.Lett. **B208** (1988), 447.

[172] J. Strathdee, *Extended Poincaré Supersymmetry*, Int.J.Mod.Phys. **A2** (1987), 273.

[173] M. Sugiura, *Unitary Representations and Harmonic Analysis*, 2nd ed ed., North-Holland, Amsterdam ; New York, 1990.

[174] L. Susskind, *Dual-Symmetric Theory of Hadrons. 1.*, Nuovo Cim. **A69S10** (1970), 457.

[175] M. Taylor, *Matching of Correlators in* $\mathrm{AdS}_3/\mathrm{CFT}_2$, JHEP **06** (2008), 010 [arXiv:0709.1838].

[176] C. B. Thorn, *A Detailed Study of the Physical State Conditions in Covariantly Quantized String Theories*, Nucl.Phys. **B286** (1987), 61.

[177] D. Tong, *String Theory*, Aug 2009 [arXiv:0908.0333].

[178] J. Troost, *Massless Particles on Supergroups and* $\mathrm{AdS}_3 \times \mathrm{S}^3$ *Supergravity*, JHEP **07** (2011), 042 [arXiv:1102.0153].

[179] D. Volkov and V. Akulov, *Possible Universal Neutrino Interaction*, JETP Lett. **16** (1972), 438.

[180] M. Walton, *Affine Kac-Moody Algebras and the Wess-Zumino-Witten Model*, (1999) [arXiv:hep-th/9911187].

[181] S. Weinberg, *The quantum theory of fields vol. 2: Modern applications*, Cambridge Univ. Press, Cambridge, 1996.

[182] J. Wess and B. Zumino, *Supergauge Transformations in Four-Dimensions*, Nucl.Phys. **B70** (1974), 39.

[183] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton Univ. Pr., Princeton, NJ, 1983.

[184] E. Witten, *Search for a Realistic Kaluza-Klein Theory*, Nucl.Phys. **B186** (1981), 412.

[185] ______, $D = 10$ *Superstring Theory*, Philadelphia 1983, Proceedings, Grand Unification, 1984, p. 395.

[186] ______, *Nonabelian Bosonization in Two Dimensions*, Commun.Math.Phys. **92** (1984), 455.

[187] ______, *Topological Sigma Models*, Commun.Math.Phys. **118** (1988), 411.

[188] ______, *Mirror Manifolds and Topological Field Theory*, Mirror Symmetry I (S. T. Yau, ed.), 1991, p. 121.

[189] ______, *Bound States of Strings and p-Branes*, Nucl.Phys. **B460** (1996), 335 [arXiv:hep-th/9510135].

[190] ______, *Anti-de Sitter Space and Holography*, Adv.Theor.Math.Phys. **2** (1998), 253 [arXiv:hep-th/9802150].

[191] ______, *Three-Dimensional Gravity Revisited*, Jun 2007 [arXiv:0706.3359].

[192] N. Wyllard, *Pure Spinor Superstrings in* $d = 2, 4, 6$, JHEP **11** (2005), 009 [arXiv:hep-th/0509165].

[193] T. Yoneya, *Connection of Dual Models to Electrodynamics and Gravidynamics*, Prog.Theor.Phys. **51** (1974), 1907.

[194] A. B. Zamolodchikov, *Infinite Extra Symmetries in Two-Dimensional Conformal Quantum Field Theory*, Teoret. Mat. Fiz. **65** (1985), no. 3, 347. MR 829902 (87j:81253)

[195] Y. M. Zou, *Categories of Finite-Dimensional Weight Modules over Type I Classical Lie Superalgebras*, J. Algebra **180** (1996), 459.