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Q-capacitor formulation of the $\mathbb{C}P^N$ nonlinear sigma model

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Abstract. A multi-component $\mathbb{C}P^N$ model's scalar electrodynamics is investigated. The model contains Q-balls/shells, which are non-topological compact solitons with time dependency $e^{i\omega t}$. Because of the compacton nature of solutions, Q-shells with another compact Q-ball or Q-shell inside their cavity can exist. Even if compactons do not overlap, they can interact with one another via the electromagnetic field. They look similar to the capacitor in the standard electromagnetism. We focus on the structure of such Q-capacitor with opposite charges.

1. Introduction

Q-balls [1, 2] are *nontopological* solitons of scalar field theory model, which circumvents the Derrick's argument with the time component $e^{i\omega t}$. The soliton carries the Noether charge Q, and the stability is formally expressed via a scaling relation for the energy $E \sim |Q|^\alpha,\ \alpha < 1$.

Compactons have scalar field configurations with finite radius $r_{\rm out}$. For $r > r_{\rm out}$, the scalar field disappears (takes its vacuum value). his feature is guaranteed by the shape of the field potential, which is sharp at its minimum (so-called V-shaped potential). The simplest such potential, seen in [3], has the form $V \sim |\phi|$. Interestingly, when the field is coupled with the electromagnetism, the inner radius emerges, i.e., the scalar field vanishes for $r < r_{\rm in}$, which is called Q-shells [4]. Such shell solutions exhibit the harbor,*i.e.*, they involve the composite of the different constituents such as the black holes or the Q-ball in the interior. It served as *a merkmal* for searching for more complex multi-shell Q-ball solutions.

The $\mathbb{C}P^N$ nonlinear sigma model [5, 6] can be defined in terms of of the $N+1$ dimensional complex vector $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \cdots, \mathcal{Z}_{N+1})$, satisfying the constraint $\mathcal{Z}^\dagger \cdot \mathcal{Z} = 1$ is formally given as

$$\mathcal{L} = \lambda_0 \mathcal{D}^\mu \mathcal{Z}^\dagger \cdot \mathcal{D}_\mu \mathcal{Z}, \quad \mathcal{D}_\mu \mathcal{Z} \equiv \partial_\mu \mathcal{Z} - (\mathcal{Z}^\dagger \cdot \partial_\mu \mathcal{Z})\mathcal{Z} \tag{1}$$

where λ_0 is a dimensional constant. To satisfy the constraint, new fields $u_k \equiv \frac{\mathcal{Z}_k}{\mathcal{Z}_{n+1}}, k = 1, 2, \cdots, N$ are useful, where the entry becomes

$$\mathcal{Z} = \frac{Z_{N+1}}{|Z_{N+1}|} \frac{(u_1, u_2, \cdots, u_N, 1)}{\sqrt{1 + |u_1|^2 + |u_2|^2 + \cdots |u_N|^2}}. \tag{2}$$

Without loss of generality, we fix the phase $\frac{Z_{N+1}}{|Z_{N+1}|} = 1$.Models with a larger number of fields can be parametrized by the principal variable which parametrizes the coset space $SU(N+1)/U(N) \sim \mathbb{C}P^N$, [7]. This parametrization was used successfully in [8, 9, 10, 11]. Sec.2 provides a brief overview of the model defined in terms of such a variable.

In this paper, we concentrate on the composite Q-balls of the U(1) gauged version of the model. In particular, a novel configuration that combines oppositely charged ball-shell solutions is extensively studied. We call it a Q-capacitor because its geometric shape resembles to that of a spherical capacitor with two concentric and oppositely charged plates. The outcomes described in this work are for flat Minkowski spacetime. The extension to the case of curved spacatime is straightforward.

2. The model of the $\mathbb{C}P^N$ compacton

The principal variable X is useful for parameterizing the target space $\mathbb{C}P^N \sim SU(N+1)/U(N)$. The technical aspects of this parameterization were discussed especially in references [7, 8], where the parameterization of group element $g \in SU(N+1)$ in terms of fields $u = (u_1, u_2, \cdots, u_N)$. It can be shown that $X(g) = g^2$ and it reads

$$X = \begin{pmatrix} \mathbb{I}_{N\times N} & 0 \\ 0 & -1 \end{pmatrix} + \frac{2}{1+u^\dagger \cdot u} \begin{pmatrix} -u \otimes u^\dagger & iu \\ iu^\dagger & 1 \end{pmatrix} \tag{3}$$

where $\mathbb{I}_{N\times N}$ is the identity $N \times N$ matrix. The $\mathbb{C}P^N$ model potential is defined by the following Lagrangian:

$$\mathcal{L} = -\frac{M^2}{2}\,\mathrm{Tr}(X^{-1}\partial_\mu X)^2 - \mu^2 V(X), \tag{4}$$

where M, μ are the coupling constants and the potential

$$V(X) = \frac{1}{2}[\mathrm{Tr}(\mathbb{I} - X)]^{\frac{1}{2}} \tag{5}$$

realizes the compact support. The Q-ball or the Q-shell $\mathbb{C}P^{2\ell+1}$ solutions are obtained within the subclass of solutions defined by the ansatz

$$u_m(t, r, \theta, \phi) = \sqrt{\frac{4\pi}{2\ell+1}} f(r) Y_{\ell m}(\theta, \phi) e^{i\omega t}, \quad m = -\ell, -\ell+1, \cdots, \ell-1, \ell. \tag{6}$$

The ansatz reduces a system of Euler Lagrange equations to a single radial equation of the form

$$f'' + \frac{2f'}{r} - \frac{n(n+1)f}{r^2} + \frac{\omega^2 f(1-f^2)}{1+f^2} - \frac{2ff'^2}{1+f^2} - \frac{1}{8}\mathrm{sgn}(f)\sqrt{1+f^2} = 0 \tag{7}$$

We impose the boundary conditions $f(R) = 0$, $f'(R) = 0$ based on the results for the signum-Gordon model (which is a low amplitude limit of this model). This condition permits the energy density to vanish at the compacton border.

$$f(R) = 0, f'(R) = 0, \tag{8}$$

Next we solve the equation in the compact region $D : 0 \leqq r \leqq R$. In fact, we numerically solve the equation (7), and the typical solution of the Q-compacton in the $\mathbb{C}P^3$ model is shown in Fig.1.

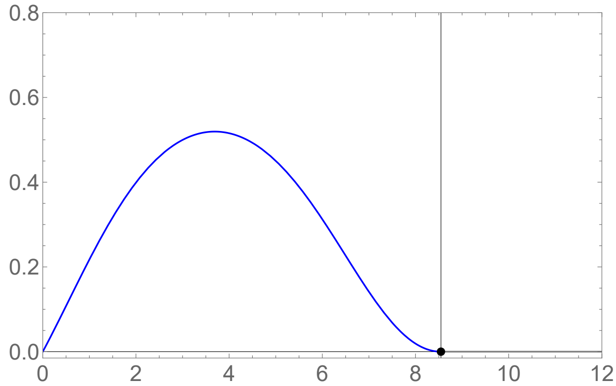

Figure 1. The $\mathbb{C}P^3$ compacton solution with $\omega = 1.0$. The profile smoothly connects to the vacuum solution at $r = 8.546$.

Here, it is worth to discuss how the compacton condition depends on the behavior of the potential in vicinity of its minimum.s Without the potential, it is easy to see that the equation has a trivial vacuum solution $f(r) = 0$. The shooting method allows for the solution to be obtained. When we integrated the field equation from infinity to the center, however, we only got a simple vacuum solution. Therefore, to get the nontrivial answer, we must incorporate any kind of potential in the equation. For the equation with the compact support (7), we assume the solution with the series expansion at the compacton radius

$$f(r \sim R^-) = \sum_{k=0}^{\infty} F_k (R-r)^k \,, \tag{9}$$

which is substituted into the equation. Eventually we obtain the explicit form

$$f(r \sim R^-) = \frac{1}{16}(R-r)^2 + \frac{1}{24R}(R-r)^3 + O((R-r)^4). \tag{10}$$

The results support the existence of a nontrivial solution that satisfies the compactness criteria (8).

3. The U(1) gauged, multi-component model

For $N \geq 2$, compact Q-balls become shells, with a spherical vacuum area that, unlike other scalar Q-shells, does not require an electromagnetic field. Coupling many $\mathbb{C}P^N$ models is a very lovely idea that, when combined with the presence of V-shaped potentials, allows for the generation of shell field configurations that surround another shell or ball field compact structure. This enables us to find the harbor type solution. There are several approaches to broaden and thin the shell. The first is to extend the model into multi-component fields of which each fields are $\mathbb{C}P^N_a$ components, $a = 1, 2, \cdots, n$. The Lagrangian of each component is labeled as follows:

$$\mathcal{L}_a(X_a, \partial_\mu X_a; M_a, \mu_a) = -\frac{M_a^2}{2} \mathrm{Tr}(X_a^{-1}\partial_\mu X_a)^2 - \mu_a^2 V(X_a), \quad a = 1, 2, \cdots, n \tag{11}$$

in which M_a, μ_a are coupling constants. The model is defined as follows

$$\mathcal{L}_{\mathbb{C}P^N} = \sum_{n=1}^{n} \mathcal{L}_a + \mathcal{L}_{\text{pot}} \tag{12}$$

where the non analytical potential is now

$$\mathcal{L}_{\text{pot}} = -\lambda\left(\frac{1}{4}\text{Tr}(\mathbb{I} - X_a)\right)^{\alpha}\left(\frac{1}{4}\text{Tr}(\mathbb{I} - X_b)\right)^{\beta}, \quad \alpha, \beta \geqq 1 \tag{13}$$

and where λ is the coupling constant.

The model is then coupled to the electromagnetic field. Compactons' size and radial form are altered by their electrostatic repulsive/attractive properties. The interaction is manifested by electric fields that significantly expands the matter fields. We assume that the model is invariant under the local transformation

$$u_j^{(a)} \to e^{iq^{(a)}\Lambda(x)} u_j^{(a)}, \qquad j = 1, 2, \cdots, N_a \tag{14}$$

where $\Lambda(x)$ is an arbitrary function of spacetime coordinates and $q^{(a)}$ are constant parameters. It is enough to introduce the covariant derivative of the form

$$D_\mu u^{(a)} = \partial_\mu u_j^{(a)} - ieq^{(a)} A_\mu u_j^{(a)} \tag{15}$$

where A_μ is the connection transforming according to

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \Lambda(x). \tag{16}$$

The model's action is divided into three sections: the $\mathbb{C}P^N$ part with minimal coupling achieved by covariant derivatives, the action for the connection field, and the action including coupling between scalar fields.

$$\mathcal{L}_{CP^N} = \int d^4x\sqrt{-g}\left(\sum_{a=1}^{n} \mathcal{L}^{(a)} + \mathcal{L}_{\text{EM}} - \sum_{a\neq b} \lambda\, W(u^{(a)}, u^{(b)})\right), \tag{17}$$

where

$$\mathcal{L}^{(a)} = 4M_a^2\, D_\mu u^{\dagger(a)} \cdot \Delta^{(a)2} \cdot D^\mu u^{(a)} - \mu_a^2 V(u^{(a)})\,, \tag{18}$$

$$\Delta_{ij}^{(a)2} \equiv \frac{(1 + u^{\dagger(a)} \cdot u^{(a)})\delta_{ij} - u_i^{(a)} u_j^{*(a)}}{(1 + u^{\dagger(a)} \cdot u^{(a)})^2}, \tag{19}$$

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{20}$$

$$W(u^{(a)}, u^{(b)}) = V(u^{(a)})^{2\alpha} V(u^{(b)})^{2\beta}, \quad V(u^{(a)}) = \left(\frac{u^{\dagger(a)} \cdot u^{(a)}}{1 + u^{\dagger(a)} \cdot u^{(a)}}\right)^{\frac{1}{2}}. \tag{21}$$

Variation of the action (17) with respect to $u^{\dagger(a)*}$ fields yields equations of motion. In turn, the variation with regard to the four potential A_μ leads to Maxwell's equations

$$\begin{aligned}
&\frac{1}{\sqrt{-g}} D_\mu\left(\sqrt{-g} D^\mu u_j^{(a)}\right) - 2\frac{\left(u^{(a)\dagger} \cdot D_\mu u^{(a)}\right) D^\mu u_j^{(a)}}{1 + u^{(a)\dagger} \cdot u^{(a)}} + \\
&\quad + \frac{1}{4}\left(1 + u^{(a)\dagger} \cdot u^{(a)}\right)\sum_{l=1}^{N_a}\left\{\left(\delta_{jl} + u_j^{(a)} u_l^{(a)*}\right)\left[\frac{\mu_a^2}{M_a^2}\frac{\delta V_a}{\delta u_l^{(a)*}} + \frac{\lambda}{M_a^2}\frac{\delta W}{\delta u_l^{(a)*}}\right]\right\} = 0,
\end{aligned} \tag{22}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = -\sum_{a=1}^{n}\frac{4ieq^{(a)}M_a^2}{(1 + u^{(a)\dagger} \cdot u^{(a)})^2}\left[u^{(a)\dagger} \cdot D^\nu u^{(a)} - D^\nu u^{(a)\dagger} \cdot u^{(a)}\right]. \tag{23}$$

In the following, we consider the two-component model, *i.e.*, $a = 1, 2$ for simplifying the discussion. The ansatz (6) of the model has the following form

$$\begin{aligned} u_{m_1}(t,r,\theta,\phi) &= \sqrt{\frac{4\pi}{2\ell_1+1}} f(r) Y_{\ell_1,m_1}(\theta,\phi) e^{i\omega_1 t}, \\ u_{m_2}(t,r,\theta,\phi) &= \sqrt{\frac{4\pi}{2\ell_2+1}} g(r) Y_{\ell_2,m_1}(\theta,\phi) e^{i\omega_2 t}, \\ A_\mu(t,r,\theta,\phi) &= (A_t(r),0,0,0). \end{aligned} \tag{24}$$

The model has global continuous symmetry. It enables us to get the Noether charges (17). The action is invariant with respect to continuous symmetry $U(1)^{N_1} \otimes U(1)^{N_2}$

$$u_s^{(1)} \to u_s^{(1)} e^{i\alpha_s},\ s = 1, 2, \cdots, N_1;\ \ u_s^{(2)} \to u_s^{(2)} e^{i\alpha_s},\ s = 1, 2, \cdots, N_2\,. \tag{25}$$

The conserved Noether current have the form

$$J_\mu^{s,(1)} = -4M_1^2\, i \sum_{k=1}^{N_1} \Big[u_s^{(1)*} \Delta_{sk}^{2(1)} D^\mu u_k^{(1)} - D^\mu u_k^{(1)*} \Delta_{ks}^{2(1)} u_s^{(1)} \Big], \qquad s = 1, \ldots, N_1 \tag{26}$$

$$J_\mu^{p,(2)} = -4M_2^2\, i \sum_{i=1}^{N_2} \Big[u_p^{(2)*} \Delta_{pi}^{2(2)} D^\mu u_i^{(2)} - D^\mu u_i^{(2)*} \Delta_{ip}^{2(2)} u_p^{(2)} \Big], \qquad p = 1, \ldots, N_2. \tag{27}$$

The ansatz (24) allows us to obtain conserved charges exclusively in terms of the radial profile function $f(r)$. They have the form

$$Q_t^{(m_1)} = \int d^3x \sqrt{-g}\, \widetilde{J}_t^{m_1,(1)} = \frac{32\pi}{2\ell_1+1} \int_0^\infty dr r^2 \frac{b_1 f^2}{(1+f^2)^2}\,. \tag{28}$$

$Q_t^{(m_2)}$ is obtained by replacing $\ell_1 \to \ell_2, f \to g$. Note that they have exactly the same form for each value of m_1, m_2.

The electric charges are defined in terms of the Noether charges. Applying the ansatz (24) to the Euler equation of the gauge field (23), we get the equation

$$A_t'' + \frac{2}{r} A_t' + \rho(r) = 0, \quad \rho(r) := 8\left(M_1 e q^{(1)} \frac{b_1 f^2}{(1+f^2)^2} + M_2 e q^{(2)} \frac{b_2 f^2}{(1+f^2)^2} \right), \tag{29}$$

$$b_a(r) \equiv \omega_a - e q^{(a)} A_t(r)\,. \tag{30}$$

We directly integrate (29) and then, we define the electric charge $\bar{Q}$ through the radial compoment of the electrostatic field $E_r(r)$ as

$$-A_t(r)' = \frac{1}{r^2} \int_0^r dr' r'^2 \rho(r') \equiv E_r(r) = \frac{\bar{Q}}{4\pi r^2}\,. \tag{31}$$

Therefore, one can relate the electric charges to the corresponding Noether charge as the following. We consider the harbor-type case. For the ball-shell solution, *i.e.*, the ball defined by $f(r)$ is located at $0 \le r \le R_1$ and the shell by $g(r)$ is at $R_2^{(\text{in})} \le r \le R_2^{(\text{out})}$, the electric charges read

$$\begin{aligned} \bar{Q} &:= \bar{Q}_1 + \bar{Q}_2 \\ \bar{Q}_1 &:= 32\pi M_1^2 e q^{(1)} \int_0^{R_1} dr' r'^2 \frac{b_1(r') f(r')^2}{(1+f(r')^2)^2} \equiv (2\ell_1+1) M_1^2 e q^{(1)} Q_1\,, \\ \bar{Q}_2 &:= 32\pi M_2^2 e q^{(2)} \int_{R_2^{(\text{in})}}^{R_2^{(\text{out})}} dr' r'^2 \frac{b_2(r') g(r')^2}{(1+g(r')^2)^2} \equiv (2\ell_2+1) M_1^2 e q^{(1)} Q_2\,. \end{aligned} \tag{32}$$

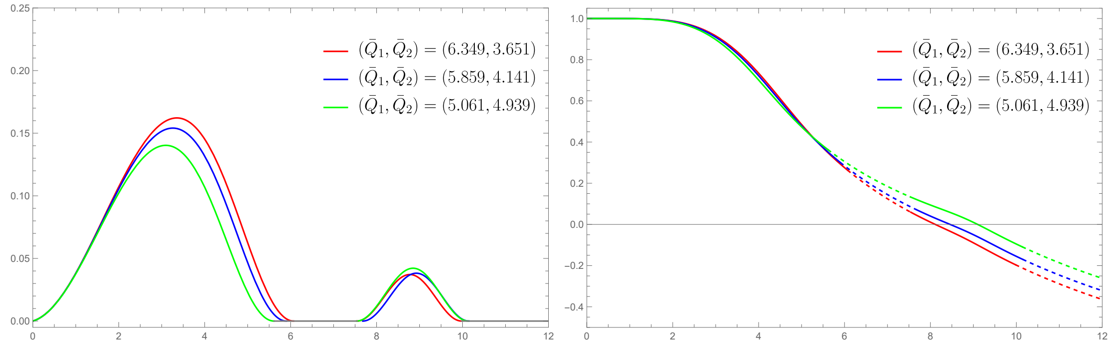

Figure 2. The $\mathbb{C}P^3 - \mathbb{C}P^7$ solutions with the total net charge $\bar{Q} = +10$. The left plot shows the profile functions and the right is gauge field.

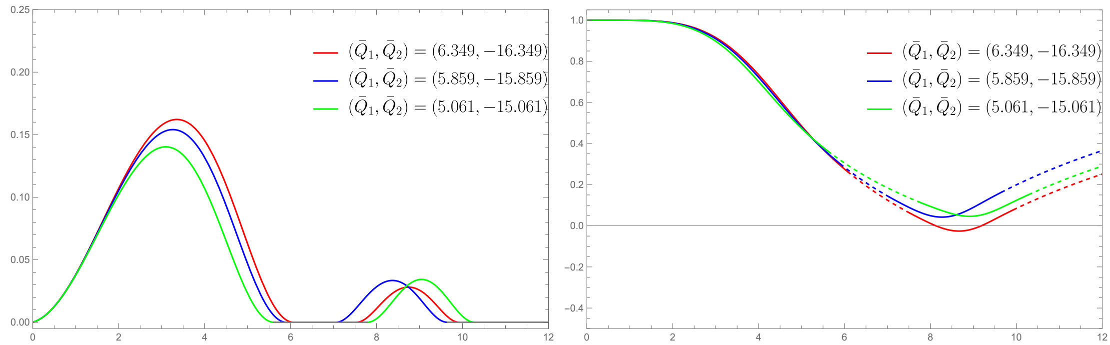

Figure 3. The $\mathbb{C}P^3 - \mathbb{C}P^7$ solutions with the total net charge $\bar{Q} = -10$. The left plot shows the profile functions and the right is gauge field.

From (32), we define the prescription for the sign of the electric charge. We define

$$\begin{aligned} &\bar{Q}_a > 0: \quad q^{(a)} > 0,\ Q_a > 0 \ \text{ or } \ q^{(a)} < 0,\ Q_a < 0, \\ &\bar{Q}_a < 0: \quad q^{(a)} > 0,\ Q_a < 0 \ \text{ or } \ q^{(a)} < 0,\ Q_a > 0, \quad a = 1, 2. \end{aligned} \tag{33}$$

The negative charged Q-ball is well expressed using the definition.

Fig.2 show the $\mathbb{C}P^3 - \mathbb{C}P^7$ solutions with net charge $\bar{Q}_1 + \bar{Q}_2 = 10$. Though all solutions have the same net charge, the structures are quite different.

On the other hand, Fig.3 is the $\mathbb{C}P^3 - \mathbb{C}P^7$ solutions with net charge $\bar{Q}_1 + \bar{Q}_2 = -10$. In terms of the attractive force between the constituents, they are slightly become compact than the cases of Fig.2.

4. Summary

In this paper, we discussed novel Q-ball configurations in the two coupled the $\mathbb{C}P^N$ models with non analytic potentials. The presence of a vacuum hole inside compact Q-shells enables the construction of harbor-type solutions, such as compactonas surrounded by compactons. The capacitor formulation is particularly interesting among them since the net charge of such solutions can take on any value between a negative and a positive number. The solutions we presented in this paper have a finite net charge. The construction of the zero net-charged Q-capacitor solution is the most crucial problem. We shall report it in our future article.

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