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Global Classical Solutions of the 1.5D Relativistic Vlasov–Maxwell–Chern–Simons System

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Abstract: We investigate the kinetic model of the relativistic Vlasov–Maxwell–Chern–Simons system, which originates from gauge theory. This system can be seen as an electromagnetic fields (i.e., Maxwell–Chern–Simons fields) perturbation for the classical Vlasov equation. By virtue of a nondecreasing function and an iteration method, the uniqueness and existence of the global solutions for the 1.5D case are obtained.

Keywords: relativistic Vlasov–Maxwell–Chern–Simons system; electromagnetic fields; Klein–Gordon equation; classical solutions; characteristic curves

MSC: 35F25; 35J05; 35Q83

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1. Introduction

In this paper, we focus on the relativistic Vlasov–Maxwell–Chern–Simons (**RVMCS**) system [1,2]

$$\partial_t f + \hat{v}_1\partial_w f + (E_1 + \hat{v}_2 B, E_2 - \hat{v}_1 B)\cdot\nabla_v f = 0, \tag{1}$$
$$\partial_t E_1 = -E_2 - j_1, \qquad \partial_w E_1 = B + \rho, \tag{2}$$
$$\partial_t E_2 = -\partial_w B + E_1 - j_2, \qquad \partial_t B = -\partial_w E_2, \tag{3}$$
$$B(0,w) = B^0(w), \;\; E(0,w) = E^0(w), \;\; f(0,w,v) = f_0(w,v), \tag{4}$$

on the whole space $(w,v) \in \mathbb{R}\times\mathbb{R}^2$ (all physical constants are normalized to unity), where at position $w \in \mathbb{R}$ and time $t \geq 0$, $f(t,w,v)$ is the density of the particles, moving with velocity $v = (v_1, v_2) \in \mathbb{R}^2$. The functions $B(t,w)$ and $E(t,w) = (E_1(t,w), E_2(t,w))$ stand for the magnetic and electric fields by the Chern–Simons theory, respectively. In addition, the current and charge densities are defined by

$$j = (\int \hat{v} f dv)4\pi, \qquad \rho = (\int f dv)4\pi,$$

where $\hat{v}_1$, $\hat{v}_2$, $\hat{v}$ denote the relativistic velocity, respectively:

$$\hat{v}_1 = \frac{v_1}{\sqrt{|v|^2+1}}, \;\; \hat{v}_2 = \frac{v_2}{\sqrt{|v|^2+1}}, \;\; \hat{v} = \frac{v}{\sqrt{|v|^2+1}}.$$

In fact, the **RVMCS** system is derived from gauge theory and can be described as the interaction between Vlasov matter and Maxwell–Chern–Simons fields. The Chern–Simons theory could explain many interesting phenomena, such as high-T_c superconductivity [3] and the quantum Hall effect [4]. If there is no Chern–Simons term, the corresponding system is known as the relativistic Vlasov–Maxwell system, i.e., the **RVM** system, which has received a lot of attention in the past decades (see, e.g., Refs. [5–9] and the references therein).

Although a great deal of mathematical results for the Chern–Simons theory (such as [10,11]) have been established, as the authors know, there are few results for the **RVMCS** system. In two dimensions, [12] obtained global classical solutions for the **RVMCS** system and deduced that the **RVMCS** system converges to the Vlasov–Yukawa equations by using the similar method of [8,13]. By virtue of the moment estimate and inhomogeneous Strichartz estimates (see [14]), ref. [2] established the global existence of a classical solution for the **RVMCS** system without compact momentum support.

Our main interest in this paper concerns the one-and-one-half-dimensional **RVMCS** system. A pioneer result by Glassey and Schaeffer [6] showed the global existence of the one-and-one-half-dimensional **RVM** system. When considering a fixed background $\eta(w) \in C_0^1(\mathbb{R})$ which is neutralizing in the way

$$\int_{-\infty}^{\infty} \rho(0,w)dw = \int_{-\infty}^{\infty} (\int f_0(p,v)dv - n(p))dp = 0,$$

from $\partial_w E_1 = \rho$, ref. [6] proved that $E_1(t,w)$ has compact support; thus, it is clear to deduce that

$$\sup_{t\in[0,T],w\in\mathbb{R}} |E_1(t,w)| < \infty,$$

which is the key point to obtain the existence result.

However, for the **RVMCS** system, we could not obtain the formula of $E_1(t,w)$ directly from $\partial_w E_1 = B + \rho$. It is well-known that Maxwell fields can be considered as wave equations, while the Maxwell–Chern–Simons fields may be supposed to be Klein–Gordon-type equations. Therefore, we briefly review the solution of the one-dimensional Klein–Gordon equations [15]:

$$\begin{aligned} &\partial_{tt}u - \partial_{ww}u + u = g(t,w), \\ &u(0,w) = u^0(w), \quad \partial_t u(0) = u_1^0(w). \end{aligned}$$

The fundamental solution of the above equations could be written as

$$\frac{1}{2}J_0(\sqrt{t^2 - |w|^2}),$$

where J_0 is the second kind of modified Bessel function of order zero. The interested readers are referred to [16] for a more detailed discussion about Bessel functions. In the present paper, we only give the following properties and asymptotic estimate:

$$J_n(z) = \begin{cases} \sqrt{\frac{2}{\pi z}} cos(z - \frac{n\pi}{2} - \frac{\pi}{4}), & z \gg 1, \\ \frac{z^n}{2^n n!}, & z \ll 1. \end{cases}$$

and

$$(J_0(z))' = -J_1(z), \quad (z^{-1}J_1(z))' = -z^{-1}J_2(z), \quad J_0(0) = 1, \quad J_1(0) = 0. \tag{5}$$

Combining function $J_n(z)$ and (5), one can show that

$$|J_0(z)|, \left|\frac{J_1(z)}{z}\right|, \left|\frac{J_2(z)}{|z|^2}\right| \leq C. \tag{6}$$

Using the fundamental solution $J_0(z)$, one can easily write the solution of the one-dimensional classical Klein–Gordon equation as follows:

$$\begin{aligned} u(t,w) &= \frac{1}{2}\Big(\int_{w-t}^{w+t} J_0(\sqrt{t^2-|w-p|^2})u_1^0(p)dp + \frac{\partial}{\partial t}\int_{w-t}^{w+t} J_0(\sqrt{t^2-|w-p|^2})u^0(p)dp\Big) \\ &+\frac{1}{2}\int_0^t \int_{w-(t-s)}^{w+(t-s)} J_0(\sqrt{-|p-w|^2+(s-t)^2})g(s,p)dsdp. \end{aligned} \tag{7}$$

Moreover, as for the Vlasov Equation (1), similarly with the **RVM** system, we can denote characteristic equations:

$$\begin{cases} \dot{X}(s) = \widehat{V}_1(s), \\ \dot{V}(s) = (E_1(s,X(s)) + \hat{V}_2(s)B(s,X(s)),\ E_2(s,X(s)) - \hat{V}_1(s)B(s,X(s))), \\ X(t) = w, \ \ V(t) = v, \end{cases} \tag{8}$$

wherein $X(s) = X(s,t,w,v)$, $V(s) = V(s,t,w,v)$. Along the characteristic curves, $f(t,w,v)$ is a constant, i.e.,

$$f(t,w,v) = f_0(X(0),V(0)).$$

In the rest of this paper, C shows a positive scalar that varies from line to line and only depends on the initial inputs. $C(t)$ stands for a positive nondecreasing function, which may vary from line to line. For the sake of simplicity, $\int_{\mathbb{R}^2} f dv$ will be written as $\int f dv$. Moreover, $\|f(t)\|$ and $\|B(t)\|$ are, respectively, denoted by

$$\begin{aligned} \|f(t)\| &= \sup_{(w,v)\in\mathbb{R}\times\mathbb{R}^2} \{|f(t,w,v)|\}, \\ \||B(t)|\| &= \sup_{0\le s\le t}\{\|B(s)\|\}, \ \ \|B(t)\| = \sup_{w\in\mathbb{R}}\{|B(t,w)|\}. \end{aligned}$$

Now, we summarize the major results of this work.

Theorem 1. *Let $f_0(w,v)$ be a nonnegative C^1 function with compact support. $E^0(w)$ and $B^0(w)$ are two C^2 functions and satisfy*

$$\partial_w E_1^0 = B^0 + \int f_0(w,v)dv,$$

and the initial data satisfy

$$\|\nabla^\alpha_{(w,v)} f_0\| + \|\nabla^\beta_w E^0\| + \|\nabla^\beta_w B^0\| < \infty, \qquad (|\alpha| \le 1,\ \ |\beta| \le 2).$$

*Then, with the **RVMCS** system exists a unique global classical solution $f(t,w,v)$, $(t,w,v) \in [0,+\infty)\times\mathbb{R}\times\mathbb{R}^2$. Furthermore, $f,\ E,\ B \in C^1$, having initial data $f_0,\ E^0,\ B^0$ satisfy*

$$f(t,w,v) = 0 \qquad for\ C(t)\ \le\ |v|,$$

and

$$\|\nabla^\alpha_{(t,w,v)} f(t)\| + \|\nabla^\alpha_{(t,w)} E(t)\| + \|\nabla^\alpha_{(t,w)} B(t)\| < C(t), \qquad (|\alpha| \le 1),$$

wherein $C(t)$ is a nondecreasing function.

The outline of the remainder of this work is structured as comes next. In Section 2, we give the representation of $E(t,w)$ and $B(t,w)$. By view of the Bessel function and the Gronwall inequality, the derivatives of $E(t,w)$ and $B(t,w)$ are controlled by ∇f. In Section 3, we obtain our main results by constructing the iteration scheme and estimating the fields more precisely.

2. Estimates of the Fields $B(t,w)$ and $E(t,w)$

In this section, combining the method given in [6,17] with the solution of Klein–Gordon Equation (7), we deduce the representations of $E(t,w)$ and $B(t,w)$ firstly.

Lemma 1. *Assume (f,E,B) is a classical resolution of the system **RVMCS** in (1)–(4). Assume that $f(t,w,v)$ has compact support in (w,v) for each t. Then, the fields $E(t,w)$ and $B(t,w)$ have representations ($k=1,2$):*

$$E_k = \widetilde{E}_k^0 + E_k^T + E_k^S + E_k^M + E_k^N,$$
$$B = \widetilde{B}^0 + B^T + B^S + B^M + B^N,$$

where

$$\begin{aligned}
\widetilde{E}_1^0 &= \frac{1}{2}\Big(\int_{w-t}^{w+t} J_0(\sqrt{t^2-|-p+w|^2})\partial_t E_1^0(p)dp + \frac{\partial}{\partial t}\int_{w-t}^{w+t} J_0(\sqrt{t^2-|-p+w|^2})E_1^0(p)dp\Big)\\
&\quad +2\pi\int_{w-t}^{w+t}\int J_0(\sqrt{t^2-|-p+w|^2})(1+\hat{v}_1)f_0(p,v)dvdp,\\
\widetilde{E}_2^0 &= \frac{1}{2}\Big(\int_{w-t}^{w+t} J_0(\sqrt{t^2-|-p+w|^2})\partial_t E_2^0(p)dp + \frac{\partial}{\partial t}\int_{w-t}^{w+t} J_0(\sqrt{t^2-|-p+w|^2})E_2^0(p)dp\Big)\\
&\quad -2\pi\int_{w-t}^{w+t}\int J_0(\sqrt{t^2-|-p+w|^2})\frac{\hat{v}_1\hat{v}_2}{1-\hat{v}_1}f_0(p,v)dvdp,\\
\widetilde{B}^0 &= \frac{1}{2}\Big(\int_{w-t}^{w+t} J_0(\sqrt{t^2-|-p+w|^2})\partial_t B^0(p)dp + \frac{\partial}{\partial t}\int_{w-t}^{w+t} J_0(\sqrt{t^2-|-p+w|^2})B^0(p)dp\Big)\\
&\quad -2\pi\int_{w-t}^{w+t}\int J_0(\sqrt{t^2-|-p+w|^2})\frac{\hat{v}_2}{\hat{v}_1-1}f_0(p,v)dvdp,\\
E_1^S &= 0,\\
E_2^S &= B^S = 2\pi\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int J_0(\sqrt{(-\varsigma+t)^2-|-p+w|^2})\nabla_v(-\frac{\hat{v}_2}{1-\hat{v}_1})(Ff)(\varsigma,p,v)dvdpd\varsigma,\\
E_1^T &= 2\pi\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int \frac{J_1(\sqrt{(-\varsigma+t)^2-|-p+w|^2})}{\sqrt{(-\varsigma+t)^2-|-p+w|^2}}[(-\varsigma+t)-(-p+w)](1+\hat{v}_1)f(\varsigma,p,v)dvdpd\varsigma,\\
E_2^T &= 2\pi\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int \frac{J_1(\sqrt{(-\varsigma+t)^2-|-p+w|^2})}{\sqrt{(-\varsigma+t)^2-|-p+w|^2}}[(-p+w)-(-\varsigma+t)]\frac{\hat{v}_1\hat{v}_2}{1-\hat{v}_1}f(\varsigma,p,v)dvdpd\varsigma,\\
B^T &= 2\pi\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int \frac{J_1(\sqrt{(-\varsigma+t)^2-|-p+w|^2})}{\sqrt{(-\varsigma+t)^2-|-p+w|^2}}[(-p+w)-(-\varsigma+t)]\frac{\hat{v}_2}{1-\hat{v}_1}f(\varsigma,p,v)dvdpd\varsigma.
\end{aligned}$$

Furthermore, the others are defined by

$$\begin{aligned}
E_1^M &= -4\pi\int_0^t\int(1+\hat{v}_1)f(\varsigma,w+(-\varsigma+t),v)dvd\varsigma,\\
E_2^M &= 4\pi\int_0^t\int\frac{\hat{v}_1\hat{v}_2}{1-\hat{v}_1}f(\varsigma,w+(-\varsigma+t),v)dvd\varsigma,\\
B^M &= 4\pi\int_0^t\int\frac{\hat{v}_2}{1-\hat{v}_1}f(\varsigma,w+(-\varsigma+t),v)dvd\varsigma,\\
E_1^N &= 2\pi\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int J_0(\sqrt{(-\varsigma+t)^2-|w-p|^2})\hat{v}_2 f(\varsigma,p,v)dvdpd\varsigma,\\
E_2^N &= -2\pi\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int J_0(\sqrt{(-\varsigma+t)^2-|w-p|^2})\hat{v}_1 f(\varsigma,p,v)dvdpd\varsigma,\\
B^N &= -2\pi\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int J_0(\sqrt{(-\varsigma+t)^2-|w-p|^2})f(\varsigma,p,v)dvdpd\varsigma,
\end{aligned}$$

where $F=(E_1+\hat{v}_2B, E_2-\hat{v}_1B)$.

Proof. As in [6], we define two operators as comes next:

$$\mathbf{S} = \partial_t + \hat{v}_1 \partial_w, \qquad \mathbf{T} = \partial_t + \partial_w.$$

Applying the calculations of [7,8,17], we can rewrite time and spatial derivatives below:

$$\partial_t = \frac{\mathbf{S} - \hat{v}_1 \mathbf{T}}{1 - \hat{v}_1}, \qquad \partial_w = \frac{\mathbf{T} - \mathbf{S}}{1 - \hat{v}_1}. \tag{9}$$

From (2) and (3), we obtain that

$$\begin{aligned}
&\partial_{tt} E_1 - \partial_{ww} E_1 + E_1 = -\partial_w \rho + j_2 - \partial_t j_1, \\
&\partial_{tt} E_2 - \partial_{ww} E_2 + E_2 = -j_1 - \partial_t j_2, \\
&\partial_{tt} B - \partial_{ww} B + B = -\rho + \partial_w j_2.
\end{aligned}$$

Because of the calculation process of the representation of E_1, E_2 and B are almost similar, we just only calculate E_2.

Actually, using (7), the E_2 field could be written as follows:

$$E_2 = \frac{1}{2}\Big(\frac{\partial}{\partial t} \int_{w-t}^{w+t} J_0\big(\sqrt{t^2 - |-p+w|^2}\big) E_2^0(p)dp + \int_{w-t}^{w+t} J_0\big(\sqrt{t^2 - |-p+w|^2}\big)\partial_t E_2^0(p)dp\Big) + \widetilde{E}_2, \tag{10}$$

where

$$\widetilde{E}_2 = \frac{1}{2}\int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} J_0\big(\sqrt{(-\varsigma+t)^2 - |w-p|^2}\big)(-j_1 - \partial_\varsigma j_2)(\varsigma, p)dpd\varsigma,$$

and in view of (9), $\widetilde{E}_2$ can be rewritten as

$$\begin{aligned}
\widetilde{E}_2 &= \int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int J_0\big(\sqrt{(-\varsigma+t)^2 - |-p+w|^2}\big)(-2\pi \hat{v}_1 f)(\varsigma, p, v)dvdpd\varsigma \\
&\quad + \int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int J_0\big(\sqrt{(-\varsigma+t)^2 - |-p+w|^2}\big)(-2\pi \hat{v}_2 \partial_\varsigma f)(\varsigma, p, v)dvdpd\varsigma \\
&= E_2^N - 2\pi \int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int J_0\big(\sqrt{(-\varsigma+t)^2 - |-p+w|^2}\big)\hat{v}_2\Big(\frac{\mathbf{S} - \hat{v}_1 \mathbf{T}}{1-\hat{v}_1}\Big) f(\varsigma, p, v)dvdpd\varsigma \\
&= E_2^N - 2\pi \int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int J_0\big(\sqrt{(-\varsigma+t)^2 - |-p+w|^2}\big)\frac{\hat{v}_2}{1-\hat{v}_1}\mathbf{S} f(\varsigma, p, v)dvdpd\varsigma \\
&\quad + 2\pi \int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int J_0\big(\sqrt{(-\varsigma+t)^2 - |-p+w|^2}\big)\frac{\hat{v}_1 \hat{v}_2}{1-\hat{v}_1}\mathbf{T} f(\varsigma, p, v)dvdpd\varsigma \\
&= E_2^N + \widetilde{E}_2 S + \widetilde{E}_2 T.
\end{aligned} \tag{11}$$

By virtue of (1), we obtain

$$\mathbf{S} f = -F \cdot \nabla_v f = -\nabla_v (F f).$$

It is now deduced that

$$
\begin{aligned}
\widetilde{E}_2 S &= 2\pi \int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int J_0\Big(\sqrt{(-\varsigma+t)^2 - |-p+w|^2}\Big) \frac{\hat{v}_2}{1-\hat{v}_1} \nabla_v (Ff)(\varsigma,p,v)\, dv dp d\varsigma \\
&= 2\pi \int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int J_0\Big(\sqrt{(-\varsigma+t)^2 - |-p+w|^2}\Big) \nabla_v \Big(-\frac{\hat{v}_2}{1-\hat{v}_1}\Big)(Ff)(\varsigma,p,v)\, dv dp d\varsigma \\
&= E_2^S. \qquad (12)
\end{aligned}
$$

For the $\widetilde{E}_2 T$ term, using the definition of $\mathbf{T}$, we obtain

$$
\widetilde{E}_2 T = 2\pi \int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int J_0\Big(\sqrt{(\varsigma-t)^2 - |-p+w|^2}\Big) \frac{\hat{v}_1 \hat{v}_2}{1-\hat{v}_1} (\partial_\varsigma + \partial_p) f(\varsigma,p,v)\, dv dp d\varsigma.
$$

Set $\varepsilon \in (0,1)$, by invoking integration by parts and (5), and we obtain

$$
\begin{aligned}
&\int_0^t \int_{w-(1-\varepsilon)(-\varsigma+t)}^{w+(1-\varepsilon)(-\varsigma+t)} \int J_0\Big(\sqrt{(-\varsigma+t)^2 - |w-p|^2}\Big)(\partial_\varsigma + \partial_p) f(\varsigma,p,v)\, dv dp d\varsigma \\
= &\int_0^t \int_{w-(1-\varepsilon)(-\varsigma+t)}^{w+(1-\varepsilon)(-\varsigma+t)} \int \frac{J_1\big(\sqrt{(-\varsigma+t)^2 - |w-p|^2}\big)}{\sqrt{(-\varsigma+t)^2 - |w-p|^2}} [(w-p) - (-\varsigma+t)] f(\varsigma,p,v)\, dv dp d\varsigma \\
&+ \int_0^t \frac{d}{d\varsigma}\left(\int_{w-(1-\varepsilon)(-\varsigma+t)}^{w+(1-\varepsilon)(-\varsigma+t)} J_0\Big(\sqrt{(-\varsigma+t)^2 - |w-p|^2}\Big) f dp \right) d\varsigma \\
&+ (2-\varepsilon) \int_0^t \int J_0\Big(\sqrt{\varepsilon(2-\varepsilon)}(-\varsigma+t)\Big) f(\varsigma, w+(1-\varepsilon)(-\varsigma+t), v)\, dv d\varsigma \\
&- \varepsilon \int_0^t \int J_0\Big(\sqrt{\varepsilon(2-\varepsilon)}(-\varsigma+t)\Big) f(\varsigma, w-(1-\varepsilon)(-\varsigma+t), v)\, dv d\varsigma. \qquad (13)
\end{aligned}
$$

As $\varepsilon \to 0$, it is easy to deduce that

$$
\widetilde{E}_2 T = -2\pi \int_{w-t}^{w+t} \int J_0\Big(\sqrt{t^2 - |-p+w|^2}\Big) \frac{\hat{v}_1 \hat{v}_2}{1-\hat{v}_1} f_0(p,v)\, dv dp + E_2^T + E_2^M. \qquad (14)
$$

Inserting inequalities (11)–(14) into (10), we obtain the representation of E_2. □

Remark 1. *Because of the different fundamental function between the Klein–Gordon equations with wave equations, the representations of fields E and B for the* ***RVMCS*** *system have some different points from the* ***RVM*** *system [6]. For example, we have additional terms E_k^S, E_k^T, B^S and B^T. Nevertheless, these additional terms can be controlled by (5) and (6).*

Now, we are devoted to estimating the fields $E(t,w)$ and $B(t,w)$.

Lemma 2. *Suppose that $(f(t,w,v), E(t,w), B(t,w))$ satisfy the same conditions as in Lemma 1 and*

$$
\|f_0\|,\ \|E^0\|,\ \|B^0\|,\ \|\partial_w E^0\|,\ \|\partial_w B^0\|
$$

are finite. If there is a nondecreasing function $C(t)$ yielding

$$
f(t,w,v) = 0 \quad if \;\; C(t) \le |v|,
$$

then

$$
\||B(t)\|| + \||E(t)\|| \le C(t).
$$

Proof. Because of $|v| < C(t)$ when $f \neq 0$, it is easy to show that

$$\begin{aligned} \left|\frac{\hat{v}_1\hat{v}_2}{1-\hat{v}_1}\right| &\leq \left|\frac{v_1\hat{v}_2}{\sqrt{|v|^2+1}-v_1}\right| = \left|\frac{v_1\hat{v}_2(\sqrt{|v|^2+1}+v_1)}{1+|v_2|^2}\right| \\ &\leq 2|v_2|\sqrt{1+|v|^2} \\ &\leq C(t), \end{aligned} \tag{15}$$

$$\begin{aligned} \left|\partial_{v_1}(-\frac{\hat{v}_2}{1-\hat{v}_1})\right| &= \left|\partial_{v_1}\left(\frac{v_2}{1+|v_2|^2}(\sqrt{1+|v|^2}+v_1)\right)\right| \\ &= \left|\frac{v_2(\hat{v}_1+1)}{|v_2|^2+1}\right| \leq 2, \end{aligned} \tag{16}$$

$$\begin{aligned} \left|\partial_{v_2}(-\frac{\hat{v}_2}{1-\hat{v}_1})\right| &= \left|\frac{1+|v_1|^2}{(1+|v|^2)^{\frac{3}{2}}(1-\hat{v}_1)} - \frac{\hat{v}_1|\hat{v}_2|^2}{(1-\hat{v}_1)^2\sqrt{|v|^2+1}}\right| \\ &\leq \sqrt{1+|v|^2}+2|v_1| \\ &\leq C(t), \end{aligned} \tag{17}$$

where we have used $\sqrt{1+|v|^2} - v_1 \geq \hat{v}_2$. Hence, for $0 \leq t \leq T$, using (6) and in consideration of Lemma 1, we obtain that

$$\begin{aligned} \|E(t)\| + \|B(t)\| &\leq C(t) + C(t)\int_0^t (\|E(\varsigma)\| + \|B(\varsigma)\|)d\varsigma \\ &\leq C(T) + C(T)\int_0^t (\|E(\varsigma)\| + \|B(\varsigma)\|)d\varsigma. \end{aligned}$$

The Gronwall's inequality implies that $\|E(t)\| + \|B(t)\| \leq C(T)$, for $0 \leq t \leq T$. This is the desired result. □

Next, we show that the derivatives of the fields are also bounded.

Lemma 3. *Assume that (f, E, B) are as in Lemma 1, and $\|\nabla_{(w,v)} f_0\|$, $\|\nabla_w^2 E^0\|$ and $\|\nabla_w^2 B^0\|$ are finite. Then,*

$$|||\nabla_w B(t)||| + |||\nabla_w E(t)||| \leq C(t)(1 + |||\nabla_w f(t)|||).$$

Proof. Firstly, we calculate every term of $\partial_w E_2$. For $\partial_w E_2^S$, using the definition of operators S and T, together with (1) and (9), we have

$$\begin{aligned} \partial_w E_2^S =& 2\pi\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int J_0(\sqrt{(-\varsigma+t)^2-|-p+w|^2})\nabla_v(-\frac{\hat{v}_2}{1-\hat{v}_1})\nabla_p(Ff)(\varsigma,p,v)dvdpd\varsigma \\ =& 2\pi\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int J_0(\sqrt{(-\varsigma+t)^2-|-p+w|^2})\nabla_v\left(\nabla_v(\frac{\hat{v}_2}{1-\hat{v}_1})\cdot\frac{F}{1-\hat{v}_1}\right)\cdot Ffdvdpd\varsigma \\ &+ 2\pi\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int J_0(\sqrt{(-\varsigma+t)^2-|-p+w|^2})\nabla_v(-\frac{\hat{v}_2}{1-\hat{v}_1})\cdot F\frac{\mathbf{T}f}{1-\hat{v}_1}dvdpd\varsigma \\ &+ 2\pi\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int J_0(\sqrt{(-\varsigma+t)^2-|-p+w|^2})\nabla_v(-\frac{\hat{v}_2}{1-\hat{v}_1})\cdot\partial_p F\cdot fdvdpd\varsigma \\ =& \partial_w E_2^{S1} + \partial_w E_2^{S2} + \partial_w E_2^{S3}. \end{aligned} \tag{18}$$

On the set $\{v : |v| < C(t)\}$, by an elementary computation as well as Lemma 1, this yields that

$$\left|\nabla_v\left(\nabla_v(\frac{\hat{v}_2}{1-\hat{v}_1})\cdot\frac{F}{1-\hat{v}_1}\right)\cdot Ff\right| \leq C(t).$$

Consequently, we obtain

$$|\partial_w E_2^{S1}| \leq C(t)\int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} J_0(\sqrt{(-\varsigma+t)^2-|-p+w|^2})dpd\varsigma \leq C(t). \tag{19}$$

Similar to the estimate of (13), we have

$$\begin{aligned}
\partial_w E_2^{S2} =& 2\pi\int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int \partial_\varsigma(-J_0(\sqrt{(-\varsigma+t)^2-|-p+w|^2})F)\nabla_v(-\frac{\hat{v}_2}{1-\hat{v}_1})\frac{f}{1-\hat{v}_1}dvdpd\varsigma \\
&+2\pi\int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int \partial_p(-J_0(\sqrt{(-\varsigma+t)^2-|-p+w|^2})F)\nabla_v(-\frac{\hat{v}_2}{1-\hat{v}_1})\frac{f}{1-\hat{v}_1}dvdpd\varsigma \\
&+2\pi\int_{w-t}^{w+t}\int \frac{J_0(\sqrt{t^2-|-p+w|^2})}{1-\hat{v}_1}\nabla_v(-\frac{\hat{v}_2}{1-\hat{v}_1})\cdot(-Ff)|_{\varsigma=0}dvdp \\
&+4\pi\int_0^t\int \nabla_v(-\frac{\hat{v}_2}{1-\hat{v}_1})\cdot\frac{Ff}{1-\hat{v}_1}|_{(\varsigma,w+(-\varsigma+t),v)}dvd\varsigma.
\end{aligned}$$

By (2) and (3) and Lemma 1, for $0 \leq \varsigma \leq t$, we have

$$|\partial_\varsigma F| \leq C(t) + \|\nabla_w E(\varsigma)\| + \|\nabla_w B(\varsigma)\|.$$

Then, combining the properties of Bessel functions (5) and (6) with the support of f, we observe that

$$|\partial_w E_2^{S2}| \leq C(t)\left(1+\int_0^t(\|\nabla_w E(\varsigma)\|+\|\nabla_w B(\varsigma)\|)d\varsigma\right). \tag{20}$$

Similarly, we have

$$|\partial_w E_2^{S3}| \leq C(t)\int_0^t(\|\nabla_w E(\varsigma)\|+\|\nabla_w B(\varsigma)\|)d\varsigma.$$

Finally, by virtue of the above inequality and (18)–(20), we obtain

$$|\partial_w E_2^{S}| \leq C(t)\left(1+\int_0^t(\|\nabla_w E(\varsigma)\|+\|\nabla_w B(\varsigma)\|)d\varsigma\right).$$

Next, using equality (9), we can estimate $\partial_w E_2^T$,

$$\begin{aligned}
\partial_w E_2^T =& 2\pi\int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int \frac{J_1(\sqrt{(-\varsigma+t)^2-|-p+w|^2})}{\sqrt{(-\varsigma+t)^2-|-p+w|^2}}[(-p+w)-(-\varsigma+t)]\frac{\hat{v}_1\hat{v}_2}{1-\hat{v}_1}\partial_p f(\varsigma,p,v)dvdpd\varsigma \\
=& -2\pi\int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int \frac{J_1(\sqrt{(-\varsigma+t)^2-|-p+w|^2})}{\sqrt{(-\varsigma+t)^2-|-p+w|^2}}[(-p+w)-(-\varsigma+t)]\frac{\hat{v}_1\hat{v}_2}{(1-\hat{v}_1)^2}\mathbf{S}fdvdpd\varsigma \\
&+2\pi\int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int \frac{J_1(\sqrt{(-\varsigma+t)^2-|-p+w|^2})}{\sqrt{(-\varsigma+t)^2-|-p+w|^2}}[(-p+w)-(-\varsigma+t)]\frac{\hat{v}_1\hat{v}_2}{(1-\hat{v}_1)^2}\mathbf{T}fdvdpd\varsigma, \\
=& \partial_w E_2^T S + \partial_w E_2^T T.
\end{aligned}$$

Then, we estimate each term in the above equality separately. For $\partial_w E_2^T S$, using (1), Lemma 2 and $|v| \leq C(t)$, we show that

$$\begin{aligned}
|\partial_w E_2^T S| &\leq 2\pi\int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int |(-p+w)-(-\varsigma+t)|\nabla_v\left(\frac{\hat{v}_1\hat{v}_2}{(1-\hat{v}_1)^2}\right)\cdot Ffdvdpd\varsigma \\
&\leq C(t)\int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}(-\varsigma+t)dpd\varsigma \leq C(t).
\end{aligned}$$

For $\partial_w E_2^T T$, similar to the estimate of (13), in view of (5), (6) and (9), we have

$$\begin{aligned}|\partial_w E_2^T T| \leq & 2\pi \int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \int \Big| \frac{J_2(\sqrt{(-\varsigma+t)^2 - |-p+w|^2})}{(-\varsigma+t)^2 - |-p+w|^2} [(-p+w) - (-\varsigma+t)]^2 \frac{\hat{v}_1 \hat{v}_2}{(1-\hat{v}_1)^2} f \Big| dv dp \\ & + \int_{w-t}^{w+t} \int \Big| \frac{J_1(\sqrt{t^2 - |-p+w|^2})}{\sqrt{t^2 - |-p+w|^2}} (t - (-p+w)) \frac{\hat{v}_1 \hat{v}_2}{(1-\hat{v}_1)^2} f_0 dv dp \Big| \\ \leq & C(t) \Big| \int_0^t \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} (-\varsigma+t)^2 dp d\varsigma + \int_{w-t}^{w+t} (t - (-p+w)) dp \Big| \leq C(t).\end{aligned}$$

Hence, combining the above estimate, we obtain $|\partial_w E_2^T| \leq C(t)$.
For $\partial_w E_2^M$, we could compute it directly:

$$|\partial_w E_2^M| \leq 4\pi |||\nabla_w f(t)||| \int_0^t \int \Big| \frac{\hat{v}_1 \hat{v}_2}{1-\hat{v}_1} \Big| dv d\varsigma \leq C(t) |||\nabla_w f(t)|||.$$

Then, for $\partial_w E_2^N$, similar to the estimates of $\partial_w E_2^T$, it is easy to show

$$|\partial_w E_2^N| \leq C(t).$$

Lastly, for $\partial_w \widetilde{E}_2^0$, we can obtain $|\partial_w \widetilde{E}_2^0| \leq C(t)$ by integration by parts. Thus, from the above estimates and Lemma 1, we have

$$|\partial_w E_2| \leq C(t) \left(1 + |||\nabla_w f(t)||| + \int_0^t (\|\nabla_w E(\varsigma)\| + \|\nabla_w B(\varsigma)\|) d\varsigma \right).$$

Again, in the same way, we can estimate $\partial_w E_1$ and $\partial_w B$. Consequently, for $0 \leq t \leq T$, we have

$$\begin{aligned}|\partial_w E| + |\partial_w B| \quad & \leq \quad C(t) \left(1 + |||\nabla_w f(t)||| + \int_0^t (\|\nabla_w E(\varsigma)\| + \|\nabla_w B(\varsigma)\|) d\varsigma \right) \\ & \leq \quad C(T) \left(1 + |||\nabla_w f(T)||| + \int_0^t (\|\nabla_w E(\varsigma)\| + \|\nabla_w B(\varsigma)\|) d\varsigma \right)\end{aligned}$$

This, together with Gronwall's inequality, completes the proof. □

Lemma 4. *Assume that $(f(t,w,v), E(t,w), B(t,w))$ are as in Lemmas 1–3 and the conditions of Lemmas 1–3 hold. Then,*

$$\|f(t)\| + \|E(t)\| + \|B(t)\| + \|\nabla_{(w,v)} f(t)\| + \|\nabla_w E(t)\| + \|\nabla_w B(t)\| \leq C(t).$$

Proof. It is similar to [7] (Theorem 4) and [12] (Lemma 3.2), so we omit it. □

3. Proof of the Main Results

This section is furnished to investigate the existence and uniqueness of classical solutions for the **RVMCS** system. First of all, we will give a conditional existence proposition.

Proposition 1. *Let $f_0(w,v)$ be nonnegative C^1 functions. Suppose that $E^0(w)$ and $B^0(w)$ are two C^2 functions, such that*

$$\partial_w E_1^0 = B^0 + 4\pi \int f_0(w,v) dv.$$

If the data satisfy

$$\|\nabla_{(w,v)}^{\alpha} f_0\| + \|\nabla_w^{\beta} E^0\| + \|\nabla_w^{\beta} B^0\| < \infty, \qquad (|\alpha| \leq 1, \ \ |\beta| \leq 2),$$

and furthermore, if there is a non-decreasing function $C(t)$ such that

$$f(t,w,v) = 0 \qquad for\ |v|\ \geq\ C(t).$$

Then, there exists a unique C^1 global classical solution for the ***RVMCS*** *system.*

Proof. In this work, we use a well-known iteration scheme method ([6,7,12,17,18]) which may be well used to prove the existence theorem. Denote $\left\{f^{(n)}(t,w,v),\ E^{(n)}(t,w), B^{(n)}(t,w)\right\}$ the iteration functions. We also take smooth initial data

$$f^{(0)}(t,w,v) = f_0(w,v) \in C^2,\ E^{(0)}(t,w) = E_0(w) \in C^3,\ B^{(0)}(t,w) = B_0(w) \in C^3.$$

After the $(n-1)^{st}$ iteration, we set that $f^{(n)}$ is the solution of the following **Vlasov** problem:

$$\partial_t f^{(n)} + \hat{v}_1 \partial_w f^{(n)} + (E_1^{(n-1)} + \hat{v}_2 B^{(n-1)}, E_2^{(n-1)} - \hat{v}_1 B^{(n-1)}) \cdot \nabla_v f^{(n)} = 0. \tag{21}$$

Hence, $f^{(n)}$ is a C^2 function if $E^{(n-1)}$ and $B^{(n-1)}$ are C^2 functions. By the theory of ordinary differential equations, along the characteristics equations in (21)

$$\dot{w} = \hat{v}_1, \quad \dot{v} = (E_1^{(n-1)} + \hat{v}_2 B^{(n-1)}, E_2^{(n-1)} - \hat{v}_1 B^{(n-1)})$$

$f^{(n)}(t,w,v)$ is a constant. Therefore, $f^{(n)}(t,w,v)$ also has compact support in v. In addition,

$$\rho^{(n)}(t,w) = 4\pi \int f^{(n)}(t,w,v)dv \in C^2, \qquad j^{(n)}(t,w) = 4\pi \int \hat{v} f^{(n)}(t,w,v)dv \in C^2$$

are well-defined. Then, we obtain $E^{(n)}$ and $B^{(n)}$ by solving the following equations

$$\begin{aligned}
\partial_{tt} E_1^{(n)} - \partial_{ww} E_1^{(n)} + E_1^{(n)} &= -\partial_w \rho^{(n)} + j_2^{(n)} - \partial_t j_1^{(n)}, \\
\partial_{tt} E_2^{(n)} - \partial_{ww} E_2^{(n)} + E_2^{(n)} &= -j_1^{(n)} - \partial_t j_2^{(n)}, \\
\partial_{tt} B^{(n)} - \partial_{ww} B^{(n)} + B^{(n)} &= -\rho^{(n)} + \partial_w j_2^{(n)}
\end{aligned}$$

with initial data $E^0(w),\ B^0(w)$. Furthermore, with Lemma 4, we can prove easily that these sequences are Cauchy in the C^1-norm and obtain the existence from Proposition 1 as in [17]. $\square$

To prove Theorem 1, we will show that the nondecreasing function in Proposition 1 exists on $[0,\infty)$. To this end, we establish a lemma for energy conservation.

Lemma 5. *Suppose $(f(t,w,v), E(t,w), B(t,w))$ are the solutions stated in Proposition 1. Then, the energy identity*

$$\partial_t e + \nabla_w \left(4\pi \int v_1 f dv + E_2 B \right) = 0 \tag{22}$$

holds, where

$$e = 4\pi \int \sqrt{1+|v|^2} f dv + \frac{1}{2}|E|^2 + \frac{1}{2}|B|^2.$$

Moreover,

$$\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} e(\varsigma,p)dp \le C, \tag{23}$$

$$\sup_{|w|<C+t, 0<t<T} \int_0^t \int \frac{|v_2|}{\sqrt{1+|v|^2}} f(\varsigma, w+(-\varsigma+t), v) dv d\varsigma \le C, \tag{24}$$

$$\sup_{|w|<C+t} \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \rho^{\frac{3}{2}}(\varsigma,p)dp \le C(-\varsigma+t), \tag{25}$$

$$\sup_{|w|<C+t} \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \left(\int \frac{f}{\sqrt{1+|v|^2}} dv \right)^3 dp \le C(-\varsigma+t). \tag{26}$$

Proof. The energy identity (22) is similar to [12] (Lemma 3.3), and its certification process is omitted (see [12] for details). It is obvious that $f(t,w,v) = 0$ if $|w| \ge C+t$ because of $\dot{w} = \hat{v}$ and $|\hat{v}| < 1$. So, we have the total energy identity by (22), i.e., (23). Similar with [8] (Lemma 1), we can obtain (24).

To prove (25), note that for each $R > 0$, we use the usual manner,

$$\rho \le \int_{|v|\le R} f dv + \int_{|v|>R} f dv \le C(R^2 + \frac{e}{R}).$$

Taking $R = e^{1/3}$, we have $\rho^{3/2} \le Ce$ and hence

$$\sup_{w\in\mathbb{R}} \int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)} \rho^{\frac{3}{2}}(\varsigma,p)dp \le C(-\varsigma+t).$$

Similarly, we can prove (26). □

Next, our goal is to deduce that $P(t) \le C(t)$ where

$$P(t) = 1 + \sup\{|v| : f(\varsigma,w,v) \ne 0 \ \ for\ some\ (\varsigma,w) \in [0,t]\times\mathbb{R}\}.$$

As the similar method in [6–8,12], by virtue of the estimate of $P(t)$, Proposition 1 can be extended to Theorem 1.

To this end, following from Lemma 5, we give more precise estimates of the fields than Lemma 2.

Lemma 6. *Let (f,E,B) be the solution furnished in Proposition 1, $0 < \widetilde{T} < T$, and then the estimate*

$$\|E(t)\| + \|B(t)\| \le C(\widetilde{T})(1+P^2(\widetilde{T}))$$

holds for $t \in [0,\widetilde{T}]$.

Proof. Combining (16) with (17), we obtain

$$\left| F\cdot\nabla_v(-\frac{\hat{v}_2}{1-\hat{v}_1}) \right| \quad \le \quad C(|E|+|B|)(1+(1+|v|^2)^{\frac{1}{2}}).$$

Then, using (6) and (15) and Lemma 1, we obtain

$$
\begin{aligned}
&\|E(t)\|+\|B(t)\| \\
\leq\ & C(t)+C\Big(\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int(|E|+|B|)(1+|v|^2)^{\frac{1}{2}}f\,dv\,dp\,d\varsigma+\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\rho\,dp\,d\varsigma \\
&+\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int(|E|+|B|)f\,dv\,dp\,d\varsigma+\int_0^t\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\int\sqrt{1+|v|^2}f\,dv\,dp\,d\varsigma \\
&+\int_0^t\int|v_2|\sqrt{1+|v|^2}f(\varsigma,w+(-\varsigma+t),v)\,dv\,d\varsigma\Big) \\
\triangleq\ & C(t)+I_1+I_2+I_3+I_4+I_5.
\end{aligned}
\tag{27}
$$

Then, we calculate I_i $(1\leq i\leq 5)$, respectively. For I_1, using the Hölder inequality, (22) and (26), we have

$$
\begin{aligned}
|I_1|\leq & C(1+P^2(\widetilde{T}))\int_0^t\left(\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}(|E|+|B|)^2dp\right)^{\frac{1}{2}}\left(\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\left(\int\frac{f}{\sqrt{1+|v|^2}}dv\right)^2dp\right)^{\frac{1}{2}}d\varsigma \\
\leq & C(\widetilde{T})(1+P^2(\widetilde{T}))\int_0^t\left(\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\left(\int\frac{f}{\sqrt{1+|v|^2}}dv\right)^3dp\right)^{\frac{1}{3}}\left(\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}dp\right)^{\frac{1}{6}}d\varsigma \\
\leq & C(\widetilde{T})(P^2(\widetilde{T})+1)\int_0^t(-\varsigma+t)^{\frac{1}{2}}d\varsigma\leq C(\widetilde{T})(P^2(\widetilde{T})+1).
\end{aligned}
$$

By (25) and the Hölder inequality again, it gives that

$$
|I_2|\leq C\int_0^t\left(\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\rho^{\frac{3}{2}}dp\right)^{\frac{2}{3}}\left(\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}dp\right)^{\frac{1}{3}}d\varsigma\leq C\int_0^t(-\varsigma+t)d\varsigma\leq C\widetilde{T}^2.
$$

With the above inequality, it is easy to deduce that

$$
\begin{aligned}
|I_3|\ \leq\ & C\int_0^t(\|E(\varsigma)\|+\|B(\varsigma)\|)\left(\int_{w-(-\varsigma+t)}^{w+(-\varsigma+t)}\rho\,dp\right)d\varsigma\leq C\int_0^t(-\varsigma+t)(\|E(\varsigma)\|+\|B(\varsigma)\|)d\varsigma \\
\leq\ & C\widetilde{T}\int_0^t(\|E(\varsigma)\|+\|B(\varsigma)\|)d\varsigma.
\end{aligned}
$$

Similar to the estimate of I_1, we also obtain

$$
|I_4|\leq C(\widetilde{T})(1+P^2(\widetilde{T})).
$$

For I_5, by (24), it yields that

$$
|I_5|\leq C(1+P^2(\widetilde{T})).
$$

Combining the above inequalities with (27), we obtain

$$
\|E(t)\|+\|B(t)\|\leq C(\widetilde{T})(1+P^2(\widetilde{T}))+C\widetilde{T}\int_0^t(\|E(\varsigma)\|+\|B(\varsigma)\|)d\varsigma.
$$

Thus, by the inequality of the Gronwall, it implies that

$$
\|E(t)\|+\|B(t)\|\leq C(\widetilde{T})(1+P^2(\widetilde{T}))\exp\{C\widetilde{T}t\}\leq C(\widetilde{T})(1+P^2(\widetilde{T})).
$$

□

Employing the characteristic curves (8) and Lemma 6, we have

$$|V(0,t,w,v)| \leq |v| + C\int_0^t (\|E(\varsigma)\| + \|B(\varsigma)\|)d\varsigma \leq C + \int_0^t C(\varsigma)(1+P^2(\varsigma))d\varsigma.$$

It follows that

$$P(t) \leq C + C(t)\int_0^t (1+P^2(\varsigma))^2 d\varsigma \leq C + C(\widetilde{T})\int_0^t (1+P^2(\varsigma))d\varsigma,$$

wherein $P(t)$ is bounded via $Q(t)$ on $[0,T]$, and also $Q(t)$ satisfies

$$Q(t) = \tan(C(\widetilde{T})t + \arctan C).$$

So, choosing $t = \widetilde{T}$, for each $\widetilde{T} \in (0,T)$, one obtains

$$P(\widetilde{T}) \leq \tan(C(\widetilde{T})\widetilde{T} + \arctan C) = C(\widetilde{T}).$$

This completes the proof of the theorem.

4. Conclusions

In this manuscript, we have considered the relativistic Vlasov–Maxwell–Chern–Simons system in the 1.5D case. Different from the well-known Vlasov–Maxwell equations, the **RVMCS** system could be seen as a set of the Klein–Gordon-type equations and Vlasov equation. However, the Vlasov–Maxwell system could be considered as a system of the linear wave equation. The fundamental solution of the one-dimensional Klein–Gordon PDE has some decaying and bounded properties; hence, we can control $B(t,w)$ and $E(t,w)$. By view of the iteration method and a nondecreasing function condition, we establish the global uniqueness and existence of the **RVMCS** system. In a forthcoming work, inspired by the work of [19–23], we may study two questions. On the one hand, we will consider establishing the well-posedness of the **RVMCS** system in Besov space with large Maxwell fields. On the other hand, we will consider the behavior of the **RVMCS** system, when the speed of light tends to infinity.

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