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The School of Graduate Studies in Basic Sciences
"GALILEO GALILEI"

PhD Thesis:

"Confinement, deconfinement and monopoles in gauge theories"

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1. Introduction

The study of the confinement properties of the Yang-Mills theories is a notoriously difficult subject, confinement being a completely nonperturbative phenomenon, and this prevents a complete understanding of the physics of the Standard Model of particles in the strongly coupled regime.

The lattice formulation of QCD introduced by Wilson [1] is an invaluable tool for studying strong coupling gauge theories both analytically and by means of numerical simulations. During the years enormous numerical evidence has been collected to support the idea that in non-abelian gauge theories the color degrees of freedom are confined, *i.e.* that only color singlet states are present in the spectrum. Nevertheless a satisfactory understanding of the physical mechanism responsible for color confinement is still lacking. Topologically stable configurations are though to be involved in the color confinement mechanism, however there is no consensus on the choice of the relevant topological defects, the two main candidates being vortices and monopoles.

The ideas behind the two proposal of vortex- or monopole-related confinement are very different in spirit although they are both aimed at explaining the presence of a linearly rising potential between a quark-antiquark static pair or, equivalently, the area law behaviour of the Wilson loops.

In the vortex-related theory the area scaling of the Wilson loops is explained by what is usually called "center disorder": if in the confined phase a large number of sufficiently randomly distributed vortices are present, a given Wilson loop will be pierced by a large number of independent vortices and, depending on the even or odd number of piercings, the sign of the Wilson loop will strongly fluctuate, with large cancellations occurring and a net exponential behaviour will result (for details see *e.g.* [2]). In the monopole-related confinement scenario the assumption is that the monopole degrees of freedom are condensed in the confined phase and the linearly rising potential is generated by the dual analogue of the Abrikosov flux tubes, *i.e.* the vacuum behaves as a dual superconductor.

While in the vortex scenario confinement is related to the existence of a percolating vortex cluster, in the monopole scenario confinement is connected to the realization of the magnetic $U(1)$ symmetry and the absence of colored asymptotic states in the spectrum is enforced by the vacuum symmetry, thus avoiding naturalness problems. This is not the only theoretically appealing feature of the dual superconductivity model, since it leaves open the door to the possibility of a duality symmetry between the electric and the magnetic degrees of freedom, *i.e.* to the possibility of describing the QCD strong coupling regime by means of an effective weakly interacting theory of monopoles.

Effective weakly interacting theories which describe the strong coupling regime of a physical systems by means of effective degrees of freedom are ubiquitous in condensed matter physics (*e.g.* the Landau theory of Fermi liquids), however there are very few

examples of systems for which the duality transformation is explicitly known. These are typically simple spin systems, like the 2d Ising model [3] or the 2d XY model [4]. A notable exception is the solution by Seiberg and Witten of the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory [5] in which the duality transformation can be explicitly performed and confinement is described by monopole condensation.

The effective degrees of freedom are typically introduced by the topologically nontrivial behaviour of the fields at spatial infinity. For example, for a Yang-Mills theory living in $D + 1$ dimension, the effective degrees of freedom would be associated to the π^{D-1} homotopy group and in the ordinary 3+1 dimensional space-time monopoles thus appear as the natural choice.

The prototype monopole configuration for gauge theories is the soliton solution of the $SU(2)$ Higgs model with the Higgs field in the adjoint representation [6, 7]. The general behaviour of this solution can easily be computed when the gauge symmetry is broken to $U(1)$ by the Higgs vacuum expectation value, the magnetic degree of freedom being the massless unbroken component of the gauge field. Since we do not know the explicit form of the duality transformation, when the gauge symmetry is unbroken it is not clear how to select the $U(1)$ magnetic subgroup of the gauge group.

In [8] the possibility was advocated that all the choices of the residual $U(1)$ magnetic gauge symmetry (abelian projections) are equivalent, motivated by the apparent absence of a preferred direction in color space. In particular a convenient way to define monopoles is to use a composite field in the adjoint representation of the gauge group: monopoles can then be identified with the points in which two eigenvalues of the composite field becomes degenerate.

After the seminal work by DeGrand and Toussaint [9], in which a method to detect monopoles in numerically generated lattice configurations was proposed, it was noted that the number and the position of the observed monopoles in a given configuration strongly depend on the abelian projection adopted. Monopoles thus seem to be gauge invariant objects. This is unacceptable from a physical point of view: for condensation of monopoles to be at the origin of color confinement, monopoles have to be gauge invariant object, independent of the projection used to define them.

While most of the numerical work related to monopoles in lattice gauge theories was aimed to detect monopoles, in order to confirm or disprove the dual superconductivity picture a better strategy is to compute the vacuum expectation value of a magnetically charged operator. To define such an operator we have to choose an abelian projection, so also this second strategy can give indications on the equivalence (or not) of the various abelian projections. The numerical results indicate that, in contrast to monopole detection, monopole condensation is a gauge invariant phenomenon. We thus have two apparently conflicting results

- monopole detection depends on the abelian projection
- monopole condensation is abelian projection independent

In order to reconcile the two points of view and, more important, to gain a better understanding of the role played by the abelian projection in the definition of monopoles,

it is convenient to investigate if a gauge covariant quantity exists that is related to the magnetic monopole. In the first chapter of this thesis we will show that such a quantity is the violation of the non-abelian Bianchi identity and by using its relation to the magnetic current we will show from a theoretical point of view that monopole condensation is indeed abelian projection independent. In the following we will also analyze the DeGrand-Toussaint recipe to detect monopoles on the lattice and we will show that the gauge dependence of the number of observed monopole is not in contradiction with the gauge independence of monopoles.

We mentioned above the possibility of constructing a magnetically charged operator to be used to detect monopole condensation. The construction of such an operator in abelian lattice gauge theory is well understood, however the generalization to the non-abelian ones turn out to be far from trivial: the operator proposed in [10] seemed to satisfy all the needed requirements, however a more accurate analysis revealed that it is not well defined. This was interpreted as a signal of the failure of the dual superconductivity picture in [11] but we will show it is just a consequence of the nonlocal nature of the operator, that requires some care in dealing with the $O(a^2)$ lattice artefacts. An improved version of the operator proposed in [10] will be presented, together with numerical simulations that show that the problem of the original formulation does not affect the improved version of the monopole operator.

When fermions are coupled to the gauge field the dual superconductivity picture of the vacuum does not require any *ad hoc* modifications, however also the chiral degrees of freedom can play a predominant role in the determination in the phase diagram, thus making the theoretical analysis more difficult.

An accurate understanding of the QCD phase diagram at non zero temperature is clearly of the utmost importance for its considerable phenomenological implications. Nevertheless there are still some points that are not settled and deserve further investigations. Among these is the determination of the order of the chiral transition for the case of two massless quark flavours. Theoretical arguments based on effective chiral Lagrangians restrict the possibilities for the transition to be first order or second order in the 3d $O(4)$ universality class; which of these two possibilities is realized in QCD is a non universal features that need to be investigated by means of numerical simulations. We will present in the following the state of the art of this problem and the investigations we are performing in order to give it a definite answer.

Simulations of QCD are however extremely computationally demanding and require dedicated machines to be performed. Since both a precise verification of the theoretical expectation and the numerical evaluation of phenomenologically relevant quantities depend on the precision of these simulations (which in turn depends on the acquired statistics), in order to speed up the computations much attention is being devoted to improve the algorithms and to find new architectures on which to perform simulations.

In the last few years it was shown that the modern graphics cards, also known as graphic processing units (GPUs), can be used in the high performance computing field with surprisingly good results. For QCD simulations they are typically used together with more traditional architectures in order to speed up just some steps of the computations, mainly the analysis of the generated configuration. We showed that it is possible

to use GPUs to perform complete simulations without the need to rely on the traditional expensive dedicated machines; in the last section we will analyze some of the problems encountered in using GPUs for QCD simulations and the solution strategies adopted to circumvent them.

2. Monopoles and the 't Hooft tensor

2.1. Monopoles in abelian gauge theories

Monopoles can be introduced in electrodynamics as classical configurations with vector potential ([12])

$$\mathbf{A} = \frac{g}{r}\,\frac{\mathbf{r}\times\mathbf{n}}{r-\mathbf{r}\cdot\mathbf{n}} \quad \text{or} \quad \mathbf{A} = g\frac{(1-\cos\theta)}{r\sin\theta}\mathbf{e}_\phi \quad \text{if} \quad \mathbf{n} = -\mathbf{e}_z \tag{2.1.1}$$

where $\mathbf{n}^2 = 1$. This potential has a singularity along the $\mathbf{n}$ line, which generates a singularity also in the magnetic field $\mathbf{B}$ (Dirac string). The magnetic field is given by (see *e.g.* [13])

$$\mathbf{B} = \nabla\times\mathbf{A} = g\frac{\mathbf{r}}{r^3} + 4\pi g\,\theta(-z)\delta(x)\delta(y)\mathbf{n}$$

and we see that the Dirac string is necessary to satisfy the Bianchi identity $\nabla\cdot\mathbf{B} = 0$. Indeed this has to be the only role played by the Dirac string, since its position is not gauge invariant: for example the transformation $\mathbf{A}\to\mathbf{A}+\frac{i}{e}(\nabla U)U^{-1}$ with $U = e^{2ieg\phi}$ moves the string from $\mathbf{n} = -\mathbf{e}_z$ to $\mathbf{n} = \mathbf{e}_z$. Since the position of the string is not gauge invariant, it can not have any physical influence, that is the equation $\exp(ie\oint_\gamma \mathbf{A}\cdot\mathbf{ds}) = 1$ must be satisfied for every curve γ; we thus obtain the Dirac quantization condition

$$2eg = n \qquad n\in\mathbb{Z} \tag{2.1.2}$$

In a compact formulation, like that on the lattice, the string is invisible and a violation of the Bianchi identity is associated to the monopole.

A mathematically neater treatment of the abelian monopole is the one introduced in [14]. We consider for the sake of simplicity just the case $\mathbf{n} = \mathbf{e}_z$: it is possible to introduce two different vector potentials, one for the northern hemisphere and one for southern one

$$\begin{cases} \mathbf{A}^N = g\dfrac{(1-\cos\theta)}{r\sin\theta}\,\mathbf{e}_\phi & 0\le\theta<\pi/2+\epsilon \\ \mathbf{A}^S = -g\dfrac{(1+\cos\theta)}{r\sin\theta}\,\mathbf{e}_\phi & \pi/2-\epsilon<\theta\le\pi \end{cases} \tag{2.1.3}$$

For this description to be consistent it is necessary that $\mathbf{A}^N$ and $\mathbf{A}^S$ be gauge equivalent on the overlapping region $\pi/2-\epsilon<\theta<\pi/2+\epsilon$. It is simple to show that by using the gauge transformation $U(\phi) = e^{-2ieg\phi}$ we have $\mathbf{A}^S\to\mathbf{A}^S+\frac{i}{e}(\nabla U)U^{-1} = \mathbf{A}^N$. For this gauge transformation to be single valued we also have to impose $U(0) = U(2\pi)$ obtaining again the Dirac quantization condition Eq. (2.1.2). In this formulation there are no singularities but the vector potential is not defined globally.

2.2. The 't Hooft-Polyakov monopole

While in abelian gauge theories monopoles have to be introduced by hand, in the non-abelian ones they are naturally present, as first shown in [6, 7] for the Georgi-Glashow model ([15]). In this model an $SU(2)$ gauge field interacts with a Higgs field in the adjoint representation through the lagrangian density

$$L = -\frac{1}{2}\mathrm{Tr}\, G_{\mu\nu}G^{\mu\nu} + \mathrm{Tr}\,(D_\mu\phi)^\dagger(D^\mu\phi) - \frac{\lambda}{4}(2\,\mathrm{Tr}\,\phi^2 - v^2)^2$$

where the group generators are normalized according to $\mathrm{Tr}\,(T^aT^b) = \frac{1}{2}\delta^{ab}$. For a static configuration the energy is given by

$$E = \int \mathrm{d}^3x \left[\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \frac{1}{2}(D_\mu\phi^a)^\dagger(D^\mu\phi^a) + \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2\right]$$

and a necessary condition for the total energy of a static configuration to be finite is

$$\phi^a\phi^a \to v^2 \quad \text{as} \quad |\mathbf{x}| \to \infty \tag{2.2.1}$$

For non zero v values the gauge symmetry is spontaneously broken to $U(1)$ and from the relation Eq. (2.2.1) it follows that to each field configuration is associated a map from the sphere at infinity S^2 to the manifold of possible vacua, which is also S^2 since it is determined by the equation $\phi^a\phi^a = v^2$; configurations can thus be classified by their winding number.

A convenient gauge to be used in order to reveal the existence of topologically stable configuration is the "hedgehog" gauge:

$$\phi^a(r) \to v\frac{r^a}{r} \quad \text{as} \quad r \to \infty \tag{2.2.2}$$

This mapping has winding number $n = 1$ and thus a configuration with this asymptotic is topologically stable against decay to trivial vacuum. To obtain a finite energy we must impose that asymptotically $D_\mu\phi^a = 0$; inserting in this equation the requirement Eq. (2.2.2) we obtain the large distance behavior of the gauge field:

$$A^a_k(r) \to \frac{1}{e}\epsilon_{ank}\frac{r_n}{r^2} \qquad A^a_0(r) \to 0 \tag{2.2.3}$$

The two previous results Eq. (2.2.2), Eq. (2.2.3) justify the t'Hooft-Polyakov ansatz

$$\xi = evr \quad \phi^a = \frac{r^a}{er^2}H(\xi) \quad A^a_n = \epsilon_{amn}\frac{r^m}{er^2}(1 - K(\xi)) \quad A^a_0 \equiv 0 \tag{2.2.4}$$

The specific form of the functions H, K will not be used in the sequel.

To show that a field configuration with asymptotic behavior Eq. (2.2.2), Eq. (2.2.3) has non vanishing magnetic charge it is first of all necessary to define an electromagnetic field strength $F_{\mu\nu}$. The identification of the e.m. field is simple in the so called unitary

gauge, defined by $\phi^a = \text{const.}$ and $\phi^a\phi^a = v^2$: because of the symmetry breaking two component of the gauge field become massive. It is then natural to identify the electromagnetic field with the A_μ component associated to the residual unbroken $U(1)$ symmetry, thus

$$A_\mu^{\text{em}} = \frac{\phi^a}{v} A_\mu^a \tag{2.2.5}$$

and the e.m. field strength is defined as usual by $F_{\mu\nu} = \partial_\mu A_\nu^{\text{em}} - \partial_\nu A_\mu^{\text{em}}$. In general $F_{\mu\nu}$ can be defined as the gauge invariant tensor which reduces to the abelian definition in the unitary gauge and its explicit expression will be derived in full generality in the following sections. For the moment we just quote the result for this specific model, introduced in [6]: if we define for the sake of simplicity $\hat{\phi}$ as the normalized Higgs field (ϕ/v) then we define

$$\frac{1}{2}F_{\mu\nu} = \text{Tr}(\hat{\phi}\, G_{\mu\nu}) - \frac{i}{e}\text{Tr}(\hat{\phi}\,[D_\mu\hat{\phi}, D_\nu\hat{\phi}]) \tag{2.2.6}$$

where the $\frac{1}{2}$ in front of $F_{\mu\nu}$ is a consequence of the normalization chosen for the generators. This expression is usually referred to as the 't Hooft tensor and it is not difficult to show that it can be rewritten as ([16])

$$\frac{1}{2}F_{\mu\nu} = \text{Tr}(\partial_\mu(\hat{\phi}A_\nu) - \partial_\nu(\hat{\phi}A_\mu)) - \frac{i}{e}\text{Tr}(\hat{\phi}[\partial_\mu\hat{\phi}, \partial_\nu\hat{\phi}]) \tag{2.2.7}$$

This form, although not manifestly gauge invariant, is best suited to understand the definition of $F_{\mu\nu}$: in the unitary gauge $\hat{\phi}$ is constant, so that the second term is identically zero and $F_{\mu\nu}$ is clearly the field strength of the residual $U(1)$ gauge symmetry. If instead we use the hedgehog gauge the fields are regular everywhere, so that only the second term in Eq. (2.2.7) contributes to the magnetic charge.

$$\text{Tr}(\hat{\phi}[\partial_\nu\hat{\phi}, \partial_\rho\hat{\phi}]) = \tfrac{1}{8}\hat{\phi}^a\partial_\nu\hat{\phi}\partial_\rho\hat{\phi}\text{Tr}(\sigma^a[\sigma^b, \sigma^c]) = \tfrac{1}{2}i\epsilon_{abc}\hat{\phi}^a\partial_\nu\hat{\phi}^b\partial_\rho\hat{\phi}^c$$
$$\partial_\mu h^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} = \tfrac{1}{2e}\epsilon^{\mu\nu\rho}\epsilon_{abc}\partial_\mu\hat{\phi}^a\partial_\nu\hat{\phi}^b\partial_\rho\hat{\phi}^c$$

Since the configuration is static we can define the dual by $\epsilon^{\mu\nu\rho}$ instead of using $\epsilon^{\mu\nu\rho\sigma}$. The winding number is defined by

$$\nu = \frac{1}{8\pi}\int \epsilon^{\mu\nu\rho}\epsilon_{abc}\partial_\mu\hat{n}^a\partial_\nu\hat{n}^b\partial_\rho\hat{n}^c\,\mathrm{d}V$$

and

$$Q_m = \frac{1}{4\pi}\int \partial_\mu h^\mu\,\mathrm{d}V = \frac{1}{4\pi}\int \frac{1}{2}\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}\,\mathrm{d}V =$$
$$= \frac{1}{8\pi e}\int \epsilon^{\mu\nu\rho}\epsilon_{abc}\partial_\mu\hat{\phi}^a\partial_\nu\hat{\phi}^b\partial_\rho\hat{\phi}^c\,\mathrm{d}V$$

The magnetic charge obtained by the flux at infinity is thus equal to $\frac{1}{e}$ times the winding number. For the 't Hooft-Polyakov monopole, defined by Eq. (2.2.4), the winding number is equal to 1 by construction, so the magnetic charge of this configuration is $Q_m = \frac{1}{e}$.

Since the elementary electric charge is equal to $Q_e = \frac{e}{2}$ we have $2Q_mQ_e = 1$, so that the 't Hooft-Polyakov monopole has a charge of 1 Dirac units (see Eq. (2.1.2)).

In the following we shell also need the expression of the 't Hooft-Polyakov monopole Eq. (2.2.4) in the unitary gauge. The gauge transformation that unwraps the hedgehog is

$$\begin{aligned} U(\phi,\theta) &= \exp\left(-i\phi\frac{\sigma^3}{2}\right)\exp\left(-i\theta\frac{\sigma^2}{2}\right)\exp\left(i\phi\frac{\sigma^3}{2}\right) \\ A_\mu^{\text{un}} &= U^\dagger A_\mu^{\text{hedg}} U - \frac{i}{e}U^\dagger\partial_\mu U \end{aligned} \tag{2.2.8}$$

This transformation is singular in the origin of the coordinate (where the monopole is located) and along the negative z axis. To control this last singularity it is convenient to use a regularized form of the θ polar angle [13]:

$$\Theta = \theta\frac{1+\cos\theta}{1+\cos\theta+\epsilon^2} \tag{2.2.9}$$

in the transformation Eq. (2.2.8). By applying this gauge transformation to the gauge field Eq. (2.2.4) we obtain (the temporal component trivially vanishes)

$$\begin{aligned} \mathbf{A} = -\frac{1}{2er}\Bigg\{ & \mathbf{e}_\phi\left(\frac{\cos\Theta-1}{\sin\theta}+(1-K)\sin(\Theta-\theta)\right)\sigma_3 \\ & + \Bigg[\mathbf{e}_\phi\left((1-K)(\cos(\Theta-\theta)-\frac{\sin\Theta}{\sin\theta}\right)\sigma_1 + \\ & + \mathbf{e}_\theta(\Theta'-1+K)\sigma_2\Bigg](\cos(\phi)+i\sigma_3\sin(\phi))\Bigg\} \end{aligned} \tag{2.2.10}$$

where $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ are the versors of the two polar coordinates. In the *naive* limit $\epsilon \to 0$ the previous expression becomes the Dirac-like potential

$$\mathbf{A} = \frac{(1-K)}{2er}\,\frac{1-\cos\theta}{\sin\theta}\mathbf{e}_\phi\sigma^3$$

The gauge potential is singular along the negative z axes, however, contrary to what happens in the abelian case, by using the complete expression Eq. (2.2.10) it is not difficult to show (see *e.g.* [13], App. C) that the magnetic field is regular everywhere except at the origin. Its only asymptotically (for large r) non-vanishing component is the radial one, which is given by

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) = \frac{1}{2er^2}\sigma^3 \tag{2.2.11}$$

In the unitary gauge, from Eq. (2.2.5) it follows that the abelian magnetic field h_μ is just the projection of the non-abelian one:

$$h_r = 2\text{Tr}\left(\frac{\sigma^3}{2}B_r\right) = \frac{1}{er^2}$$

(the 2 comes from the $\frac{1}{2}$ multiplying $F_{\mu\nu}$ in Eq. (2.2.6)) from which we have again that $Q_m = \frac{1}{e}$ for the 't Hooft-Polyakov monopole.

2.3. Monopoles in pure gauge theories

We shall now discuss the so-called dynamical or GNO (Goddard, Nuyts, Olive) [17, 18] classification of magnetic monopole configurations in gauge theories; in this framework a magnetic monopole is defined as a static solution of the equations of motion with zero non-abelian electric field whose asymptotic behaviour is $\sim \frac{1}{r}$. The starting point of this classification is a gauge configuration that satisfies

$$A_0 = 0 \quad \frac{\partial \mathbf{A}}{\partial t} = 0 \tag{2.3.1}$$

in order to guarantee that the non-abelian electric field vanishes. Some of the residual gauge freedom can be used to impose the conditions that $A_r = 0$: if we define the gauge transformation U by

$$U = P \exp\left(ie \int_r^\infty A(r, \theta, \phi) \mathrm{d}r \right) \tag{2.3.2}$$

then clearly $\frac{\partial U}{\partial r} + ieA_r U = 0$ and

$$A_r^U = -iU^\dagger \frac{\partial U}{\partial r} + U^\dagger A_r U = 0$$

The gauge transformation Eq. (2.3.2) can have a singularity in the point $r = 0$, however this is not a problem since we are interested in the asymptotic region $r \to \infty$.

In an analogous way we can impose the condition $A_\theta = 0$ by using the parallel transport along meridian (lines at fixed ϕ value) as the gauge transformation. Again it is possible for this procedure to introduce spurious singularities, this time along the negative z axes ($\theta = \pi$).

Having exploited the gauge symmetries of the problem we are left with a field whose only non-vanishing component is the one directed along the ϕ axes. It is convenient to introduce a_ϕ by

$$\mathbf{A} = \frac{a_\phi(r, \theta, \phi)}{r \sin\theta} \mathbf{e}_\phi$$

As far as the leading order term is concerned we can now neglect the dependence of a_ϕ on the r variable. With this approximation the only non-vanishing component of the field strength is $G_{\theta\phi} = \partial_\theta a_\phi$.

The equations of motion of pure Yang-Mills theory in general orthogonal coordinates are

$$\partial_\mu \left(\sqrt{-g}\, G^{\mu\nu} \right) + ie[A_\mu, \sqrt{-g}\, G^{\mu\nu}] = 0 \tag{2.3.3}$$

where g is the determinant of the metric. If we use spatial polar coordinates we have

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix} \qquad \sqrt{-g} = r^2 \sin\theta \qquad G^{\phi\theta} = \frac{1}{r^4 \sin^2\theta} G_{\phi\theta}$$

The equation with $\nu = \theta$ is

$$\partial_\theta \frac{1}{\sin\theta} \partial_\theta a_\phi = 0 \tag{2.3.4}$$

and to have a field $\mathbf{A}$ regular at $\theta = 0$ we have to impose the condition $a_\phi(\theta = 0, \phi) = 0$, obtaining

$$a_\phi(\theta, \phi) = Q(\phi)(1 - \cos\theta)$$

The other equation of motion, the one with $\nu = \phi$ in Eq. (2.3.3), is then $\partial_\phi Q(\phi) = 0$ and we conclude that the leading order of the asymptotic gauge field must be

$$\mathbf{A} = Q \frac{1 - \cos\theta}{r \sin\theta} \mathbf{e}_\phi \tag{2.3.5}$$

This is, up to the constant matrix Q, the field of an abelian Dirac monopole, Eq. (2.1.1). In the solution of Eq. (2.3.4) we could have imposed the condition $a_\phi(\theta = \pi, \phi) = 0$, which ensures the field $\mathbf{A}$ to be regular at the south pole instead of the north pole. With this choice the final solution becomes

$$\mathbf{A} = -Q \frac{1 + \cos\theta}{r \sin\theta} \mathbf{e}_\phi \tag{2.3.6}$$

As noted in Sec. (2.1), the two forms Eq. (2.3.5) and Eq. (2.3.6) are transformed into each other by the gauge transformation $U = \exp(i2eQ\phi)$. The condition for this transformation to be well defined is the Dirac condition, which is

$$2eQ = n \quad n \in \mathbb{N} \tag{2.3.7}$$

The matrix Q has to be an element of the group algebra, it can be diagonalized and its eigenvalues are the charges of the residual $U(1)^r$ symmetry, where r is the rank of the gauge group (the maximal number of commuting generators). A convenient basis for the diagonal matrices to be used in the following is the one of the fundamental weights of the gauge group algebra $\hat{\mu}_i$ (our notations are explained in Sec. (A)): to each fundamental weight we can associate the abelian monopole field obtained by projecting on it the non abelian gauge field A_μ. From Eq. (2.3.7) it follows that the charge of the monopole expressed in Dirac units (*i.e.* n) is the winding number of the monopole in the residual $U(1)$ of the unitary representation.

This construction amounts to break the gauge group G to the little group of $\hat{\mu}^i$, which we will denote by $\tilde{H}$. The configurations can thus be classified according to the homotopy group $\pi_2(G/\tilde{H})$. The general form of the little group $\tilde{H}$ of the fundamental weight $\hat{\mu}^i$ is [21]

$$\tilde{H} = H \times U(1) \tag{2.3.8}$$

where H is the group whose Dynkin diagram is obtained by erasing from the diagram of G the root $\vec{\alpha}^{\,i}$ and the links which connect it to others roots (Levi subgroup). Indeed the operators associated with the roots $\vec{\alpha}^{\,j}$, $j \neq i$, commute with $\hat{\mu}^i$.

To calculate $\pi_2(G/\tilde{H})$ it is useful to remember the relation (see *e.g.* [19, 20])

$$\pi_2(G/\tilde{H}) = \ker(\pi_1(\tilde{H}) \to \pi_1(G)) \tag{2.3.9}$$

in order to reduce the problem to the evaluation of $\pi_1(\tilde{H})$. Let us consider as an example the case of the special unitary group $SU(N)$. The Levi subgroup of the fundamental weight $\hat{\mu}^i$, $i = 1, \ldots, N-1$ is $H_i = SU(i) \times SU(N-i)$, as can be seen also from the explicit form of the $\hat{\mu}^i$ operator (see Sec. (A)). Since $SU(N)$ is simply connected, $\pi_1\big(SU(N)\big) = 0$, from Eq. (2.3.9) we have

$$\pi_2(SU(N)/\tilde{H}_i) = \pi_1(\tilde{H}_i) = \pi_1(SU(i) \times SU(N-i) \times U(1)) = \mathbb{Z}$$

A list of the possible symmetry breakings that can happen in a simple group and their homotopy groups is shown in Tab. (2.1).

2.4. The 't Hooft tensor

The 't Hooft tensor is defined as the gauge invariant tensor which coincides with the residual abelian field strength in the unitary gauge. The explicit form Eq. (2.2.6) was introduced for $SU(2)$ gauge theories and then shown to be valid also for $SU(N)$ theories, however it is not the correct one for other gauge groups, like *e.g.* for the G_2 group. The reason is that for groups different from the $SU(N)$ ones the quadratic terms present in Eq. (2.2.6) do not cancel; in particular the identity Eq. (2.2.7) is not satisfied and $F_{\mu\nu}$ is gauge invariant but it is no more the abelian field strength in the unitary gauge.

We will discuss the definition of the magnetic field of the $i-$th monopole, related to the $U(1)$ subgroup associated to the $i-$th fundamental weight $\vec{\mu}^i$ following [21]: by starting from the expression in the unitary gauge and then performing a gauge transformation we will obtain the form of the 't Hooft tensor for the general group case.

In this general setting the unitary gauge is by definition the gauge in which the operator associated to this weight, $\hat{\mu}^i$, is diagonal. In we denote by A'_μ the field in the unitary gauge, the natural generalization of Eq. (2.2.5) is

$$A^{\rm em}_\mu = \frac{1}{K}\mathrm{Tr}\big(\phi^i_0 A'_\mu\big) \qquad \phi^i_0 = \hat{\mu}^i \qquad K = \mathrm{Tr}(T^{a\,\dagger} T^a) \tag{2.4.1}$$

The fields in the unitary gauge, ϕ^i_0, A'_μ, are related to the general gauge ones by the gauge transformation b:

$$\begin{cases} A'_\mu = b A_\mu b^\dagger + \frac{i}{e}(\partial_\mu b) b^\dagger \\ \phi^i_0 = b \phi^i b^\dagger \end{cases}$$

If we introduce the notation

$$\Omega_\mu = \frac{i}{e} b^\dagger (\partial_\mu b) \tag{2.4.2}$$

it is simple to show that in a general gauge the e.m. field strength $F^i_{\mu\nu} = \partial_\mu A^{\rm em}_\nu - \partial_\nu A^{\rm em}_\mu$ can be written as

$$K\, F^i_{\mu\nu} = \mathrm{Tr}(\phi^i G_{\mu\nu}) - ie\mathrm{Tr}\big(\phi^i [A_\mu + \Omega_\mu, A_\nu + \Omega_\nu]\big) \tag{2.4.3}$$

In order to simplify the notation we introduce $V_\mu = A_\mu + \Omega_\mu$. From the simple identity

$$\mathrm{Tr}\big(\phi^i [V_\mu, V_\nu]\big) = \mathrm{Tr}\big(V_\nu [\phi^i, V_\mu]\big)$$

G	$H \times U(1)$	λ_I	$\pi_2(G/\tilde{H})$
$SU(n)$	$SU(n-m) \times SU(m) \times U(1)$	1	$\mathbb{Z}$
$SO(2n+1)$	$SO(2n-1) \times U(1)$	1	$\mathbb{Z}$
$SO(2n+1)$	$SO(2m+1) \times SU(n-m) \times U(1)$	1,4	$\mathbb{Z}$
$SO(2n+1)$	$SU(n) \times U(1)$	1,4	$\mathbb{Z}/Z_2$
$SO(2n)$	$SO(2n-2) \times U(1)$	1	$\mathbb{Z}$
$SO(2n)$	$SO(2m) \times SU(n-m) \times U(1)$	1,4	$\mathbb{Z}$
$SO(2n)$	$SU(n-2) \times SU(2) \times SU(2) \times U(1)$	1,4	$\mathbb{Z}/Z_2$
$SO(2n)$	$SU(n) \times U(1)$	1	$\mathbb{Z}/Z_2$
$Sp(2n)$	$Sp(2m) \times SU(n-m) \times U(1)$	1,4	$\mathbb{Z}$
$Sp(2n)$	$SU(n-1) \times SU(2) \times U(1)$	1,4	$\mathbb{Z}$
$Sp(2n)$	$SU(n) \times U(1)$	1	$\mathbb{Z}$
G_2	$SU(2) \times U(1)$	1,4,9	$\mathbb{Z}$
G_2	$SU(2) \times U(1)$	1,4	$\mathbb{Z}$
F_4	$Sp(6) \times U(1)$	1,4	$\mathbb{Z}$
F_4	$SU(3) \times SU(2) \times U(1)$	1,4,9	$\mathbb{Z}$
F_4	$SU(3) \times SU(2) \times U(1)$	1,4,9,16	$\mathbb{Z}$
F_4	$Spin(7) \times U(1)$	1,4	$\mathbb{Z}$
E_6	$Spin(10) \times U(1)$	1	$\mathbb{Z}$
E_6	$SU(5) \times SU(2) \times U(1)$	1,4	$\mathbb{Z}$
E_6	$SU(6) \times U(1)$	1,4	$\mathbb{Z}$
E_6	$SU(3) \times SU(3) \times SU(2) \times U(1)$	1,4,9	$\mathbb{Z}$
E_7	$Spin(12) \times U(1)$	1,4	$\mathbb{Z}$
E_7	$SU(7) \times U(1)$	1,4	$\mathbb{Z}$
E_7	$SU(6) \times SU(2) \times U(1)$	1,4,9	$\mathbb{Z}$
E_7	$SU(4) \times SU(3) \times SU(2) \times U(1)$	1,4,9,16	$\mathbb{Z}$
E_7	$SU(5) \times SU(3) \times U(1)$	1,4,9	$\mathbb{Z}$
E_7	$Spin(10) \times SU(2) \times U(1)$	1,4	$\mathbb{Z}$
E_7	$E_6 \times U(1)$	1	$\mathbb{Z}$
E_8	$Spin(14) \times U(1)$	1,4	$\mathbb{Z}$
E_8	$SU(8) \times U(1)$	1,4,9	$\mathbb{Z}$
E_8	$SU(7) \times SU(2) \times U(1)$	1,4,9,16	$\mathbb{Z}$
E_8	$SU(5) \times SU(3) \times SU(2) \times U(1)$	1,4,9,16,25,36	$\mathbb{Z}$
E_8	$SU(5) \times SU(4) \times U(1)$	1,4,9,16,25	$\mathbb{Z}$
E_8	$Spin(10) \times SU(3) \times U(1)$	1,4,9,16	$\mathbb{Z}$
E_8	$E_6 \times SU(2) \times U(1)$	1,4,9	$\mathbb{Z}$
E_8	$E_7 \times U(1)$	1,4	$\mathbb{Z}$

Table 2.1.: Symmetry breaking of the generic simple compact group G to the residual subgroup $\tilde{H}$, the corresponding value of λ_I (defined at page 14) and the homotopy group $\Pi_2(G/\tilde{H})$. (From Ref. [21]).

it follows that only the component of V_μ that does not commute with ϕ^i contributes to the second term of Eq. (2.4.3). By expanding V_μ in the form

$$V_\mu = \sum_{\vec{\alpha}} V_\mu^{\vec{\alpha}}\, bE_{\vec{\alpha}} b^\dagger + \sum_{j=1}^{r} V_\mu^j\, bH_j b^\dagger$$

and noting that

$$[\phi^i, bH_j b^\dagger] = 0 \quad [\phi^i, bE_{\vec{\alpha}} b^\dagger] = (\vec{\mu}^{\,i}\cdot\vec{\alpha}) bE_{\vec{\alpha}} b^\dagger$$

it is clear that the projection $P^i V_\mu$ of V_μ on its component that does not commute with ϕ^i is given by

$$P^i V_\mu = \left\{ 1 - {\prod_{\vec{\alpha}}}' \left(1 - \frac{[\phi^i, [\phi^i, \cdot\,]]}{(\vec{\mu}^{\,i}\cdot\vec{\alpha})^2} \right) \right\} V_\mu \tag{2.4.4}$$

where the product runs on the roots $\vec{\alpha}$ such that $\vec{\mu}^{\,i}\cdot\vec{\alpha} \neq 0$ and only one representative is taken of the set of the roots having the same value of $(\vec{\mu}^{\,i}\cdot\vec{\alpha})^2$. For the sake of notation simplicity we will denote by λ_I^i the non-zero values that $(\vec{\mu}^{\,i}\cdot\vec{\alpha})^2$ can assume.

With the new notations the expression in Eq. (2.4.3) can be rewritten as

$$K\, F_{\mu\nu}^i = \mathrm{Tr}(\phi^i G_{\mu\nu}) - ie\mathrm{Tr}(\phi^i [P^i V_\mu, V_\nu]) \tag{2.4.5}$$

To make this expression more transparent it is convenient to note that the covariant derivative of ϕ^i can be written as

$$D_\mu \phi^i = ie[V_\mu, \phi^i] \tag{2.4.6}$$

and that the projector $P^i V_\mu$ has the expansion

$$P^i V_\mu = \sum_I \frac{1}{\lambda_I} [\phi^i, [\phi^i, V_\mu]] - \sum_{I\neq J} \frac{1}{\lambda_I \lambda_J} [\phi^i, [\phi^i, [\phi^i, [\phi^i, V_\mu]]]] + \cdots$$

The contribution of the first order of the projector in Eq. (2.4.5) is

$$-\frac{ie}{\lambda_I}\mathrm{Tr}\Big\{\phi^i[[\phi^i,[\phi^i,V_\mu]],V_\mu]\Big\} = -\frac{ie}{\lambda_I}\mathrm{Tr}\Big\{[V_\nu,\phi^i][\phi^i,[\phi^i,V_\mu]]\Big\} =$$
$$= -\frac{ie}{\lambda_I(-ie)(ie)}\mathrm{Tr}\Big\{D_\nu\phi^i[\phi^i,D_\mu\phi^i]\Big\} = -\frac{i}{e\lambda_I}\mathrm{Tr}\Big\{\phi^i[D_\mu\phi^i,D_\nu\phi^i]\Big\}$$

while from the $n-$th term of the projector expansion we get (apart from the multiplicative coefficient $(\lambda_{I_1}\cdots\lambda_{I_n})^{-1}$)

$$-ie(-1)^{n+1}\mathrm{Tr}\Big\{\phi^i[\overbrace{[\phi^i,\cdots,[\phi^i}^{2n},V_\mu]\cdots],V_\nu]\Big\} =$$
$$= +ie(-1)^n\mathrm{Tr}\Big\{[V_\nu,\phi^i]\overbrace{[\phi^i,\cdots,[\phi^i}^{2n},V_\mu]\cdots]\Big\} =$$
$$= \frac{ie(-1)^n}{(-ie)(ie)}\mathrm{Tr}\Big\{D_\nu\phi^i\overbrace{[\phi^i,\cdots,[\phi^i}^{2n-1},D_\mu\phi^i]\cdots]\Big\} =$$
$$= +\frac{i(-1)^n}{e}\mathrm{Tr}\Big\{[D_\nu\phi^i,\phi^i]\overbrace{[\phi^i,\cdots,[\phi^i}^{2n-2},D_\mu\phi^i]\cdots]\Big\}$$

By iteration of the last step we obtain

$$\begin{aligned}
&\frac{i(-1)^n}{e}\mathrm{Tr}\Big\{[\cdots[D_\nu\phi^i,\overbrace{\phi]^i,\cdots,\phi^i}^{n-1}]\overbrace{[\phi^i,\cdots,[\phi^i}^{n},D_\mu\phi^i]\cdots]\Big\}=\\
&=\frac{i(-1)^n}{e}\mathrm{Tr}\Big\{\phi^i\big[\ \overbrace{[\phi^i,\cdots,[\phi^i}^{n-1},D_\mu\phi^i]\cdots],[\cdots[D_\nu\phi^i,\overbrace{\phi^i],\cdots,\phi^i}^{n-1}]\ \big]\Big\}=\\
&=\frac{i(-1)^n(-1)^{n-1}}{e}\mathrm{Tr}\Big\{\phi^i\big[\ [\cdots[D_\mu\phi^i,\overbrace{\phi^i],\cdots,\phi^i}^{n-1}],[\cdots[D_\nu\phi^i,\overbrace{\phi^i],\cdots,\phi^i}^{n-1}]\ \big]\Big\}=\\
&=-\frac{i}{e}\mathrm{Tr}\Big\{\phi^i\big[\ [\cdots[D_\mu\phi^i,\overbrace{\phi^i],\cdots,\phi^i}^{n-1}],[\cdots[D_\nu\phi^i,\overbrace{\phi^i],\cdots,\phi^i}^{n-1}]\ \big]\Big\}
\end{aligned}$$

and the 't Hooft tensor can finally be written in the form

$$\begin{aligned}
K\,F^i_{\mu\nu}=\mathrm{Tr}\Big\{\phi^i G_{\mu\nu}\Big\}&-\frac{i}{e}\sum_I\frac{1}{\lambda_I}\mathrm{Tr}\Big\{\phi^i[D_\mu\phi^i,D_\nu\phi^i]\Big\}-\\
&-\frac{i}{e}\sum_{I\neq J}\frac{1}{\lambda_I\lambda_J}\mathrm{Tr}\Big\{\phi^i\big[\ [D_\mu\phi^i,\phi^i],[D_\nu\phi^i,\phi^i]\ \big]\Big\}-\cdots-\\
&-\frac{i}{e}\sum_{I_1\neq I_2\neq\cdots\neq I_n}\frac{1}{\lambda_{I_1}\cdots\lambda_{I_n}}\mathrm{Tr}\Big\{\phi^i\big[\ [\cdots[D_\mu\phi^i,\overbrace{\phi^i],\cdots,\phi^i}^{n-1}],\\
&\qquad\qquad[\cdots[D_\nu\phi^i,\overbrace{\phi^i],\cdots,\phi^i}^{n-1}]\ \big]\Big\}
\end{aligned}\tag{2.4.7}$$

In the particular case of the group $SU(N)$ the only non-vanishing λ_I has value 1, so we get the simpler expression Eq. (2.2.6), which is thus valid not only for $SU(2)$ but for the general $SU(N)$ case, as first shown in Ref. [22].

We note that the magnetic charge expressed in Dirac units is independent of the normalization of the generators: let us change the normalization by $T^a\to cT^a$, then

$$H_i\to cH_i\quad E_{\vec\alpha}\to cE_{c\vec\alpha}\quad\vec\alpha\to c\vec\alpha\quad\vec\mu\to\frac{1}{c}\vec\mu\quad\hat\mu\to\hat\mu$$

The field A_μ has to remain unchanged since we want to check the dependence of the charge of a given field configuration on the generators normalization. The only dependence of $F^i_{\mu\nu}$ is thus through the coefficient K, which scales as $K\to c^2K$, so $Q_m\to\frac{1}{c^2}Q_m$. On the other hand $Q_e\to c^2Q_e$, so the product Q_eQ_m is invariant.

We will now show that the general form of the 't Hooft tensor, Eq. (2.4.7), is linear in the gauge field following [23] (where the argument of [22] was generalized to the case of general groups). Let us denote by $X_2(A,\phi)$ the terms of $F^i_{\mu\nu}$ which depend quadratically on the gauge field A_μ: by using the expression for the covariant derivative in the adjoint

representation we get

$$\begin{aligned} K\,X_2(A,\phi) &= ie\mathrm{Tr}\Big\{\phi^i[A_\mu,A_\nu]\Big\} - \frac{i}{e}(+ie)^2\sum_I \frac{1}{\lambda_I}\mathrm{Tr}\Big\{\phi^i\big[[A_\mu,\phi^i],[A_\nu,\phi^i]\big]\Big\}+ \\ &- \frac{i}{e}(+ie)^2\sum_{I\neq J}\frac{1}{\lambda_I\lambda_J}\mathrm{Tr}\Big\{\phi^i\big[[[A_\mu,\phi^i],\phi^i],[[A_\nu,\phi^i],\phi^i]\big]\Big\}+\cdots = \\ &= +ie\Bigg(\mathrm{Tr}\Big\{\phi^i[A_\mu,A_\nu]\Big\}+\sum_I\frac{1}{\lambda_I}\mathrm{Tr}\Big\{\phi^i\big[[A_\mu,\phi^i],[A_\nu,\phi^i]\big]\Big\}+ \\ &+\sum_{I\neq J}\frac{1}{\lambda_I\lambda_J}\mathrm{Tr}\Big\{\phi^i\big[[[A_\mu,\phi^i],\phi^i],[[A_\nu,\phi^i],\phi^i]\big]\Big\}+\cdots\Bigg) \end{aligned} \tag{2.4.8}$$

Since X_2 is gauge invariant it can be calculated in the gauge in which ϕ^i is diagonal ($\phi^i=\phi_0^i$). By developing the gauge field in the Cartan base

$$A_\mu = \sum_i a_\mu^i H_i + \sum_{\vec\alpha} a_\mu^{\vec\alpha} E_{\vec\alpha} \tag{2.4.9}$$

we get for the first term of X_2

$$\mathrm{Tr}\Big\{\phi_0^i[A_\mu,A_\nu]\Big\} = \sum_{\vec\alpha\vec\beta} a_\mu^{\vec\alpha}a_\nu^{\vec\beta}\mathrm{Tr}\big\{\phi_0^i[E_{\vec\alpha},E_{\vec\beta}]\big\} = \sum_{\vec\alpha} a_\mu^{\vec\alpha}a_\nu^{-\vec\alpha}\mathrm{Tr}\big\{\phi_0^i[E_{\vec\alpha},E_{-\vec\alpha}]\big\}$$

and for the $n-$th term

$$\begin{aligned} &\mathrm{Tr}\Big\{\phi_0^i\big[[\cdots[A_\mu,\overbrace{\phi^i],\cdots,\phi^i}^{n}],[\cdots[A_\nu,\overbrace{\phi^i],\cdots,\phi^i}^{n}]\big]\Big\} = \\ &= \sum_{\vec\alpha\vec\beta} a_\mu^{\vec\alpha}a_\nu^{\vec\beta}\,\mathrm{Tr}\Big\{\phi_0^i[\big(-(\vec\mu^{\,i}\cdot\vec\alpha)\big)^n E_\alpha,\big(-(\vec\mu^{\,i}\cdot\vec\beta)\big)^n E_\beta]\Big\} = \\ &= (-1)^n\sum_{\vec\alpha} a_\mu^{\vec\alpha}a_\nu^{-\vec\alpha}(\vec\mu^{\,i}\cdot\vec\alpha)^{4n}\mathrm{Tr}\Big\{\phi_0^i[E_{\vec\alpha},E_{-\vec\alpha}]\Big\} \end{aligned}$$

We can thus rewrite $X_2(A,\phi^i)$ as

$$\begin{aligned} K\,X_2(A,\phi^i) &= +ie\sum_{\vec\alpha} a_\mu^{\vec\alpha}a_\nu^{-\vec\alpha}\mathrm{Tr}\big\{\phi_0^i[E_{\vec\alpha},E_{-\vec\alpha}]\big\}\times \\ &\times\Big[1-\sum_I\frac{1}{\lambda_I}(\vec\mu^{\,i}\cdot\vec\alpha)^2+\sum_{I\neq J}\frac{1}{\lambda_I\lambda_J}(\vec\mu^{\,i}\cdot\vec\alpha)^4-\cdots\Big] \end{aligned}$$

and, remembering that the λ_I^i are just the non zero values of $(\vec\mu^{\,i}\cdot\vec\alpha)^2$ each taken only once (see pag. 14), it is simple to show by induction that the second line vanishes, which means that the 't Hooft tensor is linear in the gauge field.

By using this result we can now proceed to show that in the general expression for the 't Hooft tensor the covariant derivatives can be avoided, as in Eq. (2.2.7). Let us introduce the notation $\Delta_{\mu\nu}$ by

$$KF^i_{\mu\nu} = \mathrm{Tr}\{\phi^i G_{\mu\nu}\} + \Delta_{\mu\nu} \tag{2.4.10}$$

that is

$$\Delta_{\mu\nu} = -\frac{i}{e}\sum_I \frac{1}{\lambda_I}\mathrm{Tr}\{\phi^i[D_\mu\phi^i, D_\nu\phi^i]\} - \\ -\frac{i}{e}\sum_{I\neq J}\frac{1}{\lambda_I\lambda_J}\mathrm{Tr}\Big\{\phi^i\big[[D_\mu\phi^i,\phi^i],[D_\nu\phi^i,\phi^i]\big]\Big\} - \dots$$

From Eq. (2.4.6) and Eq. (2.4.8) we get

$$X_2(A+\Omega,\phi) = +ie\mathrm{Tr}\Big\{\phi^i[A_\mu+\Omega_\mu, A_\nu+\Omega_\nu]\Big\} + \\ +ie\sum_I\frac{1}{\lambda_I}\mathrm{Tr}\Big\{\phi^i\big[[A_\mu+\Omega_\mu,\phi^i],[A_\nu+\Omega_\nu,\phi^i]\big]\Big\} + \dots = \\ = +ie\mathrm{Tr}\Big\{\phi^i[A_\mu+\Omega_\mu, A_\nu+\Omega_\nu]\Big\} + \\ +ie\left(-\frac{i}{e}\right)^2\sum_I\frac{1}{\lambda_I}\mathrm{Tr}\Big\{\phi^i\big[D_\mu\phi^i, D_\nu\phi^i\big]\Big\} + \dots = \\ = +ie\mathrm{Tr}\Big\{\phi^i[A_\mu+\Omega_\mu, A_\nu+\Omega_\nu]\Big\} + \Delta_{\mu\nu}$$

where Ω_μ is defined in Eq. (2.4.2). Since we have just shown that $X_2 \equiv 0$ we thus get

$$\Delta_{\mu\nu} = -ie\mathrm{Tr}\Big\{\phi^i[A_\mu+\Omega_\mu, A_\nu+\Omega_\nu]\Big\}$$

By inserting this expression in Eq. (2.4.10) and using $\partial_\mu\phi^i = ie[\Omega_\mu,\phi^i]$ (which is just Eq. (2.4.6) with $A_\mu \equiv 0$) we finally get

$$KF^i_{\mu\nu} = \mathrm{Tr}\Big\{\partial_\mu(\phi^i A_\nu) - \partial_\nu(\phi^i A_\mu)\Big\} - ie\mathrm{Tr}\Big\{\phi^i[\Omega_\mu,\Omega_\nu]\Big\} \tag{2.4.11}$$

which is the generalization of Eq. (2.2.7). For the special case of groups with only a single value for λ_I, namely 1, we can replace Ω_μ by $[\Omega_\mu,\phi^i]$ (up to a sign) since the commutator of ϕ^i with the elements of the algebra corresponding to positive roots reproduces the element, the commutator with the elements corresponding to negative roots reproduces them up to a sign; the trace then selects the products of elements corresponding to a root and to its negative. For this class of groups (which include the $SU(N)$ groups) we can thus rewrite Eq. (2.4.11) as

$$KF^i_{\mu\nu} = \mathrm{Tr}\Big\{\partial_\mu(\phi^i A_\nu) - \partial_\nu(\phi^i A_\mu)\Big\} - \frac{i}{e}\mathrm{Tr}\Big\{\phi^i[\partial_\mu\phi^i, \partial_\mu\phi^i]\Big\}$$

We will now give some examples to elucidate the relation between the 't Hooft tensor just defined and the topological classification of the monopoles discussed in the previous

section (in particular Tab. (2.1)). Let us consider again the gauge field of the 't Hooft-Polyakv monopole in the unitary gauge: the asymptotic non-abelian gauge field was shown to be

$$B_r = \frac{1}{er^2}\frac{\sigma^3}{2}$$

By using Eq. (2.4.1) we get for the charges

$$Q_m = \frac{1}{e} \quad Q_e = \frac{1}{2}e \quad 2Q_mQ_e = 1 \tag{2.4.12}$$

If we use instead the $SO(3)$ group, the proper generalization of the 't Hooft-Polyakov field is

$$B_r = \frac{1}{er^2}\frac{H_1}{2}$$

where we used the notation introduced in Sec. (A) for the $SO(2N+1)$ generators. We thus gets

$$Q_m = \frac{1}{2e} \quad Q_e = 2e \quad 2Q_mQ_e = 2 \tag{2.4.13}$$

These result are consistent with the topological classification: for $SU(N)$ the winding number can be any integer while for the breaking of $SO(2N+1)$ related to the $N-$th root only even winding numbers are allowed.

2.5. Relation with NABIs and consequences

In the previous section we deduced the generalized expression for the 't Hooft tensor, Eq. (2.4.7), starting from the one in the unitary gauge and then explicitly performing a gauge transformation. Nevertheless Eq. (2.4.7) defines a conserved magnetic current $j_\nu = \partial^\mu \tilde{F}_{\mu\nu}$ for any choice of the operator ϕ^i in the adjoint representation, which is usually referred to as abelian projection. Since for a general gauge configuration there seems to be no preferred direction in color space, in [8] the possibility was advocated that all abelian projection are equivalent.

To study the dependence on the abelian projection it is useful to relate the 't Hooft tensor to other gauge covariant quantities; in this section we will show its connection with the violations of the non-abelian Bianchi identity following [23].

If we denote by $\tilde{G}_{\mu\nu}$ the tensor dual to the field strength, *i.e.* $\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$, we can define the current

$$J_\nu = D^\mu \tilde{G}_{\mu\nu} \tag{2.5.1}$$

This equation is gauge covariant and therefore the vanishing (or not vanishing) of J_μ is a gauge invariant property. A non zero value of J_μ is by definition a violation of the non abelian Bianchi identities (NABIs). While the current defined by the violation of the abelian Bianchi identities, $j_\mu = \partial^\nu \tilde{F}^i_{\mu\nu}$, is conserved, it is simple to show that the current defined by Eq. (2.5.1) is only covariantly conserved:

$$D^\nu J_\nu = \frac{1}{2}[D^\nu, D^\mu]\tilde{G}_{\mu\nu} \propto [G^{\mu\nu}, \tilde{G}_{\mu\nu}] = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[G^{\mu\nu}, G^{\rho\sigma}] = 0$$

where the last equality follows from the fact that $\epsilon_{\mu\nu\rho\sigma}$ is even under the exchange $(\mu\nu) \rightleftarrows (\rho\sigma)$ while the commutator is odd.

We will now show that the 't Hooft tensor Eq. (2.4.7) is related to Eq. (2.5.1) by

$$K\,\partial^\mu \tilde{F}^i_{\mu\nu} = \mathrm{Tr}\big(\phi^i J_\nu\big) \tag{2.5.2}$$

The starting point is the identity

$$\partial^\mu \mathrm{Tr}(\phi^i \tilde{G}_{\mu\nu}) = \mathrm{Tr}(\phi^i D^\mu \tilde{G}_{\mu\nu}) + \mathrm{Tr}(D^\mu \phi^i \tilde{G}_{\mu\nu})$$

that can be rewritten, by using Eq. (2.5.1), in the form

$$\partial^\mu \mathrm{Tr}(\phi^i \tilde{G}_{\mu\nu}) = \mathrm{Tr}(\phi^i J_\nu) + \mathrm{Tr}(D^\mu \phi^i\, \tilde{G}_{\mu\nu}) \tag{2.5.3}$$

In order to prove Eq. (2.5.2) we sum the divergence of the dual of the tensor $\Delta_{\mu\nu}$ (defined in Eq. (2.4.10)) to both sides of the previous equality, thus obtaining

$$K\,\partial^\mu \tilde{F}^i_{\mu\nu} = \mathrm{Tr}(\phi^i J_\nu) + R_\nu \tag{2.5.4}$$

where

$$R_\nu = \mathrm{Tr}(D^\mu \phi^i\, \tilde{G}_{\mu\nu}) + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial_\mu \Delta^{\rho\sigma} \tag{2.5.5}$$

We thus need to show that R_ν vanishes. It convenient to write the second term of the right-hand side of the previous equation as the sum of two terms B_ν and C_ν,

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial_\mu \Delta^{\rho\sigma} = B_\nu + C_\nu$$

where B_ν collects all the terms with a double covariant derivative of ϕ^i while C_ν is the sum of the terms with just one covariant derivative. Explicitly we have

$$\begin{aligned} B_\nu = &-\frac{i}{e}\sum_I \frac{1}{\lambda_I}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big\{\phi^i[D^\mu D^\rho \phi^i, D^\sigma \phi^i]\big\} - \\ &-\frac{i}{e}\sum_{I\neq J}\frac{1}{\lambda_I\lambda_J}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\Big\{\phi\big[[D^\mu D^\rho \phi^i, \phi^i], [D^\sigma \phi^i, \phi^i]\big]\Big\} - \dots \end{aligned} \tag{2.5.6}$$

and

$$\begin{aligned} C_\nu = &-\frac{i}{2e}\sum_I \frac{1}{\lambda_I}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big\{D^\mu\phi^i[D^\rho \phi^i, D^\sigma \phi^i]\big\} - \\ &-\frac{i}{2e}\sum_{I\neq J}\frac{1}{\lambda_I\lambda_J}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\Big\{D^\mu\phi^i\big[[D^\rho \phi^i, \phi^i], [D^\sigma \phi^i, \phi^i]\big]\Big\} - \\ &-\frac{i}{e}\sum_{I\neq J}\frac{1}{\lambda_I\lambda_J}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\Big\{\phi^i\big[[D^\rho \phi^i, D^\mu\phi^i], [D^\sigma \phi^i, \phi^i]\big]\Big\} - \dots \end{aligned} \tag{2.5.7}$$

Let us now analyze the first term of Eq. (2.5.5): since it gets nonzero contribution only from the components of the field strength $G^{\rho\sigma}$ that do not commute with ϕ^i we can

use again the projector introduced in Eq. (2.4.4) and replace $G^{\rho\sigma}$ by $P^i G^{\rho\sigma}$. From the identity

$$\epsilon_{\mu\nu\rho\sigma}[\phi^i, G^{\rho\sigma}] = +\frac{i}{e}\epsilon_{\mu\nu\rho\sigma}[[D^\rho, D^\sigma], \phi^i] = +\frac{2i}{e}\epsilon_{\mu\nu\rho\sigma}D^\rho D^\sigma \phi^i$$

it follows that

$$\begin{aligned}\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^i G^{\rho\sigma} = &-\frac{i}{e}\epsilon_{\mu\nu\rho\sigma}\Big\{-\sum_I \frac{1}{\lambda_I}[\phi^i, D^\rho D^\sigma \phi]+ \\ &+\sum_{I\neq J}\frac{1}{\lambda_I\lambda_J}[\phi^i,[\phi^i,[\phi^i, D^\rho D^\sigma\phi^i]]] - \dots\Big\}\end{aligned}$$

By using the simple identities

$$\begin{aligned}&\mathrm{Tr}\big(D^\mu\phi^i[\phi^i, D^\rho D^\sigma\phi^i]\big) = \mathrm{Tr}\big(\phi^i[D^\rho D^\sigma\phi^i, D^\mu\phi^i]\big)\\ &\mathrm{Tr}\big(D^\mu\phi^i[\phi^i,[\phi^i, D^\rho D^\sigma\phi^i]]\big) = -\mathrm{Tr}\big(\phi^i[[D^\rho D^\sigma\phi^i,\phi^i],[D^\mu\phi^i,\phi^i]]\big)\end{aligned}$$

and their generalizations we then arrive to

$$\begin{aligned}\mathrm{Tr}\big(D^\mu\phi^i\,\widetilde{P^iG}_{\mu\nu}\big) = &+\frac{i}{e}\epsilon_{\mu\nu\rho\sigma}\sum_I\frac{1}{\lambda_I}\mathrm{Tr}\big(\phi^i[D^\mu D^\rho\phi^i, D^\sigma\phi^i]\big)+\\ &+\frac{i}{e}\epsilon_{\mu\nu\rho\sigma}\sum_{I\neq J}\frac{1}{\lambda_I\lambda_J}\mathrm{Tr}\big(\phi^i[[D^\mu D^\rho\phi^i,\phi^i],[D^\sigma\phi^i,\phi^i]]\big)+\dots = -B_\nu\end{aligned}$$

and thus $R_\nu = C_\nu$.

The last step of the proof of Eq. (2.5.2) consists in showing that $C_\nu = 0$. This has to be checked by direct computation. For the case of just one λ_I from Eq. (2.5.7) we have

$$C_\nu = -\frac{i}{2e}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\big(D^\mu\phi^i[D^\rho\phi^i, D^\sigma\phi^i]\big)$$

and by using again the expansion in the Cartan base of the gauge field Eq. (2.4.9) it is simple to arrive in the unitary gauge to the expression

$$C^\nu = -\frac{e^2}{2}\epsilon^{\mu\nu\rho\sigma}\sum_{\ell,j,k}a_\mu^{\vec\alpha_\ell}a_\rho^{\vec\alpha_j}a_\sigma^{\vec\alpha_k}(\vec\mu^{\,i}\cdot\vec\alpha_\ell)(\vec\mu^{\,i}\cdot\vec\alpha_j)(\vec\mu^{\,i}\cdot\vec\alpha_k)\mathrm{Tr}\big(E_{\vec\alpha_\ell}[E_{\vec\alpha_j}, E_{\vec\alpha_k}]\big)$$

where $\hat\mu^i$ is the fundamental weight which coincides with ϕ^i in the unitary representation. A necessary condition for this expression to be non-vanishing is the existence of a triple of roots such that

$$\vec\alpha_\ell + \vec\alpha_j + \vec\alpha_k = 0 \tag{2.5.8}$$

Moreover, each root must contain the simple root corresponding to the fundamental weight $\vec\mu^{\,i}$ with coefficients plus or minus one, otherwise $\vec\mu^{\,i}\cdot\vec\alpha$ would vanish. Since the simple roots are linearly independent the sum $\vec\alpha_\ell+\vec\alpha_j+\vec\alpha_k$ must then contain the simple root $\vec\alpha^i$ one or three times and cannot be zero. As a consequence $C_\nu = 0$ when only one λ_I is present.

If there are two different values of λ, namely $\lambda = 1, 4$, we gets

$$C^\nu = -\frac{e^2}{2}\sum_{\ell jk}\epsilon^{\mu\nu\rho\sigma}a_\mu^{\vec{\alpha}_\ell}a_\rho^{\vec{\alpha}_j}a_\sigma^{\vec{\alpha}_k}c^i_{\vec{\alpha}_\ell}c^i_{\vec{\alpha}_j}c^i_{\vec{\alpha}_k}\mathrm{Tr}\big(E_{\vec{\alpha}_\ell}[E_{\vec{\alpha}_j},E_{\vec{\alpha}_k}]\big)\times\Pi_2 \tag{2.5.9}$$

where we introduced the short-hand notation $c^i_{\vec{\alpha}_j}\equiv\vec{\mu}^{\,i}\cdot\vec{\alpha}_j$ and

$$\Pi_2 = \sum_I\frac{1}{\lambda_I}+\sum_{I\neq J}\frac{1}{\lambda_I\lambda_J}\left(c^i_{\vec{\alpha}_j}c^i_{\vec{\alpha}_k}-2(c^i_{\vec{\alpha}_\ell})^2\right) \tag{2.5.10}$$

The only possibility for Eq. (2.5.9) to be non-vanishing is when the simple root $\vec{\alpha}_i$ appears once and with the same sign in each of two roots, say $\vec{\alpha}_j$, $\vec{\alpha}_k$ and twice with opposite sign in the third one $\vec{\alpha}_\ell$, in order to satisfy Eq. (2.5.8). Since Eq. (2.5.9) is invariant under permutations of ℓ, j, k, we can take the mean over the permutations; moreover it follows from Eq. (2.5.8) that $c^i_{\vec{\alpha}_j}c^i_{\vec{\alpha}_k}=\frac{1}{2}(\lambda_\ell-\lambda_j-\lambda_k)$. The coefficient Π_2 is thus equal to

$$\Pi_2 = \frac{5}{4}-\frac{1}{4}\times\frac{5}{2}\langle\lambda\rangle$$

and by using $\langle\lambda\rangle = 2$ we get $C_\nu = 0$.

We have shown that Eq. (2.5.2) is satisfied for the groups with no more than two different λ_I values, *i.e.* the classical groups (see Tab. (2.1)). The proof for the exceptional groups is obtained in the same way but it is computationally more involved; a sketch is presented in Sec. (B).

An immediate consequence of the relation Eq. (2.5.2) is that the existence of a monopole in a given abelian projection is related to the violation of non-abelian Bianchi identities and thus the existence of the monopole is a gauge invariant property.

We can use Eq. (2.5.2) to study the dependence of the magnetic charge on the abelian projection. Once again we use as a test case the 't Hooft-Polyakov monopole: in the unitary gauge the "natural" abelian projection is the fundamental weight $\frac{1}{2}\sigma^3$, directed along the Higgs field *v.e.v.*, the gauge field is given by Eq. (2.2.10) and the non abelian magnetic field by Eq. (2.2.11), so that

$$\nabla\cdot\mathbf{h} = \frac{4\pi}{e}\delta^3(\mathbf{x}) \qquad \mathbf{D}\cdot\mathbf{B} = \frac{2\pi}{e}\delta^3(\mathbf{x})\sigma^3$$

The monopole has charge $Q_m = \frac{1}{e}$ and Eq. (2.5.2) trivially states that

$$\frac{1}{2}\nabla\cdot\mathbf{h} = \mathrm{Tr}\left(\frac{\sigma^3}{2}\mathbf{D}\cdot\mathbf{B}\right)$$

Let us now use the abelian projection $\phi = U(\mathbf{x})\frac{\sigma_3}{2}U^\dagger(\mathbf{x})$, where $U(\mathbf{x})$ is an unitary transformation. By using Eq. (2.2.10) we simply have

$$\nabla\cdot\mathbf{h} = \frac{2\pi}{e}\delta^3(\mathbf{x})\mathrm{Tr}\left(\sigma^3U^\dagger(\mathbf{x})\sigma^3U(\mathbf{x})\right)$$

and if we parametrize the U transformation as

$$U(\mathbf{x}) = \exp\left(i\alpha(\mathbf{x})\frac{\sigma^3}{2}\right)\exp\left(i\beta(\mathbf{x})\frac{\sigma^2}{2}\right)\exp\left(i\gamma(\mathbf{x})\frac{\sigma^3}{2}\right)$$

we obtain

$$\nabla\cdot\mathbf{h} = \frac{4\pi}{e}\cos\big(\beta(\mathbf{0})\big)\delta^3(\mathbf{x}) \qquad Q_m = \frac{\cos\big(\beta(\mathbf{0})\big)}{e}$$

from which it is clear that for general $U(\mathbf{x})$ the Dirac quantization condition cannot be satisfied. We thus have to conclude that the abelian projections are not all equivalent to define the dual abelian magnetic charge, *i.e.* there exits a natural choice of the gauge fixing. This preferred gauge is the unitary gauge, that can be characterized by the property of having all the components of the NABI's violation J_ν diagonal in color space (the existence of such a gauge is guaranteed by the Coleman Mandula theorem [24]).

For general monopole configurations we can use the theorem of Sec. (2.3) to extend the conclusion reached for the 't Hooft-Polyakov monopole: we have seen that a gauge exists in which the asymptotic field of the monopole assumes the form of Eq. (2.3.5), which we will denote by $A^{(\mathrm{un})}$. We can now add and subtract to it a field $A^{('\mathrm{tHP})}$ proportional to the 't Hooft Polyakov solution Eq. (2.2.10) in such a way that the field $A^{(\mathrm{un})} - A^{('\mathrm{tHP})}$ has zero magnetic charge (for this we need the linearity of the 't Hooft tensor in the gauge field) and we get

$$A_\mu = \underbrace{A_\mu^{(0)} + A_\mu^{(\mathrm{un})} - A_\mu^{('\mathrm{tHP})}}_{A_\mu^{(\mathrm{sub})}} + A_\mu^{('\mathrm{tHP})} \tag{2.5.11}$$

Since the subleading field $A_\mu^{(\mathrm{sub})}$ has zero magnetic charge, it does not produce violations in the NABI and the dominant contribution at large distance of the NABI violation for the general field A_μ is just the one of the 't Hooft-Polyakov like term $A_\mu^{('\mathrm{tHP})}$. The previous considerations on the abelian projection dependence of the magnetic field can thus be extended to general monopole configurations without changes.

We have just seen that the definition of the abelian magnetic charge is abelian projection dependent; we will now show that nevertheless the spontaneous breaking of the dual magnetic symmetry is instead independent of the specific projection used. Let $\hat{Q}$ be the magnetic charge operator associated to the $i-$th fundamental weight in the unitary gauge; by Eq. (2.5.2) it can be written in the form

$$\hat{Q} = \int \frac{1}{K}\mathrm{Tr}\big(\hat{\mu}^i J_0^{(\mathrm{un})}(\mathbf{x})\big)\mathrm{d}^3\mathbf{x} \tag{2.5.12}$$

where $J_0^{(\mathrm{un})}(\mathbf{x})$ is the temporal component of the NABI violation in the unitary gauge. An operator $\hat{O}(\mathbf{x})$ is magnetically charged if it satisfies

$$[\hat{Q}, \hat{O}(\mathbf{x})] = m\,\hat{O}(\mathbf{x}) \qquad m \neq 0$$

and the magnetic symmetry is spontaneously broken is a charged operator exists with non vanishing vacuum expectation value: $\langle\hat{O}(\mathbf{x})\rangle \neq 0$.

The magnetic charge in the generic projection, $\hat{Q}(U)$, is related to $\hat{Q}$ by the substitution $\hat{\mu}^i \to U(\mathbf{x})\hat{\mu}^i U^\dagger(\mathbf{x})$ in Eq. (2.5.12) and, since the integrand is gauge invariant, we can compute the trace in the gauge in which $\hat{\mu}^i$ and $J_0^{(\mathrm{un})}(\mathbf{x})$ are diagonal. Since $U(\mathbf{x})\hat{\mu}^i U^\dagger(\mathbf{x})$ is an element of the group algebra it can be expanded as

$$U(\mathbf{x})\hat{\mu}^i U^\dagger(\mathbf{x}) = \sum_i C_i(\mathbf{x})\hat{\mu}^i + \sum_{\vec{\alpha}} D_{\vec{\alpha}}(\mathbf{x}) E_{\vec{\alpha}}$$

By using this expansion in the charge definition we get

$$[\hat{Q}(U), \hat{O}(\mathbf{x})] = m\, C_i(\mathbf{x})\, \hat{O}(\mathbf{x})$$

and since $C_i(\mathbf{x})$ is generically non-vanishing the operator $\hat{O}(\mathbf{x})$ is charged also in the generic abelian projection.

3. Lattice gauge theories

3.1. The pure gauge theory

In this first section we will introduce the lattice discretization of pure gauge theories, without coupling to fermions. To justify the fundamental definitions it will be nevertheless convenient to introduce a matter field ϕ transforming in a given representation of the gauge group.

To construct a lattice action invariant under local gauge transformation it is possible to follow the same line of thought as in the continuum: we start from an action for the matter fields invariant under global gauge transformations and we introduce the gauge fields in order to make it also local gauge invariant. The construction of lattice action for the matter field is a non-trivial task and it will be analyzed in the following section. It is however clear that such an action will be the sum of terms of the forms $\phi(\mathbf{x})^\dagger\phi(\mathbf{x})$ and $\phi(\mathbf{x})^\dagger\phi(\mathbf{x}+\hat{\mu})$, where $\mathbf{x}$ is a lattice point (we will always use a hyper-cubical lattice of step a in the four dimensional euclidean space) and $\hat{\mu}$ is the versor of the μ-th axis.

If we let the gauge group act on the matter field through $\phi(\mathbf{x}) \to R(\mathbf{x})\phi(\mathbf{x})$ it is clear that the term $\phi(\mathbf{x})^\dagger\phi(\mathbf{x}+\hat{\mu})$ it is invariant under global gauge transformations but not under local ones. The most natural way to make it also local gauge invariant is to associate the gauge field $U_\mu(\mathbf{x})$ to the lattice link $(\mathbf{x}, \mathbf{x}+\hat{\mu})$ and to make it transform as $U_\mu(\mathbf{x}) \to R(\mathbf{x})U_\mu(\mathbf{x})R(\mathbf{x}+\hat{\mu})^\dagger$, in order to replace $\phi(\mathbf{x})^\dagger\phi(\mathbf{x}+\hat{\mu})$ with the local gauge invariant expression $\phi(\mathbf{x})^\dagger U_\mu(\mathbf{x})\phi(\mathbf{x}+\hat{\mu})$. The lattice gauge field is thus a mapping of the (oriented) links into the gauge group, while the continuum field is a mapping of the points into the group algebra. For convenience the adjoint of the lattice field is associated to links with the opposite orientation.

We now need to introduce an action for the gauge field, which has to be a gauge invariant expression in $U_\mu(\mathbf{x})$, and to relate the lattice gauge field $U_\mu(\mathbf{x})$ to the continuum field $A_\mu(\mathbf{x})$. Since the lattice gauge field is associated to the links, it can be seen as an elementary parallel transport and it is then reasonable to define

$$U_\mu(\mathbf{x}) = \exp\Big\{ - iaeA_\mu(\mathbf{x})\Big\} \tag{3.1.1}$$

where a is the lattice spacing and e will be identified with the continuum coupling. Gauge invariant quantities are naturally associated to closed path on the lattice, since the trace of the product of the gauge field along a closed path is gauge invariant. The simplest of these quantities is the one associated to the edges of the face of an elementary cube of the lattice:

$$P_{\mu\nu}(\mathbf{x}) = \mathrm{Tr}\Big\{U_\mu(\mathbf{x})U_\nu(\mathbf{x}+\hat{\mu})U_\mu(x+\hat{\nu})^\dagger U_\nu(\mathbf{x})^\dagger\Big\} \tag{3.1.2}$$

These elementary faces are usually called plaquettes and, by extension, also the associated invariants are denoted with the same name.

An action for the gauge field can be introduced in the form (Wilson action [1])

$$S = \sum_{\square} \beta \left[1 - \frac{1}{N} \mathrm{Re}\, P_{\mu\nu}(\mathbf{x}) \right] \tag{3.1.3}$$

where β is the lattice coupling, N is the dimension of the gauge group representation used and the sum is extended on the plaquettes of the lattice. It is simple to show, by using the Hausdorff-Campbell-Baker formula $\exp(M_1)\exp(M_2) = \exp(M_1 + M_2 + \frac{1}{2}[M_1, M_2] + \cdots)$ and Eq. (3.1.1), that

$$S = \sum_{\square} \beta \left(1 - \frac{1}{N} \mathrm{ReTr} \exp\left\{ ie^2 a^2 G_{\mu\nu} + O(a^4) \right\} \right)$$

where $G_{\mu\nu}$ is the usual continuum field strength. If we normalize the group generators according to $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$ we finally get

$$S = \sum_{\mathbf{x}} \frac{\beta e^2}{8N} a^4 G^a_{\mu\nu}(\mathbf{x}) G^a_{\mu\nu}(\mathbf{x}) + O(a^6)$$

The action Eq. (3.1.3), with the identification Eq. (3.1.1), thus reduces in the formal continuum limit to the Yang-Mills action if $\beta = \frac{2N}{e^2}$.

The vacuum vacuum expectation values of functionals of the lattice gauge fields can be calculated as functional integral: if $f(U)$ is a function of the gauge fields then

$$\langle f(U) \rangle = \frac{\int f(U) e^{-S(U)} \mathscr{D}U}{\int e^{-S(U)} \mathscr{D}U}$$

where

$$\mathscr{D}U = \prod_{\mathbf{x},\mu} \mathrm{d}U_\mu(\mathbf{x})$$

and $\mathrm{d}U_\mu(\mathbf{x})$ is the Haar measure. It is possible to show that if $f(U)$ is a function with compact support, which transforms according to a non trivial representation of the gauge group then $\langle f(U) \rangle = 0$ (for a proof see *e.g.* [25]); this statement is usually referred to as Elitzur theorem.

The potential between a quark anti-quark static pair can be extracted from the vacuum expectation value of the Wilson loop $W(R,T)$, which is the path-ordered product of the gauge field $U_\mu(\mathbf{x})$ along a rectangular loop of size $R \times T$, as follows (for the details see *e.g.* [26, 27])

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \log\langle W(R,T) \rangle$$

It can be shown (see [28, 29]) that two constants c_1, c_2 exist such that for asymptotically large R the bounds $c_1 R \leq V(R) \leq c_2/R$ are satisfied. For non-abelian gauge theories the typical asymptotic behaviour is $V(r) \sim \sigma R$ and the parameter σ is known as the

string tension. This linear behaviour is interpreted as a signal of quark confinement. The potential is however not always asymptotically linear with the distance: the gauge theory based on the G_2 group has vanishing string tension [30].

While for the typical models of statistical physics the coupling is a fixed parameter and the lattice has a physical meaning (*e.g.* a crystal lattice), for gauge theories the lattice is just an ultraviolet regulator that has to be introduced to allow the computations to be performed (analytically or numerically). For this reason the physically sensible results are those obtained in the limit $a \to 0$ or, more physically, $\xi \gg 1$, where ξ is the correlation length in lattice units, and the coupling β has to be tuned in order to satisfy this requirement.

By fixing the value of some observable to its physical value (*i.e.* fixing the physical correlation length) and using renormalization group arguments it can be shown (see *e.g.* [26, 27, 31]) that for $SU(N)$ gauge theories the lattice spacing a depends on the coupling constant through the expression

$$a = \Lambda_0^{-1} \left(\frac{2N\gamma_0}{\beta} \right)^{-\gamma_1/(2\gamma_0^2)} \exp\left[-\frac{\beta}{4N\gamma_0} \right] (1 + O(1/\beta)) \tag{3.1.4}$$

where

$$\gamma_0 = \frac{1}{16\pi^2}\frac{11N}{3}; \quad \gamma_1 = \frac{1}{(16\pi^2)^2}\frac{34N^2}{3}$$

and Λ_0 is an integration constant; from Eq. (3.1.4) it is clear that the continuum limit is reached for $\beta \to \infty$. On the other hand numerical simulations can handle only lattices of finite extent and to obtain reliable results the typical dimension of the lattice must be much larger that the correlation length. If we suppose to use a lattice of $N_t \times N_s^3$ sites, the coupling constant β has to be tuned in order to satisfy the inequalities

$$1 \ll \xi \ll N_s$$

and N_t is related to the physical temperature T by $aN_t = 1/T$. While periodic spatial boundary conditions are typically used just to minimize the finite size effects, periodic conditions along the temporal direction are needed in order to guarantee the interpretation of the vacuum expectation values as mean values at finite temperature of physical observables.

3.2. The deconfinement transition

While at low temperature color is confined, soon after the discovery of the asymptotic freedom it was conjectured [32] that at high enough temperature colored stated could appear in the spectrum of QCD.

At finite temperature confinement is related to the behaviour of the free energy of a static quark anti-quark couple at large distance. If we introduce the Polyakov loop as

$$L(\mathbf{x}) = \mathrm{Tr} \prod_{i=0}^{N_t-1} U_0(\mathbf{x} + i\hat{0}) \qquad 0 = \text{time direction} \tag{3.2.1}$$

it can be shown that the free energy of a static quark and anti-quark, positioned at $\mathbf{x}$ and $\mathbf{y}$ respectively, is given by (see *e.g.* [33])

$$e^{-\frac{1}{T}F(|\mathbf{x}-\mathbf{y}|)} = \langle L(\mathbf{x})L(\mathbf{y})^* \rangle \tag{3.2.2}$$

and free color charges can exist if $\lim_{\mathbf{r}\to\infty}\langle L(\mathbf{x})L(\mathbf{x}+\mathbf{r})^*\rangle \neq 0$. If we suppose the cluster property to be valid and use translation invariance we get

$$\lim_{r\to\infty} e^{-\frac{1}{T}F(r)} = |\langle L \rangle|^2$$

The mean value of the Polyakov loop is thus an order parameter for the deconfinement transition.

If the gauge group has a nontrivial center the deconfinement transition can be associated to the breaking of a global symmetry. Let us denote by z an element of the group center $Z(G)$; if we multiply by z all the temporal component of the gauge fields of a fixed time slice, it is simple to show that the Wilson action is invariant but the Polyakov loop transforms as $L(\mathbf{x}) \to zL(\mathbf{x})$. A non-vanishing value for the mean Polyakov loop thus signals the breaking of this center symmetry.

This observation naturally led to the following conjecture [34]: the universality class of the deconfinement transition for a lattice gauge theory in D dimensions with gauge group G coincides with that of the transition of a spin system in $D-1$ dimensions with global $Z(G)$ symmetry. As an example the deconfinement transition of the 4D $SU(2)$ lattice gauge theory is expected to be in 3D Ising universality class, which is indeed the behaviour observed in numerical simulations.

This guess is obviously reliable only for second order phase transitions, to which universality arguments apply, and does not take into account the possibility of first order phase transitions. Nevertheless it can be used also to predict first order transitions: if the deconfinement transition of the 4D $SU(3)$ lattice gauge theory was second order it would be in the 3D Z_3 universality class, however no infrared fixed points are known for the 3D Z_3 systems, so this transition is expected to be (and is) a first order transition.

This Svetitsky-Yaffe conjecture clearly does not account for the groups with trivial center, like the exceptional group G_2, for which the mean Polyakov loop is nonzero also in the low temperature phase. The possibility was advocated that the low and the high temperature regions of the G_2 lattice gauge theory are not separated by a phase transition, but are instead analytically connected. This proposal was refuted by numerical simulations and it is by now known that the two phases are separated by a first order transition [35, 36].

3.3. Fermions on the lattice

We will use for the euclidean gamma matrices the conventions

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}; \quad (\gamma^\mu)^\dagger = \gamma^\mu$$

so the continuum free fermion action takes the form

$$S_F = \int \bar{\psi}(x)(\gamma^\mu \partial_\mu + M)\psi(x)\mathrm{d}^4x$$

The simplest discretization of this action is the "naive" fermion action

$$S_F = \frac{1}{2}\sum_{n,\hat{\mu}}[\bar{\psi}_n\gamma^\mu\psi_{n+\hat{\mu}} - \bar{\psi}_{n+\hat{\mu}}\gamma^\mu\psi_n] + \hat{M}\sum_n \bar{\psi}_n\psi_n \tag{3.3.1}$$

where ψ_n is the fermion field on the n-th site of the lattice (rescaled by $a^{3/2}$) and $\hat{M} = aM$. This action, however, does not describe a single fermion field, as can be seen by looking at the limit $a \to 0$ of its Green function:

$$\langle\psi(x)\bar{\psi}(y)\rangle = \lim_{a\to 0}\int_{-\pi/a}^{\pi/a}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{M - i\sum_\mu \gamma^\mu \bar{k}_\mu}{\sum_\mu \bar{k}_\mu^2 + M^2}e^{ik(x-y)}; \quad \bar{k}_\mu = \frac{1}{a}\sin(k_\mu a)$$

Since the expression $\sum_\mu \bar{k}_\mu^2$ has 2^4 zeroes in the Brillouin zone $\left(-\frac{\pi}{a}, \frac{\pi}{a}\right]^4$, the continuum limit of the "naive" action Eq. (3.3.1) describes 2^4 fermion species, a phenomenon know as "fermion doubling".

This behaviour of the "naive" fermion action can be simply explained by the existence of a symmetry of the action Eq. (3.3.1) that in momentum space connects the states with momentum $\hat{k}$ and $\hat{k} + \pi\hat{\mu}$ ($\hat{k}$ is the adimensional lattice momentum defined in $(-\pi, \pi]^4$), thus producing the noted degeneration in the spectrum (see *e.g.* [27]). The deep origin of this phenomenon is however much more physical, not just related to an accidental symmetry of the action, and it can be traced back to the axial anomaly: the action Eq. (3.3.1) is, for $M = 0$, chiral invariant and, without doubling, it would be a regularization of the fermion field with no axial anomaly. Indeed it can be explicitly checked that the doublers cancel the axial anomaly (see *e.g.* [27, 37]).

A number of no-go theorems [38–41] prevent from the possibility of constructing a discretized fermion action with chiral invariance. It is indeed simple to show that a lattice Dirac operator D_L can not satisfy all of the following requirements:

1. in the limit $a \to 0$ a zero mass fermion is present in the spectrum
2. it has no doublers
3. it is local (*i.e.* its matrix elements decay fast enough with the distance)
4. it is chiral invariant

The general form of D_L is the momentum representation is $D_L(p) = \gamma_\mu F^\mu(p) + G(p)$; from the requirement (4) it follows that $G(p) \equiv 0$, from (1) it follows that $F_\mu(p) = p_\mu + O(ap^2)$ and because of (3) the function $F_\mu(p)$ is continuous. Since the momenta on the lattice are defined only modulo $2\pi/a$ it follows that F_μ also has zeroes at non-vanishing values of the momentum, *i.e.* D_L has doublers. In order to circumvent this result the assumption of chiral invariance has been relaxed in several ways during the years.

The first solution was proposed by Wilson and it amounts to add to the "naive" action a term irrelevant in the continuum limit but capable to lift the degeneracy between the doublers, which acquire a great mass and decouple. An example of such an action is

$$S_W = S_F - \frac{r}{2}\sum \bar{\psi}_n \left(\sum_{\hat{\mu}} \psi_{n+\hat{\mu}} + \psi_{n-\hat{\mu}} - 2\psi_n \right)$$

which gives rise to the correlation functions

$$\langle \psi(x)\bar{\psi}(y)\rangle = \lim_{a\to 0} \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{M(k) - i\sum_\mu \gamma^\mu \bar{k}_\mu}{\sum_\mu \bar{k}_\mu^2 + M(k)^2} e^{ik(x-y)}$$
$$\bar{k}_\mu = \frac{1}{a}\sin(k_\mu a); \qquad M(k) = M + \frac{2r}{a}\sum_\mu \sin^2(k_\mu a/2)$$

In the limit $a \to 0$ only region of integration near the origin gives a non-vanishing contribution to the correlation function, since $M \to \infty$ for $k \neq 0$.

In the Wilson approach the chiral symmetry is completely lost and it has to be recovered only in the continuum limit. A different approach is the so called "staggered fermion" method: the action is written in term of an auxiliary spinless field, and the physical fermion can be reconstructed by using the values of this field at 2^d different points. Although less intuitive than the Wilson method, this approach has the advantage to preserve part of the chiral symmetry and to be computationally less demanding. We will illustrate the idea of the method in two dimension for the sake of the simplicity, but the generalization to higher dimension is just a simple technical problem. The starting point is the action

$$S = \sum_{i,\hat{\mu}} \bar{\chi}_i (-1)^{\phi(i,\mu)} [\chi_{i+\hat{\mu}} - \chi_{i-\hat{\mu}}] \tag{3.3.2}$$

where χ_i are Grassmann variables without spinor structure and the phases $(-1)^{\phi(i,\mu)}$ are defined by

$$(-1)^{\phi(i,0)} = +1 \quad (-1)^{\phi(i,1)} = (-1)^{|n_0|}$$

where n_0 is the temporal coordinate of i-th lattice site. Let us now group together blocks of 2×2 sites, which will be denoted by the indexes

2	$1'$
1	$2'$

and introduce a fermion doublet (ψ^a, ψ^b) by

$$\psi^a = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}; \quad \bar{\psi}^a = \begin{pmatrix} \bar{\chi}_1 & \bar{\chi}_2 \end{pmatrix}$$
$$\psi^b = i\begin{pmatrix} \chi_{1'} \\ \chi_{2'} \end{pmatrix}; \quad \bar{\psi}^b = -i\begin{pmatrix} \bar{\chi}_{1'} & \bar{\chi}_{2'} \end{pmatrix}$$

By using the notation $\nabla_\mu \chi_i \equiv \chi_{i+\hat{\mu}} - \chi_{i-\hat{\mu}}$ the action Eq. (3.3.2) can be rewritten as

$$S = \sum_{\text{blocks}} \Big\{ \bar{\psi}_1^a (\nabla_0 \psi_2^a - i\nabla_1 \psi_2^b) + \bar{\psi}_2^a (\nabla_0 \psi_1^a + i\nabla_1 \psi_1^b) + \\ + \bar{\psi}_1^b (\nabla_0 \psi_2^b - i\nabla_1 \psi_2^a) + \bar{\psi}_2{}^b (\nabla_0 \psi_1^b + i\nabla_1 \psi_1^a) \Big\}$$

This action involves both coupling between different blocks and inside the same block; to disentangle these two class of couplings it is convenient to introduce the block derivatives of first and second order:

$$\hat{\nabla}_\mu f_i = \frac{1}{2}(f_{i+3\hat{\mu}} - f_{i-\hat{\mu}}); \quad \hat{\nabla}^2_\mu f_i = f_{i+3\hat{\mu}} + f_{i-\hat{\mu}} - 2f_{i+\hat{\mu}}$$

by which the derivative ∇_μ can be rewritten as $\nabla_\mu = \hat{\nabla}_\mu - \frac{1}{2}\hat{\nabla}^2_\mu$. By introducing the doublet $\Psi = \binom{u}{d}$ by

$$\psi^a = (u + \gamma_1 d)/\sqrt{2}; \quad \bar{\psi}^a = (\bar{u} + \bar{d}\gamma_1)/\sqrt{2}; \quad \psi^b = (u - \gamma_1 d)/\sqrt{2}; \quad \bar{\psi}^b = (\bar{u} - \bar{d}\gamma_1)/\sqrt{2}$$

the staggered action Eq. (3.3.2) can finally be rewritten in the form

$$S = \sum_{\text{blocks}} \left\{ \bar{\Psi}(\gamma_0 \hat{\nabla}_0 + \gamma_1 \hat{\nabla}_1)\Psi + \frac{1}{2}\bar{\Psi}\gamma_5 T_\mu \hat{\nabla}^2_\mu \Psi \right\} \tag{3.3.3}$$

where

$$T_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad T_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

In Eq. (3.3.3) it is possible to identify the second term as a Wilson-like term, which ensure that the variable Ψ correctly displaies the physical content of the action. In this way the original 4-degeneration of the "naive" action in two dimension, is reduced to a degeneration 2. In the four dimensional case we end up with 4 species (usually called tastes) of fermion instead of the 16 species of the "naive" action.

We will now show how the action Eq. (3.3.2) retains part of the chiral symmetry invariance: this action is clearly invariant under the $U(1) \times U(1)$ transformation $\chi_n \to U_e \chi_n$ for even n and $\chi_n \to U_o \chi_n$ for odd n and it is simple to show that the action of this transformation on the fermion doublet Ψ is

$$\Psi \to U_{(+)}\Psi + U_{(-)}\gamma_5 T_3 \Psi; \quad T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad U_{(\pm)} = \frac{U_o \pm U_e}{2}$$

which is a remnant of the chiral symmetry and forbids a mass term for the u and d fermions.

The residual 4-degeneration of the staggered formulation is usually removed by using the 4-root of the fermion determinant in the action; this introduces non-localities and it is still disputed whether it is possible to theoretically justify this procedure. The final form of the partition function used in staggered fermion simulation is thus

$$Z(T) \equiv \int e^{-S_W[U]} \Big(\det M[U] \Big)^{N_f/4} \mathscr{D}U \tag{3.3.4}$$

where S_W is the Wilson action, N_f is the number of (degenerate) flavours and the fermion matrix $M_{i,j}$ (i, j being lattice sites) is the fermion matrix, defined by

$$M_{i,j} = am\delta_{i,j} + \frac{1}{2}\sum_{\mu} \eta_{i,\mu}\left(U_{i,\mu}\delta_{i,j-\hat{\mu}} - U^{\dagger}_{i-\hat{\mu},\mu}\delta_{i,j+\hat{\mu}}\right) \tag{3.3.5}$$

the $\eta_{i,\mu}$ being phase factors that generalize the $(-1)^{\phi(i,\mu)}$ of Eq. (3.3.2).

The Wilson and staggered fermions are the two historically standard methods to remove the doubling problem; more recently a new class of fermion discretization was developed: the overlap fermions [42]. The most appealing property of this new formulation is the presence at finite lattice spacing of an exact global symmetry, which in the continuum limit reduces to the usual chiral symmetry [43]. This discretization is however much more computationally demanding that the two previous ones and its use is by now limited to studies of QCD at zero temperature. A feasibility analysis for a project on finite temperature QCD with overlap fermions is reported in [44].

4. Monopoles and confinement on the lattice

4.1. Detection of monopoles on the lattice

The standard method to detect monopoles on the lattice was developed by DeGrand and Toussaint in [9] for the $U(1)$ lattice gauge theory and it is based on the Gauss's law. The $U(1)$ gauge field is given by $U_\mu(\mathbf{x}) = e^{i\Theta_\mu(\mathbf{x})}$ and if Θ_μ is sufficiently small the magnetic flux[1] through the (μ, ν) plaquette with vertex $\mathbf{x}$ is well approximated by

$$\Theta_{\mu\nu}(\mathbf{x}) \equiv \Theta_\mu(\mathbf{x}) + \Theta_\nu(\mathbf{x} + \hat{\mu}) - \Theta_\mu(\mathbf{x} + \hat{\nu}) - \Theta_\nu(\mathbf{x}) \tag{4.1.1}$$

Clearly, if we use the previous expression to estimate the flux through a closed surface we will always get zero total flux, since each link would be included in the sum twice, one with each sign. In order to obtain a non trivial result it is necessary to identify the Dirac strings, which can be done as follows. We assume that the plaquette angle $\Theta_{\mu\nu}$ consists of two terms: physical fluctuations which lie in the range $-\pi$ to π and Dirac strings which carry 2π units of flux. Defining $\bar{\Theta}_{\mu\nu} = \Theta_{\mu\nu} \bmod 2\pi$ the magnetic flux through a closed surface Σ is given by

$$\Phi = \sum_{\text{plaq} \in \Sigma} \bar{\Theta}_{\mu\nu} = 2\pi M \tag{4.1.2}$$

where M is the number of monopoles inside the surface. By computing the magnetic flux through the boundary of the elementary cubes of the lattice we can then locate the monopoles.

In the case of non-abelian gauge theories one has first to fix a gauge and then to apply the above procedure to the abelian subgroup spanned by some diagonal component of the Lie algebra. However, while for the $U(1)$ gauge theory the DeGrand-Toussaint recipe is gauge invariant, this is not the case for non-abelian theories: the result strongly depends on the choice of the gauge and, as consequence, the existence of a monopole in a location of a given lattice configuration seems to be a gauge-dependent property.

Indeed for many years people have been speaking of monopoles in a given abelian projection as different from those in the other projections and gauge transformation were thought to be capable of creating and destroying monopoles: a common statement was, for example, that in the Landau gauge there are no monopoles.

This physically unacceptable behaviour can be understood by using the results of [23], presented in Sec. (2.5): we have shown that monopoles are gauge invariant but the

[1] For the sake of the simplicity we will call just "flux" in this section what really is e times the magnetic flux, where e is the gauge coupling.

charge calculated from the magnetic flux is not and that for a general projection the Dirac quantization condition is not satisfied. On the other hand the Dirac quantization condition is a fundamental ingredient of the DeGrand-Toussaint recipe, since the flux of a Dirac string is a multiple of 2π only if Dirac condition is satisfied. In a general gauge the flux carried by a Dirac string is less than 2π and some monopoles escape detection by means of the DeGrand-Toussaint recipe.

It is thus clear that a gauge dependence exists in the detection of monopoles; we will now show that it is also possible to predict the dependence of the number of monopoles observed *à la* DeGrand-Toussaint on the gauge fixing used.

It is first of all necessary to identify the correct abelian projection in order to define monopoles in a given configuration, *i.e.* how to perform a gauge fixing to the unitary gauge in a numerical simulation. A possible strategy is to look for a differential equation which is satisfied by the field in the unitary gauge Eq. (2.2.10) and then use it to fix the gauge. It is simple to check that the field Eq. (2.2.10) obeys the condition [23]

$$\partial_\mu A^\pm_\mu + ie\left[A^3_\mu, A^\pm_\mu\right] = 0 \tag{4.1.3}$$

which is just the continuum form of the Maximal Abelian Gauge (MAG [45]). The MAG is thus the correct gauge choice for monopole detection.

Let us now suppose to have a gauge field configuration in the MAG with a monopole located in the origin and the Dirac string directed along the $\hat{z}$ direction, *i.e.* the lattice analogue of the field Eq. (2.2.10). We can now apply to this configuration a known gauge transformation and check by the DeGrand-Toussaint recipe if the monopole is still there or not after the gauge change. We will do that with a class of gauge transformation depending on a parameter α

$$U(\alpha) = \exp\left(-i\gamma(\theta,\phi,\alpha)\frac{\sigma_3}{2}\right)\exp\left(-i\beta(\theta,\phi,\alpha)\frac{\sigma_2}{2}\right)\exp\left(i\gamma(\theta,\phi,\alpha)\frac{\sigma_3}{2}\right) \tag{4.1.4}$$

with

$$\gamma(\theta,\phi,\alpha) = \phi \quad \beta(\theta,\phi,\alpha) = \alpha\Theta \tag{4.1.5}$$

where Θ is the regularized θ angle defined by Eq. (2.2.9). For $\alpha = 0$ this transformation is the identity, while for $\alpha = \pi$ it is the gauge transformation that maps the unitary gauge into the hedgehog gauge (see Eq. (2.2.8)). Applying this gauge transformation to the field Eq. (2.2.10) at large distance and projecting on the third axis gives the abelian gauge field

$$\mathbf{A}_3(\alpha) \equiv 2\mathrm{Tr}\left[\frac{\sigma_3}{2}\mathbf{A}(\alpha)\right] = \frac{1-\cos\theta\cos(\alpha\theta)}{er\sin\theta}\mathbf{e}_\phi$$

whose magnetic field is

$$\mathbf{b}(\alpha) = \mathbf{e}_r\frac{1}{r\sin\theta}\partial_\theta\Big[\sin\theta\,\mathbf{e}_\phi\cdot\mathbf{A}_3(\alpha)\Big]$$

The magnetic flux at infinity is

$$\Phi(\alpha) = r^2\int \mathrm{d}\Omega\,\hat{r}\cdot\vec{b}(\alpha) = \frac{2\pi}{e}\big[1+\cos(\alpha\pi)\big]$$

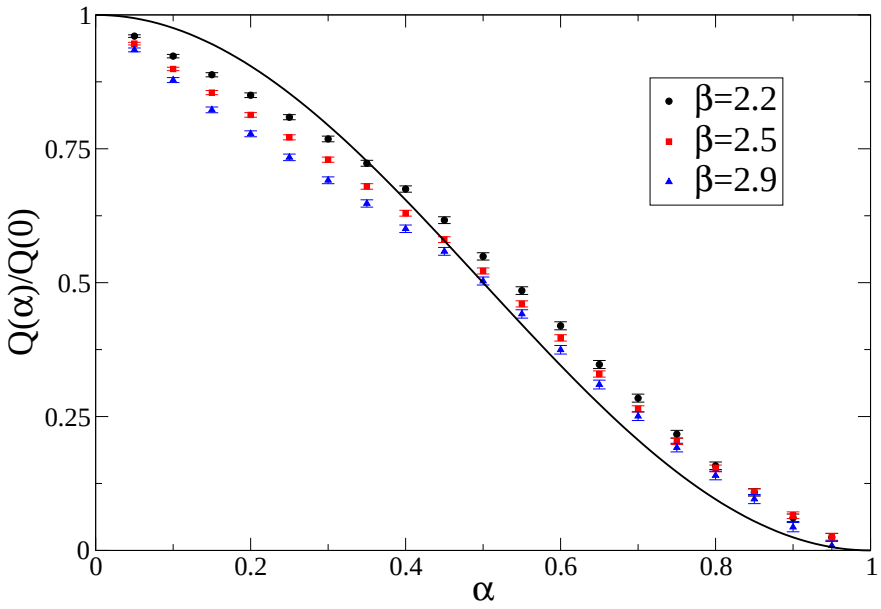

Figure 4.1.: Results and theoretical prediction (black line) for the ratio defined in (4.1.6).

and we finally obtain for the "effective" monopole charge $Q_m(\alpha)$ calculated by using the flux at infinity after the gauge transformation the following expression:

$$\frac{Q_m(\alpha)}{Q_m(0)} = \frac{1+\cos(\pi\alpha)}{2} \tag{4.1.6}$$

The DeGrand-Toussaint recipe reveals only integer monopole charges, so for a given configuration the charge measured after a gauge transformation of parameter α will be equal to the one measured in the MAG for sufficiently small values of the α parameter, while for larger values it will be zero. The relation Eq. (4.1.6) is then expected to be satisfied only by the mean values, where the average is to be calculated on an ensamble of gauge configurations.

A numerical check of Eq. (4.1.6) has been performed in [46]. The lattice formulation introduces however several sources of systematic error:

1. the monopole location in the initial MAG gauge fixed configuration is determined with the precision of a lattice spacing. We assume the monopole to be in the center of the elementary cube having non-vanishing magnetic flux through its boundary;

2. the direction of the string, *i.e.* the direction of the $\hat{z}$ axes in the previous discussion, is known with an angular precision of $\pi/4$, since it is determined by the face of the cube pierced by the Dirac string. We assume the Dirac string to be parallel to the coordinate axes;

3. Eq. (4.1.6) was deduced in the continuum, on the lattice discretization errors are to be expected.

It is also to be noted that in order to satisfy Eq. (4.1.6) we have to calculate the magnetic flux by using a surface sufficiently far away from the monopole, otherwise a residual dependence on the function K of Eq. (2.2.10) would be observed. In the DeGrand-Toussaint recipe it is tacitly assumed that the boundary of an elementary cube is "far enough" for the flux to give a reliable estimate of the monopole charge. A test to verify if this assumption is correct was performed in [47] by using a modified DeGrand-Toussaint recipe, in which, instead of elementary cells, cubes of size n^3 with increasing values of n were used. The result of this analysis was that for the plaquette and Polyakov gauges the number of observed monopoles strongly decreases by increasing the n value, indicating strong UV noise, while for the MAG it is almost independent of n. In testing Eq. (4.1.6) we thus use just the elementary cubes.

The configurations used to test Eq. (4.1.6) were numerically generated by means of a standard combination of heathbath [48, 49] and overrelaxation algorithms [50] and the MAG gauge fixing was achieved by an iterative combination of local maximization and overrelaxation steps (see *e.g.* appendix of [51]).

Since in the entire procedure we have to operate only on the elementary cubes containing the monopoles, we can use a small lattice, of size 4×8^3 (the results were nevertheless checked for consistency also on a 4×16^3 lattice). To compute mean values about 3×10^4 independent gauge configurations were generated at three different β values: one below ($\beta = 2.2$) and two above the deconfinement transition ($\beta = 2.5$ and $\beta = 2.9$), which for lattices with temporal extent $N_t = 4$ is located at $\beta_c = 2.2986(6)$.

The results of the simulations are shown in Fig. (4.1) together with the theoretical prediction, Eq. (4.1.6). Data are in good qualitative agreement with the theoretical expectation, the previously noted sources of systematic errors preventing a complete quantitative agreement. In particular for all the non-vanishing α values the ratio $Q_m(\alpha)/Q_m(0)$ is strictly lower than one, consistent with the expectation that in the MAG the number of observed monopoles is the largest.

As a last comment we note that from Eq. (4.1.6) it follows that for $\alpha = 1$ no monopoles should be observed, as indeed happens in simulations. The transformation Eq. (4.1.4)-(4.1.4) for $\alpha = 1$ is the gauge transformation from the unitary gauge, equivalently MAG, to the hedgehog gauge and it is simple to show that the gauge field in the hedgehog gauge, Eq. (2.2.4), satisfies the equation $\nabla \cdot \mathbf{A} = 0$, which defines the Landau gauge for a static configuration. As a byproduct of the previous argument we thus arrive at the conclusion that in the Landau gauge no monopoles can be detected by the DeGrand-Toussaint recipe.

It is a well known fact that in the Landau gauge no monopoles are detected, however no explanation of this phenomenon was given before [46]. This fact was usually interpreted as the absence of monopoles in the Landau gauge (see *e.g.* [52]), this being motivated by the idea that monopoles are gauge dependent object. We gave strong evidence that the origin of this behaviour is completely different: monopoles are gauge invariant quantities related to the gauge covariant violation of NABI. It is their detection by the DeGrand-Toussaint method that is gauge dependent, since it relies on the Dirac quantization condition that we have shown not to hold, for a given configuration, in all gauges. In particular in the Landau gauge all monopoles escapes detection.

4.2. The monopole operator

We have analyzed in the previous section the detection of monopoles in numerically generated gauge configurations, however the numerical density of monopoles in not a suitable observable for revealing monopole condensation. In order to detect monopole condensation the *vev* of a magnetically charged operator μ has to be computed: $\langle\mu\rangle$ is zero in the normal phase and can be different from zero in the condensed phase.

The simplest way to construct a magnetically charged operator is to define it as the operator that adds a monopole to the state to which it is applied [53] by a dual version of the Dirac gauge invariant charged operator [54]. We will start from the abelian theory, where the results are better established and we will then discuss the generalization to the non-abelian case.

An operator that adds a monopole to a configuration is easily constructed in the Schrödinger representation: if we denote by $\mathbf{\Pi}(\mathbf{x})$ the canonical momenta conjugate to the physical transverse gauge field $\mathbf{A}_\perp(\mathbf{x})$, the operator μ defined by

$$\mu(\mathbf{y}) = \exp\left(i \int b^i_\perp(\mathbf{x},\mathbf{y})\Pi_i(\mathbf{x})\mathrm{d}^3x\right) \tag{4.2.1}$$

is the translation operator of the gauge field by $\mathbf{b}_\perp(\mathbf{x},\mathbf{y})$:

$$\mu(\mathbf{y})|\mathbf{A}_\perp(\mathbf{x})\rangle = |\mathbf{A}_\perp(\mathbf{x}) + \mathbf{b}_\perp(\mathbf{x},\mathbf{y})\rangle$$

which is equivalent to the commutation relation

$$\begin{aligned} [A_i(\mathbf{x}),\mu(\mathbf{y})] &= b_i(\mathbf{x},\mathbf{y})\mu(\mathbf{y}) \\ [\Pi_i(\mathbf{x}),\mu(\mathbf{y})] &= 0 \end{aligned} \tag{4.2.2}$$

Because of the linearity with respect to A_μ of the 't Hooft tensor, it follows from the previous relations that

$$[Q,\mu(\mathbf{y})] = m\ \mu(\mathbf{y}) \tag{4.2.3}$$

where Q is the magnetic charge operator and m is the charge of the field $\mathbf{b}_\perp(\mathbf{x},\mathbf{y})$. The operator μ is thus charged if $\mathbf{b}_\perp(\mathbf{x},\mathbf{y})$ is the transverse component of the field in $\mathbf{x}$ of a monopole located in $\mathbf{y}$.

A discretization of the operator Eq. (4.2.1) to be used in lattice simulations was introduced in [55] and later improved in [56]: on the lattice the canonical momenta correspond (with the Wilson action) to the mixed spatial-temporal plaquettes and the operator in Eq. (4.2.1) can be rewritten (up to $O(a^2)$ lattice artefacts) as

$$\mu = \exp(-\beta\Delta S) \qquad \Delta S = \sum_{\mathbf{n}} \mathrm{Tr}\{\Pi_{i0}(\mathbf{n},t) - \Pi'_{i0}(\mathbf{n},t)\} \tag{4.2.4}$$

where

$$\begin{aligned} \Pi_{i0}(\mathbf{n},t) &= U_i(\mathbf{n},t)U_0(\mathbf{n}+\hat{\imath},t)U_i(\mathbf{n},t+1)^\dagger U_0(\mathbf{n},t)^\dagger \\ \Pi'_{i0}(\mathbf{n},t) &= U_i(\mathbf{n},t)U_0(\mathbf{n}+\hat{\imath},t)M_i(\mathbf{n}+\hat{\imath},t)U_i(\mathbf{n},t+1)^\dagger U_0(\mathbf{n},t)^\dagger \\ M_j(\mathbf{n},t) &= \begin{cases} \exp\left(i\,a\,b_j(\mathbf{n},\mathbf{y})\right) & \text{if} \quad t=0 \\ 0 & \text{if} \quad t\neq 0 \end{cases} \end{aligned} \tag{4.2.5}$$

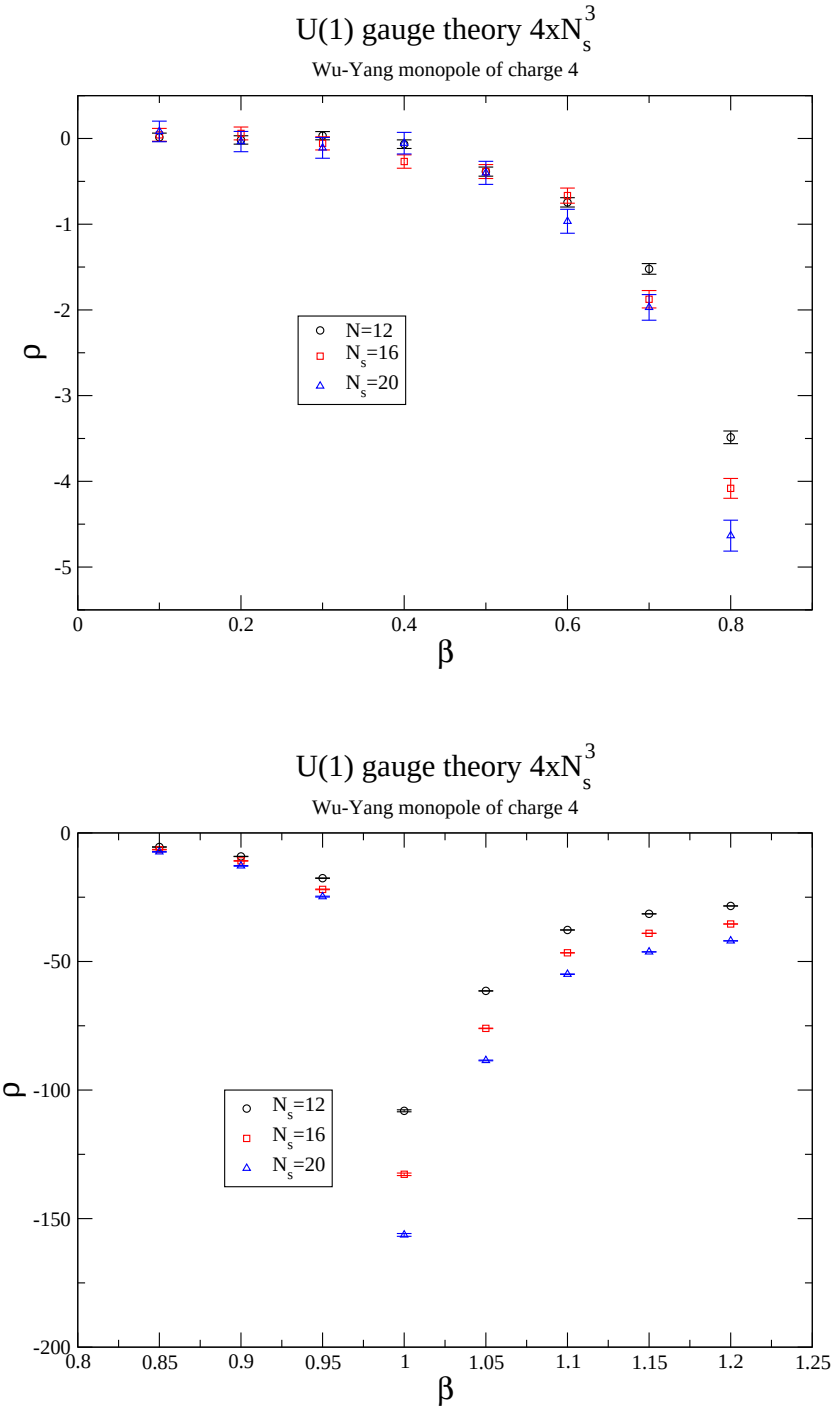

Figure 4.2.: (*upper*) Numerical calculation of ρ for $U(1)$ gauge theory on a $4 \times N_s^3$ lattice by using a Wu-Yang monopole of charge 4, for small β values and (*lower*) in the neighbourhood of the transition.

The fact that the operator Eq. (4.2.4) adds a monopole to the configuration, *i.e.* that the commutation relations Eq. (4.2.2) are preserved by the discretization, can be explicitly checked by performing the gauge transformation

$$U_j(\mathbf{n},1) \to U_j(\mathbf{n},1)\exp\left(i\,a\,b_j(\mathbf{n}+\hat{\jmath},\mathbf{y})\right) \qquad 0 \le j \le 3 \tag{4.2.6}$$

After this transformation the plaquettes of the 0-th temporal slice come back to its original Wilson form, while the spatial plaquette angles of the 1-th temporal slice $\Theta_{ij}(\mathbf{n},1)$ defined in Eq. (4.1.1), change as

$$\Theta_{ij}(\mathbf{n},1) \to \Theta_{ij}(\mathbf{n},1) + \Delta_i b_j(\mathbf{n},\mathbf{y}) - \Delta_j b_i(\mathbf{n},\mathbf{y}) \tag{4.2.7}$$

so that the field of a monopole located in $\mathbf{y}$ is added to the gauge field in the temporal slice $t = 1$. As a consequence of the transformation Eq. (4.2.6), the expressions in Eq. (4.2.5) become

$$\begin{aligned} \Pi'_{i0}(\mathbf{n},t) &= U_i(\mathbf{n},t)U_0(\mathbf{n}+\hat{\imath},t)U_i(\mathbf{n},t+1)^\dagger M_i(\mathbf{n}+\hat{\imath},t)U_0(\mathbf{n},t)^\dagger \\ M_j(\mathbf{n},t) &= \begin{cases} \exp\left(i\,a\,b_j(\mathbf{n},\mathbf{y})\right) & \text{if} \quad t = 1 \\ 0 & \text{if} \quad t \neq 1 \end{cases} \end{aligned} \tag{4.2.8}$$

Because of the commutativity of the $U(1)$ gauge theory, the previous expressions are the same of Eq. (4.2.5), just translated on the slice $t = 1$ instead of the original $t = 0$. By iterating the application of transformation Eq. (4.2.6) on the temporal slices with $t > 1$ we thus see that the effect of the operator μ is to add a monopole to the whole configuration.

In [56] it was shown that the previous construction is gauge equivalent to the one introduced in [57, 58] for the $U(1)$ gauge theory by means of the explicit duality transformation. In the same references it was proven that $\langle \mu^\dagger(\mathbf{x})\mu(\mathbf{y})\rangle$ approaches a finite value in the confining phase of 4d $U(1)$ lattice gauge theory while it vanishes in the deconfined phase as $|\mathbf{x}-\mathbf{y}| \to \infty$, *i.e.* $\langle\mu\rangle$ is an order parameter for the deconfinement transition in the $U(1)$ gauge theory.

Since μ is the exponential of an extensive quantity, it is computationally extremely difficult to directly measure $\langle\mu\rangle$ in numerical simulations. It is convenient to compute instead its logarithmic derivative

$$\rho = \frac{\mathrm{d}}{\mathrm{d}\beta}\log\langle\mu\rangle = \langle S\rangle_S - \langle S + \Delta S\rangle_{S+\Delta S} \tag{4.2.9}$$

Since $\langle\mu\rangle = 1$ at $\beta = 0$, the *vev* of the μ operator can then be reconstructed as

$$\langle\mu\rangle = \exp\left(\int_0^\beta \rho(x)\mathrm{d}x\right) \tag{4.2.10}$$

If monopole condensation is responsible for the confinement of color and we denote by β_c the deconfinement coupling, $\langle\mu\rangle$ should be different from zero for $\beta < \beta_c$ and vanish for $\beta \ge \beta_c$. In terms of the ρ observable this amounts to say that ρ is regular for

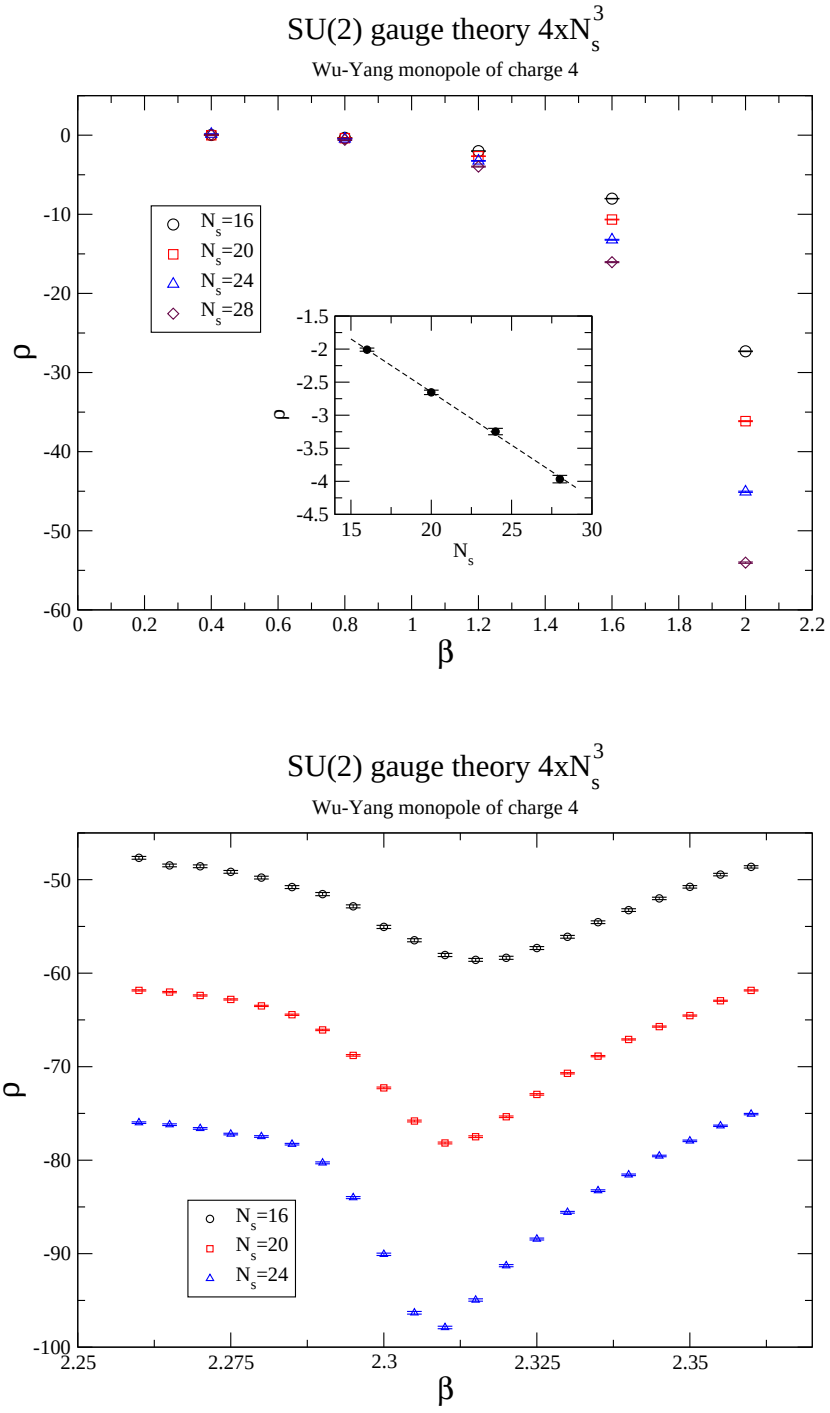

Figure 4.3.: (*upper*) Behaviour of ρ in $SU(2)$ gauge theory at low β, the inset shows the scaling with N_s for $\beta = 1.2$ and the line is a linear fit, and (*lower*) near the deconfinement transition.

$\beta < \beta_c$, it develops a negative peak for $\beta \approx \beta_c$ and for $\beta > \beta_c$ it diverges negatively in thermodynamical limit, $\lim_{N_s\to\infty} \rho = -\infty$.

For computation purposes it is convenient to look at the finite size scaling (FSS) properties of ρ. Since μ is well defined everywhere we have

$$\mu = \mu\left(\frac{\xi}{L}, \frac{a}{\xi}\right) \tag{4.2.11}$$

where a is the lattice spacing, L the lattice size and ξ the correlation length. Near a second order the correlation length diverges as (see *e.g.* [59])

$$\xi \propto \left(\beta - \beta_c\right)^{-\nu}$$

and, as a consequence, Eq. (4.2.11) is well approximated by

$$\mu = \mu\Big(L^{1/\nu}(\beta - \beta_c)\Big)$$

from which it follows that

$$\rho = L^{1/\nu} f\Big(L^{1/\nu}(\beta - \beta_c)\Big) \tag{4.2.12}$$

This behaviour can be used to determine the order and the position of the deconfinement transition in numerical simulations.

The generalization of Eq. (4.2.4)-(4.2.5) for non abelian gauge theory introduced in [10, 60, 61] uses

$$M_j(\mathbf{n}, t) = \begin{cases} \exp\left(i\, a\, b_j(\mathbf{n}, \mathbf{y})\, \hat{\Phi}\right) & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0 \end{cases} \tag{4.2.13}$$

where $\hat{\Phi}$ is the generator of the gauge group which identifies the magnetic $U(1)$. By applying again the gauge transformation Eq. (4.2.6) it is not difficult to show that Eq. (4.2.7) is again satisfied, although only to leading order in the lattice spacing. Also Eq. (4.2.8) is equal to Eq. (4.2.5) only up to lattice corrections. In order to avoid the problems related to the possible difference between the direction (in color space) of a monopole located in the configuration to which μ is applied and the direction of the monopole created by μ, it is convenient to use periodic boundary conditions, that ensure the absence of (a net numbers) of monopoles in the initial configuration.

For abelian lattice gauge theories, in numerical simulations ρ behaves as theoretically expected: in the low β region ρ is well defined and it reaches its thermodynamical limit quite fast (see Fig. (4.2)); near the deconfinement transition it develops a negative peak and for $\beta > \beta_c$ it is no more defined.

In early works the behaviour of ρ in non abelian gauge theories seemed analogous to that in the $U(1)$ theory, however more systematic studies [11, 62] have shown that the generalization of the μ operator is not as straightforward as it could seem: in Fig. (4.3) the behaviour of ρ is shown for small β values. The values of ρ do not appear to reach the thermodynamical limit even for $\beta \ll \beta_c$ and numerically they are observed to scale linearly with the size of the lattice (see the inset in Fig. (4.3)). At the deconfinement

transition a bump develops, however, due to the low β behaviour, it is superimposed on a still growing background. As a consequence, if the expectation value of the monopole operator is reconstructed by Eq. (4.2.10) we get in thermodynamical limit $\langle\mu\rangle = 0$ for every $\beta > 0$.

This behaviour of ρ was initially ascribed to lattice artefacts and, in order to reduce their effect, the Wu-Yang form for the $\mathbf{b}(\mathbf{x}, \mathbf{y})$ field was preferred to the previously adopted Dirac one. This because in the Dirac expression the field gets large near the string also far away from the monopole, while in the Wu-Yang formulation the only singularity is at the monopole location . In this way the background is reduced typically by about a factor of 3. However general behaviour does not change significantly.

The strong coupling expansion (see *e.g.* [27]) is a convenient tool in order to point out the origin of this behaviour for small β values. By using

$$\int U_{ij} U^{\dagger}_{kl} \, \mathrm{d}U \propto \delta_{il}\delta_{jk}$$

it is simple to see that the leading order of the ρ strong coupling expansion for $SU(N)$ is given by

$$\rho \propto \beta^5 \sum_{\mathbf{n}} \left(T_{xy} + T_{yz} + T_{xz} - 3N^3\right) \tag{4.2.14}$$

where we introduced the notation

$$T_{ij}(\mathbf{n}) = \mathrm{Re}\Big\{ \mathrm{Tr}\big[M_i(\mathbf{n}+\hat{\imath})\big] \mathrm{Tr}\big[M_j(\mathbf{n}+\hat{\jmath})^{\dagger}\big] \mathrm{Tr}\big[M_i(\mathbf{n}+\hat{\imath}+\hat{\jmath})^{\dagger} M_j(\mathbf{n}+\hat{\imath}+\hat{\jmath})\big] \Big\} \tag{4.2.15}$$

For the Wu-Yang monopole asymptotically $\mathbf{A}(\mathbf{r}) \approx g\mathbf{r}/|\mathbf{r}|^2$ and for $SU(2)$ theory we get

$$\begin{aligned} \sum_{\mathbf{n}} \frac{1}{8} T_{xy} = \sum_{\mathbf{n}} & \cos\left(a\, A_x(\mathbf{n}+\hat{x})\right) \cos\left(a\, A_y(\mathbf{n}+\hat{y})\right) \cos\left(a\, A_y(\mathbf{n}+\hat{x}+\hat{y}) - \right. \\ & \left. - a\, A_x(\mathbf{n}+\hat{x}+\hat{y})\right) \approx \sum_{n} \Big[1 - g^2 a^2 \frac{1}{|\mathbf{n}|^2} + O\Big(\frac{1}{|\mathbf{n}|^4}\Big)\Big] \end{aligned} \tag{4.2.16}$$

and ρ is thus expected to scale as

$$\rho \sim \beta^5 \int_{r<N_s} \frac{a^2}{r^2} \, \mathrm{d}^3x \sim \beta^5 a^2 N_s \tag{4.2.17}$$

which is indeed the behaviour seen in Fig. (4.3). For the $U(1)$ gauge theory it is instead simple to show that

$$\sum_{\mathbf{n}} T_{xy} \approx \sum_{n} \Big[1 + O\Big(\frac{1}{|\mathbf{n}|^4}\Big)\Big]$$

and as a consequence ρ has a finite thermodynamical limit.

We thus have to conclude that the μ operator as defined above satisfies the following two properties

1. it is the correct monopole operator up to $O(a^2)$ lattice corrections (as seen by using the transformation Eq. (4.2.6));

2. it is not well defined in the thermodynamical limit for fixed non-vanishing lattice spacing (as seen by Eq. (4.2.17)).

The point (1) is well known in the literature and it is related to the presence of charged field: in a given abelian projection the gauge field can be decomposed in abelian-like components (the diagonal components) and components charged with respect to the residual $U(1)$. These component are responsible for the $O(a^2)$ differences between Eq. (4.2.5) and the transformed Eq. (4.2.8).

A different point of view on this particular aspect was put forward in [63], where it was noted that the definition in Eq. (4.2.4)-(4.2.13) does not satisfy the Dirac quantization condition at small scales, because of the continuum nature of the field $\mathbf{b}(\mathbf{x}, \mathbf{y})$. The proposed way to overcome this difficulty is to use as an intermediate step integer valued string fields instead of $\mathbf{b}$ and to weight the resulting μ operator with a measure defined in order to reproduce the asymptotic monopole behaviour of $\mathbf{b}$. This procedure is however numerically very expensive since the "string measure" is defined by means of an auxiliary Higgs model and its implementation thus requires two nested Monte Carlo simulations.

The ultraviolet problem of point (1) was commonly thought to be irrelevant on large ($\gg a$) scales, however it was not considered that, due to the non-local character of the μ operator, the $O(a^2)$ terms can conspire to produce nontrivial results also on large scales, such as the observed spurious divergence of ρ. These $O(a^2)$ terms are indeed the origin of the contractions that determine the structure of the traces in Eq. (4.2.15) and the infrared divergence of ρ in Eq. (4.2.17).

We will now show a possible method to overcome this problem [64]: in order to better understand the physical significance of the previous difficulties it is convenient to study the behaviour of the *vev* $\langle \mu(\mathbf{x})^\dagger \mu(\mathbf{x}) \rangle$. The lattice form of this operator is constructed by using Eq. (4.2.4) and

$$\Pi'_{i0}(\mathbf{n}, t) = U_i(\mathbf{n}, t) M_i(\mathbf{n} + \hat{\imath}, t)^\dagger U_0(\mathbf{n} + \hat{\imath}, t) M_i(\mathbf{n} + \hat{\imath}, t) U_i(\mathbf{n}, t+1)^\dagger U_0(\mathbf{n}, t)^\dagger \quad (4.2.18)$$

Indeed by using gauge transformations of the form Eq. (4.2.6) it simple to show that this operator introduces (up to $O(a^2)$ corrections) a monopole in the positive time slices and an anti-monopole in the negative ones. To study the strong coupling expansion of $\langle \mu^\dagger \mu \rangle$ it is useful to introduce a quantity analogous to ρ, which we will denote by ρ_2 and is defined by

$$\rho_2 = \frac{\mathrm{d}}{\mathrm{d}\beta} \log \langle \mu^\dagger \mu \rangle$$

Its strong coupling expansion is given at the lowest nontrivial order by

$$\rho_2 \propto \beta^5 \sum_{\mathbf{n}} \left(T^{(2)}_{xy} + T^{(2)}_{yz} + T^{(2)}_{xz} - 3N^3 \right) \quad (4.2.19)$$

where

$$T^{(2)}_{ij}(\mathbf{n}) = \left| \mathrm{Tr}\left[M_i(\mathbf{n} + \hat{\imath}) \right] \right|^2 \left| \mathrm{Tr}\left[M_j(\mathbf{n} + \hat{\jmath})^\dagger \right] \right|^2 \times \\ \times \left| \mathrm{Tr}\left[M_i(\mathbf{n} + \hat{\imath} + \hat{\jmath})^\dagger M_j(\mathbf{n} + \hat{\imath} + \hat{\jmath}) \right] \right|^2 \quad (4.2.20)$$

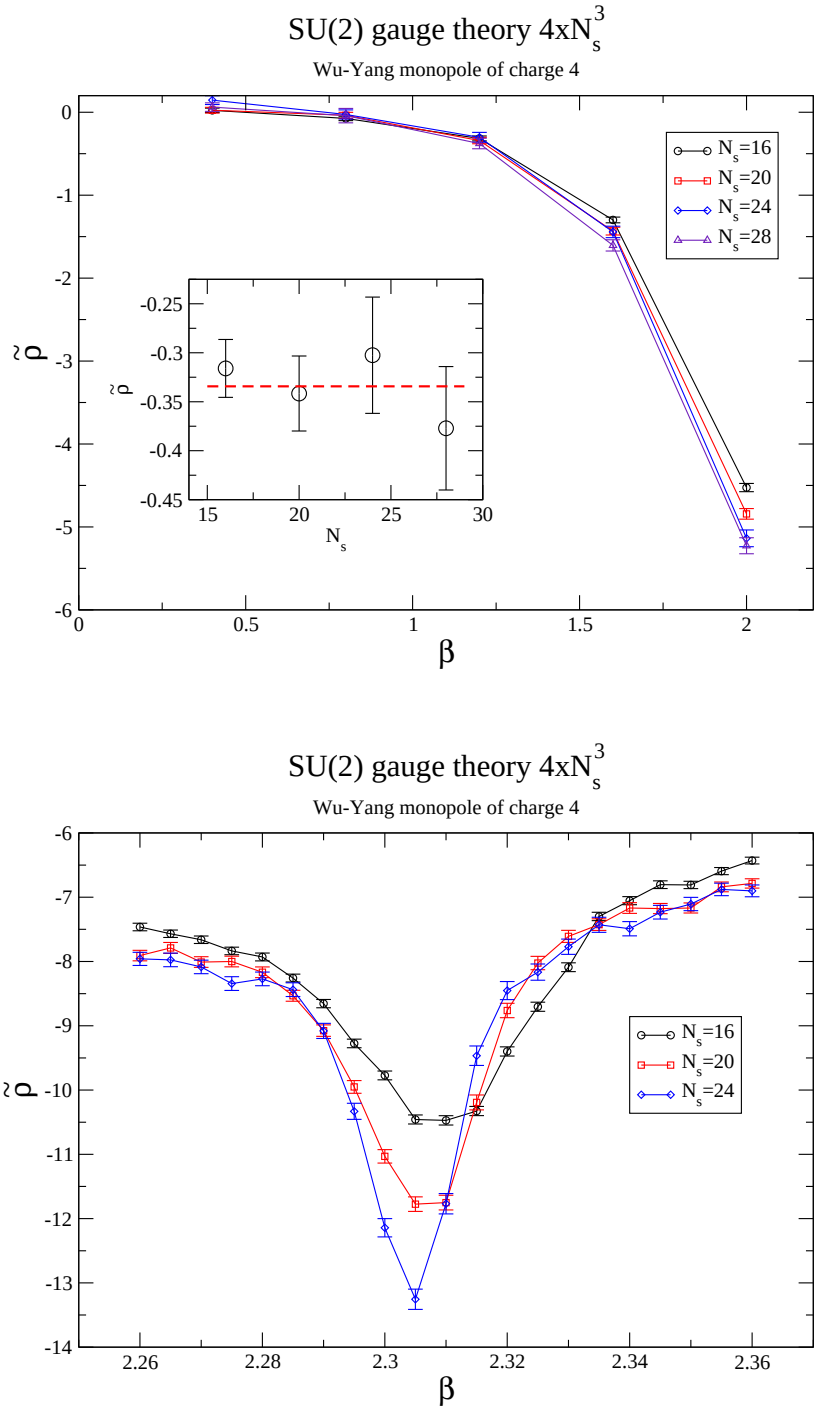

Figure 4.4.: (*upper*) Behaviour of $\tilde{\rho}$ in $SU(2)$ gauge theory at low β, the inset shows that for $\beta = 1.2$ the value of $\tilde{\rho}$ is independent of N_s (the dashed line is a constant), and (*lower*) near the deconfinement transition.

For $SU(2)$ theories we thus have the result that ρ_2, like ρ, is infrared divergent

$$\rho_2 \sim \beta^5 \int_{r<N_s} \frac{a^2}{r^2}\,\mathrm{d}^3x \sim \beta^5 a^2 N_s \tag{4.2.21}$$

since for the Wu-Yang monopole we have

$$\sum_{\mathbf{n}} \frac{1}{8} T^{(2)}_{xy} \approx \sum_{n} \Big[1 - 2g^2a^2 \frac{1}{|\mathbf{n}|^2} + O\Big(\frac{1}{|\mathbf{n}|^4}\Big)\Big] \tag{4.2.22}$$

while in the abelian case trivially $\langle \mu^\dagger \mu \rangle \equiv 1$.

If we denote by $|0\rangle$ a state of zero magnetic charge and by $|1\rangle$ the one-monopole state, a properly defined monopole operator $\mathfrak{m}$ should be an interpolating operator between these two states, in such a way that $\langle 0|1\rangle \propto \langle 0|\mathfrak{m}|0\rangle$ and the value of $\langle \mathfrak{m} \rangle$ can then be used to test if the magnetic symmetry is broken (*i.e.* both the states $|0\rangle$ and $|1\rangle$ has non zero overlap with the vacuum) or not. The μ operator previously introduced does not completely fulfill this requirement, since it adds a monopole to a given configuration (*i.e.* $\langle 1|\mu|0\rangle \neq 0$) but $\langle \mu^\dagger \mu \rangle \equiv \langle 0|\mu^\dagger \mu|0\rangle \to 0$ in the thermodynamical limit, *i.e.* $\mu|0\rangle = C(L)|1\rangle$ with $\lim_{L\to\infty} C(L) = 0$, where L is the size of the system. A simple improvement on μ is achieved by using

$$\tilde{\mu} = \frac{\mu}{\sqrt{\langle \mu^\dagger \mu \rangle}} \tag{4.2.23}$$

in such a way that $|1\rangle = \tilde{\mu}|0\rangle$ and $\langle 0|1\rangle = \langle 0|\tilde{\mu}|0\rangle = \langle \tilde{\mu} \rangle$.

The quantity analogous to ρ that can be introduced for $\tilde{\mu}$ is defined by

$$\tilde{\rho} = \frac{\mathrm{d}}{\mathrm{d}\beta} \log\langle \tilde{\mu} \rangle = \rho - \frac{1}{2}\rho_2 = \frac{1}{2}\langle S \rangle_S + \frac{1}{2}\langle S + \widetilde{\Delta S} \rangle_{S+\widetilde{\Delta S}} - \langle S + \Delta S \rangle_{S+\Delta S} \tag{4.2.24}$$

where ΔS and $\widetilde{\Delta S}$ are defined by

$$\begin{aligned}
&\Delta S = \sum_{\mathbf{n}} \mathrm{Tr}\{\Pi_{i0}(\mathbf{n},t) - \Pi'_{i0}(\mathbf{n},t)\} \qquad \widetilde{\Delta S} = \sum_{\mathbf{n}} \mathrm{Tr}\{\Pi_{i0}(\mathbf{n},t) - \tilde{\Pi}'_{i0}(\mathbf{n},t)\} \\
&\Pi_{i0}(\mathbf{n},t) = U_i(\mathbf{n},t) U_0(\mathbf{n}+\hat{\imath},t) U_i(\mathbf{n},t+1)^\dagger U_0(\mathbf{n},t)^\dagger \\
&\Pi'_{i0}(\mathbf{n},t) = U_i(\mathbf{n},t) U_0(\mathbf{n}+\hat{\imath},t) M_i(\mathbf{n}+\hat{\imath},t) U_i(\mathbf{n},t+1)^\dagger U_0(\mathbf{n},t)^\dagger \\
&\tilde{\Pi}'_{i0}(\mathbf{n},t) = U_i(\mathbf{n},t) M_i(\mathbf{n}+\hat{\imath},t)^\dagger U_0(\mathbf{n}+\hat{\imath},t) M_i(\mathbf{n}+\hat{\imath},t) U_i(\mathbf{n},t+1)^\dagger U_0(\mathbf{n},t)^\dagger \\
&M_j(\mathbf{n},t) = \begin{cases} \exp\left(i\,a\,b_j(\mathbf{n},\mathbf{y})\right) & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0 \end{cases}
\end{aligned} \tag{4.2.25}$$

The mean value $\langle \tilde{\mu} \rangle$ can be reconstructed by using the generalization of Eq. (4.2.10):

$$\langle \tilde{\mu} \rangle = \exp\left(\int_0^\beta \tilde{\rho}(x) \mathrm{d}x \right) \tag{4.2.26}$$

By using the developments Eq. (4.2.16) and Eq. (4.2.22) it is simple to show that in the strong coupling expansion of Eq. (4.2.24) the infrared divergences present at the order

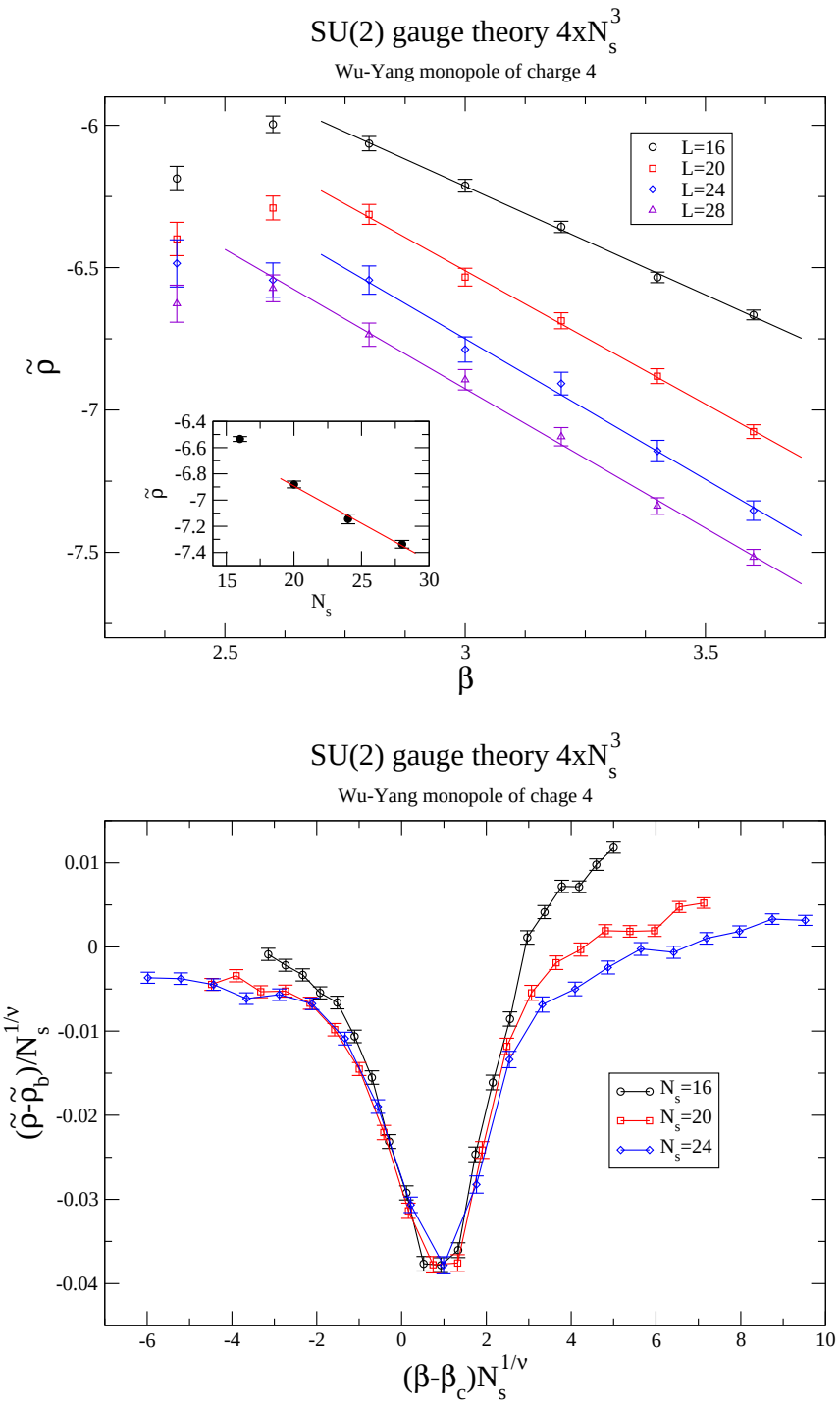

Figure 4.5.: (*upper*) Behaviour of $\tilde{\rho}$ in $SU(2)$ gauge theory at high β, the lines are linear fits and the inset shows the dependence of $\tilde{\rho}$ on N_s at fixed $\beta = 3.2$ together with a linear fit. (*lower*) Scaling of $\tilde{\rho}$ near the deconfinement transition: $\beta_c = 2.2986(6)$ is the critical coupling, $\tilde{\rho}_b \approx -7.39$ is an estimate of the analytical background and $1/\nu \approx 1.587$ for the 3d Ising universality class.

β^5 in ρ and ρ_2 cancel and the result is, to this order, well defined in the thermodynamical limit. There are some hints that this cancellation of divergences is true at all order of the $\tilde{\rho}$ strong coupling expansion, however a complete proof is still laking.

Numerical results for the new observable $\tilde{\rho}$ on lattices of temporal extent $N_s = 4$ are shown in Fig. (4.4) and Fig. (4.5): in the low β regime the values saturate, at the deconfinement transition a negative peak develops and for larger β values $\tilde{\rho}$ is no more well defined. The scaling in the neighbourhood of the deconfinement transition is consistent with the known value of the critical coupling, $\beta_c = 2.2986(6)$, and with the critical exponent $\nu = 0.6301(4)$ of the 3d Ising universality class.

In Fig. (4.6) the behaviour of $\tilde{\rho}$ on a lattice of temporal extent $N_t = 6$ is shown. This is clearly a preliminary result, however two observations are in order: a negative peak in $\tilde{\rho}$ develops in correspondence of the deconfinement transition at $\beta_c = 2.4265(30)$ and the analytical background is approximately $\tilde{\rho}_b \approx -7$, as in the $N_t = 4$ case.

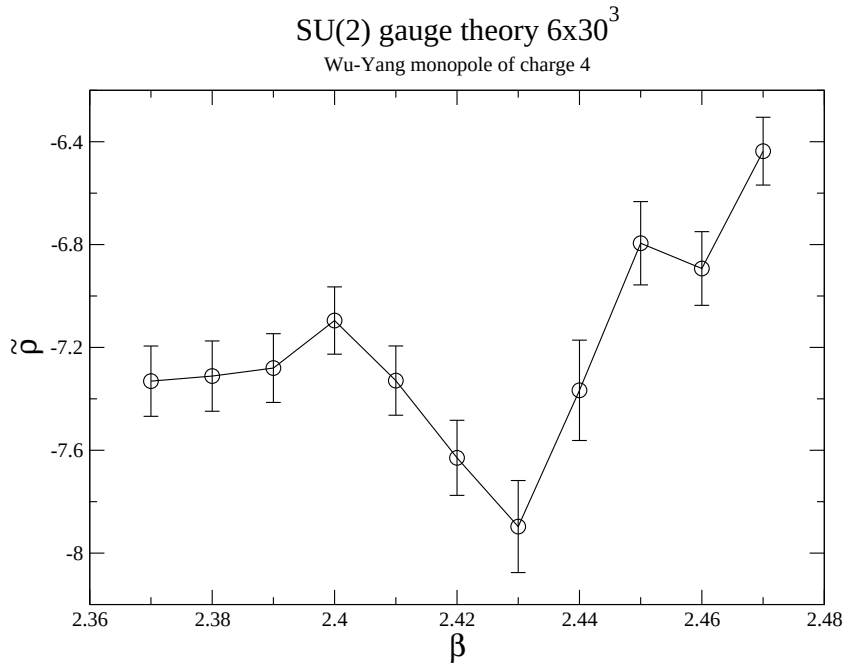

Figure 4.6.: Behaviour of $\tilde{\rho}$ on a 6×30^3 lattice.

The rescaled definition Eq. (4.2.24) thus appears to be theoretically well motivated and numerically accurate. Further studies are being carried out in order to check the dependence on the lattice temporal extent (*i.e.* the continuum limit) and to extend the previous analysis to $SU(3)$ and G_2 lattice gauge theories.

4.3. The QCD phase diagram

In $SU(N)$ gauge theory without quarks the confined low temperature and deconfined high temperature phases are separated by a phase transition, as ensured by their different symmetries: the center symmetry is realized *à la* Wigner in the low temperature confined phase while it is spontaneously broken in the deconfined phase.

One of the principal aims of the studies of QCD at finite temperature is to analyze how the introduction of the fermions modifies the order or the universality class of the deconfinement transition. A major difficulty is that when quarks are coupled to the gauge field the center symmetry is explicitly broken and the Polyakov loop cannot be used as an order parameter.

Explicit symmetries of the QCD Lagrangian are known only when quarks have infinite or vanishing masses. In the limit of infinite mass, the so-called quenched approximation, quarks decouple and the QCD Lagrangian reduces to the Lagrangian of pure gauge theory, the invariance being again related to the center symmetry. In the opposite limit of vanishing masses the invariance of the Lagrangian is associated to the chiral symmetry.

The classical invariance group of QCD Lagrangian in presence of N_f massless fermions is given by

$$G_c = SU_V(N_f) \times SU_A(N_f) \times U(1) \times U_A(1) \tag{4.3.1}$$

which, due to the chiral anomaly, is reduced in the quantum theory to

$$G_q = SU_V(N_f) \times SU_A(N_f) \times U(1) \times Z_{N_f} \tag{4.3.2}$$

At zero temperature the $SU_A(N_f)$ component is spontaneously broken while at high temperature it can be shown [65] that it has to be realized *à la* Wigner. Also the symmetry $U_A(1)$ is expected to be effectively restored at sufficiently high temperature. An order parameter for the $SU_A(N_f)$ symmetry is the $N_f \times N_f$ matrix

$$\Phi_{ij} = \langle \bar{\psi}_i (1+\gamma_5) \psi_j \rangle \tag{4.3.3}$$

where ψ_i is the field of the i-th fermion and $\bar{\psi}_i(1+\gamma_5)\psi_j$ is a color singlet. The matrix Φ_{ij} transforms under G_c as

$$\Phi \to e^{-2i\alpha} U_+ \Phi U_-$$

where $U_\pm$ are independent $SU(N_f)$ matrices and α is the parameter of the $U_A(1)$ transformation. The more general effective Lagrangian for Φ with symmetry group G_c is [66]

$$L_\phi = \frac{1}{2}\mathrm{Tr}[(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi)] + \frac{1}{2} m_\Phi^2 \mathrm{Tr}(\Phi^\dagger \Phi) + g_1 [\mathrm{Tr}(\Phi^\dagger \Phi)]^2 + g_2 \mathrm{Tr}(\Phi^\dagger \Phi)^2 \tag{4.3.4}$$

while if we use the invariance group G_q a term of the form

$$c(\det \Phi + \det \Phi^\dagger) \tag{4.3.5}$$

has to be added to Eq. (4.3.4).

The use of G_c or G_q as symmetry group of the effective Lagrangian corresponds to the two physical possibilities

1. the phase transition happens at a sufficiently high temperature, such that the classical symmetry $U_A(1)$ can be considered as effectively restored,

2. at transition the classical symmetry $U_A(1)$ is still broken to Z_{N_f} by the chiral anomaly.

In the first case the group G_c is the correct choice, otherwise G_q has to be used. The analysis performed in [66] led to the following results regarding the universality class of the chiral transition:

G_c
- if $N_f = 1$ second order in the 3d $O(2)$ universality class or first order
- if $N_f > 1$ first order[2]

G_q
- if $N_f = 1$ no transition,
- if $N_f = 2$ second order in the 3d $O(4)$ universality class or first order,
- if $N_f > 2$ first order.

For non zero quark masses no exact symmetry of the QCD Lagrangian is known that can be used in order to build an order parameter for the deconfinement or chiral transition. In particular there is no known symmetry reason to force the existence of a phase transition. If the dual superconductivity of the vacuum picture is correct, a dual symmetry could however exist, in which the degrees of freedom associated to the boundary conditions plays a prominent role.

The above general considerations give some constraints on the QCD phase diagram but are not capable to predict its complete form, which depends on the non universal features of the model.

For QCD with $2 + 1$ flavours (*i.e.* 2 degenerate quark flavours with masses m_u and a third flavour with a possibly different mass m_s) the phase diagram is usually assumed to be of the form sketched in Fig. (4.7): the tick lines represent second order phase transitions, which are expected to be in the 3d $O(4)$ and 3d Ising (Z_2) universality classes, while the dot denoted by T represent a tricritical point. In the connected regions that include the $m_u = m_s = 0$ or the $m_s = m_u = \infty$ points the transition is first order while for intermediate masses no phase transition is present, just and analytical crossover. The upper line describes the $N_f = 2$ case since for $m_s = \infty$ the third flavour decouple; analogously the right of the figure corresponds to $N_f = 1$.

The phase diagram depicted in Fig. (4.7) satisfies all of the constraints imposed by the universality arguments exposed above and it is often presented as completely established, the point corresponding to the physical quark masses being positioned in the crossover region. Nevertheless there are some points which deserves a more careful analysis:

- the position of the physical quark masses in the diagram,

[2]The more detailed analysis performed in [67] revealed the existence of an infrared stable fixed point also for the $N_f = 2$ case, with critical indices almost identical to those of the 3d $O(4)$ universality class.

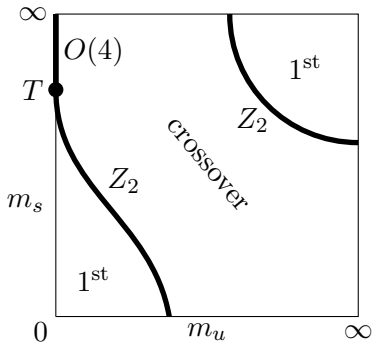

Figure 4.7.: The QCD phase diagram as it is usually presented (the so-called "Columbia plot").

- the order of the chiral transition for $N_f = 2$.

The precise location of the physical point in Fig. (4.7) is clearly of great phenomenological importance: the presence of a phase transition and its order could have observable consequences in heavy ions collisions and cosmology. Simulations at the physical mass values are however computationally extremely demanding and the conclusion that at the physical point no phase transition is present is based only on the observation that the susceptibilities of the observables do not grow, within the statistical errors, by increasing the lattice size. This is clearly a necessary condition for the absence of a phase transition, but the presence of the statistical errors prevents it from being also a sufficient condition: the transition can be weak enough that much larger volumes than the available ones are needed in order to observe a significant increase in the susceptibilities.

A theoretically safer procedure would be to map the position of the line of Z_2 transitions. This procedure was followed in [68], with the additional aim to detect the influence of the tricritical point, but did not lead to conclusive results because of the limited lattice sizes used.

The second point, the order of the chiral transition for $N_f = 2$, is at first sight less physically significant, however it can help to shed new light on the previous point, not to mention that, in order to understand the phase diagram at finite density and physical quark masses, it is convenient to have a complete understanding of the phase diagram at zero temperature also for non physical values of the quark masses. In particular the tricritical point shown in Fig. (4.7) exists only if the $N_f = 2$ transition if of second order and the presence of the tricritical point near the physical point is expected to have observable consequences [69].

In the studies previous to [70] the $O(4)$ scaling was never observed for all the measured quantities, only the dependence of the critical coupling β_c on the quark mass used seemed to favour the $O(4)$ scenario. In [70] it was shown that also this β_c scaling, when properly analyzed, does not provide statistically significant support to the claim that the $N_f = 2$ is in the 3d $O(4)$ universality class.

Taking advantage of the knowledge of the scaling field at the chiral transition, in [70] a novel simulation strategy was proposed in order to have a clear-cut distinction between

the two theoretically possible scenarios, *i.e.* a first order transition or a second order transition in the 3d $O(4)$ universality class. The idea of the method is quite simple: if the transition is second order in the $O(4)$ universality class, two relevant scaling variables have to be present, a thermal-like variable τ and a magnetic-like variable h. Because of the chiral symmetry the magnetic variable is to lowest order proportional to the quark mass m; the free energy can then be written in the scaling form (see *e.g.* [59])

$$F = L^{-4}\phi(\tau L^{y_t}, amL^{y_h}) \tag{4.3.6}$$

where y_t, y_h are the thermal and magnetic renormalization group eigenvalues ($y_t = 1/\nu$, where ν is the correlation length critical index) and L is the system size. The idea is then to vary L and m but keeping the scaling variable amL^{y_h} fixed, in order to reduce the double scaling of Eq. (4.3.6) to a single scaling problem.

This analysis was performed in [70] by using for y_h the known value of the 3d $O(4)$ model and lattice temporal extension $N_t = 4$: the results did not scale as expected from Eq. (4.3.6), moreover some indications were presented that favoured the first order scaling. The same study was repeated in [71] by using $y_h = 3$, *i.e.* the value expected for a first order transition, and a nice scaling was reported. These results raised the following objections

1. no clear signal of bistability was observed, at least to the spatial sizes studied
2. the lattice used (standard staggered fermions with $N_t = 4$) is rather coarse

As to the first point this is not really an objection, since bistability can show up at larger volumes. Clearly a well defined bistability is the most evident signal of a first order transition but the explicit observation of the discontinuity is not the only way to check for the existence a first order transition. In particular statistical systems exist (like the 2d Potts model) which are theoretically known to display a first order transition but for which it is not possible to clearly observe a bistability in simulations, the only way to numerically check the theoretical results being the scaling with the first order critical indices (see *e.g.* [72]). In any case there is no way a priori to discriminate, at finite volume, between an analytical cross-over and a weak first order transiton.

The observation that the used lattice spacing are quite coarse is clearly true, but it would be surprising if the scaling of local observables at a second order phase transition, *i.e.* a typically IR effect, were influenced by the UV lattice artefacts. This point, however, deserves further investigation and the analysis of [70, 71] is being repeated by using lattices with temporal extension $N_t = 6$, thus reducing by 3/2 the lattice spacing.

A complementary analysis is being performed in the high mass region of the $N_f = 2$ theory: the existence of an endpoint for the large mass first order transition is clearly a necessary condition for the existence of the second order chiral phase transition, however its existence has never been verified in a convincing way (some hints were presented in [73]). Simulations are being carried on in the large mass regime in order to get clear evidence of the existence (or not) of this endpoint.

4.4. The computational challenge

Simulations of lattice gauge theory coupled to fermions are much more numerically demanding than pure gauge theory ones since fermions have to be represented in the functional integration by anticommuting variables. The best available method to deal with this problem is to explicitly integrate out the fermions and substitute them with dummy bosonic field ϕ (know as pseudo-fermions) in the following way [74]:

$$\begin{aligned} Z = \int \mathscr{D}U \mathscr{D}\bar{\psi} \mathscr{D}\psi e^{-S_g[U] - \bar{\psi} M[U] \psi} \propto \int \mathscr{D}U \det(M[U]) e^{-S_g[U]} \propto \\ \propto \int \mathscr{D}U \mathscr{D}\phi \exp\Big(- S_g[U] - \phi^* \big(M[U]^\dagger M[U] \big)^{-1} \phi \Big) \end{aligned} \tag{4.4.1}$$

To use Monte Carlo methods it obviously needed that $\det M[U] \geq 0$, which is the case in the zero quark density simulations we are considering.

A convenient algorithm to simulate the action in Eq. (4.4.1) is the Hybrid Monte Carlo [75] HMC). The idea is very simple and it is conveniently exposed by using as an example the case of a single boson with action $S = V(\varphi)$: as a first step a dummy conjugate momentum p is associate to φ and the action becomes $S = \frac{1}{2}p^2 + V(\varphi)$; it is trivial that this change leaves invariant the mean values of φ observables. The HMC then proceeds as follow

1. a random initial momentum is generated with probability $\propto e^{-\frac{1}{2}p^2}$,
2. starting from the state (φ, p), a new trial state (φ', p') is generated by numerically solving the equations of motion derived from the action S,
3. the new state (φ', p') is accepted with probability $\min(1, e^{-\delta S})$ where $\delta S = S(\varphi', p') - S(\varphi, p)$ (Metropolis step).

It can be shown (see *e.g.* [27, 75, 76]) that the sequence of the φ configurations obtained in this way is distributed with the correct $e^{-V(\varphi)}$ probability provided the solution of the equation of motion satisfies the requirements

- the evolution is reversible, *i.e.* $(\varphi, p) \to (\varphi', p')$ if and only if $(\varphi', -\pi') \to (\varphi, p)$,
- the evolution preserves the measure of the phase space, *i.e.* $\det \frac{\partial(\varphi', p')}{\partial(\varphi, p)} = 1$.

A large class of integrators that satisfy these two constraints are the so-called symmetric symplectic integrators, the simplest member of this class being the leap-frog or PQP scheme (for improved schemes see *e.g.* [77–79]).

In the particular case of the action in Eq. (4.4.1) a convenient implementation of this picture is given by the Φ algorithm of [80]:

1. a vector R of complex Gaussian random numbers is generated,
2. the pseudofermion field is initialized by $\phi = M[U]^\dagger R$, in such a way that the probability distribution for ϕ is proportional to $\exp(\phi^*(M^\dagger M)^{-1}\phi)$,

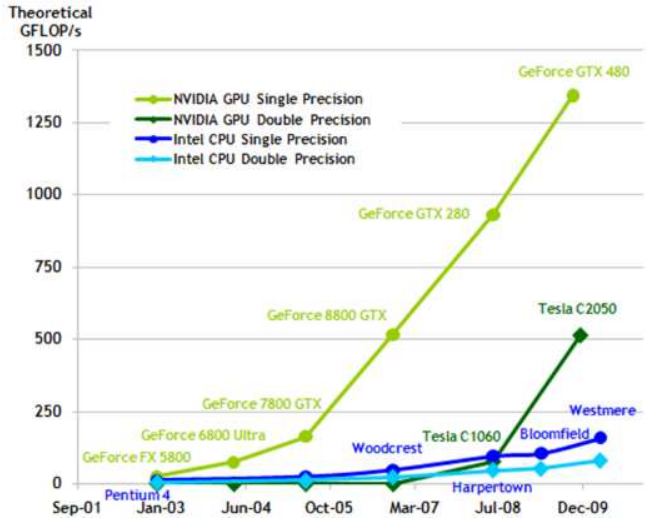

Figure 4.8.: Peack performance of Intel CPU and NVIDIA GPU.

3. the dummy momenta associated to gauge fields are initialized by Gaussian random matrices,

4. the gauge field and momenta are updated by using the equations of motion,

5. the final value of the action is computed and the Metropolis step performed.

The point (4) is the more time consuming, since the calculation of the force requires at each step the solving of the sparse linear system

$$\Big(M[U]^\dagger M[U]\Big)X = \phi \tag{4.4.2}$$

which is usually performed by means of Krylov methods (see *e.g.* [81]). In order to speed up the algorithm a number of methods have been proposed, whose aim is typically to reduce the number of inversions needed in the update [82–85].

For staggered fermions a complication is the presence of the $4-$th root of the determinant in the action: Eq. (4.4.2) becomes

$$\Big(M[U]^\dagger M[U]\Big)^{n/4} X = \phi \tag{4.4.3}$$

where $n \in \mathbb{Z}$ is related to the number of flavours. In order to overcome this problem the Rational Hybrid Monte Carlo (RHMC) was introduced in [86], in which the root of the fermion matrix is approximated by a rational function, which is then efficiently computed by means of the shifted versions of the Krylov solvers (see *e.g.* [87]).

The matrix $M[U]$ is in a typical case a $\sim 10^6 \times 10^6$ sparse matrix, whose details depend on the discretization used for the fermions, and the system Eq. (4.4.2) has to be solved $\sim$ 20 times in order to get a gauge configuration. In finite temperature simulations near phase transitions the situation is worsened by the increasing of the autocorrelation

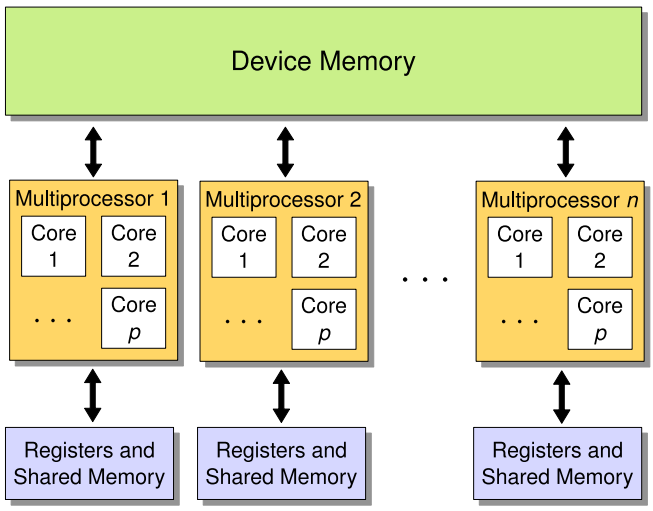

Figure 4.9.: Architecture of a modern NVIDIA graphics card.

time of the Markov chain, so that in order to obtain two statistically independent gauge configuration an increasing number of updates is necessary. In usual simulation a total amount of $\sim 10^5$ updates is a reasonable estimate. The CPU core time needed for of an update strongly depends on the physical parameters but it practically amounts to $\gtrsim 2$ hours.

From these rough estimates it clearly emerges the need for dedicated machines perform QCD simulations. The standard architectures used are CPU clusters with fast interconnections between the nodes, like the APE and the Blue Gene machines, however the increasing need for low cost computing resources led in recent years to explore less traditional architectures.

The video game market developments compelled graphic cards (Graphic Processing Units, GPUs) manufacturers to increase the floating point calculation performance of their products, by far exceeding the performance of standard CPUs (see Fig. (4.8)). The GPU architecture evolved toward programmable many-core chips that are designed to process in parallel massive amounts of data. These developments suggested the possibility of using GPUs in the field of high-performance computing as low-cost substitutes of more traditional CPU-based architectures.

The seminal work that introduced the use of GPUs in lattice QCD simulation was [88] in which the native graphics APIs were used, but the real explosion of interest in the field

GPU	Cores	Bandwidth GB/s	Gflops (peak) single	Gflops (peak) double	Device Memory GB
Tesla C1060	240	102	933	78	4
Tesla C2050/2070	448	144	1030	515	3/6

Table 4.1.: Specifications of the NVIDIA cards designed for high performance computing.

followed the introduction of NVIDIAs CUDA (Compute Unified Device Architecture) platform, that effectively disclosed the field of GPGPU (General Purpose GPU) by allowing the possibility of programming GPUs by a C-like language.

The typical architecture of a modern NVIDIA graphic card is shown in Fig. (4.9). Primary storage is provided by the device memory, which is accessible by all multiprocessors but has a relatively high latency. Within the same multiprocessor, cores have access to local registers and to shared memory, which is shared between the threads of the multiprocessor and it is orders of magnitude faster than device memory, being very close to the computing units. While the total amount of device memory is of order of GBs, the local storage is only 16KB both for the registers and for the shared memory[3], so that it is typically impossible to use just these local fast memories. The latency time of the device memory can be hidden by having a large number of threads in concurrent execution, so when data are needed from device memory for some threads, the ones ready to execute are immediately sent to computation. The highest bandwidth from device memory is achieved when a group of 16 threads accesses a contiguous memory region (coalesced memory access), because its execution requires just one instruction call, saving a lot of clock-cycles. Because of the presence of these three storage levels with different latencies, in GPU application much care has to be used in the memory management, especially in QCD simulations, where the efficiency turns out to be strongly limited by bandwidth.

The specifications of the NVIDIA cards used are reported in Tab. (4.1), where the reported bandwidth is the internal bandwidth of the GPUs. This has to be compared with the typical bandwidth of 5GB/s of the communications between the GPU and the CPU host, which are settled by a PCI express bus. This is the main bottleneck in most GPU applications.

Application of GPUs in lattice QCD simulations have been mainly limited to the inversion of the Dirac matrix (see *e.g.* [89]) to be performed in order to analyze stored gauge configurations previously obtained by using more standard architectures. Our aim [90] was instead to use GPUs to efficiently perform a complete simulation, without the need to rely on more traditional architectures.

In order to avoid the bottleneck of the communication between CPU and GPU we decided to copy the starting gauge configuration (and momenta) on the device memory at the beginning of the simulation and to perform the complete update on the GPU, instead of using it just to speed up some functions and transferring gauge field back and forth between host and device memories. In our implementation of the Dirac kernel a different thread is associated to every (even) site in the fermion update and to every link in the gauge update, so that different treads do not cooperate: shared memory is used just as a local fast memory (the cache memory is not present on GPU). This setup is forced by the high ratio between data and floating point operations per kernel.

Double precision capability was introduced with NVIDIAs GT200 generation, the first one specifically designed having in mind HPC market. While at first the performance in double precision was about an order of magnitude smaller than the one in single precision,

[3]For NVIDIA Tesla cards 10 series. The 20 series has 64KB of on-chip memory.

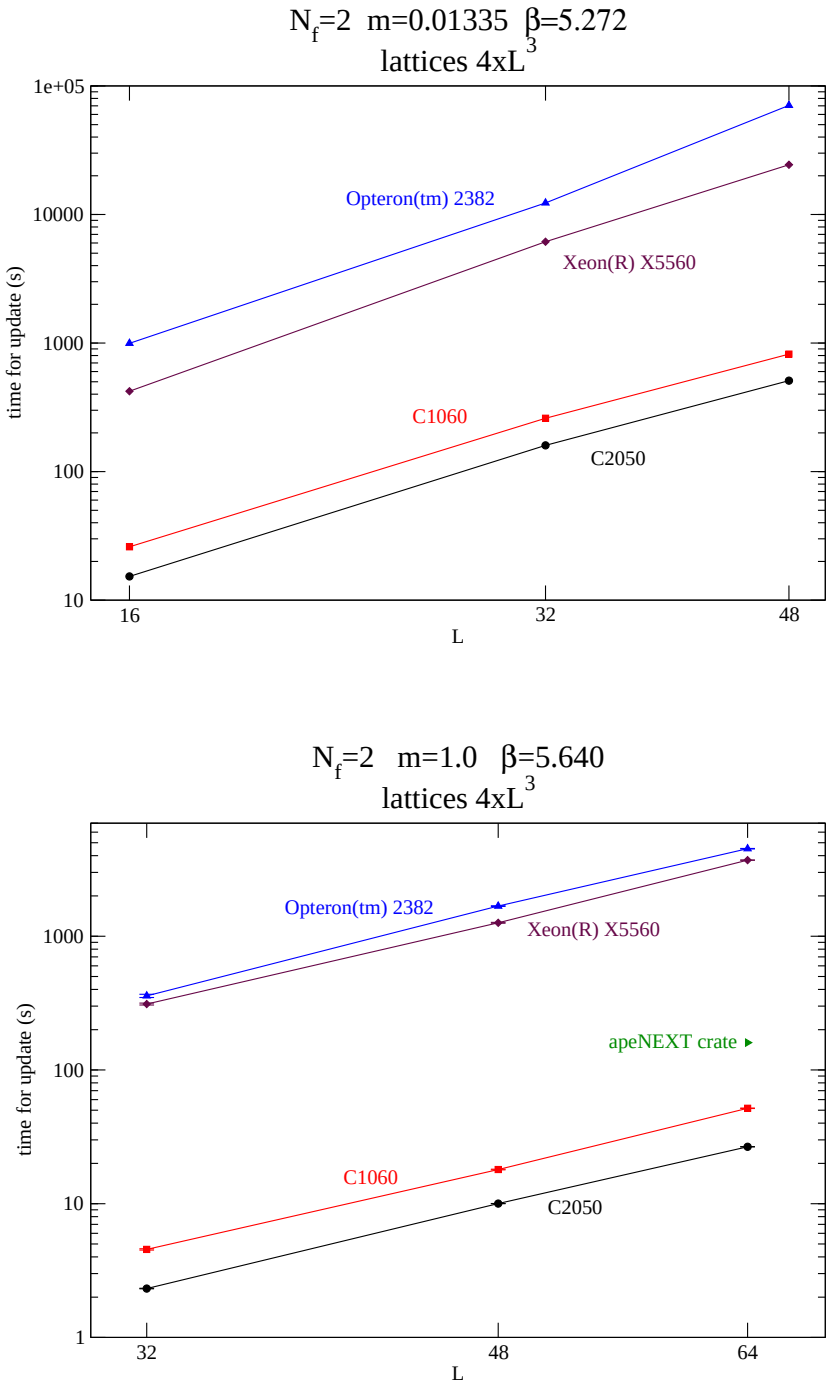

Figure 4.10.: Run times on different architectures. For the Opteron and Xeon runs a single core was used.

Lattice	Bandwidth GB/s	Gflops
4×16^3	56.84 ± 0.03	49.31 ± 0.02
4×32^3	64.091 ± 0.002	55.597 ± 0.002
4×48^3	69.94 ± 0.02	60.67 ± 0.02

Table 4.2.: Staggered Dirac operator kernel performance figures on a C1060 card (single precision).

in the Fermi generation (2010) there is only a factor 2 between the peak performance in single and double precision. Nevertheless, when there are not strong precision issues that force the use of the double precision, single precision is to be preferred, since it halves the size of the memory transfers, effectively increasing the efficiency of the program.

An example of the performance obtained for the Dirac operator is shown in Tab. (4.2), from which it is clear that the main bottleneck is the bandwidth: while using $60-70\%$ of the bandwidth, only the $5-6\%$ of the peak performance is reached. For more technical details on the implementation see [90].

The time needed for a complete update on different architectures is shown in Fig. (4.10) and the time gains of the GPUs over the CPUs are reported in Tab. (4.3) and Tab. (4.4), from which it appears clearly that GPUs can be efficiently used as substitutes of the traditional architectures. In particular a single GPU, whose cost is $\approx 10^3$€, typically outperform an APENext crate, whose cost is of order of 10^5€.

	high mass			low mass		
spatial size	32	48	64	16	32	48
Opteron (single core)	65	75	75	40	50	85
Xeon (single core)	50	50	50	15	25	30
apeNEXT crate	$\sim$3			$\sim$1		

Table 4.3.: NVIDIA C1060 time gains over CPU and apeNEXT.

	high mass			low mass		
spatial size	32	48	64	16	32	48
Opteron (single core)	115	130	140	65	75	140
Xeon (single core)	85	85	100	30	40	50
apeNEXT crate	$\sim$6			$\sim$2		

Table 4.4.: NVIDIA C2050 time gains over CPU and apeNEXT (same code as for C1060, no specific C2050 improvement implemented).

5. Conclusions

The identification of the physical mechanism responsible for color confinement is still matter of controversy in the physical community. The dual superconductivity of the vacuum picture is probably the most interesting possibility from the theoretical point of view, since it is possibly related to a dual symmetry of QCD.

The main theoretical problem in this framework is the definition of the monopole degrees of freedom, which appear to explicitly depend on the specific abelian projection used. We pointed out that the abelian magnetic current is the component in the direction of the abelian projection of the violation of the non-abelian Bianchi identities, thus providing a gauge invariant characterization of the monopoles. A direct consequence of this result is that dual superconductivity through monopole condensation is a well defined gauge invariant mechanism for color confinement.

By using the connection between the magnetic current and the gauge covariant NABIs violations, it is possible to give an explanation of the gauge dependence of the monopole detection *à la* DeGrand-Toussaint and to provide quantitative predictions for the dependence of the number of monopoles detected on the gauge fixing adopted. These predictions were checked numerically and the agreement is satisfactory. In particular in this way we explained the impossibility to detect monopole by the DeGrant-Toussaint recipe in the Landau gauge.

Io order to study monopole condensation and its relation with color confinement by numerical simulations it is necessary to compute the vacuum expectation value of a magnetically charged operator. The introduction of such an operator on the lattice presents non-trivial problems for the non-abelian gauge theories. We provided strong indications that these problems can be overcome by properly modifying the previously introduced monopole operator μ and we presented numerical results for the pure gauge $SU(2)$ theory that, although not yet conclusive, support the theoretical analysis.

The introduction of the fermions does not require any modification of the picture of color confinement through dual superconductivity, since monopole degrees of freedom are only related to the topology of the gauge configuration. Monopole condensation can thus provide valuable information in the study of the QCD phase diagram at finite temperature. Simulations are being performed in order to clarify some controversial points of the QCD phase diagram, in particular the order of the chiral transition with two massless quark flavours. Theoretically there are two available possibilities, first order or second order in the 3d $O(4)$ universality class, and which of these two possibilities is realized is a peculiar property of the QCD phase diagram that can be settled by numerical simulations.

Numerical simulations of QCD are however extremely computationally demanding and require the use of dedicated machines. In order to made these simulations more feasible

(both from the computational and the economical point of view) new computational architectures are being currently tested by various research groups. We showed that the graphics processing units are a valid and relatively cheap substitute of the traditional dedicated machines and, after an initial stage of software development, we are currently using them to study the QCD phase diagram.

A. Conventions

A.1. Gauge coupling

The fundamental electric charge will be indicated by e, while g will be the magnetic charge. The covariant derivative, the covariant derivative in the adjoint representation and the field strength are defined by the relations

$$\begin{aligned} D_\mu &= \partial_\mu + ieA_\mu \\ D_\mu^{\text{agg}} &= \partial_\mu + ie[A_\mu, \cdot\,] \\ G_{\mu\nu} &= \frac{1}{ie}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu] \end{aligned}$$

The gauge transformation acts on fermions and gauge field as

$$\psi' = U\psi \quad A'_\mu = UA_\mu U^\dagger + \frac{i}{e}(\partial_\mu U)U^\dagger$$

It is simple to check by direct computation that if ϕ is a field in the adjoint representation then

$$[D_\mu^{\text{agg}}, D_\nu^{\text{agg}}]\phi = ie[G_{\mu\nu}, \phi]$$

A.2. Roots and weights

Our notation is the standard one (see *e.g.* [91,92]): the generator of the Cartan subgroup are denoted by H_i, $i = 1, \ldots, r$ (r is the rank of the group). The eigenvalues of the H_i operators in a given representation are called the weights of the representation. A distinguished role is played by the adjoint representation and its weights are called roots; the generators of the algebra not in the Cartan subalgebra are associated to non-vanishing root values and occur in pairs with opposite root values:

$$\begin{aligned} &[H_i, H_j] = 0 && [H_i, E_{\pm\vec{\alpha}}] = \pm\alpha_i\, E_{\pm\vec{\alpha}} \\ &[E_{\vec{\alpha}}, E_{\vec{\beta}}] = N_{\vec{\alpha},\vec{\beta}}\, E_{\vec{\alpha}+\vec{\beta}} && [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \vec{\alpha}\cdot\vec{H} \end{aligned}$$

where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and $N_{\vec{\alpha},\vec{\beta}} \neq 0$ only if $\vec{\alpha} + \vec{\beta}$ is a root. A root $\vec{\alpha}$ is conventionally called positive if its first non zero component is positive and a positive root is said to be simple if it cannot be written as the sum of two other positive root. Also

$$H_i^\dagger = H_i \quad E_{-\vec{\alpha}}^\dagger = E_{\vec{\alpha}} \quad \text{Tr}(H_i H_j) = \lambda\delta_{ij} \quad \text{Tr}(E_{\vec{\alpha}}^\dagger E_{\vec{\beta}}) = \lambda\delta_{\vec{\alpha},\vec{\beta}} \quad \lambda > 0$$

The fundamental weights $\vec{\mu}^i$ are defined in terms of the simple roots $\vec{\alpha}^j$ by means of the duality

$$\vec{\alpha}^i \cdot \vec{\mu}^j = \delta^{ij}$$

In the literature the usual definition is $2\frac{\vec{\alpha}^i \cdot \vec{\mu}^j}{(\vec{\alpha}^i)^2} = \delta^{ij}$; we changed the normalization just to simplify the notation. The operators associated to fundamental weights, which will be called $\hat{\mu}^i$, are defined by $\hat{\mu}^i = \vec{\mu}^i \cdot \vec{H}$, are a base of the Cartan algebra and they satisfy the commutation rules

$$[\hat{\mu}^i, H_j] = 0 \quad [\hat{\mu}^i, E_{\vec{\alpha}^j}] = \delta_{ij} E_{\vec{\alpha}^j}$$

A.3. $SU(N)$

The group $SU(N)$ is an $N^2 - 1$ dimensional group, whose Lie algebra $\mathfrak{su}(N)$ is the algebra of hermitian traceless matrices. Its rank is $N - 1$ and a convenient base for the Cartan algebra are the generalized Gell-Mann matrices H_m defined by

$$\left(H_m\right)_{ij} = \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^{m} \delta_{ik}\delta_{jk} - m\delta_{i,m+1}\delta_{j,m+1} \right) \quad 1 \leq m \leq N-1 \quad 1 \leq i,j \leq N$$

which satisfy the normalization condition $\mathrm{Tr}(H_i H_j) = \frac{1}{2}\delta_{ij}$. For example

$$H_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 & \cdots \\ 0 & -1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad H_2 = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & -2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$H_3 = \frac{1}{\sqrt{24}} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & -3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The N weights of the fundamental representation are the $(N-1)-$vectors given by

$$\left(\vec{\nu}^j\right)_m = \left(H_m\right)_{jj} = \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^{m} \delta_{jk} - m\delta_{j,m+1} \right)$$

and it is straightforward to check that they satisfies the relations

$$\vec{\nu}^i \cdot \vec{\nu}^j = \frac{1}{2}\delta_{ij} - \frac{1}{2N}$$

The Es operators get from one weight to another, so the roots are given by the differences of the weights; the positive root are $\vec{\nu}^i - \vec{\nu}^j$ for $i < j$, while the simple roots are

$$\vec{\alpha}^i = \vec{\nu}^i - \vec{\nu}^{i+1} \quad 1 \leq i \leq N-1$$

and they satisfy the relation

$$\vec{\alpha}^{\,i} \cdot \vec{\alpha}^{\,j} = \delta_{ij} - \frac{1}{2}\delta_{i,j\pm1}$$

so that all simple roots have length 1 and the angle between two consecutive simple roots is $\frac{2}{3}\pi$. The Dynkin diagram for $SU(N)$ is shown in Fig. (A.1). It is simple to show that the fundamental weights are

$$\vec{\mu}^{\,j} = 2\sum_{k=1}^{j} \vec{\nu}^{\,k}$$

where the factor 2 is due to our normalizations. Explicitly the simple roots and the associated operators are given by

$$\big(\vec{\alpha}^{\,i}\big)_m = -\sqrt{\frac{i-1}{2i}}\delta_{i-1,m} + \sqrt{\frac{i+1}{2i}}\delta_{im} \quad 1 \le i, m \le N-1$$
$$\big(\vec{\alpha}^{\,i} \cdot \vec{H}\big)_{jk} = \frac{1}{2}\big(\delta_{ij} - \delta_{i+1,j}\big)\delta_{jk}$$

The operators associated to fundamental weights are

$$\hat{\mu}^i = \frac{1}{N}\mathrm{diag}\big(\overbrace{N-i,\ldots,N-i}^{i}, \overbrace{-i,\ldots,-i}^{N-i}\big)$$

which is easily shown by noting that $\mathrm{Tr}\big[\hat{\mu}^i\,(\vec{\alpha}^j \cdot \vec{H})\big] = \frac{1}{2}\delta^{ij}$ for every simple root.

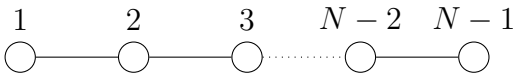

Figure A.1.: Dynkin diagram of $SU(N)$.

A.4. $SO(2N+1)$

The group $SO(2N+1)$ is an $N(2N+1)$ dimensional group, its Lie algebra $\mathfrak{so}(2N+1)$ is the algebra of the imaginary antisymmetric matrices and it has rank N. A convenient base for the algebra is the one whose elements are the matrices M_{ab} defined by

$$\big(M_{ab}\big)_{xy} = -i(\delta_{ax}\delta_{by} - \delta_{bx}\delta_{ay})$$

which satisfy the condition $\mathrm{Tr}(M_{ab}M_{cd}) = 2\delta_{ac}\delta_{bd}$. As generators of the Cartan algebra we can choose the matrices

$$H_j = M_{2j-1,2j} \quad 1 \le j \le N$$

while the operators associated to the roots are ($\eta, \eta' = \pm 1$)

$$\begin{aligned} E_{\eta\vec{\nu}^j} =& \frac{1}{\sqrt{2}}\left(M_{2j-1,2N+1} + i\eta\, M_{2j,2N+1}\right) \\ E_{\eta\vec{\nu}^j+\eta'\vec{\nu}^k} =& \frac{1}{2}\left(M_{2j-1,2k-1} + i\eta\, M_{2j,2k-1} + \right. \\ & \left. + i\eta'\, M_{2j-1,2k} - \eta\eta'\, M_{2j,2k}\right) \end{aligned}$$

where $1 \leq j,k \leq N$ and $\left(\vec{\nu}^j\right)_x = \delta_{jx}$. The simple roots are

$$\begin{aligned} &\vec{\alpha}^j = \vec{\nu}^j - \vec{\nu}^{j+1} \quad 1 \leq j \leq N-1 \\ &\vec{\alpha}^N = \vec{\nu}^N \end{aligned}$$

The first $N-1$ simple roots have lengths $\sqrt{2}$ and relative angles $\frac{2}{3}\pi$; the $N-$th root has length 1 and the angle with the $N-1-$th root is $\frac{3}{4}\pi$. These properties are summarized in the Dynkin diagram Fig. (A.2). The fundamental weights are given by the expression

$$\vec{\mu}^i = \sum_{j=1}^{i} \vec{\nu}^j$$

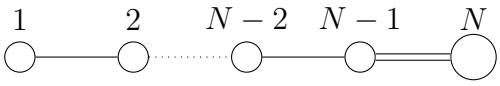

Figure A.2.: Dynkin diagram of $SO(2N+1)$.

A.5. G_2

The group G_2 is most conveniently described by using the Cayley octonions. These can be defined by means of the Cayley-Dickson process applied to quaternion, similarly to the construction of quaternion from complex numbers and complex numbers from real ones (see *e.g.* [93]). Octonions are non-commutative and non-associative, although they satisfy a weak form of associativity, namely alternativity, *i.e.* $[x,y,z] = (xy)z - x(yz)$ is completely antisymmetric.

The group G_2 is defined as the automorphism group of the octonions: $G_2 = GL(\mathbb{O})$. It can be shown (see *e.g.* [93]) that $G_2 \subset SO(7)$ and that G_2 is exactly the 14− dimensional subgroup of $SO(7)$ that fixes the completely antisymmetric 3−tensor T whose independent non-vanishing components are

$$T_{127} = T_{154} = T_{163} = T_{235} = T_{264} = T_{374} = T_{576} = 1$$

As a consequence G_2 can be described as the group of the real 7×7 matrices that satisfy the relations

$$\Omega_{ab}\Omega_{ac} = \delta_{bc} \quad \det\Omega = 1 \quad T_{abc} = T_{def}\Omega_{da}\Omega_{eb}\Omega_{fc}$$

A convenient base for the algebra of the G_2 group was introduced in [94] and it is given by the following C_i matrices, $i = 1, \ldots, 14$ ($\mathrm{Tr}(C_i C_j) = \delta_{ij}$)

$$C_1 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad C_2 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$C_3 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad C_4 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_5 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad C_6 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_7 = \frac{i}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad C_8 = \frac{i}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$C_9 = \frac{i}{2\sqrt{3}}\begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \qquad C_{10} = \frac{i}{2\sqrt{3}}\begin{pmatrix} 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$C_{11} = \frac{i}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad C_{12} = \frac{i}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_{13} = \frac{i}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad C_{14} = \frac{i}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The group G_2 has rank 2 and the Cartan subgroup is generated by C_5 and C_{11}. By simultaneous diagonalization of C_5 and C_{11} it is trivial to obtain the weights of the fundamental 7−dimensional representation of G_2, which are depicted in Fig. (A.3).

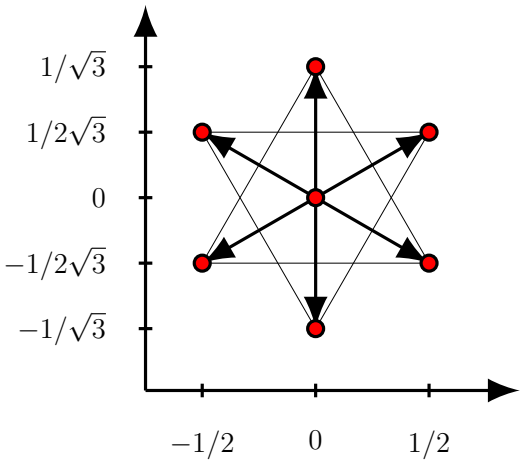

Figure A.3.: Weights diagram of the G_2 fundamental representation.

Root vectors are obtained as differences of the weights of the fundamental representations and are shown in Fig. (A.4).

The simple roots are

$$\vec{\alpha}_1 = \left(0, \frac{1}{\sqrt{3}}\right) \quad \vec{\alpha}_2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

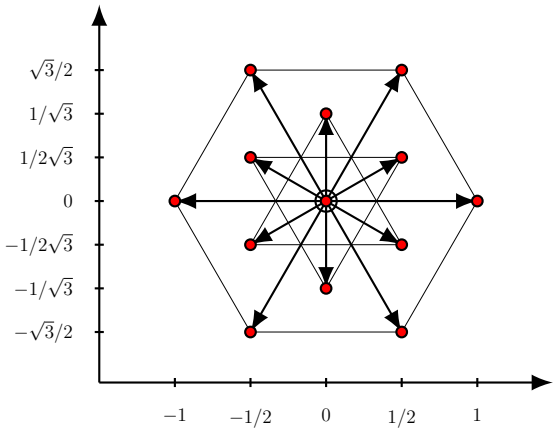

Figure A.4.: Weights diagram of the G_2 adjoint representation.

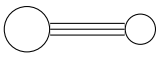

Figure A.5.: Dynkin diagram of G_2.

and the angle between them is $\frac{5}{6}\pi$, so the Dynkin diagram of the group G_2 is the one shown in Fig. (A.5) and the fundamental weights are

$$\vec{\mu}_1 = \left(3, \sqrt{3}\right) \quad \vec{\mu}_2 = (2, 0)$$

B. Proof of Eq. (2.5.2) for the exceptional groups

We will sketch how the proof of Eq. (2.5.2) given in the text can be extended to the exceptional groups. By using the definition Eq. (2.5.7) we arrive also for the exceptional groups to the equation

$$C^\nu = -\frac{e^2}{2}\sum_{\ell jk}\epsilon^{\mu\nu\rho\sigma}a_\mu^{\vec{\alpha}_\ell}a_\rho^{\vec{\alpha}_j}a_\sigma^{\vec{\alpha}_k}c^i_{\vec{\alpha}_\ell}c^i_{\vec{\alpha}_j}c^i_{\vec{\alpha}_k}\times \mathrm{Tr}(E_{\vec{\alpha}_\ell}[E_{\vec{\alpha}_j},E_{\vec{\alpha}_k}])\times\Pi_{n=3,4,5,6}$$

and the esplicit form of the Π_n terms is

$$\begin{aligned}
\Pi_3 =&\frac{49}{36}-\frac{14}{36}\times\frac{5}{2}\langle\lambda\rangle+\frac{1}{36}\Big(3\langle\lambda^2\rangle+\langle\lambda_i\lambda_j\rangle\Big)\\
\Pi_4 =&\frac{205}{144}-\frac{91}{192}\times\frac{5}{2}\langle\lambda\rangle+\frac{5}{96}\Big(3\langle\lambda^2\rangle+\langle\lambda_i\lambda_j\rangle\Big)+\\
&+\frac{1}{576}\left(-3\langle\lambda^3\rangle-3\langle\lambda_i^2\lambda_j\rangle+\frac{1}{2}\langle\lambda_i\lambda_j\lambda_k\rangle\right)\\
\Pi_5 =&\frac{5269}{3600}-\frac{1529}{2880}\times\frac{5}{2}\langle\lambda\rangle+\frac{341}{4800}\Big(3\langle\lambda^2\rangle+\langle\lambda_j\lambda_i\rangle\Big)+\\
&+\frac{11}{2880}\left(-3\langle\lambda^3\rangle-3\langle\lambda_i^2\lambda_j\rangle+\frac{1}{2}\langle\lambda_i\lambda_j\lambda_k\rangle\right)+\\
&+\frac{1}{14400}\Big(3\langle\lambda^4\rangle+2\langle\lambda_i^2\lambda_j^2\rangle+3\langle\lambda_i^3\lambda_j\rangle-\langle\lambda_i\lambda_j^2\lambda_k\rangle\Big)\\
\Pi_6 =&\frac{5369}{3600}-\frac{37037}{64800}\times\frac{5}{2}\langle\lambda\rangle+\frac{44473}{518400}\Big(3\langle\lambda^2\rangle+\langle\lambda_j\lambda_i\rangle\Big)+\\
&+\frac{1001}{172800}\left(-3\langle\lambda^3\rangle-3\langle\lambda_i^2\lambda_j\rangle+\frac{1}{2}\langle\lambda_i\lambda_j\lambda_k\rangle\right)+\\
&+\frac{91}{518400}\Big(3\langle\lambda^4\rangle+2\langle\lambda_i^2\lambda_j^2\rangle+3\langle\lambda_i^3\lambda_j\rangle-\langle\lambda_i\lambda_j^2\lambda_k\rangle\Big)+\\
&+\frac{1}{518400}\Big(-3\langle\lambda^5\rangle-3\langle\lambda_i\lambda_j^4\rangle-4\langle\lambda_i^3\lambda_j^2\rangle+\langle\lambda_i\lambda_j^3\lambda_k\rangle+\frac{1}{2}\langle\lambda_i^2\lambda_j^2\lambda_k\rangle\Big)
\end{aligned}$$

The combinations of the roots that satisfy the constraint $\vec{\alpha}_\ell+\vec{\alpha}_j+\vec{\alpha}_k=0$ are, together with generic permutations of ℓ,j,k

$$c^i_\ell=1,c^i_j=1,c^i_k=-2\;\Big|\;c^i_\ell=1,c^i_j=2,c^i_k=-3$$

if λ takes three different values,

$$\begin{array}{l|l} c^i_\ell = 1, c^i_j = 1, c^i_k = -2 & c^i_\ell = 1, c^i_j = 2, c^i_k = -3 \\ c^i_\ell = 1, c^i_j = 3, c^i_k = -4 & c^i_\ell = 2, c^i_j = 2, c^i_k = -4 \end{array}$$

if λ takes four different values,

$$\begin{array}{l|l} c^i_\ell = 1, c^i_j = 1, c^i_k = -2 & c^i_\ell = 1, c^i_j = 2, c^i_k = -3 \\ c^i_\ell = 1, c^i_j = 3, c^i_k = -4 & c^i_\ell = 2, c^i_j = 2, c^i_k = -4 \\ c^i_\ell = 1, c^i_j = 4, c^i_k = -5 & c^i_\ell = 2, c^i_j = 3, c^i_k = -5 \end{array}$$

if λ takes five different values and

$$\begin{array}{l|l} c^i_\ell = 1, c^i_j = 1, c^i_k = -2 & c^i_\ell = 1, c^i_j = 2, c^i_k = -3 \\ c^i_\ell = 1, c^i_j = 3, c^i_k = -4 & c^i_\ell = 2, c^i_j = 2, c^i_k = -4 \\ c^i_\ell = 1, c^i_j = 4, c^i_k = -5 & c^i_\ell = 2, c^i_j = 3, c^i_k = -5 \\ c^i_\ell = 1, c^i_j = 5, c^i_k = -6 & c^i_\ell = 2, c^i_j = 4, c^i_k = -6 \\ c^i_\ell = 3, c^i_j = 3, c^i_k = -6 & \end{array}$$

if λ takes six different values.

By using any of these combinations we get $\Pi_n = 0$ thus proving that $C_\nu = 0$ also for the exceptional groups.

Bibliography

[1] K. G. Wilson *Confinement of Quarks*. Phys. Rev. D **10**, 2445 (1974).

[2] M. Engelhardt, K. Langfeld, H. Reinhardt, O. Tennert *Deconfinement in $SU(2)$ Yang-Mills theory as a center vortex percolation transition*. Phys. Rev. D **61**, 054504 (2000) [arXiv:hep-lat/9904004].

[3] L. P. Kadanoff, H. Ceva *Determination of an opeator algebra for the two-dimensional Ising model*. Phys. Rev. B **3**, 3918 (1971).

[4] J. V. Jose, L. P. Kadanoff, S. Kirkpatrick, D. R. Nelson *Renormalization, vortices, and symmetry breaking perturbations on the two-dimensional planar model*. Phys. Rev. B **16**, 1217 (1977).

[5] N. Seiberg and E. Witten *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*. Nucl. Phys. B **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [arXiv:hep-th/9407087].

[6] G. 't Hooft *Magnetic monopoles in unified gauge theories*. Nucl. Phys. B **79**, 276 (1974).

[7] A. M. Polyakov *Particle spectrum in quantum field theory*. JETP Lett. **20**, 194 (1974).

[8] G. 't Hooft *Topology of the Gauge Condition and New Confinement Phases in Nonabelian Gauge Theories*. Nucl. Phys. B **190**, 455 (1981).

[9] T. A. DeGrand, D. Toussaint *Topological excitations and Monte Carlo simulation of Abelian gauge theory*. Phys Rev. D **22**, 2478 (1980).

[10] A. Di Giacomo, B. Lucini, L. Montesi, G. Paffuti *Color confinement and dual superconductivity of the vacuum. 1.* Phys. Rev. D **61**, 034503 (2000) [arXiv:hep-lat/9906024].

[11] J. Greensite, B. Lucini *Is Confinement a Phase of Broken Dual Gauge Symmetry?*. Phys. Rev. D **78**, 085004 (2008) [arXiv:0806.2117 [hep-lat]].

[12] P. A. M. Dirac *Quantized Singularities in the Electromagnetic Field*. Proc. Roy. Soc. Lond. A **133**, 60 (1931).

[13] Y. M. Shnir "Magnetic Monopoles". Springer-Verlag (2005).

[14] T. T. Wu, C. N. Yang *Concept of nonintegrable phase factors and global formulation of gauge fields*. Phys. Rev. D **12**, 3845 (1975).

[15] H. Georgi, S. L. Glashow *Unified Weak and Electromagnetic Interactions without Neutral Currents*. Phys. Rev. Lett. **28**, 1494 (1972).

[16] J. Arafune, P. G. O. Freund, C. J. Goebel *Topology of Higgs fields*. Jour. of Math. Phys. **16**, 433 (1975).

[17] P. Goddard, J. Nuyts, D. I. Olive *Gauge Theories and Magnetic Charge*. Nucl. Phys. B **125**, 1 (1977).

[18] S. Coleman *The Magnetic Monopole Fifty Years Later*, in A. Zichichi (ed.) "The unity of the fundamental interactions", Plenum Press (1983).

[19] S. Coleman *Classical lumps and their quantum descendants*, in S. Coleman "Aspects of symmetry", Cambridge University Press (1988).

[20] B. A. Dubrovin, A. T. Fomenko, I. S. Novikov "Modern Geometry − Methods And Applications. Part 2: The Geometry And Topology Of Manifolds", Springer-Verlag (1985).

[21] A. Di Giacomo, L. Lepori, F. Pucci *Homotopy, monopoles and 't Hooft tensor in QCD with generic gauge group*. JHEP **0810**, 096 (2008).

[22] L. Del Debbio, A. Di Giacomo, B. Lucini, G. Paffuti *Abelian projection in $SU(N)$ gauge theories*, [arXiv:hep-lat/0203023].

[23] C. Bonati, A. Di Giacomo, L. Lepori and F. Pucci *Monopoles, abelian projection and gauge invariance*. Phys. Rev. D **81**, 085022 (2010) [arXiv:1002.3874 [hep-lat]].

[24] S. R. Coleman, J. Mandula *All Possible Symmetries Of The S Matrix*. Phys. Rev. **159**, 1251 (1967).

[25] C. Itzykson, J.-M. Drouffe "Statistical field theory, vol. 1". Cambridge University Press (1989).

[26] H. J. Rothe "Lattice gauge theories. An introduction". World Scientific (2005).

[27] I. Montvay, G. Münster "Quantum fields on a lattice". Cambridge University Press (1993).

[28] E. Seiler *Upper bound on the color-confinig potential*. Phys. Rev. D **18**, 482 (1978).

[29] B. Simon, L. G. Yaffe *Rigorous perimeter law upper bound on Wilson loops*. Phys. Lett. B **115**, 145 (1982).

[30] K. Holland, P. Minkowski, M. Pepe, U.-J. Wiese *Exceptional Confinement in G(2) Gauge Theory*. Nucl. Phys. B **668**, 207 (2003) [arXiv:hep-lat/0302023].

[31] M. Creutz "Quarks, gluons and lattices". Cambridge University Press (1983).

[32] N.Cabibbo, G.Parisi *Exponential Hadronic Spectrum and Quark Liberation*. Phys. Lett. B **59**, 67 (1975).

[33] L. D. McLerran, B. Svetitsky *Quark liberation at high temperature: A Monte Carlo study of $SU(2)$ gauge theory*. Phys. Rev. D **24**, 450 (1981).

[34] B. Svetitsky, L. G. Yaffe *Critical behavior at finite-temperature confinement transitions*. Nucl. Phys. B **210**, 423 (1982).

[35] M. Pepe, U. -J. Wiese *Exceptional Deconfinement in $G(2)$ Gauge Theory*. Nucl. Phys. B **768**, 21 (2007) [hep-lat/0610076].

[36] G. Cossu, M. D'Elia, A. Di Giacomo, B. Lucini, C. Pica *G_2 gauge theory at finite temperature*. JHEP **0710**, 100 (2007) [arXiv:0709.0669 [hep-lat]].

[37] L. H. Karsten, J. Smit *Lattice fermions: species doubling, chiral invariance and the triangle anomaly*. Nucl Phys. B **183**, 103 (1981).

[38] H. B. Nielsen, M. Ninomiya *Absence Of Neutrinos On A Lattice. 1. Proof By Homotopy Theory*. Nucl. Phys. B **185**, 20 (1981). Erratum, *ibid.* 195, 541 (1982).

[39] H. B. Nielsen, M. Ninomiya *Absence Of Neutrinos On A Lattice. 2. Intuitive Topological Proof*. Nucl. Phys. B **193**, 173 (1981).

[40] H. B. Nielsen, M. Ninomiya *A no-go theorem for regularizing chiral fermions*. Phys. Lett. B **105**, 219 (1981).

[41] L. H. Karsten *Lattice fermions in euclidean space-time*. Phys. Lett. B **104**, 315 (1981).

[42] H. Neuberger *Exactly massless quarks on a lattice*. Phys. Lett. B **417**, 141 (1998) [hep-lat/9707022].

[43] M. Lüscher *Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation*. Phys. Lett. B **428**, 342 (1998) [hep-lat/9802011].

[44] G. Cossu, S. Aoki, S. Hashimoto, T. Kaneko, H. Matsufuru, J. Noaki, E. Shintani *Finite temperature QCD at fixed Q with overlap fermions*. [arXiv:1011.0257].

[45] A. S. Kronfeld, M. L. Laursen, G. Schierholz, U. J. Wiese *Monopole condensation and color confinement*. Phys. Lett. B **198**, 516 (1987).

[46] C. Bonati, A. Di Giacomo, M. D'Elia *Detecting monopoles on the lattice*. Phys. Rev. D **82**, 094509 (2010) [arXiv:1009.2425 [hep-lat]].

[47] L. Del Debbio, A. Di Giacomo, M. Maggiore, S. Olejnik *Confinement and monopoles in lattice QCD*. Phys. Lett. B **267**, 254 (1991).

[48] M. Creutz *Monte Carlo study of quantized $SU(2)$ gauge theory*. Phys. Rev. D **21**, 2308 (1980).

[49] A. D. Kennedy, B. J. Pendleton *Improved heatbath method for Monte Carlo calculations in lattice gauge theories*. Phys. Lett. B **156**, 393 (1985).

[50] M. Creutz *Overrelaxation and Monte Carlo simulation*. Phys. Rev. D **36**, 515 (1987).

[51] P. Cea, L. Cosmai *Dual superconductivity in the $SU(2)$ pure gauge vacuum: A lattice study*. Phys. Rev. D **52**, 5152 (1995) [arXiv:hep-lat/9504008].

[52] T. Suzuki, K. Ishiguro, Y. Mori, T. Sekido *Dual Meissner Effect and Magnetic Displacement Currents*. Phys. Rev. Lett. **94**, 132001 (2005) [arXiv:hep-lat/0410001].

[53] E. C. Marino, B. Schroer, J. A. Swieca *Euclidean Functional Integral Approach For Disorder Variables And Kinks*. Nucl. Phys. B **200**, 473 (1982).

[54] P. A. M. Dirac *Gauge invariant formulation of quantum electrodynamics*. Can. J. Phys. **33**, 650 (1955).

[55] L. Del Debbio, A. Di Giacomo, G. Paffuti *Detecting dual superconductivity in the ground state of gauge theory*. Phys. Lett. B **349**, 513 (1995) [arXiv:hep-lat/9403013].

[56] A. Di Giacomo, G. Paffuti *Disorder parameter for dual superconductivity in gauge theories*. Phys. Rev. D **56**, 6816 (1997) [arXiv:hep-lat/9707003].

[57] J. Frölich, P. A. Marchetti *Magnetic Monopoles and Charged States in Four-Dimensional, Abelian Lattice Gauge Theories*. Europhys. Lett. **2**, 933 (1986).

[58] J. Frölich, P. A. Marchetti *Soliton Quantization in Lattice Field Theories*. Comm. Math. Phys. **112**, 343 (1987).

[59] J. Cardy "Scaling and Renormalization in Statistical Physics". Cambridge University Press (2003).

[60] A. Di Giacomo, B. Lucini, L. Montesi, G. Paffuti *Color confinement and dual superconductivity of the vacuum. 2.* Phys. Rev. D **61**, 034504 (2000) [arXiv:hep-lat/9906025].

[61] J. M. Carmona, M. D'Elia, A. Di Giacomo, B. Lucini, *Color confinement and dual superconductivity of the vacuum. 3.* Phys. Rev. D **64**, 114507 (2001) [arXiv:hep-lat/0103005].

[62] G. Cossu, M. D'Elia, A. Di Giacomo, B. Lucini *unpublished* (2006).

[63] J. Frölich, P. A. Marchetti *Order parameter reconciling the Abelian and center dominance in $SU(2)$ Yang-ills theory*. Phys. Rev. D **64**, 014505 (2001).

[64] C. Bonati, G. Cossu, M. D'Elia, A. Di Giacomo, *unpublished*.

[65] E. T. Tomboulis, L. G. Yaffe *Absence of Chiral Symmetry Breaking at High Temperature*. Phys. Rev. Lett. **52**, 2115 (1984).

[66] R. D. Pisarski, F. Wilczek *Remarks on the chiral chromodynamics*. Phys. Rev. D **29**, 338 (1984).

[67] F. Basile, A. Pelisseto, E. Vicari *Finite-temperature chiral transition in QCD with quarks in the fundamental and adjoint representation*. PoS(Lattice 2005)199 [arXiv:hep-lat/0509018].

[68] F. De Forcrand, O. Philipsen *The chiral critical line of Nf = 2+1 QCD at zero and non-zero baryon density*. JHEP **0701**, 077 (2007) [hep-lat/0607017].

[69] M. A. Stephanov, K. Rajagopal, E. V. Shuryak *Signatures of the tricritical point in QCD*. Phys. Rev. Lett. **81**, 4816 (1998) [arXiv:hep-ph/9806219].

[70] M. D'Elia, A. Di Giacomo, C. Pica *Two flavor QCD and confinement*. Phys. Rev. **D72**, 114510 (2005) [arXiv:hep-lat/0503030].

[71] G. Cossu, M. D'Elia, A. Di Giacomo, C. Pica *Two flavor QCD and confinement. II*. [arXiv:0706.4470 [hep-lat]].

[72] W. Janke *Accurate first-order transition points from finite-size data without power-law corrections*. Phys. Rev. B **47**, 14757 (1993).

[73] F. Karsch, Ch. Schmidt, S. Stickan *Common features of deconfining and chiral critical points in QCD and the three state Potts model in an external field*. Comput. Phys. Commun. **147**, 451 (2002) [arXiv:hep-lat/0111059].

[74] D. H. Weingarten, D. N. Petcher *Monte Carlo integration for lattice gauge theories with fermions*. Phys. Lett. B 99, 333 (1981).

[75] S. Duane, A. D. Kennedy, B. J. Pendleton, D. Roweth *Hybrid Monte Carlo*. Phys. Lett. B 195, 216 (1987).

[76] A. D. Kennedy *Algorithms for Dynamical Fermions*. [arXiv:hep-lat/0607038].

[77] M. Campostrini, P. Rossi *A comparison of numerical algorithms for dynamical fermions*. Nuc. Phys. B **329**, 753 (1990).

[78] M. Creutz, A. Gocksch *Higher-order hybrid Monte Carlo algorithms*. Phys. Rev. Lett. **63**, 9 (1989).

[79] T. Takaishi, P. de Forcrand *Testing and tuning symplectic integrators for Hybrid Monte Carlo algorithm in lattice QCD*. Phys. Rev. E **73** 036706, (2006) [arXiv:hep-lat/0505020].

[80] S. Gottlieb, W. Liu, D. Toussaint, R. L. Renken, R. L. Sugar *Hybrid-molecular-dynamics algorithms for the numerical simulation of quantum chromodynamics.* Phys. Rev. D **35**, 2531 (1987).

[81] H. van der Vorst "Iterative Krylov Methods for Large Linear Systems". Cambridge University Press (2003).

[82] T. DeGrand, P. Rossi *Conditioning Techniques for Dynamical Fermions.* Comput. Phys. Commun. **60**, 211 (1990).

[83] J. C. Sexton, D. H. Weingarten *Hamiltonian evolution for the hybrid Monte Carlo algorithm.* Nucl Phys. B **380**, 665 (1992).

[84] M. Hasenbusch *Speeding up the hybrid Monte Carlo algorithm for dynamical fermions.* Phys. Lett. B **519**, 177 (2001) [arXiv:hep-lat/0107019].

[85] M. A. Clark, A. D. Kennedy *Accelerating Dynamical Fermion Computations using the Rational Hybrid Monte Carlo (RHMC) Algorithm with Multiple Pseudofermion Fields.* Phys. Rev. Lett. **98**, 051601 (2007) [arXiv:hep-lat/0608015].

[86] A. D. Kennedy, I. Horvath, S. Sint *A New exact method for dynamical fermion computations with nonlocal actions.* Nucl. Phys. Proc. Suppl. **73**, 834 (1999) [arXiv:hep-lat/9809092].

[87] B. Jegerlehner *Krylov space solvers for shifted linear systems.* [arXiv:hep-lat/9612014].

[88] G. I. Egri, Z. Fodor, C. Hoelbling, S. D. Katz, D. Nogradi, K. K. Szabo *Lattice QCD as a video game.* Comput. Phys. Commun. **177**, 631 (2007) [arXiv:hep-lat/0611022].

[89] M. A. Clark, R. Babich, K. Barros, R. C. Brower, C. Rebbi *Solving Lattice QCD systems of equations using mixed precision solvers on GPUs.* Comput. Phys. Commun. **181**, 1517 (2010) [arXiv:0911.3191 [hep-lat]].

[90] C. Bonati, G. Cossu, M. D'Elia, A. Di Giacomo *Staggered fermions simulations on GPUs.* PoS(Lattice 2010)324 [arXiv:1010.5433 [hep-lat]].

[91] B. Simon "Representations of Finite and Compact Groups". American Mathematical Society (1996).

[92] H. Georgi "Lie Algebras in Particle Physics". Westview Press (1999).

[93] F. Reese Harvey "Spinors and Calibrations". Acadeemic Press (1990).

[94] S. L. Cacciatori, B. L. Cerchiai, A. Della Vedova, G. Ortenzi, A. Scotti *Euler angles for G_2*, J. Math. Phys. **46**, 083512 (2005) [arXiv:hep-th/0503106].