Effective theories of phase transitions

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Abstract:

In this thesis we study systems undergoing a superfluid phase transition at finite temperature and chemical potential. We construct an effective description valid at late times and long wavelengths, using both the holographic duality and the Schwinger-Keldysh formalism for non-equilibrium field theories. In particular, in chapter 2 we employ analytic techniques to find the leading dissipative corrections to the energy-momentum tensor and the electric current of a holographic superfluid, away from criticality. Our method is based on the symplectic current of Crnkovic and Witten [1] and extends on previous results [2, 3]. We assume a general black hole background in the bulk, with finite charge density and scalars fields turned on. We express one-point functions of the boundary field theory solely in terms of thermodynamic quantities and data related to the black hole horizon in the bulk spacetime. Matching our results with the expected constitutive relations of superfluid hydrodynamics, we obtain analytic expressions for the five transport coefficients characterising superfluids with small superfluid velocities. In chapter 3 we examine the hydrodynamics of holographic superfluids arbitrarily close to the critical point. The main difference in this case is that, close to the critical point, the amplitude of the order parameter is an additional hydrodynamic degree of freedom and we have to include it in our effective theory. For simplicity, we choose to work

in the probe limit. Utilising the symplectic current once again, we find the equations that govern the critical dynamics of the order parameter and the charge density and show that our holographic results are in complete agreement with Model F of Hohenberg and Halperin [4]. Through this process, we find analytic expressions for all the parameters of Model F, including the dissipative kinetic coefficient, in terms of thermodynamics and horizon data. In addition, we perform various numerical checks of our analytic results. Finally, in chapter 4 we consider critical superfluid dynamics within the Schwinger-Keldysh formalism. As in chapter 3, we focus on the complex order parameter and the conserved current of the spontaneously broken global symmetry, ignoring temperature and normal fluid velocity fluctuations. We construct an effective action up to second order in the a-fields and compare the resulting stochastic system with Model F and our holographic results in chapter 3. A crucial role in this construction is played by a time independent gauge symmetry, called “chemical shift symmetry”. We also integrate out the amplitude mode and obtain the conventional equations of superfluid hydrodynamics, valid for energies well below the gap of the amplitude mode.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. Chapters 2, 3 and 4 are a reproduction of the following collaborative works:

[5] Aristomenis Donos and Polydoros Kailidis. 'Dissipative effects in finite density holographic superfluids'. In: *JHEP* 11 (2022), p. 053.
doi:10.1007/JHEP11(2022)053.
arXiv:2209.06893 [hep-th],

[6] Aristomenis Donos and Polydoros Kailidis. 'Nearly critical holographic superfluids'. In: *JHEP* 12 (2022), p. 028.
doi:10.1007/JHEP12(2022)028.
arXiv:2210.06513 [hep-th],

[7] Aristomenis Donos and Polydoros Kailidis. 'Nearly Critical Superfluids in Keldysh-Schwinger Formalism'. In: (Apr. 2023).
arXiv:2304.06008 [hep-th]

During my Ph.D. we also published the following work, which is not included in this thesis,

[2] Aristomenis Donos, Polydoros Kailidis and Christiana Pantelidou. 'Dissipation in holographic superfluids'. In: *JHEP* 09 (2021), p. 134.
doi:10.1007/JHEP09(2021)134.
arXiv:2107.03680 [hep-th],

[3] Aristomenis Donos, Polydoros Kailidis and Christiana Pantelidou. 'Holographic dissipation from the symplectic current'. In: *JHEP* 10 (2022), p. 058.
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arXiv:2208.05911 [hep-th]

No part of this thesis has been submitted elsewhere for any degree or qualification.

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Dedicated to

Στον μπαμπά μου Σταύρο, την
μαμά μου Ειρήνη και τον αδερφό
μου Δημήτρη.

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Chapter 1

Introduction

Several areas of physics, some of them seemingly different with each other, come together in this thesis. The purpose of this introduction is to review briefly the foundation on which the main body of this work is built.

This introductory chapter is structured as follows. In section 1.1, we discuss general aspects of continuous phase transitions and Landau's mean-field theory. Having emphasized the role of critical fluctuations, we also present the key points of the renormalisation group. In section 1.2, a short introduction to conventional relativistic hydrodynamics is given. We review the idea of the derivative expansion and linear response theory. After that, in section 1.3, we focus on the role of hydrodynamic fluctuations. We briefly mention the approach of stochastic hydrodynamics and then move on to discuss the Schwinger-Keldysh formalism. We try to motivate the formalism through a simple quantum mechanical argument and also highlight the basic features of a Schwinger-Keldysh effective field theory. Section 1.4 is a short introduction to the gauge/gravity correspondence. We discuss important elements of the holographic dictionary, the holographic computation of correlation functions and some applications of gauge/gravity duality in hydrodynamics and superfluid phase transitions.

1.1 Critical phenomena

In this section we give a very brief overview of the theory of second order phase transitions and the renormalisation group [8, 9, 10, 11]. In simple terms, one could describe a phase transition as the abrupt change in certain properties of a macroscopic system, when an external parameter like the temperature is altered. The most familiar example is the liquid-gas phase transition, the typical phase diagram being displayed below.

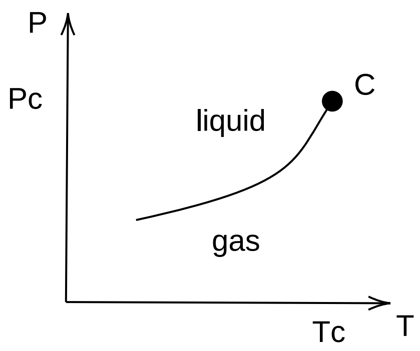

Figure 1.1: A generic liquid-gas phase diagram.

Above and below the line separating the two phases, called the coexistence curve, the fluid has a well-defined density, $\rho_{liquid}(T,p)$ and $\rho_{gas}(T,p)$ respectively. However, if by changing the temperature and/or pressure the system happens to cross the coexistence curve, the density of the fluid is going to change discontinuously from one density to the other. Then one would talk of a first order phase transition in Ehrenfest's classification.

Notice that the point C of the diagram is special: It is the endpoint of the coexistence curve, separating the two phases. Such a point, which is the endpoint of a first-order transition line, is called a critical point. The theory of second-order (or "continuous") phase transitions examines the behaviour of a system in the vicinity of such a critical point. The are numerous examples of continuous transitions in nature apart from the liquid-gas one, such as the paramagnetic-ferromagnetic phase transition in elements like iron and nickel and the superfluid phase transition of helium-4.

Various physical quantities demonstrate singular and non-analytic behaviour as the system approaches the critical point. For example, if we move across the coexistence curve towards C, the difference of the densities between the two phases vanishes as $\rho_L - \rho_G \propto (T_c - T)^\beta$ with $\beta \approx 0.33$. (See e.g. [11] for a review of experimental results.) Exponents such as β, are called critical exponents and have been the main focus of the theory of critical phenomena, both in experimental and in theoretical investigations. In the description of static phenomena there are in general six critical exponents. However, there are four relations among them, called scaling relations [9], which imply that only two critical exponents are independent.

The most interesting aspect about the critical exponents is that they have the same values, in very different materials. In particular, experiments demonstrate that the critical exponents turn out to be the same, within experimental accuracy, for all fluids. But, even more spectacular, is the observation that liquid-gas transitions share the same critical exponents with e.g. uniaxial magnetic systems. These observations strongly indicate that a notion of universality characterises critical phenomena. Systems which otherwise have completely different microscopic description, seem to behave in exactly the same way near a critical point.

A quantity that allows us to differentiate between the two competing states in a phase transition is called the order parameter. In the ferromagnetic transition the magnetization is the order parameter, whereas in the gas-liquid case the difference $\rho - \rho_c$ could serve as the order parameter. In the case of the superfluid, the order parameter is a complex field and roughly represents the "wavefunction" of the boson condensate. As a general rule, the order parameter is nonzero in the low-temperature, ordered state and zero at the high-temperature state. As the system approaches the critical point from an arbitrary direction, the order parameter goes to zero in a non-analytic, powerlike, fashion.

Any theory of critical phenomena, no matter how involved, is based around the order parameter (and any other low energy degree of freedom coupled to it). In fact, the whole critical behaviour of a system stems from the fact that the fluctuations of the order parameter become increasingly enhanced as we approach the critical point. Let us give a brief example. In the case of a uniaxial ferromagnet, a quantity that captures the correlation of magnetisations at positions $\vec{x}$ and $\vec{y}$ is the connected correlator $G(x, y) = \langle m(x)m(y)\rangle_c$. Away from the critical point, the microscopic spins are aligned in patches of typical length ξ, called the correlation length. For distances $|x - y| \gg \xi$, $G(x, y)$ is exponentially suppressed and thus the spins are considered as independent. However, as the system comes closer and closer to the critical point the correlation length increases indefinitely and exactly at the critical point, ξ diverges. In the language of hydrodynamics, we say that at criticality the order parameter becomes gapless (i.e. a hydrodynamic degree of freedom).

How does one then go on to setup a theory of the order parameter? In principle, all the information about a statistical system can be extracted starting from the microscopic Hamiltonian H_{UV} and calculating the partition function $Z = e^{-\beta F} = \mathrm{Tr}\left(e^{-\beta H_{UV}}\right)$. But as usual in theoretical physics, such an attempt is rarely fruitful analytically. A notable exception is of course Onsager's solution of the Ising model in 2 spatial dimensions [12]. Furthermore, universality indicates that systems with very different microscopic H_{UV} behave in the same manner close to criticality. Thus, universality suggests that attacking the microscopic problem directly might not be only difficult, but also unnecessary. After all, the main objective would not be to prove that a certain system undergoes a phase transition from first principles, but to predict information about the transition itself.

The natural thing to do, is to try and express the partition function in terms of an effective Hamiltonian $H_{eff}[\phi]$, which is a functional of the order parameter ϕ, namely,

$$Z = \int D\phi \, e^{-\beta H_{eff}} \,. \tag{1.1.1}$$

.

It is one of the postulates of the theory of critical phenomena (see e.g the discussion in [13]) that the microscopic system we are willing to describe is effectively captured by H_{eff}. In general, ϕ will be an n-component vector in field space. The guiding principle to construct $H_{eff}[\phi]$, in the spirit of effective theory, is to write it as a local polynomial in the order parameter and its spatial derivatives, respecting the space and internal symmetries present in the microscopic system. The simplest possible example of such a Hamiltonian, capable of describing a phase transition is the Ginzburg-Landau Hamiltonian[1],

$$\beta H_{GL} = \beta F_0 + \int d^d x \left(\frac{r}{2}\phi^2(x) + u\phi^4(x) + \frac{K}{2}(\nabla\phi(x))^2 - \beta h\phi \right) \tag{1.1.2}$$

The parameters appearing in H_{GL}, such as r and u depend implicitly on microscopic parameters(e.g. microscopic coupling constants), as well as thermodynamic parameters of the system such as temperature T and pressure p. h is the external field conjugate to ϕ. Also, the coefficients r, u, K could depend on h themselves, but such terms lead to subleading corrections which can be neglected. F_0 is independent of the order parameter and a nonsingular function, and thus its effect is ignored close to the critical point. An assumption about the phenomenological coefficients in H_{GL} is that they are smooth functions of external parameters. Moreover, one can argue that u has to be positive so that the probability distribution $e^{-\beta H_{GL}}$ is finite for large ϕ.

The simplest approach to the calculation of (1.1.1) is to evaluate the path integral in the saddle-point approximation. More specifically, we are looking for the field configuration $\phi(x)$ that minimises H_{GL}. This is going to be a constant field $\overline{\phi}$ solving $\frac{\delta H_{GL}}{\delta\phi(x)}|_{\overline{\phi}} = 0$. In order to reproduce the phase transition it is sufficient to assume that near the critical temperature the coefficients can be expanded as follows[2]

[1]This form of the Hamiltonian for $n = 2$ has been first put forward by Ginzburg and Landau [14] as an effective description of superconductors and hence the name.

[2]In fact, T_c is the "mean field" critical temperature. The real critical temperature gets corrections from fluctuations.

$$
\begin{aligned}
r(T) =& (T - T_c) r_c + \mathcal{O}(T - T_c)^2 \,, \\
u =& u_c + \mathcal{O}(T - T_c) \,, \\
K =& K_c + \mathcal{O}(T - T_c) \,,
\end{aligned}
\tag{1.1.3}
$$

with $r_0 > 0$. Then, one finds that above T_c (for $h = 0$) the order parameter $\overline{\phi}$ vanishes and below T_c, $\overline{\phi} \propto (T_c - T)^{\frac{1}{2}}$. This theory thus reproduces the qualitative picture of the phase transition: Above the critical point, we have the "normal" phase of the system with $\overline{\phi} = 0$ and below there is the ordered phase where the nonvanishing value of the order parameter spontaneously breaks the internal symmetry group. The partition function is then $Z = e^{-\beta F} \approx e^{-\beta H_{GL}}$. This approach is the Landau theory of second order phase transitions [15]. It provides the same results as the older mean field theories, such as the van der Waals equation for the liquid-gas transition and the Curie-Weiss theory of ferromagnetism.

One can then go on and calculate the critical exponents relevant to thermodynamic quantities and find the so-called "mean-field" or "classic" exponents. Since spatial correlations have been excluded altogether, one cannot calculate the respective critical exponents. To achieve that, we can work in the Gaussian approximation, writing $\phi(x) = \overline{\phi} + \delta\phi(x)$ and keep terms up to second order in $\delta\phi$.

Landau's theory evaluates the field theory path integral in the saddle-point approximation and thus essentially neglects fluctuations of the order parameter. For this reason, we cannot expect Landau's theory to provide the right answers for the critical exponents, and indeed its predictions differ from experimental results. This can be expected already from the fact that mean field predictions don't depend on the spatial dimensionality of the system. However, the Hohenberg-Mermin-Wagner theorem [16, 17] states that there cannot be spontaneous symmetry breaking of a continuous symmetry in $d_s \leq 2$ spatial dimensions a finite temperature[3]. Intuitively,

[3] Coleman's theorem [18] excludes spontaneous symmetry breaking of a continuous symmetry at zero temperature in $1 + 1$ dimensions.

we would expect that the mean-field predictions are better, as the spatial dimensionality grows. Indeed, an argument based on the Ginzburg-Landau approach, called the Ginzburg criterion [19], suggests that the mean-field predictions must fail below a critical spatial dimension, called upper critical dimension, which for the models we discuss is $d_s = 4$. For (unphysical) spatial dimensionalities $d_s > 4$ mean-field predictions are correct.

The systematic method to treat fluctuations in the theory of static critical phenomena was eventually found due to the work of Wilson, Fisher, Kadanoff and others. The main idea in Wilson's approach [20] is the following. We can split the order parameter $\phi(x)$ in two parts $\phi = \phi_{fast} + \phi_{slow}$. The "fast" modes contain only wavevectors $\frac{\Lambda}{s} < |k| < \Lambda$ $(s > 1)$, where Λ is the momentum cutoff, and the "slow" modes contain wavevectors $|k| < \frac{\Lambda}{s}$. One then integrates out ϕ_{fast}, and after a rescaling of momenta and of the remaining field variables ϕ_{slow}, one obtains an effective Hamiltonian for the slow modes with new coupling constants. These two steps constitute a renormalisation group (RG) transformation R_s. This transformation can be applied recursively in order to find the effective Hamiltonian for slower and slower modes. In general, this procedure could asymptote to a certain Hamiltonian H^*_{eff}, called a fixed point of the renormalisation group. The crucial assumption in Wilson's theory is that, at the critical temperature,

$$\lim_{s\to\infty} R_s\, H_{eff}|_{T=T_c} = H^*_{eff}\,. \tag{1.1.4}$$

In RG terminology, when $T = T_c$ the system lies on the "critical surface" of a fixed point of the renormalisation group. What (1.1.4) is telling us is that at low energies the critical behaviour is governed completely by the fixed point theory. When a specific fixed point is found, we can then linearise the transformation R_s around this point and consequently calculate the critical exponents through its eigenvalues.

The application of the renormalisation group in critical phenomena provided a framework to systematically calculate the deviations of the critical exponents from the mean-field predictions. But most importantly, it provided an explanation for the universality observed in critical phenomena. If two system lie on the critical surface of a common fixed point, even if they have completely different microscopic descriptions, they are bound to show similar critical behaviour. As a result, their long wavelength effective description is the same. Universality, thus, dictates that physical systems split into the so-called "universality classes". Within each universality class, systems have identical behaviour close to critical point (e.g. the same critical exponents). Of course, as we move away from criticality, non-universal characteristics are expected to emerge. In general, the static universality class is determined by the spatial dimensionality, the symmetry group of the Hamiltonian and the range of the forces (long-range or short-range). (See e.g.[10].)

The techniques of the renormalisation group can also be generalised and applied to the theory of dynamical (i.e. time dependent) critical phenomena [4, 13, 21]. As the system under consideration approaches criticality, the time it takes for an order-parameter fluctuation to relax back to equilibrium grows indefinitely. This is a phenomenon called "critical slowing down". In order to describe the relaxation back to equilibrium, of the order parameter and other possible slow variables such as conserved charges (see next section), one postulates certain phenomenological equations. These equations, are stochastic differential equations, which include noise variables, as a proxy for the time-dependent fluctuations which become increasingly strong close to the transition. Depending on the slow variables that couple to the order parameter, the static universality classes split into various dynamical universality classes, labelled by capital letters A to J in the review of Hohenberg and Halperin [4]. The main focus of this dissertation is Model F, i.e. the dynamical universality class of superfluids.

1.2 Hydrodynamics

The purpose of this section is to provide an introduction to some basic aspects of hydrodynamics [22, 23, 24]. Hydrodynamics can be broadly defined as the effective description of any system at finite temperature, at long distances and late times. It is valid after a local thermodynamic equilibrium has been established. Since the theory of hydrodynamics only concerns the macroscopic, coarse-grained behaviour of a system, it is applicable to various classical or quantum settings, ranging from nuclear physics to cosmology. If we denote by l_{mic} and τ_{mic} the typical length and time scale respectively, of a microscopic process of the system under consideration, then the hydrodynamic degrees of freedom are by definition the ones that survive at distances $l_{hydro} \gg l_{mic}$ and times $t_{hydro} \gg t_{mic}$.

A conserved charge is the most prominent example of a hydrodynamic variable. The associated conservation law ensures that a fluctuation of the charge cannot be destroyed locally (as it happens for any non-hydro variable)[25], but it relaxes back to equilibrium via spreading across the system. According to Noether's theorem, a conserved current is always present when the underlying microscopic theory has a continuous symmetry. For example, Poincare covariance gives rise to a symmetric energy-momentum tensor $T^{\mu\nu}$ and a $U(1)$ global symmetry implies the existence of a current J^μ. The equations of hydrodynamics, describing the spacetime evolution of the system, are precisely the conservation laws of the relevant currents, namely

$$\partial_\mu T^{\mu\nu} = 0\,, \quad \partial_\mu J^\mu = 0\,. \tag{1.2.1}$$

In d spacetime dimensions, (1.2.1) are $d+1$ equations and must be expressed in terms of $d+1$ variables for the system of equations to be closed. These $d+1$ variables can be taken to be the conserved charges, which in this setting are the energy density T^{00}, the momentum density T^{0i} and the charge density J^0. Another option, which is the most common, is to consider instead their conjugate pairs, that is the local temperature $T(x)$, the local fluid velocity $u^\mu(x)$, normalized as $u^2 = -1$ and the local chemical potential $\mu(x)$.

The equations that supplement (1.2.1) and express the conserved currents in terms of the $d+1$ hydrodynamic variables are called constitutive relations. The main assumption of hydrodynamics is that T, u^μ, μ are slowly varying functions of space-time, schematically, $\partial_t \sim \frac{1}{t_{hydro}} \ll \frac{1}{t_{mic}}$, $\partial_x \sim \frac{1}{l_{hydro}} \ll \frac{1}{l_{mic}}$. Because of this condition, it is then natural to write the constitutive relations as a derivative expansion, in the following form

$$T^{\mu\nu} = T^{\mu\nu}_{ideal} + T^{\mu\nu}_{diss}\,, \qquad J^\mu = J^\mu_{ideal} + J^\mu_{diss}\,, \tag{1.2.2}$$

where $T^{\mu\nu}_{ideal}$, J^μ_{ideal} contain no derivatives and correspond to the ideal (or inviscid) fluid, whereas $T^{\mu\nu}_{diss}$, J^μ_{diss} contain all the terms with one derivative and higher and correspond to dissipation. In order to express $T^{\mu\nu}_{diss}$, J^μ_{diss} up to order n in derivatives, one must write down all possible structures that can be built out of derivatives of T, u^μ, μ, which are consistent with Lorentz symmetry and are on-shell independent. The coefficients of these terms, called transport coefficients, contain information about the microscopic UV degrees of freedom and cannot be calculated within the framework of hydrodynamics. In the language of effective field theory we would say that all the fast modes have been integrated out from the theory and the only trace they leave is in the transport coefficients. The role of the UV cutoff in hydrodynamics is played by the microscopic quantities l_{mic}, τ_{mic}.

A crucial observation for the construction of the constitutive relations is that the fluid variables T, u^μ, μ are only defined unambiguously in static equilibrium [23, 26]. More specifically, if we have a set T, u^μ, μ of variables, another acceptable set would be

$$T' = T + \delta T\,,$$

$$\mu' = \mu + \delta\mu\,,$$
$$u^{\mu\prime} = u^\mu + \delta u^\mu\,, \tag{1.2.3}$$

as long as the two sets agree in static equilibrium. This implies that $\delta T, \delta\mu, \delta u^\mu$ must contain terms with one derivative or higher, but they are otherwise unconstrained.[4] Under the transformation (1.2.3), the microscopically defined conserved currents $T^{\mu\nu}, J^\mu$ and their ideal parts $T^{\mu\nu}_{ideal}, J^\mu_{ideal}$ remain invariant, but the dissipative parts $T^{\mu\nu}_{diss}, J^\mu_{diss}$ transform nontrivially. We can fix the 'gauge freedom' (1.2.3) (or in hydrodynamics jargon "fix a frame") by imposing $d+1$ conditions. The two most common frame choices for the case of the normal fluid are the Landau frame [22]

$$T^{\mu\nu}_{diss} u_\nu = 0\,, \quad J^\mu_{diss} u_\mu = 0\,, \tag{1.2.4}$$

and the Eckart frame[27, 28]

$$J^\mu_{diss} = 0\,, \quad T^{\mu\nu}_{diss} u_\mu u_\nu = 0\,. \tag{1.2.5}$$

For example, for the case of a normal fluid at finite chemical potential, at first order in derivatives, following these steps one finds [23] four transport coefficients, namely the shear viscosity η, the bulk viscosity ζ, the charge conductivity σ and a coefficient χ_T. All of them are, in principle, arbitrary functions of $T(x), \mu(x)$, but it turns out that they are constrained in the following way

$$\eta \geq 0, \quad \zeta \geq 0, \quad \sigma \geq 0, \quad \chi_T = 0\,. \tag{1.2.6}$$

To prove these constraints, one needs to make additional assumptions, such as the existence of an entropy current $S^\mu = s u^\mu + \mathcal{O}(\partial)$, constructed from the hydrodynamic

[4] The velocity must however remain normalised $u'^2 = u^2 = -1$.

variables T, μ, u^μ, that satisfies on-shell

$$\partial_\mu S^\mu \geq 0\,, \tag{1.2.7}$$

for any solution of the equations of motion (1.2.1). The condition (1.2.7) is simply a local form of the second law of thermodynamics. Another assumption that constrains the transport coefficients is the existence of discrete symmetries in the microscopic theory, such as parity, time reversal and charge conjugation. In particular, time reversal covariance, is expressed via the Onsager relations [29]. At the level of retarded two-point functions (for a system with spacetime translation invariance) the Onsager relations read [23]

$$G^R_{ab}(\omega,\vec{k}) = \eta_a\eta_b G^R_{ba}(\omega,-\vec{k})\,, \tag{1.2.8}$$

where η_a, η_b are the eigenvalues of the operators, denoted by a and b, under time reversal.

We must emphasise at this point that, depending on the system under consideration, there might be more hydrodynamic variables, apart from T, u^μ, μ. For instance, if the system exhibits spontaneous breaking of a continuous global symmetry, then the corresponding Goldstone bosons have to be included in the hydrodynamics, since they are gapless (i.e. they satisfy $\omega \to 0$ as $k \to 0$). Moreover, whenever we have a phase transition and the system is close to the critical point, due to the critical slowing-down, the order parameter is a hydrodynamic variable as well. These two cases are of course particularly relevant for superfluid states of matter [26, 30, 31, 32], as we are going to see next.

A powerful tool, often used in conjuction with hydrodynamics is linear response theory[33, 23]. Linear response studies the time evolution of systems that are perturbatively driven out of equilibrium by external sources, to leading order in the sources. Let's consider a time independent Hamiltonian H_0 and a equilibrium state described by a density matrix ρ_0 at time t_0. At t_0 we imagine that external sources $J_a(t,\vec{x})$, conjugate to the operators O_a, are turned on. The Hamiltonian for times $t \geq t_0$ is $H(t) = H_0 - \int d^{d-1}x J_a(t,\vec{x}) O_a(\vec{x})$. It is straightforward to show that to leading order in J

$$\begin{aligned}\delta\langle O_a(x)\rangle =& i\int d^3x' \int_{t_0}^{t} dt' \langle [O_a(t,\vec{x}), O_b(t',\vec{x}')]\rangle J_b(t',\vec{x}') \\ =& i\int d^3x' \int_{-\infty}^{\infty} dt' \theta(t-t') \langle [O_a(t,\vec{x}), O_b(t',\vec{x}')]\rangle J_b(t',\vec{x}') \\ =& \int d^4x' G^R_{ab}(x,x') J_b(x')\,, \end{aligned} \tag{1.2.9}$$

where in second step we took $t_0 \to -\infty$ and introduced the retarded Green's function

$$G^R_{ab}(x,x') = i\theta(t-t')[O_a(t,\vec{x}), O_b(t',\vec{x}')]\,. \tag{1.2.10}$$

The usefulness of this formalism lies on the observation that if we can somehow calculate (e.g. using hydrodynamics) $\delta\langle O_a(x)\rangle$, then (1.2.9) allows us to calculate the retarded Green's functions in the equilibrium state, without having to do a microscopic calculation. Moreover, combining (1.2.9) with constitutive relations we can immediately derive the so-called Kubo formulas, which express transport coefficients as a certain zero frequency, zero wavevector limit of retarded Green's functions (see e.g. [23]).

1.3 Fluctuations and Schwinger-Keldysh effective theory

Conventional hydrodynamics, as described in the previous section, is a theory of coupled partial differential equations supplemented by constitutive relations. This framework however is incomplete, since it ignores thermal (i.e. statistical) fluctuations of the hydrodynamic variables, which are crucial, not only for the nearly critical behaviour of a system as we saw earlier, but also for the long distance, late time behaviour of hydrodynamic variables. Fluctuations lead to phenomena such as long-time tails [34, 35] of correlation functions and renormalisation of transport coefficients [36] which cannot be captured by just classical hydrodynamics. The standard method to incorporate fluctuations [37] is to modify the constitutive relations according to,

$$T^{\mu\nu} = T^{\mu\nu}_{cl} + T^{\mu\nu}_{noise}\,, \quad J^{\mu} = J^{\mu}_{cl} + J^{\mu}_{noise}\,, \tag{1.3.1}$$

where $T^{\mu\nu}_{cl}$, J^{μ}_{cl} are the classical objects given by (1.2.2) up to any given order in the derivative expansion and $T^{\mu\nu}_{noise}$, J^{μ}_{noise} are random stresses and currents respectively. These noise terms are chosen to follow a Gaussian distribution with width determined by the fluctuation-dissipation theorem. This approach transforms by hand the deterministic equations of motion (1.2.1) to non-linear stochastic differential equations. The final equations are similar to the Langevin-type equations of dynamical critical phenomena, although here the physical system doesn't have to be near a critical point. Indeed, the RG methods of dynamical critical phenomena have been applied successfully in the past to hydrodynamics (see e.g. [38]).

However, even this framework, called stochastic hydrodynamics, in which fluctuations can be treated, is not completely satisfactory. As we already mentioned, various assumptions, such as the existence of an entropy current with semi-positive divergence, or the Onsager relations, have to be imposed by hand in the theory. One would ideally want everything to be derived from an action and a corresponding path integral, as it happens for zero temperature quantum field theory. In addition, one would want to find a framework that goes beyond the simplification of Gaussian white noise and includes interactions of dynamical and stochastic variables from first principles. The research towards this goal, culminated in the last decade in an effective theory for dissipative hydrodynamics, based on the Schwinger-Keldysh formalism [39, 40, 41, 42, 43] (see [25] for a review).

1.3.1 Schwinger-Keldysh basics

Before giving more information about the effective theory construction, let us review briefly the basic features of this formalism at the level of quantum field theory. The "Schwinger-Keldysh", "closed-time path", or "in-in" formalism was developed by various authors starting in the 1960's [44, 45, 46] and has found numerous applications since then, in condensed matter physics [47, 48], cosmology [49] and the construction of effective field theories [25], to name a few.

The most important aspect of the Schwinger-Keldysh formalism is that it enables us to calculate expectation values, as opposed to transition amplitudes [50, 51, 52]. Let's consider the simplest imaginable setup, following [52], quantum mechanics of a point particle. The generating functional is

$$Z[x_f, t_f, x_i, t_i; J] = \langle x_f | U_J(t_f, t_i) | x_i \rangle = \int_{x(t_i)=x_i}^{x(t_f)=x_f} Dx(t)\, e^{i \int_{t_i}^{t_f} dt (\mathcal{L}(t) + J(t) x(t))}\,, \quad (1.3.2)$$

where $U_J(t_f, t_i)$ is the time evolution operator in the presence of the time dependent source $J(t)$. Taking a functional derivative we find

$$-i \frac{\delta}{\delta J(t)} Z[x_f, t_f, x_i, t_i, J]|_{J=0} = \langle x_f, t_f | \hat{x}(t) | x_i, t_i \rangle\,,$$

which is not an expectation value. What we would like to find is, for instance, $\langle x_i, t_i | \hat{x}(t) | x_i, t_i \rangle$. The natural thing to do is to look for a different generating functional. Introducing a complete set of states

$$\langle x_i, t_i | \hat{x}(t) | x_i, t_i \rangle = \int_{-\infty}^{\infty} dx_f\, \langle x_i, t_i | x_f, t_f \rangle \langle x_f, t_f | \hat{x}(t) | x_i, t_i \rangle =$$
$$\int_{-\infty}^{\infty} dx_f\, Z[x_f, t_f, x_i, t_i; J_2]^* (-i) \frac{\delta}{\delta J_1(t)} Z[x_f, t_f, x_i, t_i; J_1]|_{(J_1=0,\, J_2=0)}$$

and similarly

$$\langle x_i, t_i|\hat{x}(t)|x_i, t_i\rangle = \int_{-\infty}^{\infty} dx_f\, i\frac{\delta}{\delta J_2(t)} Z[x_f, t_f, x_i, t_i; J_2]^* Z[x_f, t_f, x_i, t_i; J_1]|_{(J_1=0,\, J_2=0)}\,.$$

These expressions show that the generating functional we are after is the following

$$\begin{aligned} Z_{SK}[x_i, t_i, t_f; J_1, J_2] &= \int_{-\infty}^{\infty} dx_f\, Z^*[x_f, t_f, x_i, t_i; J_2] Z[x_f, t_f, x_i, t_i; J_1] \\ = \int_{-\infty}^{\infty} dx_f \int_{x_1(t_i)=x_i}^{x_1(t_f)=x_f} Dx_1 &\int_{x_2(t_i)=x_i}^{x_2(t_f)=x_f} Dx_2\, e^{iS[x_1]-iS[x_2]+i\int_{t_i}^{t_f} dt(J_1 x_1 - J_2 x_2)}\,. \end{aligned} \tag{1.3.3}$$

This last expression is precisely the Schwinger-Keldysh generating functional for a particle initially at the state $|x_i, t_i\rangle$. Already from this result we can see some novel characteristics of the formalism. First of all, there is a doubling of the degrees of freedom and a doubling of sources in the path integral. We should highlight though that in the operator formalism, only the sources are genuinely doubled, since there is still only one operator $x(t)$. The doubling $x \to x_1, x_2$ is an artefact of the path integral formalism. In addition, the time integral coming from $Z[J_2]$, written as: $-\int_{t_i}^{t_f} = \int_{t_f}^{t_i}$, can be interpreted as a contour that goes backwards in time. As a result, we could think of the theory as living on a closed time contour consisting of a forward branch C_1 (related to x_1, J_1) from t_i, when the state is specified, to a final arbitrary time t_f, and a backward branch C_2 (related to x_2, J_2) from t_f to t_i [5].

Another thing to note is that at the final expression (1.3.3) x_f is being integrated over. Hence, the final state is not specified but only the initial one. This is useful when we actually don't know the final state (e.g. when we don't have adiabatic evolution). In contrast, in quantum field theory at zero temperature [54], one focuses on time-ordered Green's functions between asymptotic "in" and "out" vacua which turn out to be the same up to a phase (due to the adiabatic control of the interactions). Because of this, there is no need for a closed-time path formalism [47].

Any operator $O(t)$ of the theory can be placed on the contour $C = C_1 \cup C_2$. On

[5] There are many variations of this contour in the literature, see for example [53].

this closed time path we can define a path-ordering operator $\mathcal{P}$, which rearranges a product of operators in the expected way,

$$\begin{aligned} \mathcal{P}(A_1(t_A)B_1(t_B)) &= \mathcal{T}(A_1(t_A)B_1(t_B)) \,, \\ \mathcal{P}(A_2(t_A)B_2(t_B)) &= \tilde{\mathcal{T}}(A_2(t_A)B_2(t_B)) \,, \\ \mathcal{P}(A_1(t_A)B_2(t_B)) &= B_2(t_B)A_1(t_A) \,. \end{aligned} \tag{1.3.4}$$

where $\mathcal{T}$ denotes time ordering and $\tilde{\mathcal{T}}$ denotes anti-time ordering. The Schwinger-Keldysh generating functional for a theory with some initial (mixed or pure) state ρ_i at t_i reads then

$$\begin{aligned} Z[\rho_i, t_i, t_f; J_1, J_2] &= \langle \mathcal{P} e^{i \int_C d\tau J(\tau) O(\tau)} \rangle = \langle \mathcal{P} e^{i \int_{t_i}^{t_f} dt (J_1 O_1 - J_2 O_2)} \rangle \\ &= \mathrm{Tr} \left(U_{J_1}(t_f, t_i) \rho_i U^\dagger_{J_2}(t_f, t_i) \right) \,. \end{aligned} \tag{1.3.5}$$

This formalism is often expressed in another set of variables, the "r/a" or "Keldysh" basis

$$\begin{aligned} O_r &= \frac{1}{2}(O_1 + O_2) \,, \quad O_a = O_1 - O_2 \,, \\ J_r &= \frac{1}{2}(J_1 + J_2) \,, \quad J_a = J_1 - J_2 \,. \end{aligned} \tag{1.3.6}$$

In these variables the generating functional is given by

$$e^{W[J_a, J_r]} = Z[J_a, J_r] = \langle \mathcal{P} e^{i \int dt (O_r J_a + J_a O_r)} \rangle \tag{1.3.7}$$

and so J_a is the source for O_r, whereas J_r is the source of O_a, with W the generating functional for connected correlation functions.

An advantage of the Schwinger-Keldysh generating functional is that it allows us to compute n-point Green's functions with various orderings, as opposed to the usual generating functional in zero-temperature field theory which gives us only time-ordered Green's functions [6]. For example, for two-point functions, taking two functional derivatives of (1.3.7) leads to [25]

$$
\begin{aligned}
&G_{ra}(x_1, x_2) = G_R(x_1, x_2)\,,\\
&G_{ar}(x_1, x_2) = G_A(x_1, x_2)\,,\\
&G_{rr}(x_1, x_2) = G_S(x_1, x_2)\,,\\
&G_{aa}(x_1, x_2) = 0\,,
\end{aligned}
\tag{1.3.8}
$$

where G_R, G_A, G_S are the retarded, advanced and symmetric Green's functions respectively.

Moreover, just from the definition (1.3.7) and the fact that the time evolution operator $U_J(t_f, t_i)$ is unitary, one can show the following relations [25]

$$
\begin{aligned}
&W[J_a = 0, J_r] = 0\,,\\
&W^*[J_a, J_r] = W[-J_a, J_r]\,,\\
&\mathrm{Re}W[J_a, J_r] \leq 0\,.
\end{aligned}
\tag{1.3.9}
$$

The first condition implies also that $G_{aaa\cdots} = 0$.

Let's now assume that the system is in global thermal equilibrium at the initial time t_i, and then we turn on an external source $J(t)$ which drives it out of equilibrium at any later time. This means that the initial density matrix is a thermal one, $\rho_i = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$ [7]. Defining the generating functional W_T through

[6] However, not all conceivable orderings are attainable through this contour [55].
[7] As usual, $\beta = 1/T$ with T the temperature.

$$e^{W_T[J_1,J_2]} = \mathrm{Tr}\left(U^\dagger_{J_2}(t_f,t_i)\rho_i U_{J_1}(t_f,t_i)\right) , \tag{1.3.10}$$

one can show that [41]

$$W[J_1(t), J_2(t)] = W_T[J_1(t), J_2(t - i\beta_0)] \, . \tag{1.3.11}$$

Equation (1.3.11) is in fact the KMS condition [56, 57].

For two-point functions (1.3.11) gives the standard fluctuation-dissipation theorem. However, for n-point functions with $n \geq 3$ the KMS condition (1.3.11) only connects the correlation functions coming from W, which are of the form $\langle \tilde{\mathcal{T}}(\cdots)\,\mathcal{T}(\cdots)\rangle$, with the correlation functions coming from W_T, which are of the form $\langle \tilde{\mathcal{T}}(\cdots)\mathcal{T}(\cdots)\rangle$. Thus, it doesn't imply a constraint on the correlation functions of W [41].

Let's now assume, apart from thermal equilibrium, that the theory is invariant under a discrete symmetry which contains time-reversal, e.g. $\mathcal{PT}$ for concreteness. Then (1.3.11) becomes [41]

$$W[J_1(t,\vec{x}), J_2(t,\vec{x})] = W[J_1(-t,-\vec{x}), J_2(-t - i\beta_0, -\vec{x})] \, . \tag{1.3.12}$$

Expanding this relation to arbitrarily high powers of the sources one can find non-linear generalisations of the fluctuation-dissipation theorem [41, 58].

1.3.2 Effective theory construction

In this subsection, we are going to outline the basic steps one has to follow in order to construct effective theories living on a closed-time path, following [25]. See also section 4.2 for additional information.

As in any effective theory, the first thing to note is that for an effective low-energy description one doesn't have to keep track of all the microscopic degrees of freedom. In the spirit of Wilson's renormalisation and Kadanoff's coarse-graining, one can imagine integrating out all the UV degrees of freedom, and keeping only the IR ones that are relevant. Of course, as usual one cannot actually perform the path integration and this is why one would have to set up an effective theory after-all. Denoting the effective degrees of freedom as χ_1, χ_2, the Schwinger-Keldysh path integral will be of the form

$$e^{W[J_1,J_2;\rho_i]} = \int_{\rho_i} D\chi_1 D\chi_2 \, e^{iS_{eff}[\chi_1,\chi_2;J_1,J_2,\rho_i]} \,. \tag{1.3.13}$$

The dependence on the initial density matrix ρ_i is encoded in principle both in the path integral measure (or alternatively in the boundary conditions of χ_1, χ_2 at t_i) and in the coefficients of the effective action S_{eff}. At the final time t_f, which we can take to be $t_f \to \infty$, according to our previous discussion we must impose the boundary condition $\chi_1(t_f) = \chi_2(t_f)$. Although the microscopic action was splittable as $S[\chi_1] - S[\chi_2]$, because of the boundary condition at t_f, the modes χ_1, χ_2 are going to be coupled in S_{eff}.

Furthermore, the unitarity constraints of the microscopic theory (1.3.9) are reflected in the effective action as follows [25]

$$\begin{aligned} S^*_{eff}[\chi_1, J_1; \chi_2, J_2] &= -S_{eff}[\chi_2, J_2; \chi_1, J_1] \,, \\ \mathrm{Im} S_{eff} &\geq 0 \,, \\ S_{eff}[\chi_a = 0, J_a = 0; \chi_r, J_r] &= 0 \,. \end{aligned} \tag{1.3.14}$$

The guiding principle in any effective theory are the symmetries present. Regarding global symmetries, due to the boundary condition at t_f, χ_1 and χ_2 must transform in the same way simultaneously. As a result, for any global symmetry of the microscopic theory, there must be a single copy of it at the level of S_{eff}. In the case of local symmetries, the transformation of χ_1 and χ_2 can be independent and consequently, for every gauge symmetry of the microscopic theory, we have to separate copies of it appearing in S_{eff}.

1.4 Gauge/Gravity duality

In this section, we introduce several key points of the gauge/gravity duality, that are relevant for the main part of this thesis. For extended reviews of the subject we refer the interested reader to the vast literature, see e.g. [59, 60, 61, 62, 63, 64, 65].

The holographic correspondence claims that certain quantum field theories with gauge invariance in d dimensions are dual to certain gravitational theories in (at least) $d+1$ dimensions. The fact that the two theories are dual means they are completely equivalent, containing the same physical information, despite appearances. The theory of gravity is imagined to 'live' in the so-called "bulk" of the $d+1$ dimensional spacetime and the gauge theory lives at its d-dimensional boundary. In this sense, the gauge/gravity correspondence is a concrete realisation of the holographic principle [66, 67].

The most well-studied example of the holographic correspondence is the duality between $\mathcal{N}=4$ super Yang-Mills theory in 4 dimensional Minkowski spacetime and type IIB superstring theory on $AdS_5 \times S_5$, first put forward by Maldacena [68], almost 30 years ago. Since then, numerous other dual pairs have been found and analysed thoroughly. In this thesis we will follow the "bottom-up" approach in holography, meaning that we will talk of a generic dual pair of theories, assuming that a precise mapping can indeed be established between them.

The robustness of the duality lies on the fact that it is a weak-strong one, i.e. when the gauge theory is strongly interacting, the gravitational theory is weakly-coupled and vice versa. In addition, taking the large N limit of the field theory corresponds to classical gravity in the bulk. Here, we are interested in cases where the field theory describes a many-body strongly coupled system and so we are going to use classical gravity to extract useful information about it.

The initial formulation of the duality, the AdS/CFT correspondence, suggests that a gravitational theory in a $d+1$ dimensional anti-de Sitter (AdS) spacetime[8],

$$ds^2 = -\frac{r^2}{L^2}dt^2 + \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}dx^i dx^i \,, \tag{1.4.1}$$

is dual to a conformal field theory (CFT) (at zero temperature), living at the d dimensional Minkowski boundary of the bulk, located at $r \to \infty$. The duality, though, between the gravitational theory and the field theory can be extended in more complicated situations, as long as the background gravitational solution is asymptotically AdS (aAdS). This corresponds to the fact that the dual field theory might not be a CFT, but it has to flow to a fixed point CFT in the UV [69].

With a view towards experimentally realised scenarios, we must most often deal with field theories at finite temperature and finite charge density. A black hole in an aAdS spacetime is dual to a quantum field theory at finite temperature, with the Hawking temperature of the black hole being equal to the temperature of the boundary theory. The prototype example [70] is the Schwarzschild-AdS black hole,

$$\begin{aligned} ds^2 &= -\,U(r)dt^2 + \frac{dr^2}{U(r)} + \left(\frac{r}{L}\right)^2 dx^i dx^i \,, \\ U(r) &= \left(\frac{r}{L}\right)^2 \left(1 - \left(\frac{r_0}{r}\right)^d\right), \end{aligned} \tag{1.4.2}$$

with the horizon located at $r = r_0$. Note that both AdS and Schwarzschild-AdS are solutions of Einstein's equations without matter, with negative cosmological constant. One can introduce a background with chemical potential and charge

[8]In fact, this coordinate system only covers the Poincare patch of the spacetime. Here, $i = 1, \ldots, d-1$ labels the spatial coordinates of the boundary.

density, associated with a $U(1)$ global symmetry in the boundary, by including a gauge field $A = A_t(r)dt$. With no other field content, this leads to the Reissner-Nordstrom AdS solution [71].

These examples highlight that the state of the system (vacuum, thermal density matrix etc.) is encoded in the background solution in the bulk. There are two other important ingredients of the the holographic dictionary that we finally wish to emphasise. The first one is that any gauge symmetry in the bulk is dual to a global symmetry in the boundary. The second one is that there is a map between gauge invariant, single-trace operators of the quantum field theory and fields propagating in the bulk of the aAdS spacetime. For example, a scalar operator $\mathcal{O}(x)$ in the field theory corresponds to a bulk scalar field $\phi(x, r)$, whereas the field theory energy-momentum tensor $T_{\mu\nu}$ is dual to the metric $g_{\mu\nu}$ of the bulk theory. If there is global symmetry with group G in the field theory side, which gives rise to a conserved current J^{μ}, then in the gravitational theory there is going to be a dual gauge symmetry with gauge group G, associated to a gauge field A_{μ}.

1.4.1 GKPW formula

The correspondence between bulk fields and boundary operators was first put forward in the form of a simple rule in [72, 73]. For the case of a single scalar operator, this so-called Gubser-Klebanov-Polyakov-Witten (GKPW) rule is,

$$\langle e^{\int d^dx J(x)\mathcal{O}(x)}\rangle_{QFT} = Z_{ST}[J]\,. \tag{1.4.3}$$

The left hand side is the generating functional for (connected) $\mathcal{O}(x)$ correlators and J is the source of $\mathcal{O}$. On the right hand side, we have the string theory partition function on the aAdS spacetime. The path integral defining $Z_{ST}[J]$ is over field configurations $\phi(x, r)$ that behave as[9] J near the AdS boundarary.

[9]Below we make this statement more precise.

Equation (1.4.3) is not yet particularly useful in order to extract information about the field theory, as it is not an easy task to calculate Z_{ST}. However, in the strong coupling, large N limit of the dual field theory, we have weakly-coupled classical gravity in the bulk and we can estimate the quantum gravity partition function via a saddle point approximation[10], $Z_{ST}[J] \approx e^{-S^{os}_{sugra}[J]}$, where $S^{os}_{sugra}[J]$ is the on-shell classical supergravity action. Then the GKPW rule becomes

$$\langle e^{\int d^d x J(x)\mathcal{O}(x)} \rangle_{QFT} \approx e^{-S^{os}_{sugra}[J]} \,. \tag{1.4.4}$$

This last formula is the backbone of the gauge/gravity duality. It tells us that in order to obtain the generating functional for the strongly interacting field theory living on the boundary, we only have to solve the equations of motion in the bulk of the spacetime, with certain asymptotic behaviour. In other words, it allows us to trade a quantum calculation in a strongly coupled field theory, which would otherwise be intractable (except when it is fixed by symmetry), with a purely classical one in the gravitational theory.

To make the aforementioned rule more precise, let's focus on massive scalar field $\phi(x,r)$ in a aAdS spacetime, coupled to the metric and other matter fields as well. The relevant part of the Euclidean action is,

$$S_b = \int d^{d+1}x \sqrt{g} \left(\frac{1}{2} \left(\partial_\mu \phi\right)^2 + \frac{m^2}{2}\phi^2 \right) + \cdots \,. \tag{1.4.5}$$

and the equation of motion reads,

$$\frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \phi\right) - m^2 \phi^2 + \cdots = 0 \,. \tag{1.4.6}$$

Close to the AdS_{d+1} boundary $r \to \infty$, plugging the ansatz $\phi \sim \frac{1}{r^\lambda}$ in (1.4.6), we find two solutions,

[10]We work in Euclidean signature.

$$\lambda_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2}\,. \tag{1.4.7}$$

We refer to λ_- as the slow fall-off mode and to λ_+ as the fast fall-off mode. Then the bulk solution near the boundary of the bulk spacetime behaves as,

$$\phi(x,r) \approx \frac{\phi_-(x)}{r^{\lambda_-}} + \cdots + \frac{\phi_+(x)}{r^{\lambda_+}} + \cdots\,. \tag{1.4.8}$$

At this point comes the crucial identification,

$$\phi_-(x) \equiv J(x)\,, \tag{1.4.9}$$

namely the slow fall-off coefficient in the near boundary expansion is equal to the source of the dual field theory operator. This means that if we calculate the on-shell action (1.4.5) and functionally differentiate it twice with respect to ϕ_- we can obtain the two-point function $\langle \mathcal{O}(x)\mathcal{O}(x')\rangle$. This computation was first presented in [72, 73] for a free massive scalar in AdS_{d+1} and the result is[11],

$$\langle \mathcal{O}(x)\mathcal{O}(x')\rangle \propto \frac{1}{|x-x'|^{2\lambda_+}}\,. \tag{1.4.10}$$

This is exactly the two-point function of a scalar primary operator of conformal dimension $\Delta = \lambda_+$. Moreover, the vacuum expectation value (vev) of the operator is related to the fast fall-off coefficient according to the relation [74, 75]

$$\langle \mathcal{O}(x)\rangle = (2\lambda_+ - d)\phi_+(x)\,. \tag{1.4.11}$$

The take home message of this calculation is that λ_+ is the scaling dimension of the dual operator $\mathcal{O}(x)$.

Note that in AdS space, as opposed to scalars in flat spacetime, negative scalar masses ($m^2 < 0$) are allowed. In particular, it turns out that to ensure stability the

[11] Here we omit the normalisation constant for convenience.

scalar masses must obey the Breitenlohner-Freedman bound [76, 77],

$$m^2 \geq -\frac{d^2}{4}\,. \tag{1.4.12}$$

This bound then constraints the conformal dimension of the dual operator to obey, $\Delta = \lambda_+ \geq \frac{d}{2}$. However, in a generic conformal field theory we expect to have scalar operators with scaling dimensions all the way down to the unitarity bound, $\Delta \geq \frac{d-2}{2}$. This contradiction was resolved in [75] , where the authors, following [76, 77], argued that for masses obeying,

$$-\frac{d^2}{4} < m^2 \leq -\frac{d^2}{4} + 1\,, \tag{1.4.13}$$

we can describe, with the same bulk theory, two different boundary field theories. More specifically, the first option (often called "standard quantisation" in the literature) is that the bulk field ϕ is dual to an operator $\mathcal{O}_{\lambda_+}$ of scaling dimension $\Delta_{reg} = \lambda_+$ and the source and vev are identified as in (1.4.9) and (1.4.11). The second possibility (called "alternative quantisation") is that the same bulk field ϕ corresponds to a different operator $\mathcal{O}_{\lambda_-}$ of scaling dimension $\Delta_{alt} = \lambda_-$, which, for masses in the range (1.4.13), obeys $\frac{d}{2} > \Delta_{alt} \geq \frac{d-2}{2}$.

In the alternative quantisation of the theory, the bulk action is modified by dropping a diverging boundary term [75], and the roles of the source and the vev are interchanged,

$$\phi_+(x) = J(x), \quad \langle \mathcal{O}(x) \rangle = (2\lambda_- - d)\phi_-(x)\,. \tag{1.4.14}$$

1.4.2 Correlation functions and Holographic renormalisation

The calculation of field theory correlation functions, according to the rule (1.4.4), involves finding the on-shell supergravity action. However, one can easily check that the bulk action is divergent, due to the noncompactness of the AdS spacetime. These divergences can be thought of as being dual to the UV divergences of the boundary field theory. In order to obtain sensible results for correlation functions, one must deal consistently with these divergences.

One way to obtain correct results (see e.g. [72, 78, 79]) without having to alter the definition of the bulk action, is to regularise the action by introducing a cutoff in the radial direction of AdS near the boundary, namely at $z = \epsilon$ [12] and impose boundary conditions of the fields on this hypersurface. The on-shell action is still divergent in the limit $\epsilon \to 0$, but it turns out that the diverging terms give rise to contact terms in the correlation functions which we can ignore anyway. So, in the end of the computation one can take $\epsilon \to 0$ and keep only the finite result.

Although this procedure provides correct results, the fact that we simply drop diverging terms is not very satisfactory. Experience from quantum field theory tells us that we can tame divergences with the addition of counterterms. This is the essence of holographic renormalisation [80, 81, 82] which has proven to be the most systematic method to deal with divergences arising from the infinite volume of the asymptotically AdS spacetime. Given a bulk action with arbitrary field content, this method entails modifying the action with the addition of a counterterm action which is a surface term and thus doesn't change the equations of motion. The addition of the counterterm action renders the full action finite and at the same time makes the variational problem in the bulk well-defined [83, 84]. It is important to note that the final counterterm action is fixed only by the bulk action (field content, interaction terms e.g. scalar potential) and is independent of the explicit solution one considers.

An important outcome of this technique is that it provides naturally all the renormalised one-point functions of the dual operators in terms of the external sources. With these objects at hand, one can in principle go on and calculate without any problem all higher-point correlation functions for any background. In addition, inside this

[12] Here we think of a different coordinate system compared to (1.4.1), namely $ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2} dx^i dx^i$, with $i = 1, \ldots, d$, where the AdS boundary is located at $z = 0$.

framework Ward identities and the associated anomaly terms of the dual field theory can be successfully reproduced. Most notably, in [85, 86] the authors found using holographic renormalisation the gravitational part of the conformal anomaly.

We will now give a brief overview of the method, following closely [82]. Holographic renormalisation works for any asymptotically AdS spacetime. For such a spacetime, one can always bring the metric in Fefferman-Graham coordinates near the boundary $z = 0$,

$$
\begin{aligned}
ds^2 &= \frac{dz^2}{z^2} + \frac{1}{z^2} g_{ij}(x,z) dx^i dx^j \,, \\
g_{ij}(z,x) &= g_{(0)ij} + g_{(1)ij} z + \cdots \,.
\end{aligned}
\tag{1.4.15}
$$

The first step of holographic renormalisation is to find the asymptotic solution, by solving perturbatively the equations of motion near the conformal boundary. As we already saw for the case of the scalar field, the equations are second order in z and so there are two possible behaviours, $\sim z^m$ and $\sim z^{m+n}$, and the most general solution for any type of field $\mathcal{F}$ near the boundary is,

$$
\mathcal{F}(x,z) = z^m (f_{(0)}(x) + f_{(1)}(x) z + \cdots + z^n (f_{(n)}(x) + \log z \tilde{f}_{(n)}(x)) + \cdots) \,. \tag{1.4.16}
$$

The equations of motion become algebraic and can be solved for the coefficients $f_{(i)}(x)\,, i < n$ and $\tilde{f}_{(n)}(x)$, in terms of the leading coefficient $f_{(0)}(x)$. The near boundary analysis cannot fix $f_{(n)}$ in terms of $f_{(0)}(x)$. This could be expected since it is a linearly independent second solution near the boundary. To find $f_{(n)}$ we will have to obtain the full bulk solution. In general, $f_{(n)}$ is a nonlocal function of $f_{(0)}(x)$.

To regularise the action, one imposes a cutoff at $z = \epsilon$. Then, the on-shell action, plugging the asymptotic solution (1.4.16), becomes a surface integral on the boundary hypersurface, of the form

$$S_{reg}[f_{(0)}, \epsilon] = \int d^d x \sqrt{g_{(0)}} \left(\epsilon^{-\nu} a_{(0)} + \epsilon^{-(\nu+1)} a_{(2)} + \cdots + \log \epsilon a_{(2\nu)} + \mathcal{O}(\epsilon) \right) \quad (1.4.17)$$

Note that in this expression, there is a finite number of diverging terms with coefficients (the a's) that are local functions of $f_{(0)}$. At this point, we can solve (1.4.16) iteratively to express $f_{(0)}$ in terms of $\mathcal{F}(x, \epsilon)$. Substituting the resulting expression in the divergent terms of (1.4.17) we obtain $S_{div}[\mathcal{F}(x, \epsilon)]$. This action, contains the same divergent terms as $S_{reg}[f_{(0)}, \epsilon]$ but it will have a different finite part in general, which will contribute eventually to the computation of correlation functions. Finally, one defines as the minimal counterterm action,

$$S_{ct} = -S_{div} \quad (1.4.18)$$

and the total, finite action, is $S_{tot} = S + S_{ct}$.

Example: Probe massive scalar in AdS

To illustrate the above methods we are going to consider the simplest example possible: A free massive scalar in pure AdS_{d+1} in the probe limit[13].

If we do the the Fourier transformation,

$$\phi(x, z) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \phi(k, z) \quad (1.4.19)$$

and then substitute $\phi(k, z) = z^{\frac{d}{2}} f(k, z)$ the equation of motion becomes

$$z^2 f'' + z f' - (z^2 k^2 + \nu^2) f = 0 \, . \quad (1.4.20)$$

This last equation is the modified Bessel equation which has two linearly independent solutions $K_\nu(kz), I_\nu(kz)$, with $k = \sqrt{k^2}$. Out of these two, we must keep only $K_\nu(kz)$

[13] Which means that, although Einstein's equations are sourced by the scalar field, we assume that it doesn't backreact onto the metric, which remains pure AdS.

as the other one blows up exponentially as $z \to \infty$. We will fix from now on $d = 4$ and $\nu = \frac{3}{2}$ for concreteness. To find the correlator in the way of [72, 78] we must normalise the solution as,

$$\phi(k,z) = z^2 \frac{K_{\frac{3}{2}}(kz)}{K_{\frac{3}{2}}(k\epsilon)\epsilon^{\frac{3}{2}}} \phi^s_k \,. \tag{1.4.21}$$

Integrating by parts the kinetic term in the action, the on-shell action is,

$$S^{os}_b = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{L^3}{z^3} \phi(-k,z) \partial_z \phi(k,z)|^{z\to\infty}_{z\to\epsilon} . \tag{1.4.22}$$

The contribution from $z \to \infty$ is vanishing due to the asymptotics of the modified Bessel function, whereas the contribution from the boundary gives,

$$S^{os}_b = -\frac{L^3}{2} \int \frac{d^4k}{(2\pi)^4} \phi^s_{-k} \phi^s_k \left(\frac{1}{2\epsilon^3} - \frac{k^2}{\epsilon} + k^3 \right) , \tag{1.4.23}$$

where we dropped terms that vanish in the $\epsilon \to 0$ limit. According to the GKPW rule, we must take the second functional derivative of the above expression to find the two-point function,

$$\langle O(k) O(k') \rangle = -(2\pi)^8 \frac{\delta S^{os}_b}{\delta \phi^s_{-k} \delta \phi^s_{-k'}} = L^3 (2\pi)^4 \delta^{(4)}(k+k') \left(\frac{1}{2\epsilon^3} - \frac{k^2}{\epsilon} + k^3 \right) . \tag{1.4.24}$$

One can observe that the diverging terms in the limit $\epsilon \to 0$ are also analytic in k and so in coordinate space they are going to contribute delta functions (and its derivatives). In other words, they give rise to contact terms, which are zero away from coincident points and we can ignore them since they are scheme dependent.

The final result, ignoring the diverging contact terms is thus

$$\langle O(k) O(k') \rangle = L^3 (2\pi)^4 \delta^{(4)}(k+k') k^3 . \tag{1.4.25}$$

Let's now compute the two-point function with holographic renormalisation. This time, we normalise the bulk solution as

$$\phi(k,z) = z^2 \frac{k^{\frac{3}{2}}}{\sqrt{\frac{\pi}{2}}} \phi_k^s K_{\frac{3}{2}}(kz), \tag{1.4.26}$$

which behaves near the boundary as,

$$\phi(k,z) = \phi_k^s z^{\frac{1}{2}} - \phi_k^s \frac{k^2}{2} z^{\frac{5}{2}} + \phi_k^v z^{\frac{7}{2}} + \mathcal{O}(z^{9/2}), \tag{1.4.27}$$

with $\phi_k^v = \frac{k^3}{3}\phi_k^s$. Note that if we solve the equation of motion perturbatively near the boundary we can specify the coefficient of the $z^{\frac{5}{2}}$ term, but not ϕ_k^v. To find ϕ_k^v one must have the full bulk solution. As we can see from this example, the expression for ϕ_k^v is a non-analytic function of the wavevector.

The minimum counterterm action needed to obtain a finite on-shell action and make the variational problem well defined is [82],

$$S_{ct} = \frac{1}{2L} \int d^4x \sqrt{\gamma} \left(\frac{\phi^2}{2} + L^2 \phi \Box \phi \right). \tag{1.4.28}$$

One can indeed check that, with the above counterterms, the total on-shell action is finite and the first variation reads,

$$\delta S_{tot} = \delta S_b^{os} + \delta S_{ct} = -3L^3 \int \frac{d^4k}{(2\pi)^4} \delta\phi_{s,-k} \phi_k^v \tag{1.4.29}$$

Hence, the renormalised one-point function is $\langle O(k) \rangle_s = 3L^3 \phi_k^v$, and taking a further functional derivative,

$$\langle O(k) O(k') \rangle = L^3 (2\pi)^4 \delta^{(4)}(k+k') k^3 \tag{1.4.30}$$

The results (1.4.25) and (1.4.30) agree with one another. However, employing holographic renormalisation no diverging terms show up in the calculation.

As one can imagine, this is one of the few cases where we can solve the equations of motion in the bulk analytically. As we are going to see in many examples later, in a typical holographic computation, one either solves the equations of motion numerically or resorts to some sort of perturbative solution, e.g. a hydrodynamic expansion for small ω, k.

Real-time correlators and Quasinormal Modes

As it is evident from this simple computation, apart from the boundary condition at the AdS boundary, we have to impose an additional one at the horizon. In the case of Euclidean signature, we saw that this additional condition came from demanding regularity of the solution in the deep interior of the bulk spacetime. However, in Minkowski spacetime both solutions are regular at the horizon. The prescription to calculate retarded two-point functions, first put forward in [87], is that we must impose incoming-wave boundary conditions at the horizon. Then the retarded Green's function is given by

$$G_R(\omega, k) = \frac{\delta\langle O(\omega, k)\rangle\rangle}{\delta\phi_s(\omega, k)}|_{\phi_s=0} \tag{1.4.31}$$

The physical intuition behind this choice of infalling boundary conditions is that "things must fall into the horizon" and not come out of it. This is in a sense a holographic version of dissipation. Similarly, in order to get advanced Green's functions we must impose outgoing boundary conditions at the horizon. The prescription of [87] was later justified in [88, 89, 90, 91], expanding on the idea of a complex time contour on the field theory side.

Suppose now that we seek a solution of the bulk equations of motion with infalling boundary conditions, with vanishing source and non-zero VEV. For a fixed wavevector $\vec{k}$ such a solution can be found, provided that the frequency ω takes values in a discrete set $\omega_i(\vec{k})\,, i = 1, 2, 3, \cdots$. Equation (1.4.31) implies that the retarded Green's function must diverge in this case and hence, the frequencies $\omega_i(\vec{k})$ correspond to poles of the retarded two-point function [92, 87]. Such source-free solutions are known as quasinormal modes (QNMs) in the literature. See [93] for a review. At finite temperature the QNM frequencies are complex and due to the analyticity of G_R, they must lie on the lower half of the complex ω plane. Their imaginary part reflects the presence of dissipation and controls the relaxation time of the respective perturbation, back to equilibrium [94]. QNMs with dispersion relations obeying $\lim_{k\to 0}\omega(k) = 0$ are dual to hydrodynamic perturbations of the field theory and govern its late-time response, in the linear response regime.

1.4.3 Hydrodynamics, Superfluids and Holography

One of the most fruitful applications of holography, since the early years of the AdS-CFT correspondence, has been the study of thermal field theories in the hydrodynamic regime (see e.g. [95, 96, 97, 98]). Given a specific gravitational background, dual to the thermal state of the field theory, we can study linearised, spacetime dependent perturbations around it, of the generic form $\mathcal{F}(r,x) = f(r)e^{-i\omega t + i\vec{q}\vec{x}}$, with infalling boundary conditions at the horizon of the black hole. In the limit $\frac{\omega}{T}, \frac{k}{T} \ll 1$ we are probing the hydrodynamics of the field theory, which as we already saw, lead to universal predictions for the hydrodynamic degrees of freedom. If we solve the linearised bulk equations, we can then obtain via (1.4.31), the retarded Green's functions and compare with the results expected from hydrodynamics. Moreover, we can calculate the relevant quasinormal modes which are dual to the hydrodynamic modes of the field theory (see e.g.[99]).

Following this line of thought, the transport coefficients can be calculated in various ways (e.g. via Kubo formulas). Recall that the transport coefficients depend on the thermal state of the system, entailing information about the UV degrees of freedom and are inaccessible within the framework of hydrodynamics. To find the values of these coefficients one would need to know the microscopic theory describing the system. Holography serves exactly this purpose.

A holographic computation for a transport coefficient, that stimulated many theoretical and experimental investigations, mainly due to the possible applications to the quark-gluon plasma, was the one for the shear viscosity over entropy density (see [100] for a review). In [101] the authors, generalising previous results [102, 95], found that this ratio, for many different backgrounds is[14]

$$\frac{\eta}{s} = \frac{1}{4\pi} \,. \tag{1.4.32}$$

They also conjectured [101] that this ratio is universal for any holographic theory and that $\frac{1}{4\pi}$ is in fact a universal lower bound for any system. (This was called the "KSS" bound.) Soon after, it was shown [103] that this bound is indeed saturated for any theory with a holographic dual description in the supergravity approximation. However, it was later found numerically in [104] that deviation from this universal value exists, still in the classical approximation, in anisotropic fluids. In addition, it turned out that certain curvature corrections violate the postulated lower bound [105, 106, 107].

The gauge-gravity duality has also been successfully used to study phases with spontaneously broken symmetries. See [69, 108, 109] for reviews of the subject. These phases are called holographic superfluids or holographic superconductors in the literature[15]. In this thesis, we focus only on s-wave superfluids, i.e. superfluids with scalar order parameter. We refer to [110, 111, 112, 113, 114] for holographic superfluids with other types of order parameters.

It was first argued in [115] and shown explicitly in [116, 117] that the spontaneous breaking of a $U(1)$ symmetry can be reproduced holographically with an Einstein-Maxwell-complex scalar bulk action of the form

$$S = \int d^{d+1}x\sqrt{-g}\left(R - \frac{1}{4}F^2 - |D_\mu\psi|^2 - V(|\psi|^2)\right) \,. \tag{1.4.33}$$

[14] In units with $\hbar = 1, k_B = 1$.

[15] Strictly speaking, only the term superfluid is correct since the symmetry on the field theory side is a global one.

Irrespective of the specific model under consideration, one must look for a solution of the bulk equations of motion with a source-free charged scalar $\psi(r)$, in order to describe the spontaneous breaking of the global $U(1)$ symmetry in the field theory side. At a certain temperature T_c, an instability occurs due to the charged scalar. Above T_c we can only have $\psi = 0$ and this is dual to the normal phase of the system. Below T_c, the solution with $\psi(r) \neq 0$ is thermodynamically favoured compared to the solution without a charged scalar and this corresponds to the formation of the charge condensate in the boundary field theory.

The phase transition described in these models is a second order one, with spontaneous symmetry breaking of the $U(1)$ continuous symmetry, and the space dimensionality is taken to be $d - 1 = 2$. But this seems to contradict the Coleman-Mermin-Wagner-Hohenberg theorem we mentioned in section 1.1. The resolution of this puzzle, as explained in [116], is that working at the classical gravity level in the bulk we are effectively taking the $N \to \infty$ limit in the field theory side. This limit suppresses all the fluctuations of the Goldstone bosons that are responsible for the destruction of the ordered phase in low dimensions. In addition, the critical exponents in this transition take their mean-field values [118]. This is also attributed to the large N limit of the classical gravity, which suppresses all fluctuations. For the study of large N corrections in gauge-gravity duality see, for example, the works [119, 120]. For more references on applications of holography, the reader should consult the introductory sections of the upcoming chapters.

Chapter 2

Dissipative effects in finite density holographic superfluids

This chapter is a reproduction of [5], written in collaboration with Aristomenis Donos.

In this paper, we derive the leading dissipative corrections of holographic superfluids at finite temperature and chemical potential by employing our recently developed techniques to study dissipative effects in the hydrodynamic limit of holographic theories. As part of our results, we express the incoherent conductivity, the shear and the three bulk viscosities in terms of thermodynamics and the black hole horizon data of the dual bulk geometries. We use our results to show that all three bulk viscosities exhibit singular behaviour close to the critical point.

2.1 Introduction

Holography provides large classes of examples of strongly coupled field theories where exact computations can be carried out [59, 73]. In a certain limit, non-trivial questions about field theory can be mapped to well defined problems in Einstein's classical theory of gravity in an asymptotically Anti de-Sitter spacetime (AdS) of dimensionality larger by one. The fall-off conditions of the classical gravitational fields close to the conformal boundary of AdS set the sources of local operators on the field theory side. In the classical limit, the holographic principle states that the partition functions of the two sides are equal, providing a powerful tool to compute expectation values of local operators.

In thermal equilibrium, the geometric dual of the thermal state is a black hole geometry with the temperature set by the Hawking temperature of the Killing horizon. Moreover, the holographic dictionary suggests that global $U(1)$ symmetries on the field theory side are gauged in the bulk. The asymptotic flux of the corresponding gauge fields set the electric charge density of the field theory making holography an invaluable laboratory to study large classes of strongly coupled systems at finite temperature and number density [63].

An interesting application of holography concerns the study of systems exhibiting spontaneous breaking of a global symmetry. In general, continuous phase transitions are driven via perturbative instabilities of black holes against fluctuations of classical bulk fields. The first examples of symmetry breaking were due to bulk fields which are charged under continuous internal symmetry groups, leading to superfluid phases of holographic matter [116, 115]. Spontaneous breaking of spacetime symmetries were realised later in [121, 122, 123, 124] making holography even more appealing for applications in condensed matter systems.

Holographic systems reach local thermal equilibrium with their long wavelength excitations obeying the laws of hydrodynamics. This has been a very active area of research over the past years [100, 125, 126, 127, 128] from which new lessons about low energy effective field theory have been learned [63, 129]. The hydrodynamic limit of broken symmetry phases where the standard hydrodynamic degrees of freedom of charged fluids combine with gapless Goldstone modes has also been considered in holography [31, 26, 30]. More recently, the amplitude mode which becomes gapless at continuous phase transitions has been realised holographically in [130]. This is a first step towards constructing effective theories which include fluctuations of the amplitude of the order parameter apart from its phase.

The hydrodynamic limit of holographic superfluids has been studied extensively since their first discovery. As standard in hydrodynamics, the stress tensor of the theory and the electric current admit a derivative expansion in terms of the local temperature, the normal fluid velocity, the chemical potential and the phase of the order parameter. The inequivalent terms that can appear in the first few orders of the hydrodynamic series of relativistic superfluids have been classified in [31, 26, 30]. This is a necessary step in order to extract the number of the transport coefficients that parametrise the different terms in the expansion series. These numbers are essentially the invariants that one can have under different choices of a fluid frame. For an isotropic relativistic fluid, the coefficients that need to be specified in the small superfluid velocity limit are the incoherent electric conductivity σ, the shear viscosity η and three bulk viscosities ζ_i. Similarly to normal fluids, conformal symmetry constrains the form of the bulk viscosities allowing only one of them to be non-zero [31, 26]. By introducing scales through relevant scalar operators, we will retain as many independent transport coefficients as in any relativistic superfluid.

A number of previous works have considered various aspects of the hydrodynamic limit of holographic superfluids. However, most of them have either resorted to numerical techniques [131, 132, 133, 134] or they have focused on specific models where analytic solutions for the gravitational problem can be obtained infinitesimally close to the transition [135, 31]. In this paper, we will employ the techniques developed in [2, 130], based on the Crnkovic-Witten symplectic current [1], to derive the first dissipative corrections in the hydrodynamic limit of holographic superfluids. A significant advantage of this approach is that an explicit solution of the gravitational fields is not required. Instead, it focuses on the universal aspects of the black holes dual to the thermal states.

As a by-product of our derivation, we will be able to fix the five non-trivial dissipative transport coefficients that we expect to have. The expressions we will obtain will be in terms of horizon data of the black holes dual to the thermal states of our system. Our results reproduce a well known result for the shear viscosity η to entropy density s ratio which is proportional to $1/4\pi$ [100, 136, 31]. However, our results for the incoherent conductivity and the bulk viscosities are new. In particular, our formula for the bulk viscosity ζ_1 generalises the results of [137] and [3] to include the bulk massive vector on the horizon. Moreover, as we will see our expression for ζ_3 generalises the result of [2] which was obtained for holographic superfluids at zero chemical potential.

Given our analytic expressions in terms of horizon data, we are able study the behaviour of the first dissipative corrections close to the phase transition. By following general arguments about the behaviour of our bulk fields near the phase transition, we are able to show that the shear viscosity and the electric conductivity are continuous functions across the transition. At the same time, we show that all three bulk viscosities diverge at the critical point.

As one might anticipate, the hydrodynamic modes of our system consist of the two longitudinal sound modes and the transverse shear mode responsible for the diffusion of momentum density. Taking the limit of the dispersion relations close to the critical temperature we find that the speed of the first sound mode remains finite while the speed for the second sound vanishes. These results can be shown through general considerations of ideal superfluid hydrodynamics. More interestingly, using the explicit expressions for the dissipative coefficients we show the the attenuation of the first sound diverges. This is in contrast to the second sound whose attenuation part remains finite.

Our paper is organised in six sections which are further divided in subsections. In section 2.2 we present the class of holographic models we wish to study along with the thermodynamic properties of the geometries dual to the field theory thermal states. In section 2.3 we discuss perturbations around our black holes and present the static perturbations which are the infinite wavelength and zero frequency limits of our hydrodynamic expansion. In section 2.4 we extract the leading dissipative corrections to the ideal superfluid based on the techniques we developed in [2, 130].

We conclude our analytic results in section 2.5 where we consider the limit of our hydrodynamic expansion close to the critical point. We also take the limit of zero chemical potential and compare our results with those in [2].

2.2 Setup

To model a holographic superfluid phase at finite density, we will consider a bulk theory which contains a Maxwell field A_μ, a neutral scalar ϕ and a complex scalar ψ which is charged under the local $U(1)$ symmetry. The neutral scalar ϕ is not a necessary ingredient but we will use it in order to introduce additional scales into the system. This will allow all the bulk viscosities we expect to find to be non-zero. The system is described by the bulk action,

$$S = \int d^4x \sqrt{-g}\left(R - V(\phi,|\psi|^2) - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}(D_\mu\psi)(D^\mu\psi)^* - \frac{1}{4}\tau(\phi,|\psi|^2)\,F^{\mu\nu}F_{\mu\nu}\right)\,, \tag{2.2.1}$$

with the covariant derivative $D_\mu\psi = \nabla_\mu\psi + iq_e A_\mu\,\psi$ and the field strength $F = dA$. It is easy to see that the above action is invariant under the local gauge transformations $A \to A + d\Lambda$ and $\psi \to e^{-iq_e\Lambda}\,\psi$. For small values of our scalar fields, we will assume that the functions τ and V behave according to,

$$\begin{aligned} V &\approx \frac{1}{2}m_\psi^2\,|\psi|^2 + \frac{1}{2}m_\phi^2\,\phi^2 + \cdots\,, \\ \tau &\approx 1 + c_\psi\,|\psi|^2 + c_\phi\,\phi + \cdots\,. \end{aligned} \tag{2.2.2}$$

Our focus will be on the superfluid phase of our system corresponding to backgrounds with a non-trivial profile for the complex bulk scalar ψ. In this case, the field redefinition $\psi = \rho\, e^{iq_e\theta}$ is well defined bringing the action to the form,

$$S = \int d^4x \sqrt{-g}\left(R - V(\phi,\rho^2) - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\rho)^2 - \frac{1}{2}q_e^2\,\rho^2\,B^2 - \frac{1}{4}\tau(\phi,\rho^2)\,F^{\mu\nu}F_{\mu\nu}\right)\,, \tag{2.2.3}$$

where we have set $B = A + \partial\theta$ and $F = dB$. The equations of motion which extremise the bulk action are,

$$
\begin{aligned}
R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}V-\frac{\tau}{2}\left(F_{\mu\rho}F_{\nu}{}^{\rho}-\frac{1}{4}g_{\mu\nu}\,F^2\right)&\\
-\frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi-\frac{1}{2}\partial_\mu\rho\,\partial_\nu\rho-\frac{1}{2}q_e^2\rho^2 B_\mu\,B_\nu &=0\,,\\
\nabla_\mu\nabla^\mu\phi-\partial_\phi V-\frac{1}{4}\partial_\phi\tau\,F^2 &=0\,,\\
\nabla_\mu\nabla^\mu\rho-2\,\partial_{\rho^2}V\,\rho-\frac{1}{2}\partial_{\rho^2}\tau\,\rho\,F^2-q_e^2\,\rho\,B^2 &=0\,,\\
\nabla_\mu\left(\rho^2B^\mu\right) &=0\,,\\
\nabla_\mu(\tau\,F^{\mu\nu})-q_e^2\,\rho^2\,B^\nu &=0\,.
\end{aligned}
\tag{2.2.4}
$$

We will consider electrically charged black brane solutions which are dual to thermal states of the deformed CFT by the relevant operator $\mathcal{O}_\phi$ dual to the bulk field ϕ. Moreover, we will assume that below a critical temperature T_c, the system exhibits spontaneous breaking of the field theory global $U(1)$.

The corresponding ansatz for the background fields is,

$$
\begin{aligned}
ds^2 &= -U(r)\,dt^2+\frac{dr^2}{U(r)}+e^{2g(r)}\left(dx^2+dy^2\right)\,,\\
B &=a(r)\,dt,\qquad \phi=\phi(r),\qquad \rho=\rho(r)\,.
\end{aligned}
\tag{2.2.5}
$$

The above choice of coordinates fixes the radial coordinate apart from a global shift. We will use this freedom to always have the event horizon sitting at $r=0$. Near the horizon, regularity implies the Taylor expansions

$$
\begin{aligned}
&U(r)\approx\,4\pi T\,r+\mathcal{O}(r^2),\qquad g(r)\approx g^{(0)}+\mathcal{O}(r),\qquad a(r)\approx r\,a^{(0)}+\cdots\,,\\
&\phi(r)\approx\phi^{(0)}+\mathcal{O}(r),\qquad \rho(r)\approx\rho^{(0)}+\mathcal{O}(r)\,,
\end{aligned}
\tag{2.2.6}
$$

where T is the Hawking temperature.

Close to the conformal boundary at $r\to\infty$ we impose the expansions,

$$
\begin{aligned}
&U(r)\approx(r+R)^2+\cdots+\frac{g_{(v)}}{r+R}+\cdots\,,\qquad g(r)\approx\ln(r+R)+\mathcal{O}(r^{-1})\,,\\
&a(r)\approx\mu_t-\frac{\varrho}{r+R}+\cdots\,,
\end{aligned}
$$

$$\phi(r) \approx \phi_{(s)}\,(r+R)^{\Delta_\phi-3} + \cdots + \phi_{(v)}\,(r+R)^{-\Delta_\phi} + \cdots\,,$$
$$\rho(r) \approx \rho_{(s)}\,(r+R)^{\Delta_\psi-3} + \cdots \rho_{(v)}\,(r+R)^{-\Delta_\psi} + \cdots\,, \qquad (2.2.7)$$

where we have defined the scalar operators sources $\phi_{(s)}$ and $\rho_{(s)}$ for the neutral and complex operators correspondingly. Moreover, we have defined the chemical potential μ_t and as we will see in the next section, the field theory charge density is given by the constant of integration ϱ. In this paper we will consider spontaneous breaking of the global $U(1)$ and we will be setting the non-perturbative complex scalar source $\rho_{(s)}$ equal to zero. Finally, the global shift in the radial coordinate which fixes the horizon at $r = 0$ is reflected by the constant of integration R.

In general, when the complex scalar source $\rho_{(s)}$ is set to zero, a perturbation of the bulk vector will admit the UV expansion [2],

$$\delta B_\alpha = \frac{\partial_\alpha \delta\theta_{(s)}}{(r+R)^{3-2\,\Delta_\rho}} + \cdots + \delta m_\alpha + \cdots + \frac{\delta j_\alpha}{r+R} + \cdots\,, \qquad (2.2.8)$$

where $m_\alpha = \partial_\alpha \theta_{(v)} + \mu_\alpha$ is a gauge invariant combination of the superfluid velocity $\partial_\alpha \theta_{(v)}$ and the source μ_α for the $U(1)$ current. As we will explain in the next section, the constant of integration $\theta_{(s)}$ is essentially a source for the complex scalar operator and the constants δj_α correspond to perturbations of the $U(1)$ electric current. As we will see these in the next section, these constants of integration are subject to a scalar constraint which is equivalent to the Ward identity satisfied by the electric current.

From the above we see that the phase θ of the complex scalar ψ has become part of the massive vector B. It will useful for us to note that, in the absence of a background source $\rho_{(s)}$ the asymptotic expansion for the bulk phase close to the conformal boundary is given by,

$$\delta\theta \approx (r+R)^{2\Delta_\psi-3}\,\delta\theta_{(s)} + \cdots + \delta\theta_{(v)} + \cdots\,, \qquad (2.2.9)$$

allowing us to write the expansion for the complex scalar perturbation,

$$\psi \approx (r+R)^{\Delta_\psi - 3}\, e^{iq_e\,\theta_{(v)}}\,(iq_e\,\rho_{(v)}\,\delta\theta_{(s)} + \delta\rho_{(s)}) \\ + \cdots + (r+R)^{-\Delta_\psi}\, e^{iq_e\,\theta_{(v)}}\,(\delta\rho_{(v)} + iq_e\,\rho_{(v)}\,\delta\theta_{(v)}) + \cdots\,. \tag{2.2.10}$$

From the above, we can read off the perturbative source for the complex operator to be,

$$\delta\lambda = e^{iq_e\,\theta_{(v)}}\,(iq_e\,\rho_{(v)}\,\delta\theta_{(s)} + \delta\rho_{(s)})\,. \tag{2.2.11}$$

2.2.1 Thermodynamics and Ward Identities

In order to extract quantities which are relevant to the field theory living on the boundary, our bulk action (2.2.1) needs to be supplemented with appropriate counterterms that will render it finite on-shell [82]. Apart from regularisation, the appropriate counterterms make the variational problem well defined [83]. For the bulk action we are interested in, a set universal terms are contained in,

$$S_{bdr} = -\int_{\partial M} d^3x\,\sqrt{-\gamma}\,(-2K + 4 + R_{bdr}) \\ -\frac{1}{2}\int_{\partial M} d^3x\,\sqrt{-\gamma}\,[(3-\Delta_\phi)\phi^2 - \frac{1}{2\Delta_\phi - 5}\,\partial_a\phi\partial^a\phi] \\ -\frac{1}{2}\int_{\partial M} d^3x\,\sqrt{-\gamma}\,[(3-\Delta_\psi)|\psi|^2 - \frac{1}{2\Delta_\psi - 5}\,D_a\psi D^a\psi^*] + \cdots\,, \tag{2.2.12}$$

where $\gamma_{\alpha\beta}$ is the induced metric on the asymptotic hypersurface ∂M of constant radial coordinate r.

The grand canonical free energy density w_{FE} for the thermal states captured by the geometries of equation (2.2.5) coincides with the value of the total action $S_{tot} = S + S_{bdr}$ after a Wick rotation to Euclidean time $t = -i\,\tau$. This yields the total Euclidean action I_{tot} such that $w_{FE} = T\,I_{tot}$ and,

$$w_{FE} = \epsilon - T\,s - \mu_t\,\varrho\,. \tag{2.2.13}$$

In the expression above, ϵ is the conserved energy density, ϱ is the electric charge density and,

$$s = 4\pi\, e^{2\, g^{(0)}}\,, \tag{2.2.14}$$

is the Bekenstein-Hawking entropy density of the system. Another horizon quantity we can define and which will become useful later is related to the flux of the one-form B_μ through the black hole horizon. We will call this the horizon charge density ϱ_h and by using our near horizon expansion (2.2.6) we can write,

$$\varrho_h = e^{2g^{(0)}}\tau^{(0)}a^{(0)}\,. \tag{2.2.15}$$

In our analysis, we are aiming to use the set of techniques that were developed in [2, 130, 3] in order to extract holographic information for the perturbations in real time. For this reason, we will perform an integration by parts in the Einstein-Hilbert in order to obtain a Lagrangian density that will only contain first order partial derivatives of the metric. As we argued in [3], the boundary counterterms $\delta S'_{bdr}$ for that action will be given by,

$$\begin{aligned}\delta S'_{bdr} = &-\int_{\partial M} d^3x\,\sqrt{-\gamma}\,\left(4 + R_{bdr}\right)\\ &-\frac{1}{2}\int_{\partial M} d^3x\,\sqrt{-\gamma}\,[(3-\Delta_\phi)\phi^2 - \frac{1}{2\Delta_\phi - 5}\,\partial_a\phi\partial^a\phi]\\ &-\frac{1}{2}\int_{\partial M} d^3x\,\sqrt{-\gamma}\,[(3-\Delta_\psi)|\psi|^2 - \frac{1}{2\Delta_\psi - 5}\,D_a\psi D^a\psi^*] + \cdots\,.\end{aligned} \tag{2.2.16}$$

After integrating by parts, the total action is given by,

$$S' = \int d^4x\,\mathcal{L}\left(g_{\mu\nu}, \partial_\lambda g_{\mu\nu}, \varphi^I, \partial_\lambda\varphi^I\right) + S'_{bdr}\,. \tag{2.2.17}$$

As we explained in [3], this action is suitable to extract real time boundary quantities as we drop a term that can potentially arise on the black holes horizon. Notice that dropping this term is standard within the holographic dictionary. However, for the Euclidean solutions this term is responsible for the entropy term in the expression (2.2.13) for the grand canonical free energy.

The VEV of the boundary stress tensor $T_{\mu\nu}$, the electric current J_μ and the scalar operators $\mathcal{O}_\phi$ and $\mathcal{O}_\psi$ are obtained via the variations,

$$\begin{aligned}
\langle T^{\mu\nu}\rangle &= \lim_{r\to\infty}\frac{2\,r^5}{\sqrt{-\gamma}}\left[\frac{\partial\mathcal{L}}{\partial(\partial_r g_{\mu\nu})}+\frac{\delta S'_{bdr}}{\delta\gamma_{\mu\nu}}\right],\\
\langle J^{\mu}\rangle &= \lim_{r\to\infty}\frac{r^3}{\sqrt{-\gamma}}\left[\frac{\partial\mathcal{L}}{\partial(\partial_r B_{\mu})}+\frac{\delta S'_{bdr}}{\delta B_{\mu}}\right],\\
\langle \mathcal{O}_\phi\rangle &= \lim_{r\to\infty}\frac{r^{\Delta_\phi}}{\sqrt{-\gamma}}\left[\frac{\partial\mathcal{L}}{\partial(\partial_r \phi)}+\frac{\delta S'_{bdr}}{\delta \phi}\right],\\
\langle \mathcal{O}_\psi\rangle &= \lim_{r\to\infty}\frac{2\,r^{\Delta_\psi}}{\sqrt{-\gamma}}\left[\frac{\partial\mathcal{L}}{\partial(\partial_r \psi^*)}+\frac{\delta S'_{bdr}}{\delta \psi^*}\right].
\end{aligned} \tag{2.2.18}$$

This form will be particularly useful to us as we will be able to read off directly the dissipative parts of the VEVs for the stress tensor and the electric current by using the techniques developed in [2, 130, 3].

In our analysis, the gravitational and vector field constraints in the bulk will be the last set of equations to be imposed. From the field theory point of view, when imposed on a hypersurface close to conformal boundary, they are equivalent to the Ward identities of diffeomorphism and gauge invariance for the sources,

$$\begin{aligned}
\nabla_a\langle T^{ab}\rangle &= F^{ba}\langle J_a\rangle+\nabla^b\varphi^I_{(s)}\,\langle\mathcal{O}_I\rangle+D^b\lambda\,\frac{\langle\mathcal{O}_{\psi^*}\rangle}{2}+D^b\lambda^*\,\frac{\langle\mathcal{O}_{\psi}\rangle}{2},\\
\nabla_a\langle J^{a}\rangle &= \frac{q_e}{2i}\,\left(\lambda\,\langle\mathcal{O}_{\psi^*}\rangle-\lambda^*\,\langle\mathcal{O}_{\psi}\rangle\right).
\end{aligned} \tag{2.2.19}$$

In the above we have defined the field strength $F = dm = d\mu$ of the external source of the current operator and the covariant derivative $D_a\lambda = \partial_a\lambda + iq_e A_a\lambda$. Our main aim is to first obtain a set of constitutive relations for the stress tensor and electric current in terms of an appropriate set of hydrodynamic variables by solving the radial equations. The Ward identities (2.2.19) will play the role of conservation laws in the theory of hydrodynamics.

The background geometries of equation (2.2.5) yield the stress tensor,

$$\langle T_{tt}\rangle=\epsilon\,,\quad \langle T_{xx}\rangle=\langle T_{yy}\rangle=p\,,\quad \langle J^t\rangle=\varrho\,, \tag{2.2.20}$$

where the energy density ϵ is given by,

$$\epsilon=-2\,g_{(v)}-\varphi_{(s)}\,\langle\mathcal{O}_\phi\rangle\,, \tag{2.2.21}$$

with $\langle\mathcal{O}_\phi\rangle = (2\Delta_\phi - 3)\,\phi_{(v)}$. Moreover, due to the translational invariance of the background thermal states under consideration, the pressure p is related to the free energy density according to $p = -w_{FE}$.

Finally, the VEV of our complex scalar operators is given by the constants of integration appearing in the asymptotic expansions (2.2.8) and (2.2.11) according to,

$$\langle\mathcal{O}_\psi\rangle = \frac{1}{2}(2\Delta_\psi - 3)\,\rho_{(v)} e^{i\,q_e\,\theta_{(v)}}\,, \tag{2.2.22}$$

which after we perturb yields,

$$\delta\langle\mathcal{O}_\psi\rangle = \langle\mathcal{O}_\psi\rangle_b\, i\, q_e\,\delta\theta_{(v)} + e^{i\,\arg(\langle\mathcal{O}_\psi\rangle_b)}\,\delta|\langle\mathcal{O}_\psi\rangle|\,. \tag{2.2.23}$$

The thermal states we are considering in equation (2.2.5) are parametrised by temperature T, the chemical potential μ_t and the neutral scalar deformation parameter $\phi_{(s)}$. However, superfluids are also characterised by a non-trivial electric current susceptibility χ_{JJ} allowing for thermal states which have a persistent electric with no heat current flow. This leads to a form of the first law,

$$dw_{FE} = -s\,dT - j^a\,dm_a - \langle\mathcal{O}_\phi\rangle\,d\phi_{(s)}\,, \tag{2.2.24}$$

where once again we have the gauge invariant combination $m_a = \mu_a + \partial_a\theta_{(v)}$. Even though we are not considering thermal background states with a persistent current, their perturbative form will become important in the construction of our hydrodynamic modes, through the supercurrent susceptibility. We will come back to this point in the next subsection when we discuss our static, thermodynamic perturbations.

For later reference, we will also define the thermodynamic susceptibilities through the variations,

$$ds = T^{-1}\,c_\mu\,dT + \xi\,d\mu_t\,, \qquad d\varrho = \xi\,dT + \chi_{QQ}\,d\mu_t\,. \tag{2.2.25}$$

2.3 Perturbations

In this section we will discuss the various perturbations around the black hole geometries (2.2.5) that will be relevant to our construction. In subsection 2.3.1 we discuss properties of general space-time dependent perturbations before considering any hydrodynamic limit. In subsections 2.3.2 and 2.3.3 we discuss static perturbations that we can obtain through variations of the thermodynamic variables parametrising our background geometries as well as perturbations that we can obtain through large coordinate transformations. As we will see, these will play a dual role in obtaining the constitutive relations. Finally, in subsection 2.3.4 we present our hydrodynamic expansion along with the leading dissipative corrections we are after.

2.3.1 Real Time Perturbations

Before discussing the derivative expansion of hydrodynamics, it will be beneficial to consider space-time dependent fluctuations from the boundary theory point of view. Due to the translational symmetry in both space and time, we find it convenient to perform a Fourier mode expansion of the general form,

$$\delta\mathcal{F}(t,x^i;r) = e^{-iw\,(t+S(r))+ik_i x^i}\,\delta f(r)\,, \tag{2.3.1}$$

where $\delta\mathcal{F}$ represents represents perturbations of the scalars as well as the metric field components. Moreover, by choosing the function $S(r)$ to approach $S(r) \to \frac{\ln r}{4\pi T} + \cdots$ close to the black hole horizon at $r = 0$, we are guaranteed to the correct infalling boundary conditions provided that $\delta f(r)$ admits a Taylor series expansion there. Finally, in order for the holographic dictionary to be solely dictated by the asymptotics of $\delta f(r)$, we will choose $S(r)$ to behave as $S(r) \to \mathcal{O}(1/r^3)$ close to the conformal boundary.

Close to the conformal boundary the radial functions that parametrise our space-time dependent perturbations will have to behave according to,

$$\begin{aligned}
\delta g_{ab}(r) &= (r+R)^2\left(\delta s_{ab}+\cdots+\frac{\delta t_{ab}}{(r+R)^3}+\cdots\right)\,,\\
\delta g_{ra}(r) &= \mathcal{O}\left(\frac{1}{r^3}\right),\quad \delta g_{rr}(r)=\mathcal{O}\left(\frac{1}{r^4}\right)\,,\\
\delta B_a(r) &= \delta m_a+\frac{\delta j_a}{r+R}+\cdots\,,\quad \delta B_r(r)=\mathcal{O}\left(\frac{1}{r^3}\right)\,,\\
\delta\phi(r) &= \frac{\delta\phi_{(v)}}{(r+R)^{\Delta_\phi}}+\cdots\,,\quad \delta\rho(r)=\frac{\delta\rho_{(v)}}{(r+R)^{\Delta_\psi}}+\cdots\,.
\end{aligned}\tag{2.3.2}$$

Notice that we do not choose to work in a particular coordinate system, we will only constrain our coordinates asymptotically through the decays of the metric components $\delta g_{r\mu}$. Moreover, the above stated decay for the one-form field components δB_r is correct provided that we impose the Ward identity for the electric current in equation (2.2.19).

Close to the black hole horizon at $r = 0$, we need to impose ingoing boundary conditions which we can achieve through the asymptotics,

$$\begin{aligned}
&\delta g_{tt}(r)=4\pi T\,r\,\delta g_{tt}^{(0)}+\cdots\,,\quad \delta g_{rr}(r)=\frac{\delta g_{rr}^{(0)}}{4\pi T\,r}+\cdots\,,\\
&\delta g_{ti}(r)=\delta g_{ti}^{(0)}+r\,\delta g_{ti}^{(1)}+\cdots\,,\quad \delta g_{ri}(r)=\frac{\delta g_{ri}^{(0)}}{4\pi T\,r}+\delta g_{ri}^{(1)}+\cdots\,,\\
&\delta g_{ij}(r)=\delta g_{ij}^{(0)}+\cdots\,,\qquad \delta g_{tr}(r)=\delta g_{tr}^{(0)}+\cdots\,,\\
&\delta B_t(r)=\delta b_t^{(0)}+\delta b_t^{(1)}\,r+\cdots\,,\qquad \delta B_r(r)=\frac{\delta b_r^{(0)}}{4\pi T\,r}+\delta b_r^{(1)}+\cdots\,,\\
&\delta B_i(r)=\delta b_i^{(0)}+\cdots\,,\\
&\delta\phi(r)=\delta\phi^{(0)}+\cdots\,,\qquad \delta\rho(r)=\delta\rho^{(0)}+\cdots\,.
\end{aligned}\tag{2.3.3}$$

In order to achieve regular infalling boundary conditions, the above need to be supplemented by additional conditions,

$$\begin{aligned}
-2\pi T(\delta g_{tt}^{(0)}+\delta g_{rr}^{(0)})=-4\pi T\,\delta g_{rt}^{(0)}&\equiv\delta T_h\,,\\
\delta g_{ti}^{(0)}=\delta g_{ri}^{(0)}&\equiv-\delta u_i\,,\\
\delta b_r^{(0)}=\delta b_t^{(0)}&\equiv\delta\mu_h\,.
\end{aligned}\tag{2.3.4}$$

In the above equations we have defined some horizon quantities which can be thought of as local temperature, fluid velocity and chemical potential and which can be defined on the black hole horizon. These will not be directly relevant to us since we will study the fluid from the boundary theory point of view.

2.3.2 Thermodynamics Perturbations

In this subsection we will consider all static perturbations which can be realised as small variations of thermodynamic variables and the Goldstone mode. As such, these will be the starting point of our hydrodynamic expansion which we will have to correct by using the techniques introduced in [2, 130, 3].

The first perturbation we will consider is generated by a variation in the temperature T of the background black holes of equation (2.2.5). However, in order to achieve the regular infalling boundary conditions of equations (2.3.3) and (2.3.4), we need to accompany the temperature variation with an infinitesimal coordinate transformation,

$$t \to t - \partial_T S\, \delta T\,, \tag{2.3.5}$$

yielding the fluctuation,

$$\begin{aligned}
&\delta_T g_{tt} = -\partial_T U\,, \quad \delta_T g_{rr} = -\frac{\partial_T U}{U^2}\,, \quad \delta_T g_{tr} = U\,\partial_T S'\,, \\
&\delta_T g_{ij} = 2\,\delta_{ij}\,e^{2g}\,\partial_T g\,, \quad \delta_T B_t = \partial_T a\,, \quad \delta_T B_r = -\partial_T S' a\,, \\
&\delta_T \phi = \partial_T \phi\,, \quad \delta_T \rho = \partial_T \rho\,.
\end{aligned} \tag{2.3.6}$$

Another thermodynamic variation which we wish to consider is with respect to the external field source $\delta\mu_a$ and superfluid velocity $\partial_a\delta\theta$. These two variables are packaged in the gauge invariant quantity $m_a = \mu_a + \partial_a\theta$. In order to generate the variations with respect to m_t, we can simply consider the derivative of our backgrounds with respect to the chemical potential to obtain,

$$\delta_{m_t} g_{tt} = -\partial_{\mu_t} U\,, \quad \delta_{m_t} g_{rr} = -\frac{\partial_{\mu_t} U}{U^2}\,, \quad \delta_{m_t} g_{ij} = 2\,\delta_{ij}\,e^{2g}\,\partial_{\mu_t} g$$

$$\delta_{m_t} B_t = \partial_{\mu_t} a \,, \quad \delta_{m_t} \phi = \partial_{\mu_t} \phi \,, \quad \delta_{m_t} \rho = \partial_{\mu_t} \rho \,. \tag{2.3.7}$$

In contrast to the case of the temperature variation, we don't need to shift the time coordinate here. The derivative of U with respect to the chemical potential results in a perturbation such that δg_{tt} and δg_{rr} exhibit a regular behaviour near the horizon, as can be seen from the asymptotics (2.2.6). In the notation of equations (2.3.3) and (2.3.4) it leads to $\delta T_h = 0$.

The perturbation with respect to the spatial components m_i will result in a static solution representing a static, isotropic current flow captured by,

$$\delta_{m_j} g_{ti} = \delta_i^j \, \delta f_g \,, \quad \delta_{m_j} B_i = \delta_i^j \, \delta f_b \,. \tag{2.3.8}$$

Since this solution is not directly generated through a derivative of our backgrounds (2.2.5), it is worth discussing it in more detail. An important point is that in order to define the current-current susceptibility χ_{JJ} through this fluctuation, we have to fix a frame where there is no heat current flow. This is necessary in order to obtain a unique solution since in the opposite case, we would be able to perform a Lorentz boost and still have a solution with a different current and heat flow. From a physics point of view, this is simply the fact that we want to consider a frame where only the superfluid component carries electric current. In this situation, there cannot be a heat current as the superfluid cannot transport it.

The absence of a heat current flow on the boundary imposes that the constant term in $\delta f_g(r)$ will vanish in the near horizon expansion,

$$\delta f_g(r) = \delta f_g^{(1)} \, r + \cdots \,, \quad \delta f_b(r) = \delta f_b^{(0)} + \cdots \,. \tag{2.3.9}$$

This can be justified through either earlier works relating the horizon data of static perturbations to boundary quantities [138, 139] or by following the techniques used in the present paper. By using our techniques, we can show this statement by e.g. considering the symplectic current constructed from the perturbation (2.3.8) and $\delta_{s_{ti}}$ discussed in the next subsection. This is also the reason that we don't need to include a perturbation for the metric component δg_{ri} since this produces a perturbation which has $\delta u_i = 0$, in the notation of the boundary conditions (2.3.3) and (2.3.4). Asymptotically we must have,

$$\delta f_g(r) = \frac{1}{3}\frac{\mu_t\,\chi_{JJ}}{r+R} + \cdots\,, \quad \delta f_b = 1 - \frac{\chi_{JJ}}{r+R} + \cdots\,, \tag{2.3.10}$$

which imposes the zero heat current condition.

The final static solution we would like to consider is generated by boundary infinitesimal Lorentz boosts with parameter δv_i. As one can easily see, at finite chemical potential this can be generated by a combination of a metric and a constant current source,

$$t \to t - \delta v_i\, x^i\,, \quad x^i \to x^i - \delta v^i\, t\,, \quad \delta m_i = \mu_t\, \delta v_i\,, \tag{2.3.11}$$

which will guarantee the absence of a net current source in the boundary theory. In order to obtain a regular solution in the bulk, we need to see the above as the asymptotics of a large coordinate transformation which is otherwise regular everywhere in the bulk. This is achieved by the coordinate transformation,

$$t \to t - \delta v_i\, x^i\,, \quad x^i \to x^i - \delta v^i\,(t + S(r))\,, \tag{2.3.12}$$

combined with the static solution (2.3.8) with the appropriate sources to cancel the external gauge field source. This leads to the bulk perturbation,

$$\begin{aligned}
&\delta_{v_j} g_{tt} = \delta_{v_j} g_{rr} = \delta_{v_j} g_{tr} = 0\,, \quad \delta_{v_j}\varphi = 0\,,\\
&\delta_{v_j} g_{ti} = \delta^j_i\,\left(U - e^{2g} + \mu_t\,\delta f_g\right)\,, \quad \delta_{v_j} g_{ri} = -\delta^j_i\,\left(e^{2g}\,S' + \mu_t\,\delta f_c\right)\,,\\
&\delta_{v_j} B_i = \delta^j_i\,\left(\mu_t\,\delta f_b - a\right)\,, \quad \delta_{v_j} B_t = 0\,, \quad \delta_{v_j} B_r = 0\,.
\end{aligned} \tag{2.3.13}$$

The perturbations discussed in this section will be the building blocks for the zero frequency, infinite wavelength limit of the hydrodynamic perturbations we will start constructing in section 2.3.4. From the boundary point of view, the perturbations we discussed here give rise to the stress tensor and electric current fluctuations of the ideal superfluid,

$$
\begin{aligned}
\delta\langle T^{tt}\rangle &= \partial_T\epsilon\,\delta T + \partial_{\mu_t}\epsilon\,\delta m_t = (c_\mu + \mu_t\,\xi)\;\delta T + (T\,\xi + \mu_t\,\chi_{QQ})\;\delta m_t\,,\\
\delta\langle T^{ij}\rangle &= \delta^{ij}\;(\partial_T p\,\delta T + \partial_{\mu_t} p\,\delta m_t) = \delta^{ij}\;(s\,\delta T + \varrho\,\delta m_t)\;,\\
\delta\langle T^{ti}\rangle &= \left(\epsilon + p - \mu_t^2\,\chi_{JJ}\right)\;\delta v^i - \mu_t\,\chi_{JJ}\,\delta m^i\,,\\
\delta\langle J^{t}\rangle &= \partial_T\varrho\,\delta T + \partial_{\mu_t}\varrho\,\delta m_t = \xi\,\delta T + \chi_{QQ}\,\delta m_t\,,\\
\delta\langle J^{i}\rangle &= (\varrho - \mu_t\,\chi_{JJ})\;\delta v^i - \chi_{JJ}\,\delta m^i\,. \qquad (2.3.14)
\end{aligned}
$$

It is also useful to check the heat current,

$$
\delta\langle Q^i\rangle = \delta\langle T^{ti}\rangle - \mu_t\,\delta\langle J^i\rangle = (\epsilon + p - \mu_t\,\varrho)\;\delta v^i = T s\,\delta v^i\,, \qquad (2.3.15)
$$

is indeed transferred by the normal fluid component which contributes to the entropy density of the system. Moreover, it will be useful to define the normal and superfluid charge densities,

$$
\begin{aligned}
\varrho_n &= \varrho - \mu_t\,\chi_{JJ}\,,\\
\varrho_s &= \mu_t\,\chi_{JJ}\,, \qquad (2.3.16)
\end{aligned}
$$

respectively.

2.3.3 Static Perturbations from Diffeomorphisms

Another class of static perturbations we will find useful are simply generated by large diffeomorphisms. Similarly with the thermodynamic perturbations of the previous subsection, their role in our construction will be twofold. Firstly, they will later be used in order to introduce sources in our hydrodynamic expansion as they will be part of the zero frequency, infinite wavelength limit. Secondly, they will be used in the construction of the symplectic current. As described in [3], this will help us read off the leading dissipative corrections to the stess tensor.

Similarly to [3], we will consider the combination of large coordinate transformations along with constant sources for the electric current sources,

$$\begin{aligned} x^a &\to x^a + \delta s^a{}_b \, (x^b + \delta^b_t \, S(r)) \, , \\ \delta\mu_a &= -\mu_t \, \delta s^t{}_a \, , \end{aligned} \tag{2.3.17}$$

where the term involving $S(r)$ takes care of regular, infalling boundary conditions on the black hole horizon at $r = 0$. It is easy to see that from the boundary theory point of view, this coordinate transformation induces a change in the metric,

$$\delta g_{ab} = \eta_{ac} \, \delta s^c{}_b + \eta_{bc} \, \delta s^c{}_a = 2 \, \delta s_{(ab)} \, , \tag{2.3.18}$$

while the explicit current sources of equation (2.3.17) combine with the transformation of the background source to give zero.

From the bulk point of view the bulk perturbation corresponding to the source δs_{tt} is realised by,

$$\begin{aligned} &\delta_{s_{tt}} g_{tt} = 2 \, U - \mu_t \, \partial_{\mu_t} U \, , \quad \delta_{s_{tt}} g_{tr} = U \, S' \, , \\ &\delta_{s_{tt}} g_{rr} = -\mu_t \, \frac{\partial_{\mu_t} U}{U^2} \, , \quad \delta_{s_{tt}} g_{ij} = 2 \, \delta_{ij} \, \mu_t \, e^{2g} \, \partial_{\mu_t} g \, , \\ &\delta_{s_{tt}} B_t = -a + \mu_t \, \partial_{\mu_t} a \, , \quad \delta_{s_{tt}} B_r = -a \, S' \, , \\ &\delta s_{tt} \phi = \mu_t \, \partial_{\mu_t} \phi \, , \quad \delta s_{tt} \rho = \mu_t \, \partial_{\mu_t} \rho \, . \end{aligned} \tag{2.3.19}$$

For the rest of the perturbations generated by (2.3.17) we have,

$$\begin{aligned} &\delta_{s_{tj}} g_{ti} = \delta^j_i \, \left(U + \mu_t \, \delta f_g \right) \, , \quad \delta_{s_{tj}} B_i = \delta^j_i \, \left(-a + \mu_t \, \delta f_b \right) \, , \\ &\delta_{s_{jt}} g_{ti} = \delta^j_i \, e^{2g} \, , \quad \delta_{s_{jt}} g_{ri} = \delta^j_i \, e^{2g} \, S' \, , \\ &\delta_{s_{ij}} g_{kl} = 2 \, \delta^{(i}_k \, \delta^{j)}_l \, e^{2g} \, . \end{aligned} \tag{2.3.20}$$

It is a simple matter to combine variations and transformations the VEVs of the background stress tensor and electric current to obtain the perturbations,

$$\begin{aligned} \delta \langle T^{tt} \rangle &= \left(2 \, \epsilon + \mu_t \, \left(T \, \xi + \mu_t \, \chi_{QQ} \right) \right) \, \delta s^{tt} \, , \\ \delta \langle T^{ij} \rangle &= \delta^{ij} \mu_t \, \varrho \, \delta s^{tt} - 2 \, p \, \delta s^{(ij)} \, , \end{aligned}$$

$$\delta\langle T^{ti}\rangle = \epsilon\,\delta s^{it} - \left(p - \mu_t^2\,\chi_{JJ}\right)\,\delta s^{ti}\,,$$
$$\delta\langle J^t\rangle = \left(\varrho + \mu_t\,\chi_{QQ}\right)\,\delta s^{tt}\,,$$
$$\delta\langle J^i\rangle = \varrho\,\delta s^{it} + \mu_t\,\chi_{JJ}\,\delta s^{ti}\,. \tag{2.3.21}$$

The above variations with respect to the sources, along with the ones coming from thermodynamics in equation (2.3.14), will become the constitutive relations for the ideal superfluid.

2.3.4 Hydrodynamic Perturbations

In this subsection we will construct the hydrodynamic modes in the bulk. In order to do this, we think of them as finite frequency and long wavelength deformations of the static modes we discussed in subsections 2.3.2 and 2.3.3. To make things technically more tractable, we will consider the Fourier decomposition (2.3.1) with $k_i = \varepsilon\, q_i$ and $w = \varepsilon\,\omega$.

Our hydrodynamic can be expanded in ε according to,

$$\begin{aligned}\delta_H f &= \varepsilon\,\delta f^{(1)} + \varepsilon^2\,\delta f^{(2)} + \cdots \\ &= \delta_T f\,\delta T + \delta_{v_i} f\,\delta v_i + \delta_{m_a} f\,\delta m_a + \delta_{s_{ab}} f\,\delta s_{ab} + \varepsilon^2\,\delta f^{(2)} + \cdots\,,\end{aligned} \tag{2.3.22}$$

where $\delta f^{(1)}$ is a linear combination of the static modes that we discussed in subsections 2.3.2 and 2.3.3. After dressing this linear combination with the exponential of the Fourier modes, we expect that the radial function will admit an ε expansion which after writing back in position space, we can think of as derivative corrections. In order to have this interpretation, we will need to show that the VEV corrections induced by the leading correction $\delta f^{(2)}$ can be indeed expressed in terms of derivatives of the local temperature δT, fluid velocity δv_i and superfluid velocity $\partial_a \delta\theta$.

To organise the expansion of the boundary stress tensor and electric current, we can write,

$$\begin{aligned}
\delta\langle T^{tt}\rangle &= (c_\mu + \mu_t\,\xi)\;\delta T + (T\,\xi + \mu_t\,\chi_{QQ})\;\delta\mu + 2\,\epsilon\,\delta s^{tt} + \varepsilon^2\,\delta\langle T^{tt}\rangle_{(2)} + \cdots\,,\\
\delta\langle T^{it}\rangle &= \delta\langle T^{ti}\rangle = (T s + \mu_t\,\varrho_n\,)\;\delta v^i - \varrho_s\,\delta m^i + \epsilon\,\delta s^{it} + \varepsilon^2\,\delta\langle T^{ti}\rangle_{(2)} + \cdots\,,\\
\delta\langle T^{ij}\rangle &= \delta^{ij}\;(s\,\delta T + \varrho\,\delta\mu) - 2\,p\,\delta s^{(ij)} + \varepsilon^2\,\delta\langle T^{ij}\rangle_{(2)} + \cdots\,,\\
\delta\langle J^t\rangle &= \xi\,\delta T + \chi_{QQ}\,\delta\mu + \varrho\,\delta s^{tt} + \varepsilon^2\,\delta\langle J^t\rangle_{(2)}\,,\\
\delta\langle J^i\rangle &= \varrho_n\,\delta v^i - \chi_{JJ}\,\delta m^i + \varrho\,\delta s^{it} + \varepsilon^2\,\delta\langle J^i\rangle_{(2)}\,,
\end{aligned} \tag{2.3.23}$$

where we have used our definitions (2.3.16) for the normal and superfluid charge densities. Moreover, we have defined the variation of the new chemical potential,

$$\delta\mu = \delta m_t + \mu_t\,\delta s^{tt}\,, \tag{2.3.24}$$

which as we will later see in section 2.4.2, it will receive dissipative corrections. In the above, the corrections $\delta\langle T^{ab}\rangle_{(2)}$ and $\delta\langle J^a\rangle_{(2)}$ are precisely the corrections due to $\delta f^{(2)}$ in the expansion (2.3.22). Already at leading order, the $\mathcal{O}(\varepsilon)$ terms in the constitutive relations (2.3.23) are enough to yield the equations of motion of the ideal superfluid, when combined with the Ward identities (2.2.19),

$$\begin{aligned}
&(T s + \mu_t\varrho_n)\;\partial_t\delta v_i + \varrho_n\,\partial_t\delta m_i + s\,\partial_i\delta T - T s\,\partial_i\delta s^{tt} = 0\,,\\
&\xi\,\partial_t\delta T + \chi_{QQ}\,\partial_t\delta\mu + \varrho_n\,\partial_i\delta v^i - \chi_{JJ}\,\partial_i\delta m^i + \varrho\left(\partial_i\delta s^{it} + \delta_{ij}\partial_t\delta s^{ij}\right) = 0\,,\\
&(c_\mu + \mu_t\,\xi)\;\partial_t\delta T + (T\xi + \mu_t\chi_{QQ})\;\partial_t\delta\mu + (T s + \mu_t\varrho_n)\;\partial_i\delta v^i\\
&\qquad\qquad - \varrho_s\,\partial_i\delta m^i + (\epsilon + p)\left(\partial_i\delta s^{it} + \delta_{ij}\partial_t\,\delta s^{ij}\right) = 0\,.
\end{aligned} \tag{2.3.25}$$

The task of the next section will be to employ the techniques of [3] and express these corrections in terms of δT, δv_i and $\partial_a\delta\theta$. As a byproduct, we will obtain specific expressions for the shear and the three bulk viscosities, along with the two independent dissipative coefficients that enter the currents or the Josephson relation, depending on the frame one would like to use. For completeness we will express our constitutive relations in the transverse frame [31, 26].

2.4 Constitutive Relations

In this section we will derive the constitutive relations for the stress tensor and electric current relevant to the hydrodynamic fluctuations of our system. We will achieve this in two steps which we discuss in detail in the following subsections. The first step is to extract the constitutive relations in terms of our hydrodynamic variables. As we will see, we will land in an unusual fluid frame which will include bulk integrals as part of its artefacts. In the second step we will change our description to the so called transverse frame from which we will be able to read off the shear viscosity η, the three bulk viscosities ζ_i and the incoherent conductivity σ.

2.4.1 Constitutive Relations in Terms of Horizon Data

The aim of this section is to write the dissipative corrections of the constitutive relations (2.3.23) in terms of derivatives of the hydrodynamic variables δT, δv^i and $\delta\theta$. In order to achieve this, we will follow closely the logic developed in [3].

The main tool in our construction will be the Crnkovic-Witten symplectic current defined for any classical theory of a collection of fields ϕ^I whose equations of motion can be obtained from a first order Lagrangian density $\mathcal{L}(\phi^I, \partial\phi^I)$. For any two perturbations $\delta_1\phi^I$ and $\delta_2\phi^I$ around a background ϕ_b^I which solve the Euler-Lagrange equations of motion, the vector density,

$$P^\mu_{\delta_1,\delta_2} = \delta_1\phi^I\,\delta_2\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^I}\right) - \delta_2\phi^I\,\delta_1\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^I}\right)\,, \tag{2.4.1}$$

is divergence free,

$$\partial_\mu P^\mu_{\delta_1,\delta_2} = 0\,. \tag{2.4.2}$$

For completeness, we write list the contributing terms to (2.4.1),

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\partial_\mu g_{\alpha\beta}} &= \sqrt{-g}\,\Gamma^\mu_{\gamma\delta}\left(g^{\gamma\alpha}\,g^{\delta\beta} - \frac{1}{2}\,g^{\gamma\delta}g^{\alpha\beta}\right) - \sqrt{-g}\,\Gamma^\kappa_{\kappa\lambda}\left(g^{\mu(\alpha}\,g^{\beta)\lambda} - \frac{1}{2}\,g^{\mu\lambda}g^{\alpha\beta}\right)\,,\\ \frac{\partial\mathcal{L}}{\partial\partial_\mu B_\nu} &= -\sqrt{-g}\,\tau\,F^{\mu\nu}\,,\quad \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = -\sqrt{-g}\,\partial^\mu\phi\,,\quad \frac{\partial\mathcal{L}}{\partial\partial_\mu\rho} = -\sqrt{-g}\,\partial^\mu\rho\,,\end{aligned} \tag{2.4.3}$$

through the derivatives of the first order bulk action (2.2.17).

Following very similar arguments with [3], one can show that the asymptotic behaviour of the radial components of the symplectic current is,

$$\begin{aligned}P^r_{\delta_1,\delta_2} =&\frac{1}{r^3}\left(\delta_1\phi_{(s)}\,\delta_2\left(\sqrt{-\gamma}\,\langle\mathcal{O}_\phi\rangle\right)-\delta_2\phi_{(s)}\,\delta_1\left(\sqrt{-\gamma}\,\langle\mathcal{O}_\phi\rangle\right)\right)\\&+\frac{1}{r^3}\left(\delta_1\rho_{(s)}\,\delta_2\left(\sqrt{-\gamma}\,\langle\mathcal{O}_\rho\rangle\right)-\delta_2\rho_{(s)}\,\delta_1\left(\sqrt{-\gamma}\,\langle\mathcal{O}_\rho\rangle\right)\right)\\&+\frac{1}{r^3}\left(\delta_1 m_a\,\delta_2\left(\sqrt{-\gamma}\,\langle J^a\rangle\right)-\delta_2 m_a\,\delta_1\left(\sqrt{-\gamma}\,\langle J^a\rangle\right)\right)\\&+\frac{1}{r^3}\frac{1}{2}\left(\delta_1\gamma_{ab}\,\delta_2\left(\sqrt{-\gamma}\,\langle T^{ab}\rangle\right)-\delta_2\gamma_{ab}\,\delta_1\left(\sqrt{-\gamma}\,\langle T^{ab}\rangle\right)\right)+\cdots\,,\end{aligned}\tag{2.4.4}$$

with the VEVs being correct up to second derivatives of the sources. This is useful to us as we are interested in obtaining the VEVs of the stress tensor and the electric current up to order $\mathcal{O}(\varepsilon^2)$. The above expression shows that by using appropriate combinations of our hydrodynamic mode (2.3.22) and the static solutions of subsections 2.3.2 and 2.3.3, we can read off the VEVs of our system.

By introducing the Fourier decomposition (2.3.1) of the symplectic current itself and by integrating the condition (2.4.2) we can express,

$$P^r_{\delta_1,\delta_2}\Big|_{r=\infty} = P^r_{\delta_1,\delta_2}\Big|_{r=0} + B_{\delta_1,\delta_2}\,,\tag{2.4.5}$$

where we have set,

$$B_{\delta_1,\delta_2} = i\int_0^\infty dr\,\left(-w\,\left(S'\,P^r_{\delta_1,\delta_2}+P^t_{\delta_1,\delta_2}\right)+k_i\,P^i_{\delta_1,\delta_2}\right)\,.\tag{2.4.6}$$

This shows that when e.g. one of the two perturbations δ_2 is chosen to be hydrodynamic mode δ_H we can expand in ε to obtain,

$$P^\mu_{\delta_1,\delta H} = \varepsilon\,P^{\mu(1)}_{\delta_1,\delta H}+\varepsilon^2\,P^{\mu(2)}_{\delta_1,\delta H}+\cdots\,.\tag{2.4.7}$$

The leading term $P^{r(1)}_{\delta_1,\delta H}$ is the symplectic current formed by the ε terms in the hydrodynamic expansion (2.3.22). As such, it will not offer us additional information apart from facts about the static perturbations since δ_1 always refers to static perturbations of the background. At second order in the ε expansion, we have,

$$P^{r(2)}_{\delta_1,\delta_2}\Big|_{r=\infty} = P^{r(2)}_{\delta_1,\delta_2}\Big|_{r=0} + B^{r(2)}_{\delta_1,\delta_2}\,, \tag{2.4.8}$$

with,

$$B^{(2)}_{\delta_1,\delta_2} = i\int_0^\infty dr\,\left(-\omega\,\left(S'\,P^{r(1)}_{\delta_1,\delta_2} + P^{t(1)}_{\delta_1,\delta_2}\right) + q_i\,P^{i(1)}_{\delta_1,\delta_2}\right)\,. \tag{2.4.9}$$

We therefore see that bulk integrals will in principle be present in our final expressions. However, after moving to a fluid frame in which the transport coefficients are of physical significance and using properties of the static perturbations, the bulk integral will disappear from the final constitutive relations.

Before we embark on our journey to fix the dissipative parts of the constitutive relations (2.3.23), we would like to point at some piece of information that will be useful and that we can obtain from the symplectic current formed entirely from our static solutions. By considering the symplectic current $P^{\mu}_{\delta v^i,\delta m^j}$ and applying equation (2.4.5) we obtain,

$$\delta f_g^{(1)} = -\frac{4\pi\varrho_h}{s}\,\delta f_b^{(0)} + \frac{4\pi\varrho_n}{s}\,. \tag{2.4.10}$$

In Appendix A we list all the constraints we can obtain by forming the symplectic current of all possible combinations of the static perturbations in subsections 2.3.2 and 2.3.3. In contrast to the case of normal fluids that were studied in [3], these bulk constraints will be used in order to show all the transport coefficients are fixed by the event horizon of our thermal states (2.2.5).

The static perturbations of subsection 2.3.3 which contain sources for the asymptotic metric, will let us express the boundary stress tensor in terms of horizon data and bulk integrals of the perturbation. Similarly, the perturbation δm_a discussed in subsection 2.3.2 will let us do the same for the electric current. As we will see, this will leave us with constants of integration on the horizon which still need to be expressed in terms of our hydrodynamic variables. Similarly to [3], we will achieve this by considering the symplectic current formed by using the hydrodynamic mode

(2.3.22) along with the temperature variations δ_T and boosts δv_i of subsection 2.3.2.

In order to read off the corrections $\delta\langle T^{ab}\rangle_{(2)}$, we will consider the symplectic current $P^{\mu}_{\delta s_{ab},\delta_H}$ to obtain,

$$\begin{aligned}
\varepsilon\,\delta\langle T^{tt}\rangle_{(2)} =& \frac{i}{4\pi}\Big(-q_i s\,\left(\delta s^{it}+\delta v^i\right)+\omega\,(s-\mu_t\,\xi)\;\delta_{ij}\delta s^{ij}+\omega\frac{c_\mu}{sT}\,(s-\mu_t\,\xi)\,\delta T\\
&+\omega\,\frac{\xi}{s}\,(s-\mu_t\,\xi)\;\delta\mu+\omega\mu_t\,\xi\delta s^{tt}+\varepsilon\,4\pi i\,\partial_\mu\varrho_h\,\mu_t\,\delta b_t^{(2)(0)}-\varepsilon\,8i\,\pi^2\,T\,\delta^{ij}\delta g_{ij}^{(2)(0)}\\
&+\varepsilon\,2\pi i\,T\,\mu_t\,\xi\,\left(\delta g_{rr}^{(2)(0)}+\delta g_{tt}^{(2)(0)}\right)+s\mu_t\,\omega\left(\partial_T\rho^{(0)}\partial_\mu\rho^{(0)}+\partial_T\phi^{(0)}\partial_\mu\phi^{(0)}\right)\,\delta T\\
&+s\mu_t\,\omega\,\left(\left(\partial_\mu\rho^{(0)}\right)^2+\left(\partial_\mu\phi^{(0)}\right)^2\right)\,\delta\mu\Big)+\varepsilon\,B^{(2)}_{\delta_{s_{tt}},\delta_H}\,,\\
\varepsilon\,\delta\langle T^{ti}\rangle_{(2)} =& -\varepsilon\,\frac{4\pi}{s}\,(\mu_t\varrho_n+Ts)\,\delta g_{ti}^{(2)(0)}+i\omega\mu_t(\delta f_b^{(0)})^2\left(\delta m^i+\mu_t\delta v^i\right)+\varepsilon\,B^{(2)}_{\delta_{s_{ti}},\delta_H}\,,\\
\varepsilon\,\delta\langle T^{ij}\rangle_{(2)} =& -\frac{is}{4\pi}\left(2\left(q^{(i}\delta s^{j)t}+q^{(i}\delta v^{j)}\right)-\delta^{ij}q_k\left(\delta s^{kt}+\delta v^k\right)\right)\\
&-\frac{i\omega}{4\pi}\left(\delta^{ij}\left(2\,s\,\delta_{kl}\delta s^{kl}-s\,\delta s^{tt}+\frac{c_\mu}{T}\delta T+\xi\,\delta\mu\right)-2s\,\delta s^{(ij)}\right)\\
&-\varepsilon\,\delta^{ij}\frac{s\,T}{2}\left(\delta g_{rr}^{(2)(0)}+\delta g_{tt}^{(2)(0)}\right)-\varepsilon\,\varrho_h\delta^{ij}\,\delta b_t^{(2)(0)}+\varepsilon\,B^{(2)}_{\delta_{s_{ij}},\delta_H}\,.
\end{aligned}\tag{2.4.11}$$

By considering the symplectic current $P^{\mu}_{\delta s_{it},\delta_H}$, we can obtain the form,

$$\begin{aligned}
\varepsilon\,\delta\langle T^{ti}\rangle_{(2)} =& -\frac{i}{4\pi T}\left(q^i\,\left(sT\delta^{kl}\delta s_{kl}+c_\mu\delta T+T\xi\delta\mu\right)+\omega\,\delta v^i\,\left(sT+\mu\varrho_n\right)\right)\\
&-\frac{i}{4\pi T}\left(\omega\,\varrho_n\,\delta m^i-\delta f_b^{(0)}\varrho_h\left(\delta m^i+\mu\delta v^i\right)\omega\right)+\varepsilon\,\frac{s}{4\pi}\delta^{ij}\left(\delta g_{tj}^{(2)(1)}-2\delta g_{tj}^{(2)(0)}g_{(1)}\right)\\
&+\varepsilon\,\frac{is}{4\pi}\omega\left(\delta m^i+\mu\delta v^i\right)\delta f_c^{(1)}+\varepsilon\,\varrho_h\delta^{ij}\delta b_j^{(2)(0)}+\varepsilon\,B^{(2)}_{\delta_{s_{it}},\delta_H}\,,
\end{aligned}\tag{2.4.12}$$

This form for $\delta\langle T^{ti}\rangle_{(2)}$ is equivalent to the one we listed in (2.4.11) because $P^{\mu}_{\delta s_{it},\delta_H}$ and $P^{\mu}_{\delta s_{ti},\delta_H}$ linearly combine to $P^{\mu}_{\delta v_i,\delta_H}$ which is source free.

The above expressions express the first dissipative correction to the ideal superfluid as functions of the our hydrodynamic variables as well as the horizon constants of integration $\delta^{ij}\delta g_{ij}^{(2)(0)}$ and $\delta b_t^{(2)(0)}$ for the correction $\delta f^{(2)}$ in our hydrodynamic expansion (2.3.22). Moreover, we have the appearance of integrals over the bulk which, as we will later see, are artifacts of the fluid frame. Moreover, the appearance of the constants of integration $\delta g_{tt}^{(2)(0)}+\delta g_{rr}^{(2)(0)}$ and $\delta g_{ti}^{(2)(0)}$ should not bother us as they can be considered as corrections to the local temperature δT and fluid velocity δv_i respectively. In other words, we will be able to absorb them in the definition of

δT and δv_i via a change of fluid frame.

In order to obtain similar expressions for the leading dissipative corrections $\delta\langle J^a\rangle$ of the electric current, we will apply equation (2.4.8) for the symplectic current $P^{\mu}_{\delta_{m_a},\delta_H}$. This allows us to write,

$$\begin{aligned}
\varepsilon\,\delta\langle J^t\rangle_{(2)} =& -\frac{i\omega}{4\pi}\xi\left(\delta^{kl}\delta s_{kl}-\delta s^{tt}\right)-\frac{i\omega}{4\pi s}\xi\left(\frac{c_\mu}{T}\delta T+\xi\delta\mu\right)\\
&+\frac{i\omega s}{4\pi}\left(\left(\partial_T\rho^{(0)}\partial_\mu\rho^{(0)}+\partial_T\phi^{(0)}\partial_\mu\phi^{(0)}\right)\delta T+\left(\left(\partial_\mu\rho^{(0)}\right)^2+\left(\partial_\mu\phi^{(0)}\right)^2\right)\delta\mu\right)\\
&-\varepsilon\,\partial_\mu\varrho_h\,\delta b_t^{(2)(0)}-\varepsilon\,\frac{T\xi}{2}\left(\delta g_{rr}^{(2)(0)}+\delta g_{tt}^{(2)(0)}\right)+\varepsilon\,B^{(2)}_{\delta_{m_t},\delta_H}\,,\\
\varepsilon\,\delta\langle J^i\rangle_{(2)} =& i\omega\left(\delta f_b^{(0)}\right)^2\left(\delta m^i+\mu_t\delta v^i\right)-\varepsilon\,\frac{4\pi}{s}\varrho_n\delta^{ij}\delta g_{tj}^{(0)}+\varepsilon\,B^{(2)}_{\delta_{m_i},\delta_H}\,.
\end{aligned} \tag{2.4.13}$$

Once again, the above expressions contain constants of integration for the perturbation which need to be fixed in terms of our hydrodynamic variables.

To achieve this, we will consider equation (2.4.8) for the symplectic currents $P^{\mu}_{\delta v^i,\delta_H}$ and $P^{\mu}_{\delta T,\delta_H}$ which don't contain any sources. Moreover, we will examine the radial component of the vector field equation of motion in (2.2.4) close to the black hole horizon. This step was also necessary in [2] and in [3] in order to eliminate the horizon degree of freedom.

By considering the symplectic current $P^{\mu}_{\delta v^i,\delta H}$, we obtain the equation,

$$\begin{aligned}
& i\omega\left(\delta f_b^{(0)}\right)^2\mu_t\left(\delta m_i+\mu_t\delta v_i\right)-\varepsilon\,\varrho_h\delta b_i^{(2)(0)}-\varepsilon\,\frac{4\pi}{s}\left(s\,T+\mu_t\varrho_n\right)\delta g_{ti}^{(2)(0)}\\
&+\frac{i}{4\pi T}\left(q_i\left(s\,T\,\delta^{kl}\delta s_{kl}+c_\mu\delta T+T\xi\,\delta\mu\right)+\omega\left(sT+\mu_t\varrho_n\right)\delta v_i\right)\\
&+\frac{i}{4\pi T}\left(\omega\varrho_n\,\delta m_i-\delta f_b^{(0)}\varrho_h\,\omega\left(\delta m_i+\mu_t\delta v_i\right)\right)\\
&-\varepsilon\,\frac{s}{4\pi}\left(\delta g_{ti}^{(2)(1)}-2\delta g_{ti}^{(2)(0)}g_{(1)}\right)-\frac{is}{16\pi^2T}\omega\,\delta f_c^{(1)}\left(\delta m_i+\mu_t\delta v_i\right)=\varepsilon\,B^{(2)}_{\delta_{v^i},\delta H}\,.
\end{aligned} \tag{2.4.14}$$

The above can used to show the equivalence between the expressions (2.4.12) and the one given in (2.4.11) after noting that,

$$B^{(2)}_{\delta_{v_i},\delta H}=B^{(2)}_{\delta_{s_{ti}},\delta H}-B^{(2)}_{\delta_{s_{it}},\delta H}\,. \tag{2.4.15}$$

Our next step is to consider equation (2.4.8) for the symplectic current $P^{\mu}_{\delta T,\delta H}$

yielding,

$$\begin{aligned}
&\frac{c_\mu^2}{sT^2}\,\omega\,\delta T+\frac{c_\mu\xi}{sT}\,\omega\,\delta\mu+\frac{c_\mu}{T}\omega\left(\delta^{ij}\delta s_{ij}-\delta s^{tt}\right)-\varepsilon\,2\pi i\,c_\mu\left(\delta g_{rr}^{(2)(0)}+\delta g_{tt}^{(2)(0)}\right)\\
&-\varepsilon\,4\pi i\,\partial_T\varrho_h\,\delta b_t^{(2)(0)}-\varepsilon\,8\,i\,\pi^2\,\delta^{ij}\delta g_{ij}^{(2)(0)}-s\omega\left(\left(\partial_T\rho^{(0)}\right)^2+\left(\partial_T\phi^{(0)}\right)^2\right)\delta T\\
&-s\,\omega\left(\partial_T\rho^{(0)}\partial_\mu\rho^{(0)}+\partial_T\phi^{(0)}\partial_\mu\phi^{(0)}\right)\delta\mu=\varepsilon\,\pi i\,B^{(2)}_{\delta_T,\delta H}\,. \qquad (2.4.16)
\end{aligned}$$

The above equation in combination with the radial component of the vector equation in (2.2.4) evaluated on the horizon,

$$\varepsilon\,\delta b_t^{(2)(0)}=\frac{4\pi i}{s\,q_e^2\left(\rho^{(0)}\right)^2}\left(\varrho_h q_i\left(\delta s^{it}+\delta v^i\right)-\omega\left(\varrho_h\delta^{ij}\delta s_{ij}+\partial_T\varrho_h\delta T+\partial_\mu\varrho_h\delta\mu\right)\right)\,, \qquad (2.4.17)$$

can be used to eliminate the constants $\delta^{ij}\delta g_{ij}^{(2)(0)}$ and $\delta b_t^{(2)(0)}$ from the constitutive relations (2.4.11) and (2.4.13). By eliminating those we can express all our constitutive relations in terms of the hydrodynamic variables δT, δv_i and $\delta\theta_{(v)}$. However, there is still a set of bulk integrals which are not equal to zero. By performing the change of frame given by,

$$\begin{aligned}
\delta T&\to\delta T-\frac{\varepsilon}{c_\mu+\mu_t\,\xi}\left(B^{(2)}_{\delta_{s_{tt}},\delta H}-T\,B^{(2)}_{\delta_T,\delta H}\right)\,,\\
\delta v^i&\to\delta v^i-\frac{\varepsilon}{s\,T+\mu_t\,\varrho_n}B^{(2)}_{\delta_{s_{ti}},\delta H}\,, \qquad (2.4.18)
\end{aligned}$$

we can eliminate the bulk integrals from the dissipative corrections $\delta\langle T^{ti}\rangle_{(2)}$ and $\delta\langle T^{tt}\rangle_{(2)}$. However, the change of fluid frame will make them appear in the dissipative corrections $\delta\langle T^{ij}\rangle_{(2)}$, $\delta\langle J^t\rangle_{(2)}$ and $\delta\langle J^i\rangle_{(2)}$. The important observation is that the combinations,

$$\begin{aligned}
-\frac{s}{c_\mu+\mu_t\,\xi}\left(B^{(2)}_{\delta_{s_{tt}},\delta H}-T\,B^{(2)}_{\delta_T,\delta H}\right)\delta^{ij}+B^{(2)}_{\delta_{s_{ij}},\delta H}&=0\,,\\
-\frac{\xi}{c_\mu+\mu_t\,\xi}\left(B^{(2)}_{\delta_{s_{tt}},\delta H}-T\,B^{(2)}_{\delta_T,\delta H}\right)+B^{(2)}_{\delta_{m_t},\delta H}&=0\,,\\
-\frac{\varrho_n}{s\,T+\mu_t\,\varrho_n}B^{(2)}_{\delta_{s_{ti}},\delta H}+B^{(2)}_{\delta_{m_i},\delta H}&=0\,, \qquad (2.4.19)
\end{aligned}$$

that would appear, are actually equal to zero. This is a non-trivial result which in order to be shown, requires the use of the constraints that we list in Appendix A along with equations of motion of the ideal fluid (2.3.25).

After cancelling out the bulk integrals, we land in a non-standard fluid frame which we need to change in order to bring our constitutive relations to a more conventional form. In the next subsection we will rewrite our theory of hydrodynamics in the so called transverse frame. This will help us read off the transport coefficients without resorting to the relevant Kubo formulae.

2.4.2 The Transverse Fluid Frame

In this subsection we wish to bring our constitutive relations to a form compatible with a well established frame in the literature. In the transverse frame that we wish to consider, the constitutive relations of our system can be decomposed to ideal and dissipative pieces according,

$$\begin{aligned} \langle T^{ab} \rangle &= \langle T^{ab} \rangle_{ideal} + \langle T^{ab} \rangle_{diss} \,, \\ \langle J^{a} \rangle &= \langle J^{a} \rangle_{ideal} + \langle J^{a} \rangle_{diss} \,, \end{aligned} \tag{2.4.20}$$

with the perturbations of the ideal parts being given by the $\mathcal{O}(\varepsilon)$ terms of equations (2.3.23). The constraints that fix the transverse frame that we wish to consider are,

$$u_a \, \langle T^{ab} \rangle_{diss} = 0 \,, \quad u_a \, \langle J^{a} \rangle_{diss} = 0 \,. \tag{2.4.21}$$

In a spacetime of $d+1$ spacetime dimensions, the above equations constitute $d+2$ constraints. These can be achieved by transforming the local temperature δT, chemical potential $\delta\mu$ and fluid velocity δv^a. Given the fact that the normal fluid velocity, u_a satisfies the normalisation condition $u^2 = -1$, we have exactly $d+2$ variables to achieve the conditions (2.4.21).

It can be shown that the most general form for the constitutive relations after imposing these constraints takes the form,

$$\langle T_{ab} \rangle_{diss} = -\, \eta \, \sigma^{n}_{ab} - \zeta_1 \, P_{ab} \, \nabla_c u^c - \zeta_2 \, P_{ab} \, \nabla_c \left(\varrho_s n^c \right) \,,$$

$$\langle J_a\rangle_{diss} = -\,\sigma\, P_{ab}\,\left(\nabla^b\left(\frac{\mu}{T}\right) - \frac{1}{T}\,F^{bc}\,u_c\right)\,, \tag{2.4.22}$$

where we have defined,

$$\begin{aligned}\sigma^n_{ab} =&P_a{}^c P_b{}^d\left(\nabla_c u_d + \nabla_d u_c\right) - P_{ab}\,P^{cd}\,\nabla_c u_d\,,\\ n_a =&-\frac{1}{\mu_s}\,P_a{}^b\,m_b\,,\\ \mu_s =&u^a\,m_a\,,\end{aligned} \tag{2.4.23}$$

and the projection operator is as usual $P_{ab} = g_{ab} + u_a\,u_b$. In order to transform to this frame, we need to perform a redefinition of the local chemical potential to obtain a Josephson relation,

$$\mu = u^a\,m_a + \mu_{diss}\,, \tag{2.4.24}$$

with dissipative corrections given by,

$$\mu_{diss} = \zeta_3\,\nabla_c\left(\varrho_s\,n^c\right) + \zeta_2\,\nabla_c u^c\,. \tag{2.4.25}$$

Notice that even though the components $\langle T_{tt}\rangle_{diss}$ and $\langle J_t\rangle_{diss}$ are trivial, the fact that the chemical potential (2.4.24) contains dissipative terms it is enough for the ideal parts to lead to dissipation.

By linearising the expression we have in fluid and source perturbations, we obtain the constitutive relations,

$$\begin{aligned}\delta\langle T^{ij}\rangle_{diss} =&\eta\left(-2\left(\partial^{(i}\delta s^{j)t} + \partial^{(i}\delta v^{j)}\right) + \delta^{ij}\partial_k\left(\delta s^{kt} + \delta v^k\right) - 2\,\partial_t\,\delta s^{(ij)} + \delta^{ij}\,\delta^{kl}\partial_t\delta s_{kl}\right)\\ &-\zeta_1\delta^{ij}\left(\partial_k\left(\delta s^{kt} + \delta v^k\right) + \partial_t\delta s_{kl}\delta^{kl}\right) + \delta^{ij}\zeta_2\,\chi_{JJ}\,\partial_k\,\left(\delta m^k + \mu_t\,\delta v^k\right)\,,\\ \delta\langle J^i\rangle_{diss} =&\frac{\sigma}{T}\left(\partial^i\left(\delta m_t - \delta\mu\right) + \frac{\mu_t}{T}\,\partial^i\delta T - \partial_t\delta m^i\right)\,,\end{aligned} \tag{2.4.26}$$

with the linear piece of the dissipative part of the chemical potential reading,

$$\delta\mu_{diss} = \zeta_2\,\left(\partial_i\left(\delta s^{it} + \delta v^i\right) + \partial_t\delta^{ij}\delta s_{ij}\right) -\,\zeta_3\,\chi_{JJ}\,\partial_i\left(\delta m^i + \mu_t\,\delta v^i\right)\,. \tag{2.4.27}$$

In order to compare to the above form for the constitutive relations, we can use the relations (2.4.16) and (2.4.17) to express the constitutive relations (2.4.11) and (2.4.13) entirely in terms of our hydrodynamic variables. The relations (2.4.19) guarantee that the bulk integrals are a frame artifact and that they can be removed from our constitutive relations. After transforming to the transverse frame, our constitutive relations take the form (2.4.20) with dissipative parts (2.4.26) and (2.4.27) fixed by the incoherent conductivity and shear viscosity,

$$\sigma = \frac{s^2 T^3}{(sT + \mu_t \varrho_n)^2} \, (\delta f_b^{(0)})^2 \, \tau^{(0)} \, ,$$
$$\eta = \frac{s}{4\pi} \, . \tag{2.4.28}$$

In order to express the three bulk viscosities ζ_i in terms of horizon data and thermodynamic quantities, it is convenient to make the change of background thermodynamic variables $(T, \mu_t) \to (s, \varrho)$. For any quantity F, using the chain rule, the thermodynamic derivatives are connected through,

$$\partial_{\mu_t} F = \chi_{QQ} \, \partial_\varrho F + \xi \, \partial_s F \, , \tag{2.4.29}$$
$$\partial_T F = \xi \, \partial_\varrho F + \frac{c_\mu}{T} \, \partial_s F \, . \tag{2.4.30}$$

After performing this change of variables, the three bulk viscosities take the form,

$$\begin{aligned}
\zeta_1 =& \frac{s}{4\pi} \left(\left(s \, \partial_s \phi^{(0)} + \varrho \, \partial_\varrho \phi^{(0)} \right)^2 + \left(s \, \partial_s \rho^{(0)} + \varrho \, \partial_\varrho \rho^{(0)} \right)^2 \right) \\
&+ \frac{4\pi}{s \, q_e^2 \left(\rho^{(0)} \right)^2} \left(\varrho_h - \varrho \, \partial_\varrho \varrho_h - s \, \partial_s \varrho_h \right)^2 \, , \\
\zeta_2 =& \frac{s}{4\pi} \left(\partial_\varrho \phi^{(0)} \left(s \, \partial_s \phi^{(0)} + \varrho \, \partial_\varrho \phi^{(0)} \right) + \partial_\varrho \rho^{(0)} \left(s \, \partial_s \rho^{(0)} + \varrho \, \partial_\varrho \rho^{(0)} \right) \right) \\
&- \frac{4\pi}{s \, q_e^2 \left(\rho^{(0)} \right)^2} \partial_\varrho \varrho_h \left(\varrho_h - \varrho \, \partial_\varrho \varrho_h - s \, \partial_s \varrho_h \right) \, , \\
\zeta_3 =& \frac{s}{4\pi} \left(\left(\partial_\varrho \phi^{(0)} \right)^2 + \left(\partial_\varrho \rho^{(0)} \right)^2 \right) + \frac{4\pi}{s \, q_e^2 \left(\rho^{(0)} \right)^2} \left(\partial_\varrho \varrho_h \right)^2 \, ,
\end{aligned} \tag{2.4.31}$$

where ϱ_h is the horizon charge density we defined in equation (2.2.15). From these expressions obviously $\zeta_1 \geq 0$. Also, using the Schwarz inequality we can show that[1] $\zeta_1 \zeta_3 \geq \zeta_2^2$.

[1]This corrects a typo in the relation previously given in [31] which was based on the positivity of entropy productioon.

In the case of a conformal superfluid, the neutral scalar ϕ vanishes. In addition, as we are going to explain below, $\rho^{(0)}$ and ϱ_h are homogeneous functions of s and ϱ. More specifically,

$$\rho^{(0)} = f_1\left(\frac{s}{\varrho}\right), \quad \varrho_h = \varrho\, f_2\left(\frac{s}{\varrho}\right), \tag{2.4.32}$$

for arbitrary functions f_1, f_2, and as a result

$$\begin{aligned} s\,\partial_s\rho^{(0)} + \varrho\,\partial_\varrho\rho^{(0)} &= 0\,, \\ s\,\partial_s\varrho_h + \varrho\,\partial_\varrho\varrho_h &= \varrho_h\,. \end{aligned} \tag{2.4.33}$$

Using (2.4.33) in (2.4.31) we can immediately arrive at the well-known fact [31] that for a conformal superfluid ζ_1 and ζ_2 are zero, whereas ζ_3 is positive definite.

Let's now prove, from the bulk point of view, our statement (2.4.32). In the absence of a neutral scalar, with a short counting argument we can see that, out of all the integration constants in the UV and IR expansions (2.2.6) and (2.2.7), only two are independent[2]. We can take these two to be the entropy density s (given by (2.2.14)) and the charge density ϱ. Based on that, we can only argue that $\rho^{(0)}$ and ϱ_h are functions of s, ϱ.

Moreover, the ansatz (2.2.5) has the scaling symmetry [117]

$$r \to \lambda r\,, \quad (t,x,y) \to (t,x,y)/\lambda\,, \quad U \to \lambda^2 U\,, \quad e^{2g} \to \lambda^2 e^{2g}\,, \quad a \to \lambda a\,. \tag{2.4.34}$$

In other words, if (2.2.5)(with trivial ϕ) is a solution of the equations of motion, so is

[2]Fixing the source of the charged scalar to zero is understood.

$$ds^2 = -\lambda^2 U(r/\lambda)\, dt^2 + \frac{dr^2}{\lambda^2 U(r/\lambda)} + \lambda^2\, e^{2g(r/\lambda)} \left(dx^2 + dy^2\right) ,$$
$$B = \lambda\, a(r/\lambda)\, dt, \quad \rho = \rho(r/\lambda)\,. \tag{2.4.35}$$

By expanding in the UV and the IR, we can check that the solution (2.4.35) has charge density $\hat{\varrho} = \lambda^2 \varrho$, entropy density $\hat{s} = \lambda^2 s$, horizon charge density $\hat{\varrho_h} = \lambda^2 \varrho_h$ and value of the amplitude at the horizon $\hat{\rho^{(0)}} = \rho^{(0)}$. Consequently, we have

$$\rho^{(0)}(\lambda^2\, s, \lambda^2\, \varrho) = \rho^{(0)}(s, \varrho)$$
$$\varrho_h(\lambda^2\, s, \lambda^2\, \varrho) = \lambda^2 \varrho_h(s, \varrho)\,, \tag{2.4.36}$$

and thus (2.4.32) follows.

2.5 Limits

In this following subsections we will consider the limit of our hydrodynamics close to the phase transition as well as at zero chemical potential. This is possible because of our explicit expressions for the five transport coefficients in terms of horizon data.

2.5.1 Near Critical Point

Here, we wish to consider the limit of the bulk viscosities and of the hydrodynamic modes close to the phase transition, as we approach it from the broken phase.

Bulk viscosities

In order to take the limit close to the critical point, we can vary either the background chemical potential μ_t, the temperature T or the scalar deformation parameter $\phi_{(s)}$. In the three dimensional space spanned by $T\,, \mu_t,\ \phi_{(s)}$, in general we expect a two dimensional surface of critical points. Suppose that we approach a specific critical point of this surface along a curve $\left(T(\epsilon)\ , \mu_t(\epsilon)\ , \phi_{(s)}(\epsilon)\right)$ with parameter ϵ defined so that $\epsilon = 0$ corresponds to the critical point.

Sufficiently close to the critical point, based on mean field arguments we can write[3]

$$\rho = \epsilon\,\rho_1 + \epsilon^3\,\rho_3 + \cdots\,, \tag{2.5.1}$$

for the bulk scalar dual to the amplitude of the complex order parameter. Similarly for its value at the horizon

$$\begin{aligned} \rho^{(0)} &= \epsilon\,\rho_1^{(0)} + \epsilon^3\,\rho_3^{(0)} + \cdots\,, \\ \partial_\varrho\rho^{(0)} &= \frac{d_{\varrho,-1}}{\epsilon} + d_{\varrho,1}\epsilon + \cdots\,, \\ \partial_s\rho^{(0)} &= \frac{d_{s,-1}}{\epsilon} + d_{s,1}\epsilon + \cdots\,. \end{aligned} \tag{2.5.2}$$

The quantities s, ϱ, ϱ_h as well as the thermodynamic derivatives $\partial_\varrho\varrho_h, \partial_s\varrho_h, \partial_\varrho\phi^{(0)}, \partial_s\phi^{(0)}$ are expected to behave regularly close to the critical point, i.e. as

$$c_0 + c_2\,\epsilon^2 + c_4\,\epsilon^4 + \cdots\,. \tag{2.5.3}$$

Finally, a simple analysis of the gauge field equation of motion in (2.2.4) reveals that close to the transition we can write $\chi_{JJ} = c_{JJ}\,\epsilon^2 + \cdots$.

The above expansions together with the expressions (2.4.31) yield the following singular behaviour for the bulk viscosities,

$$\zeta_i = \frac{\zeta_{i,-2}}{\epsilon^2} + \zeta_{i,0} + \cdots\,, i = 1, 2, 3. \tag{2.5.4}$$

[3]All the coefficients of the ϵ powers are constants, independent of ϵ.

Earlier studies regarding the divergence of the bulk viscosities in a holographic context can be found in [140, 31, 2]. In particular, the authors of [31], using the analytic solution of [135], find an expansion of the form (2.5.4) for ζ_3 close to the critical point. In order to reproduce their result using our general formula (2.4.31), we should highlight first some key points.

The analytic solution of [135] is found in the probe limit, with the background metric fixed to be five-dimensional Schwarzschild-AdS (i.e. (1.4.2) with $d = 4$). In five bulk dimensions (2.2.14) and (2.2.15) become[4]

$$\begin{aligned} s &= \frac{2\pi}{\kappa_5^2}\, e^{3g^{(0)}} = \frac{2\pi}{\kappa_5^2\, b^3}\,, \\ \varrho_h &= \frac{1}{2\kappa_5^2} e^{3g^{(0)}} a^{(0)} = \frac{a^{(0)}}{2\kappa_5^2 b^3}\,. \end{aligned} \tag{2.5.5}$$

In the second equalities we specialise[5] to the Schwarzschild-AdS spacetime of [31] with horizon at $r = \frac{1}{b}$.

As explained in section 4.5 of [31] the analytic solution close to the critical point is given by the expansions

$$\begin{aligned} B_t &= \frac{2}{q_e\, b} G_0^{(\bar{0})}(rb) + \frac{1}{q_e\, b} G_0^{(\bar{2})}(rb)\epsilon^2 + \frac{1}{q_e\, b} G_0^{(\bar{4})}(rb)\epsilon^4 + \dots\,, \\ \rho &= \frac{1}{q_e}\rho^{(\bar{1})}(br)\epsilon + \frac{1}{q_e}\rho^{(\bar{3})}(br)\epsilon^3 + \dots\,, \end{aligned} \tag{2.5.6}$$

with expansion parameter $\epsilon = \sqrt{\mu\, b\, q_e - 2}$ and the functions at each order given in Appendix C of [31]. From (2.5.6) one can easily find the horizon quantities $\rho^{(0)}$, ϱ_h and the charge density ϱ, expanded in powers of ϵ.

Moreover, since the analytic solution refers to a conformal superfluid we have $\zeta_1 = \zeta_2 = 0$ and using the chain rule,

[4]In [31] they use the same action (2.2.3) with $\tau = 1$ and with a factor $\frac{1}{2\kappa_5^2}$ in front.
[5]Note that equation (5.6) of [31] for the entropy density has a typo.

$$\zeta_3 = \frac{s}{4\pi}\left(\partial_\epsilon \rho^{(0)} \frac{1}{\frac{\partial \varrho}{\partial \epsilon}}\right)^2 + \frac{4\pi}{s\, q_e^2\, (\rho^{(0)})^2}\left(\partial_\epsilon \varrho_h \frac{1}{\frac{\partial \varrho}{\partial \epsilon}}\right)^2 . \tag{2.5.7}$$

Finally, substituting in this expression the results that follow from (2.5.6) yields

$$\zeta_3 = b^3 \kappa_5^2 \left(\frac{13}{294\,\epsilon^2} + \frac{21821 - 37152\ln(2)}{49392} + \mathcal{O}(\epsilon^2)\right) , \tag{2.5.8}$$

which agrees with equation (1.3) of [31], upon identifying the Hawking temperature $T = \frac{1}{\pi\, b}$.

Hydrodynamic modes

The first task is to find the hydrodynamic modes of our superfluid away from the critical point and then take the limit. More concretely, we would like to turn off the sources δs_{ab} for the metric and the gauge field $\delta\mu_a$ and solve the Ward identities (2.2.19) given the constitutive relations in the transverse frame that we discussed in section 2.4.2. Without loss of generality, we can take the wavevector of the fluctuation to lie entirely on the x^1 axis with components $k_1 = \varepsilon\, q$ and $k_2 = 0$ with ε the hydrodynamic expansion parameter. The goal is to fix the dispersion relations of the relevant quasi-normal modes whose frequency we can expand according to,

$$\omega = \omega_{[1]}\,\varepsilon + \omega_{[2]}\,\varepsilon^2 + \cdots . \tag{2.5.9}$$

In order to find the modes we are after, we Fourier expand our hydrodynamic variables according to,

$$\begin{aligned}
\delta T &= e^{-i\omega\, t + i\varepsilon q x^1}\, \varepsilon\left(\delta T_0 + \varepsilon\, \delta T_1 + \cdots\right) ,\\
\delta\theta_{(v)} &= e^{-i\omega\, t + i\varepsilon q x^1}\, \left(\delta\theta_0 + \varepsilon\, \delta\theta_1 + \cdots\right) ,\\
\delta v^i &= e^{-i\omega\, t + i\varepsilon q x^1}\, \varepsilon\left(\delta v_0^i + \varepsilon\, \delta v_1^i + \cdots\right) .
\end{aligned} \tag{2.5.10}$$

The leading part of the modes along with $\omega_{[1]}$ are determined by the ideal superfluid equations of motion (2.3.25). This leads to a linear algebraic system for the constants δT_0, $\delta\theta_0$ and δv_0^i which is trivial unless the frequency satisfies the zero determinant condition,

$$\omega_{[1]}\Big(\left(s\,T+\mu_t\,\varrho_n\right)\Big(T\,\xi^2-c_\mu\,\chi_{QQ}\Big)\,\omega_{[1]}^4-s^2\,T\,\chi_{JJ}\,q^4 \\ +\left(c_\mu\,\varrho\,\varrho_n+s\,T\left(s\,\chi_{QQ}+\left(c_\mu+\mu_t\,\xi\right)\chi_{JJ}-\left(\varrho+\varrho_n\right)\xi\right)\right)\omega_{[1]}^2\,q^2\Big)=0\,. \quad (2.5.11)$$

By solving the above equation for $\omega_{[1]}$, we fix the leading piece of the dispersion relations we are after. The first solution trivially yields,

$$\omega_{[1]}^{shear}=0, \quad (2.5.12)$$

which is a transverse mode with $\delta v_0^1=0$. This is essentially the shear mode describing the diffusion of momentum along a transverse direction.

The remaining four modes come in two pairs corresponding to the first and the second sounds modes of the superfluid. Here, we will only be interested in the limit of the dispersion relations near the critical point. For the first sound we have the asymptotic behaviour,

$$(\omega_{[1]}^{f.s.})^2=\frac{Z}{\left(s\,T+\mu_t\,\varrho\right)\left(c_\mu\,\chi_{QQ}-T\,\xi^2\right)}\,q^2+\cdots\,, \quad (2.5.13)$$

where for convenience we have defined the quantity,

$$Z=s^2\,T\,\chi_{QQ}+c_\mu\,\varrho^2-2s\,T\,\xi\,\varrho\,. \quad (2.5.14)$$

leading to a finite speed of sound in the small ϵ limit. In contrast, for the second sound we have the leading piece of the dispersion relation,

$$(\omega_{[1]}^{s.s.})^2=\frac{\chi_{JJ}\,s^2\,T}{Z}\,q^2+\cdots \quad (2.5.15)$$

with a speed of sound approaching zero close to the critical point.

After imposing the vanishing of the determinant of the leading order linear system of equations, the allowed modes are determined by the kernel of the linear operator multiplying the vector with components δ_0, $\delta\theta_0$ and δv_0^i. By expanding the Ward identities up to order ε^3, we can determine the higher order constants in the hydrodynamic expansion (2.5.10). However, in order for this to be possible we find that the $\omega_{[2]}$ part of the frequency expansion needs to satisfy an algebraic condition which completely fixes it in terms of $\omega_{[1]}$.

More specifically, for the shear mode we have the correction,

$$\omega_{[2]}^{shear} = -i \frac{\eta}{T\, s + \mu_t\, \varrho} q^2 \,, \tag{2.5.16}$$

leading to a purely diffusive mode, as expected. For the leading correction to the first sound mode we obtain,

$$\omega_{[2]}^{f.s.} = -\frac{i}{2} \frac{\zeta_1}{s\, T + \mu_t\, \varrho} q^2 + \cdots \,, \tag{2.5.17}$$

showing the the attenuation blows up close to the critical point for fixed wavenumber. This is due the fact that the bulk viscosity of relativistic fluids blows up close to the critical point when approached from the broken phase. This has been discussed before in the context of simpler systems [3] with scalar order parameters.

However, for the second sound we find the limiting behaviour,

$$\begin{aligned} \omega_{[2]}^{s.s.} = -\frac{i}{2} \Bigg(& \zeta_3\, \chi_{JJ} + \frac{\zeta_1\, \chi_{JJ}\, (s\, T\, \xi - c_\mu\, \varrho)^2}{Z^2} \\ & + \frac{2\, T^2\, \zeta_2\, \chi_{JJ}\, (s\, T\, \xi - c_\mu\, \varrho) + (s\, T + \mu_t\, \varrho)^2\, \sigma}{Z\, T^2} \Bigg) q^2 + \cdots \end{aligned} \tag{2.5.18}$$

for the leading attenuation part showing that it remains finite close to the phase transition even though the bulk viscosities seem to blow up close to the transition according to (2.5.4). This is in parallel to the observation first made in [2] for holographic superfluids at zero chemical potential. In fact, by taking the zero chemical potential limit, we can match the limiting behavious of the attenuation (2.5.18) to the expression of [2].

2.5.2 Zero Chemical Potential

It is interesting to consider the zero chemical potential limit of our hydrodynamics and compare with our results in [2]. For the classes of holographic theories we consider, it is easy to see that the background scalar fields ϕ and ρ are even under changing the sign of either the chemical potential or the charge density ϱ. This suggests that at zero chemical potential and therefore zero charge density, we must have vanishing $\partial_\varrho \phi^{(0)}$ and $\partial_\varrho \rho^{(0)}$. Moreover, the horizon charge density ϱ_h and the thermodynamic susceptibity ξ are identically zero for any value of the entropy of the system. This shows that $\zeta_2 = 0$ while ζ_1 becomes the bulk viscosity of relativistic holographic fluids [137, 3]. The final expressions for the bulk viscosities in the zero chemical potential limit are,

$$
\begin{aligned}
\zeta = \zeta_1 &= \frac{s^3}{4\pi}\left(\left(\partial_s \phi^{(0)}\right)^2 + \left(\partial_s \rho^{(0)}\right)^2\right) = \frac{s}{4\pi}\left(\frac{Ts}{c_\mu}\right)^2 \left(\left(\partial_T \phi^{(0)}\right)^2 + \left(\partial_T \rho^{(0)}\right)^2\right)\,, \\
\zeta_2 &= 0\,, \\
\zeta_3 &= \frac{2\pi}{s\, q_e^2\, \left(\rho^{(0)}\right)^2}\left(\partial_\varrho \varrho_h\right)^2 = \frac{2\pi}{s\, q_e^2\, \left(\rho^{(0)}\right)^2 \chi_{QQ}^2}\left(\partial_\mu \varrho_h\right)^2\,. \qquad (2.5.19)
\end{aligned}
$$

In order to compare with the constitutive relations of [2] for the electric current, we will express our constitutive relations in terms of the phase $\delta\theta_{(v)}$. Combining our expressions (2.4.20) along with (2.4.22), (2.3.23) and (2.4.27) we obtain,

$$
\begin{aligned}
\delta\langle J^t\rangle &= \chi_{QQ}\,\delta\mu = \chi_{QQ}\,\partial_t\delta\theta_{(v)} - \chi_{QQ}\,\chi_{JJ}\,\zeta_3\,\partial_i\partial^i\,\delta\theta_{(v)} = \chi_{QQ}\,\partial_t\delta\theta_{(v)} - \chi_{QQ}^2\,\zeta_3\,\partial_t^2\,\delta\theta_{(v)}\,, \\
\delta\langle J^i\rangle &= -\chi_{JJ}\,\partial^i\theta_{(v)} - \frac{\sigma}{T}\,\partial^i\partial_t\theta_{(v)} \qquad (2.5.20)
\end{aligned}
$$

where in the first line we used the ideal superfluid equations of motion (2.3.25). We see that the above agrees with [2] after matching $\Xi = \chi_{QQ}^2\,\zeta_3$ and $\sigma_d = \sigma/T$, noting that $\delta c = q_e\,\delta\theta_{(v)}$. To make the comparison more precise we note that for small chemical potential $\delta\varrho_h|_{here} = e^{2g^{(0)}}\tau^{(0)}a_t^{(0)}\,\delta\mu_t\big|_{there}$. For the thermodynamic supercurrent perturbation we have $\delta f_b^{(0)}\big|_{here} = a_x^{(0)}\big|_{there}$. Moreover, due to different normalisation of the bulk scalar we have $\rho^{(0)}\big|_{here} = \sqrt{2}\,\rho^{(0)}\big|_{there}$.

2.6 Discussion

In this paper we have used the techniques that were recently developed in [2, 130, 3] to study the hydrodynamic limit of fluctuations in a holographic superfluid at finite chemical potential. Based on general arguments, we expected organise the long wavelength limit of the stress tensor and the electric current in terms of a derivative expansion of appropriate hydrodynamic variables which have a clear interpretation in the infinite wavelength, thermodynamic limit. After fixing a specific fluid frame, we would then expect the leading dissipative corrections to be parametrised by a set of independent transport coefficients. For a relativistic fluid which preserves homogeneity and isotropy, we expect five independent coefficients, the incoherent conductivity σ, the shear viscosity η and the three bulk viscosities ζ_i.

An important by-product of our derivation was the explicit expressions for the dissipative transport coefficients in equations (2.4.28) and (2.4.31) in terms of thermodynamics and the black hole horizon data. Our results confirm [100, 136, 31] that in the leading gravitational limit and while preserving translations and isotropy, the shear viscosity η is fixed by the entropy density of the theory according to (2.4.28). The results for the incoherent conductivity and the bulk viscosities are new as far as we know.

As an application of our results, in section 2.5.1 we studied the limit of the hydrodynamic fluctuations close to the critical temperature. As we saw, the bulk viscosities ζ_i blow up close to the critical, signalling the breakdown of the hydrodynamic expansion. The two longitudinal sound modes behave differently close to the critical temperature. The first sound behaves in a way similar to the behaviour that was discusses earlier in [3]. The speed of sound remains finite while the attenuation blows up due to the leading behaviour of the bulk viscosity ζ_1. The second sound, which is due to the superfluid component, has a vanishing speed of sound close to the transition. However, the attenuation remains finite, similarly to what happens in superfluids at zero chemical potential [2].

Given the fact that our expressions for the transport coefficients are determined by data on the black hole horizon, it is possible to consider their low temperature limit for a given ground state geometry. Large classes of holographic ground states have been considered over the years in different works [141, 142, 143] in models which are sub cases of the general model in equation (2.2.1). It would be interesting to use our results to study the effects of dissipation in holographic superfluids at low temperatures by using these geometries.

The fact that the three bulk viscosities blow up close to the phase transition is due to the amplitude mode of the superfluid becoming exactly gapless. Below the phase transition this mode becomes gapped, joining the rest of the UV modes and we can safely integrate it out. In chapter 3 we are going to include this mode in the hydrodynamic description and obtain an effective theory which is valid up to energy scales which are equal to the gap of this universal mode and even beyond that. This is certainly possible given the recent progress that was made in [130].

Chapter 3

Nearly Critical Holographic Superfluids

This chapter is a reproduction of [6], written in collaboration with Aristomenis Donos.

In this paper, we study the nearly critical behaviour of holographic superfluids at finite temperature and chemical potential in their probe limit. This allows us to examine the coupled dynamics of the full complex order parameter with the charge density of the system. We derive an effective theory for the long wavelength limit of the gapless and pseudo-gapped modes by using analytic techniques in the bulk. We match our construction with Model F in the classification of Hohenberg and Halperin and compute the complex dissipative kinetic transport coefficient in terms of thermodynamics and black hole horizon data. We carry out an analysis of the corresponding modes which allows us to compare and contrast our results with earlier numerical work.

3.1 Introduction

The holographic duality provides a laboratory to analyse the behaviour of large classes of strongly coupled systems [59, 73]. In a certain large N limit, large classes of field theories become dual to classical theories of gravity. Using holography as a tool kit is particularly helpful in dealing with real time physics when finite temperature and chemical potential are involved [63]. More generally, holography is a powerful tool to study field theories deformed by relevant deformations.

The geometries dual to the field theory thermal states are black holes of Hawking temperature equal to the field theory temperature. In the most well understood case of conformal field theories, the bulk geometries asymptote to Anti de-Sitter space (AdS). The chemical potentials for the charges of global symmetries are fixed by the asymptotic behaviour of the gauge fields dual to the corresponding field theory Noether current operators. Likewise, the deformation parameters of other irrelevant operators are fixed by the boundary conditions of their bulk duals at the time-like conformal boundary of AdS.

In this chapter we will be particularly interested in the intersection of two areas that holography has already seen many applications. The first one is the study of thermal phase transitions and symmetry breaking. One of the most prominent examples is the superfluid phase transition which was pioneered in [116, 115]. With applications in condensed matter physics in mind, examples where spacetime symmetries are broken were also realised holographically [121, 122, 123, 124].

From the point of view of the bulk theory, continuous phase transitions are driven by perturbative instabilities which can lead to spontaneous symmetry breaking in the stable phase. In such a case, a new gapless mode appears in the theory, the dual of the Goldstone mode. At the same time, the mode which drives the transition acquires a small gap which closes to zero when approaching the critical point. This gapless collective degree of freedom is precisely the Higgs/amplitude mode.

The second area of applied holography that we will be interested in this paper is the effective theory governing the dynamics of low energy modes close to the critical point and incorporate the almost gapless Higgs mode. In the language of superfluids, the usual description of hydrodynamics away from the critical point captures the long wavelength behaviour of the phase of the order parameter [144, 145, 146, 147]. Our aim is to enlarge the effective theory to include fluctuations of its modulus.

Papers with similar questions have appeared in the past. However, they either involved models which can be solved exactly close to the critical point [135, 31], or numerical techniques [131, 132, 133, 134]. We chose to employ analytic techniques as we want to understand the universality of the underlying physics from a boundary theory point of view. The main tool in our construction will be the techniques we have recently developed in [3, 130, 2], as well as in chapter 2, to analyse dissipative effects in holographic theories. These will let us identify an equation of motion for the amplitude of the order parameter, a constitutive relation for the conserved electric current of the theory and a Josephson relation for a local chemical potential we will identify. In combination with the Ward identity for the global $U(1)$ of the theory, these will constitute a closed system of equations.

Later, we compare the resulting equations with those resulting from the Model F in the classification of Hohenberg and Halperin [4] and find exact agreement after a certain identification of the parameters in their model with our holographic results. As part of the matching procedure, we produce a holographic formula for the complex kinetic coefficient Γ_0 in terms of black hole horizon data and thermodynamic quantities of the state. It would be interesting to compare model F to holography beyond linear response.[1].

Using our effective theory equations, we analyse the behaviour of quasinormal modes in the broken phase. By incorporating the dynamics of the almost gapless mode, we are able to commute the limits of zero gap and infinite wavelength for the fluctuations. Moreover, we analyse interesting pole collisions in the complex plane that happen in the crossover region.

Even though the background of the broken phase is continuously connected to that of the normal phase, the mode for the fluctuations of the order parameter involve its

[1] See e.g. [148] for some recent numerical work in a direction along those lines.

phase in a singular manner close to the critical point. This was already evidenced from the analysis of [130], at infinite wavelengths. Interestingly, we find that the mode responsible for charge diffusion is continuous.

Finally, we carry out a few numerical checks in order to verify some of our analytic results. In particular, we focus on reproducing the dispersion relations for the quasinormal modes that our theory predicts. The model that we chose to apply our analysis to has been studied before in [131] and we chose to use exactly the same set of parameters that was used there. Both our analytical and numerical results indicate that the original suggestion of [131] regarding the "diffusion" constant of the pseudo-diffusive mode is not accurate for small wavevectors. Interestingly, it only holds true for wavevectors of norm much larger than the gap and below any other UV scale of the theory.

Our paper is organised in six main sections. In section 3.2 we present our holographic setup along with the necessary thermodynamics of the bulk geometries. In section 3.3 we employ our holographic techniques to extract all the necessary ingredients for our effective theory. In section 3.4 we state our theory in two equivalent ways and we write the constitutive relations of the current in terms of our hydrodynamic variables. In a separate subsection, we carry out the comparison with Model F of [4]. In section 3.5 we examine the behaviour of the quasinormal modes of our system in various limits. Section 3.6 is devoted to our numerical checks. We conclude with some discussion and conclusions in section 3.7.

3.2 Setup

Our bulk theory will have to contain a complex scalar ψ which is dual to the operator $\mathcal{O}_\psi$ whose VEV will play the role of the order parameter in our system. The global $U(1)$ under which the boundary operator $\mathcal{O}_\psi$ transforms, corresponds to a local symmetry in the bulk gauged by the one-form A_μ. Moreover, we will include a relevant operator $\mathcal{O}_\phi$ which will introduce an additional deformation parameter ϕ_s.

As we will see, the phase transition we wish to study will be driven by either tuning the chemical potential μ or the deformation parameter $\phi_{(s)}$. Our results will be valid both at finite and at zero charge density.

For our purposes, it is sufficient to consider once again the bulk action (2.2.1). However, here we assume that our matter fields decouple from the metric and so, we can drop the Ricci scalar term from the bulk action. Since we will be primarily interested in the broken phase of our probe theory, the complex scalar ψ will be non-trivial in the bulk geometry. In this case, the field redefinitions $\psi = \rho\, e^{iq_e\theta}$ and $B_\mu = \partial_\mu\theta + A_\mu$ are legitimate and bring the bulk action to the form (2.2.3). The resulting equations of motion are the same[2] as (2.2.4) and we repeat them here for convenience,

$$\begin{aligned}
\nabla_\mu\nabla^\mu\rho - \partial_\rho V - q^2\rho B^2 - \frac{1}{4}\partial_\rho\tau F^2 &= 0\,,\\
\nabla_\mu\nabla^\mu\phi - \partial_\phi V - \frac{1}{4}\partial_\phi\tau F^2 &= 0\,,\\
\nabla_\mu(\tau F^{\mu\nu}) - q_e^2\rho^2 B^\nu &= 0\,.
\end{aligned} \tag{3.2.1}$$

For the bulk geometry dual to the thermal state, we will consider a general metric which preserves the Euclidean subgroup and time translations. Without any loss of generality, this is captured by the general metric,

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + e^{2g(r)}(dx^2 + dy^2)\,. \tag{3.2.2}$$

One can arrive to this background in a variety of ways and the details will not be important to our analysis. As we will see, what matters is the general properties of the background geometry (3.2.2).

The conformal boundary is at $r \to \infty$ and we can use the coordinate invariance of the background theory to fix the horizon $r = 0$. In the asymptotic region, the

[2] Apart from Einstein's equations, which are not relevant here since the metric decouples.

functions that appear in our metric can be taken to approach,

$$U(r) = (r+R)^2 + \cdots \,, \qquad g(r) = \ln(r+R) + \cdots \,. \tag{3.2.3}$$

We will set the Hawking temperature of the horizon to be T, fixing the near horizon Taylor expansion,

$$U(r) = 4\pi T r + \cdots \,, \qquad g(r) = g^{(0)} + \cdots \,. \tag{3.2.4}$$

In our construction we will consider background solutions of (3.2.1) that correspond to deforming the theory by a chemical potential μ and scalar deformation parameter ϕ_s. To achieve this, we will consider backgrounds with

$$\rho = \rho(r)\,, \qquad \phi = \phi(r)\,, \qquad B = B_t(r)\, dt\,. \tag{3.2.5}$$

Near the horizon at $r = 0$, regularity imposes the expansion,

$$\rho(r) = \rho^{(0)} + \cdots \,, \qquad \phi(r) = \phi^{(0)} + \cdots \,, \qquad B_t(r) = B_t^{(0)} r + \cdots \,, \tag{3.2.6}$$

where $\rho^{(0)}$, $\phi^{(0)}$ and $B_t^{(0)}$ are constants of integration which need to fixed.

Close to the conformal boundary at $r \to \infty$, our physical considerations suggest the power series expansions,

$$\begin{aligned} \rho(r) &= \frac{\rho_s}{(r+R)^{3-\Delta_\psi}} + \cdots + \frac{\rho_v}{(r+R)^{\Delta_\psi}} + \cdots \,, \\ \phi(r) &= \frac{\phi_s}{(r+R)^{3-\Delta_\phi}} + \cdots + \frac{\phi_v}{(r+R)^{\Delta_\phi}} + \cdots \,, \\ B_t(r) &= \mu - \frac{\varrho}{r+R} + \cdots \,, \end{aligned} \tag{3.2.7}$$

where the conformal dimensions Δ_ψ and Δ_ϕ of the dual operators $\mathcal{O}_\psi$ and $\mathcal{O}_\phi$ are fixed by the bulk masses according to $\Delta_\psi (\Delta_\psi - 3) = m_\psi^2$ and $\Delta_\phi (\Delta_\phi - 3) = m_\phi^2$. For the purposes of this paper, we will be setting the complex scalar source ρ_s equal to zero. In terms of the backgrounds, this will guarantee that we have no explicit breaking of the $U(1)$ symmetry. The constant of integration μ is the field theory chemical potential and ϱ is the corresponding charge density. It is worth noting, that our background thermal states will eventually be parametrised by the temperature T, the chemical potential μ and the scalar deformation parameter ϕ_s.

It is now worth describing the phase diagram of the system we are considering. Its precise details will depend on the parameters of our theory. For our purposes, we will be interested in the class of theories where a thermal phase transition does take place. In this case, for some fixed values of T and scalar deformation ϕ_s, there is a critical value for the chemical potential $\mu_c(T, \phi_s)$ above which we can find solutions with a source-free non-trivial ρ. These backgrounds correspond to the thermal states of the broken phase. The hypersurface $(T, \phi_s, \mu_c(T, \phi_s))$ defines a critical surface on which the energy difference between the broken and the normal phase is exactly zero. In the following sections, we will consider phase transitions which are driven by holding the T fixed and varying ϕ_s and μ.

3.2.1 Holographic Renormalisation and Thermodynamics

In this subsection we will present some of the thermodynamic properties of our system that will be useful in our construction. The first step in extracting meaningful quantities form the boundary theory point of view is holographic renormalisation [82]. Equally important is the fact that holographic renormalisation is crucial in order to make the variational problem well defined in the bulk [83]. In order to render the bulk action finite, a suitable set of boundary counterterms is required. The precise form can depend on the details of the theory but a universal set of counterterms is given by,

$$\begin{aligned} S_{bdr} = &-\frac{1}{2}\int_{\partial M} d^3x\,\sqrt{-\gamma}\,[(3-\Delta_\phi)\phi^2 - \frac{1}{2\Delta_\phi - 5}\,\partial_a\phi\partial^a\phi] \\ &-\frac{1}{2}\int_{\partial M} d^3x\,\sqrt{-\gamma}\,[(3-\Delta_\psi)|\psi|^2 - \frac{1}{2\Delta_\psi - 5}\,D_a\psi D^a\psi^*] + \cdots\,, \end{aligned} \tag{3.2.8}$$

where $\gamma_{\alpha\beta}$ is the induced metric on the asymptotic hypersurface ∂M of constant radial coordinate r. The higher order terms can include higher derivatives of the bulk fields which in our approximation become irrelevant.

In order to find the probe's contribution to the free energy of the system, one must Wick rotate to Euclidean time $\tau = i\,t$ and evaluate the total on-shell action $I_{tot} = I_b + I_{bdr}$. More precisely, since our system is infinite, one should evaluate the probe's contribution to the free energy density $w_{FE} = T\,I_{tot}$ in the $x - y$ plane.

In this paper, we will be ignoring the backreaction of the matter fields of our probe theory on the background geometry. This makes meaningful to keep the temperature T fixed even during the real time evolution of the system. Given that we are only considering variations of the chemical potential $\delta\mu$ and the scalar deformation parameter $\delta\phi_s$, the first law for the grand canonical free energy w_{FE} becomes[3],

$$\delta w_{FE} = -\varrho\,\delta\mu - \langle\mathcal{O}_\phi\rangle\,\delta\phi_s\,, \tag{3.2.9}$$

where $\varrho = \langle J^t\rangle$ is the charge density of the theory and $\langle\mathcal{O}_\phi\rangle$ is the VEV of the neutral scalar operator. In contrast to the electric charge, the expectation value $\langle\mathcal{O}_\phi\rangle$ is not a conserved quantity. However, from the statistical physics point of view we can still consider the variation of the free energy with respect to one of the couplings of the theory. Such variations would show up as extra terms in the first law like the last term in (3.2.9).

More generally, we can define the expectation value $\langle J^\mu\rangle$ of the conserved $U(1)$ current operator. For later reference, it will also be useful to define the thermodynamic susceptibilities through variations of ϱ and $\langle\mathcal{O}_\phi\rangle$ as functions of μ and ϕ_s,

$$\begin{aligned}\delta\varrho &= \chi_{QQ}\,\delta\mu + \nu_\mu\,\delta\phi_s\,,\\ \delta\langle\mathcal{O}_\phi\rangle &= \nu_\mu\,\delta\mu + \nu_\phi\,\delta\phi_s\,.\end{aligned} \tag{3.2.10}$$

Another quantity that will prove useful later is the horizon charge density,

$$\varrho_h = e^{2g^{(0)}}\tau^{(0)}B_t^{(0)}\,. \tag{3.2.11}$$

By using the equation of motion for the vector field in (3.2.1), one can show that

[3]We should note that in the presence of persistent superfluid currents, the first law contains additional terms [31, 26, 30] which we can ignore in our case.

in the normal phase, the horizon charge density is equal to the density ϱ. However, this is not true for the broken phase black holes since the bulk vector field becomes massive and Stokes' theorem doesn't apply.

We will follow very similar techniques to those of [3] and of the previous chapter. For this reason, we note that the VEVs around which we will construct our effect theory can be extracted from,

$$\begin{aligned}\langle J^\mu \rangle &= \lim_{r\to\infty} \frac{r^3}{\sqrt{-\gamma}} \left[\frac{\partial \mathcal{L}}{\partial(\partial_r B_\mu)} + \frac{\delta S'_{bdr}}{\delta B_\mu} \right] ,\\ \langle \mathcal{O}_\phi \rangle &= \lim_{r\to\infty} \frac{r^{\Delta_\phi}}{\sqrt{-\gamma}} \left[\frac{\partial \mathcal{L}}{\partial(\partial_r \phi)} + \frac{\delta S'_{bdr}}{\delta \phi} \right] ,\\ \langle \mathcal{O}_\psi \rangle &= \lim_{r\to\infty} \frac{2\, r^{\Delta_\psi}}{\sqrt{-\gamma}} \left[\frac{\partial \mathcal{L}}{\partial(\partial_r \psi^*)} + \frac{\delta S'_{bdr}}{\delta \psi^*} \right] ,\end{aligned} \tag{3.2.12}$$

where $\mathcal{L}$ is the Lagrangian density of the bulk action (2.2.3). The above formulae are going to be directly useful to us when we consider the symplectic current of the theory. Finally, we note that the electric current satisfies the Ward identity,

$$\nabla_a \langle J^a \rangle = \frac{q_e}{2i} \left(\lambda \, \langle \mathcal{O}_{\psi^*} \rangle - \lambda^* \, \langle \mathcal{O}_\psi \rangle \right) . \tag{3.2.13}$$

In the above, the parameter λ is the source for the complex scalar operator $\mathcal{O}_\psi$.

Apart from the thermodynamic quantities defined above, we would like to define the stiffness parameter w^{ij}, in a very similar way it was done in [149, 150]. One can imagine, that instead of the homogeneous background that we consider in this paper, we could have a more general family of background which break translations with a characteristic spatial wavevector k_i. For example these broken phase backgrounds would be driven by a static mode of the form,

$$\delta\rho = \delta\rho(r)\, \cos(k_1\, x + c_x)\, \cos(k_2\, y + c_y)\, , \tag{3.2.14}$$

where c_x and c_y are the zero modes of the Goldstone modes for translations. The corresponding backgrounds will then also be parametrised by the periods $2\pi/k_x$ and $2\pi/k_y$ and so will the free energy w_{FE}. Following very similar arguments with [150], we can easily show the bulk expression,

$$w^{ij} = \partial_{k_i}\partial_{k_j} w_{FE}\Big|_{k_i=0} = \delta^{ij}\int_0^\infty dr\,\rho^2(r) = \delta^{ij}\,\gamma\,. \tag{3.2.15}$$

A defining characteristic of the superfluid phase is the appearance of persistent supercurrents. The thermodynamic conjugate variables of these is the spatial components of the source for the electric currents $\delta\mu_i$, or more precisely the gauge invariant combination $\delta\mu_i + \partial_i\,\delta\theta_v$, with $\delta\theta_v$ the phase of the complex VEV $\langle\mathcal{O}_\psi\rangle$. This, can be read off from the asymptotic behaviour of the phase,

$$\delta\theta \approx (r+R)^{2\Delta_\psi - 3}\,\delta\theta_{(s)} + \cdots + \delta\theta_{(v)} + \cdots\,. \tag{3.2.16}$$

From the point of view of the variables we have chosen to work with, the asymptotics of the phase field are encoded in the asymptotic behaviour of the gauge invariant one-form field components along the field theory directions according to,

$$\delta B_\alpha = \frac{\partial_\alpha\delta\theta_{(s)}}{(r+R)^{3-2\,\Delta_\rho}} + \cdots + \delta m_\alpha + \cdots + \frac{\delta j_\alpha}{r+R} + \cdots\,, \tag{3.2.17}$$

where $m_\alpha = \partial_\alpha\theta_{(v)} + \delta\mu_\alpha$ is the gauge invariant combination of the sources.

The final thermodynamic quantity we would now like to discuss it the current susceptibility χ_{JJ}. If we wanted to consider all possible thermal states of our superfluids, we would have to include backgrounds in (3.2.5) which contain these supercurrents. In the present work, we wish to study the effective theory around states with zero supercurrents. However, these supercurrents will be relevant to us from a perturbative point of view, as they will be involved in the hydrodynamic modes we will consider. From the bulk point of view, our broken phase backgrounds admit a non-trivial perturbation for the bulk one-form field of the form,

$$\delta B^i = \delta B_i^i(r)dx^i \tag{3.2.18}$$

which behaves near the boundary as,

$$\delta B_i^i = \delta m_i - \frac{\chi_{JJ}\,\delta m_i}{r+R} + \cdots\,, \tag{3.2.19}$$

and χ_{JJ} is precisely the current susceptibility. Near the horizon, regularity imposes the behaviour,

$$\delta B^i_i = a^{(0)}_J \delta m_i + \cdots \,. \tag{3.2.20}$$

Is is useful to note that given the perturbation δB^i_i, the equation of motion for the gauge field gives the relation,

$$\chi_{JJ}\, \delta m_i = q_e^2 \int_0^\infty dr\, \rho^2\, \delta B^i_i \,. \tag{3.2.21}$$

Given the fact that close to the phase transition we have approximately $\delta B^i_i \approx \delta m_i$ everywhere in the bulk, using the above equation it is easy to argue that close to the phase transition we must have,

$$\gamma = \frac{\chi_{JJ}}{q_e^2} + \cdots \,. \tag{3.2.22}$$

We may have used holography to show this relation but the deeper reason is gauge invariance with respect to the external source for the current.

3.3 Extracting the effective theory

In this section we will extract all the necessary ingredients to construct our effective theory. In order to achieve this, we will follow a combination of techniques and arguments developed in [3, 130, 2], as well as in chapter 2.

3.3.1 Expansions near the critical point

An important ingredient of our construction, will be the expansions of the background solutions (3.2.5) around the critical point at $(\mu_c(\phi_s), \phi_s, T)$. At exactly the critical point, the perturbative equation of motion for the amplitude ρ in (3.2.1) admits a static solution $\delta\rho_{*(0)}$. We will denote the background field at that point by $B_t = a_c$ and $\phi = \phi_c$.

In order to establish our notation, we are moving away from the critical point according to,

$$\begin{aligned} \mu(\varepsilon) &= \mu_c(\phi_s, T) + \frac{\varepsilon^2}{2}\delta\mu_{\star(2)} + \cdots\,, \\ \phi_s(\varepsilon) &= \phi_s + \frac{\varepsilon^2}{2}\delta\phi_{s\star(2)} + \cdots\,, \end{aligned} \tag{3.3.1}$$

with ε a parametrically small number. In this notation, the parameters, $\delta\mu_{\star(2)}$ and $\delta\phi_{s\star(2)}$, define the direction that we move away from the critical point in the space of thermodynamic variables. At the same time, the background will have to change with ε accordingly. By expanding the equations of motion (3.2.1), we can establish that the correction for the background will admit an ε expansion of the form,

$$\begin{aligned} \rho &= \varepsilon\delta\rho_{\star(0)} + \frac{\varepsilon^3}{3!}\delta\rho_{\star(2)} + \cdots\,, \\ \phi &= \phi_c + \frac{\varepsilon^2}{2!}\delta\phi_{\star(2)} + \cdots\,, \\ B_t &= a_c + \frac{\varepsilon^2}{2!}\delta a_{\star(2)} + \cdots\,, \end{aligned} \tag{3.3.2}$$

along the broken phase. Following the steps of [130], it will also be useful to consider the expansion of our backgrounds along the normal phase as well. The notation we will use for this case is,

$$\begin{aligned} \mu(\varepsilon) &= \mu_c(\phi_s, T) + \frac{\varepsilon^2}{2}\delta\mu_{\#(2)} + \cdots\,, \\ \phi_s(\varepsilon) &= \phi_s + \frac{\varepsilon^2}{2}\delta\phi_{s\#(2)} + \cdots\,, \end{aligned} \tag{3.3.3}$$

with the corresponding expansion for the normal phase backgrounds,

$$\begin{aligned} \rho &= 0\,, \\ \phi &= \phi_c + \frac{\varepsilon^2}{2!}\delta\phi_{\#(2)} + \cdots\,, \\ B_t &= a_c + \frac{\varepsilon^2}{2!}\delta a_{\#(2)} + \cdots\,. \end{aligned} \tag{3.3.4}$$

From the point of view of the boundary theory, it is the asymptotic behaviour of the corrections that will be important. For the broken phase backgrounds we can write,

$$
\begin{aligned}
\delta\rho_{\star(0)} &= \frac{\delta\rho^{v}_{\star(0)}}{(r+R)^{\Delta_\psi}} + \cdots\,, \\
\delta\phi_{\star(2)} &= \frac{\delta\phi_{s\star(2)}}{(r+R)^{3-\Delta_\phi}} + \cdots + \frac{\delta\phi_{v\star(2)}}{(r+R)^{\Delta_\phi}} + \cdots\,, \\
\delta a_{\star(2)} &= \delta\mu_{\star(2)} - \frac{\delta\varrho_{\star(2)}}{r+R} + \cdots\,.
\end{aligned} \tag{3.3.5}
$$

In our analysis, we will also need the behaviour of these perturbations close to the black hole horizon at $r = 0$ which reads,

$$
\begin{aligned}
\delta a_{\star(2)} &= \delta a^{(1)}_{\star(2)} r + \cdots\,, \\
\delta\phi_{\star(2)} &= \delta\phi^{(0)}_{\star(2)} + \cdots\,, \\
\delta\rho_{\star(0)} &= \delta\rho^{(0)}_{\star(0)} + \cdots\,.
\end{aligned} \tag{3.3.6}
$$

For the normal phase expansion under the variations (3.3.3), we can write very similar expressions for both the asymptotic and the near horizon expansions.

By using the definitions the thermodynamic susceptibilities of equation (3.2.10) we can write the following relations,

$$
\begin{aligned}
\delta\varrho_{\star(2)} &= \chi^{\star}_{QQ}\,\delta\mu_{\star(2)} + \nu^{\star}_{\mu}\,\delta\phi_{s\star(2)} \\
\delta\langle\mathcal{O}\rangle_{\star(2)} &= \nu^{\star}_{\mu}\,\delta\mu_{\star(2)} + \nu^{\star}_{\phi}\,\delta\phi_{s\star(2)}
\end{aligned} \tag{3.3.7}
$$

for the broken phase. For the normal phase expansion we can write the corresponding relations,

$$
\begin{aligned}
\delta\varrho_{\#(2)} &= \chi^{\#}_{QQ}\,\delta\mu_{\#(2)} + \nu^{\#}_{\mu}\,\delta\phi_{s\#(2)}\,, \\
\delta\langle\mathcal{O}\rangle_{\#(2)} &= \nu^{\#}_{\mu}\,\delta\mu_{\#(2)} + \nu^{\#}_{\phi}\,\delta\phi_{s\#(2)}\,.
\end{aligned} \tag{3.3.8}
$$

In the hydrodynamic limit we can write the expressions,

$$
\delta\langle\mathcal{O}_\phi\rangle_{\star(2)} = (2\,\Delta_\phi - 3)\;\delta\phi_{v\star(2)}\,, \qquad \delta\langle\mathcal{O}_\phi\rangle_{\#(2)} = (2\,\Delta_\phi - 3)\;\delta\phi_{v\#(2)}\,, \tag{3.3.9}
$$

but for us, it is the asymptotic form of the symplectic current that will play a direct role in our analysis. Finally, it is worth noting for the normal phase we have the additional relation $\delta\varrho_{\#(2)} = e^{2g^{(0)}}\tau^{(0)}\delta a^{(1)}_{\#(2)}$. This is nothing but our earlier statement that in the normal phase the horizon charge density (3.2.11) is equal to the field theory one.

Another important part of our strategy is the set of static perturbations we will use in the Crnkovic-Witten symplectic current. Similarly to [130], the first static perturbation we will need is can be obtained from the broken phase background expansion (3.3.2). Taking a derivative with respect to ε we find the static perturbation,

$$\begin{aligned}\delta\rho^{\star} &= \delta\rho_{\star(0)} + \frac{\varepsilon^2}{2!}\delta\rho_{\star(2)} + \cdots\,,\\ \delta\phi^{\star} &= \varepsilon\,\delta\phi_{\star(2)} + \cdots\,,\\ \delta B_t^{\star} &= \varepsilon\,\delta a_{\star(2)} + \cdots\,.\end{aligned} \tag{3.3.10}$$

This fluctuation will play a double role in our construction. The first one is as we described above, it will be used as one of the two solutions in the symplectic current. The second role is that it will be used to construct the next to leading order hydrodynamic perturbation we wish to study with our effective theory. For the same reason, we will also consider the expansion along the normal phase (3.3.4) and take a derivative with respect to ε to find the perturbation,

$$\begin{aligned}\delta\rho^{\#} &= 0\,,\\ \delta\phi^{\#} &= \varepsilon\,\delta\phi_{\#(2)} + \cdots\,,\\ \delta B_t^{\#} &= \varepsilon\,\delta a_{\#(2)} + \cdots\,.\end{aligned} \tag{3.3.11}$$

The second static solution that we will use in the symplectic current is the perturbation for the supercurrent of equation (3.2.18). Since our effort is to extract information infinitesimally close to the critical point, we will also need to consider the ε expansion of that as well,

$$\delta B_i^i = \delta B^i_{i(0)} + \varepsilon^2\delta B^i_{i(2)} + \cdots\,,$$

$$\chi_{jj} = \varepsilon^2 \chi_{jj(2)} + \cdots \,,$$
$$a_J^{(0)} = 1 + \varepsilon^2 a_{J(2)}^{(0)} + \cdots \,, \tag{3.3.12}$$

where the zeroth order solution is simply $\delta B^i_{i(0)} = \delta m_i$, as can be seen by solving the one-form field equation of motion (3.2.1) in the normal phase with $\rho = 0$. This is exactly the argument we used to show the relation (3.2.22).

3.3.2 Hydrodynamic Perturbations

Before specialising to the low frequency, long wavelength fluctuations we are interested in, it is worth setting up the problem for a generic perturbation that depends on the field theory coordinates. By exploiting the translations in space and time, we will assume a Fourier decomposition of the form,

$$\delta\mathcal{F}(r,t,x) = e^{-i\omega(t+S(r))+i\varepsilon q x}\delta f(r)\,, \tag{3.3.13}$$

for bulk fields. The function $S(r)$ a function which behaves near the horizon as $S(r) = \frac{\ln r}{4\pi T} + \cdots$ and vanishes sufficiently fast at the boundary. We will consider the quasinormal modes in the longitudinal sector and we will ignore the y component of the gauge field as it is decoupled from the rest. Our goal is to describe the system near the critical point and for this reason we take the momentum to be of order $\varepsilon q \sim \sqrt{\delta\mu}, \sqrt{\delta\phi_s}$. Given the above behaviour for the function S near the horizon, imposing regular ingoing boundary conditions near the horizon leads to the expansions,

$$\delta\rho(r) = \delta\rho^{(0)} + \cdots \,,$$
$$\delta\phi(r) = \delta\phi^{(0)} + \cdots \,,$$
$$\delta b_x(r) = \delta b_x^{(0)} + \cdots \,,$$
$$\delta b_t(r) = \delta b_t^{(0)} + \cdots \,,$$
$$\delta b_r(r) = \frac{\delta b_t^{(0)}}{4\pi T r} + \cdots \,. \tag{3.3.14}$$

By following general arguments, the generic expansion of our fluctuations close to the conformal boundary is,

$$\begin{aligned}
\delta\rho(r) &= \frac{\delta\rho_s}{(r+R)^{3-\Delta_\psi}} + \cdots + \frac{\delta\rho_v}{(r+R)^{\Delta_\psi}} + \cdots\,,\\
\delta\phi(r) &= \frac{\delta\phi_s}{(r+R)^{3-\Delta_\phi}} + \cdots + \frac{\delta\phi_v}{(r+R)^{\Delta_\phi}}\,,\\
\delta b_x(r) &= (\delta\mu_x + i\,\varepsilon\, q\,\delta\theta_v) + \frac{\delta j_x}{r+R} + \cdots\,,\\
\delta b_t(r) &= (\delta\mu_t - i\,\omega\,\delta\theta_v) + \frac{\delta j_t}{r+R} + \cdots\,.
\end{aligned} \tag{3.3.15}$$

However, for the purposes of this paper we will be interested in the source free dynamics of the low energy modes. For this reason, we will aim to set the scalar and current sources $\delta\phi_s$, $\delta\rho_s$, $\delta\mu_a$ equal to zero.

By following arguments very similar to [130], we can show that the degrees of freedom we wish to describe are captured by the expansion,

$$\begin{aligned}
\omega &= \varepsilon^2\,\omega_{[2]} + \cdots\,,\\
\delta\rho &= \delta\tilde{\rho}_{(0)} + \varepsilon\,\delta\tilde{\rho}_{(1)} + \frac{\varepsilon^2}{2}\,\delta\tilde{\rho}_{(2)}\cdots\,,\\
\delta\phi &= \varepsilon\,\delta\tilde{\phi}_{(2)} + \cdots\,,\\
\delta b_t &= \varepsilon\,\delta\tilde{a}_{(2)} + \cdots\,,\\
\delta b_x &= \delta\tilde{b}_{x(0)} + \varepsilon\,\delta\tilde{b}_{x(1)} + \varepsilon^2\,\delta\tilde{b}_{x(2)} + \cdots\,,\\
\delta\theta_v &= \frac{1}{\varepsilon}\,\delta\tilde{\theta}_v + \delta\tilde{\theta}_{v(0)} + \cdots\,.
\end{aligned} \tag{3.3.16}$$

By expanding the equations of motion in ε, we can see that the only solution regular at the horizon for $\delta\tilde{b}_{x(0)}, \delta\tilde{b}_{x(1)}$ is just a constant. Moreover, the equations of motion for the fields $\{\delta\tilde{\rho}_{(0)}, \delta\tilde{a}_{(2)}, \delta\tilde{\phi}_{(2)}\}$ are solved by a linear combination of the following solutions $\{\delta\tilde{\rho}_{(0)} = 0, \delta\tilde{a}_{(2)} = \delta a_{\#(2)}, \delta\tilde{\phi}_{(2)} = \delta\phi_{\#(2)}\}$ and $\{\delta\tilde{\rho}_{(0)} = \delta\rho_{\star(0)}, \delta\tilde{a}_{(2)} = \delta a_{\star(2)}, \delta\tilde{\phi}_{(2)} = \delta\phi_{\star(2)}\}$. Finally, note that we can add a constant (everywhere in the bulk) to $\delta\tilde{a}_{(2)}$, which we will call for convenience $-i\omega_{[2]}\delta\theta_0$, and get another solution.

As a result we can write,

$$
\begin{aligned}
\delta\tilde{\rho}_{(0)} &= \delta a\,\delta\rho_{\star(0)}\,,\\
\delta\tilde{a}_{(2)} &= \delta a\,\delta a_{\star(2)} - \delta a_{\#(2)} - i\omega_{[2]}\delta\theta_0\,,\\
\delta\tilde{\phi}_{(2)} &= \delta a\,\delta\phi_{\star(2)} - \delta\phi_{\#(2)}\,,\\
\delta\tilde{b}_{x(0)} &= iq\,\delta\tilde{\theta}_v\,.
\end{aligned}
\tag{3.3.17}
$$

An important point is that the parameters $\delta\mu_{*(2)}$ and $\delta\phi_{*(2)}$ defining the broken phase variation in the above equations are identical to the variations of the background. This can be easily seen by inspection of the equations of motion. However, the variations $\delta\phi_{s\#(2)}$ and $\delta\mu_{\#(2)}$ are left undetermined by simply looking at the equations of motion.

Regarding the neutral scalar, the net deformation for the hydrodynamic perturbation is,

$$
\delta\tilde{\phi}_{s(2)} = \delta a\,\delta\phi_{s*(2)} - \delta\phi_{s\#(2)}\,. \tag{3.3.18}
$$

The requirement for zero total deformations suggests that $\delta\phi_{s\#(2)}$ should be such that $\delta\tilde{\phi}_{s(2)} = 0$. Finally, the variation parameter $\delta\mu_{\#(2)}$ will be determined by imposing the electric current conservation Ward identity (3.2.13). Note that the Ward identity does not provide any non-trivial information about the static perturbations of the backgrounds. However, it is going to be essential in fixing $\delta\mu_{\#(2)}$.

For the electric current of the theory, the electric current follows the ε expansion,

$$
\begin{aligned}
\delta j_t &= \varepsilon\,\delta j_{t[1]} + \cdots\,,\\
\delta j_x &= \varepsilon^2\,\delta j_{x[2]} + \cdots\,,
\end{aligned}
\tag{3.3.19}
$$

which is compatible with the expansion of the vector field in equation (3.3.20). The above to the identifications of the asymptotic data,

$$
\begin{aligned}
-i\omega_{[2]}\delta\tilde{\theta}_v &= \delta a\,\delta\mu_{\star(2)} - \delta\mu_{\#(2)} - i\omega_{[2]}\delta\theta_0\\
0 &= \delta a\,\delta\phi_{s\star(2)} - \delta\phi_{s\#(2)}
\end{aligned}
$$

$$
\begin{aligned}
\delta j_{t[1]} &= -\delta a\, \delta\varrho_{\star(2)} + \delta\varrho_{\#(2)} \\
\delta\tilde{\phi}_{v(2)} &= \delta a\, \delta\phi_{v\star(2)} - \delta\phi_{v\#(2)}\,.
\end{aligned}
\tag{3.3.20}
$$

At the same time, the above lead to the near horizon behaviour for the time component of the one-form field,

$$
\delta\tilde{a}_{(2)} = -i\omega_{[2]}\delta\theta_0 + (\delta a\, \delta a^{(1)}_{\star(2)} - \delta a^{(1)}_{\#(2)})\, r + \cdots\,. \tag{3.3.21}
$$

The radial component of the one-form field admits the ε expansion,

$$
\delta b_r = \varepsilon\delta\tilde{b}_{r(1)} + \frac{\varepsilon^2}{2}\delta\tilde{b}_{r(2)} + \cdots\,. \tag{3.3.22}
$$

Near the horizon, the equation of motion (3.2.1) and regularity (3.3.14) yields the constraint,

$$
i\omega_{[2]}\left(q^2\tau_c^{(0)}e^{-2g^{(0)}} + q_e^2\left(\delta\rho^{(0)}_{\star(0)}\right)^2\right)\delta\theta_0 = i\omega_{[2]}\,\delta\tilde{\varrho}_{h(2)}e^{-2g^{(0)}} + i\tau_c^{(0)}q^2\omega_{[2]}e^{-2g^{(0)}}\delta\tilde{\theta}_v\,. \tag{3.3.23}
$$

As we will later see, this is the first equation that will be part of our effective theory and it will play the role of a Josephson relation.

3.3.3 Symplectic current

In this section we will combine the analysis we have discussed so far with the techniques developed in chapter 2 and in [3, 130, 2]. For the theory described by the bulk action (2.2.3), in the probe limit the symplectic current density is given by,

$$
\mathcal{P}^{\mu}_{\delta_1,\delta_2} = \delta_1 B_\alpha\, \delta_2\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu B_\alpha)}\Big) + \delta_1\rho\, \delta_2\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\rho)}\Big) + \delta_1\phi\, \delta_2\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big) - (1\leftrightarrow 2)\,, \tag{3.3.24}
$$

where δ_1 and δ_2 denote any two perturbations which solve the equations of motion (3.2.1). Moreover, the asymptotic behaviour of the radial component gives,

$$
P^r_{\delta_1,\delta_2} = \frac{1}{r^3}\left(\delta_1\phi_s\,\delta_2\left(\sqrt{-\gamma}\,\langle\mathcal{O}_\phi\rangle\right) - \delta_2\phi_s\,\delta_1\left(\sqrt{-\gamma}\,\langle\mathcal{O}_\phi\rangle\right)\right)
$$

$$+\frac{1}{r^3}\left(\delta_1\rho_s\,\delta_2\left(\sqrt{-\gamma}\,\langle\mathcal{O}_\rho\rangle\right)-\delta_2\rho_s\,\delta_1\left(\sqrt{-\gamma}\,\langle\mathcal{O}_\rho\rangle\right)\right)$$
$$+\frac{1}{r^3}\left(\delta_1 m_a\,\delta_2\left(\sqrt{-\gamma}\,\langle J^a\rangle\right)-\delta_2 m_a\,\delta_1\left(\sqrt{-\gamma}\,\langle J^a\rangle\right)\right)+\cdots\,, \qquad (3.3.25)$$

where we have used the expressions in equation (3.2.12) along with the fact that we work in the hydrodynamic limit. The latter allows us to drop the derivatives terms in the counterterms of equation (3.2.8).

The property which is crucial in our construction is the fact that for any two perturbations which solve the equations of motion (3.2.1), the symplectic current density is divergence free,

$$\partial_\mu\mathcal{P}^\mu_{\delta_1,\delta_2}=0\,. \qquad (3.3.26)$$

We are going to construct two symplectic currents from our hydrodynamic and two static perturbations. The first one is $P_{\delta H,\delta_\star}$, which is made out of the static perturbation (3.3.10). The second one is $P_{\delta H,\delta_{m_x}}$, made out of the static perturbation (3.2.18).

Given the fact that we can Fourier expand our modes, we find it convenient to do the same for the components of the symplectic current density according to,

$$\mathcal{P}^\mu_{\delta_1,\delta_2}=e^{-i\omega(t+S(r))+i\varepsilon qx}P^\mu_{\delta_1,\delta_2}\,. \qquad (3.3.27)$$

The divergence free condition (3.3.26) gives,

$$-i\omega P^t_{\delta_1,\delta_2}+P^{r\prime}_{\delta_1,\delta_2}-i\omega S' P^r_{\delta_1,\delta_2}+i\varepsilon q P^x_{\delta_1,\delta_2}=0$$
$$\Rightarrow\ P^r_{\delta_1,\delta_2}\Big|_{r\to\infty}-P^r_{\delta_1,\delta_2}\Big|_{r\to 0}+\int_0^\infty dr(-i\omega P^t_{\delta_1,\delta_2}-i\omega S'P^r_{\delta_1,\delta_2}+i\varepsilon qP^x_{\delta_1,\delta_2})=0\,. \qquad (3.3.28)$$

where in order to get the second line, we have integrated from the horizon to infinity.

Turning our attention to the specific examples for perturbations to be used in the symplectic current, we will first consider $\mathcal{P}_{\delta H,\delta_\star}$ which is made out of our hydrodynamic mode and the static perturbation in (3.3.2). After performing an expansion of the Fourier components in ε, we obtain,

$$P^t_{\delta H,\delta_\star} = \mathcal{O}(\varepsilon^2)\,, \qquad P^x_{\delta H,\delta_\star} = -i\, q\, \delta a\, \delta\rho^2_{\star(0)}\, \varepsilon + \mathcal{O}(\varepsilon^2)\,, \tag{3.3.29}$$

and for the radial component we have,

$$\begin{aligned} P^r_{\delta H,\delta_\star} &= -e^{2g}\Big(\delta a_{\star(2)}\delta_H(\tau W^{rt})_{[1]} - \delta\tilde{a}_{(2)}\delta_\star(\tau W^{rt})_{[1]} + \\ &\frac{aU}{2}(\delta\rho_{\star(2)}\delta\rho'_{\star(0)} - \delta\rho'_{\star(2)}\delta\rho_{\star(0)}) + \frac{U}{2}(\delta\rho_{\star(0)}\delta\tilde{\rho}'_{(2)} - \delta\rho'_{\star(0)}\delta\tilde{\rho}_{(2)}) + \\ &U(\delta\phi_{\star(2)}\delta\tilde{\phi}'_{(2)} - \delta\phi'_{\star(2)}\delta\tilde{\phi}_{(2)}) - i\omega_{[2]}S'Ua\delta\rho^2_{\star(0)}\Big)\varepsilon^2 + \mathcal{O}(\varepsilon^3)\,. \end{aligned} \tag{3.3.30}$$

Using the above relations for the symplectic current $\mathcal{P}_{\delta H,\delta_\star}$ in equation (3.3.3) we obtain our first reduced equation,

$$\begin{aligned} -\delta j_{t[1]}\delta\mu_{\star(2)} + i\omega_{[2]}\delta\varrho_{\star(2)}\delta\tilde{\theta}_v + (2\Delta_\phi - 3)\delta\tilde{\phi}_{v(2)}\delta\phi_{s\star(2)} &= \\ i\omega_{[2]}\delta\varrho_{h\star(2)}\delta\theta_0 + i\omega_{[2]}\left(\delta\rho^{(0)}_{(0)\star}\right)^2 e^{2g^{(0)}}\,\delta a - q^2\,\delta a \int_0^\infty dr\delta\rho^2_{\star(0)}\,. \end{aligned} \tag{3.3.31}$$

As we will in the next section, the above relation will become the effective equation of motion for the amplitude of the order parameter.

We will now consider the symplectic current $\mathcal{P}_{\delta H,\delta_{a_x}}$ which is constructed from the hydrodynamic and the static perturbation of equation (3.3.10). For the Fourier modes of the components along the field theory directions we have that,

$$P^t_{\delta H,\delta_{a_x}} = \mathcal{O}(\varepsilon^2)\,, \qquad P^x_{\delta H,\delta_{a_x}} = \mathcal{O}(\varepsilon^3)\,, \tag{3.3.32}$$

while for the radial component we have the non-trivial form,

$$P^r_{\delta H,\delta_{a_x}} = -U\tau_c\Big(\delta B^x_{x(0)}(-iq\delta\tilde{b}_{r(1)} + q\omega_{[2]}S'\delta\tilde{\theta}_v + \delta\tilde{b}'_{x(2)}) - iq\delta\tilde{\theta}_v\delta B^{x\prime}_{x(2)}\Big)\varepsilon^2 + \mathcal{O}(\varepsilon^3)\,. \tag{3.3.33}$$

Substituting the asymptotic and near horizon expressions in equation (3.3.3) at order ε^2 we find that,

$$\delta j_{x(2)} = -i\chi^{(2)}_{jj}\, q\, \delta\tilde{\theta}_v + \tau^{(0)}_c q\omega_{[2]}\,(\delta\theta_0 - \delta\tilde{\theta}_v)\,, \tag{3.3.34}$$

which as we will later see, will yield a constitutive relation for the electric current along with the third line of equation (3.3.20).

3.4 Effective Theory

In this section we will collect the results of section 3.3 to form our effective theory for a suitable set of hydrodynamic variables. We will do this in two different ways, using two different sets of variables. In the first approach, we will write a constitutive relation for the electric current in terms of the chemical potential and the amplitude and phase of the condensate. The effective theory will then be complete by fixing a Josephson relation and a first order time evolution equation for the amplitude of the order parameter. When combined with the Ward identity (3.2.13), we obtain a closed system of equations for the dynamics of the system. The second approach uses an effective energy potential which allows us to write the equations of motion for the charge density and the amplitude and phase of the condensate as a system of first order equations in time. This will allow us to compare with the Model F in the classification of Hohenberg and Halperin [4].

3.4.1 Hydro Description

In order to clearly state our effective theory, we need to identify its dynamical variables. For this purpose, the most straightforward set of variables is the phase of the order parameter $\delta\tilde{\theta}_v$, the variable $\delta\theta_0$ and δa which parametrises the amplitude of the order parameter according to,

$$\langle\mathcal{O}_\psi\rangle = \Delta\langle\mathcal{O}_\psi\rangle_b\,\left(1+\varepsilon^{-1}\,\delta a + i\,q_e\,\delta\theta_v\right)\,, \tag{3.4.1}$$

with $\Delta\langle\mathcal{O}_\psi\rangle = (2\,\Delta_\psi - 3)\,\varepsilon\,\delta\rho_{v*(0)} + \cdots$, the VEV of the complex scalar operator in the thermal state. For later convenience, we will define the new amplitude variable, $\delta\tilde{a} = \delta a/\varepsilon$.

However, we still have the variation variable $\delta\mu_{\#}$ in our description which can be specified by the first line of (3.3.20). An alternative description, which seems more natural from the hydrodynamics point of view, is to maintain the phase $\delta\tilde{\theta}_v$, the amplitude δa and to trade $\delta\mu_{\#(2)}$ for the chemical potential variation defined by,

$$\delta\tilde{\mu} = \varepsilon\left(\delta\mu_{*(2)}\,\delta a - \delta\mu_{\#(2)}\right)\,. \tag{3.4.2}$$

For any quantity $\mathcal{O}$ which is a function of the chemical potential μ and scalar deformation ϕ_s, we can define the difference,

$$\Delta\mathcal{O} = \mathcal{O}_{\star}(\mu_c + \delta\mu_*, \phi_s + \delta\phi_{s*}) - \mathcal{O}_{\#}(\mu_c + \delta\mu_*, \phi_s + \delta\phi_{s*})\,. \tag{3.4.3}$$

This measures the difference of the value of $\mathcal{O}$ between the normal and the broken phase at fixed chemical potential and deformation parameter.

After these definitions, we can write the constitutive relations for the current,

$$\begin{aligned}\delta J_t &= -\chi^{\#}_{QQ}\,\delta\tilde{\mu} - 2\,\Delta\varrho\,\delta\tilde{a}\,,\\ \delta J_i &= -\chi_{JJ}\,\partial_i\delta\theta_v - \sigma_d\,\partial_i\delta\tilde{\mu}\,,\end{aligned} \tag{3.4.4}$$

which are nothing but equation (3.3.34) and the third line of (3.3.20) when combined with the first line of (3.3.20). In the above we have reinstated factors of ε and Fourier transformed back to a coordinate space description and we have also introduced the incoherent conductivity $\sigma_d = \tau^{(0)}$. Recasting equation (3.3.23) in terms of our new variables then provides a Josephson relation for the chemical potential variation $\delta\tilde{\mu}$,

$$q_e^2 e^{2g^{(0)}}\left(\Delta\rho^{(0)}\right)^2\,\partial_t\delta\theta_v = \left(\chi^{\#}_{QQ}\,\partial_t - \sigma_d\,\partial_i\partial^i + q_e^2\,e^{2g^{(0)}}\left(\Delta\rho^{(0)}\right)^2\right)\,\delta\tilde{\mu} + 2\,\Delta\varrho_h\,\partial_t\delta\tilde{a}\,. \tag{3.4.5}$$

By using the Ward identity (3.2.13), this equation can be written in the form of a Josephson relation for the local chemical potential,

$$\delta\tilde{\mu} = \partial_t\delta\theta_v - \frac{\chi_{JJ}}{q_e^2\,\varpi_1}\,\partial^2\delta\theta_v + \frac{\varpi_2}{q_e\,\varpi_1}\,\partial_t\delta\tilde{a}\,, \tag{3.4.6}$$

where we have defined the coefficients,

$$\varpi_1 = \frac{s_c}{4\pi}\left(\Delta\rho^{(0)}\right)^2\,, \qquad \varpi_2 = \frac{2}{q_e}\left(\Delta\varrho - \Delta\varrho_h\right) = \frac{2}{q_e}\left(\varrho_* - \varrho_{h*}\right)\,. \tag{3.4.7}$$

In the last equality we have used that in the normal phase the field theory charge density is equal to the horizon one and therefore $\varrho_{h\#} = \varrho_{\#}$.

From the above we see that in this notation our Josephson relation (3.4.6) contains dissipative effects. It is also evident that the amplitude degree of freedom has to enter both the constitutive relations (3.4.4) as well as the Josephson relation (3.4.6). The final equation we have left in order to have a complete description is (3.3.3) which fixed the dynamics of the amplitude. In our notation it reads,

$$\varpi_1\,\partial_t\delta\tilde{a} - \left(8\,\Delta w_{FE} + \gamma\,\partial_i\partial^i\right)\delta\tilde{a} - q_e\,\varpi_2\,\partial_t\delta\theta_v - \left(2\,\Delta\varrho - q_e\,\varpi_2\right)\delta\tilde{\mu} = 0\,, \tag{3.4.8}$$

with γ as defined in (3.2.15). In order to obtain the equation above, we have used that the difference in the free energy density can be written as,

$$\Delta w_{FE} = -\frac{1}{2}\left(\chi^{\star}_{QQ} - \chi^{\#}_{QQ}\right)\delta\mu_*^2 - \frac{1}{2}\left(\nu^{\star}_{\phi} - \nu^{\#}_{\phi}\right)\delta\phi_{s*}^2 - \left(\nu^{\star}_{\mu} - \nu^{\#}_{\mu}\right)\delta\phi_{s*}\,\delta\mu_* + \cdots\,, \tag{3.4.9}$$

at leading order in the variations $\delta\mu_*$ and $\delta\phi_{s*}$. This concludes the construction of our effective theory which is comprised of the equations (3.4.6), (3.4.8) and the Ward identity (3.2.13) given the constitutive relations (3.4.4).

Finally, we would like to write the constitutive relation for the expectation value,

$$\delta\langle\mathcal{O}_\phi\rangle = 2\,\Delta\langle\mathcal{O}_\phi\rangle\,\delta\tilde{a} + \nu^{\#}_{\mu}\,\delta\tilde{\mu}\,, \tag{3.4.10}$$

of the neutral scalar operator. The above relation follows from the last line of equation (3.3.20) and the definitions (3.3.8) and (3.4.2).

3.4.2 Matching with model F

In this subsection we will compare the effective theory we finalised in the previous subsection to the Model F of Hohenberg and Halperin [4]. In order to do this, we will need to rewrite our theory in terms of the amplitude δa, the angle $\delta\theta_v$ and the charge density $\delta\tilde{\varrho} = \delta J^t$ fluctuations. To do this we can simply invert the constitutive relation for the time component of the electric current in equation (3.4.4). After solving for the time derivatives of the fields in our description by using equations (3.4.6), (3.4.8) and (3.2.13) we have the system of first order equations in time,

$$\begin{aligned}
\partial_t \delta\tilde{\varrho} &= \frac{\sigma_d}{\chi^{\#}_{QQ}}\,\partial_i\partial^i(\delta\tilde{\varrho} - 2\,\Delta\varrho\,\delta\tilde{a}) + \chi_{JJ}\,\partial_i\partial^i\delta\theta_v\,,\\
\partial_t \delta\tilde{a} &= \lambda_1\left(\gamma\,\partial_i\partial^i\delta\tilde{a} + 8\;\Delta E|_{\varrho,\phi_s}\,\delta\tilde{a} + \frac{2\,\Delta\varrho}{\chi^{\#}_{QQ}}\,\delta\tilde{\varrho}\right) + \frac{\chi_{JJ}}{q_e}\,\lambda_2\,\partial_i\partial^i\delta\theta_v\,,\\
\partial_t \delta\theta_v &= \frac{\chi_{JJ}}{q_e^2}\lambda_1\,\partial_i\partial^i\delta\theta_v + \frac{1}{\chi^{\#}_{QQ}}\,(\delta\tilde{\varrho} - 2\,\Delta\varrho\,\delta\tilde{a})\\
&\quad - \frac{\lambda_2}{q_e}\left(\gamma\,\partial_i\partial^i\delta\tilde{a} + 8\;\Delta E|_{\varrho,\phi_s}\,\delta\tilde{a} + \frac{2\,\Delta\varrho}{\chi^{\#}_{QQ}}\delta\tilde{\varrho}\right)\,. \qquad (3.4.11)
\end{aligned}$$

In order to simplify the notation we have introduced the quantity,

$$\Delta E|_{\varrho,\phi_s} = \Delta w_{FE}|_{\mu,\phi_s} - \frac{1}{2\chi^{\#}_{QQ}}\;(\Delta\varrho)^2\Big|_{\mu,\phi_s}\,, \qquad (3.4.12)$$

which is the energy density difference of the broken and the normal phase at fixed charge density and scalar deformation. The above relation is easy to show by e.g. using the results of Appendix B in [130]. Moreover, in equations (3.4.11), we have defined the two important quantities,

$$\lambda_i = \frac{\varpi_i}{\varpi_1^2 + \varpi_2^2}\,, i = 1, 2. \qquad (3.4.13)$$

We now need to match the above system of equations to the equations that one would obtain from the phenomenologically motivated equations of the Model F of [4]. In order to do that, we first need to consider the Ginzburg-Landau free energy functional close to the critical point,

$$F[\psi, m] = \int d^2x\left(\frac{w_0}{2}|\nabla\psi|^2 + \frac{\tilde{r_0}}{2}|\psi|^2 + \tilde{u_0}|\psi|^4 + \frac{1}{2\,C_0}m^2 + \gamma_0\,m\,|\psi|^2\right)\,, \qquad (3.4.14)$$

where in this notation m is the conserved charge density. Given the above energy functional, the dissipative equations of motion for the $U(1)$ order parameter ψ and the current continuity equation respectively are,

$$\begin{aligned}\partial_t\psi &= -2\Gamma_0\frac{\delta F}{\delta\psi^\star} - ig_0\psi\,\frac{\delta F}{\delta m}\,,\\ \partial_t m &= \lambda_0^m\,\nabla^2\frac{\partial F}{\partial m} + 2\,g_0\,\mathrm{Im}(\psi^\star\frac{\delta F}{\delta\psi^\star})\,.\end{aligned} \tag{3.4.15}$$

After decomposing the order parameter as $\psi = \Delta\langle\mathcal{O}_\psi\rangle\;(1+\delta\tilde{a} - i\,g_0\,\delta\theta_v)$ and the charge density as $m = m_0 + \delta\tilde{\varrho}$, we can match the resulting equations of motion provided that,

$$\begin{aligned}& g_0 = q_e\,, \qquad C_0 = \chi^{\#}_{qq}\,, \qquad \gamma_0 = -\frac{\Delta\varrho}{\chi^{\#}_{QQ}\;(\Delta\langle\mathcal{O}_\psi\rangle)^2}\,, \qquad w_0 = \frac{\chi_{JJ}}{q_e^2\;(\Delta\langle\mathcal{O}_\psi\rangle)^2}\,,\\ & m_0 = \Delta\varrho\,, \qquad \tilde{u}_0 = -\frac{1}{(\Delta\langle\mathcal{O}_\psi\rangle)^4}\;\Delta E|_{\varrho,\phi_s}\,, \qquad \tilde{r}_0 = 4\,\frac{\Delta w_{FE}}{(\Delta\langle\mathcal{O}_\psi\rangle)^2}\,,\\ & \Gamma_0 = (\Delta\langle\mathcal{O}_\psi\rangle)^2\;(\lambda_1 + i\,\lambda_2)\;, \qquad \lambda_0^m = \sigma_d = \tau^{(0)}\,.\end{aligned} \tag{3.4.16}$$

The important lesson that we extract from having explicit expressions for these constants from holography is that apart from $\tilde{r}_0$ and m_0, the rest remain finite as we approach the critical point. Notably, the dissipative kinetic coefficient Γ_0 remains complex as $\varepsilon \to 0$. These observations will play an important role in the next subsection where we discuss the hydrodynamic modes as we approach the critical point from both the normal and the broken phases.

It is important to note that the coefficients (3.4.16) are ultimately fixed by information which is held fixed either due to conservation laws, like the charge density ϱ, or because it is part of the sources in the problem, like the deformation parameter ϕ_s. We have written quantities, like the susceptibilities $\chi^{\#}_{QQ}$, $\chi^{\star}_{QQ}$ and the free energy w_{FE} which are more natural for the grand canonical ensemble. However, these are to be evaluated on states of chemical potential which is specified by the fixed charge density ϱ.

3.5 Hydrodynamic Modes

In this section we will consider the quasinormal modes of the system which is captured by the effective theory that we constructed in section 3.4. More specifically, we will be interested in three different regimes of the phase space while holding fixed the wavevector of fluctuations k_i. The first one will be the normal phase as we approach the critical point.

The two subsequent subsections are devoted to two distinct limits of the general, finite density case, for small and for large values of the gap as compared to the modulus of the wavevector. As we will see, at a fixed wavevector the dispersion relations of the quasinormal modes are continuous across the phase transition. The final regime we will examine is the broken phase at zero chemical potential. In this case, we will have a great simplification of the dispersion relations and we will be able to follow the modes in the complex plane analytically. Moreover, this section will answer some of the questions raised in [2].

3.5.1 Normal Phase

In this subsection we will consider the hydrodynamic modes of our system as we approach the critical point from the normal phase of the system. In general, at exactly the critical point the only gapless modes of our system are fluctuations of the charge density and the critical modes of the complex scalar that become gapless. Even though we are at finite charge density, the fact that we are in the probe limit suggests that the charge density fluctuations will decouple from the pressure and the momentum of system leading to a purely diffusive mode.

More specifically, in the normal phase the constitutive relations for the fluctuations of the electric current are,

$$\delta J^t = \chi_{QQ}^{\#}\,\delta\mu\,, \qquad \delta J^i = -\sigma_d\,\partial^i\delta\mu\,. \tag{3.5.1}$$

From a holographic point of view, the incoherent conductivity is given by $\sigma_d = \tau^{(0)}$ in the normal phase. Imposing the Ward identity (3.2.13) yields a mode with dispersion relation,

$$\omega = -i\,\frac{\sigma_d}{\chi^{\#}_{QQ}}\,k^2 = -i\,\frac{\tau^{(0)}}{\chi^{\#}_{QQ}}\,k^2\,. \tag{3.5.2}$$

In order to understand the mode relevant to the charged scalar, we will employ once again the symplectic current of our theory. It is important to note that in the normal phase of the system, the background value of the amplitude is vanishing, so the decomposition $\psi = \rho\, e^{iq_e\theta}$ is ill-defined. For this reason we will work with the complex field ψ instead. Recall that $\delta\rho_{\star(0)}$ is the static mode at the critical point. In order to construct the finite wavevector one we perturbatively expand it in the wavevector according to,

$$\delta\psi_H = e^{-i\omega\varepsilon^2(t+S(r))+i\varepsilon kx}\,\left(\alpha\delta\rho_{\star(0)}(r) + \varepsilon^2\,\delta\psi_{(2)} + \cdots\right)\,, \tag{3.5.3}$$

with α complex. The symplectic current expressed in terms of the complex field reads

$$\begin{aligned}\mathcal{P}^{\mu}_{\delta_1,\delta_2} = \delta_1 A_\alpha\,\delta_2\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\alpha)}\Big) + \delta_1\psi\,\delta_2\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\Big) + \delta_1\psi^*\,\delta_2\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Big)\\ +\delta_1\phi\,\delta_2\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big) - (1\leftrightarrow 2)\,.\end{aligned} \tag{3.5.4}$$

For the perturbation that we are interested in, in the above expression only terms involving the fluctuations of the complex scalar are going to contribute. The reason is that, in the normal phase, the perturbations of the complex scalar decouple from the perturbations of the neutral scalar and the gauge field. Let's now consider the symplectic current that we can form from the static mode $\delta_1\psi = c\delta\rho_{(0)}$, with c complex, and the above hydrodynamic mode $\delta_2\psi = \delta_H\psi$. Following the strategy outlined in subsection 3.3.3 we obtain two distinct modes with dispersion relations

$$\omega_1 = -i \, \frac{\int_0^\infty dr \, \delta\rho^2_{(0)}}{I} \, k^2 \,, \quad \omega_2 = -i \, \frac{\int_0^\infty dr \, \delta\rho^2_{(0)}}{I^*} \, k^2 \,, \tag{3.5.5}$$

with $I = e^{2g_c^{(0)}} \left(\delta\rho^{(0)}_{(0)}\right)^2 + 2iq_e \int_0^\infty dr \, \frac{e^{2g_c}}{U_c} \, a_c \, \delta\rho^2_{(0)}$. After using the equation of motion of the gauge field in the broken phase (3.2.1), the above dispersion relations can also be written as

$$\omega_1 = -i \, w_0 \, \bar{\Gamma}_0 \, k^2 \,, \quad \omega_2 = -\omega_1^* = -i \, w_0 \, \Gamma_0 \, k^2 \,. \tag{3.5.6}$$

3.5.2 Large Gap Limit

In this subsection we will consider the limit in which the module of the wavevector is much smaller than the expected gap of the amplitude mode. Since we are working in a probe limit, the expectation is that we will recover the regular hydrodynamics of the supercurrent. This should happen after integrating out the amplitude mode which will acquire a large gap. Before doing that we will examine what happens with the modes of our effective theory in the large gap limit.

In order to do this we perform a Fourier decomposition of our modes and recast the linearised equations of motion (3.4.15) in matrix form,

$$\mathbb{M}\,(\omega, k_i) \begin{pmatrix} \delta a_0 \\ \delta\theta_{v0} \\ \delta\tilde{\varrho}_0 \end{pmatrix} = 0 \,. \tag{3.5.7}$$

In order for the perturbation to admit non-trivial solutions, the matrix $\mathbb{M}$ must be non-invertible and it should therefore have zero determinant. This condition becomes a third order algebraic equation for ω, that fixes the dispersion relations $\omega(k_i)$ of our quasinormal modes.

The general solution of this system is quite complicated but we will consider the two limits of small and large wavevectors in this and the next subsections. For small wavevectors, we perform the expansion,

$$
\begin{aligned}
k_i &= \lambda\, q_i\,,\\
\omega &= \omega_{[0]} + \lambda\, \omega_{[1]} + \lambda^2\, \omega_{[2]} + \cdots\,,
\end{aligned}
\tag{3.5.8}
$$

and solve the equation order by order in λ. As one would expect, the three modes we find consist of two sound and one gapped diffusive mode. The first few terms in a wavevector modulus expansion are,

$$
\begin{aligned}
\omega_H &= i\,\mathrm{Re}\Gamma_0\, \frac{8\ \Delta E|_{\varrho,\phi_s}}{(\Delta\langle\mathcal{O}_\psi\rangle)^2}\\
&\quad - i\left(w_0\, \mathrm{Re}\Gamma_0 - \frac{(\Delta\varrho)^2\ \lambda_0^m}{2\left(\chi_{QQ}^{\#}\right)^2\ \Delta E|_{\varrho,\phi_s}} - \frac{w_0}{\mathrm{Re}\Gamma_0}\left(\mathrm{Im}\Gamma_0 + \frac{q_e\,\Delta\varrho\,(\Delta\langle\mathcal{O}_\psi\rangle)^2}{4\,\chi_{QQ}^{\#}\ \Delta E|_{\varrho,\phi_s}}\right)^2\right) k^2\,,\\
\omega_\pm &= \pm\sqrt{\frac{\chi_{JJ}}{\chi_{QQ}^{\star}}\,k^2}\\
&\quad - \frac{i}{2}\left(w_0\, \mathrm{Re}\Gamma_0 + \frac{\lambda_0^m}{\chi_{QQ}^{\star}} + \frac{w_0}{\mathrm{Re}\Gamma_0}\left(\mathrm{Im}\Gamma_0 + \frac{q_e\,\Delta\varrho\,(\Delta\langle\mathcal{O}_\psi\rangle)^2}{4\,\chi_{QQ}^{\#}\ \Delta E|_{\varrho,\phi_s}}\right)^2\right) k^2\,.
\end{aligned}
\tag{3.5.9}
$$

In order to obtain the above result, we have used the non-obvious relation,

$$
\Delta E|_{\varrho,\phi_s} = \frac{\chi_{QQ}^{\star}}{\chi_{QQ}^{\#}}\,\Delta w_{FE}\,,
\tag{3.5.10}
$$

which we prove in Appendix B.

It is reassuring to note that the gap of the first dispersion relation, corresponding to the Higgs/Amplitude mode, agrees with the expression of [130] even though we are only in the probe limit. Moreover, the diffusion constant of the same mode is not positive definite and it remains finite as we approach the critical point.

It is nice to see that the modes behave as we would expect them to in our limit. However, it is important to understand how to reduce the theory of section 3.4 to regular hydrodynamics and reproduce the two sound modes that we derived from the full theory. An appropriate limit to take is,

$$
\partial_t \to \lambda\,\partial_t\,,\quad \partial_i \to \lambda\,\partial_i\,,\quad \delta a \to \lambda\,\delta a\,,\quad \delta\tilde{\mu} \to \lambda\,\delta\tilde{\mu}\,,
\tag{3.5.11}
$$

with λ a small dimensionless parameter. By doing this we can find the local expression for the amplitude,

$$\delta\tilde{a} = -\frac{1}{8\,\Delta w_{FE}}\left(2\,\Delta\varrho\,\delta\tilde{\mu} + \frac{\chi_{JJ}}{q_e^2\,\lambda_1}\left(q_e\lambda_2 + \frac{q_e^2\,\Delta\varrho}{4\,\Delta w_{FE}\,\chi^\star_{QQ}}\right)\partial^2\theta_v\right)\,, \tag{3.5.12}$$

where we have included corrections up to order $\mathcal{O}(\lambda^2)$. Moreover, the Josephson relation for the redefined chemical potential $\delta\hat{\mu}$ becomes,

$$\begin{aligned}\delta\hat{\mu} &= \partial_t\delta\theta_v - \chi_{JJ}\,\zeta_3\,\partial^2\delta\theta_v \\ &= \partial_t\delta\theta_v - w_0\left(\mathrm{Re}\Gamma_0 + \frac{1}{\mathrm{Re}\Gamma_0}\left(\mathrm{Im}\Gamma_0 + \frac{q_e\,\Delta\varrho\,\left(\Delta\langle\mathcal{O}_\psi\rangle\right)^2}{4\,\Delta w_{FE}\,\chi^\star_{QQ}}\right)^2\right)\partial^2\delta\theta_v\,, \end{aligned} \tag{3.5.13}$$

where we have also used the definition of the third bulk viscosity ζ_3 for a superfluid. This is true since the new chemical potential $\delta\hat{\mu}$ is chosen so that we are in the transverse frame with,

$$\begin{aligned}\delta J_t &= -\chi^\star_{QQ}\,\delta\hat{\mu}\,,\\ \delta J_i &= -\chi_{JJ}\,\partial_i\delta\theta_v - \lambda_0^m\,\partial_i\,\delta\hat{\mu}\,.\end{aligned} \tag{3.5.14}$$

Given the above, the resulting dispersion relation,

$$\omega_\pm = \pm\sqrt{\frac{\chi_{JJ}}{\chi^*_{QQ}}\,k^2} - \frac{i}{2\,\chi^\star_{QQ}}\left(\lambda_0^m + \chi^\star_{QQ}\,\chi_{JJ}\,\zeta_3\right)\,, \tag{3.5.15}$$

matches equation (3.5.9). Finally, it is worth noting that the expression for the bulk viscosity ζ_3 in (3.5.13) agrees with ζ_3 given in (2.4.31), when we take its limit near criticality.

3.5.3 Small Gap Limit

In this subsection we would like to understand the behaviour of our modes in the limit where we keep the wavevectors fixed but we approach the critical point from the broken phase. In this limit, the gap becomes small or equivalently, the wavevector is large. The equations of our effective theory become,

$$
\begin{aligned}
\partial_t \delta\tilde{\varrho} &= \frac{\lambda_0^m}{\chi_{QQ}^{\#}}\,\partial_i\partial^i\delta\tilde{\varrho}\,,\\
\partial_t \delta\tilde{a} &= w_0\left(\mathrm{Re}\Gamma_0\,\partial_i\partial^i\delta\tilde{a} + q_e\,\mathrm{Im}\Gamma_0\,\partial_i\partial^i\,\delta\theta_v\right) + \frac{2\,\mathrm{Re}\Gamma_0\,\Delta\varrho}{\chi_{QQ}^{\#}\,(\Delta\langle\mathcal{O}_\psi\rangle)^2}\,\delta\tilde{\varrho}\,,\\
\partial_t \delta\theta_v &= w_0\left(\mathrm{Re}\Gamma_0\,\partial_i\partial^i\delta\theta_v - \frac{1}{q_e}\mathrm{Im}\Gamma_0\,\partial_i\partial^i\delta\tilde{a}\right) + \frac{1}{\chi_{QQ}^{\#}}\left(1 - \frac{2\,\mathrm{Im}\Gamma_0\,\Delta\varrho}{q_e\,(\Delta\langle\mathcal{O}_\psi\rangle)^2}\right)\delta\tilde{\varrho}\,.
\end{aligned}
\tag{3.5.16}
$$

The Jacobi form of the system immediately gives that there is a charge diffusion modes with dispersion relation,

$$
\omega = -i\frac{\lambda_0^m}{\chi_{QQ}^{\#}}\,k^2\,, \tag{3.5.17}
$$

which matches exactly the charge diffusion mode of equation (3.5.2) of the normal phase given the matching of the parameters of equation (3.4.16). It is worth noting that apart from the charge density, the phase and the amplitude will also be involved in this mode in the broken phase. This is true even in the limit close to the critical point since the constants Γ_0 and w_0 remain finite.

The second mode involves only the order parameter as in order to satisfy the first equation of the system we have to necessarily set the charge density fluctuation equal to zero. A quick computation shows that the order parameter fluctuations yields two modes obeying the dispersion relations,

$$
\omega_r = -i\,w_0\,\Gamma_0\,k^2\,, \qquad \omega_r = -i\,w_0\,\bar{\Gamma}_0\,k^2\,, \tag{3.5.18}
$$

which remain finite close to the critical point. We immediately see that this pair of modes matches the modes of oscillation of the order parameter in the normal phase (3.5.6).

3.5.4 Zero chemical potential

It is easy to see that at zero chemical potential, the matching conditions (3.4.16) suggest that,

$$\mathrm{Im}\Gamma_0 = \gamma_0 = \rho_c = 0\,. \tag{3.5.19}$$

In this case, the equation of motion for the amplitude of the order parameter is not sourced by fluctuations of the phase and the charge density leading to the pseudo-gapped mode with dispersion relation,

$$\omega_H = i\,\Gamma_0 \left(\frac{8\ \Delta E|_{\varrho,\phi_s}}{(\Delta\langle\mathcal{O}_\psi\rangle)^2} - w_0\, k^2 \right)\,. \tag{3.5.20}$$

Note that this dispersion relation agrees exactly with the dispersion (3.5.6) of the complex scalar coming from the normal phase in the limit of zero gap. This is reassuring but we are still missing one more mode with the same diffusion constant. As we will see, this will come from the sector of charge density and phase.

In order to find the quasinormal modes of this sector, it is illustrating to write down their equations of motion,

$$\begin{aligned} \partial_t \delta\tilde{\varrho} &= \frac{\lambda_0^m}{\chi_{QQ}^{\#}}\, \partial_i\partial^i \delta\tilde{\varrho} + \chi_{JJ}\, \partial_i\partial^i \delta\theta_v\,, \\ \partial_t \delta\theta_v &= w_0\,\Gamma_0\, \partial_i\partial^i\delta\theta_v + \frac{1}{\chi_{QQ}^{\#}}\delta\tilde{\varrho}\,, \end{aligned} \tag{3.5.21}$$

and note once again that the amplitude decouples entirely. The corresponding dispersion relations are,

$$\omega_\pm = \pm\frac{\sqrt{4\,k^2\,\chi_{JJ}\,\chi_{QQ}^{\#} - k^4\left(\lambda_0^m - w_0\,\Gamma_0\,\chi_{QQ}^{\#}\right)^2}}{2\chi_{QQ}^{\#}} - i\frac{\lambda_0^m + w_0\,\Gamma_0\,\chi_{QQ}^{\#}}{2\,\chi_{QQ}^{\#}}\,k^2\,. \tag{3.5.22}$$

The limit we want to examine is the small k limit in which we find the approximate dispersion relation,

$$\omega_\pm \approx \pm\sqrt{\frac{\chi_{JJ}}{\chi_{QQ}^{\#}}k^2} - i\frac{\lambda_0^m + w_0\,\Gamma_0\,\chi_{QQ}^{\#}}{2\chi_{QQ}^{\#}}\,k^2\,, \tag{3.5.23}$$

which agrees with the dispersion relations of [2] when taking the nearly critical limit.

These are therefore the standard sounds modes of neutral superfluids in the broken phase.

The second point we would like to make comes from thinking of this dispersion relation as the position of poles of Green's functions in the complex frequency plane. Using our analytic formula (3.5.22) we see that the two sound-like poles collide on the imaginary axis when,

$$k_c^2 = \frac{4\,\chi_{JJ}\,\chi^{\#}_{QQ}}{\left(\lambda_0^m - w_0 \mathrm{Re}\Gamma_0 \chi^{\#}_{QQ}\right)^2}\,. \tag{3.5.24}$$

We also see that for wavevectors with modulus squared larger than k_c^2, the two modes remain purely imaginary.

For very large value of k^2 as compared to k_c, we have the two approximate dispersion relations with leading behaviour,

$$\omega_{\pm} = -\frac{i}{2\chi^{\#}_{QQ}}\left(\lambda_0^m + w_0\,\mathrm{Re}\Gamma_0\,\chi^{\#}_{QQ} \pm \left|\lambda_0^m - w_0\,\mathrm{Re}\Gamma_0\,\chi^{\#}_{QQ}\right|\right)\,k^2\,. \tag{3.5.25}$$

The above shows that the collision of the two sound modes produce the, so far missing, second diffusive mode that matches (3.5.6) as well as the charge diffusion mode (3.5.2).

3.6 Numerical checks

In this section we check numerically the dispersion relations for the Higgs and Goldstone modes close to the critical point for various values of the wavevector. First, we give an overview of our method and then we focus on two simple cases: (i) Charged superfluids at finite density and zero scalar deformation parameter ϕ_s and (ii) Neutral superfluids with a non-trivial scalar deformation parameter ϕ_s.

3.6.1 Overview of the method

In this subsection we give a few technical details on the double sided shooting method we have used. First, we describe the technique for the background solutions and then we move on to discuss the quasinormal modes we wish to construct.

Background solution

We are going to work in the probe limit, assuming that the matter fields don't backreact onto the metric, which will be of the general form (3.2.2) with asymptotics described by the expressions (3.2.3),(3.2.4). As in the analytic calculation, the conformal boundary is located at $r \to \infty$ and the black hole horizon at $r = 0$.

The matter action is taken to be (2.2.1) (without the Einstein-Hilbert term) with:

$$
\begin{aligned}
V &= \frac{1}{2} m_\psi^2 \, |\psi|^2 + \frac{1}{2} m_\phi^2 \, \phi^2 + \lambda_\psi \, |\psi|^4 + \lambda_\phi \, \phi^4 + \lambda_{\psi\phi} \, |\psi|^2 \phi^2 \,, \\
\tau &= 1 + \zeta_\psi \, |\psi|^2 + \zeta_\phi \, \phi^2 \,.
\end{aligned}
\tag{3.6.1}
$$

A background matter solution describing both the broken and normal phase of the system involves an ansatz of the form,

$$
B = a(r) dt \,, \qquad \rho = \rho(r) \,, \qquad \phi = \phi(r) \,. \tag{3.6.2}
$$

Plugging this ansatz in the equations of motion we find 3 second order ODEs, which implies that we need to fix six integration constants. The behaviour of the fields near the conformal boundary is,

$$
\begin{aligned}
a &= \mu - \frac{\varrho}{(r+R)} + \cdots \,, \\
\rho &= \frac{\rho_v}{(r+R)^{\Delta_\psi}} + \cdots \,, \\
\phi &= \frac{\phi_s}{(r+R)^{3-\Delta_\phi}} + \cdots + \frac{\phi_v}{(r+R)^{\Delta_\phi}} + \cdots \,.
\end{aligned}
\tag{3.6.3}
$$

Close to the horizon, the analytic expansion yields,

$$
a = a^{(0)} r + \cdots \,, \quad \rho = \rho^{(0)} + \cdots \,, \quad \phi = \phi^{(0)} + \cdots \,. \tag{3.6.4}
$$

Fixing the values of the chemical potential μ and the neutral scalar's source ϕ_s leaves six free integration constants ϱ, ρ_v, $\phi_v, a^{(0)}$, $\rho^{(0)}, \phi^{(0)}$ that will be fixed via double-sided shooting. For an appropriate choice of the parameters $\lambda_\psi, \lambda_\phi, \lambda_{\psi\phi}, \zeta_\psi, \zeta_\phi$ we can find background solutions describing a second order phase transition. As expected, the field ρ is going to be trivial in the normal phase of the system and non-trivial in the broken phase.

With the background solutions at hand, we can construct the functions $\varrho(\mu, \phi_s)$ and $\phi_v(\mu, \phi_s)$ numerically. By taking the appropriate partial derivatives of these functions we can calculate the static susceptibilities $\chi_{QQ}, \nu_\phi, \nu_\mu$ that appear in our analytic results. To calculate the current susceptibility χ_{JJ} we need to construct a static perturbation for the one-form field of the form

$$\delta B = \delta b_x(r)\, dx\,, \tag{3.6.5}$$

yielding a second order ODE for the function $\delta b_x(r)$ which requires fixing of two constants of integration in order to find a unique solution. This is the bulk dual of a field theory perturbation involving the supercurrent.The expansion near the conformal boundary at $r \to \infty$ is,

$$\delta b_x = \delta b_x^s + \frac{\delta b_x^v}{r+R} + \cdots\,. \tag{3.6.6}$$

Close to the black hole horizon we have the analytic expansion,

$$\delta b_x = \delta b_x^{(0)} + \cdots\,. \tag{3.6.7}$$

In total we have three free integration constants $(\delta b_x^s, \delta b_x^v, \delta b_x^{(0)})$ and because the equation for δb_x is linear and homogeneous we can set one of them to unity. This argument shows that there are indeed two free constants of integration and we can find a unique solution. Then we are able to find χ_{JJ} using the definition:

$$\chi_{JJ} = -\frac{\delta b_x^v}{\delta b_x^s}\,. \tag{3.6.8}$$

At this point, we remind the reader that this static perturbation is part of our black hole thermodynamics. However, we only consider perturbations of it as we wish to study the hydrodynamics of superfluid thermal states with zero supercurrent.

Quasinormal modes

Our ultimate goal is to calculate the dispersion relations $\omega(k)$ of the hydrodynamic modes and for this reason we need to study black hole perturbations which are source free from the boundary point of view. To achieve this, we consider perturbations of the form,

$$\delta\mathcal{F}(r,t,x) = e^{-i\omega(t+S(r))+ikx}\delta f(r)\,, \tag{3.6.9}$$

with the choice,

$$S(r) = \int_{\infty}^{r} \frac{dy}{U(y)}\,. \tag{3.6.10}$$

The longitudinal sector that we are interested in involves the fields $\delta b_t, \delta b_x, \delta b_r, \delta\rho, \delta\phi$. The radial component equation for the gauge field allows us to eliminate δb_r in terms of δb_t and δb_x. This leaves us with four second order ODEs which require the fixing of eight constants of integration. Once again, we will achieve this by implementing a double-sided shooting method.

In order to find identify the constants of integration, we consider the asymptotic behaviour of our functions close to the boundaries of our computational domain. In the IR we impose in-falling boundary conditions and solving the equations of motion we find the expansions,

$$\delta b_t = c_t + \cdots\,,\quad \delta b_x = c_x + \cdots\,,\quad \delta\rho = c_\rho + \cdots\,,\quad \delta\phi = c_\phi + \cdots\,, \tag{3.6.11}$$

where the constants c_t, c_x, c_ρ, c_ϕ are unfixed at this stage. On the other side of our domain, in the UV, we have the expansions,

$$\begin{aligned}
\delta b_t &= \delta b_t^s - i\omega\delta c + \frac{\delta b_t^v}{r+R} + \cdots\,,\\
\delta b_x &= \delta b_x^s + ik\delta c + \frac{\delta b_x^v}{r+R} + \cdots\,,\\
\delta\rho &= \frac{\delta\rho_s}{(r+R)^{3-\Delta_\psi}} + \cdots + \frac{\delta\rho_v}{(r+R)^{\Delta_\psi}} + \cdots\,,
\end{aligned}$$

$$\delta\phi = \frac{\delta\phi_s}{(r+R)^{3-\Delta_\phi}} + \cdots + \frac{\delta\phi_v}{(r+R)^{\Delta_\phi}} + \cdots , \tag{3.6.12}$$

where the constant δb_x^v is fixed in terms of the others, due to the current conservation (3.2.13).

In order to compute the quasinormal modes we have to set the sources to zero,

$$\delta\rho_s = \delta\phi_s = \delta b_t^s = \delta b_x^s = 0 \,. \tag{3.6.13}$$

In total, for fixed a fixed value of k, we have 4 independent constants from the IR $(c_t, c_x, c_\rho, c_\phi)$ and 4 independent constants from the UV $(\delta b_t^v, \delta\rho_v, \delta\phi_v, \delta c)$. Because the equations of the perturbations are linear and homogeneous we can set one of those constants to unity, so we are left with 7 free constants plus the frequency ω, which match exactly the 8 constants of integration that we need.

3.6.2 Results for charged superfluids

In this model we take the metric to be AdS-Schwarzschild with unit radius. In this case, the background geometry (3.2.2) is specified by the functions,

$$U(r) = (r+R)^2 - \frac{R^3}{r+R} , \qquad g(r) = \log(r+R) \,. \tag{3.6.14}$$

which give Hawking temperature is given by $T = \frac{3R}{4\pi}$ and entropy density $s = 4\pi R^2$. For our charged superfluids, we have chosen to set the neutral scalar's background source ϕ_s to zero which allows us to consistently set the bulk scalar ϕ equal to zero in our equations of motion. Moreover, we will choose $m_\psi^2 = -2, \zeta_\psi = 0, \lambda_\psi = 0$ and also set $T = \frac{3}{4\pi}$, without loss of generality. For these parameters the system exhibits a second order phase transition with critical chemical potential $\mu_c \approx 4.06371366$.

In Figure 3.1 we plot the quantity $\frac{1}{2}\frac{\partial^2 \mathrm{Im}\,\omega_H}{\partial k^2}$ for the Higgs mode as a function of k, for three different values of the chemical potential above μ_c. We plot the results coming from the numerical calculation (dashed lines) together with the analytic results (solid lines) that we can find for the frequency as a function of the wavevector. The latter is one of the roots of the cubic polynomial resulting from demanding that (3.5.7) has non-trivial solutions.

As we can observe, sufficiently close to the critical point the Higgs mode interpolates between two regions where it is diffusive, in accordance with the discussion in sections 3.5.2 and 3.5.3. In particular, for k much smaller than the gap, the first equation of (3.5.9) yields for our model: $\omega_H \approx \omega_{gap} - 0.43374\, i\, k^2$ and for k much greater than the gap equation (3.5.17) gives $\omega_H \approx -i\, k^2$.

This interpolating behaviour of the amplitude mode has also been described in [135]. In particular, equations (3.22) and (3.24) of [135] capture the limiting behaviour of the amplitude mode for k much smaller and much larger than the gap of the amplitude mode respectively, similarly to our equations (3.5.9) and (3.5.17). The numerical discrepancy in the coefficients of k^2 is due to the difference in bulk dimensions[4].

The authors of [131], working with exactly the same setup, argued that this mode is diffusive with $\omega = \omega_{gap} - i\, k^2$. As we showed here, this is indeed the correct behaviour but only for values of momentum much larger than the gap. The reason they didn't find the interpolation is that their numerical calculation was done only for k larger than the gap, as we can see from Figure 9 of their paper.

In Figure 3.2 we plot $\frac{\partial \mathrm{Re}\omega_+}{\partial k}$, $\frac{1}{2}\frac{\partial^2 \mathrm{Re}\omega_+}{\partial k^2}$ and $\frac{1}{2}\frac{\partial^2 \mathrm{Im}\omega_+}{\partial k^2}$ for the Goldstone mode as a function of k for $\frac{\mu_c}{\mu} = 0.999998932$. We present the results coming from the numerical calculation (dashed lines) together with the analytic results (solid lines). The latter is the second root of the cubic that also fixed the dispersion relation of the Higgs mode. Apart from showing the agreement with the full solution, these plots also confirm the asymptotic behaviour given by equations (3.5.9) and (3.5.18). In particular these tell us that for k much smaller than the gap the dispersion relation approaches the behaviour $\omega_+ \approx 0.001748k - 0.3541\, i\, k^2$ while for k much larger than the gap: $\omega_+ \approx (0.218982 - 0.07098\, i)\, k^2$. Overall, for both modes there is a very good quantitative agreement with the analytic predictions, for all values of $\frac{k}{T} \ll 1$.

[4]We work in four bulk dimenions, whereas in [135], the solution refers to a five dimensional bulk spacetime.

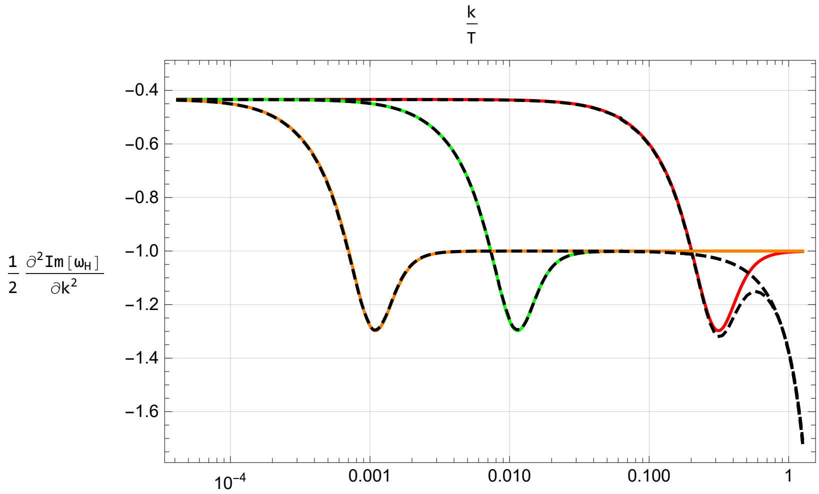

Figure 3.1: Plot of $\frac{1}{2}\frac{\partial^2 \text{Im}\,\omega_H}{\partial k^2}$ for the Higgs mode as a function of k for three values of the chemical potential $\frac{\mu_c}{\mu_{orange}} = 0.99999999$, $\frac{\mu_c}{\mu_{green}} = 0.999998932$, $\frac{\mu_c}{\mu_{red}} = 0.9991919498$. The dashed lines correspond to the numerical results and the solid lines to the analytic predictions.

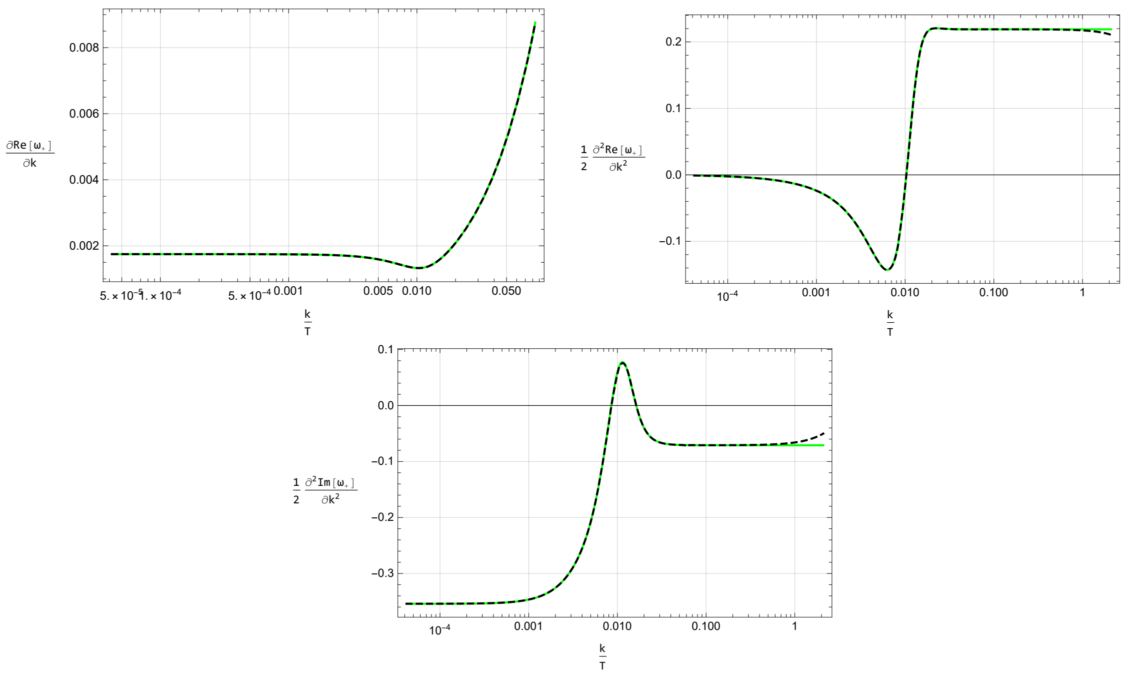

Figure 3.2: Plots of $\frac{\partial \text{Re}\,\omega_+}{\partial k}$, $\frac{1}{2}\frac{\partial^2 \text{Re}\,\omega_+}{\partial k^2}$ and $\frac{1}{2}\frac{\partial^2 \text{Im}\,\omega_+}{\partial k^2}$ for the Goldstone mode as a function of k for $\frac{\mu_c}{\mu} = 0.999998932$. The dashed lines correspond to the numerical results and the solid lines to the analytic predictions.

3.6.3 Results for neutral superfluids

Here we will study the quasi-normal modes of neutral superfluids which undergo a phase transition driven by the scalar deformation parameter $\phi_{(s)}$. We will once again choose the background metric (3.2.2) to be AdS-Schwarzschild with the functions given by (3.6.14) and Hawking temperature $T = \frac{3}{4\pi}$. For the backgrounds we will set $\mu = 0$ and we will also choose $m_\psi^2 = -2$, $m_\phi^2 = -2$, $\zeta_\psi = 1$, $\zeta_\phi = 1$, $\lambda_\psi = \frac{1}{2}$, $\lambda_\phi = \frac{1}{2}$ and $\lambda_{\psi\phi} = -\frac{3}{2}$. For this choice of parameters we find a second order phase transition with the critical value of the neutral scalar source being $\phi_{(s)c} \approx 2.5646887676$. An important observation is that for neutral superfluids we have two decoupled sectors, namely $(\delta b_t, \delta b_x)$ and $(\delta\rho, \delta\phi)$. As one might expect, the first sector will capture the two quasinormal modes with relevant to the Goldstone mode while the second sector will capture the gapped Higgs mode. This is in accordance with the discussion in section 3.5.4

In Figure 3.3 we plot our numerical results for the Higgs mode ω_H as a function of k for $\frac{\phi_{(s)c}}{\phi_{(s)}} = 0.999999519$. In addition to that, we plot the corresponding analytic result that we find using equation (3.5.20). The dashed horizontal line shows the gap of the Higgs mode, which is $\frac{\omega_{gap}}{T} \approx -6.01 \cdot 10^{-6}\, i$.

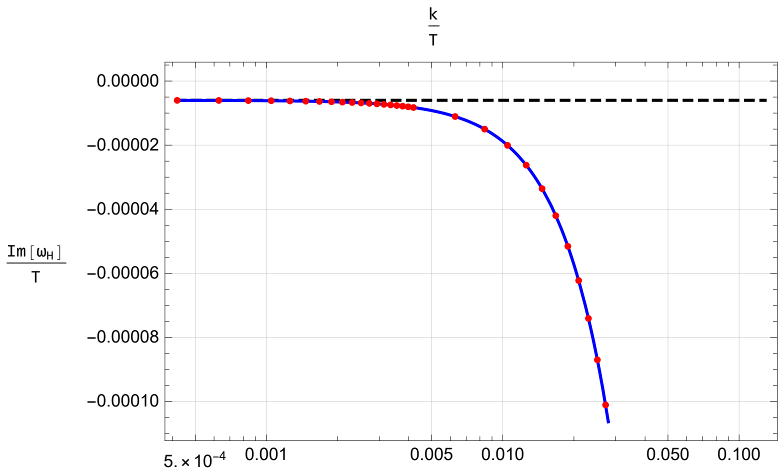

Figure 3.3: Plot of $\frac{\omega_H}{T}$ as a function of k for $\frac{\phi_{(s)c}}{\phi_{(s)}} = 0.999999519$. The dots are the numerical results and the solid line is the analytic result. The dashed horizontal line marks the analytic prediction for the gap of the Higgs mode.

For the Goldstone mode, we plot $\frac{\mathrm{Re}\,\omega_\pm}{k}$ and $\frac{T\,\mathrm{Im}\,\omega_\pm}{k^2}$, as a function of k for $\frac{\phi_{(s)c}}{\phi_{(s)}} = 0.9999996754$ in Figure 3.4. We also include plots of the analytic predictions (solid lines) of equation (3.5.22). The dashed lines on these plots mark the point of collision for the two Goldstone modes at momentum $\frac{k_c}{T} = 0.0212842$, as computed from equation (3.5.24). These plots clearly show that for $\frac{k}{k_c} \ll 1$ we can observe the usual second sound modes of superfluids, whereas for $\frac{k}{k_c} \gg 1$ these modes become purely diffusive. As we explained in section 3.5.4 one of these two diffusive modes pairs nicely with the Higgs mode of Figure 3.3 agreeing with the doublet of modes of the charged scalar in the normal phase. Once again we find very good quantitative agreement with our analytic predictions.

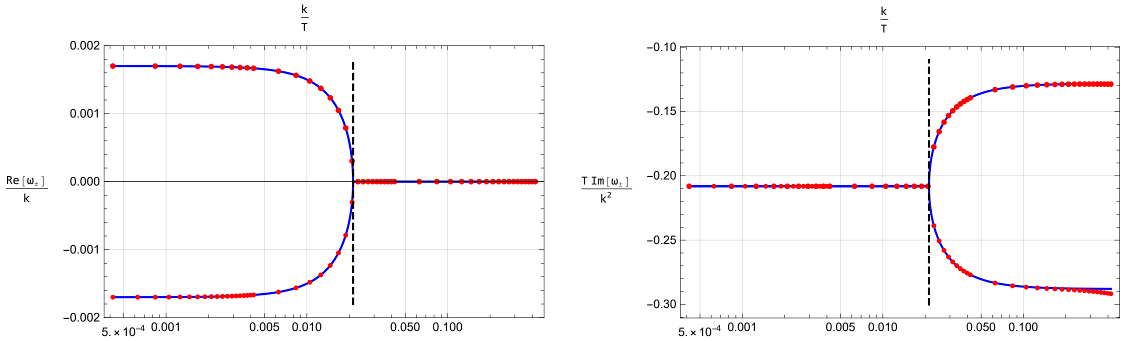

Figure 3.4: Plots of $\frac{Re\omega_{\pm}}{k}$ and $\frac{Im\omega_{\pm}}{k^2}$ as a function of k for $\frac{\phi_{(s)c}}{\phi_{(s)}} = 0.9999996754$. The dots are the numerical results and the solid lines are the analytic predictions. The dashed lines are at $\frac{k}{T} = 0.0212842$.

3.7 Discussion and outlook

In this chapter we have analysed the low energy dynamics of holographic superfluids close to their critical point. As part of our analysis, we have constructed an effective theory for the collective degrees of freedom involved in the problem. For the standard description of superfluids away from the critical point, the hydrodynamic degrees of freedom are captured by the phase of the order parameter and the local chemical potential [144, 145]. Close to the critical point, the amplitude mode driving the transition becomes gapless and has to be included in the description.

By using analytic techniques, we have constructed an effective theory consisting of the conservation law (3.2.13), the Josephson relation (3.4.6) and the equation governing the amplitude dynamics (3.4.8). By performing a change of variables, we managed to show that our system is equivalent to Model F in the classification of

Hohenberg and Halperin [4]. This allowed us to give explicit expressions for the various constants that appear in that description including the dissipative kinetic coefficient. With this information in hand, we carried out a somewhat detailed analysis of our modes in different limits for the gap of the Higgs mode.

Our work can be extended in several different directions. An obvious direction would be to enlarge our minimal description to accommodate for the temperature and fluid velocity of the normal component of our system. In order to do this, we would need to move away from the probe limit and include the backreaction of the background metric along with the coupling of the complex scalar one-form field fluctuations with the metric. In such a more complicated scenario our techniques would still produce usable results. In a sense, our analysis in chapter 2 would need to be extended with the inclusion of the Higgs mode that we discussed in the present simplifying case.

Chapter 4

Nearly Critical Superfluids in Keldysh-Schwinger Formalism

This chapter is a reproduction of [7], written in collaboration with Aristomenis Donos.

In this paper, we examine the effective theory of critical dynamics near superfluid phase transitions in the framework of the Keldysh-Schwinger formalism. We focus on the sector capturing the dynamics of the complex order parameter and the conserved current corresponding to the broken global symmetry. After constructing the theory up to quadratic order in the α-fields, we compare the resulting stochastic system with Model F as well as with holography. We highlight the role of a time independent gauge symmetry of the effective theory also known as "chemical shift". Finally, we consider the limiting behaviour at energies much lower than the gap of the amplitude mode by integrating out the high energy degrees of freedom to reproduce the effective theory of superfluids.

4.1 Introduction

The dynamics of nearly critical systems is a fascinating topic with a long history. Early theories of critical phenomena aiming to understand the small relaxation rates in systems close to criticality, started with the conventional theory of critical slow down [151, 152]. This turned out to have limited range of applicability [153] to real systems and a significant improvement was the mode-coupling theories [154, 155, 156, 157]. Mode-coupling took into account the coupling of the order parameter to other hydrodynamic slow modes that exist at wavelengths larger than the correlation length of the system. The dynamics of the coupled system is described by stochastic equations with Gaussian noise. In parallel to static renormalisation group techniques for systems undergoing a phase transition [20], a systematic treatment of interactions was carried out in [4].

Very recently, a systematic study of low energy effective field theories was proposed in the context of the Keldysh-Schwinger closed time path formalism [42, 41]. The fluctuation dissipation theorem is built in the formalism due to the Kubo-Martin-Schwinger symmetry [56, 57] providing an appropriate framework to consider thermal fluctuations beyond Gaussian noise [158, 159, 160]. This is in parallel to the standard approach of stochastic systems [161] where the variance of the noise fields is fixed again in a way that the probability distribution of the system will always relax to the Boltzmann distribution [162] and the fluctuation dissipation theorem will be satisfied.

In this paper we will construct an effective action for systems close to a superfluid phase transition up to quadratic order in the α-fields. As we will see, at this order in the α-fields, our system is essentially described by Model F in the classification of Hohenberg and Halperin [4]. Setting all our α-sources to zero will be necessary as the equations for stochastic hydrodynamics are written in terms of r-field sources.

An interesting aspect of our construction is that by choosing an appropriate set of dynamical variables, our effective theory can be viewed as a simple coupling between the $U(1)$ symmetric version of Model A and the diffusive model for current conservation [41, 25]. Both models possess their own global $U(1)$ symmetry which in the case of charge diffusion is gauged by the external electromagnetic field. Interestingly, the diffusive model for current conservation possesses an additional time independent gauge symmetry, also known as the "chemical shift" symmetry [163, 41]. Our final effective theory can be obtained by simply promoting the global $U(1)$ symmetry of Model A to a local, time independent one, which is identified with the chemical shift symmetry of charge diffusion.

By considering fluctuations of frequencies much lower than the gap of the amplitude mode, we end up with an effective description of the conserved current in the broken phase. At the level of classical hydrodynamics, this system is parametrised by the charge and current susceptibilities as well as the third bulk viscosity and the incoherent conductivity [146, 31, 26, 164]. As one might expect, the low energy theory is described by the phase of the complex order parameter. We express the third bulk viscosity of the superfluid in terms of bare thermodynamic quantities and the complex kinetic coefficient of the order parameter.

4.2 The Effective Theory

The basic ingredient of the Keldysh-Schwinger formalism [44, 45] is the acceptance that time evolution of quantum fields has to happen along a closed time path [44]. In its full generality, the formalism can be used to extract powerful statements about the dynamics of quantum systems around a state captured by a generic density matrix. More recently, significant progress has been made in realising effective theories describing the long wavelength, small frequency limit within the same framework [41]. After integrating out the fast, short wavelength modes of the system in the closed time path integral we are left with the effective theory fields χ and the sources ϕ. In terms of these, the closed time path generating functional reads,

$$e^{W[\phi_1,\phi_2]} = \int D\chi_1\, D\chi_2\, e^{i\, I_{EFT}[\chi_1,\phi_1;\chi_2,\phi_2]}\,, \tag{4.2.1}$$

where I_{EFT} is the effective action, with χ_1 and χ_2 corresponding to the forward and backward time evolution branch respectively. In order for the path integral in (4.2.1) to represent the trace of a closed time path, the dynamical fields χ_1 and χ_2 must satisfy the gluing conditions,

$$\lim_{t\to+\infty}\left(\chi_1(t)-\chi_2(t)\right)=\lim_{t\to-\infty}\left(\chi_1(t)-\chi_2(t-i\beta)\right)=0\,, \qquad (4.2.2)$$

where β is the inverse temperature of the thermal state. Due to the integration over the UV modes, the action I_{EFT} will carry dependence on the thermodynamic variables fixing the thermal state.

It is convenient to introduce the r and α-fields through,

$$\chi_r=\frac{1}{2}\left(\chi_1+\chi_2\right)\,,\quad \chi_\alpha=\chi_1-\chi_2\,,$$

and similarly for the sources ϕ_r and ϕ_α. The constraints implied by unitarity read [41],

$$\begin{aligned}&I^\star_{EFT}[\chi_r,\phi_r;\chi_\alpha,\phi_\alpha]+I_{EFT}[\chi_r,\phi_r;-\chi_\alpha,-\phi_\alpha]=0\,,\\&I_{EFT}[\chi_r,\phi_r;\chi_\alpha=0,\phi_\alpha=0]=0\,,\\&\mathrm{Im}\,I_{EFT}[\chi_r,\phi_r;\chi_\alpha,\phi_\alpha]\geq 0\,.\end{aligned} \qquad (4.2.3)$$

When both the Hamiltonian of our system and the thermal state are invariant under a symmetry group G, we would anticipate that the effective theory will possess the same symmetry. Due to the doubling of fields, one might suspect that the symmetry group should be enhanced to $G\times G$. This is true for time dependent gauge transformations which preserve the boundary conditions (4.2.2). However, for time independent symmetry transformations, it is only the diagonal subgroup that preserves these boundary conditions. This key difference will play a crucial role in our construction.

The first degree of freedom that we need to include in our effective theory is the complex order parameter ψ_i which is charged under a global $U(1)$ transformation. Moreover, we assume that this global symmetry is gauged by the external electromagnetic potential $A_{i\,\mu}$. The second observable we need to include is the associated Noether current J_μ due to the $U(1)$ symmetry and which couples to the order parameter ψ. In the context of the effective theory it contributes a scalar degree of freedom ϕ_i, the Stueckelberg field of the electromagnetic gauge transformations [41].

In the normal phase, the scalar ϕ_i describes the diffusive dynamics of charge density [41]. For this reason, the effective field theory must also be invariant under the separate diagonal time independent gauge transformation [1] [163, 41],

$$\phi'_r = \phi_r + \sigma\,, \quad \phi'_\alpha = \phi_\alpha\,, \quad \partial_t \sigma = 0\,. \tag{4.2.4}$$

Note that this is a distinct transformation from the electromagnetic gauge transformations as it does not involve neither the order parameter nor the external gauge field. This is known as a "chemical shift" symmetry [163] and it is imposed in normal phase effective theories in order to exclude the presence of supercurrents. However, in nearly critical systems this symmetry is stronger than imposing the absence of supercurrents in their normal phase. Moreover, it will be necessary in order to match with both Model F [4] as well as with the equivalent holographic system of chapter 3.

It will be useful to write down the symmetry transformations that our effective theory should be invariant under in the notation of the r- and α-fields. Keeping only the leading order terms in the α-fields we obtain,

$$\begin{aligned}
\psi'_r &= e^{iq\lambda_D}\psi_r\,, \quad \psi'_\alpha = e^{iq\lambda_D}(\psi_\alpha + iq\lambda_A\psi_r)\,,\\
\phi'_r &= \phi_r + \lambda_D + \sigma\,, \quad \phi'_\alpha = \phi_\alpha + \lambda_A\,,\\
A'_{r\mu} &= A_{r\mu} - \partial_\mu\lambda_D\,, \quad A'_{\alpha\mu} = A_{\alpha\mu} - \partial_\mu\lambda_A,
\end{aligned} \tag{4.2.5}$$

where we have defined the diagonal $\lambda_D = \frac{1}{2}(\lambda_1 + \lambda_2)$ and the anti-diagonal $\lambda_A = \lambda_1 - \lambda_2$ gauge transformation parameters. The transformations of (4.2.5) have been linearised in λ_A since the latter is of order $\mathcal{O}(\beta)$ in the semiclassical limit we are interested in. Given our discussion around symmetries, the gauged symmetry should

[1] A recent construction [165] of critical models for superfluidity has used a similar condition. However, the reasoning in footnote 4 of [165] would, in principle, allow for the extra terms we discuss below equation (4.2.14). Our model agrees with [165] as long as we impose the constraint $\kappa_0 = -q\, c_6^I$.

exist on the forward as well as on the backward branch of the time evolution path. However, we would only have to worry about the global part of the diagonal λ_D. Finally, we will also consider source fields $s_{r,\alpha}$ for the charged scalar transforming in the same way with the fields $\psi_{r,\alpha}$ in (4.2.5).

The appearance of more that one field transforming non-trivially under the gauge transformations allow us to write down the gauge invariant combinations,

$$\begin{aligned} \hat{\psi}_r &= e^{-iq\phi_r}\psi_r\,, \qquad \hat{\psi}_a = e^{-iq\phi_r}(\psi_a - iq\phi_a\psi_r)\,, \\ B_{r\mu} &= \partial_\mu\phi_r + A_{r\mu}\,, \qquad B_{a\mu} = \partial_\mu\phi_a + A_{a\mu}\,. \end{aligned} \tag{4.2.6}$$

A natural choice for the local chemical potential is the time component of the gauge invariant r-vector component $\mu = B_{r\,0}$ which coincides with the thermodynamic chemical potential in equilibrium. For spacetime dependent configurations, this choice will fix our hydrodynamic frame. In terms of our gauge invariant variables, the time independent gauge transformation (4.2.4) takes the form,

$$\begin{aligned} \hat{\psi}'_r &= e^{-iq\sigma}\hat{\psi}_r\,, \quad \hat{\psi}'_\alpha = e^{-iq\sigma}\hat{\psi}_\alpha\,, \\ B'_{ri} &= B_{ri} + \partial_i\sigma\,, \quad B'_{rt} = B_{rt}\,, \quad B'_{\alpha\mu} = B_{\alpha\mu}\,. \end{aligned} \tag{4.2.7}$$

leading to the natural covariant derivative,

$$D_i\hat{\psi}_{r,\alpha} = \partial_i\hat{\psi}_{r,\alpha} + i\,qB_{r\,i}\,\hat{\psi}_{r,\alpha}\,, \quad D_t\hat{\psi}_{r,\alpha} = \partial_t\hat{\psi}_{r,\alpha}\,. \tag{4.2.8}$$

The above discussion shows that the effective theory we are after is essentially a combination of the $U(1)$ symmetric Model A with the charge diffusion model of [41]. The degrees of freedom of the two systems are then coupled because the diagonal parts of the respective global $U(1)$ symmetries are identified. Moreover, the global $U(1)$ symmetry of the Model A sector is gauged by the time independent gauge parameter of charge diffusion.

In addition to the continuous symmetry transformations, our effective theory should also be invariant under a set of KMS transformations. Our thermal state involves a finite chemical potential, making simpler to consider the discrete transformation $\Theta = \hat{P}\,\hat{T}$ involving parity and time reversal. The KMS transformations then read[2],

$$\begin{aligned} \tilde{B}_{r\mu}(-x) = B_{r\mu}(x)\,, \quad \tilde{B}_{a\mu}(-x) = B_{a\mu}(x) + i\beta\,\partial_t B_{r\mu}(x)\,, \\ \tilde{\hat{\psi}}_r(-x) = \hat{\psi}^\star_r(x)\,, \quad \tilde{\hat{\psi}}_a(-x) = \hat{\psi}^\star_a(x) + i\beta\,\partial_t\hat{\psi}^\star_r(x)\,. \end{aligned} \tag{4.2.9}$$

The KMS transformation rules (4.2.9) preserve the total number of α-field factors and time derivatives of r-fields. It therefore makes sense to write an expansion for the effective Lagrangian density according to,

$$\mathcal{L}_{EFT} = \mathcal{L}^{[1]} + \mathcal{L}^{[2]} + \cdots\,, \tag{4.2.10}$$

where the term $\mathcal{L}^{[n]}$ contains terms with a total of n factors of α-fields and time derivatives of r-fields. In this notation, the second line of equation (4.2.3) guarantees that $\mathcal{L}^{[0]}$ vanishes.

Here, we will give the highlights of a more detailed argument for the construction, which can be found in Appendix C. Keeping only linear terms in the scalar field sources which are invariant under the symmetry as well as the KMS transformations, the leading term in (4.2.10) takes the form,

$$\begin{aligned} \mathcal{L}^{[1]} =& 2\,\mathrm{Re}\left[e^{iq\phi_r} s^\star_r\,\hat{\psi}_\alpha + e^{-iq\phi_r}\,\hat{\psi}^\star_r s_\alpha + iq\,e^{iq\phi_r}\,\hat{\psi}_r\,s^\star_r\,\phi_\alpha\right] \\ &+ \frac{\delta S_0}{\delta B_{r\,\mu}}\,B_{\alpha\,\mu} + 2\,\mathrm{Re}\left(\frac{\delta S_0}{\delta\hat{\psi}_r}\,\hat{\psi}_\alpha\right)\,. \end{aligned} \tag{4.2.11}$$

The functional S_0 takes the general form,

$$S_0[\hat{\psi}_r, \hat{\psi}^\star_r, \phi_r; A_{r\,\mu}] = -\int d^{d-1}x\,F\,, \tag{4.2.12}$$

with F being a function of quantities which are invariant under the transformation (4.2.7), such as $\hat{\psi}_r\,\hat{\psi}^\star_r$, $D_i\hat{\psi}_r\,D^i\hat{\psi}^*_r$ and $B_{r\,t}$.

Close to the phase transition, the function F should be close to the free energy density

[2]For notational simplicity, we only give the parity transformation rules in odd number of spatial dimensions. In general, we should only be flipping the sign of only one spatial coordinate. Our conclusions would remain unchanged.

F_0 of the normal phase. Introducing a perturbative parameter ε and assuming the scaling behaviour,

$$\partial_t \propto \mathcal{O}(\varepsilon^2)\,, \quad \partial_i \propto |\hat{\psi}_r| \propto \mathcal{O}(\varepsilon)\,, \quad \phi_r \propto \mathcal{O}(\varepsilon^0)\,. \tag{4.2.13}$$

we can write an expansion,

$$F = F_0 - \rho_n\, B_{r\,t} - \frac{1}{2}\chi_n\, B_{r\,t}^2 + r_0\, |\hat{\psi}_r|^2 + \frac{1}{2}u_0\, |\hat{\psi}_r|^4 \\ + \kappa_0\, B_{r\,t}\, |\hat{\psi}_r|^2 + w_0\, D_i\hat{\psi}_r\, D^i\hat{\psi}_r^\star\,, \tag{4.2.14}$$

with u_0, w_0 and κ_0 finite while $r_0 \propto \mathcal{O}(\varepsilon^2)$. Moreover, we have shifted $B_{r\,t}$ by the chemical potential of the thermal state μ_0 and all the bare constants appearing in (4.2.14) are now functions of β and μ_0. The symmetry (4.2.4) is stronger than demanding that the effective theory should be independent of purely spatial derivatives of ϕ_r in the absence of the condensate. This would allow for terms of the form $|\psi_r|^2\, B_{r\,i}\, B_r{}^i$ and $B_r{}^i\, \mathrm{Im}(\hat{\psi}_r^\star\, D_i\hat{\psi}_r)$ which are again of order $\mathcal{O}(\varepsilon^4)$ and vanish in the normal phase.

For the second term in the expansion (4.2.10) we find the gauge invariant expression,

$$\mathcal{L}^{[2]} = -\,2\,i\,\beta^{-1}c_6^R\,\hat{\psi}_\alpha\hat{\psi}_\alpha^\star + c_6\,\hat{\psi}_\alpha\,\partial_t\hat{\psi}_r^\star + c_6^\star\,\hat{\psi}_\alpha^\star\,\partial_t\hat{\psi}_r \\ -\,i\beta^{-1}c_5\,B_{\alpha\,i}^2 + c_5\,B_{\alpha\,i}\,\partial_t B_{r\,i}\,, \tag{4.2.15}$$

which also respects the KMS symmetry (4.2.9). Moreover, the third condition of constraints (4.2.3) implies that $c_6 = c_6^R + i\,c_6^I$ can be complex with $c_6^R \leq 0$ and $c_5 \leq 0$. Our effective action is invariant under charge conjugation $\hat{C}$ since as we show in Appendix C, e.g. the constant κ_0 is odd under $\hat{C}$.

Finally, the Noether current of the global symmetry that rotates the α fields can be obtained be differentiating the effective action with respect to the α-vector fields,

$$\hat{J}_r^i = -2q\,\mathrm{Im}\left(\hat{\psi}_r^\star\,\frac{\partial F}{\partial D_i\hat{\psi}_r^\star}\right) + c_5\left(\partial^i\mu - E^i\right) - i\frac{2\,c_5}{\beta}B_\alpha{}^i\,, \\ \hat{J}_r^t = -\frac{\partial F}{\partial B_{r\,t}}\,, \tag{4.2.16}$$

where $E_i = \partial_i A_{r\,t} - \partial_t A_{r\,i}$ is the external electric field.

4.3 Model F from Keldysh-Schwinger

An interesting fact about Keldysh-Schwinger effective theories at quadratic level in α-fields is that they are equivalent to a set of kinetic equations for the r-fields [41]. In particular, we will recast our theory as the stochastic Model F of [4] which describes the nearly critical dynamics of a complex order parameter and the corresponding conserved current.

As we show in Appendix D, by using standard path integral techniques, we can recast the path integral (4.2.1) in terms of noise fields instead of the α-fields. More specifically, we can trade $\hat{\psi}_\alpha$ and $B_{\alpha\,\mu}$ for the complex noise field z and the real vector noise field ζ_μ respectively following a Gaussian distribution with zero mean.

Written in terms of the function F and the chemical potential μ, the system of stochastic equations becomes,

$$
\begin{aligned}
c_6^\star\, \partial_t \hat{\psi}_r &= \frac{\partial F}{\partial \hat{\psi}_r^\star} - D_i \left(\frac{\partial F}{\partial D_i \hat{\psi}_r^\star} \right) - e^{-iq\phi_r}\, s_r - z\,, \\
\partial_\mu J_r^\mu &= \mathrm{Im}\left[\hat{\psi}_r^\star\, e^{-iq\phi_r}\, s_r \right] - \partial_i \zeta^i\,,
\end{aligned}
\tag{4.3.1}
$$

where the current is given by (4.2.16) with trivial $B_{\alpha\,i}$. The real and imaginary parts of z and the components ζ_i have variance $-c_6^R/\beta$ and $-2\,c_5/\beta$ respectively. In the equations above, we have set the α-sources equal to zero since we are interested in comparing with a theory which is meant to compute only the retarded Green's functions and the r-sources are sufficient for this purpose.

We now turn our attention to Model F of [4] which we will match with the stochastic system (4.3.1). The effective degrees of freedom in that description are the charge density excitation m above the normal phase and the complex order parameter ψ which satisfy the stochastic system of equations,

$$
\partial_t \psi = -2\,\Gamma_0\, \frac{\delta W}{\delta \psi^\star} - i\, g_0\, \frac{\delta W}{\delta m}\, \psi + \theta\,,
$$

$$\partial_t m = \lambda_0^m \, \partial_i \partial^i \left(\frac{\delta W}{\delta m} \right) + 2 g_0 \, \mathrm{Im} \left[\psi^\star \, \frac{\delta W}{\delta \psi^\star} \right] + \zeta_H \,, \tag{4.3.2}$$

where Γ_0 is a complex parameter with positive real part, λ_0^m is a positive constant, θ and ζ_H are noise fields. More specifically, the real and imaginary parts of the random field θ and the field ζ_H follow Gaussian distributions of zero mean and variances $\mathrm{Re}\,\Gamma_0/\beta$ and $-2\lambda_0^m/\beta \, \partial_i \partial^i$ respectively [166].

The variance of the z-distribution was fixed by the KMS conditions. In contrast, the variance of the field θ for Model F, is fixed by demanding that the fluctuation-dissipation theorem is satisfied. Equivalently, the relevant Fokker-Planck equation admits the grand canonical probability distribution $P[\psi] = Z^{-1} \, e^{-\beta \, W[\psi]}$ as a fixed point. The latter highlights that W plays the role of Ginzburg-Landau-Wilson free energy.

Assuming that the free energy W depends analytically on the fields ψ, m and the corresponding classical sources h and h_m, we can write the leading order terms,

$$\begin{aligned} W =& W_0 - \int d^{d-1}x \; \left[h_m \, m + \mathrm{Re}(h \, \psi^\star) \right] \,, \\ W_0 =& \int d^{d-1}x \; \left[\frac{1}{2} \tilde{r}_0 \, |\psi|^2 + \frac{1}{2} \tilde{w}_0 \, |\nabla \psi|^2 + \tilde{u}_0 \, |\psi|^4 \right. \\ & \left. + \frac{1}{2\, C_0} \, m^2 + \gamma_0 \, m \, |\psi|^2 \right] \,. \end{aligned} \tag{4.3.3}$$

We note that [4] has $\tilde{w}_0 = 1$ but this is not a significant difference as w_0 can be set to unity via a field redefinition.

In order to compare the stochastic systems (4.3.1) and (4.3.2), we need to perform a change of variables from the charge density difference m to the chemical potential $\mu_h = \frac{\delta W_0}{\delta m}$. To achieve this, we consider the Legendre transformation of the energy potential W_0 according to $\tilde{W}_0 = W_0 - \int \mu_h \, m$.

To match the system of equations (4.3.2) to the one obtained from the Keldysh-Schwinger effective action (4.3.1), we identify,

$$\begin{aligned} &\psi = e^{iq\phi_r} \hat{\psi}_r \,, \quad \mu_h = \mu \,, \quad h = 2\, s_r \,, \quad h_m = A_t \,, \\ &A_i = 0 \,, \quad z = -c_6^\star \, e^{-iq\phi_r} \, \theta \,, \quad \zeta_H = -\partial_i \zeta^i \,, \quad q = -g_0 \,. \end{aligned}$$

Then the matching requires that,

$$\tilde{W}_0 = \int d^{d-1}x \, \left(F - F_0(\mu_0) + \rho_n \, B_{r\,t}\right) \, ,$$
$$c_6^\star = -\frac{1}{2\,\Gamma_0} \, , \quad c_5 = -\lambda_0^m \, . \tag{4.3.4}$$

In the previous chapter, we matched the dynamics of Model F with a suitable class of holographic models of superfluidity. At the level of mean field theory, we showed that the holographic theories capture the same dynamics near criticality with Model F. However, given that holography provides a weakly coupled description of a strongly coupled microscopic theory, the transport coefficients Γ_0 and λ_0^m (or equivalently c_6 and c_5) where fixed in terms of black hole horizon invariants.

4.4 Superfluid Hydrodynamics at Low Energies

In this section we will consider the limit of our effective theory at energies much lower than the gap of the order parameter amplitude mode. It is useful to examine the spectrum of fluctuations of our r-fields around the vacuum solution of equations (4.3.1) in the mean field limit setting the sources s_r and s_α for the order parameter to zero. To proceed, we consider perturbative fluctuations around the background with $B_{r\,0} = B_{r\,i} = 0$ and $\hat{\psi}_r = \psi_0$. The analysis is very similar to that of chapter 3 revealing a gapped mode along with a pair of propagating sound modes for wavevectors much smaller than the gap,

$$\omega_H = -i\,\omega_g - i\,D_H\,k^2 + \cdots \, ,$$
$$\omega_\pm = \pm\sqrt{\frac{\chi_{JJ}}{\chi_b}}\,k - i\,D_s\,k^2 + \cdots \, , \tag{4.4.1}$$

where we have set,

$$\omega_g = -2\,\frac{c_6^R}{|c_6|^2}\,|\psi_0|^2\,u_0\,\frac{\chi_b}{\chi_n} = \frac{c_6^R}{|c_6|^2}\,\frac{4\,\Delta E_0}{|\psi_0|^2} \, ,$$
$$\chi_b = \chi_n + \frac{\kappa_0^2}{u_0} = \chi_n\,\frac{\Delta E_0}{\Delta F_0} \, . \tag{4.4.2}$$

In the above equations, χ_b is the charge susceptibility of the broken phase (see also chapter 3), ΔE_0 and ΔF_0 are the mean field energy and the free energy difference between the broken and normal phase. The low energy limit we wish to derive, by integrating out the gapped, amplitude mode ω_H, captures the sound mode $\omega_\pm$. We will postpone an explicit expression for D_s until we have written down the low energy limit theory.

We introduce a perturbative parameter λ and take the time and spatial derivatives to be of order $\partial_t \approx \partial_i \approx \mathcal{O}(\lambda)$. The mean field theory part of the stochastic system of equations (4.3.1) suggests that our r-fields scale according to,

$$\begin{aligned} \hat{\psi}_r &= \rho_r + \delta\rho_r + i\, q\, \rho_r\, \delta\theta_r + \cdots\,, \\ \delta\rho_r &\approx \delta\theta_r \approx B_{r\,\mu} \approx \mathcal{O}(\lambda)\,, \end{aligned} \tag{4.4.3}$$

where we have taken the background VEV $\psi_0 = \rho_r$ of the complex scalar to be real.

As we show in Appendix E, by expanding the generating functional (4.2.1) to order $\mathcal{O}(\lambda^4)$, we can perform the path integration over the fields $\delta\rho_r$, $\delta\theta_r$ and $\hat{\psi}_\alpha$ to find the low energy effective Lagrangian density,

$$\begin{aligned} \mathcal{L}_{sf} = \rho_b\, C_{\alpha\,0} - c_5 \left(\frac{i}{\beta}\, C^2_{\alpha\, i} - C_{\alpha\, i}\, \partial_t C_{r\, i} \right) - \chi_{JJ}\, C_r{}^i\, C_{\alpha\, i} \\ + \chi_b\, C_{\alpha\, 0}\, C_{r\, 0} + \chi_b^2\, \zeta_3 \left(\frac{i}{\beta}\, C^2_{\alpha\, 0} - C_{\alpha\, 0}\, \partial_t C_{r\, 0} \right)\,, \end{aligned} \tag{4.4.4}$$

with the gauge invariant vectors $C_{r,\alpha} = \partial\varphi_{r,\alpha} + A_{r,\alpha}$. We have used the identification (4.4.2) and defined the broken phase charge density and current susceptibility,

$$\rho_b = \rho_n - \kappa_0\, \rho_r^2\,, \quad \chi_{JJ} = 2\, w_0\, q^2\, \rho_r^2\,. \tag{4.4.5}$$

Following the details in Appendix E, the angle φ_r coincides with the phase of the original order parameter variable ψ_r that we introduced in section 4.2. The real constant ζ_3 can be expressed as,

$$\zeta_3 = \frac{(\chi_b\, u_0 + q\, \kappa_0\, c_6^I)^2 + q^2\, \kappa_0^2\, (c_6^R)^2}{\omega_g\, q^2\, \chi_b\, \chi_n\, |c_6|^2\, u_0}\,. \tag{4.4.6}$$

Our low energy degrees of freedom follow the same gauge transformation rules with the phase variables ϕ_r and ϕ_α in equation (4.2.5). However, the resulting action does not possess an analog of the time independent gauge symmetry (4.2.4). It is interesting to point out that the hydrodynamic frame we seem to have landed in is the same with the natural frame of holography [2].

Following the details in Appendix E we can show that the dispersion relations of the perturbative modes of our low energy theory yield the mode $\omega_\pm$ of equation (4.4.1) with,

$$D_s = \frac{1}{2\,\chi_b}\left(\chi_{JJ}\,\chi_b\,\zeta_3 - c_5\right) . \tag{4.4.7}$$

According to the ε expansion of section 4.2, the bulk viscosity ζ_3 in equation (4.4.6) is of order $\mathcal{O}(\varepsilon^{-2})$. This is certainly to be expected since we have integrated out a mode whose gap behaves like $\mathcal{O}(\varepsilon^2)$ close to the phase transition. However, the constant D_s remains finite since the current-current susceptibility behaves like $\mathcal{O}(\varepsilon^2)$ close to the transition as we can see from the expression (4.4.5).

As we pointed out in the previous chapter, in the normal phase the original effective theory has two gapped modes as well as a gapless mode describing charge diffusion with diffusion constant $D_e = -c_5/\chi_n$. This suggests a rearrangement of the degrees of freedom across the phase transition, in agreement with earlier observations concerning holographic models [2]. An interesting observation is that the product $\chi_{JJ}\,\zeta_3$ in equation (4.4.7) remains finite close to the transition. Given that statistical fluctuations become strong near the critical point, it would be interesting to explore how non-linearities could affect this conclusion.

4.5 Discussion

We constructed the effective theory for the critical dynamics close to a superfluid phase transition in the framework of the Keldysh-Schwinger formalism. An important ingredient to match with Model F [4] as well as with the holographic computation of chapter 3 was the "chemical shift" symmetry [163, 41] of equation (4.2.4), which excluded a number of terms from the effective action and which would otherwise be allowed in our ε expansion of the free energy in equation (4.2.14).

In this chapter we focused entirely on the coupled sector of the complex order parameter and the corresponding conserved current. This has simplified the problem as it allowed us to focus on some of the crucial aspects of the construction. This is in direct analogy with the probe limit of holographic theories which decouple the metric fluctuations from the charged and condensed degrees of freedom. However, at finite density this sector also couples to the energy-momentum of the system leading to a much larger description involving the local temperature and normal fluid velocity. We will leave this construction for future investigation.

As we observed at the end of section 4.4, even though the superfluid effective theory is expected to break down close to the phase transition, the dispersion relations (4.4.1) remain finite. These conclusions were drawn from the superfluid phase point of view leaving open the question of the effect of higher order noise interactions [158, 159].

Chapter 5

Conclusion

In this thesis we have studied superfluid phase transitions at finite temperature and chemical potential in the hydrodynamic regime. In order to provide an effective description of such systems we used both holographic methods and the Schwinger-Keldysh formalism for effective theories.

In chapter 1 we reviewed several concepts that are indispensable for an overall understanding of our investigations in the main body of the thesis. We provided a brief overview of critical phenomena, conventional hydrodynamics, fluctuating hydrodynamics and the Schwinger-Keldysh formalism, and gauge/gravity duality.

In chapter 2 we analysed dissipative effects in holographic superfluids at finite temperature and density. The novelty of our approach is in the use of the symplectic current of Crnkovic and Witten [1], which allows us to examine analytically linear, spacetime dependent perturbations in the bulk, in a general black hole background. The symplectic current is built out of two solutions of the linearised gravitational problem and is conserved in the bulk. This property is a crucial one, since upon integrating the continuity equation obeyed by the symplectic current along the radial direction of the aAdS spacetime, we can relate boundary quantities to horizon quantities directly.

One of the solutions entering the symplectic current is related to static fluctuations of the background and the other one is the hydrodynamic perturbation we want to study. In this particular problem there are two separate classes of static perturbations that we need to consider, namely thermodynamic perturbations and perturbations related to large diffeomorphisms of the background. The hydrodynamic perturbation is then constructed around the static solutions. More specifically, in the limit of zero wavevector and frequency, the hydrodynamic solution is a linear combination of the static perturbations we mentioned. A technical complication we encountered is the appearance of bulk integrals, after integrating the continuity equation of the symplectic current. As we showed, they can be eliminated after choosing a hydrodynamic frame. Thus, their appearance is directly related to frame invariance in hydrodynamics.

The upshot of this calculation is that we managed to confirm holographically the constitutive relations of superfluid hydrodynamics. In the regime of small superfluid velocities we found, as expected, that to leading in the derivative expansion we need five transport coefficients, the incoherent conductivity, the shear viscosity and the three bulk viscosities. For the shear viscosity over entropy density we found the universal value $\frac{1}{4\pi}$. For the other transport coefficients we found explicit expressions in terms of thermodynamics and horizon data, and as far as we know, these results are new.

The aforementioned calculation is valid far away from the critical point. This can be seen in the singular behaviour of various quantities as we approach the critical point, which in turn is attributed to the fact that the amplitude of the complex order parameter is integrated out. This observation naturally led us to apply the symplectic current technique to holographic superfluids in the critical regime in chapter 3, extending the results of [130]. In this case, we chose to work in the probe limit for simplicity. An important role in this discussion is played by the zero mode of the order parameter which drives the phase transition. Other static perturbations that are relevant here are the variations of the background, in both the broken and the normal phase of the system, with respect to the “distance” from the critical point.

Using the symplectic current, we managed to find an equation governing the dynamics of the amplitude mode, constitutive relations for the components of the conserved current and a Josephson relation for the local chemical potential. These equations, together with the conservation of the current yield a closed system, describing the effective dynamics of the holographic superfluid close to criticality. In addition, we recast these resulting equations in a different form, in order to compare with model F [4], with which we found complete agreement, in the limit where the noise fluctuations are ignored. At the same time, we derived an analytic expression for the complex kinetic coefficient of model F and considered the behaviour of the quasinormal modes in various limits, confirming some of our results with numerics. We should note that our approach differs from previous work in similar problems, which either resorted solely to numerics [131, 132, 133, 134] or dealt with models solvable near the critical point [135, 31].

Moving on to chapter 4, we tackled the problem of superfluid critical dynamics from a completely different perspective. In this chapter, we applied the recently developed formalism of Schwinger-Keldysh effective theories (for references see chapter 4) to the superfluid phase transition. As in the previous chapter, we worked in the probe limit, neglecting fluctuations of the temperature and of the normal fluid velocity. We constructed the effective action in the r/a basis, working to second order in the a-fields and to leading order in derivative corrections. A time independent gauge symmetry called "chemical shift symmetry" constrained significantly the allowed terms in the effective action. As it is well known, working to second order in the a-fields is equivalent to stochastic hydrodynamics with Gaussian noise. Indeed, in this chapter we showed explicitly the equivalence for this particular problem and the resulting equations were those of model F. Another notable output of chapter 4 was that, for energies well below the gap of the amplitude mode, we integrated it out of the theory and landed on conventional superfluid hydrodynamics, away from criticality.

Following our work in chapters 2 and 3, the obvious next step we left for future investigation, is to tackle the full problem near criticality, away from the probe limit. This would entail including the metric fluctuations in the bulk, which couple to the gauge field and order parameter fluctuations. In this way we could find an effective description of the dynamics of the phase transition, where all conserved densities and the order parameter are included. Similarly, we could extend our results in chapter 4 away from the probe limit and compare with those we would find from holography. Another possible root to follow, would be to extend our holographic results to second order hydrodynamics. All these problems, given the work presented in this thesis, are rather straightforward to deal with, although they are naturally expected to be much more involved, technically.

Another stimulating question we would like to pursue in the future regards the notoriously difficult problem of fluctuations in this setting. More specifically, all of our holographic results, including the behaviour of the quasinormal modes, necessarily presuppose the suppression of all noise fluctuations, since we are working in classical gravity and thus, they are essentially mean field predictions. On the other hand, the Schwinger-Keldysh formalism, as it has been shown in recent years, provides a systematic, unified framework to construct effective theories and, most importantly, it provides a way to go beyond Gaussian noise. It has already been applied to several problems, with interesting findings [158, 159, 160]. The pressing question is to investigate whether these new methods alter, either qualitatively or quantitatively, the established theory of dynamical critical phenomena and fluctuating hydrodynamics and to examine what holography can teach us in this respect.

Appendix A

Constraints for static perturbations

In this Appendix we will list the constraints resulting from the symplectic current when constructed from the static pairs of perturbations discussed in subsections 2.3.2 and 2.3.3. In particular, given that for those pairs of perturbations only the radial component is non-trivial, the divergence free condition (2.4.2) yields the radial constraint,

$$P^r_{\delta_1,\delta_1} = P^r_{\delta_1,\delta_1}\Big|_{r=0} = P^r_{\delta_1,\delta_1}\Big|_{r=\infty} \,. \tag{A.0.1}$$

The inequivalent constraints we can obtain read,

$$\begin{aligned}
&-e^{2g}a'\delta f_b + (aa' - U' + 2e^{2g}g')\delta f_g + aU\delta f_b' + (U - e^{2g})\delta f_g' = -\varrho + \mu_t\,\chi_{jj}\,,\\
&e^{2g}a'\delta f_b - 2e^{2g}g'\delta f_g + e^{2g}\delta f_g' = \varrho - \mu_t\,\chi_{jj}\,,\\
&-a'\partial_{\mu_t}a + U\phi'\partial_{\mu_t}\phi + U\rho'\partial_{\mu_t}\rho + 2U\partial_{\mu_t}g' + \partial_{\mu_t}U' = 0\,,\\
&e^{2g}\Big(2(U' - 2Ug' - aa')\partial_{\mu_t}g - 2g'\partial_{\mu_t}U - U\phi'\partial_{\mu_t}\phi - U\rho'\partial_{\mu_t}\rho - 4U\partial_{\mu_t}g' - a\partial_{\mu_t}a'\Big) = \xi\,T\,,\\
&e^{2g}\Big(2(U' - 2Ug' - aa')\partial_T g - 2g'\partial_T U - U\phi'\partial_T\phi - U\rho'\partial_T\rho - 4U\partial_T g' - a\partial_T a'\Big) = c_\mu\,,\\
&e^{2g}\Big(-a'\partial_T a + U\phi'\partial_T\phi + U\rho'\partial_T\rho + 2U\partial_T g' + \partial_T U'\Big) = s\,,\\
&e^{2g}\,(U' - 2Ug' - aa') = s\,T\,.
\end{aligned} \tag{A.0.2}$$

The above equations where obtained by considering the symplectic currents in the order $P^{\mu}_{\delta_{v_i},\delta_{m_i}}$, $P^{\mu}_{\delta_{s_{it}},\delta_{m_i}}$, $P^{\mu}_{\delta_{s_{ii}},\delta_{m_t}}$, $P^{\mu}_{\delta_{s_{tt}},\delta_{m_t}}$, $P^{\mu}_{\delta_T,2\delta_{s_{ii}}-\mu_t\,\delta_{m_t}}$, $P^{\mu}_{\delta_{s_{ii}},\delta T}$ and $P^{\mu}_{\delta_{s_{it}},\delta_{v_i}-\mu_t\,\delta_{m_i}}$. Finally, from the symplectic current $P^{\mu}_{\delta_T,\delta_{m_t}}$ we obtain the bulkier constraint,

$$
\begin{aligned}
&-2e^{2g}\left(-a'\partial_T a + U\phi'\partial_T\phi + U\rho'\partial_T\rho + 2U\partial_T g' + \partial_T U'\right)\partial_{\mu_t} g \\
&+2e^{2g}\left(-a'\partial_{\mu_t} a + U\phi'\partial_{\mu_t}\phi + U\rho'\partial_{\mu_t}\rho + 2U\partial_{\mu_t} g' + \partial_{\mu_t} U'\right)\partial_T g \\
&-e^{2g}\left(2\partial_T g' + \rho'\partial_T\rho + \phi'\partial_T\phi\right)\partial_{\mu_t} U + e^{2g}\left(2\partial_{\mu_t} g' + \rho'\partial_{\mu_t}\rho + \phi'\partial_{\mu_t}\phi\right)\partial_T U \\
&+e^{2g}U\left(\partial_T\rho'\partial_{\mu_t}\rho - \partial_T\rho\partial_{\mu_t}\rho'\right) + e^{2g}U\left(\partial_T\phi'\partial_{\mu_t}\phi - \partial_T\phi\partial_{\mu_t}\phi'\right) \\
&+e^{2g}\left(\partial_{\mu_t} a'\partial_T a - \partial_{\mu_t} a\partial_T a'\right) = -\xi\,. \qquad \text{(A.0.3)}
\end{aligned}
$$

Appendix B

Susceptibility relations

The aim of this Appendix is to prove equation (3.5.10). In order to do this, we remind the reader a couple of facts about the free energy difference between the normal and broken phases $\Delta w_{FE}(\mu, \phi_s)$. Since we don't vary temperature in our probe model, we have suppressed the dependence on it. The basic property that this function satisfies is that it vanishes on the critical hypersurface $(\mu_c(\phi_s), \phi_s)$,

$$\Delta w_{FE}\left(\mu_c(\phi_s), \phi_s\right) = 0\,. \tag{B.0.1}$$

Moreover, with the transition being second order, the normal derivative with respect to the hypersurface also vanishes. These two statement imply that we indeed have,

$$\nabla \Delta w_{FE}\left(\mu_c(\phi_s), \phi_s\right) = 0\,, \tag{B.0.2}$$

and that the points $(\mu_c(\phi_s), \phi_s)$ are not extrema.

This shows that the Hessian matrix of $\Delta w_{FE}(\mu, \phi_s)$ evaluated on the critical surface should only have one non-zero eigenvalue which should also be positive. As a consequence, the determinant of the Hessian should vanish showing the relation,

$$\left(\nu_\mu^\star - \nu_\mu^\#\right)^2 = \left(\nu_\phi^\star - \nu_\phi^\#\right)\,\left(\chi_{QQ}^\star - \chi_{QQ}^\#\right)\,. \tag{B.0.3}$$

Using this relation, it is then easy to show that,

$$\Delta w_{FE} = -\frac{1}{\chi^{\star}_{QQ} - \chi^{\#}_{QQ}} \frac{(\Delta \varrho)^2}{2}\,, \tag{B.0.4}$$

which is equivalent to (3.5.10) given equation the relation (3.4.12).

Appendix C

Effective Action Construction

In this Appendix we will discuss some of the details for the derivation of the first couple of terms in the effective Lagrangian expansion (4.2.10). We start by discussing the source s_i of the complex scalar $\hat{\psi}_i$ and how it appears in equation (C.0.3). The reason we would like to include such sources in our description is twofold. The first and most obvious is that they would allow us to compute correlation functions for the order parameter. The second is that static, background sources will allow us to study the effects of explicit symmetry breaking in our system. In order to preserve the gauge symmetry transformations, the source s needs to transform in the same way with the order parameter ψ_i according to,

$$s_i' = e^{iq\lambda_i}\, s_i\,, \qquad s_i^{\star\,\prime} = e^{-iq\lambda_i}\, s_i^\star\,. \tag{C.0.1}$$

It is also useful to note that under a KMS transformation, the complex scalar sources transform as,

$$\tilde{s}_r(-x) = s_r^\star(x)\,, \qquad \tilde{s}_\alpha(-x) = s_\alpha^\star(x) + i\beta\,\partial_t s_r^\star(x)\,. \tag{C.0.2}$$

For the purposes of our paper we will only be interested in perturbations of the complex scalar sources and we will only include terms which are linear in s_r and s_α.

The first property listed in equation (4.2.3) suggests that we can write,

$$\mathcal{L}^{[1]} = a^{\mu}\, B_{\alpha\,\mu} + 2\,\mathrm{Re}\left[a_{\psi}\,\hat{\psi}_{\alpha} + e^{iq\phi_r} s_r^{\star}\,\hat{\psi}_{\alpha} \right. \\ \left. + e^{-iq\phi_r}\,\hat{\psi}_r^{\star} s_{\alpha} + iq\, e^{iq\phi_r}\,\hat{\psi}_r\, s_r^{\star}\,\phi_{\alpha}\right] , \tag{C.0.3}$$

with the coefficients a_{ψ} and a^{μ} being functions of the r-fields. This form satisfies all properties of equation (4.2.3), provided that a^{μ} is real. Notice that the terms involving the complex scalar source $s_{r,\alpha}$ are invariant under the symmetry transformations (4.2.5),(4.2.7) and (C.0.1) as well as the KMS transformations (4.2.9) and (C.0.2). Imposing that the rest of the expression (C.0.3) is invariant under the KMS transformations (4.2.9) gives that,

$$a_{\psi} = \frac{\delta S_0}{\delta \hat{\psi}_r} , \quad a^{\mu} = \frac{\delta S_0}{\delta B_{r\,\mu}} , \tag{C.0.4}$$

for some functional S_0 of our r-fields,

$$S_0[\hat{\psi}_r, \hat{\psi}_r^{\star}, \phi_r; A_{r\,\mu}] = -\int d^{d-1}x\, F\, . \tag{C.0.5}$$

The final step is to ensure that (C.0.3) is also invariant under the transformations given in equations (4.2.7) and (C.0.1). Considering first derivatives of our fields, it is easy to check that any function F of the invariant quantities[1],

$$\hat{\psi}_r\,\hat{\psi}_r^{\star} , \quad D_i\hat{\psi}_r\, D^i\hat{\psi}_r^{*} , \quad B_{r\,0} , \tag{C.0.6}$$

satisfy our criteria. This allows us to write the most general function that satisfies our constraints as,

$$F = F\left(\hat{\psi}_r\,\hat{\psi}_r^{\star}, D_i\hat{\psi}_r\, D^i\hat{\psi}_r^{*}, B_{r\,0}\right) . \tag{C.0.7}$$

Before constraining the function F further, we will turn our attention to the second term in the expansion of the effective action in (4.2.10). The most general expression with single time derivative terms we can write and which are invariant under the

[1] In fact, we could include gauge invariant scalars of the form $F_{r\,ij}F_r^{ij}$ with $F_{r\,ij} = \partial_i B_{r\,j} - \partial_j B_{r\,i}$. However, this scalar depends entirely on the source gauge field and we ignore it since we don't consider external magnetic fields.

transformations (4.2.7) reads,

$$
\begin{aligned}
\mathcal{L}^{[2]} =& i\, c_1\, \hat{\psi}_\alpha \hat{\psi}^\star_\alpha + i\, c_2\, B^2_{\alpha\, 0} + i c_3\, B^2_{\alpha\, i} + c_4\, B_{\alpha\, 0}\, \partial_t B_{r\, 0} \\
&+ c_5\, B_{\alpha\, i}\, \partial_t B_{r\, i} + c_6\, \hat{\psi}_\alpha\, \partial_t \hat{\psi}^\star_r + c^\star_6\, \hat{\psi}^\star_\alpha\, \partial_t \hat{\psi}_r\, . \qquad \text{(C.0.8)}
\end{aligned}
$$

The first line of the properties in equation (4.2.3) demand that $c_1, \ldots, c_5$ are real and $c_6 = c^R_6 + i\, c^I_6$ can be complex. Moreover, these can be functions of the invariant quantities (C.0.6). Imposing KMS invariance, one can easily show that these coefficients are constrained in a way such that,

$$
\begin{aligned}
\mathcal{L}^{[2]} =& -\frac{2\, i\, c^R_6}{\beta} \hat{\psi}_\alpha \hat{\psi}^\star_\alpha - i \frac{c_4}{\beta} B^2_{\alpha\, 0} + c_4\, B_{\alpha\, 0}\, \partial_t B_{r\, 0} \\
& - i \frac{c_5}{\beta} B^2_{\alpha\, i} + c_5\, B_{\alpha\, i}\, \partial_t B_{r\, i} + 2\, \mathrm{Re}\left(c_6\, \hat{\psi}_\alpha\, \partial_t \hat{\psi}^\star_r \right) \, . \qquad \text{(C.0.9)}
\end{aligned}
$$

Finally, imposing the third property of equation (4.2.3), we conclude that our dissipative coefficients must satisfy the inequalities $c^R_6, c_4, c_5 \leq 0$.

The off-shell conserved current reads,

$$
\begin{aligned}
\hat{J}^i_r =& \frac{\delta I_{EFT}}{\delta A_{\alpha\, i}} = J^i_r - i \frac{2\, c_5}{\beta} B_\alpha{}^i \\
=& -2q\, \mathrm{Im}\left(\hat{\psi}^\star_r\, \frac{\partial F}{\partial D_i \hat{\psi}^\star_r} \right) + c_5\, \partial_t B_r{}^i - i \frac{2\, c_5}{\beta} B_\alpha{}^i \, , \\
\hat{J}^0_r =& \frac{\delta I_{EFT}}{\delta A_{\alpha\, 0}} = J^0_r - i \frac{2\, c_4}{\beta} B_{\alpha\, 0} \\
=& -\frac{\partial F}{\partial B_{r\, 0}} + c_4\, \partial_t B_{r\, 0} - i \frac{2\, c_4}{\beta} B_{\alpha\, 0} \, , \qquad \text{(C.0.10)}
\end{aligned}
$$

where J^μ_r denotes the classical part of the electric current. The above shows that in thermodynamic equilibrium the charge density is given by the derivative of the function $-F$ with respect to $B_{r\, 0}$. In order to understand the role of the function F, it will be enlightening to consider the classical equations of motion of our system. These can obtained from the effective action by taking derivatives with respect to the α-fields and setting them equal to zero. Doing so reveals the the derivatives of F with respect to the complex order parameter is fixed by the classical source s_r. The above show that we can treat F as the Ginzburg-Landau-Wilson potential since

$\mu = B_{r\,0}$ is the chemical potential of the system. This suggests that, at the level of thermodynamics, the energy density of the system is $E = F + \mu\, J_r^0$.

Expanding the function F in powers of the order parameter near criticality we obtain,

$$\begin{aligned} F =& F_0(\mu) + r(\mu)\,|\hat{\psi}_r|^2 + \frac{1}{2} u(\mu)\,|\hat{\psi}_r|^4 \\ &+ w(\mu)\,D_i\hat{\psi}_r\,D^i\hat{\psi}_r^\star + \cdots \,. \end{aligned} \tag{C.0.11}$$

In the above expansion we introduced the constant F_0, which is identified as the normal phase free energy. Its first derivative gives minus the normal phase charge density ρ_n and its second derivative will therefore yield minus the susceptibility of the normal phase χ_n, so that in a semi-classical approximation,

$$\rho_n = -\left.\frac{\partial F_0}{\partial \mu}\right|_{\mu=\mu_0}, \qquad \chi_n = -\left.\frac{\partial^2 F_0}{\partial \mu^2}\right|_{\mu=\mu_0}, \tag{C.0.12}$$

where μ_0 is the value of the chemical potential in the thermal state.

It is useful to note that our functions F_0, r, u and w will in general depend on temperature as well as the the deformation parameters and coupling constants of the microscopic theory. For example, one can imagine that the system is deformed by a relevant operator which doesn't have to be included in our low energy description. However, the corresponding deformation parameters will in general enter in the effective action.

Being interested in the dynamics of our system close to the phase transition, it is reasonable to perform the shift,

$$B_{r\,0} \to \mu_0 + B_{r\,0}\,, \tag{C.0.13}$$

allowing us to treat $B_{r\,\mu}$ as a fluctuation around the thermal state. At the same time, close to the transition we will take our derivatives, fields and constants to scale according to equation (4.2.13). The KMS transformation rules (4.2.9) then give the α-field scaling rules,

$$\phi_\alpha \propto \mathcal{O}(\varepsilon^2)\,, \quad \hat{\psi}_\alpha \propto \mathcal{O}(\varepsilon^3)\,, \quad s_\alpha \propto \mathcal{O}(\varepsilon^5)\,. \tag{C.0.14}$$

Assuming that the constant c_4 behaves as a regular function of ε, the above scalings suggest that we can drop the corresponding term in (C.0.9) by setting $c_4 = 0$. This leads us to the expression quoted in equation (4.2.15).

Keeping terms up to order ε^4 in F (or order ε^6 in I_{EFT}) we arrive at the expression of equation (4.2.14). It is useful to understand the bare constants that appear in the expansion of equation (4.2.14) for the function F in the context of mean field theory. Demanding that the free energy is extremised by the mean field value ψ_0 of the order parameter in the undeformed theory, we can identify,

$$r_0 = \frac{2\,\Delta F_0}{|\psi_0|^2}\,, \qquad u_0 = -\frac{2\,\Delta F_0}{|\psi_0|^4}\,, \tag{C.0.15}$$

where ΔF_0 is the free energy density difference between the broken and the normal phase. In terms of mean field theory, our previous arguments lead to,

$$\kappa_0 = \left.\frac{\partial r}{\partial \mu}\right|_{\mu=\mu_0} = -\frac{\Delta\rho_0}{|\psi_0|^2}\,. \tag{C.0.16}$$

In the above expression, ψ_0 is the mean field value of the order parameter in the broken phase and $\Delta\rho_0$ is the charge density difference between the broken and the normal phase close to the transition.

For convenience, it is useful to note that the energy density difference between the broken and the normal phase is,

$$\Delta E_0 = \Delta F_0 - \frac{1}{2\,\chi_n}\,(\Delta\rho_0)^2\,, \tag{C.0.17}$$

where on the left hand side we have fixed charge density and on the right hand side we have fixed background chemical potential μ_0.

Appendix D

Derivation of the Stochastic System

In this Appendix we will give some of the details needed to derive the stochastic equations of motion that we have quoted in the main text in equation (4.3.1) along with the correlation functions for the noise fields.

In order to find the equations of motion with the appropriate noise terms from the effective theory of section 4.2, we will make use of the following well-known identity [1],

$$\int D\phi \, e^{-\int d^dx \, d^dy \, \phi(x) \, K(x,y) \, \phi(y) + i \int d^dx \, J(x) \, \phi(x)} = \det\left(\frac{K}{\pi}\right)^{-\frac{1}{2}} e^{-\frac{1}{4}\int d^dx \, d^dy \, J(x) \, K^{-1}(x,y) \, J(y)} . \qquad \text{(D.0.1)}$$

In what follows, numerical constants such as the determinant factor in the above expression will be absorbed in the integration measure as they carry no dependence on the interesting part which is the sources $J(x)$.

In order to obtain the stochastic equation of motion (4.3.1) for the complex order parameter $\hat{\psi}_r$, we first split the field $\hat{\psi}_\alpha$ in real and imaginary parts: $\hat{\psi}_\alpha = \hat{\psi}^R_\alpha + i\,\hat{\psi}^I_\alpha$.

[1] This is simply a path integral generalisation of the one-dimensional Gaussian integral.

We then introduce two real fields z_1, z_2 and apply(D.0.1) to obtain,

$$e^{\int d^dx\,\frac{2\,c_6^R}{\beta}(\hat{\psi}_\alpha^R)^2} = \int Dz_1\, e^{\int d^dx\,\frac{\beta}{2\,c_6^R} z_1^2 + 2\,i\,\hat{\psi}_\alpha^R\, z_1}\,,$$
$$e^{\int d^dx\,\frac{2\,c_6^R}{\beta}(\hat{\psi}_\alpha^I)^2} = \int Dz_2\, e^{\int d^dx\,\frac{\beta}{2\,c_6^R} z_2^2 + 2\,i\,\hat{\psi}_\alpha^I z_2}\,. \quad \text{(D.0.2)}$$

Using these identities, $\hat{\psi}_\alpha$ appears linearly in the effective action. As a result, the path integral over $\hat{\psi}_\alpha$ and $\hat{\psi}_\alpha^\star$ gives two delta functions, which constrain the r-fields $\hat{\psi}_r, \hat{\psi}_r^\star$ to be on-shell, obeying the complex scalar stochastic equation,

$$c_6^\star\,\partial_t\hat{\psi}_r = -\frac{\delta S_0}{\delta\hat{\psi}_r^\star} - e^{-iq\phi_r}\, s_r - z\,. \quad \text{(D.0.3)}$$

The complex noise field $z = z_1 + i\, z_2$ then satisfies,

$$\langle z(x)\, z(y)\rangle = \int Dz\, Dz^\star z(x)\, z(y)\, e^{\int d^dx\,\frac{\beta}{2\,c_6^R}|z|^2} = 0\,,$$
$$\langle z^\star(x)\, z(y)\rangle = \int Dz\, Dz^\star z^\star(x)\, z(y)\, e^{\int d^dx\,\frac{\beta}{2\,c_6^R}|z|^2}$$
$$= -\frac{2\,\mathrm{Re}(c_6)}{\beta}\,\delta^{(d)}(x-y)\,. \quad \text{(D.0.4)}$$

For the noise related to the ϕ_α field we have essentially two choices. The first one is to introduce a real vector noise field ζ^μ according to the identities,

$$e^{\int d^dx \frac{c_4}{\beta} B_{\alpha\,0}^2} = \int D\zeta^0\, e^{\int d^dx\,\frac{\beta}{4\,c_4}\zeta_0^2 + i\,\zeta^0 B_{\alpha\,0}}\,,$$
$$e^{\int d^dx \frac{c_5}{\beta} B_{\alpha\,i}^2} = \int D\zeta^i\, e^{\int d^dx\,\frac{\beta}{4\,c_5}\zeta_i^2 + i\,\zeta^i B_{\alpha\,i}}\,. \quad \text{(D.0.5)}$$

The variable ϕ_α then appears linearly in the effective action and can be integrated over, yielding a delta functional with argument,

$$-\,\partial_\mu\, J_r^\mu - \partial_\mu\zeta^\mu + 2\,\mathrm{Re}(i\,q\,e^{iq\phi_r}\hat{\psi}_r s_r^\star)\,, \quad \text{(D.0.6)}$$

with J_r^μ as defined in equation (C.0.10). However, the weight of the path integral now involves the “source” term factor $e^{\int d^dx\, i\,\zeta^\mu A_{\alpha\,\mu}}$. To absorb it, we can make the shift,

$$\zeta^0 \to \zeta^0 - \frac{2\,i\,c_4}{\beta}\,A_{\alpha\,0}\,,$$
$$\zeta_i \to \zeta_i - \frac{2\,i\,c_5}{\beta}\,A_{\alpha\,i}\,. \tag{D.0.7}$$

The argument of the delta functional (D.0.6) implies then the current continuity equation with noise,

$$\partial_\mu J^\mu_r = -2\,q\,\mathrm{Im}\left[\hat{\psi}_r\, e^{iq\phi_r}\, s^\star_r\right] - \partial_\mu \zeta^\mu + \frac{2\,i}{\beta}\left(c_5\,\partial_i A^i_\alpha + c_4\,\partial_t A_{\alpha\,0}\right)\,. \tag{D.0.8}$$

For the correlation function of the noise field we have,

$$\langle \zeta^0(x)\,\zeta^0(y)\rangle = \int D\zeta^\mu\,\zeta^0(x)\,\zeta^0(y)\,e^{\int d^dx\,\frac{\beta}{4\,c_4}(\zeta^0)^2} = -\frac{2\,c_4}{\beta}\delta^{(d)}(x-y)\,,$$
$$\langle \zeta^i(x)\,\zeta^j(y)\rangle = \int D\zeta^\mu\,\zeta^i(x)\,\zeta^j(y)\,e^{\int d^dx\,\frac{\beta}{4\,c_5}\zeta_k^2} = -\delta^{ij}\,\frac{2\,c_5}{\beta}\delta^{(d)}(x-y)\,.$$

An alternative way to write the stochastic equation for the current is to introduce a scalar noise field ζ through the identity,

$$e^{\int d^dx\,\phi_\alpha(-\frac{c_4}{\beta}\partial_t^2 - \frac{c_5}{\beta}\partial_i^2)\phi_\alpha} = \int D\zeta\, e^{\int d^dx\,(-\frac{\beta}{4}\,\zeta\,(c_4\,\partial_t^2 + c_5\,\partial_i^2)^{-1}\zeta + i\,\zeta\,\phi_\alpha)}\,. \tag{D.0.9}$$

The integral over ϕ_α constraints the current to be on-shell, obeying the constraint equation of motion,

$$\partial_\mu J^\mu_r = -\,2\,q\,\mathrm{Im}\left[\hat{\psi}_r\, e^{iq\phi_r}\, s^\star_r\right] + \zeta + \frac{2\,i}{\beta}\left(c_5\,\partial_i A^i_\alpha + c_4\,\partial_t A_{\alpha\,0}\right)\,, \tag{D.0.10}$$

with the correlation function for the scalar noise field obeying,

$$\langle \zeta(x)\zeta(y) \rangle = \frac{2}{\beta}(c_4\, \partial_t^2 + c_5\, \partial_i^2)\delta^{(d)}(x-y)\,, \tag{D.0.11}$$

and J_r^μ as defined in equation (C.0.10).

Appendix E

Integrating out the Amplitude Mode

In this Appendix we provide some of the necessary technical details regarding the derivation of section 4.4.

For convenience, we introduce the real and imaginary parts,

$$\hat{\psi}_\alpha = \hat{\psi}_\alpha^R + i\,\hat{\psi}_\alpha^I\,, \tag{E.0.1}$$

obeying the perturbative KMS transformation rule,

$$\begin{aligned}\tilde{\hat{\psi}}_\alpha^R(-x) &= \hat{\psi}_\alpha^R + i\,\beta\,\partial_t\delta\rho_r\,,\\ \tilde{\hat{\psi}}_\alpha^I(-x) &= -\hat{\psi}_\alpha^I - i\,\beta\,q\,\rho_r\,\partial_t\delta\theta_r\,.\end{aligned} \tag{E.0.2}$$

Consistency of the KMS transformations (E.0.2) then leads to,

$$\hat{\psi}_\alpha \approx \hat{\psi}_\alpha^\star \approx B_{\alpha\,\mu} \approx \mathcal{O}(\lambda^2)\,. \tag{E.0.3}$$

Based on the discussion around equations (4.4.3) and (E.0.3), we can expand the effective action in the parameter λ. Indeed, the derivative expansion scheme above has similar flavour to a semi-classical approximation. In this expansion we aim to retain quadratic terms in α-fields, suggesting that we maintain terms up to order

$\mathcal{O}(\lambda^4)$, to obtain,

$$\mathcal{L}^{[1]} = \left(-4\,u_0\,\rho_r^2\,\delta\rho_r - 2\,\kappa_0\,\rho_r\,B_{r\,0}\right)\,\hat{\psi}_\alpha^R + \left(\rho_n - \kappa_0\,\rho_r^2 - 2\,\kappa_0\,\rho_r\,\delta\rho_r + \chi_n\,B_{r\,0}\right)\,B_{\alpha\,0}$$
$$+2\,w_0\,q\,\rho_r\,\partial^i\left(\partial_i\delta\theta_r + B_{r\,i}\right)\,\hat{\psi}_\alpha^I - 2\,w_0\,q^2\,\rho_r^2\,\left(\partial^i\delta\theta_r + B_r{}^i\right)\,B_{\alpha\,i}\,, \tag{E.0.4}$$
$$\mathcal{L}^{[2]} = 2\,\left(c_6^R\,\hat{\psi}_\alpha^R - c_6^I\,\hat{\psi}_\alpha^I\right)\partial_t\delta\rho_r + 2\,q\,\rho_r\left(c_6^R\,\hat{\psi}_\alpha^I + c_6^I\,\hat{\psi}_\alpha^R\right)\partial_t\delta\theta_r - \frac{2\,i\,c_6^R}{\beta}\left[\left(\hat{\psi}_\alpha^R\right)^2 + \left(\hat{\psi}_\alpha^I\right)^2\right]$$
$$-\frac{i\,c_5}{\beta}\,B_\alpha{}^i\,B_{\alpha\,i} + c_5\,B_\alpha{}^i\,\partial_t B_{r\,i}\,. \tag{E.0.5}$$

We now observe that $\delta\rho_r$ appears linearly in the effective action terms (E.0.4) and (E.0.5). Performing the path integration over this variable yields a delta functional imposing a constraint which can be solved perturbatively up to third order in λ according to,

$$\hat{\psi}_\alpha^R = -\frac{\kappa_0}{2\,u_0\,\rho_r}B_{\alpha\,0} + \frac{\kappa_0\,c_6^R}{4\,u_0^2\,\rho_r^3}\partial_t B_{\alpha\,0} + \frac{c_6^I}{2\,u_0\,\rho_r^2}\partial_t\hat{\psi}_\alpha^I\,.$$

In order for this to make sense, we must ensure that the domain of integration in the path integral does not include the kernel of the operator we are trying to invert. This is guaranteed by the fact that we are assuming that the gap of the mode with dispersion relation ω_H in equation (4.4.1) is larger than the UV cut off scale Λ of our effective theory. By substituting the above in the effective action we find the expressions,

$$\mathcal{L}^{[1]} = B_{r\,0}\left(\chi_b\,B_{\alpha\,0} - \frac{\kappa_0\,c_6^I}{u_0\,\rho_r}\partial_t\hat{\psi}_\alpha^I - \frac{\kappa_0^2\,c_6^R}{2\,u_0^2\,\rho_r^2}\partial_t B_{\alpha\,0}\right) + \rho_b\,B_{\alpha\,0} - 2\,w_0\,q^2\,\rho_r^2(\partial_i\delta\theta_r + B_{ri})B_{\alpha\,i}$$
$$-\,2\,w_0\,q\,\rho_r(\partial_i\delta\theta_r + B_{r\,i})\partial_i\hat{\psi}_\alpha^I\,, \tag{E.0.6}$$
$$\mathcal{L}^{[2]} = -\,\frac{2\,i\,c_6^R}{\beta}(\hat{\psi}_\alpha^I)^2 - i\frac{c_6^R\,\kappa_0^2}{2\,\beta\,u_0^2\,\rho_r^2}B_{\alpha\,0}^2 + 2\,q\,\rho_r\,c_6^R\,\hat{\psi}_\alpha^I\partial_t\delta\theta_r - \frac{\kappa_0\,c_6^I\,q}{u_0}B_{\alpha\,0}\,\partial_t\delta\theta_r$$
$$-\frac{i\,c_5}{\beta}\,B_\alpha{}^i\,B_{\alpha\,i} + c_5\,B_\alpha{}^i\,\partial_t B_{r\,i}\,. \tag{E.0.7}$$

In the above expressions we have defined χ_b and ρ_b, the charge susceptibility and the charge density respectively of the broken phase, given by,

$$\chi_b = \chi_n + \frac{\kappa_0^2}{u_0}\,, \qquad \rho_b = \rho_n - \kappa_0\,\rho_r^2\,. \tag{E.0.8}$$

At this point, it is useful to make a change of variables in the path integral [1]:

$$\begin{aligned} \phi_r &\to \varphi_r = \phi_r + \delta\theta_r \,, \\ \phi_\alpha &\to \varphi_\alpha = \phi_\alpha + \frac{1}{q\,\rho_r}\hat{\psi}^I_\alpha \,. \end{aligned} \tag{E.0.9}$$

As in the main text, we introduce the gauge invariant vectors $C_r = \partial\varphi_r + A_r$ and $C_\alpha = \partial\varphi_\alpha + A_\alpha$. Note that the last term of $\mathcal{L}^{[1]}$ will change by a total time derivative term, which we will drop.

The next step is to observe that the variable $\delta\theta_r$ appears linearly in the resulting effective action as well. Performing the corresponding path integration over it we obtain the delta functional,

$$\delta\left(-2\,q\,\rho_r\,c_6^R\,\partial_t\hat{\psi}^I_\alpha + (\chi_b + \frac{q\,\kappa_0}{u_0}c_6^I)\partial_t C_{\alpha\,0}\right) = \frac{1}{\det(-2\,q\,\rho_r\,c_6^R\,\partial_t)}\delta\left(\hat{\psi}^I_\alpha - \frac{1}{2\,q\,\rho_r c_6^R}(\chi_b + \frac{q\,\kappa_0}{u_0}c_6^I)C_{\alpha\,0}\right).$$

For the equality to make sense, ∂_t acting on $\hat{\psi}^I_\alpha$ has to be an invertible operator. The reason this is the case, is because its domain of definition contains only functions that vanish at $t \to +\infty$. This is true for all α fields due to the boundary condition (4.2.2). As a result, the ∂_t has an empty kernel on the space of functions we are integrating over. The above shows that the operation is meaningful. The final step is to integrate over $\hat{\psi}^I_\alpha$ using the delta functional in order to arrive to our final result of equation (4.4.4) for the effective action, after identifying the current susceptibility χ_{JJ} through (4.4.5).

In order to express the constant ζ_3 of equation (4.4.6) in terms of the variables appearing in Model F, we can use the matching relations (4.3.4),(C.0.15) and (C.0.16) to write the expression,

$$\zeta_3 = \frac{1}{q^2\,\rho_r^2\,\mathrm{Re}\Gamma_0}\left(\mathrm{Re}\Gamma_0^2 + \left(\mathrm{Im}\Gamma_0 + \frac{q\,\rho_r^2\,\Delta\rho_0}{4\,\chi_b\,\Delta F_0}\right)^2\right). \tag{E.0.10}$$

This expression matches precisely the previously obtained result in [6]. Moreover,

[1] These changes are just shifts of the integrated variables and so have unit Jacobian.

we can write the KMS transformation rules for our low energy fields,

$$\tilde{C}_{r\,\mu}(-x) = C_{r\,\mu}(x)\,,\ \tilde{C}_{\alpha\,\mu}(-x) = C_{\alpha\,\mu} + i\,\beta\,\partial_t C_{r\,\mu}(x)\,,$$

and check that the effective action (4.4.4) is indeed invariant.

In order to make contact with classical superfluid hydrodynamics, it is useful to write down the expression for the conserved current,

$$\begin{aligned}\hat{J}_r{}^0 =& \frac{\partial \mathcal{L}_{sf}}{\partial C_{\alpha\,0}} = \rho_b + \chi_b\,C_{r\,0} - \chi_b^2\,\zeta_3\,\partial_t C_{r\,0} + \frac{2\,i}{\beta}\,\chi_b^2\,\zeta_3\,C_{\alpha\,0}\,,\\ \hat{J}_r{}^i =& \frac{\partial \mathcal{L}_{sf}}{\partial C_{\alpha\,i}} = -\chi_{JJ}\,C_r{}^i + c_5\,\partial_t C_r{}^i - \frac{2\,i\,c_5}{\beta}\,C_\alpha{}^i\,,\end{aligned} \tag{E.0.11}$$

The mean field theory part of the above constitutive relations is consistent with superfluid hydrodynamics [146, 31, 26, 2] after identifying ζ_3 with the third bulk viscosity.

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