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Bosonic Fluctuations in Semiclassically Quantized Strings

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Abstract

The overall focus of this thesis is the calculation of observables such as Wilson loops using string theory. In the first part of the thesis, Wilson loops are introduced. Then, the quark-antiquark potential in an $AdS_5 \times S^5$ background is calculated using string theory, reproducing the result by Juan Maldacena in 1998. Then, the vacuum expectation value (vev) of a circular Wilson loop is calculated in the same ten-dimensional background. The second part of the project investigates bosonic fluctuations over a classical string solution in a general curved background and interprets them in terms of intrinsic and extrinsic geometric invariants. The objective is to extend the results of the 2015 paper by Forini et al. to the action involving the antisymmetric 2-tensor. An expression for the bosonic mass matrix is obtained. The second-order fluctuations thus obtained can be used to precisely calculate the expectation value of observables such as the Wilson loop and the free energy beyond the classical order.

Ágrip

Efni þessarar ritgerðar snýst um að reikna mælistærðir svo sem Wilson lykkjur með hjálp strengjafræðinnar. Í fyrsta hluta ritgerðarinnar kynnum við Wilson lykkjur til sögunnar og tengjum það við mættið á milli tveggja kvarka. Við rifjum upp reikning á þessu mætti sem Juan Maldacena framkvæmdi árið 1998 með því að nota hina þekktu $AdS_5 \times S^5$ lausn strengjafræðinnar. Því næst reiknum við væntigildi Wilson lykkja í nokkrum 10-víðum tímarúmslausnum. Í öðrum hluta ritgerðarinnar einbeitum við okkur að truflunarreikning fyrir streng í almennri tímarúmslausn. Við útvíkkum fyrri niðurstöður frá 2015 með því að taka andsamhverfa B-þininn með í reikninginn og reiknum massafylki truflananna. Niðurstöður þessarar ritgerðar má meðal annars nota til þess að reikna væntigildi Wilson lykkja í söðulpunktsnálgun.

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Acknowledgements

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1 Introduction and Motivation

According to the (AdS/CFT) correspondence [1, 2, 3] (see, for example, [4] and [5]), an n-dimensional conformal field theory (CFT) or a quantum field theory (QFT) is dual to a gravity theory in $n+1$ dimensions which is asymptotically anti-de Sitter (AdS) or has a constant negative curvature at infinity. The best-understood example of this duality, first proposed by Maldacena in 1998 [1], is the $AdS_5 \times S^5$ space-time and 4-dimensional $\mathcal{N}=4$ $U(N)$ Super Yang-Mills at large N [6]. Basically, the bulk is a region of dynamic gravity whereas the boundary is a conformal field theory without gravity. A very important of this holographic duality is that when the gauge theory is strongly coupled and cannot be studied using perturbation theory, the gravitational side is described by essentially a classical theory and vice versa.

What does it mean to have a duality? Two theories are said to have a duality if we can find a map between all parameters and observables of the corresponding theories, and if we can develop a precise computational framework for dynamical computations on both sides. These observables should naturally be independent of the chosen gauge. Since the discovery of this correspondence, efforts have been put to develop an AdS/CFT dictionary that relates observables on both sides. The parameters are found to have the following matching:

$$g_{YM}^2 = 4\pi g_s, \quad R = (4\pi g_s N)^{1/4} l_s, \quad \lambda = g_{YM}^2 N,$$

where, g_{YM} is the gauge theory coupling, g_s is the string coupling, R is the radius of the AdS_5 and S^5, and λ is the 't Hooft coupling. In AdS/CFT correspondence, we concern ourselves with the limit $N \to \infty$ while keeping λ finite but large ($\gg 1$) and look at expansions around it. The AdS/CFT correspondence can be generalized to a general duality, called the gauge/gravity duality or holography, between quantum field theories in n-dimensions which are not necessarily conformal or supersymmetric, and gravitational theories on AdS spacetimes in $n+1$-dimensions.

One such observable that we can study is the Wilson loop operator which is manifestly gauge invariant. It was first proposed by Kenneth G Wilson [7]. The calculation for the Wilson loop vev was initially proposed holographically in [8]. The Wilson loop operator corresponds to an open string living in an AdS spacetime with certain boundary conditions, and its vacuum expectation value (vev) at the leading order is given by the area of minimal two-dimensional surface swept by the dual open string in AdS. What makes Wilson Loops an excellent candidate for studying holography or comparing it with gravitational predictions is that we can calculate its vev exactly on the field theory side using the methods of supersymmetric localization [9, 10, 11].

The natural next step would be to calculate the vev of these operators next to the leading order. The attempts to study this duality beyond the leading order were initiated for $\mathcal{N}=4$ in [12, 13, 14, 15] and the matching between the QFT and the string theory calculation was finally achieved in [16, 17, 18, 19, 20] when a ratio of observables is considered. Recently, attempts have also been made to study holography for cases where the field theory is not conformal. Thus, making the gravitational dual a bit more complicated and the calculations of vevs a bit more involved. For examples, see [21, 22] where Free energy and Wilson loop vev calculations are done beyond the leading order for Dp-branes and 5D maximally supersymmetric Yang-Mills, respectively, using string theory and found to match the supersymmetric localization results present in the literature. An important step in calculating the next-to-leading order contribution, on the gravity side, is to calculate the quadratic fluctuations around a classical string background. These fluctuations can be expressed in terms of the geometric invariants of the theory as shown in [16]. In this thesis, we will see how we can calculate the vev of observables like the Wilson loop operator (partition function, free energy, etc.) at the leading and then beyond the leading order in a general string background extending the results of [16] to the string action with the antisymmetric tensor, that is, where the string action is given by:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma} \left(\gamma^{\alpha\beta} g_{mn} + \varepsilon^{\alpha\beta} B_{mn} \right) \partial_\alpha X^m \partial_\beta X^n. \tag{1.1}$$

The thesis is divided into two broad parts: In the first part, we calculate the vev of the Wilson loop operators on the gravity side using string theory and reproduce established results; In the second part, we take a general string action (1.1) as mentioned above and calculate the bosonic fluctuations and express them in terms of the intrinsic and extrinsic geometric invariants of the theory. We will also see how these fluctuations can be used to calculate the vev of observables such as the Wilson loop to the next to the leading order on the gravity side.

The structure of the thesis is as follows: We will first review gauge invariance, the field derivative in a gauge theory, and the construction of Wilson lines/Wilson loops in the context of abelian and non-abelian gauge theories. Then, we look at the conserved currents in a non-abelian gauge theory and find that the global symmetry does not correspond to a gauge-invariant conserved current and thus, cannot be used to define a potential. To circumvent this problem, we propose that the vev of the Wilson loop operator provides a useful definition for the potential. We also prove that this is indeed the case for Quantum Electrodynamics (QED). We then see how we can calculate the vev of the Wilson Loop operator to the leading order on the gravity side and reproduce existing results. First among these is the quark-antiquark potential separated by a distance L first calculated in [1] by Maldacena. Then, we calculate the vev of the Circular Wilson loop [23] which argues that this vev is given by an appropriate Legendre transform of the area of the minimal surface. We then see how this method is applicable, not just to $AdS_5 \times S^5$, and apply it to find the counter-term in the calculation of vev of a Wilson loop for a general Dp-brane metric.

We want to calculate the vev of observables beyond the leading order for a general string background. We start with reviewing the Polyakov path integral and the Faddeev Popov method. Then, we use the background field method [24] to obtain a manifestly covariant form for the coefficients in the expansion of the field around a classical background as a local power series in spacetime vectors. Then, we decompose our quadratic fluctuations into tangential and normal directions to the worldsheet and finally prove that the quadratic Lagrangian can be written as a sum of longitudinal and transverse quadratic Lagrangians. Then we see how these quadratic fluctuations can be used to calculate the expectation value of the Wilson loop operator to the next-to-leading order.

2 Gauge Invariance and Wilson lines/loops

In this section, we revisit gauge invariance and discuss the construction and motivation behind the Wilson line, define a derivative, and a connection, and then construct the Wilson loop in the context of abelian and non-abelian gauge theories. We will also see that the conserved currents in non-abelian gauge theories are not gauge invariant, meaning, we now need a new definition for the potential. We then show how the vev of the manifestly gauge invariant quantity, the Wilson loop, can be used as potential and prove that this definition works really well for QED. References [25] and [26] are used for the analysis that follows.

2.1 Abelian gauge theories

Let us start by considering a complex scalar field $\phi(x)$ that transforms as follows:

$$\phi(x) \longrightarrow e^{i\alpha}\phi(x).$$

We can construct Lagrangians that are invariant under this transformation and have terms such as

$$\partial_\mu \phi(x)\partial^\mu \bar{\phi}(x), \qquad m\phi(x)\bar{\phi}(x), \ldots$$

Let us see what happens if we push this to be a local symmetry instead. Say

$$\phi(x) \longrightarrow e^{i\alpha(x)}\phi(x).$$

where $\alpha(x)$ is a function of x. The term $\partial_\mu \phi \partial^\mu \phi$ will not be invariant under the transformation above. We can immediately see a problem. It lies in the transformation of the derivative. To see how the derivatives transform, let us first try to compare fields at two distinct points x^μ and y^μ. Since our theory is local, the phase convention $\alpha(x)$ should be independent of $\alpha(y)$. Let us see how the difference $\phi(y) - \phi(x)$ transforms under this transformation:

$$\phi(y) - \phi(x) \longrightarrow e^{i\alpha(y)}\phi(y) - e^{i\alpha(x)}\phi(x).$$

We can see that $|\phi(y) - \phi(x)|$ depends on the local choice of our phase. Hence, we cannot compare fields at different points in a convention-independent way. Since we cannot calculate field differences that transform nicely, we cannot obviously calculate the derivative $\partial_\mu \phi(x)$ because this transformation will depend on the choice of phase, too. To compare field values at different points x^μ and y^μ, we define a new bilocal field $W(x,y)$ called a **Wilson-line** that depends on both x and y. We demand this field transforms as follows:

$$W(x,y) \longrightarrow e^{i\alpha(x)}W(x,y)e^{-i\alpha(y)}.$$

We now define the difference between the field values at x^μ and y^μ as $W(x,y)\phi(y) - \phi(x)$. This field must satisfy $W(x,x) = 1$. We see that it transforms as follows:

$$\begin{aligned} W(x,y)\phi(y) - \phi(x) &\longrightarrow e^{i\alpha(x)}W(x,y)e^{-i\alpha(y)}e^{i\alpha(y)}\phi(y) - e^{i\alpha(x)}\phi(x) \\ &= e^{i\alpha(x)}[W(x,y)\phi(y) - \phi(x)]. \end{aligned}$$

Now, as we can see, $W(x,y)\phi(y) - \phi(x)$ is independent of our choice of a local phase convention. Consider two points x^μ and $x^\mu + \delta x^\mu$, where δx^μ is very small. From the first principle and using our definition of differences, we define the field derivative at x^μ as follows:

$$D_\mu \phi(x) = \lim_{\delta x^\mu \to 0} \frac{W(x, x+\delta x)\phi(x+\delta x) - \phi(x)}{\delta x^\mu}.$$

For small δx^μ, we can expand $W(x, x+\delta x)$ as

$$W(x, x+\delta x) = 1 - ig\delta x^\mu A_\mu(x) + O(\delta x^2),$$

for some $A_\mu(x)$. Here, we have taken out a factor of $-ig$. Using the transformation of $W(x,y)$, the transformation of $A_\mu(x)$ will be

$$A_\mu(x) \longrightarrow A_\mu(x) + \frac{1}{g}\partial_\mu \alpha(x).$$

Plugging into the expression for our derivative, we can write

$$D_\mu \phi(x) = \partial_\mu \phi(x) - igA_\mu \phi(x).$$

We can see that the derivative, now called the covariant derivative, transforms as

$$D_\mu \phi(x) \longrightarrow e^{i\alpha(x)} D_\mu \phi(x).$$

Now, we have introduced a gauge field $A_\mu(x)$ as a **connection** that allows us to compare field values at different points despite their arbitrary local phases. We can write a closed-form expression for the Wilson-line $W(x,y)$ as follows:

$$W_P(x,y) = \exp\left(ig \int_y^x A_\mu(z) dz^\mu \right). \tag{2.1}$$

This is a line integral along the path P connecting y^μ to x^μ. Since the transformation of the Wilson line is independent of the path and just depends on the endpoints, if we set $x = y$, we get a contour integral:

$$W_P^{\text{loop}} = \exp\left(ig \oint_P A_\mu dx^\mu \right). \tag{2.2}$$

This is known as a **Wilson loop**. Wilson loops are manifestly gauge invariant. Using Stokes' theorem, we can write the above integral as

$$\begin{aligned} W_P^{\text{loop}} &= \exp\left(i\frac{g}{2} \int_\Sigma F_{\mu\nu} d\sigma^{\mu\nu} \right) \\ &= 1 + i\frac{g}{2} \int_\Sigma F_{\mu\nu} d\sigma^{\mu\nu} + O(g^2), \end{aligned} \tag{2.3}$$

over the surface Σ with surface element $d\sigma^{\mu\nu}$ bounded by the path P. So, the Wilson loop only depends on the gauge invariant field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Also, we notice

$$\begin{aligned} [D_\mu, D_\nu]\phi(x) &= ([\partial_\mu, \partial_\nu] - ig[\partial_\mu, A_\nu] + ig[\partial_\nu, A_\mu])\phi(x) \\ &= -igF_{\mu\nu}\phi(x). \end{aligned}$$

The operator $[D_\mu, D_\nu]$ turns out to just be a function. We can define the field strength as

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu].$$

We also see that this operator is gauge invariant.

$$[D_\mu, D_\nu]\phi(x) \longrightarrow e^{i\alpha(x)}[D_\mu, D_\nu]\phi(x).$$

Geometric Interpretation: We can think of the field $F_{\mu\nu}$ as the difference between the derivative of the field in the ν direction followed by the derivative in the μ direction ($D_\mu D_\nu$) and vice versa ($D_\nu D_\mu$). Equivalently, we can say that its value is obtained as the result of comparing fields around an infinitesimal closed loop in the $\mu - \nu$ plane. We can see that this is nothing but the value of the Wilson loop around a rectangular path.

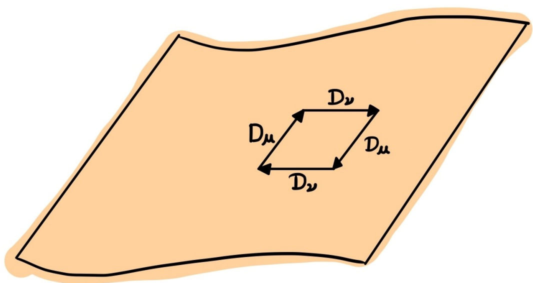

Figure 2.1. Field strength from commutator of covariant derivatives

2.2 Non-Abelian gauge theories

Let us now consider non-abelian gauge theories. Consider the kinetic Lagrangian with N Dirac fermions:

$$\mathscr{L} = \sum_{j=1}^{N} \bar{\psi}_j(i\partial\!\!\!/ - m)\psi_j.$$

The Lagrangian is invariant under a global $SU(N)$ symmetry where the fields transform as

$$\psi_i \longrightarrow (e^{i\alpha^a T^a})_{ij}\psi_j.$$

Here, T^a are the $SU(N)$ generators in the fundamental representation. This is a global symmetry because α^a does not depend on x. Here, we want to make a well-defined derivative to compare field values at different points but we have an issue because the generators do not commute. We construct the Wilson line the same way as before:

$$W_P(x,y) = P\left\{\exp\left(ig\int_y^x A_\mu^a(z)T^a dz^\mu\right)\right\}. \tag{2.4}$$

Here, the P outside the curly brackets $\{\dots\}$ implies path ordering. Taylor expanding this and applying path ordering to all terms, we get:

$$W_P(x,y) = 1 + ig\int_0^1 \frac{dz^\mu(\lambda)}{d\lambda} A_\mu^a(z(\lambda))T^a d\lambda \\ -\frac{1}{2}g^2\int_0^1 d\lambda \int_0^1 d\tau \frac{dz^\mu(\lambda)}{d\lambda}\frac{dz^\mu(\tau)}{d\tau} \\ \times A_\mu^a(z(\lambda))A_\nu^b(z(\tau))[T^aT^b\theta(\lambda-\tau) - T^bT^a\theta(\tau-\lambda)] + \dots,$$

where we have defined the integral in terms of parameters $\lambda, \tau \in [0,1]$. Let us see what happens under a gauge transformation:

$$W_P(x,y) \longrightarrow e^{i\alpha^a(x)T^a} W_P(x,y) e^{-i\alpha^a(y)T^a}.$$

Using $T^{a\dagger} = T^a$ for $SU(N)$. We will represent the gauge field as a Lie-algebra-valued field given by

$$\mathbf{A}_\mu \equiv A_\mu^a T^a.$$

Then, we can write the Wilson line as

$$W_P(x,y) = P\left\{\exp\left(ig\int_y^x \mathbf{A}_\mu(z)dz^\mu\right)\right\}.$$

This looks very similar to the Wilson line in the Abelian case. Here, A_μ^a are the components of Lie-Algebra-valued one-form $\mathbb{A} = \mathbf{A}_\mu dx^\mu$. Expanding the Wilson line infinitesimally gives

$$W(x^\mu, x^\mu + \delta x^\mu) = \mathbb{1} - ig\mathbf{A}_\mu \delta x^\mu.$$

Let us now look at local transformations. A local transformation can be expressed in terms of

$$U(x) = e^{i\alpha^a(x)T^a} \quad \in \quad SU(N).$$

This is the group element for the transformation at point x. Then

$$\vec{\psi}(x) \longrightarrow U(x)\cdot\vec{\psi}(x),$$

And

$$W(x,y) = U(x)W(x,y)U^\dagger(y),$$

using $U^\dagger(y) = U^{-1}(y)$ in $SU(N)$. We expand the transformation of W to determine the transformation of A_μ^a. We find

$$\mathbf{A}'_\mu = U\mathbf{A}_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}.$$

In terms of components, we can write

$$A_\mu^a(x) \longrightarrow A_\mu^a(x) + \frac{1}{g}\partial_\mu\alpha^a(x) - f^{abc}\alpha^b(x)A_\mu^c(x) + O(\alpha^2).$$

Let us now see how the commutators of the derivatives D_μ transform:

$$[D_\mu, D_\nu]\psi = (-ig(\partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu) - g^2[\mathbf{A}_\mu, \mathbf{A}_\nu])\psi.$$

We can see that there are no derivatives acting on $\vec{\psi}(x)$. Then, the field strength is then given by

$$\mathbf{F}_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = (\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu) - ig[\mathbf{A}_\mu, \mathbf{A}_\nu],$$

where the components of $\mathbf{F}_{\mu\nu} = F^a_{\mu\nu} T^a$ can be written as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu.$$

In the abelian case, $f^{abc} = 0$. Additionally, $F^a_{\mu\nu}$ is antisymmetric and its transformation law is given by

$$F^a_{\mu\nu} \longrightarrow F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu},$$

regardless of whether α is local or global. Now, we can write a locally $SU(N)$ invariant Lagrangian:

$$\mathscr{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \sum_{i,j=1}^{N} \bar{\psi}_i(\delta_{ij} i\partial\!\!\!/ + g A\!\!\!/^a T^a_{ij} - m\delta_{ij})\psi_j. \tag{2.5}$$

Let us now see what the conserved quantities are for this Lagrangian.

2.3 Conserved Currents

The equation of motion corresponding to the above Lagrangian is

$$\partial_\mu F^{a\mu\nu} + g f^{abc} A^b_\mu F^{c\mu\nu} = -g\bar{\psi}_i \gamma^\nu T^a_{ij} \psi_j,$$

for the gauge fields and

$$(i\partial\!\!\!/ - m)\psi_i = -g A\!\!\!/^a T^a_{ij}\psi_j,$$

for the spinors. Here, repeated indices are summed over regardless of where they appear. Since the Lagrangian has gauge symmetry, it will also have a global symmetry such that

$$\psi_i \longrightarrow \psi_i + i\alpha^a T^a_{ij}\psi_j,$$

and

$$A^a_\mu \longrightarrow A^a_\mu - f^{abc}\alpha^b A^c_\mu,$$

for an infinitesimal α. Since there is a global symmetry, there will be a conserved current given by

$$J_\mu = \sum_n \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha}.$$

In the non-abelian case, there will be a current for each symmetry direction α^a thus giving a total of $N^2 - 1$. Summing over all matter fields and gauge fields ($\phi_n = \psi_i$, A^a_μ), we get

$$J^{a\mu} = -\bar{\psi}_i \gamma^\mu T^a_{ij}\psi_j + f^{abc} A^b_\nu F^{c\mu\nu}.$$

We can see that this current is neither gauge invariant nor covariant. Thus making it unphysical. This means there is not a well-defined charge that can be measured. But the quantities

$$Q^a = \int d^3x J^a_0,$$

are conserved, that is $\partial_t Q^a = 0$. The problem, however, is that these charges depend on the choice of gauge. Thus, in non-abelian gauge theory, for example, Quantum Chromodynamics (QCD), we do not have a classical current as we do in QED. Unlike QED, the gauge fields in QCD are bound up with the matter field in an intricate and nonlinear way. The matter current only constructed out of fermions can be defined as

$$j^a_\mu = -\bar{\psi}_i \gamma_\mu T^a_{ij} \psi_j.$$

This, as we can see, is gauge invariant. It satisfies

$$D_\mu j^{a\mu} = \partial_\mu j^{a\mu} + g f^{abc} A^b_\mu j^{c\mu} = 0.$$

Thus, the matter current is not conserved (since $\partial_\mu j^a_\nu \neq 0$) and hence, there is no associated conserved charge. These observations and results also follow from the **Weinberg-Witten Theorem** [27]: *A theory with a global non-abelian symmetry under which massless spin-1 particles are charged does not admit a gauge-invariant conserved current.*

But now we run into a problem. We can see that $\langle \Omega | T\{J^a_\mu(r) J^b_\nu(0)\} | \Omega \rangle$ is not gauge invariant in QCD and therefore does not provide a useful definition of a potential. How do we define potential for non-abelian gauge theories? Wilson loops come to the rescue.

2.4 Potential from Wilson Loops

Having run into problems with the usual definition of potential in the case of non-abelian gauge theories, we now propose the expectation value of a Wilson loop for the definition of potential:

$$V(r) = \lim_{T\to\infty} \frac{1}{iT} \ln \langle \Omega | \text{tr}\, \{W_P^{\text{loop}}\} | \Omega \rangle, \tag{2.6}$$

where

$$W_P^{\text{loop}} = P\left\{ \exp\left[ig \oint_P A^a_\mu T^a_{ij} dx^\mu \right] \right\}.$$

The trace here is a color trace (trace over indices i and j) for $SU(N)$. Here, $P\{\dots\}$ denotes path ordering and P denotes the path of the loop. We take this path to be a large rectangle in the $t-z$ plane as given in figure 2.2. What is special about this definition is that it is

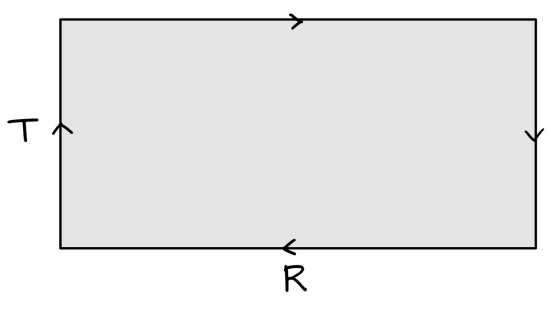

Figure 2.2. Wilson loop rectangle

manifestly gauge invariant. But the question we still have not answered is why does this work? For example, let us consider modifying the pure QED action by adding an $gA_\mu J^\mu$ term with

$$J_0(x) = \delta(x)\delta(y)\delta(z-R) - \delta(x)\delta(y)\delta(z),$$

This represents two charges separated by a distance of R. We want to adiabatically turn on this current at time $t = -\frac{T}{2}$ and then turn it off at $t = \frac{T}{2}$ with $T \gg R$ such that, at $T \to \infty$, the vacuum is unchanged. Since we are adding this term to the Lagrangian, it adds directly to the Hamiltonian density. Then, the vacuum in the background has non-zero energy E for time T. As $T \to \infty$, transient fluctuations drop out and we have

$$e^{-iET} = \langle\Omega|e^{-iHT}|\Omega\rangle \tag{2.7}$$

$$= \frac{\int \mathscr{D}A \ \exp\left[i\int d^4x\left(-\frac{1}{4}F_{\mu\nu}^2 + gA_\mu J^\mu\right)\right]}{\int \mathscr{D}A \ \exp\left[i\int d^4x\left(-\frac{1}{4}F_{\mu\nu}^2\right)\right]}. \tag{2.8}$$

If we identify $E = V(r)$ as the energy of the two charges separated by a distance R, we seem to have justified our proposition for the definition of the potential for the Abelian case. Let us see it by evaluating this path integral. We find

$$e^{-iET} = \exp\left[i\int d^4x \int d^4y \frac{g^2}{2} J^\mu(x) D_{\mu\nu}(x,y) J^\nu(y)\right]. \tag{2.9}$$

Here, $iD_{\mu\nu}(x,y)$ is the gauge boson propagator in position space given by

$$iD^{\mu\nu}(x,y) = \langle\Omega|T\{A^\mu(x)A^\nu(y)\}|\Omega\rangle = \frac{1}{4\pi^2}\frac{g^{\mu\nu}}{(x-y)^2 - i\varepsilon}, \tag{2.10}$$

where the second integral is evaluated in the Feynman gauge. These integrals diverge if both the currents are at $z = R$ or both at $z = 0$. But these contributions give no R dependence. It comes from x and y on the opposite sides of the loop. That is

$$-iET = -\frac{g^2}{4\pi^2}\int_{-T/2}^{T/2} dx^0 \int_{-\infty}^{\infty} dy^0 \frac{1}{(x_0 - y_0)^2 - R^2 - i\varepsilon} \tag{2.11}$$

$$= \frac{ig^2}{4\pi R}\int_{-T/2}^{T/2} dx^0 \tag{2.12}$$

$$= \frac{ig^2 T}{4\pi R} \tag{2.13}$$

$$\Rightarrow \ E = -\frac{g^2}{4\pi R} = V(R). \tag{2.14}$$

In the y^0 integral, we have taken T to infinity to extract the leading T behavior. Our result confirms the result for QED. Thus, we can see that the expectation value of the Wilson loop provides us with a good definition of the potential and is also gauge invariant by construction. How can this help us with QCD? This can help us prove confinement in QCD. If the QCD potential grows linearly with distance, that is, if

$$\ln\langle W_{\text{loop}}\rangle \sim TR, \tag{2.15}$$

then, it would take an infinite amount of energy to separate quarks asymptotically and would explain why we have never seen free quarks. From a lattice perspective, $\ln \langle W_{\text{loop}} \rangle$ scales as the area (TR) of the loop at strong coupling. Confinement has been confirmed in a lattice gauge theory setting using numerical simulations and holds well in any gauge theory but it is yet to be proved in the continuum limit.

2.5 Holographic Wilson Loop

Consider a p-dimensional spacetime with X^m as its embedding coordinates ($m = 0, 1, \ldots, p$) and g_{mn} as its metric. The Nambu-Goto action for a relativistic string with worldsheet coordinates σ^α ($\alpha = 1, 2$) is given by (in Euclidean signature)

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{h}, \tag{2.16}$$

where $h = \det h_{\mu\nu}$ and $h_{\mu\nu}$ is the induced metric, given by the pullback of the target space metric (g_{mn}):

$$h_{\mu\nu} = \frac{\partial X^m}{\partial \sigma^\alpha} \frac{\partial X^n}{\partial \sigma^\beta} g_{mn}. \tag{2.17}$$

This is nothing but the area of the string worldsheet which we need to minimize for the classical solution. The Nambu-Goto action has 2 symmetries, Poincare invariance, and reparametrization invariance. The equations of motion are

$$\partial_\alpha (\sqrt{h} h^{\alpha\beta} \partial_\beta X^m) = 0. \tag{2.18}$$

Now, we want to quantize it. But the square root makes it rather difficult to calculate the path integral. So, we use another string action which is equivalent to the Nambu-Goto action, called the Polyakov String action (in Euclidean signature) [28, 29]:

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma} \gamma^{\alpha\beta} g_{mn} \partial_\alpha X^m \partial_\beta X^n. \tag{2.19}$$

Here, $\gamma^{\alpha\beta}$ ($\alpha, \beta = 0, 1$) is a dynamical 2-dimensional metric on the string worldsheet, g_{mn} ($m, n = 0, 1, \ldots, 9$) is the ten-dimensional metric on the target space, X^ms are the embedding coordinates, and $\gamma = \det \gamma_{\alpha\beta}$.

The vev of the Wilson loop using string theory was first calculated by Maldacena [8]. He proposed that the vev be given by the string partition function of a string with its worldsheet ending along the loop on the boundary of space as we can see in the figure 2.3 This is given by

$$\langle W(C) \rangle \sim e^{-S_{\text{string}}}, \tag{2.20}$$

where C is the loop on the boundary that the string worldsheet encloses, and S_{string} is the string action. Classically, the vev of the Wilson loop operator is determined by a minimal surface. In [8] and [30], the Nambu-Goto action is assumed, and the vacuum expectation value (vev) of the Wilson loop operator is given as the area (A) of the minimal surface in the large N limit:

$$\langle W(C) \rangle \sim e^{-A}. \tag{2.21}$$

But this does not, however, specify the vev of the Wilson loop completely. As we move close to the boundary of the metric, divergences in the classical action show up in the result. Now,

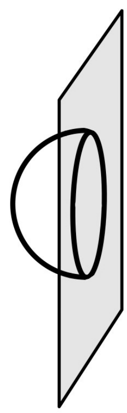

Figure 2.3. Holographic Wilson loop: The shaded region is the boundary of the space and the circular dome is the worldsheet of the string

the question is how do we get rid of these? One option is to isolate the divergent piece by regularization and shove it under the rug. But soon we will run into other issues. Since there can be many different actions having the same equation of motion differing by only a surface term or total derivatives. We can also find a way to systematically determine the counterterm to be added to the String action so our calculations do not blow up. Following this, [23] argues that the vev of the Wilson loop instead of the area of the minimal surface should be given by an appropriate Legendre transform of it.

We will now consider a few examples, starting with Maldacena's result for the quark-antiquark potential. Then, we will look at the vev of the circular Wilson loop.

3 Calculated Examples

Here, we discuss some examples to see how the calculation for the expectation value of Wilson lines and loops is performed on the gravity side.

3.1 Quark-Antiquark Potential Using Supergravity

We want to calculate the Quark-Antiquark potential kept at L distance apart on the boundary of the spacetime. This result was first obtained by Juan Maldacena in 1998 [8]. We start with the $AdS_5 \times S^5$ metric in Poincaré coordinates:

$$ds^2 = R^2\left[\frac{dU^2}{U^2} + U^2 dx_\mu dx^\mu\right] + R^2 d\Omega_5^2. \tag{3.1}$$

We can see that, as $U \to \infty$, the metric diverges but is conformally flat. This is going to be our boundary. Identifying the string coordinates such that:

$$t = \tau, x = \sigma, U = U(\sigma) = U(x). \tag{3.2}$$

Now, we can calculate the pullback on the string world sheet as follows:

$$g_{\tau\tau} = R^2 U^2 \tag{3.3}$$

$$g_{\sigma\sigma} = R^2\left(U^2 + \frac{(\partial_x U)^2}{U^2}\right). \tag{3.4}$$

Now, we write the string action using the Nambu-Goto action (2.16):

$$S_{NG} = \frac{1}{2\pi\alpha'}\int dt dx \sqrt{P[g_{\mu\nu}]} \tag{3.5}$$

$$= \frac{R^2}{2\pi\alpha'}\int dt dx \sqrt{U^4 + (\partial_x U)^2}. \tag{3.6}$$

We place a quark at $x = -\frac{L}{2}$ and an anti-quark at $x = +\frac{L}{2}$ as shown in Figure 3.1. The string connecting the two extends all the way to $U \to 0$ where gravity is weak. And this string moves in time. Now, the tension in this string will correspond to the energy of the pair. Essentially, we need to minimize the area of the string worldsheet. Our string action now looks like this:

$$S_{NG} = \frac{R^2}{2\pi\alpha'}\int_0^T dt \int_{-L/2}^{L/2} dx \sqrt{U^4 + (\partial_x U)^2} \tag{3.7}$$

$$= \frac{TR^2}{2\pi\alpha'}\int_{-L/2}^{L/2} dx \sqrt{U^4 + (\partial_x U)^2}. \tag{3.8}$$

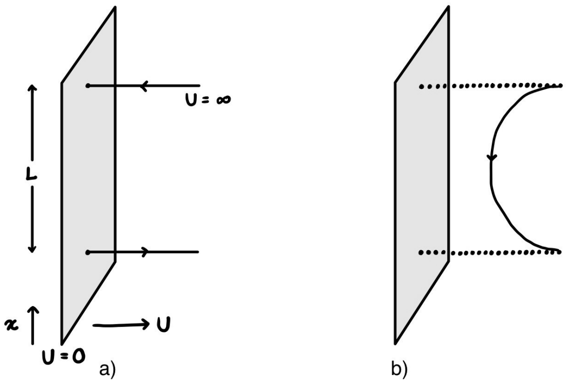

Figure 3.1. Pair of W-bosons with their coupling to the U(N) gauge theory a) turned off b) turned on. The configuration in b) minimizes the action and the vev of the Wilson line is then given by the difference of the two configurations.

Since the Lagrangian does no explicitly depend on x or t, the following quantity is conserved:

$$C = \frac{\partial L}{\partial(\partial_x U)}(\partial_x U) - L \tag{3.9}$$

$$= -\frac{TR^2}{2\pi\alpha'}\frac{U^4}{\sqrt{U^4 + (\partial_x U)^2}}. \tag{3.10}$$

Now, when $U \to 0$, the metric blows up, so, we will take it to be some finite value $U = \varepsilon$ and then take the limit $\varepsilon \to 0$. From symmetry, we can see that the string turns around at $x = 0$ and we have, $U(0) = \varepsilon$, and $U'(0) = 0$, where $U'(x) = \partial_x U$. Using this in the above equation, we get $C = -\varepsilon^2 \frac{TR^2}{2\pi\alpha'}$. Now, we solve the following differential equation for U:

$$\varepsilon^2 = \frac{U^4}{\sqrt{U^4 + (\partial_x U)^2}} \tag{3.11}$$

$$\Rightarrow \qquad 1 + \frac{(\partial_x U)^2}{U^4} = \frac{U^4}{\varepsilon^4} \tag{3.12}$$

$$\Rightarrow \qquad \partial_x U = \pm U^2 \sqrt{\frac{U^4}{\varepsilon^4} - 1} \tag{3.13}$$

$$\Rightarrow \qquad \int_\varepsilon^U \frac{dU}{U^2\sqrt{\frac{U^4}{\varepsilon^4} - 1}} = \int dx \tag{3.14}$$

$$\Rightarrow \qquad x = \frac{1}{\varepsilon}\int_1^{U/\varepsilon} \frac{dy}{y^2\sqrt{y^4 - 1}}. \tag{3.15}$$

Here, in the last step, we have made the substitution $y = U/\varepsilon$. In the limit $U \to \infty$, we evaluate the above integral to obtain ε:

$$\frac{L}{2}\varepsilon = \frac{\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)}. \tag{3.16}$$

This gives us the value of $\varepsilon = \frac{2\sqrt{\pi}\Gamma(3/4)}{L\Gamma(1/4)}$. We can see that the string will approach $x = \frac{L}{2}$ quickly for large U. Plugging (3.11) into (3.7), we evaluate the area to be

$$S = \frac{TR^2}{2\pi\alpha'} \times 2\varepsilon \int_1^{U/\varepsilon} \frac{y^2 dy}{\sqrt{y^4-1}}. \tag{3.17}$$

We can see that the integral is linearly divergent. If we subtract the self-energy contribution of the pair of quarks, we should have a finite result. To estimate the energy of a quark, we need only consider a long linear string from $U \to \infty$ to $U = 0$. Alternatively, we can identify the divergent part of the above integral and subtract it from the total potential. We find,

$$S = \frac{TR^2}{\pi\alpha'} \times \varepsilon \int_1^{U/\varepsilon} \frac{y^2 dy}{\sqrt{y^4-1}} \tag{3.18}$$

$$= \frac{TR^2\varepsilon}{\pi\alpha'} \times \left(-\frac{\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} + \frac{A}{\varepsilon}\right). \tag{3.19}$$

Here, A is some quantity linearly divergent with U. Ignoring the divergent quantity, we are now left with the energy of the quark-antiquark pair:

$$E = V(L) \sim \frac{S}{T} = \frac{R^2}{\pi\alpha'} \frac{2\sqrt{\pi}\Gamma(3/4)}{L\Gamma(1/4)} \times -\frac{\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} \tag{3.20}$$

$$= -\frac{R^2}{\alpha'} \frac{2\Gamma(3/4)^2}{\Gamma(1/4)^2} \frac{1}{L}. \tag{3.21}$$

Using $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we get $\Gamma(3/4)\Gamma(1/4) = \sqrt{2}\pi$. This gives the energy as

$$E = V(L) = -\frac{R^2}{\alpha'} \frac{4\pi^2}{\Gamma(1/4)^4} \frac{1}{L}. \tag{3.22}$$

Identifying $R^2 = \sqrt{2g_{YM}^2 N}\alpha'$, we get

$$V(L) = -\frac{4\pi^2\sqrt{2g_{YM}^2 N}}{\Gamma(1/4)^4} \frac{1}{L}. \tag{3.23}$$

Thus reproducing Maldacena's result.

3.2 Circular Wilson Loop

We start with the global metric for $AdS_5 \times S^5$:

$$ds^2 = R^2(-\cosh\rho^2 dt^2 + d\rho^2 + \sinh\rho^2 d\Omega_3^2 + d\Omega_5^2). \tag{3.24}$$

Here,

$$d\Omega_3^2 = d\theta_1^2 + \sin\theta_1{}^2 d\theta_2^2 + \sin\theta_1{}^2 \sin\theta_2{}^2 d\theta_3^2. \tag{3.25}$$

We have our boundary when $\rho \to \infty$. We wrap our string around the sphere $d\Omega_3^2$ at the equator ($\theta_1 = \theta_2 = \frac{\pi}{2}$). Taking $\tau = \theta_3$ and $\sigma = \rho$, we find the pullback on the string to be

$$g_{\tau\tau} = R^2 \sinh \rho^2 \tag{3.26}$$

$$g_{\rho\rho} = R^2 \tag{3.27}$$

$$\Rightarrow g = R^4 \sinh \rho^2. \tag{3.28}$$

The Nambu Goto action is then given by

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d\tau \int d\rho \sqrt{g} \tag{3.29}$$

$$= \frac{R^2}{\alpha'} \int_0^\infty d\rho \sinh \rho \tag{3.30}$$

$$= \frac{R^2}{\alpha'} \left[\frac{e^\rho + e^{-\rho}}{2} \Big|_{\rho\to\infty} - 1 \right]. \tag{3.31}$$

We can see the first term is divergent, so, we will ignore it. Now, we have the expectation value of the Wilson loop as

$$\langle W \rangle = e^{-S} = e^{R^2/\alpha'} = e^{\sqrt{2g_{YM}^2 N}}. \tag{3.32}$$

if we identify $R^2 = \sqrt{2g_{YM}^2 N}\alpha'$.

Let us see what happens when $\theta_1 \equiv \theta_1(\sigma)$ and $\theta_2 \equiv \theta_2(\sigma)$ depend on σ. The pullback is given by

$$g_{\tau\tau} = R^2 \sinh \rho^2 (\sin \theta_1{}^2 \sin \theta_2{}^2) \tag{3.33}$$

$$g_{\rho\rho} = R^2 [1 + \sinh \rho^2 ((\partial_\rho \theta_1)^2 + \sin \theta_1{}^2 (\partial_\rho \theta_2)^2)] \tag{3.34}$$

$$\Rightarrow g = R^4 \sinh \rho^2 (\sin \theta_1{}^2 \sin \theta_2{}^2)[1 + \sinh \rho^2 ((\partial_\rho \theta_1)^2 + \sin \theta_1{}^2 (\partial_\rho \theta_2)^2)]. \tag{3.35}$$

The Nambu-Goto action is then given by

$$S_{NG} = \frac{R^2}{\alpha'} \int d\rho \sinh \rho \sin \theta_1 \sin \theta_2 \sqrt{1 + \sinh \rho^2 [(\partial_\rho \theta_1)^2 + \sin \theta_1{}^2 (\partial_\rho \theta_2)^2]}. \tag{3.36}$$

The terms $\sinh \rho^2 [(\partial_\rho \theta_1)^2 + \sin \theta_1{}^2 (\partial_\rho \theta_2)^2]$ is positive. To minimize the above, we can take θ_1 and θ_2 to be constant. Then, we have $\partial_\rho \theta_1 = \partial_\rho \theta_2 = 0$, and the above equation becomes

$$S_{NG} = \frac{R^2 \sin \theta_1 \sin \theta_2}{\alpha'} \int d\rho \sinh \rho \tag{3.37}$$

$$= \frac{R^2 \sin \theta_1 \sin \theta_2}{\alpha'} \left[\frac{e^\rho + e^{-\rho}}{2} - 1 \right]. \tag{3.38}$$

Range of θ_1 and θ_2 is $(0, \pi)$.

3.2.1 An alternative method

Starting with the metric:

$$ds^2 = \frac{R^2}{Z^2}(dZ^2 + dX_\mu^2) + R^2 d\Omega_5^2, \tag{3.39}$$

the minimal surface is parametrized by $\tau \in [0, 2\pi]$ as follows [23, 31]:

$$X^\mu = (\sqrt{a^2 - Z^2}\cos\tau, \sqrt{a^2 - Z^2}\sin\tau, 0, 0), \tag{3.40}$$

and the rest of the coordinates be given as:

$$\theta^i = (Z, 0, 0, 0, 0, 0). \tag{3.41}$$

Assuming the S^5 sphere to just be a point. We can write the pullback as

$$g_{\tau\tau} = \frac{R^2}{Z^2}(a^2 - Z^2) \tag{3.42}$$

$$g_{ZZ} = \frac{R^2}{Z^2} \times \frac{a^2}{a^2 - Z^2} \tag{3.43}$$

$$\Rightarrow g = \frac{R^4}{Z^4}a^2. \tag{3.44}$$

Here, Z goes from 0 to a. The Nambu-Goto action will then be

$$S_{NG} = \frac{R^2}{\alpha'}\int_\varepsilon^a \frac{a}{Z^2} \tag{3.45}$$

$$= \frac{R^2 a}{\alpha'}\left[-\frac{1}{Z}\right]_\varepsilon^a \tag{3.46}$$

$$= \frac{R^2 a}{\alpha'}\left[-\frac{1}{a} + \frac{1}{\varepsilon}\right] \tag{3.47}$$

$$= -\frac{R^2}{\alpha'} + \frac{R^2 a}{\alpha'}\frac{1}{\varepsilon}. \tag{3.48}$$

We can see that the last term diverges. Hence, the expectation value of Wilson Loop is given by

$$\langle W \rangle = e^{-S} = e^{R^2/\alpha'} = e^{\sqrt{2g_{YM}^2 N}}, \tag{3.49}$$

if we identify $R^2 = \sqrt{2g_{YM}^2 N}\alpha'$. This also matches our previous result 3.32.

3.2.2 Legendre transformation method

Starting with the metric for $AdS_5 \times S^5$:

$$ds^2 = \frac{R^2}{Z^2}(dZ^2 + dX_\mu dX^\mu) + R^2 d\Omega_5^2. \tag{3.50}$$

Let us combine the Z coordinate with the 5-sphere metric and write

$$ds^2 = \frac{R^2}{Z^2}(dX_\mu dX^\mu + dY_i dY^i), \tag{3.51}$$

where $Y^i = Z\theta^i$. The boundary of the AdS is at $Z = 0$. !e choose the string worldsheet coordinates to be σ^α $(\alpha = 1,2)$ such that the boundary of the worldsheet is at $\sigma^2 = 0$. Let our string coordinates be

$$\sigma^1 = \tau, \;\; \sigma^2 = Z, \tag{3.52}$$

with $\tau = [0, 2\pi]$. Since Xs form the CFT coordinates, we impose Dirichlet boundary conditions on X^μ:

$$X^\mu(\sigma^1, 0) = x^\mu(\sigma^1), \tag{3.53}$$

Say the minimal surface is given as

$$X^\mu = (x^1(\tau, Z), x^2(\tau, Z), 0, 0), \tag{3.54}$$

and the rest of the coordinates are

$$Y^i = (Z, 0, 0, 0, 0, 0). \tag{3.55}$$

We can write the pullback as follows:

$$g_{\tau\tau} = \frac{R^2}{Z^2}(\dot{x}^2) \tag{3.56}$$

$$g_{ZZ} = \frac{R^2}{Z^2} \tag{3.57}$$

$$\Rightarrow g = \frac{R^4}{Z^4}\dot{x}^2 \tag{3.58}$$

$$\Rightarrow \sqrt{g} = \frac{R^2}{Z^2}|\dot{x}|, \tag{3.59}$$

$$\tag{3.60}$$

where $\dot{x} = \dfrac{\partial x}{\partial \tau}$ and Neumann boundary conditions on Y^i

$$\frac{1}{\sqrt{h}} h_{1\beta} \varepsilon^{\beta\alpha} \partial_\alpha Y^i(\sigma^1, 0) = \dot{y}^i(\sigma^1) = P^i(\sigma^1, 0). \tag{3.61}$$

We want to write our Lagrangian in terms of coordinate X^μ and the conjugate moment P^i of coordinates Y^i. So, we Legendre transform it

$$\tilde{L} = L - \partial_2(P_i Y^i), \tag{3.62}$$

or

$$\tilde{A} = A - \oint d\sigma^1 P_i Y^i. \tag{3.63}$$

This gives, for us

$$\tilde{A} = A - R^2 \oint d\sigma^1 \frac{\dot{y}^1}{Z^2} Y^1 \tag{3.64}$$

$$= A - R^2 \oint d\sigma^1 \frac{|\dot{y}|}{Z} \tag{3.65}$$

$$= A - R^2 \frac{1}{\varepsilon} \oint d\sigma^1 |\dot{y}|. \tag{3.66}$$

In the last step, we have put $Z = \varepsilon$. This gives

$$\tilde{A} = R^2 \int d\tau dZ \frac{|\dot{x}|}{Z^2} - \frac{R^2}{\varepsilon} \oint d\tau(|\dot{y}|) \tag{3.67}$$

$$= \frac{R^2}{\varepsilon} \int d\tau(|\dot{x}| - |\dot{y}|) + \text{finite term.} \tag{3.68}$$

We see that this works when the constraint $\dot{x}^2 = \dot{y}^2$ is satisfied. We can also see the application of this method in a general Dp-brane setting and find the counter-term to find the renormalized area to calculate the Wilson loop vev.

3.3 D-p Branes

The discussion that follows is a study of the paper [21].

3.3.1 Details of the metric

The metric for a general Dp-brane is given by

$$ds_{10}^2 = \frac{e^{\eta}}{\sqrt{Q}} \left(ds_{p+2}^2 + \frac{e^{\frac{2(p-3)}{6-p}\eta}}{g^2} \left(d\theta^2 + P\cos^2\theta d\tilde{\Omega}_2^2 + Q\sin^2\theta d\Omega_{5-p}^2 \right) \right). \tag{3.69}$$

g is the gauge coupling of the $(p+2)$-dimensional supergravity theory. We can relate it to the ten-dimensional string theory parameters as follows:

$$(2\pi l_s g)^{p-7} = \frac{g_s N}{2\pi V_{6-p}}. \tag{3.70}$$

Here,

$$\begin{aligned} N &= \text{ number of } Dp-\text{branes}, \\ g_s &= \text{ string coupling}, \\ l_s &= \text{ string length}, \\ V_n &= \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}, \text{ volume of the unit radius } n-\text{sphere} \end{aligned}$$

$d\Omega_{5-p}^2$ is the metric on the $5-p$-sphere of unit radius, $d\tilde{\Omega}_2^2$ is the metric on the 2-dimensional de Sitter space of unit radius given by

$$d\tilde{\Omega}_2^2 = -dt^2 + \cosh^2 t d\psi^2, \tag{3.71}$$

and ds_{p+2}^2 is given by

$$ds_{p+2}^2 = dr^2 + e^{2A(r)} d\Omega_{p+1}^2. \tag{3.72}$$

The function $A(r)$, can be determined in terms of scalars $\eta(r)$, $X(r)$, and $Y(r)$. P and Q are given by

$$P = \begin{cases} X(X\sin^2\theta + (X^2 - Y^2)\cos^2\theta)^{-1} & \text{for } p < 3 \\ X(\cos^2\theta + X\sin^2\theta)^{-1} & \text{for } p > 3 \end{cases}, \tag{3.73}$$

$$Q = \begin{cases} X(\sin^2\theta + X\cos^2\theta)^{-1} & \text{for } p > 3 \\ X(X\cos^2\theta + (X^2 - Y^2)\sin^2\theta)^{-1} & \text{for } p < 3 \end{cases}. \tag{3.74}$$

3.3.2 Holographic Wilson Loop

To compute the vev of the supersymmetric Wilson loop holographically, we use the Maldacena conjecture [8]. It is given by

$$\ln\langle W\rangle = -S_{\text{string}}. \tag{3.75}$$

Here, S_{string} is the on-shell action that is given by the Nambu-Goto string action

$$S_{\text{string}} = \frac{1}{2\pi\alpha'}\int_{\Sigma} d^2\sigma\sqrt{\det P[g_{\mu\nu}]}, \tag{3.76}$$

in general, where $g_{\mu\nu}$ is the 10 dimensional metric and is given by (3.69). Let us now parametrize the worldsheet by $\sigma_1 = r$ and $\sigma_2 = \zeta \in [0,2\pi]$. The pullback on the metric (3.69) becomes

$$P[ds_{10}^2] = \frac{e^{\eta}}{\sqrt{Q}}\left[\left(1+g_{ij}\frac{\partial\Theta^i}{\partial r}\frac{\partial\Theta^j}{\partial r}\right)dr^2 + e^{2A}d\zeta^2\right], \tag{3.77}$$

where the functions $\Theta^i(r)$ describe the coordinates in the internal directions. The determinant is given by

$$\det P[ds_{10}^2] = \frac{e^{2\eta+2A}}{Q}\left(1+g_{ij}\frac{\partial\Theta^i}{\partial r}\frac{\partial\Theta^j}{\partial r}\right). \tag{3.78}$$

Now, in the Euclidean signature, all g_{ij}s are positive definite. Thus, the quantity in brackets, $\left(1+g_{ij}\frac{\partial\Theta^i}{\partial r}\frac{\partial\Theta^j}{\partial r}\right)$ is naturally positive. Now, to minimize the string action, we just minimize the determinant, which can be done by considering constant $\Theta^i(r)$s. We now have

$$\det P[ds_{10}^2] = \frac{e^{2\eta+2A}}{Q} \tag{3.79}$$

$$= \begin{cases} \frac{e^{2\eta+2A}}{X}(\sin^2\theta + X\cos^2\theta) & \text{for } p<3, \\ \frac{e^{2\eta(r)+2A(r)}}{X}(X\cos^2\theta + (X^2-Y^2)\sin^2\theta) & \text{for } p>3. \end{cases} \tag{3.80}$$

We can see that the above function will have extrema at $\theta = \frac{n\pi}{2}$ for $n \in \mathbb{Z}$. As explained in [21], only $\theta = 0$ corresponds to a Wilson loop that we are interested in. So, we find the string action to be

$$S_{\text{string}} = \frac{1}{2\pi\alpha'}\int dr d\zeta\sqrt{\det P[ds_{10}^2]} \tag{3.81}$$

$$= \frac{1}{\alpha'}\int dr e^{\eta+A}. \tag{3.82}$$

In the last step, we have integrated over ζ. Now, this diverges close to the UV boundary ($r \to \infty$) and we need to renormalize it which can either be done using the Legendre transformation method discussed above using which, the counterterm is found to be [21]

$$S_{\text{string,counterterm}} = \frac{1}{g\alpha'}e^{A+\frac{3}{6-p}\eta}\Big|_{r\to\infty}, \tag{3.83}$$

or one can isolate the divergent piece and drop it.

4 Strings and the Polyakov Path Integral

The discussion that follows is from [32, 33, 34].

For the purpose of this discussion and calculations thereafter, we will be using the Polyakov string action. According to Feynman's path integral formulation, the amplitude of going from an initial state to a final state is given by summing over all the possible paths that can lead to the final state weighted by $e^{\frac{i}{\hbar}S[x]}$. This notion can be extended to String theory as well. We will basically sum over all the allowed worldsheet configurations given initial and final curves. Examples of allowed interactions are given in Figure 4.1 Examples of interactions that are not

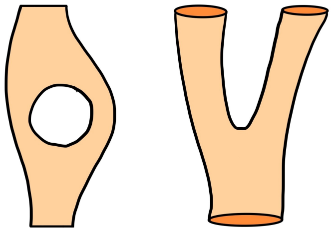

Figure 4.1. The first figure is an open string breaking into two and rejoining. The second figure is a closed string breaking into two closed strings

allowed are given in Figure 4.2 These are not allowed since they cannot be added without

(a) *(b)*

Figure 4.2. Two closed loops interacting in a) close contact or b) long-range

breaking symmetries. The allowed interactions are implicit in the worldsheets or their sums and can be seen as quantum corrections involving intermediate states.

4.1 Symmetries of the Polyakov Action

The Polyakov String action is given by (in Euclidean signature) [28, 29]:

$$S_P = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{\alpha\beta}g_{mn}\partial_\alpha X^m\partial_\beta X^n. \tag{4.1}$$

Here, $\gamma^{\alpha\beta}$ $(\alpha,\beta = 0,1)$ is a dynamical 2-dimensional metric on the string worldsheet, g_{mn} $(m,n = 0,1,\dots,9)$ is the ten-dimensional metric on the target space, X^ms are the embedding coordinates, and $\gamma = \det\gamma_{\alpha\beta}$. The Polyakov action enjoys the following symmetries:

1. Poincaré invariance. The action is invariant under the following transformation:

$$X^m \to \Lambda^m_n X^n + c^m,$$

where Λ^m_n is the Lorentz transformation matrix and c^m is a translation. This is a global symmetry.

2. Reparametrization invariance, also called diffeomorphism invariance. If we redefine the coordinates on the worldsheet as $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$, the fields X^ms and the worldsheet metric $\gamma_{\alpha\beta}$ will transform as follows:

$$X^m(\sigma) \to \tilde{X}^m(\tilde{\sigma}) = X^m(\sigma),$$

$$\gamma_{\alpha\beta}(\sigma) \to \tilde{\gamma}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial \sigma^\lambda}{\partial \tilde{\sigma}^\alpha} \frac{\partial \sigma^\rho}{\partial \tilde{\sigma}^\beta} \gamma_{\lambda\rho}(\sigma).$$

The action remains invariant under this local transformation.

3. Weyl Invariance. The action is invariant under a rescaling transformation:

$$\gamma_{\alpha\beta}(\sigma) \to \Omega^2(\sigma)\gamma_{\alpha\beta}(\sigma).$$

This transformation preserves angles. But it is special to 2 dimensions. Weyl invariance, even in two dimensions, gets broken if we add a potential term ($\int d^2\sigma\sqrt{\gamma}V(x)$) or a cosmological constant term ($\mu \int d^2\sigma\sqrt{\gamma}$).

4.2 The Path Integral

We will now try to write the path integral for the Euclidean Polyakov action given in 4.1. We integrate over all Euclidean worldsheet metrics $\gamma_{\alpha\beta}$ and the fields $X^m(\sigma)$s. In general, we can write the path integral as follows:

$$Z = \frac{1}{\text{Vol}} \int \mathscr{D}X \mathscr{D}\gamma e^{-S_P[X,\gamma]}.$$

Now, the Vol term accounts for the overcounting because as we know, the configurations related by Weyl transformation or diffeomorphism represent the same physical state. We thus need to divide by the volume of this local symmetry group or the gauge action on fields. That is, we want to find out

$$Z = \int \frac{\mathscr{D}X \mathscr{D}\gamma}{\text{Vol}_{\text{diff.}\times\text{Weyl}}} e^{-S_P[X,\gamma]}. \tag{4.2}$$

To achieve this, we will integrate over a slice (**gauge slice**) that cuts through each gauge equivalence class with the appropriate Jacobian. Each equivalence class represents a physically distinct state. The gauge orbits represent families of equivalent configurations. This Jacobian for the change of variables is nothing but the Faddeev-Popov determinant.

4.2.1 Fixing the metric

Both Reparametrization invariance and Weyl invariance are local symmetries. These are also gauge symmetries. Thus, two metrics related by Weyl transformation or diffeomorphism must

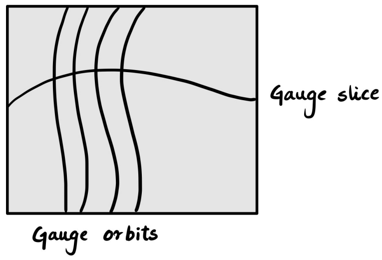

Figure 4.3. A schematic of how we pass through each gauge orbit once

be considered the same physical state. These symmetries will allow us to fix a gauge. We know that the worldsheet metric has three independent components. The reparametrization invariance allows us to choose two of them. Using this, we bring our worldsheet to a conformally flat form (this is locally true unless we are operating on surfaces with genus zero: a cylinder, or a sphere). Let us call this the fiducial metric:

$$\hat{\gamma}_{\alpha\beta} = e^{2\phi(\sigma)}\eta_{\alpha\beta}. \tag{4.3}$$

Here, $\eta_{\alpha\beta}$ is Minkowski metric, and $\phi(\sigma)$ is some function on the worldsheet. We can use Weyl invariance to set $\phi(\sigma) = 0$ and get $\hat{\gamma}_{\alpha\beta} = \eta_{\alpha\beta}$ locally.

4.2.2 The Faddeev-Popov Determinant

We use the Faddeev-Popov method to obtain the correct measure. As mentioned earlier, we plan to separate the path integral into an integral over the gauge group times an integral along the gauge slice and then divide out the former. The Faddeev-Popov determinant is essentially the Jacobian for this change of variables. Let us denote by ζ the combined coordinate and Weyl transformation. So, we write

$$\gamma_{\alpha\beta}(\sigma) \longrightarrow \gamma^{\zeta}_{\alpha\beta}(\sigma') = e^{2\phi(\sigma)}\frac{\partial\sigma^{\mu}}{\partial\sigma^{\alpha\prime}}\frac{\partial\sigma^{\nu}}{\partial\sigma^{\beta\prime}}\gamma_{\mu\nu}(\sigma).$$

Consider the integral over the gauge orbit of our fiducial metric, $\hat{\gamma}$. Now, for a certain transformation ζ, the configuration $\hat{\gamma}^{\zeta}$ will coincide with our worldsheet metric γ. Thus, we define the Faddeev-Popov measure as follows:

$$1 = \Delta_{FP}(\gamma)\int \mathscr{D}\zeta\ \delta(\gamma - \hat{\gamma}^{\zeta}). \tag{4.4}$$

We now show that this measure is gauge invariant:

$$\Delta_{FP}(\gamma^{\zeta})^{-1} = \int \mathscr{D}\zeta'\ \delta(\gamma^{\zeta} - \hat{\gamma}^{\zeta'}) \tag{4.5}$$

$$= \int \mathscr{D}\zeta'\ \delta(\gamma - \hat{\gamma}^{\zeta^{-1}\cdot\zeta'}) \tag{4.6}$$

$$= \int \mathscr{D}\zeta''\ \delta(\gamma - \hat{\gamma}^{\zeta''}) \tag{4.7}$$

$$= \Delta_{FP}(\gamma)^{-1}. \tag{4.8}$$

Here, $\zeta'' = \zeta^{-1}.\zeta'$. In the second step, we used the invariance of the delta function, and in the penultimate step, we used the invariance of the measure. Plugging (4.4) into our expression (4.2), we get

$$Z[\hat{\gamma}] = \int \frac{\mathscr{D}\zeta\mathscr{D}X\mathscr{D}\gamma}{\text{Vol}_{\text{diff.}\times\text{Weyl}}} \Delta_{FP}(\gamma)\ \delta(\gamma - \hat{\gamma}^{\zeta})\ e^{-S_P[X,\gamma]} \tag{4.9}$$

$$= \int \frac{\mathscr{D}\zeta\mathscr{D}X}{\text{Vol}_{\text{diff.}\times\text{Weyl}}} \Delta_{FP}(\hat{\gamma}^{\zeta})\ e^{-S_P[X,\hat{\gamma}^{\zeta}]}. \tag{4.10}$$

We write $Z[\hat{\gamma}]$ since our path integral here depends on the choice of the fiducial metric. In the second step, we have integrated over γ making use of the delta function. Now, if our theory is truly Weyl invariant, we could write

$$Z[\hat{\gamma}] = \int \frac{\mathscr{D}\zeta\mathscr{D}X}{\text{Vol}_{\text{diff.}\times\text{Weyl}}} \Delta_{FP}(\hat{\gamma})\ e^{-S_P[X,\hat{\gamma}]}.$$

We see that in this case, nothing on the right-hand side depends on ζ and thus, the integral over ζ gives us the gauge-group volume exactly canceling the denominator. This gives

$$Z[\hat{\gamma}] = \int \mathscr{D}X\ \Delta_{FP}(\hat{\gamma})\ e^{-S_P[X,\hat{\gamma}]}. \tag{4.11}$$

We see that Δ_{FP} is indeed the Jacobian associated with this transformation and thus gives the correct measure. Now, we calculate the value of Δ_{FP}. Let us go back to (4.4). Recall that for exactly one value of ζ, that is, for $\zeta = 0$, the value of the delta function $\delta(\gamma - \hat{\gamma}^{\zeta})$ is non-zero. Consider an infinitesimal Weyl transformation parametrized by $\phi(\sigma)$ and an infinitesimal diffeomorphism $\sigma^\alpha \to \sigma^\alpha + v^\alpha(\sigma)$. We can write the change in metric under this transformation as:

$$\delta\gamma_{\alpha\beta} = 2\phi\gamma_{\alpha\beta} - \nabla_\alpha v_\beta - \nabla_\beta v_\alpha.$$

We define a differential operator P as follows:

$$[Pv]_{\alpha\beta} = \nabla_\alpha v_\beta + \nabla_\beta v_\alpha - \nabla_\sigma v^\sigma \gamma_{\alpha\beta}. \tag{4.12}$$

The change in the metric then becomes

$$\delta\gamma_{\alpha\beta} = (2\phi - \nabla_\sigma v^\sigma)\gamma_{\alpha\beta} - [Pv]_{\alpha\beta}.$$

We can see that P generates traceless symmetric tensors since left hand side is a tensor and the first term on the right hand side contains the trace and is symmetric. Plugging all this back into (4.4), we get

$$1 = \Delta_{FP}(\hat{\gamma}) \int \mathscr{D}\phi\mathscr{D}v\ \delta(-(2\phi - \hat{\nabla}_\sigma v^\sigma)\hat{\gamma}_{\alpha\beta} + [\hat{P}v]_{\alpha\beta}). \tag{4.13}$$

Here, we have replaced $\mathscr{D}\zeta$ by $\mathscr{D}\phi\mathscr{D}v$. A hat on the operators implies that they contain the fiducial metric $\hat{\gamma}$. Now, Fourier transforming the delta-functional in the integral, we get

$$\Delta_{FP}^{-1}(\hat{\gamma}) = \int \mathscr{D}\phi\mathscr{D}v\mathscr{D}\beta\ \exp\left[2\pi \int d^2\sigma \sqrt{\hat{\gamma}}\, \beta^{\alpha\beta}(-(2\phi - \hat{\nabla}_\sigma v^\sigma)\hat{\gamma}_{\alpha\beta} + [\hat{P}v]_{\alpha\beta})\right],$$

where $\beta^{\alpha\beta}$ is a symmetric tensor field on the worldsheet. Now, let us perform the $\mathscr{D}\phi$ integral. It produces a delta functional forcing $\beta^{\alpha\beta}$ to be traceless. That is

$$\beta^{\alpha\beta}\hat{\gamma}_{\alpha\beta} = 0.$$

Then, we are finally left with

$$\Delta_{FP}^{-1}(\hat{\gamma}) = \int \mathscr{D}v\mathscr{D}\beta\ \exp\left[2\pi \int d^2\sigma \sqrt{\hat{\gamma}}\, \beta^{\alpha\beta}[\hat{P}v]_{\alpha\beta}\right].$$

Now, we have the value of the inverse determinant. We want to be able to invert it to get Δ_{FP}. Now, we recall from the path integrals in fermionic field theories that the determinant can be written as a path integral over Grassmanian or anti-commuting fields. We are going to do the same here. We replace each bosonic field with a corresponding Grassmann field (We will call them ghost fields):

$$\beta_{\alpha\beta} \longrightarrow b_{\alpha\beta}, \tag{4.14}$$

$$v^{\alpha} \longrightarrow c^{\alpha}, \tag{4.15}$$

where $b_{\alpha\beta}$ is traceless, just like $\beta_{\alpha\beta}$. Our final expression for the Faddeev-Popov determinant becomes (we have carried out the Wick rotation)

$$\Delta_{FP}(\hat{\gamma}) = \int \mathscr{D}b\mathscr{D}c\ \exp\left[-S_{\text{ghost}}\right],$$

where the ghost action S_{ghost} is defined as

$$S_{\text{ghost}} = \frac{1}{2\pi}\int d^2\sigma\ \sqrt{\hat{\gamma}} b_{\alpha\beta}[\hat{P}c]^{\alpha\beta}, \tag{4.16}$$

where

$$[Pc]_{\alpha\beta} = \nabla_{\alpha}c_{\beta} + \nabla_{\beta}c_{\alpha} - \gamma_{\alpha\beta}\nabla_{\sigma}c^{\sigma}.$$

Plugging our expression into the path integral (4.11), we get

$$Z[\hat{\gamma}] = \int \mathscr{D}X\mathscr{D}b\mathscr{D}c\ e^{-S_P[X,\hat{\gamma}]-S_{\text{ghost}}[b,c,\hat{\gamma}]}.$$

Since the action is quadratic in the field, we can perform a Gaussian integral over X and the ghost fields b and c, and write the final result as

$$Z[\hat{\gamma}] = (\det\hat{\nabla}^2)^{-D/2}\det\hat{P}. \tag{4.17}$$

5 Calculation of Bosonic Fluctuations

In this section, we generalize the calculations in [16] to the string action involving the antisymmetric tensor. We first derive the equations of motion for the background fields. Then, we find the quadratic action for fluctuations along the worldsheet and transverse to it, and find an expression for the bosonic mass matrix.

5.1 Equations of motion

The first order of business is to find out the equations of motion in the presence of a general B-field. We start with the action

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{\alpha\beta}g_{mn}\partial_\alpha X^m\partial_\beta X^n + \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\varepsilon^{\alpha\beta}B_{mn}\partial_\alpha X^m\partial_\beta X^n. \tag{5.1}$$

Here, $\gamma_{\alpha\beta}$ $(\alpha,\beta = 0,1)$ is the metric on the string worldsheet, g_{mn} $(m,n = 0,1,\ldots,d-1)$ is the target space metric, X^m are the embedding functions B_{mn} is the antisymmetric background field in the target space, and $\varepsilon^{\alpha\beta}$ is the Levi-Civita tensor ($\varepsilon^{\alpha\beta} = \dfrac{\varepsilon^{\alpha\beta}}{\sqrt{\gamma}}$ where $\varepsilon^{\alpha\beta}$ is the tensor density). Solving the Euler-Lagrange equation for the Embedding functions $X^m(\sigma^1,\sigma^2)$,

$$\frac{\partial\mathscr{L}}{\partial X^p} - \partial_\gamma\left(\frac{\partial\mathscr{L}}{\partial(\partial_\gamma X^p)}\right) = 0,$$

we get

$$\Box X^l + \gamma^{\alpha\beta}\Gamma^l{}_{mn}\partial_\alpha X^m\partial_\beta X^n = \frac{\varepsilon^{\alpha\beta}}{2}H^l{}_{mn}\partial_\alpha X^m\partial_\beta X^n, \tag{5.2}$$

Here, $\Box = \dfrac{1}{\sqrt{\gamma}}\partial_\alpha(\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\beta)$ is the covariant Laplacian on the worldsheet metric, and

$$\Gamma^l{}_{mn} = \frac{g^{pl}}{2}[\partial_m g_{pn} + \partial_n g_{mp} - \partial_p g_{mn}], \tag{5.3}$$

$$H^l{}_{mn} = g^{pl}[\partial_p B_{mn} + \partial_n B_{pm} + \partial_m B_{np}]. \tag{5.4}$$

The calculations are explicitly done in Appendix A. Expanding the covariant derivative in terms of the worldsheet metric $\gamma_{\alpha\beta}$ and related Christoffel symbols $\Lambda^\rho{}_{\alpha\beta}$, we write,

$$\gamma^{\alpha\beta}(\partial_\alpha\partial_\beta X^l - \Lambda^\rho_{\alpha\beta}\partial_\rho X^l + \Gamma^l{}_{mn}\partial_\alpha X^m\partial_\beta X^n) = \frac{\varepsilon^{\alpha\beta}}{2}H^l_{mn}\partial_\alpha X^m\partial_\beta X^n \tag{5.5}$$

$$\Rightarrow\ \gamma^{\alpha\beta}K^l{}_{\alpha\beta} = \frac{\varepsilon^{\alpha\beta}}{2}H^l{}_{mn}\partial_\alpha X^m\partial_\beta X^n \tag{5.6}$$

$$\Rightarrow\ K^l \equiv \gamma^{\alpha\beta}K^l{}_{\alpha\beta} = \frac{\varepsilon^{\alpha\beta}}{2}H^l{}_{mn}\partial_\alpha X^m\partial_\beta X^n. \tag{5.7}$$

Here, we introduce the second fundamental form, the extrinsic curvature, ${K^l}_{\alpha\beta}$, and K^l, the mean curvature. Also, we have

$$g_{lp}\partial_\sigma X^p {K^l}_{\alpha\beta} = \frac{{\varepsilon^\rho}_\alpha}{2}{H^l}_{mn}\partial_\rho X^m \partial_\beta X^n g_{lp}\partial_\sigma X^p \tag{5.8}$$

$$= \frac{{\varepsilon^\rho}_\alpha}{2}H_{pmn}\partial_\rho X^m \partial_\beta X^n \partial_\sigma X^p = 0. \tag{5.9}$$

Thus, the extrinsic curvature is orthogonal to the vectors $t^m_\alpha = \partial_\alpha X^m$. This means that among the equations of motion (5.5), only $d-2$ are independent. These are the transverse degrees of freedom.

5.2 Bosonic Fluctuations

In this section, we will discuss the second-order quantum fluctuations around the classical background (5.1). To calculate these fluctuations, we use the background field method [24] for non-linear sigma models and expand in Riemann normal coordinates. Then we will write these fluctuations in terms of the intrinsic and extrinsic geometric invariants of the classical solution. We will start with our action for fields $\tilde{X}^m$,

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\left[\gamma^{\alpha\beta}g_{mn}(\tilde{X}) + \varepsilon^{\alpha\beta}B_{mn}(\tilde{X})\right](\partial_\alpha\tilde{X}^m\partial_\beta\tilde{X}^n), \tag{5.10}$$

X^m will be our classical solution, and we express $\delta X^m = \tilde{X}^m - X^m$ will be our fluctuations around this classical solution. As discussed in [24], the fluctuations if written as a power series do not produce a manifestly covariant expression for the series coefficients. To obtain a manifestly covariant expression, we use Riemann normal coordinates and express the fluctuations as a local power series in spacetime vectors tangent to the spacetime geodesic connecting X^m and $X^m + \delta X^m$. We parametrize a geodesic $X^m(t)$ with parameter t as follows:

$$X^m(0) = X^m, \quad X^m(1) = \tilde{X}^m. \tag{5.11}$$

The geodesic equation for $X^m(t)$ is given by

$$\ddot{X}^m(t) + \Gamma^m_{np}\dot{X}^n(t)\dot{X}^p(t) = 0. \tag{5.12}$$

Writing $\zeta^m \equiv \dot{X}^m(0)$ and Taylor expanding $X^m(t)$ around $t=0$, we get

$$X^m(t) = X^m(0) + t\dot{X}^m(0) - \frac{1}{2}t^2\Gamma^m_{np}\dot{X}^m(0)\dot{X}^n(0) + O(t^3). \tag{5.13}$$

Here, we have used (5.12) to write $\ddot{X}^m(0) = -\Gamma^m_{np}\dot{X}^n(0)\dot{X}^p(0)$. This gives, for $t=1$,

$$\tilde{X}^m = X^m + \zeta^m - \frac{1}{2}\Gamma^m_{np}\zeta^m\zeta^n + O(t^3). \tag{5.14}$$

Differentiating the above with respect to the worldsheet coordinates, we have

$$\partial_\alpha\tilde{X}^m = \partial_\alpha X^m + \nabla_\alpha\zeta^m - \Gamma^m_{np}\partial_\alpha X^n\zeta^p$$
$$- \frac{1}{2}\partial_\alpha X^r(\partial_r\Gamma^m_{np} - 2\Gamma^m_{nl}{\Gamma^l}_{rp})\zeta^n\zeta^p - \Gamma^m_{np}\zeta^n\nabla_\alpha\zeta^p + O(\zeta^3).$$

where $\nabla_\alpha \zeta^m \equiv \partial_\alpha \zeta^m + \Gamma^m_{np}\partial_\alpha X^n \zeta^p$. Similarly, we expand our target metric and the antisymmetric 2-form as follows:

$$g_{mn}(\tilde{X}) = g_{mn}(X) + \zeta^r \partial_r g_{mn} - \frac{1}{2}\Gamma^r{}_{pq}\zeta^p\zeta^q\partial_r g_{mn} + \frac{1}{2}\zeta^r\zeta^s\partial_r\partial_s g_{mn} + O(\zeta^3), \tag{5.15}$$

$$B_{mn}(\tilde{X}) = B_{mn}(X) + \zeta^r \partial_r B_{mn} - \frac{1}{2}\Gamma^r{}_{pq}\zeta^p\zeta^q\partial_r B_{mn} + \frac{1}{2}\zeta^r\zeta^s\partial_r\partial_s B_{mn} + O(\zeta^3). \tag{5.16}$$

We can write the action given in (5.10) as

$$S = S^{(0)} + S^{(2)}_g + S^{(2)}_B + O(\zeta^3),$$

where, $S^{(0)}$ denotes the classical action (5.1), $S^{(2)}_g$ and $S^{(2)}_B$ denote the contributions to the quadratic fluctuations coming from the target space metric and the antisymmetric tensor respectively, and are calculated as follows:

$$S^{(2)}_g = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{\alpha\beta}[\nabla_\alpha\zeta^m\nabla_\beta\zeta^n g_{mn} - R_{rmsn}\zeta^r\zeta^s\partial_\alpha X^m\partial_\beta X^n], \tag{5.17}$$

and

$$S^{(2)}_B = \frac{1}{4\pi\alpha'}\int d^2\sigma\ \sqrt{\gamma}\varepsilon^{\alpha\beta}\Big[B_{mn}\nabla_\alpha\zeta^m\nabla_\beta\zeta^n + 2\partial_r B_{mn}\partial_\alpha X^m\nabla_\beta\zeta^n\zeta^r \tag{5.18}$$

$$+\frac{1}{2}\left(\partial_r\partial_s B_{mn} + B_{mp}R^p{}_{rsn} + B_{pn}R^p{}_{rsm}\right)\partial_\alpha X^m\partial_\beta X^n\zeta^r\zeta^s\Big] \tag{5.19}$$

$$= \frac{1}{4\pi\alpha'}\int d^2\sigma\ \sqrt{\gamma}\varepsilon^{\alpha\beta}\Big[H_{mnr}\partial_\alpha X^m\nabla_\beta\zeta^n\zeta^r + \frac{1}{2}\nabla_s H_{mnr}\partial_\alpha X^m\partial_\beta X^n\zeta^r\zeta^s\Big]. \tag{5.20}$$

To have a canonically normalized kinetic term, we introduce a set of vielbein E^A_m $(A, B, \cdots = 0, \ldots, d-1)$ for the target metric, given by

$$g_{mn} = \eta_{AB}E^A_m E^B_n = E^A_m E_{An}, \tag{5.21}$$

with inverse E^m_A such that

$$E^A_m E^m_B = \delta^A_B, \tag{5.22}$$

and a set of zweibein e^a_α $(a, b = 0, 1)$ for the worldsheet metric, given by

$$\gamma_{\alpha\beta} = \eta_{ab}e^a_\alpha e^b_\beta = e^a_\alpha e_{a\beta}, \tag{5.23}$$

with inverse e^α_a such that

$$e^a_\alpha e^\alpha_b = \delta^a_b. \tag{5.24}$$

Here, η_{AB} and η_{ab} are the d dimensional and 2 dimensional flat metrics. We redefine the fluctuation fields as follows:

$$\xi^A = E^A_m \zeta^m. \tag{5.25}$$

Equation (5.17) becomes

$$S^{(2)}_g = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}[\gamma^{\alpha\beta}D_\alpha\xi^A D_\beta\xi_A - R_{AMBN}t^{aM}t^N_a\xi^A\xi^B]. \tag{5.26}$$

Here,

$$D_\alpha\xi^A = \partial_\alpha\xi^A + \Omega^A{}_{nB}\xi^B\partial_\alpha X^n, \quad t^A_a = E^A_m e^\alpha_a\partial_\alpha X^m, \tag{5.27}$$

where $\Omega^A_{\ nB}$ is the spin connection that replaces the usual Christoffel symbols. Also, we have,

$$t_a^A t_b^B \eta_{AB} = \eta_{ab}. \tag{5.28}$$

Equation (5.18) becomes

$$S_B^{(2)} = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \sqrt{\gamma}\varepsilon^{\alpha\beta}\left[H_{mAB}\partial_\alpha X^m D_\beta \xi^A \xi^B + \frac{1}{2}\nabla_A H_{mnB}\partial_\alpha X^m \partial_\beta X^n \xi^A \xi^B\right]. \tag{5.29}$$

See Appendix B for explicit calculations that lead to the results (5.26) and (5.29). Combining equations (5.26) and (5.29), we get

$$S^{(2)} = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{\gamma}\Big[\gamma^{\alpha\beta} D_\alpha \xi^A D_\beta \xi_A - R_{AMBN} t^{aM} t_a^N \xi^A \xi^B$$
$$+ \varepsilon^{\alpha\beta} H_{mAB}\partial_\alpha X^m D_\beta \xi^A \xi^B + \varepsilon^{\alpha\beta}\frac{1}{2}\nabla_A H_{mnB}\partial_\alpha X^m \partial_\beta X^n \xi^A \xi^B\Big].$$

or

$$S^{(2)} = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \sqrt{\gamma}\Big[\gamma^{\alpha\beta} D_\alpha \xi^A D_\beta \xi_A + \varepsilon^{\alpha\beta} H_{mAB}\partial_\alpha X^m D_\beta \xi^A \xi^B - M_{AB}\xi^A \xi^B\Big]. \tag{5.30}$$

Here,

$$M_{AB} = R_{AMBN} t^{aM} t_a^N - \frac{\varepsilon^{\alpha\beta}}{2}\nabla_A H_{mnB}\partial_\alpha X^m \partial_\beta X^n. \tag{5.31}$$

We introduce $(d-2)$ orthonormal vector fields N_i^A orthogonal to the worldsheet, satisfying

$$N_i^A N_j^B \eta_{AB} = \delta_{ij}, \tag{5.32}$$
$$t_a^A N_i^B \eta_{AB} = 0, \tag{5.33}$$
$$t_a^A t_b^B \eta^{ab} + N_i^A N_j^B \delta^{ij} = \eta^{AB}, \tag{5.34}$$

and decompose the redefined fluctuation field ξ^A tangential (x^a) and orthogonal (y^i) to the worldsheet. We can now write ξ^A as

$$\xi^A = x^a t_a^{\ A} + y^i N_i^{\ A}. \tag{5.35}$$

Here, $a,b = 0,1$, and $i,j = 2,\ldots,d-1$. Carrying over this decomposition to the covariant derivatives, one finds

$$t^a_{\ A} D_\alpha \xi^A = \mathscr{D}_\alpha x^a - K^a_{\ A\alpha} N_i^{\ A} y^i, \tag{5.36}$$
$$N^i_{\ A} D_\alpha \xi^A = \mathscr{D}_\alpha y^i + x^a N^i_{\ A} K^A_{a\beta}. \tag{5.37}$$

Here, $K^a_{A\alpha} = E_{Am} e^{a\beta} K^m_{\alpha\beta}$ is the extrinsic curvature in mixed basis. Since, x^a belongs to the tangent bundle and y^i belongs to the normal bundle, $\mathscr{D}_\alpha$ acts on x^a and y^i differently and as follows:

$$\mathscr{D}_\alpha x^a \equiv \partial_\alpha x^a + \omega^a_{\ b\alpha} x^b, \tag{5.38}$$
$$\mathscr{D}_\alpha y^i \equiv \partial_\alpha y^i + A^i_{\ j\alpha} y^j. \tag{5.39}$$

The connection ${A^i}_{j\alpha}$ on the normal bundle is given by

$$ {A^i}_{j\alpha} \equiv {N^i}_B D_\alpha {N_j}^B = {N^i}_B(\partial_\alpha {N_j}^B + {N_j}^C \Omega^B_{\ C\alpha}), \tag{5.40} $$

and the connection $\omega^a_{\ b\alpha}$ on the tangent bundle is given by

$$ \omega^a_{\ b\alpha} \equiv t^a_{\ A} D_\alpha {t_b}^A = t^a_{\ A}(\partial_\alpha {t_b}^A + {t_b}^B \Omega^A_{\ B\alpha}). \tag{5.41} $$

Using *Gauss-Codazzi* equation [35]

$$ R_{ACBD} t_\alpha^{\ A} {t_\rho}^C {t_\beta}^B {t_\sigma}^D = {}^{(2)}R_{\alpha\rho\beta\sigma} + \eta_{AB} K^A_{\ \rho\beta} K^B_{\ \sigma\alpha} - \eta_{AB} K^A_{\ \rho\sigma} K^B_{\ \beta\alpha}, \tag{5.42} $$

and the *Codazzi-Mainardi* equation [35]

$$ \mathscr{D}_\alpha {K^i}_{\beta\gamma} - \mathscr{D}_\beta {K^i}_{\alpha\gamma} = R_{MNRS} t_\alpha^{\ M} t_\beta^{\ N} {t_\gamma}^S N^{Ri}, \tag{5.43} $$

where ${K^i}_{\alpha\beta} \equiv K^A_{\ \alpha\beta} {N^i}_A = K^m_{\alpha\beta} E^A_m N^i_A$, along with the equation of motion (5.5), we find the quadratic Lagrangian (5.30) to be

$$ \mathscr{L}^{(2)} = \mathscr{L}^{(2)}_g + \mathscr{L}^{(2)}_B. $$

where,

$$ \begin{aligned} \mathscr{L}^{(2)}_g = \sqrt{\gamma}\Big[& \gamma^{\alpha\beta}\mathscr{D}_\alpha x^a \mathscr{D}_\beta x_a - {}^{(2)}R_{ab} x^a x^b + \gamma^{\alpha\beta}\mathscr{D}_\alpha y^i \mathscr{D}_\beta y_i \\ & - 2\gamma^{\alpha\beta}(\mathscr{D}_\alpha x^a K_{i,a\beta} y^i - \mathscr{D}_\alpha y^i x^a K_{i,a\beta}) + 2\gamma^{\alpha\beta}\nabla_\alpha K_{i,a\beta} x^a y^i \\ & - (R_{AMBN} t^{cM} t^N_c N^A_i N^B_j - \gamma^{\alpha\beta}\gamma^{\rho\sigma} K_{i,\alpha\rho} K_{j,\beta\sigma}) y^i y^j \Big]. \end{aligned} \tag{5.44} $$

and

$$ \mathscr{L}^{(2)}_B = \sqrt{\gamma}\varepsilon^{\alpha\beta}\left[\frac{1}{2}\nabla_i H_{\alpha\beta j} y^i y^j + H_{\alpha ij}\mathscr{D}_\beta y^i y^j + H_{\alpha\lambda i}(\mathscr{D}_\beta x^\lambda y^i - \mathscr{D}_\beta y^i x^\lambda)\right]. \tag{5.45} $$

Here,

$$ \nabla_i H_{\alpha\beta j} = \nabla_r H_{mns} E^r_A N^A_i E^s_B N^B_j t^m_\alpha t^n_\beta, \tag{5.46} $$

$$ H_{mij} = H_{mns} t^m_\alpha E^n_A N^A_i E^s_B N^B_j, \tag{5.47} $$

$$ H_{\alpha\lambda i} = H_{mns} t^m_\alpha t^n_\lambda E^s_A N^A_i. \tag{5.48} $$

Combining the two, we get

$$ \begin{aligned} \mathscr{L}^{(2)} = \sqrt{\gamma}\Bigg[& \gamma^{\alpha\beta}\mathscr{D}_\alpha x^\sigma \mathscr{D}_\beta x_\sigma + \gamma^{\alpha\beta}\mathscr{D}_\alpha y^i \mathscr{D}_\beta y_i - R_{\lambda\sigma} x^\lambda x^\sigma \\ & - \left(R_{AMBN} t^{cM} t^N_c N^A_i N^B_j - \gamma^{\alpha\beta}\gamma^{\rho\sigma} K_{i,\alpha\rho} K_{j,\beta\sigma} - \frac{\varepsilon^{\alpha\beta}}{2}\nabla_i H_{\alpha\beta j} \right) y^i y^j \\ & + 2\left(\gamma^{\alpha\beta}\nabla_\alpha K_{i,\lambda\beta}\right) x^\lambda y^i + \varepsilon^{\alpha\beta} H_{\alpha ij}\mathscr{D}_\beta y^i y^j \\ & - 2\left[\gamma^{\alpha\beta} K_{i,\lambda\alpha} - \frac{\varepsilon^{\alpha\beta}}{2} H_{\alpha\lambda i}\right](\mathscr{D}_\beta x^\lambda y^i - \mathscr{D}_\beta y^i x^\lambda) \Bigg]. \end{aligned} \tag{5.49} $$

Now, we shall introduce a new quantity $\mathscr{K}^l_{\alpha\beta}$ as follows:

$$\mathscr{K}^l_{\alpha\beta} \equiv K^l_{\alpha\beta} - \frac{\varepsilon^\rho_\alpha}{2} H^l{}_{mn} \partial_\rho X^m \partial_\beta X^n = K^l_{\alpha\beta} - \frac{\varepsilon^\rho_\alpha}{2} H^l{}_{mn} t^m_\rho t^n_\beta. \tag{5.50}$$

Our equation of motion (5.5) thus becomes:

$$\gamma^{\alpha\beta} \mathscr{K}^l_{\alpha\beta} = \mathscr{K}^l = 0. \tag{5.51}$$

with the mean modified curvature ($\mathscr{K}^l$) being zero. Plugging this into our results, we get

$$\begin{aligned}\mathscr{L}^{(2)} = \sqrt{\gamma}\Bigg[& \gamma^{\alpha\beta} \mathscr{D}_\alpha x^\sigma \mathscr{D}_\beta x_\sigma + \gamma^{\alpha\beta} \mathscr{D}_\alpha y^i \mathscr{D}_\beta y_i - R_{\lambda\sigma} x^\lambda x^\sigma \\ & - \Big(R_{AMBN} t^{cM} t^N_c N^A_i N^B_j - \gamma^{\alpha\beta} \gamma^{\rho\sigma} \mathscr{K}_{i,\alpha\rho} \mathscr{K}_{j,\beta\sigma} \\ & + \frac{1}{4} \gamma^{\rho\sigma} \gamma^{\varepsilon\lambda} H_{i\varepsilon\rho} H_{j\lambda\sigma} - \frac{\varepsilon^{\alpha\beta}}{2} (\nabla_A H_{mnB}) N^A_i N^B_j t^m_\alpha t^n_\beta \Big) y^i y^j \\ & + 2 \left(\gamma^{\alpha\beta} \nabla_\alpha \mathscr{K}_{i,\beta\lambda} \right) x^\lambda y^i + \varepsilon^{\alpha\beta} H_{\alpha ij} \mathscr{D}_\beta y^i y^j \\ & - 2\gamma^{\alpha\beta} \mathscr{K}_{i,\alpha\lambda} (\mathscr{D}_\beta x^\lambda y^i - \mathscr{D}_\beta y^i x^\lambda) \Bigg]. \end{aligned} \tag{5.52}$$

To decouple the longitudinal fluctuations (x^σ) from the transverse ones (y^i), we derive the equation of motion from (5.52) for x^σ [36]:

$$\frac{\partial \mathscr{L}}{\partial x^\mu} = \mathscr{D}_\nu \left(\frac{\partial \mathscr{L}}{\partial (\mathscr{D}_\nu x^\mu)} \right).$$

This gives

$$\boxed{\gamma^{\alpha\beta} \mathscr{D}_\beta \mathscr{D}_\alpha x_\sigma + R_{\alpha\sigma} x^\alpha = \mathscr{D}^\beta \left[2\mathscr{K}_{i,\sigma\beta} y^i \right]}. \tag{5.53}$$

Now, we see that

$$\gamma^{\alpha\beta} \left(y^i \mathscr{K}_{i,\alpha\beta} \right) = y^i \gamma^{\alpha\beta} \mathscr{K}_{i,\alpha\beta} \tag{5.54}$$

$$= 0. \tag{5.55}$$

using equations of motion (5.51). This implies $C_{\alpha\beta} \equiv 2y^i \mathscr{K}_{i,\alpha\beta}$ is traceless. The equations of motion for the fluctuations parallel to the worldsheet can also be written as

$$\Box x_\alpha + R_{\alpha\beta} x^\beta = \mathscr{D}^\beta C_{\alpha\beta}. \tag{5.56}$$

We can also write this equation as

$$P_1(x)_{\alpha\beta} = \mathscr{D}_\beta x_\alpha + \mathscr{D}_\alpha x_\beta - \gamma_{\alpha\beta} \mathscr{D}_\rho x^\rho = C_{\alpha\beta}, \tag{5.57}$$

since the operator P_1 generates traceless symmetric tensors. We make the shift $x \to \bar{x} + x$ with $\bar{x}$ satisfying (5.56) and (5.57) in the Lagrangian (5.52) and find

$$\begin{aligned}\mathscr{L}^{(2)} = \sqrt{\gamma}\Bigg[&\gamma^{\alpha\beta}\mathscr{D}_\alpha x^\sigma \mathscr{D}_\beta x_\sigma - R_{\alpha\beta}x^\alpha x^\beta + \gamma^{\alpha\beta}\mathscr{D}_\alpha y^i \mathscr{D}_\beta y_i \\ &- \Big(R_{AMBN}t^{cM}t_c^N N_i^A N_j^B - \gamma^{\alpha\beta}\gamma^{\rho\sigma}\mathscr{K}_{i,\alpha\rho}\mathscr{K}_{j,\beta\sigma} \\ &+ \frac{1}{4}\gamma^{\rho\sigma}\gamma^{\varepsilon\lambda}H_{i\varepsilon\rho}H_{j\lambda\sigma} - \frac{\varepsilon^{\alpha\beta}}{2}(\nabla_A H_{mnB})N_i^A N_j^B t_\alpha^m t_\beta^n\Big)y^i y^j \\ &+ \gamma^{\alpha\beta}\mathscr{D}_\alpha(2y^i\mathscr{K}_{i\sigma\beta})\bar{x}^\sigma\Bigg]. \end{aligned} \tag{5.58}$$

We see that the longitudinal and the transverse fluctuations decouple except for the last term. Now it can be written as

$$\gamma^{\alpha\beta}\mathscr{D}_\alpha(2y^i\mathscr{K}_{i\sigma\beta})\bar{x}^\sigma = \gamma^{\alpha\beta}\mathscr{D}_\alpha(C_{\sigma\beta})\bar{x}^\sigma \tag{5.59}$$

$$= \mathscr{D}^\beta(C_{\sigma\beta}\bar{x}^\sigma) - C_{\sigma\beta}\mathscr{D}^\beta\bar{x}^\sigma. \tag{5.60}$$

The first term being a surface term will vanish. Looking at the second term, we do the following:

$$-C_{\sigma\beta}\mathscr{D}^\beta\bar{x}^\sigma = -\frac{1}{2}C_{\sigma\beta}(\mathscr{D}^\beta\bar{x}^\sigma + \mathscr{D}^\sigma\bar{x}^\beta) \tag{5.61}$$

$$= -\frac{1}{2}C_{\sigma\beta}(C^{\beta\sigma} + \gamma^{\sigma\beta}\mathscr{D}_\rho\bar{x}^\rho) \tag{5.62}$$

$$= -\frac{1}{2}C_{\sigma\beta}C^{\beta\sigma} = -2y^i y^j\mathscr{K}_{i,\sigma\beta}\mathscr{K}_j{}^{\beta\sigma}. \tag{5.63}$$

using (5.57) and the tracelessness of $C_{\alpha\beta}$. Plugging back into (5.52), we get

$$\begin{aligned}\mathscr{L}^{(2)} = \sqrt{\gamma}\Bigg[&\gamma^{\alpha\beta}\mathscr{D}_\alpha x^\sigma \mathscr{D}_\beta x_\sigma - R_{\alpha\beta}x^\alpha x^\beta + \gamma^{\alpha\beta}\mathscr{D}_\alpha y^i \mathscr{D}_\beta y_i \\ &- \Big(R_{AMBN}t^{cM}t_c^N N_i^A N_j^B + \gamma^{\alpha\beta}\gamma^{\rho\sigma}\mathscr{K}_{i,\alpha\rho}\mathscr{K}_{j,\beta\sigma} \\ &+ \frac{1}{4}\gamma^{\rho\sigma}\gamma^{\varepsilon\lambda}H_{i\varepsilon\rho}H_{j\lambda\sigma} - \frac{\varepsilon^{\alpha\beta}}{2}(\nabla_A H_{mnB})N_i^A N_j^B t_\alpha^m t_\beta^n\Big)y^i y^j\Bigg]. \end{aligned} \tag{5.64}$$

We can thus write the quadratic Lagrangian as a sum of transverse and longitudinal Lagrangians as follows:

$$\mathscr{L}^{(2)} = \mathscr{L}_{\text{long}} + \mathscr{L}_{\text{trans}}, \tag{5.65}$$

with

$$\mathscr{L}_{\text{long}} = \sqrt{\gamma}[\gamma^{\alpha\beta}\mathscr{D}_\alpha x^a \mathscr{D}_\beta x_a - {}^{(2)}R_{ab}x^a x^b], \tag{5.66}$$

and

$$\mathscr{L}_{\text{trans}} = \sqrt{\gamma}[\gamma^{\alpha\beta}\mathbb{D}_\alpha y^i \mathbb{D}_\beta y_i - \mathscr{M}_{ij}^2 y^i y^j], \tag{5.67}$$

where

$$\mathbb{D}_\alpha y^i = \mathscr{D}_\alpha y^i + \frac{\varepsilon_\alpha^\beta}{2}H_\beta{}^i{}_j, \tag{5.68}$$

and

$$\mathscr{M}_{ij}^2 = \Big(R_{AMBN} t^{cM} t_c^N N_i^A N_j^B + \gamma^{\alpha\beta} \gamma^{\rho\sigma} \mathscr{K}_{i,\alpha\rho} \mathscr{K}_{j,\beta\sigma}$$
$$+ \frac{1}{4} \gamma^{\alpha\beta} \left(\gamma^{\varepsilon\lambda} H_{i\varepsilon\alpha} H_{j\lambda\beta} + \delta^{kl} H_{i\alpha l} H_{j\beta k} \right) - \frac{\varepsilon^{\alpha\beta}}{2} (\nabla_A H_{mnB}) N_i^A N_j^B t_\alpha^m t_\beta^n \Big). \quad (5.69)$$

6 Results and Conclusions

We have successfully shown the usefulness of the vev of the Wilson loop operator as field potential and found the potential of a quark-antiquark pair separated by a distance L on the gravity side, reproducing the result in [8]. We have also calculated the vev of a circular Wilson loop reproducing the result in [23]. We see that this vev needs to be normalized. In this process of renormalization, we have also seen how instead of using the area of the minimal surface for the vev of the Wilson loop operator, one should use an appropriate Legendre transform of it [23]. We saw how the Legendre transforms can help in obtaining a finite value of vev of Wilson loop in $AdS_5 \times S^5$ metric and can be applied to calculate the vev of Wilson loops in a general Dp brane metric as shown in [21].

Then we discussed how one can calculate observables like the Wilson loop operator beyond the leading order using geometrical properties of the underlying geometry of the spacetime in which the string moves. Extending the work in [16], we considered a general string action involving the antisymmetric 2-tensor

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma} \left(\gamma^{\alpha\beta} g_{mn} + \varepsilon^{\alpha\beta} B_{mn} \right) \partial_\alpha X^m \partial_\beta X^n.$$

We calculated the net quadratic Lagrangian for this case expanding the fields (X^ms) around their classical values using the background field method and found it to be

$$\begin{aligned} \mathscr{L} = \sqrt{\gamma} \Bigg[& \gamma^{\alpha\beta} \mathscr{D}_\alpha x^\sigma \mathscr{D}_\beta x_\sigma - R_{\alpha\beta} x^\alpha x^\beta + \gamma^{\alpha\beta} \mathscr{D}_\alpha y^i \mathscr{D}_\beta y_i \\ & - \Big(R_{AMBN} t^{cM} t_c^N N_i^A N_j^B + \gamma^{\alpha\beta} \gamma^{\rho\sigma} \mathscr{K}_{i,\alpha\rho} \mathscr{K}_{j,\beta\sigma} \\ & + \frac{1}{4} \gamma^{\rho\sigma} \gamma^{\varepsilon\lambda} H_{i\varepsilon\rho} H_{j\lambda\sigma} - \frac{\varepsilon^{\alpha\beta}}{2} (\nabla_A H_{mnB}) N_i^A N_j^B t_\alpha^m t_\beta^n \Big) y^i y^j \Bigg]. \end{aligned}$$

We showed the tangential fluctuations (x^a) and the transverse fluctuations (y^i) to the string worldsheet can be considered separately and we can write the net quadratic Lagrangian as a sum of these contributions:

$$\mathscr{L}^{(2)} = \mathscr{L}_{\text{long}} + \mathscr{L}_{\text{trans}},$$

where, $\mathscr{L}_{\text{long}}$ and $\mathscr{L}_{\text{trans}}$ are given by (5.66) and (5.67), respectively. We have also obtained a general expression for the bosonic mass matrix $\mathscr{M}_{ij}$ given by

$$\begin{aligned} \mathscr{M}_{ij}^2 = \Big(& R_{AMBN} t^{cM} t_c^N N_i^A N_j^B + \gamma^{\alpha\beta} \gamma^{\rho\sigma} \mathscr{K}_{i,\alpha\rho} \mathscr{K}_{j,\beta\sigma} \\ & + \frac{1}{4} \gamma^{\alpha\beta} \left(\gamma^{\varepsilon\lambda} H_{i\varepsilon\alpha} H_{j\lambda\beta} + \delta^{kl} H_{i\alpha l} H_{j\beta k} \right) - \frac{\varepsilon^{\alpha\beta}}{2} (\nabla_A H_{mnB}) N_i^A N_j^B t_\alpha^m t_\beta^n \Big), \end{aligned}$$

which now gets contributions from the antisymmetric 2-tensor, B_{mn}. For the case $B_{mn} = 0$, our results reduce to the results in [16]. Explicit examples of calculation of the vev of the Wilson loop operator beyond the leading order can be seen in [12, 16, 22].

For future work, we also want to look at the fermionic fluctuations and obtain a general expression for the fermionic Lagrangian and mass matrix and check against known examples.

References

[1] Juan Martin Maldacena. “The Large N limit of superconformal field theories and supergravity.” In: *Adv. Theor. Math. Phys.* 2 (1998), pp. 231–252. DOI: 10.4310/ATMP.1998.v2.n2.a1. arXiv: hep-th/9711200.

[2] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. “Gauge theory correlators from noncritical string theory.” In: *Phys. Lett. B* 428 (1998), pp. 105–114. DOI: 10.1016/S0370-2693(98)00377-3. arXiv: hep-th/9802109.

[3] Edward Witten. “Anti-de Sitter space and holography.” In: *Adv. Theor. Math. Phys.* 2 (1998), pp. 253–291. DOI: 10.4310/ATMP.1998.v2.n2.a2. arXiv: hep-th/9802150.

[4] Ofer Aharony et al. “Large N field theories, string theory and gravity.” In: *Phys. Rept.* 323 (2000), pp. 183–386. DOI: 10.1016/S0370-1573(99)00083-6. arXiv: hep-th/9905111.

[5] A. Zaffaroni. “Introduction to the AdS-CFT correspondence.” In: *Class. Quant. Grav.* 17 (2000), pp. 3571–3597. DOI: 10.1088/0264-9381/17/17/306.

[6] Gerard ’t Hooft. “A Planar Diagram Theory for Strong Interactions.” In: *Nucl. Phys. B* 72 (1974). Ed. by J. C. Taylor, p. 461. DOI: 10.1016/0550-3213(74)90154-0.

[7] Kenneth G. Wilson. “Confinement of Quarks.” In: *Phys. Rev. D* 10 (1974). Ed. by J. C. Taylor, pp. 2445–2459. DOI: 10.1103/PhysRevD.10.2445.

[8] Juan Martin Maldacena. “Wilson loops in large N field theories.” In: *Phys. Rev. Lett.* 80 (1998), pp. 4859–4862. DOI: 10.1103/PhysRevLett.80.4859. arXiv: hep-th/9803002.

[9] Edward Witten. “Topological Quantum Field Theory.” In: *Commun. Math. Phys.* 117 (1988), p. 353. DOI: 10.1007/BF01223371.

[10] Vasily Pestun. “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops.” In: *Commun. Math. Phys.* 313 (2012), pp. 71–129. DOI: 10.1007/s00220-012-1485-0. arXiv: 0712.2824 [hep-th].

[11] N. A. Nekrasov. “Localizing gauge theories.” In: *14th International Congress on Mathematical Physics*. July 2003, pp. 645–654.

[12] Nadav Drukker, David J. Gross, and Arkady A. Tseytlin. “Green-Schwarz string in AdS(5) x S**5: Semiclassical partition function.” In: *JHEP* 04 (2000), p. 021. DOI: 10.1088/1126-6708/2000/04/021. arXiv: hep-th/0001204.

[13] M. Kruczenski and A. Tirziu. “Matching the circular Wilson loop with dual open string solution at 1-loop in strong coupling.” In: *JHEP* 05 (2008), p. 064. DOI: 10.1088/1126-6708/2008/05/064. arXiv: 0803.0315 [hep-th].

[14] Charlotte Kristjansen and Yuri Makeenko. “More about One-Loop Effective Action of Open Superstring in $AdS_5 \times S^5$.” In: *JHEP* 09 (2012), p. 053. DOI: 10.1007/JHEP09(2012)053. arXiv: 1206.5660 [hep-th].

[15] E. I. Buchbinder and A. A. Tseytlin. “1/N correction in the D3-brane description of a circular Wilson loop at strong coupling.” In: *Phys. Rev. D* 89.12 (2014), p. 126008. DOI: 10.1103/PhysRevD.89.126008. arXiv: 1404.4952 [hep-th].

[16] V. Forini et al. “Remarks on the geometrical properties of semiclassically quantized strings.” In: *J. Phys. A* 48.47 (2015), p. 475401. DOI: 10.1088/1751-8113/48/47/475401. arXiv: 1507.01883 [hep-th].
[17] Alberto Faraggi et al. “Toward precision holography with supersymmetric Wilson loops.” In: *JHEP* 04 (2016), p. 053. DOI: 10.1007/JHEP04(2016)053. arXiv: 1601.04708 [hep-th].
[18] V. Forini, A. A. Tseytlin, and E. Vescovi. “Perturbative computation of string one-loop corrections to Wilson loop minimal surfaces in $AdS_5 \times S^5$.” In: *JHEP* 03 (2017), p. 003. DOI: 10.1007/JHEP03(2017)003. arXiv: 1702.02164 [hep-th].
[19] Alessandra Cagnazzo, Daniel Medina-Rincon, and Konstantin Zarembo. “String corrections to circular Wilson loop and anomalies.” In: *JHEP* 02 (2018), p. 120. DOI: 10.1007/JHEP02(2018)120. arXiv: 1712.07730 [hep-th].
[20] Daniel Medina-Rincon, Arkady A. Tseytlin, and Konstantin Zarembo. “Precision matching of circular Wilson loops and strings in $AdS_5 \times S^5$.” In: *JHEP* 05 (2018), p. 199. DOI: 10.1007/JHEP05(2018)199. arXiv: 1804.08925 [hep-th].
[21] Nikolay Bobev et al. “Supersymmetric Yang-Mills, Spherical Branes, and Precision Holography.” In: *JHEP* 03 (2020), p. 047. DOI: 10.1007/JHEP03(2020)047. arXiv: 1910.08555 [hep-th].
[22] Friðrik Freyr Gautason and Valentina Giangreco M. Puletti. “Precision holography for 5D Super Yang-Mills.” In: *JHEP* 03 (2022), p. 018. DOI: 10.1007/JHEP03(2022)018. arXiv: 2111.15493 [hep-th].
[23] Nadav Drukker, David J. Gross, and Hirosi Ooguri. “Wilson loops and minimal surfaces.” In: *Phys. Rev. D* 60 (1999), p. 125006. DOI: 10.1103/PhysRevD.60.125006. arXiv: hep-th/9904191.
[24] Luis Alvarez-Gaume, Daniel Z. Freedman, and Sunil Mukhi. “The Background Field Method and the Ultraviolet Structure of the Supersymmetric Nonlinear Sigma Model.” In: *Annals Phys.* 134 (1981), p. 85. DOI: 10.1016/0003-4916(81)90006-3.
[25] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, Mar. 2014. ISBN: 978-1-107-03473-0, 978-1-107-03473-0.
[26] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. Reading, USA: Addison-Wesley, 1995. ISBN: 978-0-201-50397-5.
[27] Steven Weinberg and Edward Witten. “Limits on Massless Particles.” In: *Phys. Lett. B* 96 (1980), pp. 59–62. DOI: 10.1016/0370-2693(80)90212-9.
[28] L. Brink, P. Di Vecchia, and Paul S. Howe. “A Locally Supersymmetric and Reparametrization Invariant Action for the Spinning String.” In: *Phys. Lett. B* 65 (1976), pp. 471–474. DOI: 10.1016/0370-2693(76)90445-7.
[29] Stanley Deser and B. Zumino. “A Complete Action for the Spinning String.” In: *Phys. Lett. B* 65 (1976). Ed. by A. Salam and E. Sezgin, pp. 369–373. DOI: 10.1016/0370-2693(76)90245-8.
[30] Soo-Jong Rey and Jung-Tay Yee. “Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity.” In: *Eur. Phys. J. C* 22 (2001), pp. 379–394. DOI: 10.1007/s100520100799. arXiv: hep-th/9803001.
[31] J. K. Erickson, G. W. Semenoff, and K. Zarembo. “Wilson loops in N=4 supersymmetric Yang-Mills theory.” In: *Nucl. Phys. B* 582 (2000), pp. 155–175. DOI: 10.1016/S0550-3213(00)00300-X. arXiv: hep-th/0003055.
[32] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Dec. 2007.

ISBN: 978-0-511-25227-3, 978-0-521-67227-6, 978-0-521-63303-1. DOI: 10.1017/CBO9780511816079.
[33] K. Becker, M. Becker, and J. H. Schwarz. *String theory and M-theory: A modern introduction*. Cambridge University Press, Dec. 2006. ISBN: 978-0-511-25486-4, 978-0-521-86069-7, 978-0-511-81608-6. DOI: 10.1017/CBO9780511816086.
[34] David Tong. *Lectures on String Theory*. 2012. arXiv: 0908.0333 [hep-th].
[35] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, Volume 2*. A Wiley Publication in Applied Statistics. Wiley, 1996. ISBN: 9780471157328. URL: https://books.google.is/books?id=ajN0aD65Ez8C.
[36] Clinton L. Lewis. "Explicit gauge covariant Euler-Lagrange equation." In: *Am. J. Phys.* 77 (2009), pp. 839–843. DOI: 10.1119/1.3153503. arXiv: 0907.2301 [physics.class-ph].

Appendices

A Equations of motion

To calculate the equations of motion, we vary the action (5.1) with respect to the embedding functions X^p on the target space. The left-hand side of Euler-Lagrange equations is

$$\frac{\partial \mathscr{L}}{\partial X^p} = \frac{\sqrt{\gamma}}{4\pi\alpha'}[\gamma^{\alpha\beta}\partial_p g_{mn} + \varepsilon^{\alpha\beta}\partial_p B_{mn}](\partial_\alpha X^m \partial_\beta X^n), \tag{A.1}$$

and the right-hand side is

$$\begin{aligned}\partial_\sigma\left(\frac{\partial \mathscr{L}}{\partial(\partial_\sigma X^p)}\right) &= \frac{1}{4\pi\alpha'}\partial_\sigma[\sqrt{\gamma}(\gamma^{\sigma\beta}g_{pn} + \varepsilon^{\sigma\beta}B_{pn})\partial_\beta X^n \\ &\quad \sqrt{\gamma}(\gamma^{\alpha\sigma}g_{mp} + \varepsilon^{\alpha\sigma}B_{mp})\partial_\alpha X^m] \\ &= \frac{1}{4\pi\alpha'}\Big[(\partial_r g_{pn})(\sqrt{\gamma}\gamma^{\sigma\beta}\partial_\beta X^n) \\ &\quad + (\partial_r g_{mp})(\sqrt{\gamma}\gamma^{\alpha\sigma}\partial_\alpha X^m) \\ &\quad + g_{pn}\partial_\sigma(\sqrt{\gamma}\gamma^{\sigma\beta}\partial_\beta X^n) \\ &\quad + g_{mp}\partial_\sigma(\sqrt{\gamma}\gamma^{\alpha\sigma}\partial_\alpha X^m) \\ &\quad + \partial_\sigma[\sqrt{\gamma}\varepsilon^{\sigma\beta}B_{pn}\partial_\beta X^n + \sqrt{\gamma}\varepsilon^{\alpha\sigma}B_{mp}\partial_\alpha X^m]\Big].\end{aligned}$$

Multiplying and dividing by $\sqrt{\gamma}$, rewriting $\Box = \frac{1}{\sqrt{\gamma}}\partial_\alpha(\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\beta)$, and combining both, we get

$$
\begin{aligned}
[\gamma^{\alpha\beta}\partial_p g_{mn} + \varepsilon^{\alpha\beta}\partial_p B_{mn}](\partial_\alpha X^m \partial_\beta X^n) = [&\partial_\sigma X^m \partial_m g_{pn}\gamma^{\sigma\beta}\partial_\beta X^n \\
&\partial_\sigma X^n \partial_n g_{mp}\gamma^{\alpha\sigma}\partial_\alpha X^m \\
&2\Box X_p + \varepsilon^{\sigma\beta}\partial_m B_{pn}\partial_\sigma X^m \partial_\beta X^n \\
&\partial^{\alpha\sigma}\partial_n B_{mp}\partial_\sigma X^n \partial_\alpha X^m]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow & 2\Box X_p + \partial_\sigma X^m \partial_m g_{pn}\gamma^{\sigma\beta}\partial_\beta X^n + \partial_\sigma X^n \partial_n g_{mp}\gamma^{\alpha\sigma}\partial_\alpha X^m - \gamma^{\alpha\beta}\partial_p g_{mn}\partial_\alpha X^m \partial_\beta X^n \\
&\quad = [-\varepsilon^{\sigma\beta}\partial_m B_{pn}\partial_\sigma X^m \partial_\beta X^n - \varepsilon^{\alpha\sigma}\partial_n B_{mp}\partial_\sigma X^n \partial_\alpha X^m + \varepsilon^{\alpha\beta}\partial_p B_{mn}\partial_\alpha X^m \partial_\beta X^n] \\
\Rightarrow & 2\Box X_p + \gamma^{\alpha\beta}[\partial_m g_{pn} + \partial_n g_{mp} - \partial_p g_{mn}]\partial_\alpha X^m \partial_\beta X^n \\
&\quad = \varepsilon^{\alpha\beta}[\partial_p B_{mn} - \partial_n B_{mp} - \partial_m B_{pn}]\partial_\alpha X^m \partial_\beta X^n \\
\Rightarrow & \Box X^l + \gamma^{\alpha\beta}\Gamma^l{}_{mn}\partial_\alpha X^m \partial_\beta X^n = \frac{\varepsilon^{\alpha\beta}}{2}H^l{}_{mn}\partial_\alpha X^m \partial_\beta X^n,
\end{aligned}
$$

with

$$
\begin{aligned}
\Gamma^l{}_{mn} &= \frac{g^{pl}}{2}[\partial_m g_{pn} + \partial_n g_{mp} - \partial_p g_{mn}], \\
H^l{}_{mn} &= g^{pl}[\partial_p B_{mn} + \partial_n B_{pm} + \partial_m B_{np}].
\end{aligned}
$$

The covariant laplacian can also be written as

$$\Box X^l = \frac{1}{\sqrt{\gamma}}\partial_\alpha(\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\beta X^l) = \gamma^{\alpha\beta}(\partial_\alpha\partial_\beta X^l - \Lambda^\sigma{}_{\alpha\beta}\partial_\sigma X^l). \tag{A.2}$$

The equations of motion become

$$\gamma^{\alpha\beta}(\partial_\alpha\partial_\beta X^l - \lambda^\sigma{}_{\alpha\beta}\partial_\sigma X^l + \Gamma^l{}_{mn}\partial_\alpha X^m \partial_\beta X^n) = \frac{\varepsilon^{\alpha\beta}}{2}H^l{}_{mn}\partial_\alpha X^m \partial_\beta X^n \tag{A.3}$$

$$\Rightarrow \gamma^{\alpha\beta}K^l{}_{\alpha\beta} = \frac{\varepsilon^{\alpha\beta}}{2}H^l{}_{mn}\partial_\alpha X^m \partial_\beta X^n \tag{A.4}$$

$$\Rightarrow K^l = \frac{\varepsilon^{\alpha\beta}}{2}H^l{}_{mn}\partial_\alpha X^m \partial_\beta X^n. \tag{A.5}$$

Here, $K^l{}_{\alpha\beta} = \partial_\alpha\partial_\beta X^l - \lambda^\sigma{}_{\alpha\beta}\partial_\sigma X^l + \Gamma^l{}_{mn}\partial_\alpha X^m \partial_\beta X^n$ is the extrinsic curvature and $K^l = \gamma^{\alpha\beta}K^l{}_{\alpha\beta}$ is the mean curvature.

B Bosonic Fluctuations

Starting equation (5.30), we have 3 types of terms to calculate:

$$I = D_\alpha \xi^A D_\beta \xi_A, \tag{B.1}$$

$$II = D_\alpha \xi^A \xi^B, \tag{B.2}$$

and

$$III = \xi^A \xi^B. \tag{B.3}$$

Also, using $D_\alpha \xi^A = \partial_\alpha \xi^A + \Omega^A{}_{Bm} \xi^B \partial_\alpha X^m$ and the decomposition of ξ^A into longitudinal (x^a) and transverse (y^i) fluctuations:

$$\xi^A = x^a t_a^A + y^i N_i^A. \tag{B.4}$$

The covariant derivatives of the vectors tangent (t_a^A) and orthogonal (N_i^A) to the worldsheet are given by

$$D_\alpha t_a^A = \partial_\alpha t_a^A + \Omega^A{}_{Bm} t_a^B \partial_\alpha X^m, \tag{B.5}$$

and

$$D_\alpha N_i^A = \partial_\alpha N_i^A + \Omega^A{}_{Bm} N_i^B \partial_\alpha X^m. \tag{B.6}$$

We can write $D_\alpha \xi^A$ as

$$\begin{aligned}
D_\alpha \xi^A &= D_\alpha (x^a t_a^A + y^i N_i^A) \\
&= \partial_\alpha (x^a t_a^A + y^i N_i^A) + \Omega^A{}_{Bm} \partial_\alpha X^m (x^a t_a^B + y^i N_i^B) \\
&= (\partial_\alpha x^a) t_a^A + x^a (\partial_\alpha t_a^A) + (\partial_\alpha y^i) N_i^A + y^i (\partial_\alpha N_i^A) \\
&\quad \Omega^A{}_{Bm} \partial_\alpha X^m x^a t_a^B + \Omega^A{}_{Bm} \partial_\alpha X^m y^i N_i^B \\
&= (\partial_\alpha x^a) t_a^A + (\partial_\alpha y^i) N_i^A + x^a (\partial_\alpha t_a^A + \Omega^A{}_{Bm} \partial_\alpha X^m t_a^B) \\
&\quad y^i (\partial_\alpha N_i^A + \Omega^A{}_{Bm} \partial_\alpha X^m N_i^B) \\
&= (\partial_\alpha x^a) t_a^A + (\partial_\alpha y^i) N_i^A + x^a (D_\alpha t_a^A) + y^i (D_\alpha N_i^A).
\end{aligned}$$

Obtaining

$$\boxed{D_\alpha \xi^A = (\partial_\alpha x^a) t_a^A + (\partial_\alpha y^i) N_i^A + x^a (D_\alpha t_a^A) + y^i (D_\alpha N_i^A)}. \tag{B.7}$$

Now, the covariant derivative acts differently on the longitudinal (x^a) and the transverse (y^i) components:

$$\mathscr{D}_\alpha x^a = \partial_\alpha x^a + \omega^a_{b\alpha} x^b, \tag{B.8}$$

and

$$\mathscr{D}_\alpha y^i = \partial_\alpha y^i + A^i{}_{j\alpha} y^j. \tag{B.9}$$

Equation (B.7) becomes

$$D_\alpha \xi^A = (\mathscr{D}_\alpha x^a - \omega^a_{\ b\alpha} x^b) t^A_a + (\mathscr{D}_\alpha y^i - A^i_{\ j\alpha} y^j) N^A_i + x^a (D_\alpha t^A_a) + y^i (D_\alpha N^A_i)$$
$$= t^A_a \mathscr{D}_\alpha x^a - \omega^a_{\ b\alpha} x^b t^A_a + N^A_i \mathscr{D}_\alpha y^i - A^i_{\ j\alpha} y^j N^A_i + x^a (D_\alpha t^A_a) + y^i (D_\alpha N^A_i).$$

Calculating type I term:

$$\begin{aligned}\gamma^{\alpha\beta} D_\alpha \xi^A D_\beta \xi_A = \gamma^{\alpha\beta} \Big[& t^A_a \mathscr{D}_\alpha x^a - \omega^a_{\ b\alpha} x^b t^A_a + N^A_i \mathscr{D}_\alpha y^i \\ & - A^i_{\ j\alpha} y^j N^A_i + x^a (D_\alpha t^A_a) + y^i (D_\alpha N^A_i) \Big] \\ & \times \Big[t_{bA} \mathscr{D}_\beta x^b - \omega^b_{\ c\beta} x^c t_{bA} + N_{jA} \mathscr{D}_\beta y^j \\ & - A^j_{\ k\beta} y^k N_{jA} + x^b (D_\beta t_{bA}) + y^j (D_\beta N_{jA}) \Big]. \end{aligned} \tag{B.10}$$

We have the following terms:

- $DxDx$:

$$= \gamma^{\alpha\beta} (t^A_a \mathscr{D}_\alpha x^a)(t_{bA} \mathscr{D}_\beta x^b)$$
$$= \gamma^{\alpha\beta} \mathscr{D}_\alpha x^a \mathscr{D}_\beta x_a.$$

- $DxDy$:

$$= \gamma^{\alpha\beta} [t^A_a \mathscr{D}_\alpha x^a N_{jA} \mathscr{D}_\beta y^i + N^A_i \mathscr{D}_\alpha y^i t_{bA} \mathscr{D} \beta x^b]$$
$$= 0.$$

- $DyDy$:

$$= \gamma^{\alpha\beta} (N^A_i \mathscr{D}_\alpha y^i)(N_{jA} \mathscr{D}_\beta y^j)$$
$$= \gamma^{\alpha\beta} \mathscr{D}_\alpha y^i \mathscr{D}_\beta y_i.$$

- $Dx.x$:

$$= \gamma^{\alpha\beta} \Big[t^A_a \mathscr{D}_\alpha x^a (-\omega^b_{\ c\beta} x^c t_{bA} + x^b (D_\beta t_{bA})) - \omega^a_{\ c\alpha} x^c t^A_a t_{bA} \mathscr{D}_\beta x^b + x^a (D_\alpha t^A_a) t_{bA} \mathscr{D}_\beta x^b \Big]$$
$$= \gamma^{\alpha\beta} \Big[-\mathscr{D}_\alpha x^a \omega_{ac\beta} x^c + (\mathscr{D}_\alpha x^a) x^b (t^A_a D_\beta t_{bA}) - \omega^a_{\ c\alpha} x^c \mathscr{D}_\beta x_a + x^a (D_\alpha t^A_a) t_{bA} \mathscr{D}_\beta x^b \Big]$$
$$= -(\mathscr{D}^\alpha x^a) \omega_{ac\alpha} x^c + (\mathscr{D}^\alpha x^a) x^b t^A_a D_\alpha t_{bA} - \omega_{ac\alpha} x^c (\mathscr{D}^\alpha x^a) + x^b (D_\alpha t_{bA}) t^A_a (\mathscr{D}^\alpha x^a)$$
$$= 2(\mathscr{D}^\alpha x^a) x^b [t^A_a D_\alpha t_{bA} - \omega_{ab\alpha}] = 0.$$

- $Dy.y$:

$$= \gamma^{\alpha\beta} [N^A_i \mathscr{D}_\alpha y^i (-A^j_{\ k\beta} y^k N_{jA} + y^j (D_\beta N_{jA})) - A^i_{\ j\alpha} y^j N^A_i N_{jA} \mathscr{D}_\beta y^j + y^i (D_\alpha N^A_i) N_{jA} \mathscr{D}_\beta y^j]$$
$$= \gamma^{\alpha\beta} [-\mathscr{D}_\alpha y_i A^i_{\ k\beta} y^k + \mathscr{D}_\alpha y^i y^j N^A_i (D_\beta N_{jA}) - A^i_{\ j\alpha} y^j \mathscr{D}_\beta y_i + y^i (D_\alpha N^A_i) N_{jA} \mathscr{D}_\beta y^j]$$
$$= -\mathscr{D}^\alpha y_i A^i_{\ j\alpha} y^j + \mathscr{D}^\alpha y_i y^j N^i_A D_\alpha N^A_j - \mathscr{D}^\alpha y_i A^i_{\ j\alpha} y^j + \mathscr{D}^\alpha y_i y^j N^i_A D_\alpha N^A_j$$
$$= 2(\mathscr{D}^\alpha y_i) y^j [-A^i_{\ j\alpha} + N^i_A D_\alpha N^A_j] = 0.$$

- xx:

$$
\begin{aligned}
&= \gamma^{\alpha\beta}[-\omega^{a}_{\ b\alpha}x^b t^A_a + x^a(D_\alpha t^A_a)][-\omega^{b}_{\ c\beta}x^c t_{bA} + x^b(D_\beta t_{bA})]\\
&= \gamma^{\alpha\beta}[\omega^{a}_{\ b\alpha}\omega_{ac\beta}x^b x^c - \omega^{a}_{\ b\alpha}x^b t^A_a x^c(D_\beta t_{cA}) - x^a(D_\alpha t^A_a)\omega^{b}_{\ c\beta}x^c t_{bA} + x^a D_\alpha t^A_a x^b D_\beta t_{bA}]\\
&= \gamma^{\alpha\beta}[-x^a(D_\alpha t^A_a)\omega^{b}_{\ c\beta}x^c t_{bA} + x^a D_\alpha t^A_a x^b D_\beta t_{bA}].
\end{aligned}
$$

- yy :

$$
\begin{aligned}
&= \gamma^{\alpha\beta}[-A^{i}_{\ j\alpha}y^j N^A_i + y^i(D_\alpha N^A_i)] \times [-A^{j}_{\ k\beta}y^k N_{jA} + y^j(D_\beta N_{jA})]\\
&= \gamma^{\alpha\beta}[A^{i}_{\ j\alpha}A_{ik\beta}y^j y^k - A^{i}_{\ j\alpha}y^j N^A_i y^k D_\beta N_{kA} - y^i D_\alpha N^A_i A^{j}_{\ k\beta}y^k N_{jA} + y^i y^j D_\alpha N^A_i D_\beta N_{jA}].
\end{aligned}
$$

- $x.y$:

$$
\begin{aligned}
&= \gamma^{\alpha\beta}[(-\omega^{a}_{\ b\alpha}x^b t^A_a + x^a D_\alpha t^A_a)(-A^{j}_{\ k\beta}y^k N_{jA} + y^j D_\beta N_{jA})\\
&\quad + (-A^{i}_{\ j\alpha}y^j N^A_i + y^i D_\alpha N^A_i)(-\omega^{b}_{\ c\beta}x^c t_{bA} + x^b D_\beta t_{bA})]\\
&= \gamma^{\alpha\beta}[-\omega^{a}_{\ b\alpha}x^b t^A_a y^j D_\beta N_{jA} - x^a D_\alpha t^A_a A^{j}_{\ k\beta}y^k N_{jA} + x^a y^j D_\alpha t^A_a D_\beta N_{jA}\\
&\quad - A^{i}_{\ j\alpha}y^j N^A_i x^b D_\beta t_{bA} - y^i D_\alpha N^A_i \omega^{b}_{\ c\beta}x^c t_{bA} + x^b y^i D_\alpha N^A_i D_\beta t_{bA}]\\
&= 2\gamma^{\alpha\beta}x^a y^i D_\alpha N^A_i D_\beta t_{aA}.
\end{aligned}
$$

- $Dx.y$:

$$
\begin{aligned}
&= \gamma^{\alpha\beta}[t^A_a \mathscr{D}_\alpha x^a(-A^{j}_{\ k\beta}y^k N_{jA} + y^j D_\beta N_{jA}) + (-A^{i}_{\ j\alpha}N^A_i + y^i D_\alpha N^A_i)t_{bA}\mathscr{D}_\beta x^b]\\
&= \gamma^{\alpha\beta}[\mathscr{D}_\alpha x^a y^i t^A_a D_\beta N_{iA} + \mathscr{D}_\beta x^a y^i t^A_a D_\alpha N_{iA}]\\
&= 2\gamma^{\alpha\beta}\mathscr{D}_\alpha x^a y^i t^A_a D_\beta N_{iA}.
\end{aligned}
$$

- $Dy.x$:

$$
\begin{aligned}
&= \gamma^{\alpha\beta}[N^A_i \mathscr{D}_\alpha y^i(-\omega^{b}_{\ c\beta}x^c t_{bA} + x^b D_\beta t_{bA}) + (-\omega^{a}_{\ b\alpha}x^b t^A_a + x^a D_\alpha t^A_a)N_{jA}\mathscr{D}_\beta y^j]\\
&= 2\gamma^{\alpha\beta}\mathscr{D}_\alpha y^i x^a N^A_i D_\beta t_{aA}\\
&= -2\gamma^{\alpha\beta}\mathscr{D}_\alpha y^i x^a t^A_a D_\beta N_{iA}.
\end{aligned}
$$

Calculating type II term:

$$
\begin{aligned}
D_\beta \xi^A \xi^B = \Big[t^A_a \mathscr{D}_\beta x^a - \omega^{a}_{\ b\beta}x^b t^A_a + N^A_i \mathscr{D}_\beta y^i - A^{i}_{\ j\beta}y^j N^A_i + x^a(D_\beta t^A_a)\\
+ y^i(D_\beta N^A_i)\Big] \times \Big[x^b t^B_b + y^j N^B_j\Big]. \quad \text{(B.11)}
\end{aligned}
$$

We have the following terms:

- $Dx.x$:

$$= t^A_a \mathscr{D}_\beta x^a x^b t^B_b.$$

- $Dx.y$:

$$= t_a^A \mathscr{D}_\beta x^a y^j N_j^B.$$

- $Dy.x$:

$$= N_i^A \mathscr{D}_\beta y^i x^b t_b^B.$$

- $Dy.y$:

$$= N_i^A \mathscr{D}_\beta y^i y^j N_j^B.$$

- $x.x$:

$$\begin{aligned} &= (-\omega^a_{\ b\beta} x^b t_a^A + x^a (D_\alpha t_a^A)) x^c t_c^B \\ &= x^a x^b t_b^B D_\beta t_a^A - \omega^a_{\ b\beta} t_a^A t_c^B x^b x^c. \end{aligned}$$

- $y.y$:

$$\begin{aligned} &= (-A^i_{\ j\beta} y^j N_i^A + y^i D_\beta N_i^A) y^j N_j^B \\ &= y^i y^j N_j^B D_\beta N_i^A - A^i_{\ j\beta} y^j y^k N_i^A N_k^B. \end{aligned}$$

- $x.y$:

$$\begin{aligned} &= (-\omega^a_{\ b\beta} x^b t_a^A + x^a (D_\alpha t_a^A)) y^j N_j^B + (-A^i_{\ j\beta} y^j N_i^A + y^i D_\beta N_i^A) x^c t_c^B \\ &= -\omega^a_{\ b\beta} x^b t_a^A y^j N_j^B + x^a (D_\alpha t_a^A) y^j N_j^B - A^i_{\ j\beta} y^j N_i^A x^c t_c^B + y^i D_\beta N_i^A x^c t_c^B. \end{aligned}$$

Calculating type III term:

$$\xi^A \xi^B = (x^a t_a^A + y^i N_i^A)(x^b t_b^B + y^j N_j^B). \tag{B.12}$$

We get the following terms:

- $x.x$:

$$= x^a x^b t_a^A t_b^B.$$

- $y.y$:

$$= y^i y^j N_i^A N_j^B.$$

- $x.y$:

$$= x^a y^j t_a^A N_j^B + y^i x^b t_b^B N_i^A.$$

Going back to equation 5.30, we can now find the following terms:

1. Term 1 can be calculated as follows: $\gamma^{\alpha\beta} D_\alpha \xi^A D_\beta \xi_A$

$$\begin{aligned}\gamma^{\alpha\beta} D_\alpha \xi^A D_\beta \xi_A = {} & \gamma^{\alpha\beta} \mathscr{D}_\alpha x^a \mathscr{D}_\beta x_a + \gamma^{\alpha\beta} \mathscr{D}_\alpha y^i \mathscr{D}_\beta y_i \\ & + \gamma^{\alpha\beta} [-x^a (D_\alpha t_a^A) \omega^b{}_{c\beta} x^c t_{bA} + x^a D_\alpha t_a^A x^b D_\beta t_{aA}] \\ & + \gamma^{\alpha\beta} [-y^i D_\alpha N_i^A A^j{}_{k\beta} y^k N_{jA} + y^i y^j D_\alpha N_i^A D_\beta N_{jA}] \\ & + 2\gamma^{\alpha\beta} x^a y^i D_\alpha N_i^A D_\beta t_{bA} + 2\gamma^{\alpha\beta} \mathscr{D}_\alpha x^a y^i t_a^A D_\beta N_{iA} \\ & - 2\gamma^{\alpha\beta} \mathscr{D}_\alpha y^i x^a t_a^A D_\beta N_{iA}. \end{aligned} \quad \text{(B.13)}$$

2. Now we calculate the first term in the equation 5.31, we get

$$\begin{aligned} R_{AMBN} t^{cM} t_c^N \xi^A \xi^B = {} & R_{AMBN} t^{aM} t_a^N (x^a x^b t_a^A t_b^B + y^i y^j N_i^A N_j^B \\ & + x^a y^j t_a^A N_j^B + y^i x^b t_b^B N_i^A) \\ = {} & R_{AMBN} t^{cM} t_c^N t_a^A t_b^B x^a x^b \\ & + R_{AMBN} t^{cM} t_c^N N_i^A N_j^B y^i y^j \\ & + R_{AMBN} t^{cM} t_c^N t_a^A N_j^B x^a y^j \\ & + R_{AMBN} t^{cM} t_c^N t_b^B N_i^A y^i x^b \\ = {} & \left(R_{AMBN} t_\alpha^A t_\rho^M t_\beta^B t_\sigma^N \right) e^{c\rho} e_c^\sigma e_a^\alpha e_b^\beta x^a x^b \\ & + R_{AMBN} t^{cM} t_c^N N_i^A N_j^B y^i y^j \\ & + (R_{AMBN} t_\alpha^A t_\rho^M t_\sigma^N N^{Bi}) e^{c\rho} e_c^\sigma e_a^\alpha x^a y_i \\ & + (R_{AMBN} t_\beta^B t_\rho^M t_\sigma^N N^{Ai}) e^{c\rho} e_c^\sigma e_b^\beta x^b y_i. \end{aligned}$$

   (a) Looking at the first term:

$$\begin{aligned} \left(R_{AMBN} t_\alpha^A t_\rho^M t_\beta^B t_\sigma^N \right) e^{c\rho} e_c^\sigma e_a^\alpha e_b^\beta x^a x^b = {} & ({}^{(2)}R_{\alpha\rho\beta\sigma} + \eta_{AB} K^A_{\rho\beta} K^B_{\sigma\alpha} \\ & - \eta_{AB} K^A_{\rho\sigma} K^B_{\beta\alpha}) e^{c\rho} e_c^\sigma e_a^\alpha e_b^\beta x^a x^b \\ = {} & ({}^{(2)}R_{\rho\alpha\sigma\beta} + \eta_{AB} K^A_{\rho\beta} K^B_{\sigma\alpha} \\ & - \eta_{AB} K^A_{\rho\sigma} K^B_{\beta\alpha}) e^{c\rho} e_c^\sigma e_a^\alpha e_b^\beta x^a x^b \\ ={}^{(2)} & R^c_{acb} x^a x^b + \eta_{AB} (K^A_{\rho\beta} K^B_{\sigma\alpha} \\ & - K^A_{\rho\sigma} K^B_{\beta\alpha}) e^{c\rho} e_c^\sigma e_a^\alpha e_b^\beta x^a x^b \\ ={}^{(2)} & R_{ab} x^a x^b + \eta_{AB} (K^A_{\rho\beta} K^B_{\sigma\alpha} \\ & - K^A_{\rho\sigma} K^B_{\beta\alpha}) e^{c\rho} e_c^\sigma e_a^\alpha e_b^\beta x^a x^b. \end{aligned}$$

   (b) The third and fourth terms become:

$$\begin{aligned} & (R_{AMBN} t_\alpha^A t_\rho^M t_\sigma^N N^{Bi}) e^{c\rho} e_c^\sigma e_a^\alpha x^a y_i + R_{AMBN} t_\beta^B t_\rho^M t_\sigma^N N^{Ai}) e^{c\rho} e_c^\sigma e_b^\beta x^b y_i \\ & = (R_{AMBN} t_\alpha^A t_\rho^M t_\sigma^N N^{Bi} + R_{BNAM} t_\alpha^B t_\rho^M t_\sigma^N N^{Ai}) e^{c\rho} e_c^\sigma e_a^\alpha x^a y_i \\ & = [\mathscr{D}_\alpha K^i_{\rho\sigma} - \mathscr{D}_\rho K^i_{\alpha\sigma} + \mathscr{D}_\alpha K^i_{\sigma\rho} - \mathscr{D}_\sigma K^i_{\alpha\rho}] e^{c\rho} e_c^\sigma e_a^\alpha x^a y_i \\ & = [\mathscr{D}_\alpha K^i_{\rho\sigma} - \mathscr{D}_\rho K^i_{\alpha\sigma} + \mathscr{D}_\alpha K^i_{\sigma\rho} - \mathscr{D}_\sigma K^i_{\alpha\rho}] \gamma^{\rho\sigma} e_a^\alpha x^a y_i \\ & = 2[\mathscr{D}_\alpha K^i_{\rho\sigma} - \mathscr{D}_\rho K^i_{\alpha\rho}] \gamma^{\rho\sigma} e_a^\alpha x^a y_i. \end{aligned}$$

Combining our recent results, we get

$$\gamma^{\alpha\beta}D_\alpha\xi^A D_\beta\xi_A - R_{AMBN}t^{aM}t_a^N\xi^A\xi^B = \underline{\gamma^{\alpha\beta}\mathscr{D}_\alpha x^a\mathscr{D}_\beta x_a} + \underline{\gamma^{\alpha\beta}\mathscr{D}_\alpha y^i\mathscr{D}_\beta y_i}$$
$$+\gamma^{\alpha\beta}[-x^a(D_\alpha t_a^A)\omega^b_{c\beta}x^c t_{bA} + x^a D_\alpha t_a^A x^b D_\beta t_{bA}]$$
$$+\gamma^{\alpha\beta}[-y^i D_\alpha N_i^A A^j{}_{k\beta}y^k N_{jA} + y^i y^j D_\alpha N_i^A D_\beta N_{jA}]$$
$$+2\gamma^{\alpha\beta}x^a y^i D_\alpha N_i^A D_\beta t_{aA} + \underline{2\gamma^{\alpha\beta}\mathscr{D}_\alpha x^a y^i t_a^A D_\beta N_{iA}}$$
$$-\underline{2\gamma^{\alpha\beta}\mathscr{D}_\alpha y^i x^a t_a^A D_\beta N_{iA}} - \underline{{}^{(2)}R_{ab}x^a x^b}$$
$$-\eta_{AB}(K^A_{\rho\beta}K^B_{\sigma\alpha} - K^A_{\rho\sigma}K^B_{\beta\alpha})e^{c\rho}e_c^\sigma e_a^\alpha e_b^\beta x^a x^b$$
$$-2[\mathscr{D}_\alpha K^i_{\rho\sigma} - \mathscr{D}_\rho K^i_{\alpha\rho}]\gamma^{\rho\sigma}e_a^\alpha x^a y_i$$
$$-R_{AMBN}t^{cM}t_c^N N_i^A N_j^B y^i y^j. \quad \text{(B.14)}$$

Combining like terms, we find

$$\gamma^{\alpha\beta}D_\alpha\xi^A D_\beta\xi_A - R_{AMBN}t^{aM}t_a^N\xi^A\xi^B = \gamma^{\alpha\beta}\mathscr{D}_\alpha x^a\mathscr{D}_\beta x_a + \gamma^{\alpha\beta}\mathscr{D}_\alpha y^i\mathscr{D}_\beta y_i - {}^{(2)}R_{ab}x^a x^b$$
$$+2\gamma^{\alpha\beta}[\mathscr{D}_\alpha x^a y^i t_a^A D_\beta N_{iA} - \mathscr{D}_\alpha y^i x^a t_a^A D_\beta N_{iA}]$$
$$+2\gamma^{\alpha\beta}\nabla_\alpha K_{i,a\beta}x^a y^i - (R_{AMBN}t^{cM}t_c^N N_i^A N_j^B$$
$$-\gamma^{\alpha\beta}\gamma^{\rho\sigma}K_{i,\alpha\rho}K_{j,\beta\sigma})y^i y^j.$$

This can also be written as

$$\gamma^{\alpha\beta}D_\alpha\xi^A D_\beta\xi_A - R_{AMBN}t^{aM}t_a^N\xi^A\xi^B = \gamma^{\alpha\beta}\mathscr{D}_\alpha x^a\mathscr{D}_\beta x_a + \gamma^{\alpha\beta}\mathscr{D}_\alpha y^i\mathscr{D}_\beta y_i - {}^{(2)}R_{ab}x^a x^b$$
$$-2\gamma^{\alpha\beta}K_{i,a\alpha}[\mathscr{D}_\beta x^a y^i - \mathscr{D}_\beta y^i x^a]$$
$$+2\gamma^{\alpha\beta}\nabla_\alpha K_{i,a\beta}x^a y^i - (R_{AMBN}t^{cM}t_c^N N_i^A N_j^B$$
$$-\gamma^{\alpha\beta}\gamma^{\rho\sigma}K_{i,\alpha\rho}K_{j,\beta\sigma})y^i y^j,$$

or

$$\gamma^{\alpha\beta}D_\alpha\xi^A D_\beta\xi_A - R_{AMBN}t^{aM}t_a^N\xi^A\xi^B = \gamma^{\alpha\beta}\mathscr{D}_\alpha x^\sigma\mathscr{D}_\beta x_\sigma + \gamma^{\alpha\beta}\mathscr{D}_\alpha y^i\mathscr{D}_\beta y_i$$
$$-{}^{(2)}R_{\sigma\lambda}x^\sigma x^\lambda - 2\gamma^{\alpha\beta}K_{i,\lambda\alpha}[\mathscr{D}_\beta x^\lambda y^i - \mathscr{D}_\beta y^i x^\lambda] + 2\gamma^{\alpha\beta}\nabla_\alpha K_{i,\lambda\beta}x^\lambda y^i$$
$$-(R_{AMBN}t^{cM}t_c^N N_i^A N_j^B - \gamma^{\alpha\beta}\gamma^{\rho\sigma}K_{i,\alpha\rho}K_{j,\beta\sigma})y^i y^j. \quad \text{(B.15)}$$

Here,

$$x^\sigma = x^a e_a^\sigma$$

Now, we calculate the remaining terms of equation 5.30, that is, we now find

$$\varepsilon^{\alpha\beta}\left[H_{mAB}\partial_\alpha X^m D_\beta\xi^A\xi^B + \frac{1}{2}\nabla_A H_{mnB}\partial_\alpha X^m\partial_\beta X^n\xi^A\xi^B\right]. \quad \text{(B.16)}$$

The first term gives:

$$
\begin{aligned}
H_{mAB}\partial_\alpha X^m D_\beta \xi^A \xi^B = H_{mAB}\partial_\alpha X^m [t_a^A \mathscr{D}_\beta x^a x^b t_b^B + t_a^A \mathscr{D}_\beta x^a y^j N_j^B \\
N_i^A \mathscr{D}_\beta y^i y^j N_j^B + N_i^A \mathscr{D}_\beta y^i x^b t_b^B \\
x^a x^b t_b^B D_\beta t_a^A - \omega^a_{\ b\beta} t_a^A t_c^B x^b x^c \\
y^i y^j N_j^B D_\beta N_i^A - A^i_{\ j\beta} y^j y^k N_i^A N_k^B \\
-\omega^a_{\ b\beta} x^b t_a^A y^j N_j^B + x^a (D_\beta t_a^A) y^j N_j^B \\
-A^i_{\ j\beta} y^j N_i^A x^c t_c^B + y^i D_\beta N_i^A x^c t_c^B]. \quad \text{(B.17)}
\end{aligned}
$$

This gives

$$
\begin{aligned}
H_{mab}\partial_\alpha X^m \mathscr{D}_\beta x^a x^b + H_{mij}\partial_\alpha X^m \mathscr{D}_\beta y^i y^j + H_{maj}\partial_\alpha X^m (\mathscr{D}_\beta x^a y^j - \mathscr{D}_\beta y^j x^a) \\
+ H_{mAB}\partial_\alpha X^m x^a x^b (t_b^B D_\beta t_a^A - \omega^c_{\ b\beta} t_c^A t_a^B) \\
+ H_{mAB}\partial_\alpha X^m y^i y^j (N_j^B D_\beta N_i^A - A^k_{\ j\beta} N_k^A N_i^B) \\
+ H_{mAB}\partial_\alpha X^m x^a y^i (-\omega^b_{\ a\beta} t_b^A N_i^B + (D_\beta t_a^A) N_i^B \\
- A^j_{\ i\beta} N_j^A t_a^B + D_\beta N_i^A t_a^B), \quad \text{(B.18)}
\end{aligned}
$$

or

$$
\begin{aligned}
H_{mpq} t_\alpha^m t_\lambda^p t_\sigma^q \mathscr{D}_\alpha x^\lambda x^\sigma + H_{mnr} t_\alpha^m E_A^n E_B^r N_i^A N_j^B \mathscr{D}_\beta y^i y^j \\
+ H_{mpr} t_\alpha^m \partial_\lambda^p E_A^r N_i^A [\mathscr{D}_\beta x^\lambda y^i - \mathscr{D}_\beta y^i x^\lambda].
\end{aligned}
$$

The second term gives:

$$
\begin{aligned}
\frac{1}{2}\nabla_A H_{mnB}\partial_\alpha X^m \partial_\beta X^n \xi^A \xi^B = \frac{1}{2}\nabla_A H_{mnB}\partial_\alpha X^m \partial_\beta X^n [x^a x^b t_a^A t_b^B + y^i y^j N_i^A N_j^B \\
+ x^a y^j t_a^A N_j^B + y^i x^b t_b^B N_i^A]. \quad \text{(B.19)}
\end{aligned}
$$

This gives

$$
\begin{aligned}
\frac{1}{2}\nabla_r H_{mns} E_A^r E_B^s \partial_\alpha X^m \partial_\beta X^n x^a x^b t_a^A t_b^B \\
+ \frac{1}{2}\nabla_r H_{mns} E_A^r E_B^s N_i^A N_j^B \partial_\alpha X^m \partial_\beta X^n y^i y^j \\
+ \nabla_r H_{mns} E_A^r E_B^s \partial_\alpha X^m \partial_\beta X^n x^b y^i t_b^B N_i^A,
\end{aligned}
$$

or

$$
\nabla_r H_{mns} t_\alpha^m t_\beta^n \left(\frac{1}{2} t_\lambda^r t_\sigma^s x^\lambda x^\sigma + \frac{1}{2} E_A^r N_i^A E_B^s N_j^B y^i y^j + t_\lambda^s N_i^A E_A^r y^i x^\lambda \right). \quad \text{(B.20)}
$$

Combining the results from the previous two equations, we get

$$
\begin{aligned}
\varepsilon^{\alpha\beta} \Bigg[\nabla_r H_{mns} t_\alpha^m t_\beta^n \left(\frac{1}{2} t_\lambda^r t_\sigma^s x^\lambda x^\sigma + \frac{1}{2} E_A^r N_i^A E_B^s N_j^B y^i y^j + t_\lambda^s N_i^A E_A^r y^i x^\lambda \right) \\
H_{mpq} t_\alpha^m t_\lambda^p t_\sigma^q \mathscr{D}_\alpha x^\lambda x^\sigma + H_{mnr} t_\alpha^m E_A^n E_B^r N_i^A N_j^B \mathscr{D}_\beta y^i y^j \\
+ H_{mpr} t_\alpha^m t_\lambda^p E_A^r N_i^A [\mathscr{D}_\beta x^\lambda y^i - \mathscr{D}_\beta y^i x^\lambda] \Bigg]. \quad \text{(B.21)}
\end{aligned}
$$

Now, we combine our results. We have:

- $x.x$:

$$\left(-R_{\lambda\sigma}+\frac{\varepsilon^{\alpha\beta}}{2}\nabla_r H_{mns}t^m_\alpha t^n_\beta t^r_\lambda t^s_\sigma\right)x^\lambda x^\sigma=-\left(R_{\lambda\sigma}-\frac{\varepsilon^{\alpha\beta}}{2}\nabla_\lambda H_{\alpha\beta\sigma}\right)x^\lambda x^\sigma=-R_{\lambda\sigma}x^\lambda x^\sigma.$$

Here,

$$\nabla_\lambda H_{\alpha\beta\sigma}=\nabla_r H_{mns}t^m_\alpha t^n_\beta t^r_\lambda t^s_\sigma.$$

- $y.y$:

$$\left(-m_{ij}+\frac{\varepsilon^{\alpha\beta}}{2}\nabla_i H_{mnj}t^m_\alpha t^n_\beta\right)y^i y^j=-\left(m_{ij}-\frac{\varepsilon^{\alpha\beta}}{2}\nabla_i H_{\alpha\beta j}\right)y^i y^j.$$

Here,

$$\nabla_i H_{\alpha\beta j}=\nabla_r H_{mns}E^r_A N^A_i E^s_B N^B_j t^m_\alpha t^n_\beta,$$

and

$$m_{ij}=R_{AMBN}t^{cM}t^N_c N^A_i N^B_j-\gamma^{\alpha\beta}\gamma^{\rho\sigma}K_{i,\alpha\rho}K_{j,\beta\sigma}.$$

- $x.y$:

$$\left(2\gamma^{\alpha\beta}\nabla_\alpha K_{i,\lambda\beta}+\varepsilon^{\alpha\beta}\nabla_i H_{mns}t^m_\alpha t^n_\beta t^s_\lambda\right)x^\lambda y^i$$
$$=2\left(\gamma^{\alpha\beta}\nabla_\alpha K_{i,\lambda\beta}+\frac{\varepsilon^{\alpha\beta}}{2}\nabla_i H_{\alpha\beta\lambda}\right)x^\lambda y^i=2\gamma^{\alpha\beta}\nabla_\alpha K_{i,\lambda\beta}x^\lambda y^i.$$

Here,

$$\nabla_i H_{\alpha\beta\lambda}=\nabla_r H_{mns}E^r_A N^A_i t^m_\alpha t^n_\beta t^s_\lambda.$$

- $Dx.x$:

$$\varepsilon^{\alpha\beta}H_{mns}t^m_\alpha t^n_\lambda t^s_\sigma \mathscr{D}_\beta x^\lambda x^\sigma=\varepsilon^{\alpha\beta}H_{\alpha\lambda\sigma}\mathscr{D}_\beta x^\lambda x^\sigma.$$

Here,

$$H_{\alpha\lambda\sigma}=H_{mns}t^m_\alpha t^n_\lambda t^s_\sigma.$$

- $Dy.y$:

$$\varepsilon^{\alpha\beta}H_{mij}t^m_\alpha \mathscr{D}_\beta y^i y^j=\varepsilon^{\alpha\beta}H_{\alpha ij}\mathscr{D}_\beta y^i y^j$$

Here,

$$H_{\alpha ij}=H_{mns}t^m_\alpha E^n_A N^A_i E^s_B N^B_j.$$

- $Dx.y$:

$$-2\left[\gamma^{\alpha\beta}K_{i,\lambda\alpha}-\frac{\varepsilon^{\alpha\beta}}{2}H_{mni}t^m_\alpha t^n_\lambda\right](\mathscr{D}_\beta x^\lambda y^i-\mathscr{D}_\beta y^i x^\lambda)$$
$$=-2\left[\gamma^{\alpha\beta}K_{i,\lambda\alpha}-\frac{\varepsilon^{\alpha\beta}}{2}H_{\alpha\lambda i}\right](\mathscr{D}_\beta x^\lambda y^i-\mathscr{D}_\beta y^i x^\lambda).$$

Here,

$$K_{i,\lambda\alpha}=K_{m,\alpha\lambda}E^m_A N^A_i.$$

and

$$H_{\alpha\lambda i}=H_{mns}t^m_\alpha t^n_\lambda E^s_A N^A_i$$

- $Dx.Dx + Dy.Dy$:

$$\gamma^{\alpha\beta}(\mathscr{D}_\alpha x^\sigma \mathscr{D}_\beta x_\sigma + \mathscr{D}_\alpha y^i \mathscr{D}_\beta y_i).$$

The quadratic Lagrangian looks like

$$\begin{aligned}\sqrt{\gamma}\Bigg[\gamma^{\alpha\beta}\mathscr{D}_\alpha x^\sigma \mathscr{D}_\beta x_\sigma + \gamma^{\alpha\beta}\mathscr{D}_\alpha y^i \mathscr{D}_\beta y_i - \left(R_{\lambda\sigma} - \frac{\varepsilon^{\alpha\beta}}{2}\nabla_\lambda H_{\alpha\beta\sigma}\right)x^\lambda x^\sigma \\ - \left(R_{AMBN}t^{cM}t_c^N N_i^A N_j^B - \gamma^{\alpha\beta}\gamma^{\rho\sigma}K_{i,\alpha\rho}K_{j,\beta\sigma} - \frac{\varepsilon^{\alpha\beta}}{2}\nabla_i H_{\alpha\beta j}\right)y^i y^j \\ + 2\left(\gamma^{\alpha\beta}\nabla_\alpha K_{i,\lambda\beta} + \frac{\varepsilon^{\alpha\beta}}{2}\nabla_i H_{\alpha\beta\lambda}\right)x^\lambda y^i \\ + \varepsilon^{\alpha\beta}H_{\alpha\lambda\sigma}\mathscr{D}_\beta x^\lambda x^\sigma + \varepsilon^{\alpha\beta}H_{\alpha ij}\mathscr{D}_\beta y^i y^j \\ - 2\left[\gamma^{\alpha\beta}K_{i,\lambda\alpha} - \frac{\varepsilon^{\alpha\beta}}{2}H_{\alpha\lambda i}\right](\mathscr{D}_\beta x^\lambda y^i - \mathscr{D}_\beta y^i x^\lambda)\Bigg]. \quad \text{(B.22)}\end{aligned}$$

After making the modification 5.50, the terms involving the modified curvature become the following:

- $y.y$:

$$\begin{aligned}-\gamma^{\alpha\beta}\gamma^{\rho\sigma}K_{i,\alpha\rho}K_{j,\beta\sigma}y^i y^j &= -\gamma^{\alpha\beta}\gamma^{\rho\sigma}\left[\mathscr{K}_{i,\alpha\rho} + \frac{\varepsilon^{\varepsilon}_{\ \alpha}}{2}H_{imn}t_\varepsilon^m t_\rho^n\right]\times \\ &\qquad \left[\mathscr{K}_{j,\beta\sigma} + \frac{\varepsilon^{\lambda}_{\ \beta}}{2}H_{jpq}t_\lambda^p t_\sigma^q\right]y^i y^j \\ &= -\gamma^{\alpha\beta}\gamma^{\rho\sigma}\Bigg[\mathscr{K}_{i,\alpha\rho}\mathscr{K}_{j,\beta\sigma} + \frac{\varepsilon^{\lambda}_{\ \beta}}{2}\mathscr{K}_{i,\alpha\rho}H_{jpq}t_\lambda^p t_\sigma^q + \frac{\varepsilon^{\varepsilon}_{\ \alpha}}{2}\mathscr{K}_{j,\beta\sigma}H_{imn}t_\varepsilon^m t_\rho^n \\ &\qquad + \frac{\varepsilon^{\varepsilon}_{\ \alpha}}{2}\frac{\varepsilon^{\lambda}_{\ \beta}}{2}H_{imn}H_{jpq}t_\varepsilon^m t_\rho^n t_\lambda^p t_\sigma^q\Bigg]y^i y^j \\ &= -\gamma^{\alpha\beta}\gamma^{\rho\sigma}\mathscr{K}_{i,\alpha\rho}\mathscr{K}_{j,\beta\sigma}y^i y^j - \gamma^{\rho\sigma}\varepsilon^{\lambda\alpha}\mathscr{K}_{i,\alpha\rho}H_{jmn}t_\lambda^m t_\sigma^n y^i y^j \\ &\qquad + \frac{1}{4}\gamma^{\rho\sigma}\gamma^{\varepsilon\lambda}H_{imn}H_{jpq}t_\varepsilon^m t_\rho^n t_\lambda^p t_\sigma^q y^i y^j.\end{aligned}$$

- $Dx.y$:

$$\left[\gamma^{\alpha\beta}K_{i,\lambda\alpha} - \frac{\varepsilon^{\alpha\beta}}{2}H_{\alpha\lambda i}\right] = \gamma^{\alpha\beta}\mathscr{K}_{i,\alpha\lambda}.$$

- $x.y$:

$$\left(\gamma^{\alpha\beta}\nabla_\alpha K_{i,\lambda\beta}+\frac{\varepsilon^{\alpha\beta}}{2}\nabla_i H_{\alpha\beta\lambda}\right)x^\lambda y^i=\gamma^{\alpha\beta}\nabla_\alpha\mathscr{K}_{i,\beta\lambda}x^\lambda y^i$$
$$+\gamma^{\alpha\beta}\frac{\varepsilon^\sigma_{\ \beta}}{2}\nabla_\alpha H_{imn}t^m_\sigma t^n_\lambda x^\lambda y^i+\gamma^{\alpha\beta}\frac{\varepsilon^\sigma_{\ \beta}}{2}H_{imn}\nabla_\alpha t^m_\sigma t^n_\lambda x^\lambda y^i$$
$$+\gamma^{\alpha\beta}\frac{\varepsilon^\sigma_{\ \beta}}{2}H_{imn}t^m_\sigma\nabla_\alpha t^n_\lambda x^\lambda y^i$$
$$=\gamma^{\alpha\beta}\nabla_\alpha\mathscr{K}_{i,\beta\lambda}x^\lambda y^i-\frac{\varepsilon^{\alpha\sigma}}{2}\nabla_i H_{\alpha\sigma\lambda}x^\lambda y^i$$
$$+\frac{\varepsilon^{\sigma\alpha}}{2}H_{imn}\nabla_\alpha t^m_\sigma t^n_\lambda x^\lambda y^i+\frac{\varepsilon^{\sigma\alpha}}{2}H_{imn}t^m_\sigma\nabla_\alpha t^n_\lambda x^\lambda y^i$$
$$+\frac{\varepsilon^{\alpha\beta}}{2}\nabla_i H_{\alpha\beta\lambda}x^\lambda y^i$$
$$=\gamma^{\alpha\beta}\nabla_\alpha\mathscr{K}_{i,\beta\lambda}x^\lambda y^i+\frac{\varepsilon^{\sigma\alpha}}{2}H_{imn}K^m_{\alpha\sigma}t^n_\lambda x^\lambda y^i$$
$$+\frac{\varepsilon^{\sigma\alpha}}{2}H_{imn}t^m_\sigma K^n_{\alpha\lambda}x^\lambda y^i$$
$$=\left(\gamma^{\alpha\beta}\nabla_\alpha\mathscr{K}_{i,\beta\lambda}+\frac{\varepsilon^{\alpha\beta}}{2}H_{imn}t^m_\alpha K^n_{\ \beta\lambda}\right)x^\lambda y^i$$
$$=\left(\gamma^{\alpha\beta}\nabla_\alpha\mathscr{K}_{i,\beta\lambda}+\frac{\varepsilon^{\alpha\beta}}{2}H_{imn}t^m_\alpha\mathscr{K}^n_{\ \beta\lambda}+\frac{\varepsilon^{\alpha\beta}}{2}H_{imn}t^m_\alpha\frac{\varepsilon^\sigma_{\ \beta}}{2}H^n_{\sigma\lambda}\right)x^\lambda y^i$$
$$=\left(\gamma^{\alpha\beta}\nabla_\alpha\mathscr{K}_{i,\beta\lambda}+\frac{\varepsilon^{\alpha\beta}}{2}H_{imn}t^m_\alpha\mathscr{K}^n_{\ \beta\lambda}-\frac{\gamma^{\alpha\sigma}}{4}H_{imn}H^n_{pq}t^m_\alpha t^p_\sigma t^q_\lambda\right)x^\lambda y^i.$$

where $\frac{\varepsilon^{\sigma\alpha}}{2}H_{imn}K^m_{\alpha\sigma}t^n_\lambda=0$ for symmetry reasons. Plugging these into the 5.49, we get

$$\sqrt{\gamma}\Big[\gamma^{\alpha\beta}\mathscr{D}_\alpha x^\sigma\mathscr{D}_\beta x_\sigma+\gamma^{\alpha\beta}\mathscr{D}_\alpha y^i\mathscr{D}_\beta y_i-(R_{\lambda\sigma})x^\lambda x^\sigma$$
$$-\Big(R_{AMBN}t^{cM}t^N_c N^A_i N^B_j-\gamma^{\alpha\beta}\gamma^{\rho\sigma}\mathscr{K}_{i,\alpha\rho}\mathscr{K}_{j,\beta\sigma}$$
$$+\frac{1}{4}\gamma^{\rho\sigma}\gamma^{\varepsilon\lambda}H_{i\varepsilon\rho}H_{j\lambda\sigma}-\frac{\varepsilon^{\alpha\beta}}{2}\nabla_i H_{\alpha\beta j}\Big)y^i y^j$$
$$+2\left(\gamma^{\alpha\beta}\nabla_\alpha\mathscr{K}_{i,\beta\lambda}+\frac{\varepsilon^{\alpha\beta}}{2}H_{imn}t^m_\alpha\mathscr{K}^n_{\ \beta\lambda}-\frac{\gamma^{\alpha\sigma}}{4}H_{imn}H^n_{\ pq}t^m_\alpha t^p_\sigma t^q_\lambda\right)x^\lambda y^i$$
$$+\varepsilon^{\alpha\beta}H_{\alpha ij}\mathscr{D}_\beta y^i y^j-2\gamma^{\alpha\beta}\mathscr{K}_{i,\alpha\lambda}(\mathscr{D}_\beta x^\lambda y^i-\mathscr{D}_\beta y^i x^\lambda)\Big].\quad\text{(B.23)}$$

We propose that the covariant derivative on the tangential and transverse fluctuations will also get modified in the presence of the antisymmetric tensor. Let us assume it gets modified to

$$\mathbb{D}_\alpha y^i=\mathscr{D}_\alpha y^i+M^i_{\ j\alpha}y^j.\quad\text{(B.24)}$$

We want to find out what this $M^i{}_{j\alpha}$ is. So, let's find out. We are hoping to get a term that looks like $\mathbb{D}_\alpha y^i \mathbb{D}_\beta y_i$, so, let's just start with that

$$\begin{aligned}\gamma^{\alpha\beta}\mathbb{D}_\alpha y^i \mathbb{D}_\beta y_i &= \gamma^{\alpha\beta}(\mathscr{D}_\alpha y^i + M^i{}_{k\alpha}y^k)(\mathscr{D}_\beta y_i - M^j{}_{i\beta}y_j) \\ &= \gamma^{\alpha\beta}\left(\mathscr{D}_\alpha y^i \mathscr{D}_\beta y_i + 2\mathscr{D}_\beta y^i M_{ij\alpha}y^j - M^i{}_{k\alpha}M^j{}_{i\beta}y^k y_j\right).\end{aligned}$$

Comparing with B.23, we find

$$\begin{aligned}2\gamma^{\alpha\beta}\mathscr{D}_\beta y^i y^j M_{ij\alpha} &= \varepsilon^{\alpha\beta}H_{\alpha ij}\mathscr{D}_\beta y^i y^j \\ \Rightarrow M_{ij\alpha} &= \frac{\varepsilon^\beta_\alpha}{2}H_{\beta ij}.\end{aligned}$$

Thus, we find that we can replace $\gamma^{\alpha\beta}\mathscr{D}_\alpha y^i \mathscr{D}_\beta y_i + \varepsilon^{\alpha\beta}H_{\alpha ij}\mathscr{D}_\beta y^i y^j$ by

$$\gamma^{\alpha\beta}\mathbb{D}_\alpha y^i \mathbb{D}_\beta y_i + \frac{\gamma^{\alpha\beta}}{4}H_{\alpha i}^{\ \ k}H_{\beta jk}y^i y^j$$

Plugging into B.23, we get

$$\begin{aligned}\sqrt{\gamma}\Bigg[&\gamma^{\alpha\beta}\mathscr{D}_\alpha x^\sigma \mathscr{D}_\beta x_\sigma + \gamma^{\alpha\beta}\mathbb{D}_\alpha y^i \mathbb{D}_\beta y_i - (R_{\lambda\sigma})x^\lambda x^\sigma \\ &- \Big(R_{AMBN}t^{cM}t_c^N N_i^A N_j^B - \gamma^{\alpha\beta}\gamma^{\rho\sigma}\mathscr{K}_{i,\alpha\rho}\mathscr{K}_{j,\beta\sigma} \\ &+ \frac{1}{4}\gamma^{\rho\sigma}\gamma^{\varepsilon\lambda}H_{i\varepsilon\rho}H_{j\lambda\sigma} + \frac{\varepsilon^{\alpha\beta}}{2}(\nabla_A H_{mnB})N_i^A N_j^B t_\alpha^m t_\beta^n + \frac{1}{4}\gamma^{\alpha\beta}H_{\alpha i}^{\ \ k}H_{\beta jk}\Big)y^i y^j \\ &+ 2\left(\gamma^{\alpha\beta}\nabla_\alpha \mathscr{K}_{i,\beta\lambda}\right)x^\lambda y^i - 2\gamma^{\alpha\beta}\mathscr{K}_{i,\alpha\lambda}(\mathscr{D}_\beta x^\lambda y^i - \mathscr{D}_\beta y^i x^\lambda)\Bigg]. \quad \text{(B.25)}\end{aligned}$$