DISPERSION OF BETATRON OSCILLATION FREQUENCIES IN LARGE CYCLIC ACCELERATORS AND STORAGE RINGS

V.P.Belov, A.A.Makarov, I.A.Shukeilo
D.V.Efremov Scientific Research Institute of
Electrophysical Apparatus, Leningrad, USSR

Summary

The report presented is a further development of the works performed by G.Parzen, P.Morton et al. [1, 2] and dealing with the effect of magnetic field errors on the motion of a beam with finite emittance and momentum spread in large accelerators and storage rings. The starting point is a Hamiltonian of transverse motion, in which complete field distortion is included. Expanding such a Hamiltonian in multipoles-which is adequate to the magnetic measurement technique-leads to the equations of coupled motion, from which the dispersion of betatron oscillation frequences can be found. Considered are all the multipoles, all the powers of betatron oscillation amplitudes in two transverse directions, and all the powers of the closed orbit deviation from the ring center-line. The dispersion of betatron oscillation frequences is presented in a compact form due to the use of special functions such as the Legendre polynomials. The calculation procedure developed is valid for large machines, i.e. when one can neglect the ratio of transverse deviation to orbit radius.

In large cyclic machines especially in storage rings a particle beam circulates for a long time (up to several days). In this case, the effect of the high order resonances on the transverse particle motion becomes important. There is an experience of correction of the magnet field distortions responsible for some nonlinear resonances. It is difficult, however, to correct all resonances being dangerous. Therefore, rigid limitations are imposed on betatron frequency dispersion with the aim to avoid high order resonance crossing.

Betatron oscillation frequency dispersion far from resonance stopbands depends on magnet field nonlinearities, beam emittance, closed orbit distortions and momentum spread. Near the resonance stopbands the dispersion depends also on the amplitudes and phases of the betatron oscillations.

Assume, that the curvature radius R of the charged particle trajectory in the magnetic field is much larger of the particle deviation from the equilibrium orbit. Then, only two magnet field components may be taken into account in the betatron oscillation equations: B_z is for radial motion and B_x is for axial motion. The solution of these equations is fairly simplified and it can be shown, that the transverse coupled motion close to an arbitrary order resonance $sm\nu_x + sn\nu_z = sq + \delta$ (m, n, q - integers, s - positive integers, m, n have no common integers, $\delta \ll 1$) is described by the Hamiltonian

$$\mathcal{H} = -\frac{\delta}{2s}\left(\frac{I_x}{m} + \frac{I_z}{n}\right) - \frac{1-\varkappa}{8\pi W K_0^2 B_{z_0}} \int_0^{2\pi} d\vartheta' K(\vartheta') \left\{ \frac{1}{\pi^2} \int_0^{\pi} d\sigma \cdot \int_0^{\pi} d\gamma \int dX \left(B_z^+(X,Z) + B_z^-(X,Z) \right) \right\} - \frac{1-\varkappa}{8\pi W K_0^2 B_{z_0}} \int_0^{2\pi} K(\vartheta') d\vartheta' \cdot \left\{ \frac{1}{\pi^2} \int_0^{\pi} d\sigma \cos ms\sigma \int_0^{\pi} d\gamma \cos ns\gamma \int dX \left(B_z(X,Z) + (-1)^{ns} B_z(X,Z) \right) \right\} \cos\Psi(\Phi_{x,z};\vartheta)$$

and the canonical equations

$$\frac{dI_{x,z}}{d\vartheta} = \frac{\partial \mathcal{H}}{\partial \Phi_{x,z}}, \quad \frac{d\Phi_{x,z}}{d\vartheta} = -\frac{\partial \mathcal{H}}{\partial I_{x,z}}, \qquad (2)$$

where slowly-changeable variables are:

$$I_{x,z} = |a_{x,z}|^2, \quad \Phi_x = \alpha_x + \frac{\delta}{2sm}\vartheta, \quad \Phi_z = \alpha_z + \frac{\delta}{2sn}\vartheta$$

$K_0 = 1/R_0$ - mean, $K = 1/R = -\frac{eB_z}{P_0 c}$ - local curvature of the central line $\varkappa = \frac{\Delta P/P_0}{1+\Delta P/P_0}$,

ΔP - momentum deviation from the equilibrium value P_0, $X = x + \xi$, $x = \psi\frac{\Delta P}{P_0} + \Delta x$, $Z = z + \zeta$, ψ - dispersion function, Δx, z - horisontal and vertical distortion of the closed orbit due to field errors in the magnet units and position errors. $\xi(\vartheta)$, $\zeta(\vartheta)$ describe betatron oscillations with respect to the distorted closed orbit $x(\vartheta)$, $z(\vartheta)$; $|a_{x,z}|$, $\alpha_{x,z}$ - the amplitude and phase of the betatron oscillations, respectively;

$$\Psi(\Phi_{x,z};\vartheta') = sq\vartheta' + s\left[m\chi_x(\vartheta') + n\chi_z(\vartheta')\right] + s\left[m\Phi_x(\vartheta) + n\Phi_z(\vartheta)\right].$$

The magnetic field B_z in (1) is taken at the point of the particle position:

$$B_z^{\pm}(X,Z) = B\left(x(\vartheta') + 2\sqrt{I_x(\vartheta)}\,|f_x(\vartheta')|\cos\sigma;\ \pm z(\vartheta') + 2\sqrt{I_z(\vartheta)}\,|f_z(\vartheta')|\cos\gamma\right) \qquad (3)$$

$|f_{x,z}(\vartheta)|$, $\chi_{x,z}(\vartheta)$ - module and phase of the Floquet function, respectively,

$\varphi_{x,z}(\vartheta) = |f_{x,z}(\vartheta)| e^{i\chi_{x,z}(\vartheta)}$, $\varphi\varphi'^* - \varphi'\varphi^* = -2iW$ is normalization condition

W - Wronskian.

The derivatives of the Hamiltonian second term with respect to the variables $I_{x,z}$ are responsible for nonresonance shift of betatron oscillation frequencies and those of the third term are responsible for resonance shift of betatron oscillation frequencies. The latter amplitude determines the stopband width.

The Hamiltonian (1) and canonical equations (2) describe two-fold nonlinear oscillations of the beam particles with finite emittance and momentum spread close to the resonance $sm\nu_x + sn\nu_z = sq+\delta$ in the total accelerator field including random and regular distortions of the closed orbit. The dispersion of the betatron oscillation frequencies is considered in detail below. According to (1), (2) in a nonresonance case it is presented by the formulae:

$$\Delta\nu_x(\varepsilon_{x,z};x,z)=\frac{1-æ}{4\pi B_{z_0}K_0^2\sqrt{\varepsilon_x}}\int_0^{2\pi}d\vartheta' K(\vartheta')\beta_x^{1/2}(\vartheta') \cdot\left\{\frac{1}{\pi^2}\int_0^{\pi}d\sigma\cos\sigma\int_0^{\pi}d\gamma\left(B_z^+(X,Z)+B_z^-(X,Z)\right)\right\} \quad (4)$$

$$\Delta\nu_z(\varepsilon_{x,z};x,z)=-\frac{1-æ}{4\pi B_{z_0}K_0^2\sqrt{\varepsilon_z}}\int_0^{2\pi}d\vartheta' K(\vartheta')\beta_z^{1/2}(\vartheta') \cdot\left\{\frac{1}{\pi^2}\int_0^{\pi}d\sigma\int_0^{\pi}d\gamma\cos\gamma\left(B_z^+(X,Z)+B_z^-(X,Z)\right)\right\} \quad (5)$$

Here, $\varepsilon_{x,z}=4WI_{x,z}$ - beam emittance, $\beta_{x,z}=|f_{x,z}|^2/W$. For single-fold radial motion ($z=0$) without closed orbit distortion ($x=0$) the formula (4) is in agreement with [2].

When $B_z^{\pm}(X,Z)$ is given by (3) in an analytical form according to (4),(5) the effect of the total magnet field of the accelerator on the frequency dispersion can be taken into account. These formulae differ from those commonly used because they include the complete field in an aperture of the machine, taking into account random and regular distortions of the closed orbit rather than separate terms of the magnetic field Taylor expansion. The calculations give the most complete data on the effects mentioned.

While considering tolerances on multipole amplitudes it is convenient to set $B_z(X,Z)$ as a Taylor expansion:

$$\frac{B_z(X,Z)-B_{z_0}}{B_{z_0}}=\frac{1}{B_{z_0}}\sum_{k,\ell=1}^{\infty}\frac{1}{k!\,\ell!}\frac{\partial^{k+\ell}B_z}{\partial X^k\partial Z^\ell}X^kZ^\ell \quad (6)$$

In this case, using the relation

$$\pi\int_0^{\pi}(a\pm b\cos\varphi)^\ell\cos m\varphi\, d\varphi=\frac{(\pm 1)^m\ell!}{(\ell+m)!}(a^2-b^2)^{\ell/2}P_\ell^m\left(\frac{a}{\sqrt{a^2-b^2}}\right) \quad (7)$$

where $P_\ell^m(x)$ is an associated Legendre polynomial we find:

$$\Delta\nu_x(\varepsilon_{x,z};x,z)=\frac{1-æ}{2\pi K_0^2\sqrt{\varepsilon_x}}\sum_{\ell,m=1}^{\infty}\frac{1}{(\ell+1)!(2m)!} \cdot\int_0^{2\pi}d\vartheta'\frac{1}{B_{z_0}}\frac{\partial^{\ell+2m}B_z}{\partial X^\ell\partial Z^{2m}}K(\vartheta')\beta_x^{1/2}(\vartheta') \cdot\left(x^2(\vartheta')-\varepsilon_x\beta_x(\vartheta')\right)^{\ell/2}\left(z^2(\vartheta')-\varepsilon_z\beta_z(\vartheta')\right)^m P_\ell^1(\eta_x)P_{2m}(\eta_z) \quad (8)$$

$$\Delta\nu_z(\varepsilon_{x,z};x,z)=-\frac{1-æ}{2\pi K_0^2\sqrt{\varepsilon_z}}\sum_{\ell,m=1}^{\infty}\frac{1}{\ell!(2m-1)!} \cdot\int_0^{2\pi}d\vartheta'\frac{1}{B_{z_0}}\frac{\partial^{\ell+2m-1}B_z}{\partial X^\ell\partial Z^{2m-1}}K(\vartheta')\beta_z^{1/2}(\vartheta') \cdot\left(x^2(\vartheta')-\varepsilon_x\beta_x(\vartheta')\right)^{\ell/2}\left(z^2(\vartheta')-\varepsilon_z\beta_z(\vartheta')\right)^{m-1/2}P_\ell(\eta_x)P_{2m-1}^1(\eta_z) \quad (9)$$

$$\eta_x=\frac{x(\vartheta')}{\sqrt{x^2(\vartheta')-\varepsilon_x\beta_x(\vartheta')}}\ ,\quad \eta_z=\frac{z(\vartheta')}{\sqrt{z^2(\vartheta')-\varepsilon_z\beta_z(\vartheta')}}$$

Let's consider a simple and practically interesting case, namely: axial distortions of the closed orbit are equal zero, and $x(\vartheta)$ is included in a linear approximation. It is convenient to represent these expressions for even and odd magnet field nonlinearities and use the azimuthal variable $s=\vartheta R_0$:

$$\Delta^{2n}\nu_x=\frac{R_0 n}{2^{2n}(n!)^2\varepsilon_x R}\left\langle\frac{1}{B_{z_0}}\frac{\partial^{2n}B_z}{\partial X^{2n}}x(s)\cdot\left(\varepsilon_x\beta_x(s)+\varepsilon_z\beta_z(s)\right)^n\left[P_n(\lambda(s))+P_{n-1}(\lambda(s))\right]\right\rangle$$

$$\Delta^{2n+1}\nu_x=\frac{R_0(n+1)}{2^{2n+1}((n+1)!)^2\varepsilon_x R}\left\langle\frac{1}{B_{z_0}}\frac{\partial^{2n+1}B_z}{\partial X^{2n+1}}\cdot\left(\varepsilon_x\beta_x(s)+\varepsilon_z\beta_z(s)\right)^{n+1}\left[P_n(\lambda(s))+P_{n+1}(\lambda(s))\right]\right\rangle$$

$$\Delta^{2n}\nu_z=\frac{R_0 n}{2^{2n}(n!)^2\varepsilon_z R}\left\langle\frac{1}{B_{z_0}}\frac{\partial^{2n}B_z}{\partial X^{2n}}x(s)\cdot\left(\varepsilon_x\beta_x(s)+\varepsilon_z\beta_z(s)\right)^n\left[P_n(\lambda(s))-P_{n-1}(\lambda(s))\right]\right\rangle \quad (10)$$

$$\Delta^{2n+1}\nu_z=\frac{R_0(n+1)}{2^{2n+1}((n+1)!)^2\varepsilon_z R}\left\langle\frac{1}{B_{z_0}}\frac{\partial^{2n+1}B_z}{\partial X^{2n+1}}\cdot\left(\varepsilon_x\beta_x(s)+\varepsilon_z\beta_z(s)\right)^{n+1}\left[P_{n+1}(\lambda(s))-P_n(\lambda(s))\right]\right\rangle$$

where

$$\lambda(s)=\frac{\varepsilon_x\beta_x(s)-\varepsilon_z\beta_z(s)}{\varepsilon_x\beta_x(s)+\varepsilon_z\beta_z(s)}\ ,\quad \left\langle\mathcal{F}(s)\right\rangle=\frac{1}{2\pi R_0}\int_0^{2\pi R_0}\mathcal{F}(s)\,ds$$

$$n=1,\ 2,\ \ldots$$

The expressions (10) have been used to calculate the betatron oscillation frequency spread in the first ring of UNK [3]:

$$\Delta\nu_x = 393\frac{\Delta B_3}{B} + 170\frac{\Delta B_4}{B} + 63\frac{\Delta B_5}{B} + 51\frac{\Delta B_6}{B} + 24\frac{\Delta B_7}{B}$$

$$+16\frac{\Delta B_8}{B} + 6\frac{\Delta B_9}{B} + 6\frac{\Delta B_{10}}{B} + 2\frac{\Delta B_{11}}{B} + 2\frac{\Delta B_{12}}{B}, \quad (11)$$

$$\Delta\nu_z = 385\frac{\Delta B_3}{B} + 177\frac{\Delta B_4}{B} + 66\frac{\Delta B_5}{B} + 67\frac{\Delta B_6}{B} + 27\frac{\Delta B_7}{B}$$

$$+25\frac{\Delta B_8}{B} + 7\frac{\Delta B_9}{B} + 10\frac{\Delta B_{10}}{B} + 3\frac{\Delta B_{11}}{B} + 4\frac{\Delta B_{12}}{B}$$

Here $\frac{\Delta B}{B} = \frac{1}{n!}\frac{r^n}{B_{z_0}}\frac{\partial^n B_z}{\partial x^n}$, $r = 3{,}5\cdot10^{-2}$ m – normalization radius, $\Delta P/P_0 = \pm 2\cdot10^{-3}$, $\varepsilon_x = \varepsilon_z = 2{,}66\cdot10^{-6}$ m·rad.

Resonance shifts of the betatron oscillation frequencies are well calculated from the (1), (2).

The formula (4) has been used to calculate the radial betatron oscillation frequency spread associated with the orbit dispersion in UNK. According to the numerical simulation results [4] the UNK dipole field is well approximated by the dependence:

$$B_z(x) = B_{z_0}\left[1 + ax^2 + bx^4 + cx^2 e^{d|x|}\right] \quad (12)$$

where $a = 0{,}136667$ m^{-2}, $b = 0{,}149171\cdot10^3$ m^{-4}, $c = -1{,}42581\cdot10^{-3}$ m^{-2}, $d = 132{,}3464$ m^{-1}.

The integration in (4) is simple. The frequency shift is expressed in an analytical form through incomplete Bessel functions of an imaginary argument. The orbit deviation due to momentum spread is regular. Assume, that extra beam deviation from the chamber axis, induced by various magnet field distortions, is random and is subjected Gaussian distribution, then the frequency spread is $|\Delta\nu_x| \simeq 0.092$ and the spread dispersion is $2\Delta\nu_{x\,r.m.s.} = 0{,}02$. It is assumed in this case, that the maximum deviation of the closed orbit from the center line after correction does not exceed $2\Delta x_{r.m.s.} = 5$ mm.

Magnet lattice chromaticity correction provides sufficiently lower value of $|\Delta\nu_x|$.

References

1. G.Parzen. IEEE Trans. on Nucl.Sci., NS-28, No.3, 2549 (1981).
2. A.Chao, M.Lee, P.Morton. IEEE Trans. on Nucl.Sci., NS-22, 1878 (1975).
3. V.P.Belov. Preprint NIIEFA P-B-0587, L., 1983.
4. N.I.Doinikov, V.S.Kashikhin et al. Proc. of the VII All-Union Meeting on Charged Particle Accelerators. Dubna, JINR, 1981, v.1, p.331.