Instituto de Física Teórica
Universidade Estadual Paulista

IFT–D.007/2011

TESE DE DOUTORAMENTO

Some Mathematical Aspects and Scattering Amplitudes in the Pure Spinor Formalism.

Humberto Gomez Zuñiga

Advisor

Nathan Jacob Berkovits

Coadvisor

Abraham Hirsz Zimerman

Oct 2011

Acknowledgments

Primero agradezco a Dios, por haber conseguido terminar este trabajo, el cual pense que no iba a terminar nunca. Agradezco ante todo a mis padres, a mi querida madre, Silvia Nuvia, a quien le debo todo, su cariño, su comprension y apoyo; a mi papá, Miguel Humberto Gomez, siempre al lado mio, respaldando mis decisiones y queriendo siempre lo mejor para mi; A mi abuela, Marina Mora, por todo el Amor que me a dado, a mis hermanos Yamileth y Juan Carlos y a toda mi familia en general.

Agradezco mucho a todos los profesores del Instituto, en especial a mi orientador Nathan Berkovits, quien siempre ha creido en mis capacidades, siempre se ha procupado y me ha apoyado en mis trabajos. Agradezco mucho todas sus explicaciones ademas de sus regaños en las horas ciertas.
Tambien agradezco a mi coorientador Abrahan Zimerman y al profesor Andrei Mikhailov. Tambien estoy muy agradecido con el grupo de cuerdas del IFT, donde mis trabajos fueron presentados y sus preguntas fueron de mucha ayuda.

Agradezco con mucho cariño a todos mis amigos en São Paulo, Cristhian Said Osorio, Oscar Andres Bedoya, Camilo Andres, Juan Pablo Beltran, David, Alexis y el ultimo que llego Cristian Manuel. Por las discuciones, el interes que mostraron por mi trabajo y por no dejarme solo en los tiempos dificiles.
Tambien deseo agradecer a mis amigo Carlos Andres Cardona y Juan Pablo Gutierrez, siempre es muy divertido compartir el tiempo con ellos.

Con especial cariño agradezco a Carlos Roberto Mafra, con quuien consegui publicar un articulo y siempre estuvo interesado en mis otros trabajos. Muchas gracias por todos sus consejos y referencias.

Quiero Agradecer a todas las Universidades que visite, en especial *The Simons Center for Geometry and Physics*, donde siempre hubo un ambiente amigable y confortable de trabajo. Quiero agradecer tambien a Victor Manuel Bastidas quien me trato demasiadamente bien en su casa en Berlin.

Finalmente agradezco a la FAPESP por el apoyo financiero que me dio y al IFT por darme la oportunidad de hacer este Doctorado. Proceso: 07/54623-8.

Resumo

Primeiro vamos dar uma breve revisão sobre o artigo de Nekrasov " Lectures on curved beta-gamma systems, pure spinors, and anomalies", a fim de mostrar que o formalismo de espinor puro é livre de anomalia quando a origem é removido do espaço espinor puro.
Desta forma, damos uma nova proposta para os operadores de imagem no formalismo de espinor puro mínimo. Nós calculamos amplitudes de espalhamento a nível de árvore, realizando a integração no espaço espinor puro como uma integral de Cauchy tipo multidimensional. A amplitude é escrita em termos de variáveis do espaço de espinor puro projetivo, o que é muito útil na hora de relacionar rigorosamente as versões mínima e não mínima do formalismo de espinor puro. A linguagem natural para relacionar esses formalismos é o isomorfismo de Čech-Dolbeault. Além disso, o cociclo de Dolbeault correspondente à amplitude de espalhamento a nível de árvore deve ser avaliada no espaço compacto $SO(10)/SU(5)$ em vez de tudo o espaço de espinor puro, o que significa que a origem é removido neste espaço.

Nós também obtimos uma relação entre a função de Green para um campo escalar sem massa em dez dimensões e as amplitude de espalhamento a nível de árvore.

Os fatores globais constantes nas amplitudes de espalhamento são muito importante, porque eles precisam satisfazer as condições de unitariedade e S-dualidade [66]. Estes coeficientes não tinham sido computados no formalismo espinor puro, devido à dificuldade para resolver as integrais no espaço de espinores puro. Nós calculamos estas integrais usando o formalismo de espinor puro não mínimo. Assim, encontramos os coeficientes das amplitudes de um e dois-"loop" para quatro pontos sem massa. Contrastando com as dificuldades matemáticas no formalismo RNS, em que o desconhecimento das normalizações dos determinantes quirais são força que o coeficiente de dois-"loop" deve ser determinado apenas indiretamente, por meio de fatoração, o cálculo no formalismo de espinor puro pode ser facilmente realizado.

Palavras Chaves: Supersimetria; Supercordas; Espinor Puro; Cohomologia.

Áreas do conhecimento: Física de Partículas e Campos.

Abstract

First, we give a brief review about the Nekrasov's paper "Lectures on curved beta-gamma systems, pure spinors, and anomalies" in order to show the pure spinor formalism is anomaly free when the origin is removed from the pure spinor space.
In this way we give a new proposal for the "picture lowering" operators in the minimal pure spinor formalism. We compute the tree level scattering amplitude by performing the integration over the pure spinor space as a multidimensional Cauchy-type integral. The amplitude is written in terms of the projective pure spinor variables, which turns out to be useful to relate rigorously the minimal and non-minimal versions of the pure spinor formalism. The natural language for relating these formalisms is the Čech-Dolbeault isomorphism. Moreover, the Dolbeault cocycle corresponding to the three-level scattering amplitude must be evaluated in $SO(10)/SU(5)$ instead of the whole pure spinor space, which means that the origin is removed from this space.

We also relate the Green's function for the massless scalar field in ten dimensions to the tree-level scattering amplitude and comment about the scattering amplitude at higher orders.

The overall constant factors in the scattering amplitudes are very important because they need to satisfy the unitarity and S-duality conditions [66]. These coefficients have not been computed in the pure spinor formalism due to the difficulty to solve the integrals on the pure spinors space. We compute these integrals by using the non-minimal pure spinor formalism. So, we find the coefficients of the massless one and two-loop four-point amplitude from first principles. Contrasting with the mathematical difficulties in the RNS formalism where unknown normalizations of chiral determinant formulæ force the two-loop coefficient to be determined only indirectly through factorization, the computation in the pure spinor formalism can be smoothly carried out.

Key Words: Supersymmetry; Superstring; Pure Spinor; Cohomology.

Areas of Knowledge: Fields and Particles Physics.

Index

Chapter 1

Introduction

The description of Physics in terms of fields dates back to the 19th century and had as origin the study of the electric and magnetic phenomena. Since then, the field language has seem appropriate to describe electromagnetism, gravitation, and the remaining two type of interaction discovered in the 20th century; namely the weak and strong interactions. The Standard Model of particle physics, which describes all except for gravitational phenomena, is a beautiful example of a *unified* description for various fundamental interactions in terms of quantum fields. Nevertheless the Standard Model can be thought of as a built theory, which can be adjusted if some minor changes are required by the experiments. Furthermore, there are ingredients, in the philosophy of constructing, that are put by hand instead of deduced from more fundamental principles, for example, the way various particles are accommodated in the standard model multiplets. In some way, the ability to adjust such a theory also leaves the unsatisfactory taste of not having the right core from where to extract it in a *unique* manner.

Although the gravitational field has a well established classical field description, its quantum description has been elusive for quite long, as well as its incorporation, together with the other three interactions, in a single framework. Perhaps this first fact is an indication that the right type of description has not been used.

An important step in the direction of a quantum theory of gravity has been provided by precisely changing the type of description used in particle physics, namely Quantum Field Theory. String theory, which was *accidentally* discovered by studying an apparently singular behavior of the mass and the spin of some heavy particles in the late sixties; is a different proposal for describing particle physics. Since that time, the term string theory has developed to incorporate any of a group of related superstring theories. Five major string theories were formulated. All of them appeared to be correct. However in the mid 1990s a unification of all previous superstring theories, called M-theory, was proposed, which asserted that strings are really 1-dimensional slices of a 2-dimensional membrane vibrating in 11-dimensional spacetime.

A string is a one-dimensional object, this can be closed or open, which expands a two-dimensional surface as it evolves in time, called the *worldsheet*. In this theory the appearance of extra dimensions is "natural". In fact superstring theory is a consistent theory in ten space-time dimensions.

Among other things, superstring theory has provided us with a consistent quantum description of the gravitational force. One particular oscillation mode of the closed string has the right properties to be the quantum messenger of the gravitational force, the graviton. And its interactions are described precisely by the Einstein-Hilbert action,

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N} \int \mathrm{d}^{10}x \sqrt{-g} R \tag{1.1}$$

plus quantum and superstring corrections to be described below [1].

Also present in the open superstring spectrum is a massless string with spin one which describes the Yang-Mills gluons (or photons), whose interactions in the low energy limit are described by the standard Yang-Mills action,

$$S_{\mathrm{YM}} = \frac{1}{g_{YM}^2} \int \mathrm{d}^{10}x \mathrm{Tr}(F_{mn}F^{mn}) \tag{1.2}$$

together with other quantum or superstring corrections.

One of the most fundamental questions which naturally arise when studying the low energy properties of the superstring interactions is to understand what are the perturbative corrections to these two actions predicted by the theory. That question automatically leads us to contemplate the fact that superstring perturbation theory is finite to all loop orders [2]. Therefore besides unifying all forces of nature, superstring theory does it in such a way as to be *finite*. No renormalization is ever needed when deriving quantum corrections to the effective action.

One of the standard procedures to obtain these quantum corrections is through the computation of scattering amplitudes. The computation of these various scattering amplitudes have been traditionally done using two different prescriptions, encompassed in the so-called Ramond-Neveu-Schwarz (RNS) [3] or Green-Schwarz (GS) formalism [4][5].

1.1 The RNS Formalism and Low Energy Contribution

In the RNS formalism the superstring theory has a manifestly worldsheet supersymmetry. In this formalism the worldsheet is a Super-Riemann surface embedded on a ten dimensional space-time. In this work the space-time will always be a flat space (we can think in it as $\mathbb{R}^{10}$ after a Wick rotation). The action of the superstring (Type II) in this formalism is given by [69]

$$S_m = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2\mathbf{z}\, \mathbf{E}\, \mathcal{D}_- \mathbf{X}^m\, \mathcal{D}_+ \mathbf{X}_m + \lambda\, \chi(\Sigma_g) \tag{1.3}$$

where $\mathbf{z}^M = (\xi, \overline{\xi}; \theta, \overline{\theta})$ is the supercoordinate on the super-Riemann surface (M is an Einstein index, $I = (i, \mu)$), the measure $\mathrm{d}^2\mathbf{z}$ is defined as $\mathrm{d}^2\mathbf{z} = \mathrm{d}^2\xi \mathrm{d}\theta \mathrm{d}\overline{\theta}$, $\mathbf{E}$ is the superdeterminant of the super-zweibein E_I^A (A is a Lorentz index, $A = (a, \alpha) = (z, \bar{z}; +, -)$), i.e $\mathbf{E} = \mathrm{sdet} E_M^A$, $\mathcal{D}_\pm$ are the supercovariant derivatives, $\mathbf{X}^m$ is the superscalar field

$$\mathbf{X}^m = x^m + \theta\psi^m + \overline{\theta\psi}^m + i\theta\overline{\theta}F^m \tag{1.4}$$

and finally λ is a coupling constant and $\chi(\Sigma_g) = 2 - 2g$ is the Euler characteristic of the compact Riemann surface Σ_g of genus g. From the worldsheet point of view , type II theories are formulated as $N = 1$ two dimensional supergravity with matter multiplets x^m and $\psi^m, \overline{\psi}^m$ and a supergravity multiplet consisting of a zweibein e_m^a and a two-dimensional spin 3/2 gravitino field χ_m.
This action have the following symmetries:
(i) Local $U(1)$ transformation.
(ii) Super-reparametrizations.
(iii) Super-Weyl transformation.
However, since these symmetries are not relevant to the development of this thesis then we do not write them (see [69]).

After a geometry quantization the scattering amplitude prescription in the RNS formalism is given by [69]

$$\langle\, V_1(k_1)...V_n(k_n)\, \rangle_g = \int_{s\mathcal{M}_g} \prod_J \mathrm{d}\tau_J \sum_\nu c_\nu \frac{\mathrm{sdet}(\mu_L, \Phi_K)}{\mathrm{sdet}(\Phi_L, \Phi_K)} \left(\frac{8\pi^2 \mathrm{sdet}'\Box_0}{\mathrm{sdet}(\Psi_\alpha, \Psi_\beta)} \right)^{-5} (\mathrm{sdet}\mathcal{P}_1^\dagger \mathcal{P}_1)^{1/2}$$
$$\langle\, \langle\, V_1(k_1)...V_n(k_n)\, \rangle\, \rangle\, . \tag{1.5}$$

The double bracket $\langle\, \langle\, \cdot\, \rangle\, \rangle$ means integration by the superfield $\mathbf{X}^m$. The space $s\mathcal{M}_g$ is the supermoduli space of the super Riemann surface with genus g and τ_J is the super Teichmüller parameter. μ_L is the super Beltrami differential , which in the Wess Zumino gauge may be decomposed as

$$\mu_L = \mu_L^1 \bar{\theta} + \mu_L^0 \bar{\theta}\theta \tag{1.6}$$

where

$$\mu_L^1 = e_{\bar{z}}^m \frac{\partial e_m^z}{\partial \tau_L}, \qquad \mu_L^0 = -\frac{\partial \chi_{\bar{z}}^+}{\partial \tau_L}.$$

Ψ_α is the superconformal Killing vectors, i.e these fields are an base of the Kernel of the operator $\mathcal{P}_1$ given by

$$(\mathcal{P}_1 \delta \mathbf{V})_-^z = \mathcal{D}_- \delta \mathbf{V}^z.$$

The operator $\mathcal{P}_1^\dagger$ is the Hermitian conjugate operator of $\mathcal{P}_1$ and the holomorphic superquadratic differentials Φ_K's expand of Kernel of this operator. The operator $\Box_0$ is defines as $\Box_0 = \mathcal{D}_- \mathcal{D}_+$. The index ν is running by the number of spin structures of the Riemann surface of genus g and the c_ν's are constants which must be found by the modular invariance.

The tree-level and one-loop 4 point scattering amplitudes are not difficult computations, for details see [69][77]. These amplitudes have the form

$$\mathcal{A}_0 = C_0 e^{-2\mu} \left(\frac{\alpha'}{2}\right)^8 K_0 \bar{K}_0 C(s,t,u), \tag{1.7}$$

$$\mathcal{A}_1 = C_1 K_0 \bar{K}_0 \left(\frac{\alpha'}{2}\right)^8 \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau^5} \prod_{i=2}^4 \int d^2 z_i e^{-\alpha' \sum_{i,j} k_i \cdot k_j G_1(z_i, z_j)}$$

where K_0 is the kinematic factor which will be given in the chapter 4, $C(s,t,u)$ is the following combination of the Gamma functions

$$C(s,t,u) = \frac{\Gamma(-s\alpha'/4)\Gamma(-t\alpha'/4)\Gamma(-u\alpha'/4)}{\Gamma(1+s\alpha'/4)\Gamma(1+t\alpha'/4)\Gamma(1+u\alpha'/4)} \tag{1.8}$$

and s,t,u are the Mandelstam variables. $G_1(z,y)$ is the Green function of the torus and τ is the Teichmüller parameter.
The vertex operators used were the NS-NS vertex operators given by*

$$V(\epsilon, \bar{\epsilon}, k) = \epsilon^\mu \bar{\epsilon}^\nu \int_\Sigma d^2 z (\partial x^\mu + i k^\rho \psi_+^\rho \psi_+^\mu)(\bar{\partial} x^\nu + i k^\sigma \psi_-^\sigma \psi_-^\nu) e^{ik\cdot x}. \tag{1.9}$$

where $\epsilon\bar{\epsilon}$ is the polarization and k the momentum.
In [66] the overall coefficients were computed carefully and they found

$$C_0 = \frac{\sqrt{2}}{2^{12}\pi^6(\alpha')^5}, \qquad C_1 = \frac{1}{2^7\pi^2(\alpha')^5}. \tag{1.10}$$

In the RNS formalism the two loop computation is a hard work. D'Hoker and Phong published a serie of 7 seven papers about this computation [95] and the result is

$$\mathcal{A}_2 = C_2 e^{2\mu} K_0 \bar{K}_0 \left(\frac{\alpha'}{2}\right)^{10} \int_{\mathcal{M}_2} \frac{\mathrm{d}^2\Omega_{IJ}}{(\det \mathrm{Im}\Omega_{IJ})^5} \int \prod_{l=1}^4 \mathrm{d}z_l |\mathcal{Y}_s|^2 e^{-\alpha' \sum_{i,j} k_i \cdot k_j G_2(z_i, z_j)}. \tag{1.11}$$

*In RNS formalism is only possible to compute scattering amplitudes with external bosons at loop level. The problem is the spin fields on Riemann surfaces with genus $g \geq 1$

Ω_{IJ} is the period matrix given by

$$\int_{a_I} \mathrm{d}z\, w_J = \delta_{IJ}, \qquad \int_{b_I} \mathrm{d}z\, \omega_J = \Omega_{IJ}, \quad I, J = 1, 2, \tag{1.12}$$

where a_I and b_J are the non trivial cycles. The function $\mathcal{Y}_s$ is defined as

$$3\mathcal{Y}_s = (t-u)\Delta(1,2)\Delta(3,4) + (s-t)\Delta(1,3)\Delta(4,2) + (u-s)\Delta(1,4)\Delta(2,3) \tag{1.13}$$

where

$$\Delta(z,y) = \epsilon^{IJ} w_I(z) w_J(y). \tag{1.14}$$

In the RNS formalism the overall coefficient C_2 can not be computed in a straightforward way due to the occurrence of an unknown overall normalization constant in the chiral normalization formulas of [96].

1.2 The Pure Spinor formalism

The pure spinor formalism was born at the dawn of the new millennium [47], which has shown to be a powerful framework in two branches. The first one is the computation of scattering amplitudes and the second one is the quantization of the superstring in curved backgrounds which can include Ramond-Ramond flux. The strength of the pure spinor formalism resides precisely in the fact that it can be quantized in a manifestly super-Poincaré manner, so this covariance is not lost neither in the scattering amplitudes computation nor in the quantization of the superstring in curved backgrounds.

Since the present thesis is about the first branch, we will give a brief description of what has been done in scattering amplitudes, not attempting to give a complete list of references.

Right after the formalism came into light, the tree-level amplitudes were shown to be equivalent with the RNS computations in [39], for amplitudes containing any number of bosons and up to four fermions. Years later, Berkovits spelled out the multiloop prescription [48][52] and paved the way to show the equivalence of his formalism up to the two-loop level.

However all these computations [50][62][89][90][17] were done up to overall coefficient due to the difficulty to solve the integrals on the pure spinors space. In 2009 a technique was developed to solve these integrals [32] and in 2010 the unitarity condition (C.22) was shown in a straightforward way using the nonminimal pure spinor formalism [33].

In 2011 a new proposal for computing the tree-level scattering amplitudes in the minimal pure spinor approach was published [34]. In contrast with that given in [48], this new proposal decoupled the non physical states and the singular point of the pure spinor space is removed.

One key ingredient in this formalism is a bosonic ghost λ^α, constrained to satisfy Cartan's pure spinor condition in 10 space-time dimensions [14] †. The prescription for computing multiloop amplitudes was given in [48], where as in the RNS formalism, it was necessary to introduce picture changing operators (PCO's) in order to absorb the zero-modes of the pure spinor variables. Up to two-loops, various amplitudes were computed in [49], [50] and [17]. Later on, by introducing a set of non-minimal variables $\bar{\lambda}_\alpha$ and r_α, an equivalent prescription for computing scattering amplitudes was formulated in [57] and [52]. This last superstring description is known as the "non-minimal" pure spinor formalism, in order to distinguish it from the former "minimal" pure spinor formalism. With the non-minimal formalism, also were computed scattering amplitudes up to two-loops [62], [89]. Because of its topological nature, in the non-minimal version it is not necessary to introduce PCO's. Nevertheless, it is necessary to use a regulator. The drawback of having to introduce this regulator appears beyond two-loops, since it gets more complicated due to the divergences coming from the poles contribution of the b ghost [55] [53].

Before we give a brief introduction about the minimal and non-minimal approach, it is useful to talk a little bit about the Green-Schwarz formalism in order to understand better the pure spinor formalism.

1.2.1 The Green-Schwarz formalism

The Green-Schwarz formulation (GS) is manifestly supersymmetric [4][5] and it is based on the worldsheet fields X^m and θ^α, which are spacetime vectors and spinors, respectively. The drawback in this formalism comes from the fact that it has a complicated action,

$$S = \frac{1}{\pi}\int d^2z \Big[\frac{1}{2}\partial X^m \bar{\partial} X_m - i\partial X^m \theta_L \gamma_m \bar{\partial}\theta_L - i\bar{\partial} X^m \theta_R \gamma_m \partial \theta_R$$
$$- \frac{1}{2}(\theta_L \gamma^m \bar{\partial}\theta_L)(\theta_L\gamma_m\partial\theta_L + \theta_R\gamma_m\partial\theta_R) - \frac{1}{2}(\theta_R\gamma^m\partial\theta_R)(\theta_L\gamma_m\bar{\partial}\theta_L + \theta_R\gamma_m\bar{\partial}\theta_R)\Big],$$

which is impossible to quantize preserving manifest Lorentz covariance. The big problem of this action is that it has constraints of first and second class encoded in a single spinor, which are impossible to split in a covariant way. For instance, the conjugate momentum of θ_L is

$$p^L_\alpha = \pi\frac{\delta S}{\delta \partial_0 \theta^\alpha_L} = \frac{i}{2}\Pi^m(\gamma_m\theta_L)_\alpha - \frac{1}{4}(\theta_L\gamma^m\partial_1\theta_L)(\gamma_m\theta_L)_\alpha, \tag{1.15}$$

†Even before pure spinor were incorporated in the description for the superstring, Howe showed that integrability along pure spinor lines allowed to find the super Yang-Mills and supergravity equations of motion in ten dimensions [15].

so we get the constraint[‡] $d^L_\alpha = p^L_\alpha - \frac{i}{2}\Pi^m(\gamma_m\theta_L)_\alpha + \frac{1}{4}(\theta_L\gamma^m\partial_1\theta_L)(\gamma_m\theta_L)_\alpha \approx 0$. Since p^L_α is the conjugate momentum of θ^α_L then it is simple to see

$$\{d^L_\alpha, d^L_\beta\} = (\gamma^m)_{\alpha\beta}\Pi_m. \tag{1.16}$$

As $\Pi^m\Pi_m = 0$, then a half of these constraints are the first class and the other 8 constraints are the second class. The only way to split these constraints is to use the light cone gauge.

So, by breaking $SO(1,9)$ covariance to $SO(8)$ with the light cone gauge choice the action simplifies [35]

$$S = \frac{1}{4\pi}\int d^2z\left(\partial X^i\bar\partial X^i + S^a_L\bar\partial S^a_L + S^b_R\partial S^b_R\right).$$

where $i, a, b = 1, ..., 8$.

In this gauge the construction of vertex operators is possible and the computation of scattering amplitudes can be done. However, the need of a non-covariant gauge and restricted kinematics are features which reduce the power of this manifestly supersymmetric approach. For example, in the light cone gauge one looses the conformal symmetry of the original theory and therefore can not use the powerful methods of conformal field theory. Furthermore it is not always possible to impose those restrictions simultaneously.

1.2.2 Siegel's modification of the GS formalism

In 1986 Warren Siegel [36] tried to solve the problem of the GS fermionic constraints. He proposed a new approach where the conjugate momentum of θ^α is an independent variable, i.e (left sector)

$$S = \frac{1}{2\pi}\int \mathrm{d}^2z\left(\frac{1}{2}\partial X^m\bar\partial X_m + p_\alpha\bar\partial\theta^\alpha\right). \tag{1.17}$$

Siegel argued that it should be added an appropriate set of first class constraints to this action. In this set must be the Virasoro constraints, $T = -\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha$ and the generators of the kappa symmetry[§], $G^\alpha = \Pi^m(\gamma_m d)^\alpha$. In this approach the variable

$$d_\alpha = p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha\partial x_m - \frac{1}{8}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta) \tag{1.18}$$

is not zero.

Nevertheless, the total number of the constraints to get the superstring spectrum could not be found.

[‡] Π^m is the supersymmetry momentum $\Pi^m = \partial X^m - i\theta_L\gamma^m\partial\theta_L - i\theta_R\gamma^m\partial\theta_R$

[§] The kappa symmetry is a local fermionic symmetry of the GS action.

Note also that since the conformal weights of the fields p_α and θ^α are (1,0) and (0,0) respectively the the stress tensor

$$T(z) = -\frac{1}{2}\partial X^m \partial X_m - p_\alpha \partial\theta^\alpha$$

has central charge -22.

1.2.3 The Berkovits' approach: The Pure Spinor Formalism

Using the Noether theorem we can get in a easy way the spin Lorentz currents of the action (1.17)

$$\Sigma^{mn}(z) = \frac{1}{2} : (p\gamma^{mn}\theta) : (z).$$

From the OPE's

$$\begin{aligned} X^m(z)X^n(w) &\rightarrow -\eta^{mn} \ln|z-w| \\ p_\alpha(z)\theta^\beta(w) &\rightarrow \frac{\delta_\alpha^\beta}{(z-w)} \end{aligned} \tag{1.19}$$

we can check the algebra

$$\Sigma^{mn}(w)\Sigma^{pq}(z) = \frac{\eta^{p[n}\Sigma^{m]q} - \eta^{q[n}\Sigma^{m]p}}{w-z} + 4\frac{\eta^{m[q}\eta^{p]n}}{(w-z)^2}. \tag{1.20}$$

Recalling that in the RNS formalism the OPE of the Lorentz currents for the fermionic variables $\Sigma_{\text{RNS}} = \psi^m\psi^n$ satisfies

$$\Sigma^{mn}_{RNS}(w)\Sigma^{pq}_{RNS}(z) \rightarrow \frac{\eta^{p[n}\Sigma^{m]q}_{RNS} - \eta^{q[n}\Sigma^{m]p}_{RNS}}{w-z} + \frac{\eta^{m[q}\eta^{p]n}}{(w-z)^2} \tag{1.21}$$

the different double pole coefficient in (1.20) and (1.21) would make the computations of scattering amplitudes using RNS or the Siegel approach not agree with each other.

This fact and the inconsistency of the central charge is different to zero led to Berkovits to modify the Siegel's approach. Berkovits introduce a new set of variables with central charge 22 such that the total central charge vanishes.

BRST Operator, Stress Tensor, Lorentz current and Ghost Number

The next element of the pure spinor formalism is the construction of the BRST operator, which is given by the linear combination of the fermionic constraints d_α

$$Q_{BRST} = \int \mathrm{d}z\, \lambda^\alpha d_\alpha, \tag{1.22}$$

where the λ^α's are bosonic coefficients. The physical states are defined to be in the cohomology of the operator Q_{BRST}, therefore it must satisfy the consistency condition $Q^2_{BRST} = 0$. Computing $\{Q_{BRST}, Q_{BRST}\}$ we get

$$Q^2_{BRST} = \frac{1}{2}\{Q_{BRST}, Q_{BRST}\} = -\frac{1}{2}\int \mathrm{d}z\, (\lambda\gamma^m\lambda)\Pi_m, \tag{1.23}$$

so the bosonic coefficients must satisfy

$$(\lambda\gamma^m\lambda) = 0 \tag{1.24}$$

This condition is known as the *the pure spinor condition in dimension* $d = 10$.
The components of the $SO(10)$ spinor λ^α (after Wick rotation) can be written in the $U(5)$ variables [47][67]

$$\lambda^\alpha = (\lambda^+, \lambda_{ab}, \lambda^a), \qquad a, b = 1, ..5 \tag{1.25}$$

where $\lambda_{ab} = -\lambda_{ba}$. In these $U(5)$ variables the 10 constraints (1.24) become

$$\begin{aligned} &2\lambda^+\lambda^a - \frac{1}{4}\epsilon^{abcde}\lambda_{bc}\lambda_{de} = 0, \qquad a,\ b,\ c,\ d,\ e = 1,\ 2, ..., 5 \\ &2\lambda^b\lambda_{ab} = 0. \end{aligned} \tag{1.26}$$

These equations are solved by the parametrization

$$\lambda^+ = \gamma, \quad \lambda_{ab} = \gamma u_{ab}, \quad \lambda^a = \frac{1}{8}\gamma\epsilon^{abcde}u_{bc}u_{de}, \tag{1.27}$$

where $a, b, c, d, e = 1, ..., 5$ and $u_{ab} = -u_{ba}$. This means the number of the degree of freedom of the spinor λ^α is eleven complex degree (it is because λ^α is a $SO(10)$ spinor).
The action for these bosonic variables was proposed by Berkovits [47] and it given by (left sector)

$$S_{PS} = \int \mathrm{d}^2z \left(\beta\overline{\partial}\gamma + \frac{1}{2}v^{ab}\overline{\partial}u_{ab}\right) \tag{1.28}$$

where $v^{ab} = -v^{ba}$ and β and v^{ab} have conformal weights (1,0) and γ and u_{ab} have conformal weights $(0, 0)$. This action is called the beta-gamma system, which will be discussed in detail the next chapter.
From the previous action it is easy to get the local OPE's

$$\beta(z)\gamma(w) \to (z - w)^{-1}, \qquad v^{ab}(z)u_{cd}(w) \to \delta^a_{[c}\delta^b_{d]}(z - w)^{-1}, \tag{1.29}$$

The stress tensor

$$T(z) = \frac{1}{2}v^{ab}\partial u_{ab} + \beta\partial\gamma, \tag{1.30}$$

has central charge $c = 22$ and therefore the total theory has central charge zero and so it is conformal anomaly free¶. However the previous action is not useful to obtain the Lorentz currents, since λ^α is a spinor, so the pure spinor action can be written in a covariant way as

$$S_{PS} = \int \mathrm{d}^2z\, \omega_\alpha\overline{\partial}\lambda^\alpha, \tag{1.31}$$

¶Since the Lorentz currents, which we define later, are primary fields then it is necessary to introduce an additional term to this stress tensor given by $-\frac{7}{2}\partial^2\ln\gamma$. In the next chapter we will give a geometrical interpretation for this term.

where λ^α satisfies the constraint (1.24). This constraint means the (1.31) has the gauge symmetry

$$\delta\omega_\alpha = \Lambda_m(z,\bar{z})(\gamma^m\lambda)_\alpha, \tag{1.32}$$

so ω_α has 11 degree of freedom. Note that using the parametrization (1.27) and fixing the gauge $\omega_\alpha = (\omega_+, \omega^{ab}, \omega_a) = (\beta - \frac{1}{2\gamma}v^{ab}u_{ab}, \frac{v^{ab}}{\gamma}, 0)$ then the action (1.31) becomes in (1.28).

Since the fields ω_α and λ^α are bosonic spinors then under a Lorentz transformation these fields change

$$\delta\lambda^\alpha = \frac{1}{4}\varepsilon_{mn}(\gamma^{mn})^\alpha_\beta\lambda^\beta, \qquad \delta\omega_\alpha = \frac{1}{4}\varepsilon_{mn}(\gamma^{mn})^\beta_\alpha\omega_\beta. \tag{1.33}$$

Applying the Noether method we get the Lorentz currents

$$N^{mn}(z) = \frac{1}{2} : (\omega\gamma^{mn}\lambda) : (z). \tag{1.34}$$

However as ω_α is not strictly the conjugate momentum of λ^α because λ^α satisfies the pure spinor constrain, i.e the OPE between ω_α and λ^β goes as $(z-w)^{-1}$ plus some corrections, then to obtain the algebra of these currents it is useful to break the generators N^{mn} in the $U(5)$ variables

$$N^{mn} \to (N, N^a_b, N^{ab}, N_{ab})$$

and so to use the OPE's (1.29). Since it is not a easy work, the details are given in [67], then we write just the results

$$N^{mn}(y)N^{pq}(z) \to \frac{\eta^{p[n}N^{m]q}(z) - \eta^{q[n}N^{m]p}(z)}{y-z} - 3\frac{\eta^{q[m}\eta^{n]p}}{(y-z)^2} \tag{1.35}$$

$$N_{mn}(y)\lambda^\alpha(z) \to \frac{1}{2}\frac{(\gamma_{mn}\lambda)^\alpha(z)}{y-z}. \tag{1.36}$$

As the fields $(p_\alpha, \theta^\alpha)$ are not coupled to the fields $(\omega_\alpha, \lambda^\alpha)$ then the OPE

$$N^{mn}(y)\Sigma^{pq}(z) \to \text{reg} \tag{1.37}$$

is regular, therefore the total Lorentz current $M^{mn} = \Sigma^{mn} + N^{mn}$ satisfies the same algebra as RNS (1.21)

$$M^{mn}(y)M^{pq}(z) \to \frac{\eta^{p[n}N^{m]q}(z) - \eta^{q[n}N^{m]p}(z)}{y-z} + \frac{\eta^{q[m}\eta^{n]p}}{(y-z)^2}, \tag{1.38}$$

as it was desired.

The action (1.31) has also a global $U(1)$ symmetry given by

$$\lambda'^\alpha = e^{ic}\lambda^\alpha, \qquad \omega'_\alpha = e^{ic}\omega_\alpha, \tag{1.39}$$

which is known as ghost number. The Noether current for this symmetry is

$$J(z) =: \omega_\alpha \lambda^\alpha : (z). \tag{1.40}$$

This symmetry will be used frequently to compute scattering amplitudes.
Some important OPE's (see [67]) are

$$J(y)J(z) \to \frac{-4}{(y-z)^2}, \qquad J(y)N^{mn}(z) \to regular,$$

$$N_{mn}(y)T(z) \to \frac{N_{mn}(z)}{(y-z)^2}, \qquad T(z)J(y) \to \frac{8}{(z-y)^3} + \frac{J(y)}{(z-y)^2} + \frac{\partial J(y)}{(z-y)}.$$

Using the Sugawara construction we can represent the stress tensor in terms of the currents N^{mn} and J[48]

$$T = -\frac{1}{2}\partial x^m \partial x_m - p_\alpha \partial \theta^\alpha + \frac{1}{10} N^{mn} N_{mn} - \frac{1}{8} JJ + \partial J. \tag{1.41}$$

where we have included the X^m and p_α, θ^α sector, i.e this is the total stress tensor of the action‖ (left sector)

$$S = \frac{1}{2\pi} \int_{\Sigma_g} \mathrm{d}^2 z \, \left(\frac{1}{2} \partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial} \theta^\alpha - \omega_\alpha \bar{\partial} \lambda^\alpha \right). \tag{1.42}$$

This presentation is very useful to find the $b-$ghost in order to compute scattering amplitudes at loop level.

1.2.4 Massless vertex operators

The physical states in the pure spinor formalism are defined to be in the cohomology of the BRST operator

$$Q = \frac{1}{2\pi i} \oint \lambda^\alpha d_\alpha \tag{1.43}$$

which satisfy $Q^2 = 0$ due to the pure spinor condition. Therefore we can define the unintegrated and integrated massless vertex operators for the super-Yang-Mills states as follows

$$V = \lambda^\alpha A_\alpha(x, \theta) \tag{1.44}$$

$$U(z) = \partial \theta^\alpha A_\alpha(x, \theta) + A_m(x, \theta) \Pi^m + d_\alpha W^\alpha(x, \theta) + \frac{1}{2} N_{mn} \mathcal{F}^{mn}(x, \theta), \tag{1.45}$$

where the superfields A_α, A_m, W^α and $\mathcal{F}_{mn}$ describe the super-Yang-Mills theory in $d = 10$ and have the following θ expansion

$$A_\alpha(x, \theta) = \frac{1}{2} a_m (\gamma^m \theta)_\alpha - \frac{1}{3} (\xi \gamma_m \theta)(\gamma^m \theta)_\alpha - \frac{1}{32} F_{mn} (\gamma_p \theta)_\alpha (\theta \gamma^{mnp} \theta) + \ldots \tag{1.46}$$

‖This approach is known as the minimal pure spinor formalism.

$$A_m(x,\theta) = a_m - (\xi\gamma_m\theta) - \frac{1}{8}(\theta\gamma_m\gamma^{pq}\theta)F_{pq} + \frac{1}{12}(\theta\gamma_m\gamma^{pq}\theta)(\partial_p\xi\gamma_q\theta) + \ldots$$

$$W^\alpha(x,\theta) = \xi^\alpha - \frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn} + \frac{1}{4}(\gamma^{mn}\theta)^\alpha(\partial_m\xi\gamma_n\theta) + \frac{1}{48}(\gamma^{mn}\theta)^\alpha(\theta\gamma_n\gamma^{pq}\theta)\partial_m F_{pq} + \ldots$$

$$\mathcal{F}_{mn}(x,\theta) = F_{mn} - 2(\partial_{[m}\xi\gamma_{n]}\theta) + \frac{1}{4}(\theta\gamma_{[m}\gamma^{pq}\theta)\partial_{n]}F_{pq} + \ldots,$$

where $a_m(x) = e_m \mathrm{e}^{ik\cdot x}$, $\xi^\alpha(x) = \chi^\alpha \mathrm{e}^{ik\cdot x}$ and $F_{mn} = 2\partial_{[m}a_{n]}$.

In the RNS formalism the unintegrated vertex operator satisfies $QU = \partial V$, as one can check by recalling that $U = \{\oint b, V\}$ and $T = \{Q, b\}$. The proof then follows from the Jacobi identity

$$QU = [Q, \{\oint b, V\}] = -[V, \{Q, \oint b\}] - [\oint b, \{V, Q\}] = \partial V \tag{1.47}$$

because the cohomology condition requires $\{V, Q\} = 0$ and the conformal weight zero of V implies $[\oint T, V] = \partial V$.
In the pure spinor formalism the integrated vertex (1.45) also satisfies (1.47). To see this it is necessary to use the OPE's of the previous subsection and

$$d_\alpha(y) f(x(z), \theta(z)) \to \frac{D_\alpha f(x(z), \theta(z))}{y - z}, \tag{1.48}$$

where $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\theta\gamma^m)_\alpha \frac{\partial}{\partial x^m}$ is the supersymmetric derivative, and obviously also the equations of motion for the SYM superfields given by

$$D_{(\alpha}A_{\beta)} = \gamma^m_{\alpha\beta}A_m,\ D_\alpha A_m = (\gamma_m W)_\alpha + k_m A_\alpha,\ D_\alpha W^\beta = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathcal{F}_{mn}.$$

The unintegrated vertex operator satisfies $QV = 0$ if the superfield A_α is on-shell, i.e if equation

$$\gamma^{\alpha\beta}_{mnpqr}\left(D_\alpha A_\beta + D_\beta A_\alpha + \{A_\alpha, A_\beta\}\right) = 0. \tag{1.49}$$

is obeyed,

$$QV = \oint \lambda^\alpha(z) d_\alpha(z) \lambda^\beta(w) A_\beta(x,\theta) = \lambda^\alpha\lambda^\beta D_\alpha A_\beta = 0, \tag{1.50}$$

where we used that $\lambda^\alpha\lambda^\beta = (1/3840)(\lambda\gamma^{mnpqr}\lambda)\gamma^{\alpha\beta}_{mnpqr}$ for pure spinors λ^α.

For details about the pure spinors vertex operator we can see [47][43][44].

1.3 Organization of the Thesis

This thesis is based on the papers [32][33][34].

The chapter 2 discusses briefly a general beta-gamma system [18]. We show that the first Pontryagin class of the target space is the obstruction for a global definition of the model. We also show that the first Chern class of the target space is the obstruction to get a global stress tensor $T(z)$. We analyze the particular case when the target space is the pure spinor space.We discuss the geometry of the projective pure spinor space in order to show that pure spinor space is anomaly free when the point $\lambda = 0$ is removed from this space.

In the chapter 3, following the previous ideas, we make a new proposal for the lowering picture changing operators. We discuss the restriction that must be imposed on these operators in order to have a well defined tree level scattering amplitude. This restriction will result in the condition that the integration cycles go around the anomalous point of the theory $\lambda^\alpha = 0$.
We will compute the tree-level scattering amplitude. We start by formally defining the integration contours. Then, we proceed to write the amplitude using the projective pure spinor coordinates. Using these coordinates we analyze the poles structure and express the result of the scattering amplitude in terms of the *degree* of the projective pure spinor space, which is useful to relate the minimal and non-minimal formalism.
we establish also a direct relation between pure spinor scattering amplitudes and Green's functions for massless scalar fields in ten dimensions.
Finally we comment about what should be done in order to have a genus g formulation for the scattering amplitude.

In the chapter 4 we justify the normalization of the path integral measures. After, we will compute the integral on the pure spinors space, we suggest the reader check the appendix C.1 beforehand. We arrive to the following result

$$\boxed{\int_{\mathcal{O}(-1)}[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]\, e^{-a\,\lambda\overline{\lambda}} = (2\pi)^{11}(a^8\cdot 12\cdot 5)^{-1}\,, \qquad a\in\mathbb{R}^+}$$

where $\mathcal{O}(-1)$ is the line bundle blow-up at the origin with base space $SO(10)/U(5)$.

The computations of the three- and four-point amplitudes at tree-level are performed to show that the conventions match the RNS ones of [66]. Then we use the very same machinery of the tree-level computation to obtain also the full supersymmetric one- and two-loop amplitudes – including their precise coefficients – which explicitly shows that with the pure spinor formalism those coefficients follow directly from a first principles computation. But we find disagreement with the RNS results reported by [66]. We argue that [66] forgot the two factors of 1/2 from the GSO projection in the left- and right-moving sectors in their measure. This observation will also explain the $1/2^4$ mismatch at two-loops.

The last chapter presents the relationship between the minimal and non-minimal formalisms by using the Čech-Dolbeault language. We prove also that the scattering amplitude, in the minimal pure spinor formalism with the new PCO's, is independent of the constant spinors C^I's. First we consider the simplest non-trivial case, i.e pure spinor in four dimensions. Then, we proceed to consider the ten dimensional case. The two cases are studied differently; in four dimensions is straightforward and it teaches us what should be done. Extending the four dimensional proof to ten dimensions would be difficult, so we present a more elegant demonstration using the Čech-Dolbeault language.

Finally we give some conclusions.
In the appendix A we define two mathematical concepts, curvature on holomorphic vector bundle and spectral sequences, which are very useful.

The appendix B contains several simple examples cited through the thesis, as well as some demonstration of statements.

In the appendix C we apply the tools used to compute the integral in the pure spinors space in lower dimensions ($D = 2n < 10$). The aim is to be more familiar with the concepts of algebraic geometry involved in the computation.

We present also the detailed covariant computation of the two-loop kinematic factor. This appendix can be regarded as a fully $SO(10)$-covariant proof of the 2-loop equivalence between the non-minimal and minimal pure spinor Kinematic factors, and is analogous to the covariant proof of [67] for the 1-loop case.
The appendix C.3 is devoted to proving a formula which is used to rewrite the two-loop amplitude in terms of integrals in the period matrix instead of in the Teichmüller parameters.

Chapter 2

Beta-Gamma System and The Pure Spinor Space

The main goal of this chapter is to find the potential anomalies of the pure spinor action

$$S_{PS} = \int \mathrm{d}^2 z \; \omega_\alpha \overline{\partial} \lambda^\alpha, \tag{2.1}$$

which is a beta-gamma system [18][19] on pure spinor space. However before to work on the pure spinor space it is useful to study a general beta-gamma system and its properties.

We show that in this system the conformal invariance may be broken unless the target space has certain topological features. We will also show the coordinate transformations relating different coordinate system on the target space may act non-trivially on the operator fields of the system and moreover there are obstructions for gluing the patches of the target.

The chapter is organized as follows. The section 2.1 gives a brief classical and quantum introduction to the beta-gamma system. Quantum effects about this system lead us a wrong definition of the fields, which must be corrected. It is presented in the section 2.2. Using this result, in 2.3, we show the obstruction to get a global conformal invariant system is given by the first Chen class of the target space. In order to obtain the obstruction for the global definition of the fields of the theory then in the section 2.4 we give a short introduction to the Čech language. In the section 2.5 we find this obstruction and show it is the topological invariant called the first Pontryagin class of the target space.

The following sections are devoted to pure spinors. The section 2.6 argues why the whole pure spinor space has Chern and Pontryagin anomalies. We use the ghost anomaly and the holomorphic top form to show that. This means the pure spinor space must be restricted to a subspace such that the anomalies vanish. In fact, in the following two section, 2.7 and 2.8, we show that removing the singular point of the pure spinor space (the origin) the beta-gamma system is well defined, i.e it is anomaly free. We compute the de-Rham cohomology of this space, in a rigorous way, and show the first Chern class and the first Pontryagin class are trivial.

This chapter is based on the Nekrasov's lectures [18].

2.1 Introduction to the Beta-Gamma System

First of all we make a simple classical analysis about the beta-gamma system and later we will find its anomalies.

Let γ^i be the local holomorphic coordinates over the complex manifold M of complex dimension d, i.e $i = 1, ..., d$ and let $\beta = \beta_i \mathrm{d}\gamma^i$ be a $(1,0)$-form over M. Then the beta-gamma system (more precisely, its chiral part) is given by the action

$$S_{\beta\gamma} = \frac{1}{2\pi}\int_{\Sigma} \mathrm{d}^2 z \beta_i \bar{\partial}\gamma^i. \tag{2.2}$$

where Σ is a Riemann surface embeds on M. The field γ^i has conformal weight (0,0) and the field β_i has conformal weight (1,0), so the previous action is conformal invariant at classical level.
This action is invariant by the symmetries:
1. Diffeomorphisms

$$\begin{aligned} \delta\gamma^i &= \epsilon V^i(\gamma), \\ \delta\beta_i &= -\epsilon\beta_j \partial_i V^j(\gamma). \end{aligned} \tag{2.3}$$

2.

$$\begin{aligned} \delta\gamma^i &= 0, \\ \delta\beta_i &= \epsilon(\partial_i B_j(\gamma) - \partial_j B_i(\gamma))\partial\gamma^j. \end{aligned} \tag{2.4}$$

where ϵ is a constant parameter, V^i is a holomorphic vector and B_i is a (1-0)-form on M.

The equations of motion are simple

$$\begin{aligned} \bar{\partial}\gamma^i &= 0, \;\Rightarrow\; \gamma^i = \gamma^i(z) \quad , \\ \bar{\partial}\beta_i &= 0, \;\Rightarrow\; \beta_i = \beta_i(z) \quad . \end{aligned} \tag{2.5}$$

The stress tensor is given by

$$T(z) = \beta_i(z)\partial\gamma^i(z) \quad . \tag{2.6}$$

and the conserved currents of the previous symmetries are
1. Diffeomorphisms

$$J_V = \beta_i(z) V^i(\gamma(z)). \tag{2.7}$$

2.

$$C_B = B_i(\gamma(z))\partial\gamma^i. \tag{2.8}$$

2.1.1 OPE's

From the action (2.2) the OPE's of the fundamental fields are trivial

$$\begin{aligned} \gamma^i(z)\beta_j(w) &\sim \frac{\delta^i_j}{z-w} , \\ \gamma^i(z)\gamma^j(w) &= reg , \\ \beta_i(z)\beta_j(w) &= reg . \end{aligned} \tag{2.9}$$

So, using these OPE's the currents have the form

$$\begin{aligned} J_V =: \beta_i V^i(\gamma) : (z) &\equiv \lim_{\epsilon\to 0}\left[\beta_i(z+\epsilon)V^i(\gamma(z)) - \widehat{\beta_i(z+\epsilon)V^i}(\gamma(z))\right] \\ &= \lim_{\epsilon\to 0}\left[\beta_i(z+\epsilon)V^i(\gamma(z)) + \frac{\partial_i V^i(\gamma(z))}{\epsilon}\right] , \end{aligned} \tag{2.10}$$

$$C_B =: B_i(\gamma)\partial\gamma^i : (z) = B_i(\gamma(z))\partial\gamma^i(z). \tag{2.11}$$

Note that the current J_V, which generates the diffeomorphism symmetry, is not invariant by diffeomorphism because the term $\partial_i V^i(\gamma(z))$ is not a scalar. It also is easy to see that

$$C_B(z+\epsilon)C_D(z) = reg ,$$

and that the currents J_V and C_B do not commute

$$\begin{aligned} J_V(z+\epsilon)C_B(z) &\sim -\frac{(V^i\partial_i B_j + \partial_j V^i B_i)\partial\gamma^j(z)}{\epsilon} - \frac{V^i B_i(z)}{\epsilon^2} , \\ &\sim -\frac{i_V B(z)}{\epsilon^2} - \frac{C_{L_V B}(z)}{\epsilon} , \end{aligned} \tag{2.12}$$

where i_V is the contraction operator and L_V is the Lie derivative [18]. The last term was expected since J_V is the generator of the diffeomorphism and the infinitesimal change of a p-form is its Lie derivative. The first term is the central term.
In addition the algebra of the current J_V is not closed

$$\begin{aligned} J_{V_a}(z+\epsilon)J_{V_b}(z) \sim & -\frac{1}{2}\frac{\left(\partial_j V^i_a \partial_i V^j_b(z+\epsilon) + \partial_j V^i_a \partial_i V^j_b(z)\right)}{\epsilon^2} + \frac{1}{2}\frac{\beta_i\left[V_a, V_b\right]^i(z)}{\epsilon} \\ & + \frac{1}{2}\frac{\left(\partial_j V^i_a \partial_k \partial_i V^j_b - \partial_j V^i_b \partial_k \partial_i V^j_a\right)\partial\gamma^k(z)}{\epsilon} \\ \sim & -\frac{1}{2}\frac{\Sigma_{ab}(z+\epsilon) + \Sigma_{ab}(z)}{\epsilon^2} - \frac{J_{[V_a,V_b]}(z)}{\epsilon} + \frac{C_{\Omega_{as}}(z)}{\epsilon} , \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \mathcal{V}_a &= \partial_i V^j_a , \\ \Sigma_{ab} &= tr\mathcal{V}_a\mathcal{V}_b , \\ \Omega_{ab} &= \frac{1}{2}tr\left(\mathcal{V}_a \mathrm{d}\mathcal{V}_b - \mathcal{V}_b \mathrm{d}\mathcal{V}_a\right) \text{ (d is exterior derivative on } M) . \end{aligned}$$

Nevertheless, we can combine the symmetries J_V and C_B to get a closed algebra, i.e we define

$$v_* \equiv V_* + B_* \ \in \ TM \oplus TM^* \quad , \quad \mathcal{O}_{v_a} = J_{V_a} + C_{B_a} \quad \text{and} \quad \mathcal{O}_{v_b} = J_{V_b} + C_{B_b} \ ,$$

so the OPE's between two currents $\mathcal{O}$ is

$$\mathcal{O}_{v_a}(z+\epsilon)\mathcal{O}_{v_b}(z) = J_{V_a}(z+\epsilon)J_{V_b}(z) + J_{V_a}(z+\epsilon)C_{B_b}(z) + C_{B_a}(z+\epsilon)J_{V_a}(z) \ .$$

For to do the computation it is necessary the following OPE

$$C_B(z+\epsilon)J_V(z) \sim -\frac{1}{2}\frac{i_V B(z+\epsilon)}{\epsilon^2} - \frac{1}{2}\frac{i_V B(z)}{\epsilon^2} - \frac{1}{2}\frac{[\mathrm{d}(i_V B)]_k \partial\gamma^k(z)}{\epsilon} + \frac{(L_V B)_j \partial\gamma^j(z)}{\epsilon} \ ,$$

so we have

$$\begin{aligned}
\mathcal{O}_{v_a}(z+\epsilon)\mathcal{O}_{v_b}(z) \sim \ & - \ \frac{1}{\epsilon^2}\left\{\frac{1}{2}\left[\Sigma_{ab}(z+\epsilon) + i_{V_a}B_b(z+\epsilon) + i_{V_b}B_a(z+\epsilon)\right]\right. \\
& \qquad \left. + \frac{1}{2}\left[\Sigma_{ab}(z) + i_{V_a}B_b(z) + i_{V_b}B_a(z)\right]\right\} \\
& - \ \frac{1}{\epsilon}\left\{J_{[V_a,V_b]}(z) + [L_{V_a}B_b - L_{V_b}B_a - \frac{1}{2}\mathrm{d}(i_{V_a}B_b - i_{V_b}B_a)]_k \partial\gamma^k(z)\right\} \\
& + \ \frac{1}{\epsilon}C_{\Omega_{ab}}(z) \\
\sim \ & - \ \frac{g(v_a,v_b)(z+\epsilon) + g(v_a,v_b)(z)}{\epsilon^2} - \frac{\mathcal{O}_{[[v_a,v_b]]}(z) - C_{\Omega_{ab}}(z)}{\epsilon}
\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
2g(v_a,v_b) &= \Sigma_{ab} + i_{V_a}B_b + i_{V_b}B_a \ , \\
[[v_a,v_b]] &= [V_a,V_b] + L_{V_a}B_b - L_{V_b}B_a - \frac{1}{2}\mathrm{d}(i_{V_a}B_b - i_{V_b}B_a) \quad .
\end{aligned}$$

The double bracket is known as the Courant bracket, which is the natural bracket over a space with a generalized complex structures. This bracket was expected since that the local space (β_i, γ^i) is a complex manifold with a symplectic structure (because β_i is the conjugate momenta of γ^i) and therefore this space is locally a generalized complex manifold [13].
The level of the current algebra $g(v_a, v_b)$ is the standard symmetric bilinear form on $TM \oplus TM^*$ with a quantum correction Σ_{ab} [13]. The terms Σ_{ab} and Ω_{ab} in the OPE of the operators $\mathcal{O}_{v_a}\mathcal{O}_{v_b}$ does not have an obvious geometric interpretation, because these correction do not transform covariantly under the coordinates changes. However, this is in accordance with the non-trivial nature of the β-field, which do not transform naively. We now turn to the exact determination of the transformation properties of the β-fields.

2.2 Coordinate Transformations

Let γ^i be the coordinates over the open ball $U \subset M$ and $\tilde{\gamma}^a$ the coordinates over $V \subset M$ such that $U \cap V \neq \emptyset$. So in the intersection we have a holomorphic local transformation $f : \tilde{\gamma} \to \gamma$. Since the field β is a (1,0)-form on M then it transform in the following way (classically)

$$\tilde{\beta}_a = \beta_i g^i_a(\gamma), \tag{2.15}$$

where

$$g^i_a(\gamma) = \frac{\partial \gamma^i}{\partial \tilde{\gamma}^a} , \qquad (g^{-1})^a_i = g^a_i(\gamma) = \frac{\partial \tilde{\gamma}^a}{\partial \gamma^i} . \tag{2.16}$$

To quantum level the expression (2.15) has the same problem as the current J_V. Therefore to get a transformation of the β-field invariant by diffeomorphism it is necessary to add a term with conformal weight (1,0). So the more general transformation is

$$\tilde{\beta}_a = \beta_i g^i_a(\gamma) + B_{ai}(\gamma)\partial\gamma^i, \tag{2.17}$$

where we must find $B_{ai}(\gamma)$.
It is trivial to verify that the OPE's between $\tilde{\gamma}^a\tilde{\gamma}^b$ and $\tilde{\beta}_a\tilde{\gamma}^b$ are the same as (2.9) therefore we must compute $\tilde{\beta}_a\tilde{\beta}_b$. This computation is similar to (2.14) and so we obtain

$$\begin{aligned} \tilde{\beta}_a(z+\epsilon)\tilde{\beta}_b(z) \sim \; & - \frac{1}{2}\frac{\left[tr(\mathcal{G}_a\mathcal{G}_b) + i_{g_a}B_b + i_{g_b}B_a\right](z)}{\epsilon^2} \\ & - \frac{\left[-\Omega_{ab} + i_{g_a}\mathrm{d}B_b - i_{g_b}\mathrm{d}B_a + \mathrm{d}\mu_{ab}\right]_k \partial\gamma^k(z)}{\epsilon} \\ & + \frac{1}{2}\frac{\partial_k\left[tr(\mathcal{G}_a\mathcal{G}_b) + i_{g_a}B_b + i_{g_b}B_a\right]\partial\gamma^k(z)}{\epsilon} , \end{aligned} \tag{2.18}$$

where:

$$g_a = g^i_a\partial_i = \partial_a , \tag{2.19}$$
$$[\mathcal{G}_a]^i_k = \partial_k g^i_a , \tag{2.20}$$
$$\mu_{ab} = i_{g_a}B_b - i_{g_b}B_a , \tag{2.21}$$
$$\Omega_{ab} = \frac{1}{2}tr(\mathcal{G}_a\mathrm{d}\mathcal{G}_b - \mathcal{G}_b\mathrm{d}\mathcal{G}_a) . \tag{2.22}$$

This OPE is zero to get a well defined theory, this means we have the followings constrains

$$tr(\mathcal{G}_a\mathcal{G}_b) + i_{g_a}B_b + i_{g_b}B_a = 0 , \tag{2.23}$$

$$-\Omega_{ab} + i_{g_a}\mathrm{d}B_b - i_{g_b}\mathrm{d}B_a + \mathrm{d}\mu_{ab} = 0 . \tag{2.24}$$

Note that $d(d+1)/2$ conditions come from (2.23) and (2.24) gives $d(d-1)/2$ conditions. So we have d^2 conditions ($dim_{\mathbb{C}}M = d$) and d^2 variables (B_{ai}).

Now, we define

$$\sigma_{ab} = i_{g_a} B_b + i_{g_b} B_a \ \ , \tag{2.25}$$

then from (2.23) we have

$$\sigma_{ab} = -tr(\mathcal{G}_a \mathcal{G}_b) = -\partial_k g_a^i \partial_i g_a^k \ \ , \tag{2.26}$$

and

$$i_{g_b} B_a = g_b^i B_{ai} = \frac{1}{2}(\sigma_{ab} - \mu_{ab}) \ \ . \tag{2.27}$$

So we get

$$B_{aj} = \frac{1}{2}(\sigma_{ab} - \mu_{ab})(g^{-1})_j^b \ \ . \tag{2.28}$$

Notice that we can write B_{aj} like a (1,0)-form in the coordinates γ^i

$$\begin{aligned} B_{aj} &= B_{ak}\mathrm{d}\gamma^k(\partial_j) = \frac{1}{2}(\sigma_{ab} - \mu_{ab})(g^{-1})_k^b \mathrm{d}\gamma^k(\partial_j) \ \ , \\ B_a &= B_{ak}\mathrm{d}\gamma^k = \frac{1}{2}(\sigma_{ab} - \mu_{ab})(g^{-1})_k^b \mathrm{d}\gamma^k = \frac{1}{2}(\sigma_{ab} - \mu_{ab})\mathrm{d}\tilde{\gamma}^b \ . \end{aligned} \tag{2.29}$$

Now we must find μ_{ab} to determine B_a. For to do that we multiply (2.24) by g_c^i

$$2 \ g_c^i \left[i_{g_a}\mathrm{d}B_b - i_{g_a}\mathrm{d}B_a + \frac{1}{2}\mathrm{d}\mu_{ab} \right]_i = tr \ g_c^i(\mathcal{G}_a \partial_i \mathcal{G}_b - \mathcal{G}_b \partial_i \mathcal{G}_a) \ \ . \tag{2.30}$$

Using the identity

$$L_X = i_X \mathrm{d} + \mathrm{d} i_X \ , \ X \in TM \ \ ,$$

where L_X is the Lie derivative then (2.30) becomes

$$L_{g_c}\mu_{ab} + 2 \ L_{g_c}\mu_{ba} + 2 \ i_{g_c} L_{g_a} B_b - 2 \ i_{g_c} L_{g_b} B_a = tr \ (\mathcal{G}_a L_{g_c} \mathcal{G}_b - \mathcal{G}_b L_{g_c} \mathcal{G}_a) \ \ . \tag{2.31}$$

Permuting the labels a, b y c and adding the expressions we obtain

$$\begin{aligned} &L_{g_c}\mu_{ba} + L_{g_a}\mu_{cb} + L_{g_b}\mu_{ac} + 2 \ i_{g_c} L_{g_a} B_b - 2 \ i_{g_c} L_{g_b} B_a + 2 \ i_{g_a} L_{g_b} B_c \\ &-2 \ i_{g_a} L_{g_c} B_b + 2 \ i_{g_b} L_{g_c} B_a - 2 \ i_{g_b} L_{g_a} B_c = tr \ \left(\mathcal{G}_a L_{g_{[c}} \mathcal{G}_{b]} + \mathcal{G}_b L_{g_{[a}} \mathcal{G}_{c]} + \mathcal{G}_c L_{g_{[b}} \mathcal{G}_{a]} \right) \end{aligned} \tag{2.32}$$

where $L_{g_{[b}} \mathcal{G}_{a]} = L_{g_b}\mathcal{G}_a - L_{g_a}\mathcal{G}_b$. Finally to get μ_{ab} it is useful to prove the identities

$$1) \quad L_{g_{[a}} \mathcal{G}_{b]} = [\mathcal{G}_b, \mathcal{G}_a], \qquad 2) \quad g^{-1} L_{g_a} g = \mathcal{G}_a,$$

$$3) \quad i_{g_b} L_{g_c} B_a - i_{g_c} L_{g_b} B_a = L_{g_c} i_{g_b} B_a - L_{g_b} i_{g_c} B_a.$$

1.

$$\begin{aligned} (L_{g_{[a}} \mathcal{G}_{b]})_i^j &= g_a^k \partial_k \partial_i g_b^j - g_b^k \partial_k \partial_i g_a^j \ , \\ &= \partial_i (g_a^k \partial_k g_b^j - g_b^k \partial_k g_a^j) + [\mathcal{G}_b, \mathcal{G}_a]_i^j \ , \\ &= \partial_i (\partial_a g_b^j - \partial_b g_a^j) + [\mathcal{G}_b, \mathcal{G}_a]_i^j \ , \\ &= \partial_i (\partial_a g_b^j - \partial_a g_b^j) + [\mathcal{G}_b, \mathcal{G}_a]_i^j \ , \\ &= [\mathcal{G}_b, \mathcal{G}_a]_i^j \ . \end{aligned}$$

2.

$$\begin{aligned} \left[g^{-1}L_{g_a}g\right]_i^j &= (g^{-1})_i^b g_a^k \partial_k g_b^j = (g^{-1})_i^b \partial_a \partial_b \gamma^j \ , \\ &= (g^{-1})_i^b \partial_b \partial_a \gamma^j = (g^{-1})_i^b g_b^k \partial_k g_a^j \ , \\ &= \partial_i g_a^j = [\mathcal{G}_a]_i^j \ , \end{aligned}$$

and we can write

$$\mathcal{G}_a = g^{-1}\mathrm{d}g\ (g_a) = g^{-1}\mathrm{d}g\ (\partial_a)\ . \tag{2.33}$$

3.

$$\begin{aligned} i_{g_b}L_{g_c}B_a - i_{g_c}L_{g_b}B_a &= g_b^i g_c^k \partial_k B_{ai} + g_b^i(\partial_i g_c^k)B_{ak} - g_c^i g_b^k \partial_k B_{ai} - g_c^i(\partial_i g_b^k)B_{ak} \ , \\ &= g_b^i \partial_c B_{ai} + (\partial_b g_c^k)B_{ak} - g_c^i \partial_b B_{ai} - (\partial_c g_b^k)B_{ak} \ , \\ &= \partial_c(g_b^i B_{ai}) - \partial_b(g_c^i B_{ai}) = g_c^k \partial_k(g_b^i B_{ai}) - g_b^k \partial_k(g_c^i B_{ai}) \ , \\ &= L_{g_c} i_{g_b} B_a - L_{g_b} i_{g_c} B_a \ . \end{aligned}$$

Applying 1 and 3 to (2.32) we get

$$\begin{aligned} L_{g_c}\mu_{ba} + L_{g_a}\mu_{cb} + L_{g_b}\mu_{ac} &= \frac{1}{3}tr\ \left(\mathcal{G}_a[\mathcal{G}_b,\mathcal{G}_c] + \mathcal{G}_b[\mathcal{G}_c,\mathcal{G}_a] + \mathcal{G}_c[\mathcal{G}_a,\mathcal{G}_b]\right) \\ &= tr\ \left(\mathcal{G}_a[\mathcal{G}_b,\mathcal{G}_c]\right) . \end{aligned} \tag{2.34}$$

Since that μ_{ab} is antisymmetric in a, b then we can write μ like a (2,0)-form on $V \subset M$

$$\mu = \frac{1}{2}\mu_{ab}\,\mathrm{d}\tilde{\gamma}^a \wedge \mathrm{d}\tilde{\gamma}^b \ . \tag{2.35}$$

Note also that

$$L_{g_c}\mu_{ab} = g_c^i \partial_i \mu_{ab} = \partial_c \mu_{ab} \ , \tag{2.36}$$

so we have

$$L_{g_c}\mu_{ba} + L_{g_a}\mu_{cb} + L_{g_b}\mu_{ac} = \mathrm{d}\mu\,(\partial_c, \partial_b, \partial_a)\ . \tag{2.37}$$

From the identity 2 it is easy to see

$$tr\ \left(\mathcal{G}_a[\mathcal{G}_b,\mathcal{G}_c]\right) = -\frac{1}{3}tr\ (g^{-1}\mathrm{d}g)^3\,(\partial_c,\partial_b,\partial_a)\ , \tag{2.38}$$

therefore μ is determined by the equation

$$\mathrm{d}\mu = -\frac{1}{3}tr\ (g^{-1}\mathrm{d}g)^3 \ . \tag{2.39}$$

Clearly μ is determined up to (2,0)-exact form $\mathrm{d}f$, where f is a (1,0)-form. Finally we get the transformation of β with its quantum correction

$$\begin{aligned} \tilde{\beta}_a &= \beta_i g_a^i + B_{ai}\partial\gamma^i \\ &= \beta_i g_a^i - \frac{1}{2}\left(\partial_j g_a^i \partial_i g_b^j\right)\partial\tilde{\gamma}^b + \frac{1}{2}\mu_{ab}\partial\tilde{\gamma}^b . \end{aligned} \tag{2.40}$$

This transformation is the key to find the obstructions in order to get a well defined beta-gamma system. In the next subsection we find the obstruction to get a global stress tensor.

2.3 The Stress Tensor

The stress tensor is the generator of the conformal symmetry. The beta-gamma system has conformal anomaly, in the particular case of the pure spinor variables the central charge is $c = 22$. This central charge is necessary to cancel the contributions which come from the others fields of the theory such that the total charge is zero. So, we are not interested in the conformal anomaly, actually our focus is to know how the stress tensor changes under a coordinate transformation on M.
Classically, the stress tensor is given by

$$T(z) = \beta_i \partial\gamma^i \ . \tag{2.41}$$

In the quantum level we have

$$T(z) =: \beta_i \partial\gamma^i : (z) \equiv \lim_{\epsilon\to 0}\left(\beta_i(z+\epsilon)\partial\gamma^i(z) + \frac{d}{\epsilon^2}\right) \ . \tag{2.42}$$

Doing a coordinate transformation $\gamma \to \tilde{\gamma}$ and using (2.40) the stress tensor changes in the following way

$$\begin{aligned}
\tilde{T} &= :\tilde{\beta}_a \partial\tilde{\gamma}^a :\equiv \lim_{\epsilon\to 0}\left(\tilde{\beta}_a(z+\epsilon)\partial\tilde{\gamma}^a(z) + \frac{d}{\epsilon^2}\right) \ , \\
&= \lim_{\epsilon\to 0}\left(\beta_i g^i_a(z+\epsilon) g^a_j(z)\partial\gamma^j(z) + B_{ai}\partial\gamma^i(z+\epsilon) g^a_j(z)\partial\gamma^j(z) + \frac{d}{\epsilon^2}\right) \ , \\
&= \lim_{\epsilon\to 0}\left[\lim_{\delta\to 0}\beta_i(z+\epsilon+\delta) g^i_a(z+\epsilon) g^a_j(z)\partial\gamma^j(z) + \frac{\partial_i g^i_a(z+\epsilon)}{\delta} g^a_j(z)\partial\gamma^j(z)\right. \\
&\qquad \left. + B_{ai}\partial\gamma^i g^a_j \partial\gamma^j(z) + \frac{d}{\epsilon^2}\right] \ , \\
&= \lim_{\epsilon\to 0}\left[\lim_{\delta\to 0} :\beta_i g^i_a g^a_j \partial\gamma^j : (z) - \frac{\partial_i g^i_a(z+\epsilon)}{\delta} g^a_j(z)\partial\gamma^j(z) + \frac{\partial_i g^i_a(z+\epsilon)}{\delta} g^a_j(z)\partial\gamma^j(z)\right. \\
&\qquad \left. - \frac{g^i_a(z+\epsilon)\partial_i g^a_j(z)\partial\gamma^j(z)}{\epsilon+\delta} - \frac{g^i_a(z+\epsilon) g^a_j(z)\delta^j_i}{(\epsilon+\delta)^2} + B_{ai}\partial\gamma^i g^a_j\partial\gamma^j(z) + \frac{d}{\epsilon^2}\right] \\
&= \lim_{\epsilon\to 0}\left[:\beta_i\partial\gamma^i : (z) - \frac{g^i_a(z)\partial_j g^a_i(z)\partial\gamma^j(z)}{\epsilon} - \partial g^i_a \partial_j g^a_i \partial\gamma^j(z) - \frac{d}{\epsilon^2} - \frac{\partial g^i_a g^a_i}{\epsilon} - \frac{1}{2}\partial^2 g^i_a g^a_i\right. \\
&\qquad \left. + B_{ai}\partial\gamma^i g^a_j \partial\gamma^j + \frac{d}{\epsilon^2}\right] \\
&= T - \partial g^i_a \partial g^a_i - \frac{1}{2}\partial^2 g^i_a g^a_i + B_{ai} g^a_j \partial\gamma^i\partial\gamma^j \\
&= T - \partial g^i_a \partial g^a_i - \frac{1}{2}\partial^2 g^i_a g^a_i - \frac{1}{2}\partial_j g^i_a \partial_i g^j_b \partial\tilde{\gamma}^a\partial\tilde{\gamma}^b \\
&= T - \frac{1}{2} g^a_i \partial^2 g^i_a - \partial g^i_a \partial g^a_i + \frac{1}{2}\partial g^a_i \partial g^i_a \\
&= T - \frac{1}{2} g^a_i \partial^2 g^i_a - \frac{1}{2}\partial g^i_a \partial g^a_i \\
&= T - \frac{1}{2}\partial^2 Log\,(det\, g^i_a) \ .
\end{aligned}$$

Clearly the stress tensor is not invariant by a local coordinate transformation, so it is not globally well defined. In order to obtain an invariant tensor then we should introduce a new term such that its transformation cancels $\frac{1}{2}\partial^2 Log\,(det\, g^i_a)$. This term is well known and it is given by

$$T_D = \frac{1}{2}\partial^2\, Log\, \omega(\gamma) \ \ , \tag{2.43}$$

where $\omega(\gamma)$ is the function

$$\Omega = \omega(\gamma)\mathrm{d}\gamma^1 \wedge ... \wedge \mathrm{d}\gamma^d \ \ . \tag{2.44}$$

and Ω is the holomorphic top for over M. Note that if M is a Kalher manifold and Ω exists then M is a Calibi-Yau manifold.
This term was also found by Berkovits [16] from the condition that the Lorentz currents must be primary fields.
Now under a local transformation of coordinates the term $\omega(\gamma)$ becomes

$$\tilde{\omega}(\tilde{\gamma}) = \omega(\gamma)\, det\, g^i_a \ \ , \tag{2.45}$$

therefore T_D turns

$$\tilde{T}_D = \frac{1}{2}\partial^2\, Log\, \tilde{\omega}(\gamma) = T_D + \frac{1}{2}\partial^2\, Log\,(det\, g^i_a) \ \ . \tag{2.46}$$

This fact implies that the total stress tensor

$$T = \beta_i \partial\gamma^i + \frac{1}{2}\partial^2 Log\, \omega(\gamma) \tag{2.47}$$

is invariant under local transformations.
However, in order to obtain a global stress tensor it is necessary that the holomorphic top form Ω is well defined in whole space M, this means it must be global and non-zero section of the canonical line bundle "$\bigwedge^d TM^*$". The obstruction to get this section is the first Chern class of "$\bigwedge^d TM^*$", which is minus the first Chern class of the holomorphic tangent bundle [94]

$$c_1(\bigwedge^d TM^*) = -c_1(TM). \tag{2.48}$$

Finally, we can say the stress tensor of the beta-gamma system is well defined if the first Chern class of the Tangent bundle is zero.

Before to find the diffeomorphism anomaly it is useful to give a shor review about the Čech cohomology.

2.4 Čech Language

In this section we give a simple introduction to the Čech formalism which turns out to be very useful for this work, for instance to get the diffeomorphism anomaly, to relate the minimal and non-minimal pure spinor formalism from the tree level scattering amplitude as we will show in the chapter 5, and to check the BRST, Lorentz and SUSY symmetries in the section 3.5.

Let $\underline{U} = \{U_I\}$ be a finite cover of a complex manifold M, i.e

$$M = \bigcup_I U_I \tag{2.49}$$

then a Čech k-cochain, denoted by $\psi_{I_1 \dots I_{k+1}}$, is an holomorphic $(p, 0)$-form in the intersection $U_{I_1 \dots I_{k+1}} = U_{I_1} \cap U_{I_2} \cap \dots \cap U_{I_{k+1}}$ (the generalization to the set of (p, q)-forms or any other abelian group is trivial). I.e $\psi_{I_1 \dots I_{k+1}} \in \Omega^p(U_{I_1 \dots I_{k+1}})$ where $\Omega^p(U)$ is the abelian group of the holomorphic p-forms over U. We choose the abelian group of p-forms because it will be one of the most important group used in this thesis.
The Čech cochains must be antisymmetric in the Čech labels, for instance,
$\psi_{I_1 .. I_i \dots I_j \dots I_{k+1}} = -\psi_{I_1 .. I_j \dots I_i \dots I_{k+1}}$. This is related to the orientation of the manifold, which in our case is the pure spinor space PS.
We define the set of the 0-cochains on $PS \setminus \{0\}$ with values in the holomorphic p-forms as

$$C^0(\underline{U}, \Omega^p) = \bigoplus_{I=1}^{11} \Omega^p(U_I). \tag{2.50}$$

Similarly, the 1-cochains are elements of the set

$$C^1(\underline{U}, \Omega^p) = \bigoplus_{I<J} \Omega^p(U_{IJ}) \tag{2.51}$$

and so on. We define the Čech operator as the map $\delta : C^k(\underline{U}, \Omega^p) \to C^{k+1}(\underline{U}, \Omega^p)$ given by

$$(\delta\psi)_{I_1 \dots . I_{k+2}} \equiv \psi_{I_2 I_3 \dots . I_{k+2}} - \psi_{I_1 I_3 \dots . I_{k+2}} + \dots + (-1)^{k+1} \psi_{I_1 I_2 \dots . I_{k+1}}. \tag{2.52}$$

It is easy to show that δ is a nilpotent operator, $\delta^2 = 0$. If $(\delta\psi)_{I_1 \dots . I_{k+2}} = 0$ then $\psi_{I_1 \dots . I_{k+1}}$ is called a cocycle and the set of all cocycles in $C^k(\underline{U}, \Omega^p)$ is an abelian subgroup denoted by $Z^k(\underline{U}, \Omega^p)$. If $\psi_{I_1 \dots . I_{k+1}} = (\delta\rho)_{I_1 \dots . I_{k+1}}$ then $\psi_{I_1 \dots . I_{k+1}}$ is called a coboundary and the set of all coboundary in $C^k(\underline{U}, \Omega^p)$ is denoted by $B^k(\underline{U}, \Omega^p)$. Clearly every coboundary is a cocycle since $\delta^2 = 0$, then we can define the coset

$$H^k(M, \Omega^p) = \frac{Z^k(\underline{U}, \Omega^p)}{B^k(\underline{U}, \Omega^p)} \tag{2.53}$$

known as the k-Čech cohomology group with values in the Abelian group of holomorphic p-forms Ω^p on M. We refer the reader to [94][29] for more details about this topic.

2.5 Diffeomorphism Anomaly

In this section we present the obstruction to get a beta-gamma system invariant by diffeomorphism using the Čech language [18].

Remember that in the section 2.2 we got

$$-\frac{1}{3}tr\,(g^{-1}\mathrm{d}g)^3 = \mathrm{d}\mu \ , \tag{2.54}$$

where $g^i_a = \frac{\partial \gamma^i}{\partial \tilde{\gamma}^a}$ is the transition function from the patches* $U_\alpha \to U_\beta$. So, (2.54) is defined in $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$.
We define $[g^i_a] \equiv g_{\alpha\beta} : U_{\alpha\beta} \to GL(d, \mathbb{C})$ and $g_{\beta\alpha} = g^{-1}_{\alpha\beta}$, so (2.54) in the Čech language is

$$-\frac{1}{3}tr\,(g_{\beta\alpha}\mathrm{d}g_{\alpha\beta})^3 = \mathrm{d}\mu_{\alpha\beta} \ .$$

Clearly the 2-form $\mu_{\alpha\beta}$ is determined up to a closed 2-form, i.e $\mu_{\alpha\beta}$ and $\mu_{\alpha\beta} + b_{\alpha\beta}$ are equivalents if and only if $\mathrm{d}b_{\alpha\beta} = 0$. Therefore $b_{\alpha\beta}$ is an element of the abelian group of holomorphic closed 2-forms, which is denoted as $\mathcal{Z}^2(U_{\alpha\beta}) \subset \Omega^2(U_{\alpha\beta})$ [94]. If in the intersection $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ the 2-form $b_{\alpha\beta}$ satisfies the cocycle condition

$$b_{\alpha\beta} + b_{\beta\gamma} + b_{\gamma\alpha} = 0 \ , \tag{2.55}$$

then it belongs to the Čech cohomology group $H^1(M, \mathcal{Z}^2)$.

Now we compute the diffeomorphism anomaly.
In order to get the diffemorphism obstruction it is sufficient to analyze a triple intersection.
Let (γ^i) be the coordinates in U_α, $(\tilde{\gamma}^a)$ in U_β, $(\widehat{\gamma}^A)$ in U_γ and $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.
We want to know how the field $\beta \in U_{\alpha\beta\gamma}$ changes following the two different paths

$$\gamma^i \to \tilde{\gamma}^a \to \widehat{\gamma}^A,$$

$$\gamma^i \to \widehat{\gamma}^A.$$

Remember that the field β transforms

$$\begin{aligned}
\tilde{\beta}_a &= \beta_i g^i_a + \frac{1}{2}tr\,(\mathcal{G}_a g\partial g^{-1}) + \frac{1}{2}\mu_{ab}\partial\tilde{\gamma}^b\,, &\quad \gamma^i \to \tilde{\gamma}^a\,,\\
\widehat{\beta}_A &= \tilde{\beta}_a\tilde{g}^a_A + \frac{1}{2}tr\,(\tilde{\mathcal{G}}_A\tilde{g}\partial\tilde{g}^{-1}) + \frac{1}{2}\tilde{\mu}_{AB}\partial\widehat{\gamma}^B\,, &\quad \tilde{\gamma}^a \to \widehat{\gamma}^A\,,\\
\widehat{\beta}^\circ_A &= \beta_i\widehat{g}^i_A + \frac{1}{2}tr\,(\widehat{\mathcal{G}}_A\widehat{g}\partial\widehat{g}^{-1}) + \frac{1}{2}\widehat{\mu}_{AB}\partial\widehat{\gamma}^B\,, &\quad \gamma^i \to \widehat{\gamma}^A\,,
\end{aligned}$$

*Note that in the previous section we denoted the patch index with the latin letters $I, J, K, ..,$ in this section we denote the Čech index with the Greek letters $\alpha, \beta, \gamma, ...$

where

$$g^i_a = \frac{\partial\gamma^i}{\partial\tilde\gamma^a}\,, \quad \tilde g^a_A = \frac{\partial\tilde\gamma^a}{\partial\widehat\gamma^A}\,, \quad \widehat g^i_A = \frac{\partial\gamma^i}{\partial\widehat\gamma^A}\,,$$
$$[\mathcal{G}_a]^i_j = \partial_j g^i_a\,, \quad \left[\tilde{\mathcal{G}}_A\right]^a_b = \tilde\partial_b \tilde g^a_A\,, \quad \left[\widehat{\mathcal{G}}_A\right]^i_j = \partial_j \widehat g^i_A\,.$$

So, to obtain the obstruction of the β-γ system we must compare $\widehat\beta^\circ_A$ with $\widehat\beta_A$, i.e to compute $\widehat\beta^\circ_A - \widehat\beta_A$. Nevertheless, before we do this computation it is useful to show the following identities

$$\widehat g^i_A = g^i_a \tilde g^a_A \tag{2.56}$$

$$\begin{aligned} \widehat{\mathcal{G}}_A &= \partial_j \widehat g^i_A = \partial_j (g^i_a \tilde g^a_A) \\ &= \partial_j g^i_a \tilde g^a_A + g^i_a \partial_j \tilde g^a_A \\ &= \tilde g^a_A \mathcal{G}_a + g^i_a g^b_j \tilde\partial_b \tilde g^a_A \\ &= \tilde g \mathcal{G}_a + g^{-1} \tilde{\mathcal{G}}_A g. \end{aligned} \tag{2.57}$$

and

$$:: \beta_i g^i_a : \tilde g^a_A := \beta_i \widehat g^i_A - (\partial g^i_a)(\partial_i \tilde g^a_A)\,. \tag{2.58}$$

To prove (2.58) we can note that

$$: \beta_i g^i_a : (z) \tilde g^a_A(w) \;=\; :: \beta_i g^i_a : (z) \tilde g^a_A(w) : + : \beta_i \; \widehat{g^i_a \; : \; \tilde g^a_A} \;,$$

and

$$\begin{aligned} : \beta_i g^i_a : (z) \tilde g^a_A(w) &= \beta_i g^i_a(z) \tilde g^a_A(w) - \widehat{\beta_i \; g^i_a}(z) \tilde g^a_A(w) \\ &= \; : \beta_i g^i_a(z) \tilde g^a_A(w) : + \widehat{\beta_i \; g^i_a}(z) \tilde g^a_A(w) + g^i_a \widehat{\beta_i (z) \tilde g^a_A}(w) - \widehat{\beta_i \; g^i_a}(z) \tilde g^a_A(w) \\ &= \; : \beta_i g^i_a(z) \tilde g^a_A(w) : + \beta_i \widehat{(z)\gamma^j}(w) g^i_a(z) \partial_j \tilde g^a_A(w) \\ &= \; : \beta_i g^i_a(z) \tilde g^a_A(w) : - \frac{g^i_a(z) \partial_i \tilde g^a_A(w)}{z-w} \\ &= \; : \beta_i g^i_a(z) \tilde g^a_A(w) : - \frac{g^i_a(w) \partial_i \tilde g^a_A(w)}{z-w} - \partial g^i_a(w) \partial_i \tilde g^a_A(w) - \mathcal{O}(z-w)\;. \end{aligned}$$

From the first and last line we have

$$:: \beta_i g^i_a : (z) \tilde g^a_A(w) :=: \beta_i g^i_a(z) \tilde g^a_A(w) : - \partial g^i_a(w) \partial_i \tilde g^a_A(w) - \mathcal{O}(z-w)\;.$$

So, in the limit $z \to w$ we get (2.58):

$$\lim_{z\to w} :: \beta_i g^i_a : \tilde g^a_A :=: \beta_i \widehat g^i_A : - \partial g^i_a \partial_i \tilde g^a_A\;.$$

Now we are ready to compute $\widehat{\beta}^\circ_A - \widehat{\beta}_A$. Replacing $\tilde{\beta}_a$ in $\widehat{\beta}_A$ and using the previous identities we obtain

$$\begin{aligned}
\widehat{\beta}^\circ_A - \widehat{\beta}_A &= \beta_i \widehat{g}^i_A + \frac{1}{2} tr\,(\widehat{\mathcal{G}}_A \widehat{g} \partial \widehat{g}^{-1}) + \frac{1}{2}\widehat{\mu}_{AB}\partial\widehat{\gamma}^B - \beta_i \widehat{g}^i_A + (\partial g^i_a)(\partial_i \tilde{g}^a_A) \\
&\quad -\frac{1}{2} tr\,(\tilde{g}^a_A \mathcal{G}_a g \partial g^{-1}) - \frac{1}{2} tr\,(\tilde{\mathcal{G}}_A \tilde{g}\partial\tilde{g}^{-1}) - \frac{1}{2}(\mu_{ab}\tilde{g}^a_A \partial\tilde{\gamma}^b + \tilde{\mu}_{AB}\partial\widehat{\gamma}^B) \\
&= \frac{1}{2} tr\, \left\{ (\widehat{\mathcal{G}}_A \widehat{g}\partial\widehat{g}^{-1}) + 2(\partial g g^{-1} \tilde{\mathcal{G}}_A) - (\tilde{g}^a_A \mathcal{G}_a g \partial g^{-1}) - (\tilde{\mathcal{G}}_A \tilde{g} \partial \tilde{g}^{-1}) \right\} \\
&\quad + \frac{1}{2}(\widehat{\mu}_{AB} - \tilde{\mu}_{AB} - \mu_{ab}\tilde{g}^a_A \tilde{g}^b_B)\partial\widehat{\gamma}^B \ .
\end{aligned}$$

Using the traces

* $\widehat{\mathcal{G}}_A \widehat{g} \partial \widehat{g}^{-1} = -(\tilde{g}^{-1}\partial\tilde{g})(\widehat{\partial}_A g g^{-1}) + (\widehat{\partial}_A \tilde{g})(\partial\tilde{g}^{-1}) + (\widehat{\partial}_A g)(\partial g^{-1}) - \partial_i \tilde{g}^b_A \partial g^i_b$
* $2(\partial g g^{-1} \tilde{\mathcal{G}}_A) = (\partial_i \tilde{g}^b_A)\partial g^i_b + (\tilde{g}^{-1}\widehat{\partial}_A \tilde{g})(\partial g g^{-1})$
* $-\tilde{g}^a_A \mathcal{G}_a g \partial g^{-1} = -(\widehat{\partial}_A g)(\partial g^{-1})$
* $-\tilde{\mathcal{G}}_A \tilde{g}\partial\tilde{g}^{-1} = -(\widehat{\partial}_A \tilde{g})(\partial\tilde{g}^{-1})$,

the computation (2.59) is

$$\begin{aligned}
\widehat{\beta}^\circ_A - \widehat{\beta}_A &= \frac{1}{2} tr\, \left\{ (\tilde{g}^{-1}\widehat{\partial}_A \tilde{g})(\partial g g^{-1}) - (\tilde{g}^{-1}\partial\tilde{g})(\widehat{\partial}_A g g^{-1}) \right\} \\
&\quad + \frac{1}{2}(\widehat{\mu}_{AB} - \tilde{\mu}_{AB} - \mu_{ab}\tilde{g}^a_A \tilde{g}^b_B)\partial\widehat{\gamma}^B \ . \qquad (2.59)
\end{aligned}$$

In the Čech and differential forms language the previous terms are

$$\begin{aligned}
-tr\, \left\{ (\tilde{g}^{-1}\widehat{\partial}_A \tilde{g})(\partial g g^{-1}) - (\tilde{g}^{-1}\partial\tilde{g})(\widehat{\partial}_A g g^{-1}) \right\} &= tr\,(g_{\alpha\beta}\mathrm{d}g_{\beta\gamma} \wedge \mathrm{d}g_{\gamma\alpha}) \left(\widehat{\partial}_A, \partial\widehat{\gamma}^B \widehat{\partial}_B\right) \\
-(\widehat{\mu}_{AB} - \tilde{\mu}_{AB} - \mu_{ab}\tilde{g}^a_A \tilde{g}^b_B)\partial\widehat{\gamma}^B &= (\mu_{\alpha\beta} + \mu_{\beta\gamma} + \mu_{\gamma\alpha}) \left(\widehat{\partial}_A, \partial\widehat{\gamma}^B \widehat{\partial}_B\right) \ ,
\end{aligned}$$

where

$$\begin{aligned}
\mu_{\alpha\beta} &= \frac{1}{2}\mu_{ab}\mathrm{d}\tilde{\gamma}^a \wedge \mathrm{d}\tilde{\gamma}^b \\
\mu_{\beta\gamma} &= \frac{1}{2}\tilde{\mu}_{AB}\mathrm{d}\widehat{\gamma}^A \wedge \mathrm{d}\widehat{\gamma}^B \\
\mu_{\alpha\gamma} &= \frac{1}{2}\widehat{\mu}_{AB}\mathrm{d}\widehat{\gamma}^A \wedge \mathrm{d}\widehat{\gamma}^B \ ,
\end{aligned}$$

and

$$g_{\alpha\beta} = [g]^i_a\,, \quad g_{\beta\gamma} = [\tilde{g}]^a_A\,, \quad g_{\alpha\gamma} = [\widehat{g}]^i_A\,.$$

Therefore $\widehat{\beta}^\circ - \widehat{\beta}$ is

$$-2(\widehat{\beta}^\circ - \widehat{\beta}) = \psi_{\alpha\beta\gamma}(\cdot, \partial\widehat{\gamma}^B \widehat{\partial}_B) \equiv \{\mu_{\alpha\beta} + \mu_{\beta\gamma} + \mu_{\gamma\alpha} + tr\,(g_{\alpha\beta}\mathrm{d}g_{\beta\gamma} \wedge \mathrm{d}g_{\gamma\alpha})\}(\cdot, \partial\widehat{\gamma}^B \widehat{\partial}_B) \ .$$

Since the β-γ system is anomaly free if $\widehat{\beta}^\circ - \widehat{\beta} = 0$, then the obstruction to get a anomaly free theory is $\psi_{\alpha\beta\gamma} \equiv \mu_{\alpha\beta} + \mu_{\beta\gamma} + \mu_{\gamma\alpha} + tr\,(g_{\alpha\beta}\mathrm{d}g_{\beta\gamma} \wedge \mathrm{d}g_{\gamma\alpha})$. So we

can say that the theory is anomaly free if $\psi_{\alpha\beta\gamma} = 0$. However, since the $(2,0)$-form μ is defined up to a closed $(2,0)$-form b then the condition $\psi_{\alpha\beta\gamma} = 0$ becomes $\psi_{\alpha\beta\gamma} = (\delta b)_{\alpha\beta\gamma}$.
Now, note that the $(2,0)$-form $\psi_{\alpha\beta\gamma}$ is a Čech cocycle

$$(\delta\psi)_{\alpha\beta\gamma\delta} = 0 \tag{2.60}$$

and it also is a closed (2,0)-form†

$$\mathrm{d}\psi_{\alpha\beta\gamma} = 0, \tag{2.61}$$

therefore $[\psi_{\alpha\beta\gamma}]$ is an element of the Čech cohomology group $H^2(M, \mathcal{Z}^2)$, where $\mathcal{Z}^2 \subset \Omega^2$ is the abelian group of the (2,0)-forms which are closed.
So the condition $\psi_{\alpha\beta\gamma} = (\delta b)_{\alpha\beta\gamma}$ means the beta-gamma system is anomaly free if the cohomology class $[\psi_{\alpha\beta\gamma}]$ is the trivial element of the cohomology group $H^2(M, \mathcal{Z}^2)$.
Finally we can say the obstruction to get a beta-gamma system invariant by diffeomorphism is given by the cohomology group $H^2(M, \mathcal{Z}^2)$.

Using the "*zig-zag*" method [29] we cam map $H^2(M, \mathcal{Z}^2)$ to the Dolbeault cohomology group $H^{(2,2)}_{\bar\partial}(M) \subset H^4_{DR}(M)$ (DR means de-Rham cohomology). So, in the following, we show that the Dolbeault cocycle corresponding to $[\psi_{\alpha\beta\gamma}]$ is the first Pontryagin class of the holomorphic tangent bundle TM^+

$$p_\alpha = \frac{1}{8\pi^2} tr\,(F_\alpha \wedge F_\alpha) \equiv p_1(TM^+) \tag{2.62}$$

where F_α is the (1,1)-curvature form of TM^+ given by

$$F_\alpha = -(\partial\bar\partial h_\alpha)h_\alpha^{-1} + (\partial h_\alpha)h_\alpha^{-1} \wedge (\bar\partial h_\alpha)h_\alpha^{-1}\ , \tag{2.63}$$

where h is the hermitian form over TM^+ (see appendix A.1). The idea of the zig-zag method is to find the 3-form CS_α and 2-form $\rho_{\alpha\beta}$ such that these satisfy the sequence

$$\begin{aligned}
\mathrm{d}\psi_{\alpha\beta\gamma} &= 0 \\
(\delta\psi)_{\alpha\beta\gamma\delta} &= 0 \\
\psi_{\alpha\beta\gamma} &= (\delta\rho)_{\alpha\beta\gamma} \\
\mathrm{d}\rho_{\alpha\beta} &= (\delta CS)_{\alpha\beta} \\
\mathrm{d}CS_\alpha &= p_\alpha \\
\mathrm{d}p_\alpha &= 0 \\
(\delta p)_{\alpha\beta} &= 0.
\end{aligned} \tag{2.64}$$

†To show that it is necessary to use the property $\mathrm{d}\mu_{\alpha\beta} = -\frac{1}{3} tr\,(g_{\beta\alpha}\mathrm{d}g_{\alpha\beta})^3$

We define $\rho_{\alpha\beta}$ as

$$\rho_{\alpha\beta} = \mu_{\alpha\beta} - m_{\alpha\beta} \equiv \mu_{\alpha\beta} - tr(g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge \omega_\alpha) \tag{2.65}$$

where $\omega_\alpha = (\partial h_\alpha)h_\alpha^{-1}$ is the connection in TM^+ (appendix A.1). Since ω_α is a connection then under a transformation of coordinates it changes

$$\omega_\alpha = g_{\alpha\beta}\omega_\beta g_{\beta\alpha} - g_{\alpha\beta}\partial g_{\beta\alpha} \tag{2.66}$$

and hence $\rho_{\alpha\beta}$ is antisymmetric in its Čech index, $\rho_{\alpha\beta} = -\rho_{\beta\alpha}$. To prove $(\delta\rho)_{\alpha\beta\gamma} = \psi_{\alpha\beta\gamma}$ it is enough to show $-(\delta m)_{\alpha\beta\gamma} = tr(g_{\alpha\beta}\mathrm{d}g_{\beta\gamma} \wedge \mathrm{d}g_{\gamma\alpha})$, so we have

$$\begin{aligned}
-(\delta m)_{\alpha\beta\gamma} &= m_{\beta\alpha} + m_{\alpha\gamma} + m_{\gamma\beta} \\
&= tr(g_{\beta\alpha}\mathrm{d}g_{\alpha\beta} \wedge \omega_\beta) + tr(g_{\alpha\gamma}\mathrm{d}g_{\gamma\alpha} \wedge \omega_\alpha) + tr(g_{\gamma\beta}\mathrm{d}g_{\beta\gamma} \wedge \omega_\gamma) \\
&= tr(g_{\beta\alpha}\mathrm{d}g_{\alpha\beta} \wedge \omega_\beta) + tr(g_{\alpha\gamma}\mathrm{d}g_{\gamma\alpha} \wedge (g_{\alpha\beta}\omega_\beta g_{\beta\alpha} - g_{\alpha\beta}\mathrm{d}g_{\beta\alpha})) \\
&\quad + tr(g_{\gamma\beta}\mathrm{d}g_{\beta\gamma} \wedge (g_{\gamma\beta}\omega_\beta g_{\beta\gamma} - g_{\gamma\beta}\mathrm{d}g_{\beta\gamma})) \\
&= tr(g_{\beta\alpha}\mathrm{d}g_{\alpha\beta} \wedge \omega_\beta) + tr(g_{\alpha\gamma}\mathrm{d}g_{\gamma\alpha} \wedge g_{\alpha\beta}\omega_\beta g_{\beta\alpha}) + tr(g_{\gamma\beta}\mathrm{d}g_{\beta\gamma} \wedge g_{\gamma\beta}\omega_\beta g_{\beta\gamma}) \\
&\quad - tr(g_{\alpha\gamma}\mathrm{d}g_{\gamma\alpha} \wedge g_{\alpha\beta}\mathrm{d}g_{\beta\alpha}) \\
&= -tr(\mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge \omega_\beta) + tr(g_{\beta\gamma}\mathrm{d}(g_{\gamma\beta}g_{\beta\alpha}) \wedge g_{\alpha\beta}\omega_\beta) + tr(\mathrm{d}g_{\beta\gamma} \wedge g_{\gamma\beta}\omega_\beta) \\
&\quad + tr(g_{\alpha\beta}\mathrm{d}(g_{\beta\gamma}g_{\gamma\alpha}) \wedge g_{\alpha\gamma}\mathrm{d}g_{\gamma\alpha}) \\
&= tr(g_{\alpha\beta}\mathrm{d}g_{\beta\gamma} \wedge \mathrm{d}g_{\gamma\alpha}) + tr(g_{\alpha\beta}g_{\beta\gamma}\mathrm{d}g_{\gamma\alpha} \wedge g_{\alpha\gamma}\mathrm{d}g_{\gamma\alpha}) \\
&= tr(g_{\alpha\beta}\mathrm{d}g_{\beta\gamma} \wedge \mathrm{d}g_{\gamma\alpha}),
\end{aligned}$$

therefore $(\delta\rho)_{\alpha\beta\gamma} = \psi_{\alpha\beta\gamma}$.
Now we define the 3-form CS_α as

$$CS_\alpha = tr(\omega_\alpha \wedge \mathrm{d}\omega_\alpha - \frac{2}{3}\omega_\alpha \wedge \omega_\alpha \wedge \omega_\alpha). \tag{2.67}$$

First we show $(\delta CS)_{\alpha\beta} = \mathrm{d}\rho_{\alpha\beta}$ (in order to get a compact notation we remove the letters tr in the following computation)

$$\begin{aligned}
CS_\alpha &= \omega_\alpha \wedge \mathrm{d}\omega_\alpha - \frac{2}{3}\omega_\alpha \wedge \omega_\alpha \wedge \omega_\alpha \\
&= (g_{\alpha\beta}\omega_\beta g_{\beta\alpha} - g_{\alpha\beta}\mathrm{d}g_{\beta\alpha}) \wedge (\mathrm{d}g_{\alpha\beta} \wedge \omega_\beta g_{\beta\alpha} + g_{\alpha\beta}\mathrm{d}\omega_\beta g_{\beta\alpha} - g_{\alpha\beta}\omega_\beta \wedge \mathrm{d}g_{\beta\alpha} - \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}g_{\beta\alpha}) \\
&\quad -\frac{2}{3}\{(g_{\alpha\beta}\omega_\beta g_{\beta\alpha} - g_{\alpha\beta}\mathrm{d}g_{\beta\alpha}) \wedge (g_{\alpha\beta}\omega_\beta g_{\beta\alpha} - g_{\alpha\beta}\mathrm{d}g_{\beta\alpha}) \wedge (g_{\alpha\beta}\omega_\beta g_{\beta\alpha} - g_{\alpha\beta}\mathrm{d}g_{\beta\alpha})\} \\
&= g_{\alpha\beta}\omega_\beta g_{\beta\alpha} \wedge \mathrm{d}g_{\alpha\beta} \wedge \omega_\beta g_{\beta\alpha} + g_{\alpha\beta}\omega_\beta g_{\beta\alpha} \wedge g_{\alpha\beta}\mathrm{d}\omega_\beta g_{\beta\alpha} - g_{\alpha\beta}\omega_\beta g_{\beta\alpha} \wedge g_{\alpha\beta}\omega_\beta \wedge \mathrm{d}g_{\beta\alpha} \\
&\quad -g_{\alpha\beta}\omega_\beta g_{\beta\alpha} \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}g_{\beta\alpha} - g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge \mathrm{d}g_{\alpha\beta} \wedge \omega_\beta g_{\beta\alpha} - g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge g_{\alpha\beta}\mathrm{d}\omega_\beta g_{\beta\alpha} \\
&\quad +g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge g_{\alpha\beta}\omega_\beta \wedge \mathrm{d}g_{\beta\alpha} + g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}g_{\beta\alpha} \\
&\quad -\frac{2}{3}\{(g_{\alpha\beta}\omega_\beta g_{\beta\alpha} \wedge g_{\alpha\beta}\omega_\beta g_{\beta\alpha} - g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge g_{\alpha\beta}\omega_\beta g_{\beta\alpha} - g_{\alpha\beta}\omega_\beta g_{\beta\alpha} \wedge g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \\
&\quad +g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge g_{\alpha\beta}\mathrm{d}g_{\beta\alpha}) \wedge (g_{\alpha\beta}\omega_\beta g_{\beta\alpha} - g_{\alpha\beta}\mathrm{d}g_{\beta\alpha})\} \\
&= \omega_\beta \wedge \omega_\beta \wedge g_{\beta\alpha}\mathrm{d}g_{\alpha\beta} + \frac{2}{3}\omega_\beta \wedge \omega_\beta \wedge \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} - \omega_\beta \wedge \omega_\beta \wedge \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} + \frac{2}{3}\omega_\beta \wedge \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge \omega_\beta \\
&\quad +\frac{2}{3}\mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge \omega_\beta \wedge \omega_\beta - \omega_\beta \wedge g_{\beta\alpha}\mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} - \mathrm{d}g_{\beta\alpha} \wedge \mathrm{d}g_{\alpha\beta} \wedge \omega_\beta \\
&\quad -\mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge \mathrm{d}\omega_\beta + \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge \omega_\beta - \frac{2}{3}\omega_\beta \wedge \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \\
&\quad -\frac{2}{3}\mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge \omega_\beta \wedge \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} - \frac{2}{3}\mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge \omega_\beta \\
&\quad +g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}g_{\beta\alpha} + \frac{2}{3}g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \\
&\quad +\omega_\beta \wedge \mathrm{d}\omega_\beta - \frac{2}{3}\omega_\beta \wedge \omega_\beta \wedge \omega_\beta \\
&= -\omega_\beta \wedge \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge g_{\beta\alpha}\mathrm{d}_{\alpha\beta} - \mathrm{d}g_{\beta\alpha} \wedge \mathrm{d}g_{\alpha\beta} \wedge \omega_\beta - \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge \mathrm{d}\omega_\beta - \mathrm{d}g_{\beta\alpha} \wedge \mathrm{d}g_{\alpha\beta} \wedge \omega_\beta \\
&\quad +\frac{2}{3}\omega_\beta \wedge \mathrm{d}g_{\beta\alpha} \wedge \mathrm{d}g_{\alpha\beta} + \frac{2}{3}\mathrm{d}g_{\alpha\beta} \wedge \omega_\beta \wedge \mathrm{d}g_{\beta\alpha} + \frac{2}{3}\mathrm{d}g_{\beta\alpha} \wedge \mathrm{d}g_{\alpha\beta} \wedge \omega_\beta \\
&\quad -g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} + \frac{2}{3}g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} \wedge g_{\alpha\beta}\mathrm{d}g_{\beta\alpha} + CS_\beta \\
&= -\mathrm{d}g_{\beta\alpha} \wedge \mathrm{d}g_{\alpha\beta} \wedge \omega_\beta - \mathrm{d}g_{\beta\alpha}g_{\alpha\beta} \wedge \mathrm{d}\omega_\beta - \frac{1}{3}(g_{\alpha\beta}\mathrm{d}g_{\beta\alpha})^3 + CS_\beta \\
&= -\{\frac{1}{3}(g_{\alpha\beta}\mathrm{d}g_{\beta\alpha})^3 + \mathrm{d}(g_{\beta\alpha}\mathrm{d}g_{\alpha\beta} \wedge \omega_\beta)\} + CS_\beta \\
&= -\mathrm{d}\{\mu_{\alpha\beta} + tr(g_{\beta\alpha}\mathrm{d}g_{\alpha\beta} \wedge \omega_\beta)\} + CS_\beta. \qquad (2.68)
\end{aligned}$$

So from (2.68) we can see $(\delta CS)_{\alpha\beta} = CS_\beta - CS_\alpha = \mathrm{d}(\mu_{\alpha\beta} - m_{\alpha\beta}) = \mathrm{d}\rho_{\alpha\beta}$.
Finally we must show $\mathrm{d}CS_\alpha = p_\alpha$, which is a simple computation

$$\begin{aligned}
tr(F_\alpha \wedge F_\alpha) &= tr\{(\mathrm{d}\omega_\alpha - \omega_\alpha \wedge \omega_\alpha) \wedge (\mathrm{d}\omega_\alpha - \omega_\alpha \wedge \omega_\alpha)\} \\
&= tr\{\mathrm{d}\omega_\alpha \wedge \mathrm{d}\omega_\alpha - \mathrm{d}\omega_\alpha \wedge \omega_\alpha \wedge \omega_\alpha - \omega_\alpha \wedge \omega_\alpha \wedge \mathrm{d}\omega_\alpha\} \\
&= \mathrm{d}\{tr(\omega_\alpha \wedge \mathrm{d}\omega_\alpha - \frac{2}{3}\omega_\alpha \wedge \omega_\alpha \wedge \omega_\alpha)\}, \qquad (2.69)
\end{aligned}$$

therefore we have $\mathrm{d}CS_\alpha = 8\pi^2 p_\alpha$. The overall factor $8\pi^2$ is not important since our interest is to know if $[\psi_{\alpha\beta\gamma}]$ is a trivial element of $H^2(M, \mathcal{Z}^2)$, i.e if $[8\pi^2 p_\alpha]$ is trivial in $H^4_{DR}(M)$, which will imply that $[p_\alpha]$ is also trivial.

In summary we can conclude that the obstructions to get a beta-gamma system well defined globally and conformal invariant (i.e a well defined stress tensor) are given by the topological invariants $c_1(TM^+) \in H^2_{DR}(M)$ (the first Chen class) and $p_1(TM^+) \in H^4_{DR}(M)$ (the first Pontryagin class).

In the next section we show the beta-gamma system over pure spinor space is anomaly free when its singular point (the origin) is removed of the space.

2.6 The Pure Spinor Space

We now apply the previous description of the beta-gamma system to the pure spinor space. Our aim is to show the pure spinor action proposed by Berkovits [47] is well defined globally when the origin is removed from the pure spinor space PS.

2.6.1 Integration measure in the PS space

The integration measure in the PS space, $[\mathrm{d}\lambda]$, was found by Berkovits in [48][22][32] which is given by the expression

$$[\mathrm{d}\lambda](\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4 a_5} = \frac{2^3}{11!}\epsilon_{\alpha_1 ... \alpha_5 \beta_1 ... \beta_{11}} \mathrm{d}\lambda^{\beta_1} \wedge ... \wedge \mathrm{d}\lambda^{\beta_{11}}, \quad (2.70)$$

where this is simple to verify that the tensor $(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4 a_5}$ is totally antisymmetric. It is a holomorphic *global* top form on the PS space. Since the OPE between the ghost current and stress tensor

$$T(z)J(0) \sim \frac{8}{z^3} + \frac{J(0)}{z^2} + \frac{\partial J(0)}{z} \quad (2.71)$$

means that the theory has ghost anomaly 8 then the holomorphic top form $[\mathrm{d}\lambda]$ must have ghost number 8, so the only covariant top form with ghost number 8 is (3.132).
Nevertheless, in order to see the anomalies it is useful to write $[\mathrm{d}\lambda]$ in a local coordinate system, so let us remember that the PS space has the following parametrization

$$(\lambda^+, \lambda_{ab}, \lambda^a) = \gamma(1, u_{ab}, \frac{1}{8}\epsilon^{abcde}u_{bc}u_{de}), \qquad a, b, c, d, e = 1, ..., 5 \quad (2.72)$$

on the patch $U_\alpha = \{\lambda^+ \neq 0\}$, where $\lambda_{ab} = -\lambda_{ba}$ and $u_{ab} = -u_{ba}$. Since λ^α has ghost number 1 then we can see that γ has ghost number 1 and the u_{ab}'s have ghost number 0. In this way we can naively write the holomorphic top form as

$$[\mathrm{d}\lambda] = \gamma^7 \mathrm{d}\gamma \wedge \mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45} \quad \rightarrow \quad \text{ghost number } 8, \quad (2.73)$$

nevertheless since u_{ab}'s have ghost number zero then we can get a more general top form

$$[\mathrm{d}\lambda] = \gamma^7 f(u_{ab})\, \mathrm{d}\gamma \wedge \mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45} \quad \rightarrow \quad \text{ghost number } 8, \quad (2.74)$$

where $f(u_{ab})$ is a holomorphic function of the u_{ab}'s variables. However, from concepts of globality on the projective pure spinor space, which we will discuss later, the only possibility for the function $f(u_{ab})$ is to be a constant.
Now we show, using the the antiholomorphic top form $[\mathrm{d}\overline{\lambda}]$, (3.132) and (2.73) are the same, up to constant phase.
Since the PS space is a complex cone[‡] embedded in $\mathbb{C}^{16}$ by the quadratic constrains $(\lambda\gamma^m\lambda) = 0$, then we can define the antiholomorphic coordinates $\overline{\lambda}_\alpha$'s with the constrains[§] $\overline{\lambda}_\alpha(\gamma^m)^{\alpha\beta}\overline{\lambda}_\beta = 0$, so the antiholomorphic top form is

$$[\mathrm{d}\overline{\lambda}](\overline{\lambda}\gamma^q)^{\alpha_1}(\overline{\lambda}\gamma^r)^{\alpha_2}(\overline{\lambda}\gamma^s)^{\alpha_3}(\gamma_{qrs})^{\alpha_4 a_5} = \frac{2^3}{11!}\epsilon^{\alpha_1\ldots\alpha_5\beta_1\ldots\beta_{11}}\mathrm{d}\overline{\lambda}_{\beta_1}\wedge\ldots\wedge\mathrm{d}\overline{\lambda}_{\beta_{11}}. \tag{2.75}$$

But (2.75) is not very useful, so we can contract it with the term $(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4 a_5}$ and then to get [32]

$$[\mathrm{d}\overline{\lambda}] = \frac{(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4 a_5}}{2^3 5!\, 11!(\lambda\overline{\lambda})^3}\epsilon^{\alpha_1\ldots\alpha_5\beta_1\ldots\beta_{11}}\mathrm{d}\overline{\lambda}_{\beta_1}\wedge\ldots\wedge\mathrm{d}\overline{\lambda}_{\beta_{11}}, \tag{2.76}$$

where $(\overline{\lambda}\gamma^q)^{\alpha_1}(\overline{\lambda}\gamma^r)^{\alpha_2}(\overline{\lambda}\gamma^s)^{\alpha_3}(\gamma_{qrs})^{\alpha_4 a_5}(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4 a_5} = 2^6 5!(\lambda\overline{\lambda})^3$.
Our idea is to show the following product

$$[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}] = (\gamma\overline{\gamma})^7\mathrm{d}\gamma\wedge\mathrm{d}u_{12}\wedge\ldots\wedge\mathrm{d}u_{45}\wedge\mathrm{d}\overline{\gamma}\wedge\mathrm{d}\overline{u}^{12}\wedge\ldots\wedge\mathrm{d}\overline{u}^{45}, \tag{2.77}$$

which means that the measure $[\mathrm{d}\lambda]$ is the same as (2.73) up to constant overall phase factor.
From (3.132) and (2.76) we have

$$\begin{aligned}[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}] &= \frac{1}{5!(11!)^2(\lambda\overline{\lambda})^3}\epsilon_{\alpha_1\ldots\alpha_5\beta_1\ldots\beta_{11}}\epsilon^{\alpha_1\ldots\alpha_5\delta_1\ldots\delta_{11}} \\ &\quad \mathrm{d}\lambda^{\beta_1}\wedge\ldots\wedge\mathrm{d}\lambda^{\beta_{11}}\wedge\mathrm{d}\overline{\lambda}_{\delta_1}\wedge\ldots\wedge\mathrm{d}\overline{\lambda}_{\delta_{11}} \\ &= \frac{1}{11!(\lambda\overline{\lambda})^3}\partial\overline{\partial}(\lambda\overline{\lambda})\wedge\ldots\wedge\partial\overline{\partial}(\lambda\overline{\lambda}).\end{aligned} \tag{2.78}$$

Replacing (2.72) and

$$(\overline{\lambda}_+,\overline{\lambda}^{ab},\overline{\lambda}_a) = \overline{\gamma}(1,\overline{u}^{ab},\frac{1}{8}\epsilon_{abcde}\overline{u}^{bc}\overline{u}^{de}) \tag{2.79}$$

in (2.78) we get (2.77), therefore we have shown the measures (3.132) and (2.73) are the same up to constant phase.

Since we have found the holomorphic top form in a local coordinate system then we can see the stress tensor in this system, as in (2.47). In the coordinates (2.72) the pure spinor action becomes

$$S_{PS} = \int \mathrm{d}^2 z\ \omega_\alpha\overline{\partial}\lambda^\alpha = \int \mathrm{d}^2 z\ \beta\overline{\partial}\gamma + \frac{1}{2}v^{ab}\overline{\partial}u_{ab} \tag{2.80}$$

[‡]It is simple to see that the point $\lambda^\alpha = 0$ is a singular point; because in this point it is not possible to define a tangent space.

[§]Note that the variables $\overline{\lambda}_\alpha$ are the same as the non-minimal formalism.

where we have used the gauge transformation $\delta\omega_\alpha = \Lambda^m(z,\bar{z})(\lambda\gamma_m)_\alpha$ to fix $\omega_a = 0$ and the following parametrization for ω_α

$$\omega_\alpha = (\omega_+, \omega^{ab}, \omega_a) = (\beta - \frac{1}{2\gamma}v^{ab}u_{ab}, \frac{v^{ab}}{\gamma}, 0). \tag{2.81}$$

Note that (2.80) is the same action as in (2.2), so the stress tensor in this coordinates is given by

$$T_{PS} = \beta\partial\gamma + \frac{1}{2}v^{ab}\partial u_{ab} + \frac{7}{2}\partial^2 \ln\gamma. \tag{2.82}$$

The stress tensor is well defined in everywhere except at the origin, i.e $\gamma = 0 \Rightarrow \lambda^\alpha = 0$. It means that the beta-gamma system on PS suffers from Chern anomaly. This is not difficult to check that the holomorphic top form $[\mathrm{d}\lambda] = \gamma^7 \mathrm{d}\gamma \bigwedge \mathrm{d}u_{ab}$ is a global section of the canonical line bundle $\bigwedge^d (TM^+)^*$, i.e in the Čech language it is

$$(\delta[\mathrm{d}\lambda])_{\alpha\beta} = [\mathrm{d}\lambda]_\beta - [\mathrm{d}\lambda]_\alpha = 0, \qquad \alpha, \beta \text{ are Čech index,} \tag{2.83}$$

but since $[\mathrm{d}\lambda] = 0$ in the point $\gamma = 0$ then the canonical line bundle is not a trivial bundle therefore its first Chern class does not vanish. In addition the Pontryagin anomaly is present there as well [18].
However, by removing the locus $\gamma = 0$, i.e deleting the point $\lambda^\alpha = 0$ on PS we obtain the space $PS \setminus \{0\}$, which is anomaly free as we show in the next sections.

In the next section we compute the de Rham cohomology of the projective pure spinor space, which will be very useful to compute the de Rham cohomology of the $PS \setminus \{0\}$ space and to show that it is anomaly free.
However, before that, it is useful to understand a little bit about the geometry of the pure spinor space.

2.7 The geometry of the pure spinors: a short review

Since we are only interested in pure spinors in even dimension then we write the space-time dimension as $d = 2n$, i.e $\mathbb{R}^{2n}$.
A spinor λ^α of $SO(2n)$ is defined as a pure spinor if it satisfies

$$\lambda^\alpha\lambda^\beta = \frac{1}{n!2^n}(\gamma_{m_1\ldots m_n})^{\alpha\beta}(\lambda\gamma^{m_1\ldots m_n}\lambda) \tag{2.84}$$

where $m_i = 1, .., 2n$, $\alpha = 1, ..., 2^{n-1}$, $(\gamma^{m_1\ldots m_n})_{\alpha\beta}$ is the antisymmetrized product of n Pauli matrices and the 2^{n-1} components of λ^α are complex numbers. We denote the set of pure spinors as PS. The condition (2.7) means the number of complex degree of freedom of the pure spinor λ^α is $\frac{1}{2}n(n-1)+1$. Note that if λ^α is a pure spinor then $\tilde{\lambda}^\alpha = \rho\lambda^\alpha$ is also a pure spinor, where $\rho \in \mathbb{C}^* \equiv \mathbb{C} \setminus \{0\}$.So, we define the projective pure spinor space, $\mathbb{P}(PS_{2n})$ or $\mathcal{Q}_{2n}$, as the space of pure spinors such that they are related by the equivalence relation $\lambda^\alpha \sim \rho\lambda^\alpha$, $\rho \in \mathbb{C}^*$ i.e

$$Q_{2n} \equiv \{[\lambda^\alpha]\}, \qquad \text{where} \quad [\lambda^\alpha] = \{\tilde{\lambda}^\alpha \in PS \,:\, \tilde{\lambda}^\alpha = \rho\lambda^\alpha,\, \rho \in \mathbb{C}^*\} \tag{2.85}$$

and obviously $\lambda^\alpha \neq 0$.

The multivector $(\lambda\gamma^{m_1...m_n}\lambda)$ defines a complex plane of dimension n in $\mathbb{C}^{2n} = \mathbb{R}^{2n} \otimes \mathbb{C}$, since $(\lambda\gamma^{m_1...m_n}\lambda) = a_1^{[m_1}a_2^{m_2}...a_n^{m_n]}$ for some complex linearly independent vectors $a_1, ..., a_n$ in $\mathbb{C}^{2n}$ [14]. Using the properties of the Pauli matrices we can check that this complex plane $\mathbb{C}^n$ is isotropic, i.e

$$\delta_{m_1p_1}(\lambda\gamma^{m_1...m_n}\lambda)(\lambda\gamma^{p_1...p_n}\lambda) = 0. \tag{2.86}$$

It is simple see that if λ^α and $\tilde{\lambda}^\alpha$ are two pure spinors (non-zero) such that $\tilde{\lambda}^\alpha = \rho\lambda^\alpha$, $\rho \in \mathbb{C}^*$, then the multivectors $(\lambda\gamma^{m_1...m_n}\lambda)$ and $(\tilde{\lambda}\gamma^{m_1...m_n}\tilde{\lambda})$ define the same complex plane $\mathbb{C}^n$. Therefore we have a map one to one from the projective pure spinor space to isotropic complex n-planes in $\mathbb{C}^{2n}$.
Since the multivector index $m_1, .., m_n$ are vectorial index of $SO(2n)$ then this group acts transitively over the isotropic complex n-planes and as these planes are invariant by the action of $U(5)$ then we can see the projective pure spinor space as the coset

$$\mathcal{Q}_{2n} = SO(2n)/U(n). \tag{2.87}$$

This coset is also identified with set the complex structures of $\mathbb{R}^{2n}$ which preserve the orientation.
This means the following:
if J and J' are complex structures of $\mathbb{R}^{2n}$, i.e

$$\begin{aligned} &J : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \qquad J \text{ is a isomorphism such that } J^2 = -1_{2n}, \\ &J' : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \qquad J' \text{ is a isomorphism such that } J'^2 = -1_{2n}, \end{aligned} \tag{2.88}$$

and the two orthonormal frames of $\mathbb{R}^{2n}$

$$\{e_1, Je_1, ..., e_n, Je_n\}\ , \qquad \{e'_1, J'e'_1, ..., e'_n, J'e'_n\} \tag{2.89}$$

have the same orientation then J and J' are related by the map

$$J' = g^T\, J\, g, \tag{2.90}$$

where g is some element of $SO(2n)$. So, since $SO(10)$ acts transitively over the set the complex structures then we can choose a particular complex structure and from the action of $SO(2n)$ we can get the whole complex structure space. In particular we choose

$$J_0 = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}\ ,$$

so

$$J = g^T\, J_0\, g, \tag{2.91}$$

where $g \in SO(2n)$, parametrizes the space of the complex structures. Clearly the fix points of (2.91) ($J_0 = g^T\, J_0\, g$) are matrices $g \in SO(2n)$ given by the expression

$$g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} , \tag{2.92}$$

where A and B are real matrices $n \times n$ which satisfy $AA^T + BB^T = 1_n$ and $BA^T = AB^T$. Note that (2.92) can be written in the way $\Lambda = A + iB$, where the previous conditions are obtained simply imposing the constrain $\Lambda\Lambda^\dagger = 1_n$. It means the matrix (2.92) is a element of $U(n)$ and therefore we can conclude that (2.91) parametrizes the coset $SO(2n)/U(n)$.

To conclude this subsection we want to interpret the pure spinor space PS as a line bundle. It will be very useful for this thesis.

2.7.1 The Pure Spinor Space As A Line Bundle

From the definitions (2.84) and (2.85) we can see that every pure spinor λ^α can be written in the way

$$\lambda^\alpha = \gamma\tilde{\lambda}^\alpha, \tag{2.93}$$

where $\gamma \in \mathbb{C}$ and $\tilde{\lambda}^\alpha$ is a projective pure spinor. Note that $\lambda^\alpha = 0$ if and only if $\gamma = 0$. This is the only singular point of the pure spinor space since its associated multivector vanishes, which implies the isotropic complex plane has dimension $d \leq (n-1)$.

The expression (2.93) means the pure spinor space PS is a line bundle over the base space $SO(2n)/U(n)$, where γ is the fiber of the bundle. Schematically we have the following

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & PS \\ & & \big\downarrow \pi \\ & & SO(2n)/U(n) \end{array} \quad . \tag{2.94}$$

This bundle is known as the $\mathcal{O}(-1)$ line bundle.

In order to get an anomaly free theory (as it will be shown on the next sections) we must remove the singular point from the pure spinor space PS, this means the point $\gamma = 0$ must be removed from $\mathbb{C}$ and so we have the diagram

$$\begin{array}{ccc} \mathbb{C}^* & \longrightarrow & PS \setminus \{0\} \\ & & \downarrow \pi \\ & & SO(2n)/U(n) \end{array} \quad . \tag{2.95}$$

Since the space $\mathbb{C}^*$ can be contracted to circle S^1 (homotopically) then the pure spinor space $PS \setminus \{0\}$ can be contracted to compact space $SO(2n)/SU(n)$. It is import when we will compute the cohomology of the pure spinor space $PS \setminus \{0\}$, because in this case we can change the previous fibration by

$$\begin{array}{ccc} S^1 & \longrightarrow & SO(2n)/SU(n) \\ & & \downarrow \pi \\ & & SO(2n)/U(n) \end{array} \tag{2.96}$$

and the cohomologies of the two diagrams are the same.

As the cohomology of S^1 is very well known then in the next section we compute the cohomology of the projective pure spinor space, i.e $SO(2n)/U(n)$, and so applying techniques of spectral sequences we will get the cohomology of $SO(2n)/SU(n)$, in particular of the space $SO(10)/SU(5)$.

2.8 The Projective Pure Spinor Space and Morse Theory

In order to show the pure spinor space without the origin is anomaly free ($PS \setminus \{0\}$), we compute the cohomology of the projective pure spinor space using the Morse theory [51] [18].

2.8.1 Morse Theory

In this subsection we find the de-Rham cohomology of the projective pure spinor space from the Morse Theory.

First of all, we define a real function

$$H_\Phi : SO(2n)/U(n) \to \mathbb{R}, \tag{2.97}$$

given by the expression

$$H_\Phi(g) = tr\,(g^T\, J_0\, g\, \Phi) \tag{2.98}$$

where Φ is a fixed element of the algebra $so(2n)$.
The principal idea of Morse theory is to find the critical points of the scalar function $H_\Phi(g)$, which must be nondegenerate[¶], and later to compute the index of each critical point, i.e the number of negative eigenvalues of the Hessian matrix. This index contains valuable information about the topology of the manifold.

The critical points are given by the condition

$$\frac{\partial H_\Phi}{\partial g_{ij}} = 0 \;\Rightarrow\; [J,\Phi] = 0 \ . \tag{2.99}$$

In the spinorial language this condition means $\varphi \cdot \sigma = 0$, where $\varphi \in spin(2n)$ is related to Φ and σ is the projective pure spinor corresponding to J. Using this language it is possible to show that the number of critical points are in one to one correspondence with the spinorial weights, i.e $\alpha = (\alpha_1, ..., \alpha_n),\ \alpha_i = \pm 1$, with the condition $\prod_{i=1}^{n} \alpha_i = 1$ because the pure spinors are chiral spinors. To check this result it is useful to introduce the concepts of Pin group and the maximal Torus, however as this will not be relevant to the development of this thesis then we do not prove it here, we recommend viewing [30].
Since the number of critical points are in one to one correspondence with the spinorial weights then we can write the critical points as

$$J = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix} \ , \tag{2.100}$$

where w is the $n \times n$ matrix $w = diag\,(\alpha) = diag\,(\pm 1, ..., \pm 1)$ such that $\prod \alpha_i = 1$.
These 2^{n-1} complex structures are critical points of H_Φ for Φ

$$\Phi = \begin{pmatrix} 0 & \phi \\ -\phi & 0 \end{pmatrix} \ , \tag{2.101}$$

where ϕ is the $n \times n$ matrix $\phi = diag\,(\phi_1, ..., \phi_n)$. For convenience we choose $\phi_1 > \phi_2 > ... > \phi_n > 0$.

In order to get the index of the Hessian matrix it is necessary to make a Taylor expansion around each critical point J, then we can write the function H_Φ as

$$H_\Phi(g) = tr\,(g^T\, J\, g\, \Phi) \tag{2.102}$$

and take $g = 1 + \Lambda$, where $\Lambda \in so(10)$. However we can note the following:
since J can be written in the way $J = A^T\, J_0\, A$, where A is given by

$$A = \begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix} \ , \tag{2.103}$$

[¶]This means the Hessian matrix is invertible. A necessary condition is that the critical points must be isolated.

so the function (2.102) becomes

$$H_\Phi(g) = tr\,((Ag)^T\,J_0\,(Ag)\,\Phi). \tag{2.104}$$

Because $A, g \in SO(2n)$ then $\forall g$ it is always possible to find an element $M \in SO(2n)$ such that $A\,g = M\,A$, in this way H_Φ turns

$$H_\Phi(M) = tr\,((MA)^T\,J_0\,(MA)\,\Phi) = tr\,((M^T\,J_0\,M)(A\,\Phi A^T)) \tag{2.105}$$

where

$$A\,\Phi\,A^T = \begin{pmatrix} 0 & w\phi \\ -w\phi & 0 \end{pmatrix}\,, \quad \text{and} \quad [w\phi]_{ij} = \alpha_i\phi_i\delta_{ij}\,,\;\; i,j = 1,...,n\;. \tag{2.106}$$

Therefore we can say to expand around of J is the same as to expand around of J_0 and after to change ϕ_i by $\alpha_i\phi_i$.
Using this result we will compute the index in a simple way.

2.8.2 Cell Decomposition of The Projective Pure Spinor Space

From the previous analysis it is enough to expand H_Φ around to J_0, so we have

$$H_\Phi(g) = tr\,(g^T\,J_0\,g\,\Phi) \tag{2.107}$$

where

$$g = 1 + \Lambda\,,\;\; \Lambda = \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix}\,, \quad \text{with}\;\; A^T = -A,\;\; C^T = -C,\;\; \text{i.e}\;\Lambda \in so(2n)\;.$$

If $C = A$ and $B^T = B$ then $\Lambda \in u(n) \subset so(2n)$. Since H_Φ is a function from $SO(2n)/U(n)$ to $\mathbb{R}$, this is sufficient to take a representative Λ of the coset $so(2n)/u(n)$, so choosing

$$\Lambda = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}\,, \quad \text{com}\;\; A^T = -A\,,\;\; B^T = -B\;, \tag{2.108}$$

it implies $\Lambda \in so(2n)$ and $\Lambda \notin u(n)$, besides that $\dim_{\mathbb{R}}(\Lambda) = n^2 - n = \dim_{\mathbb{R}}(so(2n)/u(n))$. We define the complex matrix u as $u = A + iB$, so the conditions $A^T = -A$ and $B^T = -B$ can be written in the simple way $u^T = -u$ and the matrix Λ becomes[‖]

$$\Lambda = \begin{pmatrix} Re\,u & Im\,u \\ Im\,u & -Re\,u \end{pmatrix}\;. \tag{2.109}$$

[‖]Note that the variables of the u matrix, i.e $u_{ab} = -u_{ba}$, are the same coordinates used in (2.72).

Then the second order expansion of the function H_Φ around J_0 is

$$\begin{aligned} H_\Phi = tr\,(M^T J_0 M \Phi) &\approx -2\sum_a \phi_a + \frac{4}{2!} tr\,(uu^\dagger \phi) \\ &= -2\sum_a \phi_a + 2\sum_{a<b} (\phi_a + \phi_b)|u_{ab}|^2 \;. \end{aligned} \tag{2.110}$$

In order to obtain the index of the Hessian matrix it is useful to make the following identifications

$$\begin{array}{llll} u_{12} = x_1 + iy_1 & & & \\ u_{13} = x_2 + iy_2 & u_{23} = x_n + iy_n & & \\ u_{14} = x_3 + iy_3 & u_{24} = x_{n+1} + iy_{n+1} & u_{34} = x_{2n-2} + iy_{2n-2} & \\ \quad\;\; . & & & \\ \quad\;\; . & & & \dots \\ \quad\;\; . & & & \\ u_{1n} = x_{n-1} + iy_{n-1} & u_{2n} = x_{2n-3} + iy_{2n-3} & u_{3n} = x_{3n-6} + iy_{3n-6} & u_{n-1,n} = x_{\frac{n^2-n}{2}} + iy_{\frac{n^2-n}{2}} \,. \end{array}$$

In this new coordinates the function H_Φ turns

$$\begin{aligned} H_\Phi = -2\sum_a \phi_a \;+\; 2\Bigg[&\sum_{b=1}^{n-1} (\phi_1 + \phi_{b+1})(x_b^2 + y_b^2) + \sum_{b=1}^{n-2} (\phi_2 + \phi_{b+2})(x_{b+n-1}^2 + y_{b+n-1}^2) + \\ &\sum_{b=1}^{n-3} (\phi_3 + \phi_{b+3})(x_{b+n-1+n-2}^2 + y_{b+n-1+n-2}^2) + \\ &\sum_{b=1}^{n-4} (\phi_4 + \phi_{b+4})(x_{b+n-1+n-2+n-3}^2 + y_{b+n-1+n-2+n-3}^2) + \\ &. \\ &. \\ &. \\ &+(\phi_{n-1} + \phi_n)(x_{\frac{n^2-n}{2}}^2 + y_{\frac{n^2-n}{2}}^2) \Bigg] \;, \end{aligned} \tag{2.111}$$

so from (2.106) the Hessian matrix in a general critical point J is

$$[\partial^2 H_\Phi] = 4 \begin{pmatrix} \alpha_1\phi_1 + \alpha_2\phi_2 & 0 & \dots\dots & 0 & 0 \\ 0 & \alpha_1\phi_1 + \alpha_2\phi_2 & \dots\dots & 0 & 0 \\ . & & & & \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & \dots\dots & \alpha_{n-1}\phi_{n-1} + \alpha_n\phi_n & 0 \\ 0 & 0 & \dots\dots & 0 & \alpha_{n-1}\phi_{n-1} + \alpha_n\phi_n \end{pmatrix}$$

where we have changed ϕ_i by $\alpha_i\phi_i$, and the order of the coordinates is $(x_1, y_1, ..., x_{(n^2-n)/2}, y_{(n^2-n)/2})$.

Cell Decomposition

The Morse theory say the index of the Hessian matrix is the dimension of the **cell** which we must glue to obtain the manifold. A $n-$cell is denoted by e^n and it has the following definition

$$e^n = \{X \in \mathbb{R}^n \, / \, ||X|| \leq 1\} \ , \tag{2.112}$$

where $||\cdot||$ means the Euclidean norm.

For instance, in the most simple case, i.e $SO(2)/U(1)$, the Hessian matrix is given by

$$[\partial^2 H_\Phi] = 4\,(\alpha_1 \phi_1) \ . \tag{2.113}$$

Therefore there is only one critical point, $\alpha_1 = 1$, and it has index zero, so $SO(2)/U(1) \approx e^0$, which was the expected result.

The following example is $d = 4$, i.e $SO(4)/U(2) \approx SU(2)/U(1)$. In this case the Hessian matrix is

$$[\partial^2 H_\Phi] = 4 \begin{pmatrix} \alpha_1\phi_1 + \alpha_2\phi_2 & 0 \\ 0 & \alpha_1\phi_1 + \alpha_2\phi_2 \end{pmatrix} \ . \tag{2.114}$$

So there are two critical points

$$\begin{aligned} \alpha = (1,1) \ &\rightarrow \ \text{index } 0 \\ \alpha = (-1,-1) \ &\rightarrow \ \text{index } 2 \ , \end{aligned}$$

and therefore

$$SO(4)/U(2) \approx e^0 \cup e^2 \approx \mathbb{C}P^1 \ . \tag{2.115}$$

In dimension $d = 6$ the projective pure spinor space is $SO(6)/U(3) \approx SU(4)/U(3)$. The critical points and their respective indices are

$$\begin{aligned} \alpha = (1,1,1) \ &\rightarrow \ \text{index } 0 \\ \alpha = (1,-1,-1) \ &\rightarrow \ \text{index } 2 \\ \alpha = (-1,1,-1) \ &\rightarrow \ \text{index } 4 \\ \alpha = (-1,-1,1) \ &\rightarrow \ \text{index } 6 \ , \end{aligned}$$

so

$$SO(6)/U(3) \approx e^0 \cup e^2 \cup e^4 \cup e^6 \approx \mathbb{C}P^3 \ . \tag{2.116}$$

In dimension $d = 8$ we have

$$\begin{aligned}
\alpha = (1,1,1,1) &\rightarrow \text{ index } 0\\
\alpha = (1,1,-1,-1) &\rightarrow \text{ index } 2\\
\alpha = (1,-1,1,-1) &\rightarrow \text{ index } 4\\
\alpha = (1,-1,-1,1) &\rightarrow \text{ index } 6\\
\alpha = (-1,1,1,-1) &\rightarrow \text{ index } 6\\
\alpha = (-1,1,-1,1) &\rightarrow \text{ index } 8\\
\alpha = (-1,-1,1,1) &\rightarrow \text{ index } 10\\
\alpha = (-1,-1,-1,-1) &\rightarrow \text{ index } 12 \ ,
\end{aligned}$$

therefore the projective pure spinor space has the cell decomposition

$$SO(8)/U(4) \approx e^0 \cup e^2 \cup e^4 \cup e^6 \cup e^6 \cup e^8 \cup e^{10} \cup e^{12} \ . \tag{2.117}$$

Finally, the case of our interest $d = 10$. The number of critical points is 16

$$\begin{aligned}
\alpha = (1,1,1,1,1) &\rightarrow \text{ index } 0\\
\alpha = (1,1,1,-1,-1) &\rightarrow \text{ index } 2\\
\alpha = (1,1,-1,1,-1) &\rightarrow \text{ index } 4\\
\alpha = (1,1,-1,-1,1) &\rightarrow \text{ index } 6\\
\alpha = (1,-1,1,1,-1) &\rightarrow \text{ index } 6\\
\alpha = (-1,1,1,1,-1) &\rightarrow \text{ index } 8\\
\alpha = (1,-1,1,-1,1) &\rightarrow \text{ index } 8\\
\alpha = (-1,1,1,-1,1) &\rightarrow \text{ index } 10\\
\alpha = (1,-1,-1,1,1) &\rightarrow \text{ index } 10\\
\alpha = (-1,1,-1,1,1) &\rightarrow \text{ index } 12\\
\alpha = (1,-1,-1,-1,-1) &\rightarrow \text{ index } 12\\
\alpha = (-1,-1,1,1,1) &\rightarrow \text{ index } 14\\
\alpha = (-1,1,-1,-1,-1) &\rightarrow \text{ index } 14\\
\alpha = (-1,-1,1,-1,-1) &\rightarrow \text{ index } 16\\
\alpha = (-1,-1,-1,1,-1) &\rightarrow \text{ index } 18\\
\alpha = (-1,-1,-1,-1,1) &\rightarrow \text{ index } 20 \ .
\end{aligned}$$

So the cell decomposition is

$$\begin{aligned}
SO(10)/U(5) \approx \quad & e^0 \cup e^2 \cup e^4 \cup e^6 \cup e^6 \cup e^8 \cup e^8 \cup e^{10} \cup e^{10} \cup e^{12}\\
& \cup e^{12} \cup e^{14} \cup e^{14} \cup e^{16} \cup e^{18} \cup e^{20} \ .
\end{aligned} \tag{2.118}$$

Now, using these cell decompositions we compute the de-Rham cohomology of the projective pure spinor space in a trivial way.
From the Morse inequalities [51]

$$b_\lambda - b_{\lambda-1} + ... \pm b_0 \leq C_\lambda - C_{\lambda-1} + ... \pm C_0, \tag{2.119}$$

where b_λ is the λ-*th* Betti number and C_λ is the number of critical points with index λ, the following Lemma is immediate
Lemma.
If $C_{\lambda+1} = C_{\lambda-1} = 0$ then $b_\lambda = C_\lambda$ and $b_{\lambda+1} = b_{\lambda-1} = 0$.

Therefore the cohomology of the projective pure spinor space in dimension $d = 10$ is

$$\begin{aligned} H^{2i}(SO(10)/U(5)) = H^{20-2i}(SO(10)/U(5)) = \mathbb{Z}\,, \quad i = 0, 1, 2\,. \\ H^{2i}(SO(10)/U(5)) = H^{20-2i}(SO(10)/U(5)) = \mathbb{Z}^2\,, \quad i = 3, 4, 5\,. \end{aligned} \tag{2.120}$$

2.9 The de Rham Cohomology of The Pure Spinor Space

The goal of this section is to show that the cohomology groups $H^2(PS \setminus \{0\})$ and $H^2(PS \setminus \{0\})$ are trivial, this implies the Chern and Pontryagin anomalies vanish. Since the pure spinor space without the origin has the same homotopic equivalence as the space $SO(10)/SU(5)$, it was shown in the subsection 2.7.1, then we can use the fibration

$$\begin{array}{ccc} S^1 & \longrightarrow & SO(10)/SU(5) \\ & & \downarrow \pi \\ & & SO(10)/U(5) \end{array} \tag{2.121}$$

to compute the cohomology of $PS \setminus \{0\}$.
In the previous section we have got the de Rham cohomology of the projective pure spinor space, moreover as the cohomology of S^1 is very well known

$$H^q(S^1) = \begin{cases} \mathbb{Z}\,, & \text{if } q = 0, 1. \\ 0\,, & q > 1 \end{cases}\,, \tag{2.122}$$

then, using the fibration (2.121), we can apply the technique of spectral sequences to compute the cohomology of $SO(10)/SU(5)$.

The spectral sequences are not an easy subject and probably the most people are not familiar with it, therefore we give a short introduction in the appendix A.2.

As it was shown in the appendix A.2 the term

$$E_2^{p,q} = H^p(SO(10)/U(5), H^q(S^1)) = H_\delta H_{\rm d}^{p,q}(SO(10)/SU(5)), \tag{2.123}$$

where δ is the Čechc operator and d is the de Rham operator, of the spectral sequence is given by the table

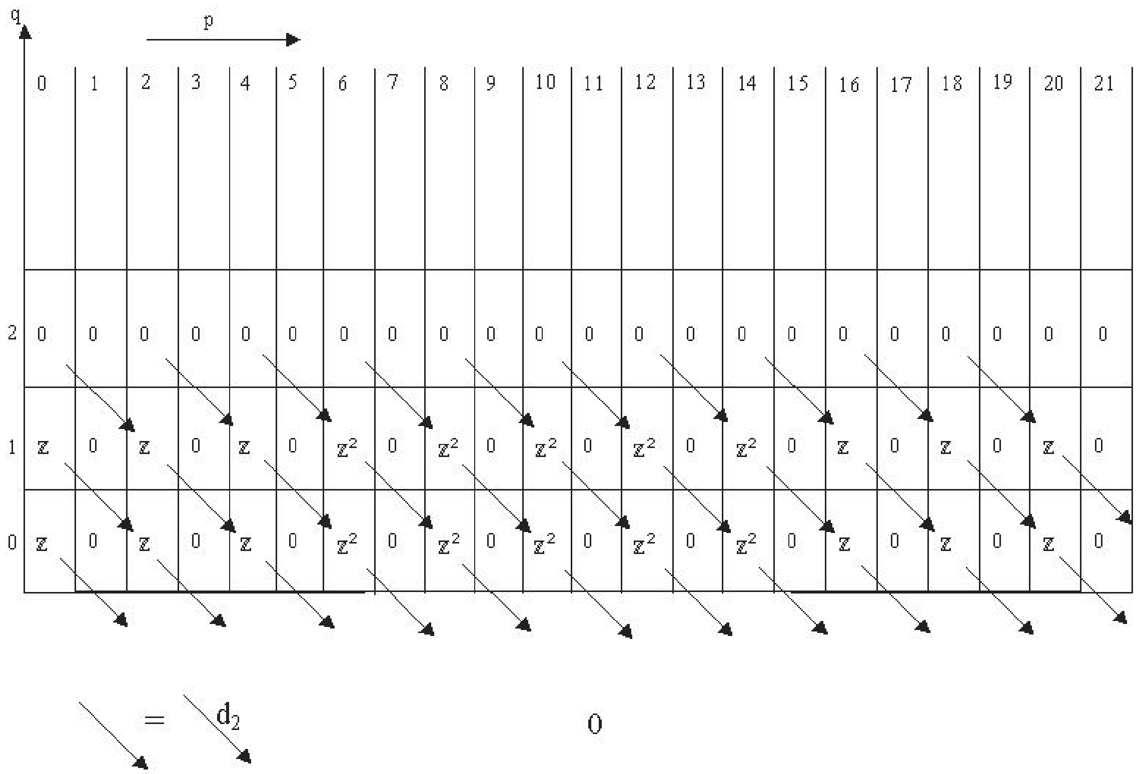


Figure 2 : Term E_2.

where the arrows mean the non trivial action of the operator $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$. As $E_2^{p,2} = 0$, then the cohomology of the operator d_2 on $E_2^{p,2} = 0$ is trivial so the term $E_3^{p,2} = 0$. Since the operator d_3 moves down two steps then $d_3 = 0$. Similarly $d_4 = d_5 = ... = 0$. Therefore we have

$$E_3 = H^*(SO(10)/SU(5)).$$

Now let us compute the cohomology of d_2. Let us remember d_2 is a linear operator from E_2 to itself, where

$$d_2 : E_2^{p,1} \to E_2^{p+2,0}, \quad p = 2k,\ k = 1, ..., 10$$

are the possible non trivial actions. Since we are just interested in the cohomology groups

$$H^2(SO(10)/SU(5)) = E_3^{2,0} \oplus E_3^{1,1} \oplus E_3^{0,2} = E_3^{2,0}$$

and

$$H^4(SO(10)/SU(5)) = E_3^{4,0} \oplus E_3^{3,1} \oplus E_3^{2,2} \oplus E_3^{3,1} \oplus E_3^{0,4} = E_3^{4,0}$$

then we must compute the image (or kernel) of

$$d_2 : E_2^{0,1} \to E_2^{2,0} \quad \text{and} \quad d_2 : E_2^{2,1} \to E_2^{4,0}.$$

As the two previous maps are applications from $\mathbb{Z}$ to $\mathbb{Z}$, i.e $d_2 : \mathbb{Z} \to \mathbb{Z}$ so our analysis will be the same for both.
Let η be the only generator of $E_2^{0,1} = \mathbb{Z}$ ($E_2^{2,1}$), then from (2.123) η satisfies

$$\mathrm{d}\eta_\alpha = 0, \qquad (\delta\eta)_{\alpha\beta} = \mathrm{d}c_{\alpha\beta}$$

where $c_{\alpha\beta}$ is some 0-form and 1-Čech cocycle (for $E_2^{2,1}$ we have $\eta_{\alpha\beta\gamma}$). Now, the operator d_2 is defined as $d_2[\eta_\alpha] = [(\delta c)_{\alpha\beta\gamma}] \in E_2^{2,0}$. If d_2 is a null operator then $(\delta c)_{\alpha\beta\gamma} = \mathrm{d}k_{\alpha\beta\gamma}$, but since $c_{\alpha\beta}$ is a 0-form then $(\delta c)_{\alpha\beta\gamma} = 0$. Therefore if d_2 is a null operator we have

$$\mathrm{d}\eta_\alpha = 0, \qquad (\delta\eta)_{\alpha\beta} = \mathrm{d}c_{\alpha\beta}, \qquad (\delta c)_{\alpha\beta\gamma} = 0.$$

However, as the Čech cohomology on the abelian group of differential forms is trivial [94][29], then we can write

$$\eta_\alpha = \mathrm{d}m_\alpha$$

where $c_{\alpha\beta} = (\delta m)_{\alpha\beta}$, which is a contradiction because η_α is the generator of $E_2^{0,1}$ ($E_2^{2,1}$), i.e it is a non trivial element of $E_2^{0,1}$ ($E_2^{2,1}$). Note that in the group $E_2^{2,1}$ we have $\eta_{\alpha\beta\gamma} = \mathrm{d}m_{\alpha\beta\gamma} + (\delta k)_{\alpha\beta\gamma}$, where $k_{\alpha\beta}$ is some 2-Čech cochain 1-form. Since the cohomology class of $\eta_{\alpha\beta\gamma}$ is the same as $\eta_{\alpha\beta\gamma} - \mathrm{d}m_{\alpha\beta\gamma}$ then this implies that η is trivial, i.e $\eta_{\alpha\beta\gamma} = (\delta k)_{\alpha\beta\gamma}$.
So the operator $d_2 : \mathbb{Z} \to \mathbb{Z}$ is a non null linear map and therefore its image is the whole space $\mathbb{Z}$.
Finally we can conclude the second and fourth group of the de Rham cohomology are trivial

$$H^2(SO(10)/SU(5)) = 0$$

$$H^4(SO(10)/SU(5)) = 0$$

and the space $PS \setminus \{0\}$ is anomaly free.

Chapter 3

A New Proposal for the Picture Changing Operators in the Minimal Pure Spinor Formalism

In this chapter we will make a new proposal for the lowering picture changing operators, so in the following, we will discuss some facts which led us to them. First of all, the pure spinor condition defines a space, also called the pure spinor cone. In the geometric treatment presented on the previous chapter [18] it was found that the pure spinor space has non-vanishing first Pontryagin class, as well as non-vanishing first Chern class; leading to anomalies in the pure spinor space diffeomorphism and worldsheet conformal symmetry respectively. Nevertheless, the careful analysis showed that these anomalies are canceled by removing the tip of the cone i.e the point $\lambda^\alpha = 0$. Therefore, in order to have a well defined theory, one should remove this point from the pure spinor space. Secondly, according to Berkovits' prescription for computing scattering amplitudes [48], in order to match the 11 pure spinor zero-modes in the minimal formalism, one should introduce 11 lowering PCO's defined by

$$Y^I_{Old} = C^I_\alpha \theta^\alpha \delta(C^I_\alpha \lambda^\alpha), \quad \text{for } I = 1 \ldots 11, \tag{3.1}$$

where C^I_α are constant spinors. The integration over the pure spinor zero-modes is performed without removing the point $\lambda^\alpha = 0$ *. A third consideration that suggests for another treatment for the PCO's comes from the higher dimensional twistor transform using pure spinor; which allowed to obtain higher-dimensional scalar Green's functions [21], [22]. As shown in [22], in order to integrate over the projective pure spinor space when $d > 6$, it was necessary to develop integration techniques because of the non-linearity of the pure spinor conditions. Those integrations are always integrations over cycles. These three considerations lead us to define a new lowering PCO, given by $Y^I_{New} = \frac{C^I_\alpha \theta^\alpha}{C^I_\alpha \lambda^\alpha}$. In this way the integration over the pure spinor zero-modes is performed as a multidimensional Cauchy integral, where the integration contours go around the anomalous point $\lambda^\alpha = 0$. As we will

*Here is worth to mention that the geometric treatment of [18] was posterior to the multiloop scattering amplitude prescription of [48].

discuss in this chapter, the new PCO fulfill our requirement and as a bonus, allows to establish elegant relationships between the minimal formalism and the twistor space, as well as between the minimal and non-minal formalisms, which will be discussed in the chapter 5. Furthermore, as was shown explicitly by tree and one-loop computations, given the distributional definition of the PCO's Y^I_{Old}, the scattering amplitudes depends on the constant spinors C^I; so for some choices of these C^I's, the theory is non-Lorentz invariant and the unphysical states do not decouple [23]. These issues were solved by integrating over the C^I's [23], [24]. In contrast, with our PCO's proposal there is no need to integrate over them. We will also formally prove that at tree level the unphysical states decouple and that the scattering amplitude does not depend on the constant spinors C^I's.

Although we only consider tree-level scattering amplitudes in this thesis, we hope to make some progress at the loop level in the future, by also redefining the raising PCO's.

The organization of this chapter is as follows. In section 3.1 we briefly review the minimal pure spinor formalism, where we focus in introducing the basic notation in order to write down the tree-level scattering amplitude prescription of [48].

In the following section we make our proposal for the new set of PCO's and discuss the restriction that must be imposed in order to have a well defined multidimensional Cauchy-type integral, which will result in the condition that the integration cycles go around the anomalous point of the theory $\lambda^\alpha = 0$. It happens that this condition is related to the specific choice of the constant spinors C^I's; so we will give two examples, one where the C^I's choice does not allows to define contours around the origin and another one which does. It turns out that the first choice is the same made in [23], which will allow to make some comparisons.

In section 3.3 we will compute the tree-level scattering amplitude. We start by formally defining the integration contours. Then, we proceed to write the amplitude using the projective pure spinor coordinates. Using these coordinates we analyze the poles structure and express the result of the scattering amplitude in terms of the "degree" of the projective pure spinor space, which will be useful to relate the minimal and non-minimal formalism and to prove that the scattering amplitude is independent of the constant spinors C^I's in the chapter 5. Due to the -Čech-Dolbeault formulation will be extremely useful, both for proving the invariances as well as the decoupling of unphysical states we include a section about this subject (section 3.4).

In 3.5 we show that the scattering amplitude is invariant under BRST, Lorentz and supersymmetry transformations. Also we show the decoupling of unphysical states.

In section 3.6 we will establish a direct relation between pure spinor scattering amplitudes and Green's functions for massless scalar fields in ten dimensions.

In section 3.7 we will comment about what should be done in order to have a genus g formulation for the scattering amplitude. In particular, we define a product for Čech cochains which would allow to get a well defined scattering amplitude from the Čech point of view.

3.1 Review of Tree-level Prescription in The Minimal Pure Spinor Formalism

In this section we will review the tree-level N-point amplitude prescription given in [48]. As noted in [23], the picture changing operators are not BRST closed inside the correlators, leading to a more careful treatment for decoupling the unphysical states.

Since the pure spinor λ^α have 11 zero modes† (one for each degree of freedom) in any Riemann surface Σ_g, it is necessary to absorb them when computing scattering amplitudes. The manner that they are absorbed is by introducing 11 PCO's

$$Y_C^I = C_\alpha^I \theta^\alpha \delta(C_\alpha^I \lambda^\alpha), \quad I = 1, \ldots 11, \tag{3.2}$$

inside the scattering amplitude [48], which for N-points at tree level is

$$\mathcal{A} = \langle V_1(z_1) V_2(z_2) V_3(z_3) \int dz_4 U_4(z_4) \ldots \int dz_N U_N(z_N) Y_C^1(y_1) \ldots Y_C^{11}(y_{11}) \rangle, \tag{3.3}$$

where the $V_i(z_i)$'s are three unintegrated vertex operators and the $U_j(z_j)$ are N-3 integrated vertex operators. The three unintegrated vertex operator are necessary to fix the residual symmetry of the conformal Killing vectors over the sphere.
To perform this computation the OPE's

$$N^{mn}(y) N^{pq}(z) \to \frac{\eta^{p[n} N^{m]q}(z) - \eta^{q[n} N^{m]p}(z)}{y - z} - 3 \frac{\eta^{q[m} \eta^{n]p}}{(y - z)^2} \tag{3.4}$$

$$N_{mn}(y) \lambda^\alpha(z) \to \frac{1}{2} \frac{(\gamma_{mn} \lambda)^\alpha(z)}{y - z}, \tag{3.5}$$

and

$$d_\alpha(z) d_\beta(w) \sim -\frac{2}{\alpha'} \frac{\gamma_{\alpha\beta}^m \Pi_m}{z - w}, \quad d_\alpha(z) \Pi^m(w) \sim \frac{\gamma_{\alpha\beta}^m \partial\theta^\beta}{z - w},$$

$$d_\alpha(z) f(\theta(w), x(w)) \sim (z - w)^{-1} D_\alpha f(\theta(w), x(w)),$$

where (see section 1.2)

$$d_\alpha = p_\alpha - \frac{1}{\alpha'} \gamma_{\alpha\beta}^m \theta^\beta \partial x_m - \frac{1}{4\alpha'} \gamma_{\alpha\beta}^m \gamma_{m\,\gamma\delta} \theta^\beta \theta^\gamma \partial\theta^\delta, \quad \Pi^m = \partial x^m + \frac{1}{2} \theta \gamma^m \partial\theta,$$

†the spinors ω_α, p_α and θ^α have $11g$, $16g$ and 16 zero modes respectively for the Riemann surface of genus g

and

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\theta^\beta\gamma^m_{\alpha\beta}\partial_m,$$

is the covariant super-derivate on $\mathbb{R}^{10}$, are used to integrate over the non-zero modes, remaining an integral over the zero modes of the pure spinor and $\theta^{\alpha\ddagger}$.

Since the integration measure $[\mathrm{d}\lambda]$, given in (3.132), has ghost number 8 and the product between the 11 PCO's has ghost number -11 then the product between the all vertex operator in the scattering amplitude must have a contribution with ghost number 3 (this contribution comes from the three unintegrated vertex operators). Now, as the 11 PCO's have 11 grassmann variables θ^α's then the vertex operators must contribute with 5 θ^α's in order to obtain an amplitude different to zero. So a non-vanishing tree-level amplitude in the pure spinor formalism is proportional to $\lambda^3\theta^5$. In [48] was shown there is only one scalar generator with ghost number 3 in the cohomology of Q and it can be written in the following way

$$(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta),$$

therefore, before to integrate the zero modes of the variables λ^α and θ^α, the contribution of the vertex operators must be always proportional to this scalar.

Note that the C^I_α is a constant projective spinor, which can be thought as a point in the $\mathbb{C}P^{15}$ space. Although in [48] it was argued that the scattering amplitude was independent of the constant spinors C^I_α, it was later found in [23] that indeed the amplitude depends on the choice of C^I_α and also that Q exact states do not decouple. In the next section, we propose a new picture operators, which does not have that disadvantage.

3.2 The New Picture Changing Operators

In this section we introduce the new lowering picture changing operators. In particular, we will discuss why with this new proposal for the picture changing operators, the origin must be removed from the pure spinor space. This will allow to write the tree-level scattering amplitude in terms of the projective pure spinor variables in the following section, and also, to find a relationship with the twistor space in section 3.6. In the end of the present section we give examples of choices for the constant spinors C^I's and discuss their implications.

‡The integration over the fields x is treated in detail in D'Hoker and Phong [69], we will not focus in those integrals.

3.2.1 The New Proposal for the PCO's

In this subsection we discuss some motivations which led us to define new lowering PCO's.
The bosonic spinor λ^α, constrained to satisfy the pure spinor condition $\lambda\gamma^m\lambda = 0$, constitutes an interesting and non-trivial complex space, which will be denoted through this paper as the pure spinor space PS or the pure spinor cone. Since the coordinates λ^α of such space are holomorphic, the integral

$$\int [\mathrm{d}\lambda]\delta(C\lambda)f(\lambda) \tag{3.6}$$

is only well defined if the domain of integration, i.e the cycles around which we integrate are known. Moreover, as it was shown by Nekrasov [18] (see previous chapter), the tip of the pure spinor cone $\lambda^\alpha = 0$ introduces anomalies. Then, by removing this point of the pure spinor space, the theory is anomaly free [§]. This is simple to see if one computes the de-Rham cohomology of the pure spinor minus the origin space

$$H^i\left(PS \setminus \{0\}\right) = \mathbb{R}, \quad \text{for } i = 0, 6, 15, 21, \tag{3.7}$$

so the first Chern class and the first Pointryagin class both vanish, $c_1(PS \setminus \{0\}) = p_1(PS \setminus \{0\}) = 0$ and therefore the theory is anomaly free, we described this outcome in the previous chapter. This motivates us to make a new proposal for the PCO's, in such a way that the tip of the cone is naturally excluded. Furthermore, Skenderis and Hoogeveen [23] showed that the scattering amplitude, as formulated in [48], depends on the choice of the constant spinors C^I_α, having to integrate over them in order to obtain a manifestly Lorentz invariant prescription. Nevertheless, as we will show in the chapter 5, the scattering amplitude will not depend on the constant spinors using the new PCO's.
Our proposal, which seems to be the most natural, is to define the PCO's as

$$Y^I_C = \frac{C^I_\alpha\theta^\alpha}{C^I_\alpha\lambda^\alpha}, \qquad I = 1, \ldots 11, \tag{3.8}$$

where C^I_α are again constant spinors. Just like the standard PCO's (3.2), this new PCO's are not manifestly Lorentz invariant. Note that Y^I_C is in fact θ^α times the Y-operator introduced in [60]. Also, since $QY^I_C = 1$, they are not BRST closed. So, any BRST closed state Ψ will be BRST exact because $\Psi = (QY^I_C)\Psi = Q(Y^I_C\Psi)$. However, it will be necessary to modify the usual BRST charge of the minimal formalism by adding the Čech operator δ, as will be done in the section 3.5. It turns out that with this modification, the description is global and we will be able to show

[§]This is the unique singular point of the pure spinor space because it is a complex cone over the smooth manifold $SO(10)/U(5) \subset \mathbb{C}P^{15}$.

in that section that the scattering amplitude is BRST, Lorentz and supersymmetric invariant.

Since we want to integrate over the pure spinor zero modes, basically as a multi-dimensional Cauchy's integral, we will start by considering the analogous of the poles. This role will be played by the denominators of the PCO's, so we start by defining the functions

$$f^I(\lambda) \equiv C^I_\alpha \lambda^\alpha, \tag{3.9}$$

which map the pure spinor space to the complex numbers for each value of $I = 1, ..., 11$, i.e $f^I : PS \rightarrow \mathbb{C}$. Given these functions, secondly we define the hypersurface "D_I" as the subspace $f^I = 0$

$$D_I = \{\lambda^\alpha \in PS : \, C^I_\alpha \lambda^\alpha = 0\}. \tag{3.10}$$

In order to have a well defined integration over the pure spinor space inside the scattering amplitude, it is necessary to impose the condition that the intersection between the D_I's satisfies $D_1 \cap D_2 ... \cap D_{11} = \{\text{finite number of points}\}$ in order to have a Cauchy like integral over PS. Just to be more explicit, using the $U(5)$ decomposition [47] for writing the pure spinor constraint, we require that the 16 equations

$$\begin{aligned} &f^I = 0, \quad \text{and} \quad \chi^a = \lambda^+ \lambda^a - \frac{1}{8}\epsilon^{abcde}\lambda_{bc}\lambda_{de} = 0, \\ &\text{with} \quad I = 1, 2, ..., 11; \quad a, b, c, d, e = 1, 2, ..., 5, \end{aligned} \tag{3.11}$$

intersect in a finite number of points. However, the five equations $\chi^a = 0$ must be taken carefully because with only this condition, there are more singular points besides $\lambda^\alpha = 0$. Therefore, a second set of equations $\zeta_a = \lambda^b \lambda_{ba} = 0$, must be taken into account. Both set of conditions $\chi^a = 0$ and $\zeta_a = 0$ come from the $U(5)$ decomposition of the pure spinor condition [47]. Although the first one implies in the second one when $\lambda^+ \neq 0$, as will be explained with one example in appendix B.1.1, disregarding the second one could lead to a not well defined tangent space at every point of PS. Therefore, both conditions will be considered when we construct an example for the C^I's in subsection 3.2.2.

To demand that the constant spinors C^I's are linearly independent in $\mathbb{C}^{16}$ is not enough to obtain an intersection in a finite number of points. However, clearly the origin $\{0\}$ is a common point in the intersection of all the hypersurfaces D_I. We claim that the only common point between the hypersurfaces D_I's is the origin because precisely, it is the unique anomalous or singular point of the theory. Therefore, the integration contours are those that go around the origin of the pure spinor space.

In the following subsection we give an example for the constant spinors C^I_α which allow for such a type of intersection.

3.2.2 Some Examples for the Constant Spinors C^I_α

In this subsection we will consider two examples. One where the C^I's are linearly independent, although do not allow for an intersection of the hypersurfaces D_I in a finite number of points. In the second example, we construct a set of C^I's which intersect just in the origin.

First Example We will make the same choice for the C^I's as in [23], so we consider this example basically to establish a comparison with this reference. Let the C^I_α's be in the $U(5)$ representation:

$$C^I_\alpha = (C^I_+, C^{I,ab}, C^I_a) \qquad a, b = 1, ..., 5,$$

where $C^{I,ab} = -C^{I,ba}$. Making the choice of [23]

$$C^1_\alpha = \delta^+_\alpha, \quad C^{2,ab} = \delta^{[a}_1 \delta^{b]}_2, \cdots, C^{11,ab} = \delta^{[a}_4 \delta^{b]}_5, \quad \text{all other} \quad C^I_\alpha = 0, \tag{3.12}$$

the functions f^I's are

$$f^1 = \lambda^+, \quad f^2 = \lambda_{12}, \quad f^3 = \lambda_{13}, \cdots, \; f^{11} = \lambda_{45}. \tag{3.13}$$

With the conditions $f^I = 0$ the pure spinor constraints are satisfied identically, but the parameters λ^a's are free, therefore the intersection is the space $\mathbb{C}^5$, in contrast with our requirement of intersecting just in the origin. With this choice we can "naively" compute the three point tree level amplitude only locally ($\lambda^+ \neq 0$), obtaining the same result as in [23] as we will review below. The answer will not be Lorentz invariant. For 3-points the computation is as follows:

$$\begin{aligned}
\mathcal{A} &= \langle \lambda^\alpha A_{1\alpha}(z_1) \lambda^\beta A_{2\beta}(z_2) \lambda^\gamma A_{3\gamma}(z_3) Y^1(z) \ldots Y^{11}(z) \rangle \\
&= \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \frac{C^1\theta}{C^1\lambda} \cdots \frac{C^{11}\theta}{C^{11}\lambda} \\
&= \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \frac{\theta^+}{\lambda^+} \frac{\theta_{12}}{\lambda_{12}} \cdots \frac{\theta_{45}}{\lambda_{45}} \\
&= \int_\Gamma \frac{\mathrm{d}\lambda^+ \wedge \mathrm{d}\lambda_{12} \wedge \ldots \wedge \mathrm{d}\lambda_{45}}{(\lambda^+)^3} \int \mathrm{d}^{16}\theta \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \frac{\theta^+}{\lambda^+} \frac{\theta_{12}}{\lambda_{12}} \cdots \frac{\theta_{45}}{\lambda_{45}}.
\end{aligned} \tag{3.14}$$

where Γ is defined as $\Gamma = \{\lambda \in PS : |f^I| = \epsilon^I, I = 1, ..., 11, \epsilon^I \in \mathbb{R}^+\}$ and $[\mathrm{d}\lambda] = \mathrm{d}\lambda^+ \wedge \mathrm{d}\lambda_{12} \wedge \ldots \wedge \mathrm{d}\lambda_{45}/(\lambda^+)^3$ [48]. Note that naively $\lambda^+ = 0$ is a singularity, but we do not have access to it since we are on the patch $\lambda^+ \neq 0$. So, for this coordinate is possible that the cycle of integration is not well defined. Formally, we should choose a patch which allows to access the singularity. In this particular example, the singularity is $\mathbb{C}^5$, which is a non-compact and infinite space, that can

not be contoured with a compact space defined by some cycle Γ. Therefore, in the Cauchy's sense this is a not well defined integral. That is what we meant with naively computing the integral.
The only contribution to the integral above will come from $\alpha = \beta = \gamma = +$. In our case, in contrast with [23], there are no subtleties with the integrals coming from the other choices, which are of the form $\int_\Gamma \mathrm{d}\lambda_{ab} \frac{\lambda_{ab}}{\lambda_{ab}}$. For example,

$$\int_\Gamma [\mathrm{d}\lambda](\lambda^+)^2 \lambda_{cd} \frac{1}{\lambda^+} \frac{1}{\lambda_{12}} \cdots \frac{1}{\lambda_{45}} = \int_\Gamma \mathrm{d}\lambda^+ \mathrm{d}^{10}\lambda_{ab} \frac{\lambda_{cd}}{\lambda^+} \frac{1}{\lambda^+} \frac{1}{\lambda_{12}} \cdots \frac{1}{\lambda_{45}} \tag{3.15}$$

will give zero because there is a double pole in λ^+ and any choice of λ_{cd} will kill one of the poles λ_{ab}. Choosing $\alpha = \beta = \gamma = +$ we obtain

$$\mathcal{A} = \int \mathrm{d}^{16}\theta f_{+++}(\theta)\theta^+\theta_{12} \ldots \theta_{45}, \tag{3.16}$$

which is exactly the same answer found by Skenderis and Hoogeveen in [23], as in their case, it is not Lorentz invariant. Now we give a geometrical explanation of why it is not Lorentz invariant. Remember that the intersection between the hypersurfaces is $\mathbb{C}^5$, $D_1 \cap \ldots \cap D_{11} = \mathbb{C}^5$, so the scattering amplitude is defined on the space

$$PS \setminus \mathbb{C}^5. \tag{3.17}$$

Since the $SO(10)$ group acts transitively up to scalings on the the pure spinor space PS, then it is always possible to have an element $g \in SO(10)$ such that if $\lambda \in (PS \setminus \mathbb{C}^5)$, then $(g\lambda) \notin (PS \setminus \mathbb{C}^5)$, i.e $(g\lambda) \in \mathbb{C}^5$. This argument implies that the scattering amplitude is not Lorentz invariant, since it is not invariant under $SO(10)$, and it is not globally defined on PS, because we can make a transformation from $(PS \setminus \mathbb{C}^5)$ to PS where the scattering is not defined. In the appendix B.1.4 we give further simple examples.
Note that the origin is the only fixed point under $SO(10)$ transformations acting on the pure spinor space¶, this means that the condition for the intersection of the D_I's in the origin, $D_1 \cap \ldots \cap D_{11} = \{0\}$, it is not just a sufficient condition, but actually it is necessary condition in order to get a well defined scattering amplitude, i.e that the scattering amplitude is invariant under the BRST, supersymmetry and Lorentz transformations, see section 3.5. Summarizing, we showed that this specific choice for the C^I's is not allowed, since it does not obey our requirement of the hypersurfaces intersecting at the origin.

Second Example Now, we show how to construct a set of C^I's which allow to satisfy $D_1 \cap \ldots \cap D_{11} = \{0\}$. This geometrical construction is as follows: Take eleven points satisfying the conditions $\chi^a = 0$ and $\zeta_a = 0$. Then, evaluate each one of the

¶This is because the origin is the unique singular point in PS.

10 gradient vectors V^a and A_a, corresponding to χ^a and ζ_a respectively, at each one of those eleven points (see the appendix B.1.1 for more details). With this vectors, we construct 11 planes through the origin, such that at each one of the 11 points in PS, the 10 gradients belong to the planes. We present the answer as an 11×16 matrix

$$C = \begin{pmatrix} 1 & 2 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -4 & -1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & -2 & 4 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -2 & 3 & 4 & 3 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 2 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -3 & 0 & -3 & 0 & 0 & -3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & -2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 2 & -1 & 1 & 0 & 1 \end{pmatrix}. \tag{3.18}$$

We computed $C^I_\alpha \lambda^\alpha$ and using Mathematica, we found the intersections of the 11 planes with the pure spinor condition

$$\chi^a = \lambda^+ \lambda^a - \frac{1}{8}\epsilon^{abcde}\lambda_{bc}\lambda_{de} = 0. \tag{3.19}$$

The answer is 12 times the tip of the cone: $\lambda^\alpha = 0$. This number 12 is the *multiplicity* or number of times the hypersurfaces intersect. This will be further discussed in the subsection (3.3.4). Nevertheless, there are 5 additional non-zero solutions[‖]. This is not an issue, since these non-zero solutions are not in the remaining pure spinor equations $\zeta_a = \lambda^b \lambda_{ba} = 0$, therefore, we can discard them safely. Note that the 11 C^I's form a $\mathbb{C}^5$ space in $\mathbb{C}^{16}$, which is invariant by $U(5)$ group, so applying elements of $U(5)$ to the matrix C^I_α (3.18) we get an infinite numbers of C'^I's, for which the intersection with PS is the origin.

Instead of computing the scattering amplitude in this second example as we did in the first one, we will show in the next section how to find the answer without an explicit form for the C^I's. In conclusion, what we wanted to show with this example is that we can indeed find a set of constant spinors fulfilling our requirement of intersection of the planes and PS only at the origin.

[‖]For these non-zero solution $\lambda^+ = 0$. Those are precisely the points for which the constrains $\chi_a = 0$ are not enough to describe the pure spinor space.

3.3 The Tree Level Scattering Amplitude

In the present section we will compute the scattering amplitude in a covariant way. We start by defining the scattering amplitude and the integration contours. Then, we proceed to perform the scattering amplitude computation in the projective pure spinor space coordinates, where the singular point is explicitly removed and will be useful to get a relationship with the twistor space. This computation will introduce the notion of degree of the projective pure spinor space, which will be to key to relate in a simple way the minimal and non-minimal formalisms in the chapter 5.

3.3.1 Integration Contours

Before attempting to compute the tree level scattering amplitude, we must discuss which are the integration contours. This will allow to have a well defined amplitude. The contours will be given by the homology cycles. In our case, they are naturally defined as

$$\Gamma = \{\lambda^\alpha \in PS : |f^I(\lambda)| = |C^I\lambda| = \varepsilon^I\}. \tag{3.20}$$

Clearly, Γ is an 11-cycle, i.e it has real dimension 11. Except for the integration contour, the tree level scattering amplitude corresponding to the zero modes has the same form as in the first example in the subsection 3.2.2

$$\mathcal{A} = \int \mathrm{d}^{16}\theta \int_\Gamma W, \tag{3.21}$$

where W is given by

$$W = [\mathrm{d}\lambda] Y_C^1 ... Y_C^{11} \lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta) \tag{3.22}$$

and Y_C^I's are the new PCO's (3.8). Since the integrand W satisfies $\mathrm{d}(W) = (\partial + \bar{\partial})(W) = 0$ then it belongs to the de-Rham cohomology group $H^{11}_{DR}(PS \setminus D)$, where $PS \setminus D$ is the space in which the W-form is defined, i.e D is the hypersurface on PS given by $D = D_1 \cup \ldots \cup D_{11}$.
Then, the cycle Γ belongs to the homology group $H_{11}(PS \setminus D, \mathbb{Z})$. We will illustrate this with the following example. Consider for instance the integral $\int_\gamma \mathrm{d}z/z$, where γ is the circle $\gamma = \{z \in \mathbb{C} : |z| = \epsilon\}$. So any circle C around the origin is related to γ since $\gamma - C$ is the boundary of some annulus U, i.e $\partial(U) = \gamma - C$, therefore γ is an element of the homology group $H_1(\mathbb{C} \setminus \{0\}, \mathbb{Z})$ and by the Stokes theorem $\int_\gamma \mathrm{d}z/z = \int_C \mathrm{d}z/z$. In $\mathbb{C}^2$ we have an analogous situation, for example consider the integral $\int_\varphi \mathrm{d}z_1\mathrm{d}z_2/(z_1z_2)$. Here the torus φ, defined by $\varphi = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = \epsilon_1, |z_2| = \epsilon_2\}$ is an element of the homology group $H_2(\mathbb{C}^2 \setminus \{(0, z_2)\, \text{and}\, (z_1, 0) : z_1, z_2 \in \mathbb{C}\}, \mathbb{Z}) = H_2((\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}), \mathbb{Z})$ and the integral depends only of the class of the torus φ. The same is true for the integral (3.21).

Therefore, the integral (3.21) will depend only on the homology class cycle and the cohomology class cocycle. This is the principle that will allow us to show that the scattering amplitude is independent of the C^I's, which will be discussed in the chapter5, section 5.2.

3.3.2 Contours for the Amplitude in the Projective Pure Spinor Coordinates

As we will show in this subsection, in the projective coordinates we can make a simple analysis of the poles in the scattering amplitude integral. The cycle Γ previously defined will be used to obtain the integration contours in the projective pure spinor space.

We can write the pure spinor coordinates as $\lambda^\alpha = \gamma\,\tilde{\lambda}^\alpha$, where $\gamma \in \mathbb{C}$ and $\tilde{\lambda}^\alpha$ are global coordinates for the $SO(10)/U(5)$ space**. That is, $\tilde{\lambda}^\alpha$ satisfies the constraints $\tilde{\lambda}\gamma^m\tilde{\lambda} = 0$ and has the equivalence relation $\tilde{\lambda}^\alpha \sim c\tilde{\lambda}^\alpha$, where $c \in \mathbb{C}^*$. When $\gamma = 0$ then $\lambda^\alpha = 0$, but $\tilde{\lambda}^\alpha$ can take any value in the projective pure spinor space, i.e $SO(10)/U(5)$, also known as the twistor space [22]. In these coordinates the poles take the form

$$
\begin{aligned}
\gamma\tilde{f}^1 &\equiv \gamma\, C^1_\alpha \tilde{\lambda}^\alpha = 0, \\
\gamma\tilde{f}^2 &\equiv \gamma\, C^2_\alpha \tilde{\lambda}^\alpha = 0, \\
&\;\;\vdots \\
\gamma\tilde{f}^{11} &\equiv \gamma\, C^{11}_\alpha \tilde{\lambda}^\alpha = 0.
\end{aligned}
\tag{3.23}
$$

When $\gamma \neq 0$, we have 11 constraints and 10 degrees of freedom for the projective pure spinor space, so, it is not possible to find a solution for the 11 constraints. On the other hand, when $\gamma = 0$, naively all the constraints behave as being zero. Nevertheless, we must consider this case inside the scattering amplitude. In the numerator of W there are 7 γ's coming from the integration measure plus 3 coming from the vertex operators, contributing in total γ^{10} in the numerator. Therefore, only one of the 11 γ's will remain in the denominator of W. This remaining γ kills one of the 11 functions $\tilde{f}^I$. Therefore, now the cycle Γ is given by

$$
\Gamma = C \times \tilde{\Gamma}, \tag{3.24}
$$

where C is the cycle $C = \{\gamma \in \mathbb{C} : |\gamma| = \epsilon\}$ and $\tilde{\Gamma}$ is a 10-cycle which we define in the following. After integrating around the contour $|\gamma| = \varepsilon$, which belongs to Γ and

**Actually γ is the fiber of the $\mathcal{O}(-1)$ line bundle over $SO(10)/U(5)$ [18].

excludes the origin of the space, the denominator of W will have 11 $\tilde{f}^I$'s. However, remember that one of the $\tilde{f}^I$'s was killed by γ. Therefore, the cycle $\tilde{\Gamma}$ must be given by

$$\tilde{\Gamma} = \{|\tilde{f}^i| = \epsilon_i, \text{where the } i\text{'s are ten numbers between 1 to 11}\}. \tag{3.25}$$

After this simple analysis, now we proceed to compute the scattering amplitude.

3.3.3 The Tree Level Scattering Amplitude in the Projective Pure spinor Space

In the last two subsections we have defined the integration contours in the pure spinor space and projective pure spinor space in order to have a well defined tree level scattering amplitude. Now, in this subsection we will proceed to compute the tree-level scattering amplitude.

The tree-level scattering amplitude has the form

$$\mathcal{A} = \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta Y_C^1 ... Y_C^{11} \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta). \tag{3.26}$$

As discussed in [48], the term $\lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta)$ can always be written in the following form

$$\lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \propto (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)K, \tag{3.27}$$

up to BRST exact and global terms, which are decoupled as we will show later in section 3.5. K is the kinematic factor, which is a function of the polarizations and momenta[††]. Then the amplitude takes the form

$$\begin{aligned} \mathcal{A} &= \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \frac{C^1\theta}{C^1\lambda} ... \frac{C^{11}\theta}{C^{11}\lambda} (\lambda\gamma^m)_{\alpha_1} (\lambda\gamma^n)_{\alpha_2} (\lambda\gamma^p)_{\alpha_3} (\gamma_{mnp})_{\alpha_4\alpha_5} \theta^{\alpha_1}\theta^{\alpha_2}\theta^{\alpha_3}\theta^{\alpha_4}\theta^{\alpha_5} K \\ &= \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \frac{\epsilon^{\alpha_1...\alpha_5\beta_1...\beta_{11}} C^1_{\beta_1} ... C^{11}_{\beta_{11}}}{C^1\lambda ... C^{11}\lambda} \\ &\qquad (\lambda\gamma^m)_{\alpha_1} (\lambda\gamma^n)_{\alpha_2} (\lambda\gamma^p)_{\alpha_3} (\gamma_{mnp})_{\alpha_4\alpha_5} \theta^1 ... \theta^{16} K. \end{aligned} \tag{3.28}$$

In the coordinates $\lambda^\alpha = \gamma\tilde{\lambda}^\alpha$, we can choose the following parametrization for the projective pure spinor in the patch $\tilde{\lambda}^+ \neq 0$

$$\tilde{\lambda}^\alpha = (1, u_{ab}, \frac{1}{8}\epsilon^{abcde} u_{bc} u_{de}). \tag{3.29}$$

So, as shown in [18], the integration measure becomes $[\mathrm{d}\lambda] = \gamma^7 \mathrm{d}\gamma \wedge \mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45}$ and the amplitude locally can be written as

$$\begin{aligned} \mathcal{A} = \int_\Gamma \frac{\mathrm{d}\gamma}{\gamma} \wedge \frac{\mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45} \epsilon^{\alpha_1...\alpha_5\beta_1...\beta_{11}} C^1_{\beta_1} ... C^{11}_{\beta_{11}}}{C^1\tilde{\lambda} ... C^{11}\tilde{\lambda}} \\ (\tilde{\lambda}\gamma^m)_{\alpha_1} (\tilde{\lambda}\gamma^n)_{\alpha_2} (\tilde{\lambda}\gamma^p)_{\alpha_3} (\gamma_{mnp})_{\alpha_4\alpha_5} K, \end{aligned} \tag{3.30}$$

[††] In general, when there are more than 3 vertex operators in the scattering amplitude, it must include integrals of the worldsheet coordinates $(z, \bar{z})$. However, we are not taking care of those terms.

where the θ^α variables have been integrated. The integral around the contour $|\gamma| = \varepsilon$ is trivial, then

$$\mathcal{A} = (2\pi i)\int_{\tilde{\Gamma}} \frac{\mathrm{d}u_{12}\wedge ... \wedge \mathrm{d}u_{45}\epsilon^{\alpha_1...\alpha_5\beta_1...\beta_{11}}C^1_{\beta_1}...C^{11}_{\beta_{11}}}{C^1\tilde{\lambda}...C^{11}\tilde{\lambda}} (\tilde{\lambda}\gamma^m)_{\alpha_1}(\tilde{\lambda}\gamma^n)_{\alpha_2}(\tilde{\lambda}\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}K, \tag{3.31}$$

where the contour $\tilde{\Gamma}$ was defined in (3.25). Note that up to a sign, the scattering amplitude is independent of the choice of the 10-cycle out of the 11 possibilities. To illustrate that we can consider the simplest and non trivial case of the projective pure spinor space, i.e the projective pure spinor space in $d = 4$, in this case the integral is (see also the appendix B.1.3)

$$\int_{\tilde{\gamma}} \frac{\epsilon_{ab}\tilde{\lambda}^a \mathrm{d}\tilde{\lambda}^b \epsilon^{cd} C^1_c C^2_d}{(C^1\tilde{\lambda})(C^2\tilde{\lambda})}, \tag{3.32}$$

where $\tilde{\lambda}^a = (\tilde{\lambda}^1, \tilde{\lambda}^2)$ are the homogeneous coordinates of $\mathbb{C}P^1$. In this case we have two choices. First we can take $\tilde{\gamma} = \{\tilde{\lambda} \in \mathbb{C}P^1 : |C^1\tilde{\lambda}| = \epsilon\}$ and for simplicity we set $C^1 = (1, 0)$ and $C^2 = (0, 1)$. In the patch $\tilde{\lambda}^2 \neq 0$ we have the parametrization $\tilde{\lambda}^a = (u, 1)$, therefore, the contour $\tilde{\gamma}$ is well defined and the integral (3.32) is

$$\int_{|u|=\epsilon} \frac{\mathrm{d}u}{u}. \tag{3.33}$$

Note that in the patch $\tilde{\lambda}^1 \neq 0$ the cycle γ is not well defined. The second choice is $\tilde{\gamma} = \{\tilde{\lambda} \in \mathbb{C}P^1 : |C^2\tilde{\lambda}| = \epsilon\}$. Here we must take the patch $\tilde{\lambda}^1 \neq 0$, where the parametrization is given by $\tilde{\lambda}^a = (1, v)$. Then, the integral (3.32) becomes

$$-\int_{|v|=\epsilon} \frac{\mathrm{d}v}{v}. \tag{3.34}$$

So, we have shown for the $d = 4$ projective pure spinor space, that different choices of the cycle $\tilde{\gamma}$ results just in changing the sign of (3.32). This fact can also be seen in a covariant manner. From the identity $\epsilon_{ab}\epsilon^{cd} = \delta^c_a\delta^d_b - \delta^c_b\delta^d_a$, we obtain

$$\int_{\tilde{\gamma}} \frac{\epsilon_{ab}\tilde{\lambda}^a \mathrm{d}\tilde{\lambda}^b \epsilon^{cd} C^1_c C^2_d}{(C^1\tilde{\lambda})(C^2\tilde{\lambda})} = \int_{\tilde{\gamma}} \frac{d(C^1\tilde{\lambda})(C^2\tilde{\lambda})}{(C^1\tilde{\lambda})(C^2\tilde{\lambda})} - \int_{\tilde{\gamma}} \frac{d(C^2\tilde{\lambda})(C^1\tilde{\lambda})}{(C^2\tilde{\lambda})(C^1\tilde{\lambda})} = \int_{\tilde{\gamma}} \frac{d(C^1\tilde{\lambda})}{C^1\tilde{\lambda}} - \int_{\tilde{\gamma}} \frac{d(C^2\tilde{\lambda})}{C^2\tilde{\lambda}}. \tag{3.35}$$

Now, considering the first contour $\tilde{\gamma} = \{\tilde{\lambda} \in \mathbb{C}P^1 : |C^1\tilde{\lambda}| = \epsilon\}$ as above, we obtain

$$\int_{\tilde{\gamma}} \frac{\epsilon_{ab}\tilde{\lambda}^a \mathrm{d}\tilde{\lambda}^b \epsilon^{cd} C^1_c C^2_d}{(C^1\tilde{\lambda})(C^2\tilde{\lambda})} = \int_{\tilde{\gamma}} \frac{d(C^1\tilde{\lambda})}{C^1\tilde{\lambda}}, \tag{3.36}$$

where the second term vanishes since $\tilde{\gamma}$ does not contour the pole $C^2\tilde{\lambda} = 0$. Similarly, choosing the second contour $\tilde{\gamma} = \{\tilde{\lambda} \in \mathbb{C}P^1 : |C^2\tilde{\lambda}| = \epsilon\}$ we obtain

$$\int_{\tilde{\gamma}} \frac{\epsilon_{ab}\tilde{\lambda}^a \mathrm{d}\tilde{\lambda}^b \epsilon^{cd} C^1_c C^2_d}{(C^1\tilde{\lambda})(C^2\tilde{\lambda})} = -\int_{\tilde{\gamma}} \frac{d(C^2\tilde{\lambda})}{C^2\tilde{\lambda}}, \tag{3.37}$$

therefore, the $d = 4$ pure spinor integral does not vanish and the choice of the contour only affects the sign. The value of the integral was also computed in the appendix B.1.3, giving $2\pi i$. The same argument about the change of sign with the choice of the contours holds for the ten dimensional projective pure spinor space.

The integration measure in (3.31), $\mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45}$, is the same found in a covariant manner by Berkovits and Cherkis in [22]. Therefore, we have the following identity.

Identity If $\tilde{\lambda}^\alpha$ is an element of the projective pure spinor space in 10 dimensions, i.e. if $\tilde{\lambda}^\alpha \in SO(10)/U(5)$, then the integration measure $[\mathrm{d}\tilde{\lambda}]$ defined by [22]

$$[\mathrm{d}\tilde{\lambda}](\tilde{\lambda}\gamma^m)_{\alpha_1}(\tilde{\lambda}\gamma^n)_{\alpha_2}(\tilde{\lambda}\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} = \frac{2^3}{10!}\epsilon_{\alpha_1...\alpha_5\beta_1...\beta_{11}}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge...\wedge\mathrm{d}\tilde{\lambda}^{\beta_{10}}\tilde{\lambda}^{\beta_{11}}, \quad (3.38)$$

written in the parametrization $\tilde{\lambda}^\alpha = (\tilde{\lambda}^+, \tilde{\lambda}_{ab}, \tilde{\lambda}^a) = (1, u_{ab}, \frac{1}{8}\epsilon^{abcde}u_{bc}u_{de})$ is

$$[\mathrm{d}\tilde{\lambda}] = \mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45}. \quad (3.39)$$

This identity is proved in the appendix B.2
With this identity in mind, the amplitude (3.31) can be written in a covariant manner with respect to $SO(10)/U(5)$

$$\begin{aligned}\mathcal{A} &= (2\pi i)\int_{\tilde{\Gamma}} \frac{[\mathrm{d}\tilde{\lambda}]\epsilon^{\alpha_1...\alpha_5\beta_1...\beta_{11}}C^1_{\beta_1}...C^{11}_{\beta_{11}}}{C^1\tilde{\lambda}...C^{11}\tilde{\lambda}} \\ &\qquad (\tilde{\lambda}\gamma^m)_{\alpha_1}(\tilde{\lambda}\gamma^n)_{\alpha_2}(\tilde{\lambda}\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}K.\end{aligned} \quad (3.40)$$

This integral is the same as the 10 dimensional integral found in [22], so it is possible to have a twistor type version for the scattering amplitude at tree level. In [22] the integral is solved up to a proportionality factor. However, we will find a rigorous solution. Using (3.39) in (3.40) we get

$$\begin{aligned}\mathcal{A} &= (2\pi i)2^3\int_{\tilde{\Gamma}}\frac{1}{10!}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge...\wedge\mathrm{d}\tilde{\lambda}^{\beta_{10}}\tilde{\lambda}^{\beta_{11}}\epsilon_{\alpha_1...\alpha_5\beta_1...\beta_{11}}\frac{\epsilon^{\alpha_1...\alpha_5\gamma_1...\gamma_{11}}C^1_{\gamma_1}...C^{11}_{\gamma_{11}}}{C^1\tilde{\lambda}...C^{11}\tilde{\lambda}}K \\ &= (2\pi i)2^3\int_{\tilde{\Gamma}}\frac{5!}{10!}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge...\wedge\mathrm{d}\tilde{\lambda}^{\beta_{10}}\tilde{\lambda}^{\beta_{11}}\delta^{\gamma_1}_{[\beta_1}\delta^{\gamma_2}_{\beta_2}...\delta^{\gamma_{11}}_{\beta_{11}]}\frac{C^1_{\gamma_1}...C^{11}_{\gamma_{11}}}{C^1\tilde{\lambda}...C^{11}\tilde{\lambda}}K.\end{aligned} \quad (3.41)$$

Without loss of generality, we take $C^1\tilde{\lambda}, ..., C^{10}\tilde{\lambda}$ to define $\tilde{\Gamma}$, then (3.41) becomes

$$\begin{aligned}\mathcal{A} &= (2\pi i)2^3\int_{\tilde{\Gamma}}5!\frac{(\mathrm{d}C^1\tilde{\lambda})\wedge...\wedge(\mathrm{d}C^{10}\tilde{\lambda})(C^{11}\tilde{\lambda})}{C^1\tilde{\lambda}...C^{11}\tilde{\lambda}}K \\ &= (2\pi i)2^3 5!\int_{\tilde{\Gamma}}\frac{(\mathrm{d}\tilde{f}^1)\wedge...\wedge(\mathrm{d}\tilde{f}^{10})}{\tilde{f}^1...\tilde{f}^{10}}K\end{aligned} \quad (3.42)$$

where $\tilde{f}^I = C^I\tilde{\lambda}$. The others terms, like

$$\int_{\tilde{\Gamma}}\frac{(C^1\tilde{\lambda})(\mathrm{d}C^2\tilde{\lambda})\wedge...\wedge(\mathrm{d}C^{10}\tilde{\lambda})\wedge(\mathrm{d}C^{11}\tilde{\lambda})}{C^1\tilde{\lambda}...C^{11}\tilde{\lambda}}$$

do not contribute since one of the poles $(C^1\tilde{\lambda}, ..., C^{10}\tilde{\lambda})$ is canceled, in this case, $(C^1\tilde{\lambda})$. Another choice of the C^I's just change the sign of (3.42).
Naively, it can be thought that the integral in (3.42) gives $(2\pi i)^{10}$. However, remember that $\tilde{f}^I$ are functions over the projective pure spinor space and $\tilde{\Gamma}$ is a 10-cycle in the projective pure spinor space. Therefore, this integral is non-trivial as in the flat space. Despite that the answer will just differ from this trivial case by a number, to know the formal answer will be extremely useful for relating the minimal and non-minimal pure spinor formalisms.

3.3.4 The Projective Pure Spinor Degree

The last step (3.42) in the computation of the scattering amplitude with the projective pure spinor space variables was

$$\mathcal{A} \;=\; (2\pi i)2^3 \int_{\tilde{\Gamma}} 5! \frac{(\mathrm{d}\tilde{f}^1) \wedge ... \wedge (\mathrm{d}\tilde{f}^{10})}{\tilde{f}^1 ... \tilde{f}^{10}} K, \tag{3.43}$$

This integral is known [94][37] and its result is given by the intersection theory

$$\int_{\tilde{\Gamma}} \frac{(\mathrm{d}\tilde{f}^1) \wedge ... \wedge (\mathrm{d}\tilde{f}^{10})}{\tilde{f}^1 ... \tilde{f}^{10}} = (2\pi i)^{10} \sum_{\nu} (\tilde{D}_1, ..., \tilde{D}_{10})_{p_\nu} \tag{3.44}$$

where ν is the number of points p_ν where the hypersurfaces $\tilde{D}_I$ were defined by $\tilde{D}_I = \{\tilde{\lambda}^\alpha \in SO(10)/U(5) : C^I\tilde{\lambda} = 0,\ I = 1, \ldots 10\}$ and $(\tilde{D}_1, ..., \tilde{D}_{10})_{p_\nu} \equiv m_\nu$ is the multiplicity[‡‡] in p_ν. Remember that the coordinates $\tilde{\lambda}^\alpha$, $\alpha = 1, ..., 16$ can be thought as coordinates of $\mathbb{C}^{16} \setminus \{0\}$ with the equivalence relation $\tilde{\lambda}^\alpha \sim c\tilde{\lambda}^\alpha$, $c \neq 0 \in \mathbb{C}$, satisfying the constraints $\tilde{\lambda}\gamma^m\tilde{\lambda} = 0$, so the projective pure spinor space $SO(10)/U(5)$ is embedded in $\mathbb{C}P^{15} = \mathbb{C}^{16} \setminus \{0\}/(\tilde{\lambda}^\alpha \sim c\tilde{\lambda}^\alpha)$, $c \in \mathbb{C}^*$. Therefore the hypersurface $\tilde{D}_I \subset SO(10)/U(5)$ is the intersection between the linear subspace $C^I_\alpha\tilde{\lambda}^\alpha = 0$ and $SO(10)/U(5)$, where now $\tilde{\lambda}^\alpha \in \mathbb{C}P^{15}$, i.e $\tilde{D}_I = \{\{C^I_\alpha\tilde{\lambda}^\alpha = 0\} \cap SO(10)/U(5)$, where $\tilde{\lambda}^\alpha \in \mathbb{C}P^{15}\}$. Note that the intersection of the 10 linear subspaces $C^I\tilde{\lambda} = 0$, $I = 1, ..., 10$ in $\mathbb{C}P^{15}$ is the linear subspace $\mathbb{C}P^5$ embedded in $\mathbb{C}P^{15}$, therefore the intersection of the hypersurfaces $\tilde{D}_I$'s is just the intersection between $\mathbb{C}P^5$ and $SO(10)/U(5)$, this means

$$\tilde{D}_1 \cap ... \cap \tilde{D}_{10} = \mathbb{C}P^5 \cap SO(10)/U(5)\Big|_{\mathbb{C}P^{15}}. \tag{3.45}$$

Since $SO(10)/U(5)$ is a smooth manifold on $\mathbb{C}P^{15}$, the multiplicity in each intersection point of (3.45) is one [94]. So, the sum of the multiplicity at each intersection point p_ν is the number of intersection points among $\mathbb{C}P^5$ and $SO(10)/U(5)$, denoted

[‡‡]the multiplicity can be understood in the same way as in the solutions of a system of algebraic equations

by ${}^{\#}(SO(10)/U(5)\cdot\mathbb{C}P^5)$

$$\sum_\nu(\tilde{D}_1,...,\tilde{D}_{10})_{p_\nu} = {}^{\#}(SO(10)/U(5)\cdot\mathbb{C}P^5). \qquad (3.46)$$

This number is called the degree of the projective pure spinor space
$\deg(SO(10)/U(5)) \equiv {}^{\#}(SO(10)/U(5)\cdot\mathbb{C}P^5)$.

Notice that using the pure spinor measure [32]

$$[\mathrm{d}\lambda](\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} = \frac{2^3}{11!}\epsilon_{\alpha_1...\alpha_5\beta_1...\beta_{11}}\mathrm{d}\lambda^{\beta_1}\wedge...\wedge\mathrm{d}\lambda^{\beta_{11}}, \quad (3.47)$$

and replacing this measure in the amplitude (3.28) we obtain (after to integrate by θ^α) *

$$\mathcal{A} = 2^3\int_\Gamma 5!\frac{(\mathrm{d}f^1)\wedge...\wedge(\mathrm{d}f^{11})}{f^1...f^{11}}K \qquad (3.48)$$

where $f^I = C^I\lambda$, $I = 1,..,11$. In the same way as (3.43) this integral is given by the intersection theory [94][37]

$$\int_\Gamma\frac{(\mathrm{d}f^1)\wedge...\wedge(\mathrm{d}f^{11})}{f^1...f^{11}} = (2\pi i)^{11}(D_1,...,D_{11})_{\{0\}}, \qquad (3.49)$$

where the origin is the unique point of intersection between the hypersurfaces D_I's given by (3.55), $D_I = \{\lambda^\alpha\in PS : f^I = 0\}, I = 1,...,11$, i.e $D_1\cap...\cap D_{11} = \{0\}$ as we claimed, and $(D_1,...,D_{11})_{\{0\}}$ means the multiplicity of this intersection. So using (3.43), (3.44), (3.46) and (3.49) we can conclude

$$(D_1,...,D_{11})_{\{0\}} = \deg(SO(10)/U(5)). \qquad (3.50)$$

Since the multiplicity in the intersection point $\{0\}$ is 12 then we can conclude

$$\deg(SO(10)/U(5)) = 12. \qquad (3.51)$$

This will be confirmed in the next chapter using the character of the pure spinor [54].

Therefore we get that the zero modes contribution is given by

$$\mathcal{A} = (2\pi i)^{11}2^3\,12\,5!\,K. \qquad (3.52)$$

$$(3.53)$$

*The normalization factor 2^3 in the measure comes from the fact that $(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}(\gamma^q\overline{\lambda})^{\alpha_1}(\gamma^r\overline{\lambda})^{\alpha_2}(\gamma^s\overline{\lambda})^{\alpha_3}(\gamma_{qrs})^{\alpha_4\alpha_5} = 2^6 5!(\lambda\overline{\lambda})^3$.

3.4 Čech and Dolbeault Language

Due to the behavior $(1/\lambda)$ in the new lowering picture changing operators, they are defined locally in the pure spinor space. However, it will be interesting to have a global description, i.e patch independent, which can be achieved by introducing the Čech language. We have gave a simple introduction to the Čech formalism on the previous chapter, section 2.4. In this section we define the patches that we use on the pure spinor space and the Čech-Dolbeault isomorphism, which turns out to be useful to check the BRST, Lorentz and SUSY symmetries in the section 3.5 and for relating the minimal and non-minimal pure spinor formalism from the tree level scattering amplitude as we will show in the chapter 5 5. We refer the reader to [52][18][94][29] for review about the Čech cohomology.

Given the new formulation for the PCO's

$$Y_C^I = \frac{C^I\theta}{C^I\lambda}, \qquad I = 1, ..., 11, \tag{3.54}$$

it is clear that Y_C^I is just defined in the patch $PS \setminus D_I$ where D_I is the hypersurface given by $f^I = C^I_\alpha\lambda^\alpha = 0$. Because 11 PCO's are needed in order to compute the tree level scattering amplitude, it is sufficient to have 11 patches to cover the pure spinor space at this order. Each patch is defined by the denominator of the picture operator, i.e we define the patch U_I as

$$U_I = PS \setminus D_I, \qquad D_I = \{\lambda \in PS : f^I \equiv C^I_\alpha\lambda^\alpha = 0\}. \tag{3.55}$$

The set $\underline{U} = \{U_I\}$ is a cover of the pure spinor space without the origin since we claimed that $D_1 \cap ... \cap D_{11} = \{0\}$. This means

$$PS \setminus \{0\} = U_1 \cup ... \cup U_{11} = \bigcup_{I=1}^{11} U_I, \tag{3.56}$$

so the index I is a Čech label. This is as desired because the singular point is removed from the theory. Note that in the papers [52][26] the authors take the patches $\mathcal{U}_\alpha = PS \setminus \mathcal{D}_\alpha$ where $\mathcal{D}_\alpha = \{\lambda \in PS : \lambda^\alpha = 0\}$, $\alpha = 1, ..., 16$. Clearly these $\mathcal{D}_\alpha$'s satisfy $\mathcal{D}_1 \cap ... \cap \mathcal{D}_{16} = \{0\}$, therefore $PS \setminus \{0\} = \mathcal{U}_1 \cup ... \cup \mathcal{U}_{16}$ and we can define the PCO's as

$$Y_\alpha = \frac{\theta^\alpha}{\lambda^\alpha}, \qquad \alpha = 1, ..., 16. \tag{3.57}$$

Actually, for tree level scattering amplitudes this notation is not very convenient, as is explained in the appendix B.1.2.

Note that the PCO's are elements of $C^0(\underline{U}, \mathcal{O})$, where $\Omega^0 \equiv \mathcal{O}$ is a group of holomorphic functions, for instance

$$Y^I_C = \frac{C^I\theta}{C^I\lambda} \in \mathcal{O}(U_I) \tag{3.58}$$

is an holomorphic function on the patch U_I. It is easy to see that Y^I_C is not a cocycle

$$(\delta Y_C)^{IJ} = \left(\frac{C^J\theta}{C^J\lambda} - \frac{C^I\theta}{C^I\lambda}\right)\Bigg|_{U_{IJ}} = -\frac{C^{[I}\theta C^{J]}\lambda}{(C^I\lambda)(C^J\lambda)}\Bigg|_{U_{IJ}} \neq 0 \tag{3.59}$$

and therefore Y^I_C is not in the Čech cohomology. The PCO's have the particular property that the product of different PCO's is a Čech cochain, for example

$$\zeta^{I_1 \ldots I_k} \equiv Y^{I_1}_C \ldots Y^{I_k}_C = \frac{C^{I_1}\theta \ldots C^{I_k}\theta}{(C^{I_1}\lambda)\ldots(C^{I_k}\lambda)} \in \mathcal{O}(U_{I_1 \ldots I_k}), \qquad k \leq 11 \tag{3.60}$$

is an element of $C^{k-1}(\underline{U}, \mathcal{O})$ because $\zeta^{I_1 \ldots I_k}$ is antisymmetry in its Čech labels. This happens because the variables θ^α are grassmann numbers, $\theta^\alpha\theta^\beta = -\theta^\beta\theta^\alpha$. When $k = 11$ we have

$$\zeta^{I_1 \ldots I_{11}} = \epsilon^{I_1 \ldots I_{11}} \frac{C^1\theta \ldots C^{11}\theta}{(C^1\lambda)\ldots(C^{11}\lambda)} \in \mathcal{O}(U_1 \cap \ldots \cap U_{11}). \tag{3.61}$$

This element is important because it is inside to the scattering amplitude. Since the cover $\underline{U}$ just has 11 patches and $(\delta\zeta)^{I_1 \ldots I_{11} I_{12}}$ is antisymmetric in all its Čech labels then

$$(\delta\zeta)^{I_1 \ldots I_{11} I_{12}} = 0 \tag{3.62}$$

so $\zeta^{I_1 \ldots I_{11}}$ belongs to Čech cohomology.

3.4.1 Čech-Dolbeault Isomorphism

Now we give a simple explanation about the Čech-Dolbeault isomorphism. There is a simple way to relate the Čech and Dolbeault cocycles using the so called the partition of unity [94][29][52]. We can take the partition of unity as

$$\rho_I = \frac{f^I \bar{f}_I}{(|f^1|^2 + \ldots + |f^{11}|^2)}, \qquad I = 1, \ldots, 11 \tag{3.63}$$

where $f^I = C^I\lambda$, $\bar{f}_I$ is its complex conjugate: $\bar{f}_I = \bar{C}_I\bar{\lambda}$, and $\bar{\lambda}_\alpha = (\lambda^\alpha)^*$. It is clear that this partition of unity is subordinated to the cover $\underline{U}$, i.e, $\rho_I \neq 0$ only when $\lambda^\alpha \in U_I$, outside of the patch U_I the partition of unity is identically zero. Obviously this partition of unity satisfies the condition

$$\sum_{I=1}^{11} \rho_I = 1. \tag{3.64}$$

Let $\psi_{I_1...I_{k+1}}$ be a k-Čech cocycle ($\psi_{I_1...I_{k+1}} \in Z^k(\underline{U}, \Omega^p)$), then we define the corresponding η_ψ Dolbeault cocycle of type (p,k) as

$$\eta_\psi = \frac{1}{k!} \sum_{I_1...I_{k+1}=1}^{11} \psi_{I_1...I_{k+1}} \rho_{I_1} \wedge \bar{\partial}\rho_{I_2} \wedge ... \wedge \bar{\partial}\rho_{I_{k+1}}. \tag{3.65}$$

Note that η_ψ is a (p,k) form, which is p holomorphic and k antiholmorphic. As expected, $\bar{\partial}\eta_\psi = \mathrm{d}\overline{\lambda}_\alpha \wedge \frac{\partial}{\partial\overline{\lambda}_\alpha}\eta_\psi = 0$ because $\psi_{I_1...I_{k+1}}$ is a cocycle. Also $\psi_{I_1...I_{k+1}}$ is a coboundary, $\psi_{I_1...I_{k+1}} = (\delta\tau)_{I_1...I_{k+1}}$, then η_ψ is $\bar{\partial}$-exact, i.e $\eta_\psi = \bar{\partial}\eta_\tau$, where η_τ is the corresponding Dolbeault cochain to $\tau_{I_1...I_k}$

$$\eta_\tau = \frac{1}{(k-1)!} \sum_{I_1...I_k=1}^{11} \tau_{I_1...I_k} \rho_{I_1} \wedge \bar{\partial}\rho_{I_2} \wedge ... \wedge \bar{\partial}\rho_{I_k}, \tag{3.66}$$

i.e $\eta_\psi = \eta_{(\delta\tau)} = \bar{\partial}\eta_\tau$. Therefore we have a map between the Čech and Dolbeault cohomology groups $H^k(PS \setminus \{0\}, \Omega^p)$ and $H^{(p,k)}_{\bar{\partial}}(PS \setminus \{0\})$. Actually this map is an isomorphism but we do not show that statement here [94][29]. In particular we can consider the Čech cocycle

$$\beta_{I_1...I_{11}} = \epsilon_{I_1...I_{11}} \frac{\mathrm{d}(C^1\lambda) \wedge ... \wedge \mathrm{d}(C^{11}\lambda)}{(C^1\lambda)...(C^{11}\lambda)} \in \Omega^{11}(U_1 \cap ... \cap U_{11}) \tag{3.67}$$

which appeared in the subsection 3.3.4. Clearly $\beta_{I_1...I_{11}}$ is an element of $H^{10}(PS \setminus \{0\}, \Omega^{11})$ so we can find its corresponding $\eta_\beta \in H^{(11,10)}_{\bar{\partial}}(PS \setminus \{0\})$. Applying the map (3.65) to $\beta_{I_1...I_{11}}$ we get

$$\begin{aligned} \eta_\beta &= \frac{1}{10!} \sum_{I_1...I_{11}=1}^{11} \beta_{I_1...I_{11}} \rho_{I_1} \wedge \bar{\partial}\rho_{I_2} \wedge ... \wedge \bar{\partial}\rho_{I_{11}} \\ &= (-1)^{i-1} \frac{\mathrm{d}(C^1\lambda) \wedge ... \wedge \mathrm{d}(C^{11}\lambda) \wedge \bar{\partial}\rho_1 \wedge ... \wedge \widehat{\bar{\partial}\rho_i} \wedge ... \wedge \bar{\partial}\rho_{11}}{(C^1\lambda)...(C^{11}\lambda)} \end{aligned} \tag{3.68}$$

where $\widehat{\bar{\partial}\rho_i}$ means that it must be removed from (3.68). The C^I dependence is eliminated by a global transformation from the projective pure spinor space to itself, as will be done in the chapter 5. †.

Since the pure spinor space without the origin ($PS \setminus \{0\}$) is contractible to $SO(10)/SU(5)$, i.e $PS \setminus \{0\}$ is deformed to $SO(10)/SU(5)$ *, where one can think of $SO(10)/SU(5)$ as the boundary of the $PS \setminus \{0\}$ space, then the topological invariants of these two spaces are the same [31], in particular the following two groups are isomorphic

$$H^{(11,10)}_{\bar{\partial}}(PS \setminus \{0\}) \approx H^{21}_{DR}(SO(10)/SU(5)) \tag{3.69}$$

† We recommend to see the example in the appendix B.1.3 to get more information about this computation.

* For example the space $\mathbb{C} \setminus \{0\}$ can be deformed to S^1.

where DR means the de-Rham cohomology [29]. For the purposes of this paper it is enough to show that the map

$$i^* : H^{(11,10)}_{\bar{\partial}}(PS \setminus \{0\}) \longrightarrow H^{21}_{DR}(SO(10)/SU(5)) \tag{3.70}$$

is an injective homomorphism, i.e for any element $\eta \in H^{(11,10)}_{\bar{\partial}}(PS \setminus \{0\})$ there is just one element $i^*(\eta) \in H^{21}_{DR}(SO(10)/SU(5))$, where "$i$" is the map which embeds the $SO(10)/SU(5)$ space in the $PS \setminus \{0\}$ space and "i^*" is the pull back of the differential forms.

Proof

Let $\lambda = (\lambda^1, ..., \lambda^{16}) = (\lambda^\alpha) \in \mathbb{C}^{16}$ be a point of the pure spinor space, $PS \setminus \{0\}$, i.e $\lambda\gamma^m\lambda = 0$ and $\lambda \neq 0$, then the $SO(10)/SU(5)$ space is embedded in $PS \setminus \{0\}$ by

$$SO(10)/SU(5) = \{(\lambda^\alpha) \in PS\setminus\{0\} : \lambda^\alpha\overline{\lambda}_\alpha = r^2\}, \qquad r \text{ is a positive constant, } r \in \mathbb{R}^+, \tag{3.71}$$

where $\overline{\lambda}_\alpha$ is the conjugate complex of λ^α † [52][30]. Therefore (3.71) defines the injective map

$$i : SO(10)/SU(5) \longrightarrow PS \setminus \{0\}. \tag{3.72}$$

Now we must prove two statements in order to show that (3.70) is an injective homomorphism:

1. First, we need to verify that the map (3.70) is well defined, in others words, if η is an (11,10)-form on $PS \setminus \{0\}$ which is $\bar{\partial}$ closed, i.e $\bar{\partial}\eta = 0$, then the 21-form on $SO(10)/SU(5)$ given by "$i^*\eta$" is "d" closed, i.e $\mathrm{d}(i^*\eta) = 0$.

   Since the exterior derivate operator d commutes with pull back, then we have

   $$\mathrm{d}(i^*\eta) = i^*(\mathrm{d}\eta) = i^*[(\partial + \bar{\partial})\eta]. \tag{3.73}$$

   Remember that η is a (11,10)-form, this means $\partial\eta = 0$ because the complex dimension of the pure spinor space is 11, $\dim_{\mathbb{C}}(PS \setminus \{0\}) = 11$, so we have $\mathrm{d}(i^*\eta) = i^*(\bar{\partial}\eta)$. As $\bar{\partial}\eta = 0$ then we have shown $\mathrm{d}(i^*\eta) = 0$.

2. Finally, we must show that the homomorphism i^* is injective. To show this, it is sufficient to prove that i^* maps the zero to the zero. In others words, if η is a (11,10)-form on $PS \setminus \{0\}$ which is $\bar{\partial}$ exact, i.e $\eta = \bar{\partial}\tau$, where τ is a (11,9)-form on $PS \setminus \{0\}$, then the 21-form on $SO(10)/SU(5)$ given by "$i^*\eta$" is "d" exact, i.e $(i^*\eta) = \mathrm{d}(i^*\tau)$.

   Since τ is a (11,9)-form then $\eta = \bar{\partial}\tau = (\partial + \bar{\partial})\tau = \mathrm{d}\tau$, because $\partial\tau = 0$. So we have

   $$i^*\eta = i^*(\mathrm{d}\tau) = \mathrm{d}(i^*\tau). \tag{3.74}$$

   Therefore we showed that the map (3.70) is an injective homomorphism.

†Note that when $r \to \infty$ we can think the $SO(10)/SU(5)$ space like the boundary of the $PS \setminus \{0\}$.

To see more information about this topic we refer to [94][31].
This isomorphism will be very useful to show that the scattering amplitude is invariant under BRST, Lorentz and supersymmetry transformations in the next section and to obtain the equivalence between the minimal and non-minimal pure spinor formalism in the chapter 5.

3.5 Symmetries of the Scattering Amplitude

In this section we analyze the symmetries of the scattering amplitude with the new PCO's. Namely, we will show that the scattering amplitude is invariant under BRST, Lorentz and supersymmetry transformations. Here we will often use the Čech language and the Čech-Dolbeault isomorphism presented in the subsection 3.4.

3.5.1 BRST Invariance

We will show that the tree level scattering amplitude is BRST invariant and that the Q exact states are decoupled.
As we discussed in the subsection 3.4, the PCO's are defined locally because they behave like $1/\lambda$: $Y_C^I = \frac{C^I\theta}{C^I\lambda}$ and they are well defined only in $U_I = PS \setminus D_I$. Therefore, as proposed in [52] one must add to the old BRST charge

$$Q = \oint \mathrm{d}z\, \lambda^\alpha d_\alpha, \tag{3.75}$$

where d_α is defined as

$$d_\alpha = p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha \partial x_m - \frac{1}{8}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta),$$

the Čech operator δ given in the section 2.4. The δ operator play an important role in the construction of the b-ghost, as we will discuss in the section 3.7. So the total BRST charge is

$$Q_T = \oint \lambda^\alpha d_\alpha + \delta \equiv Q + \delta. \tag{3.76}$$

By definition, if the tree level scattering amplitude $\mathcal{A}$ is physical then it must be Q_T closed, i.e $Q_T\,\mathcal{A} = 0$.
In the following we will show that the amplitude is Q_T closed. First of all, remember that in the tree level scattering amplitude the vertex operators can always be written as a global function in $PS \setminus \{0\}$ given by $\lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i)$ [48], where the k_i's are the momenta and the e_i's are the polarizations of the vertex operators.
Since the tree level scattering amplitude is given by

$$\mathcal{A} = \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \prod_{I=1}^{11} Y_C^I\, \lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i). \tag{3.77}$$

and the measure $[\mathrm{d}\lambda]$ is globally defined on $PS \setminus \{0\}$ [18], then the scattering amplitude is δ closed.
Proof:
The scattering amplitude (3.28) can be written as

$$\begin{aligned}
\mathcal{A} &= \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \, \frac{C^1\theta ... C^{11}\theta}{C^1\lambda ... C^{11}\lambda} \lambda^\alpha \lambda^\beta \lambda^\gamma \, f_{\alpha\beta\gamma}(\theta, \, k_i, \, e_i) \qquad (3.78) \\
&= \frac{1}{11!} \sum_{I_1 ... I_{11}} \epsilon_{I_1 ... I_{11}} \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \, \frac{C^{I_1}\theta ... C^{I_{11}}\theta}{C^{I_1}\lambda ... C^{I_{11}}\lambda} \lambda^\alpha \lambda^\beta \lambda^\gamma \, f_{\alpha\beta\gamma}(\theta, \, k_i, \, e_i) \\
&\equiv \frac{1}{11!} \sum_{I_1 ... I_{11}} \epsilon_{I_1 ... I_{11}} \int_\Gamma \beta^{I_1 ... I_{11}} = \int_\Gamma \beta^{1,...,11}.
\end{aligned}$$

Clearly $\beta^{I_1 ... I_{11}}$ is a Čech cochain*

$$\beta^{I_1 ... I_{11}} \in C^{10}(\underline{U}, \Omega^{11}) \qquad (3.79)$$

where $\underline{U}$ is the cover of the $PS \setminus \{0\}$ space, which was defined in the section 3.4, i.e $\underline{U} = \{U_I\}$, $I = 1, ..., 11$, and the patches U_I's are given by $U_I = PS \setminus D_I$, where D_I is the hypersurface $D_I = \{\lambda^\alpha \in PS : C^I_\alpha \lambda^\alpha = 0\}$. Remember that $PS \setminus \{0\} = U_1 \cup ... \cup U_{11}$. Since there are 11 patches to cover $PS \setminus \{0\}$ then $\beta^{I_1 ... I_{11}}$ is in the Čech cohomology because $C^{11}(\underline{U}, \Omega^{11}) = \{0\}$ and $(\delta\beta)^{I_1 ... I_{12}} \in C^{11}(\underline{U}, \Omega^{11})$, so $(\delta\beta)^{I_1 ... I_{12}} = 0$, so we can write

$$\beta^{I_1 ... I_{11}} \in H^{10}(PS \setminus \{0\}, \Omega^{11}). \qquad (3.80)$$

Now it remains to show $Q \, \mathcal{A} = 0$. Because

$$Q \, Y^I_C = 1, \qquad (3.81)$$

therefore we have

$$\begin{aligned}
Q(\mathcal{A}) &= Q \left(\int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \prod_{I=1}^{11} Y^I_C \, \lambda^\alpha \lambda^\beta \lambda^\gamma \, f_{\alpha\beta\gamma}(\theta, \, k_i, \, e_i) \right) \qquad (3.82) \\
&= \frac{1}{11!} \sum_{I_1,...,I_{11}=1}^{11} \epsilon_{I_1,...,I_{11}} Q \left(\int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \frac{C^{I_1}\theta ... C^{I_{11}}\theta}{C^{I_1}\lambda ... C^{I_{11}}\lambda} \, \lambda^\alpha \lambda^\beta \lambda^\gamma \, f_{\alpha\beta\gamma}(\theta, \, k_i, \, e_i) \right) \\
&= \frac{1}{11!} \sum_{I_1,...,I_{11}=1}^{11} \epsilon_{I_1,...,I_{11}} \int_\Gamma (\delta\tau)^{I_1,...,I_{11}}
\end{aligned}$$

where $\tau^{I_1,...,I_{10}}$ is the holomorphic 11-form

$$\tau^{I_1,...,I_{10}} = [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \frac{C^{I_1}\theta ... C^{I_{10}}\theta}{C^{I_1}\lambda ... C^{I_{10}}\lambda} \, \lambda^\alpha \lambda^\beta \lambda^\gamma \, f_{\alpha\beta\gamma}(\theta, \, k_i, \, e_i) \in C^9(\underline{U}, \Omega^{11}) \qquad (3.83)$$

*In [18] was shown that the measure $[\mathrm{d}\lambda]$ is defined globally on $PS \setminus \{0\}$, so, the Čech indices come only from the PCO's.

where $\underline{U}$ is the cover of the $PS \setminus \{0\}$ space given in the subsection 3.4. Clearly $(\delta\tau)^{I_1,\dots,I_{11}}$ is a trivial element of the Čech cohomology group $H^{11}(PS \setminus \{0\}, \Omega^{11})$, so its corresponding Dolbeault cocycle

$$\eta_{(\delta\tau)} = \frac{1}{10!} \sum_{I_1,\dots,I_{11}=1}^{11} (\delta\tau)^{I_1,\dots,I_{11}} \rho_{I_1} \wedge \bar{\partial}\rho_{I_2} \wedge \dots \wedge \bar{\partial}\rho_{I_{11}}, \tag{3.84}$$

where ρ_I is partition of unity (3.63), is a trivial element of the Dolbeault cohomology group $H^{(11,10)}_{\bar{\partial}}(PS \setminus \{0\})$, i.e

$$\eta_{(\delta\tau)} = \bar{\partial}(\eta_\tau) \qquad \text{(11,10)-form on } PS \setminus \{0\}, \tag{3.85}$$

where η_τ is the (11,9)-form given by

$$\eta_\tau = \frac{1}{9!} \sum_{I_1,\dots,I_{10}=1}^{11} \tau^{I_1,\dots,I_{10}} \rho_{I_1} \wedge \bar{\partial}\rho_{I_2} \wedge \dots \wedge \bar{\partial}\rho_{I_{10}}, \tag{3.86}$$

as explained in the subsection 3.4.1. So we can write (3.82) as

$$\int_\Gamma (\delta\tau)^{1,\dots,11} = \int_{SO(10)/SU(5)} i^*(\bar{\partial}(\eta_\tau)) = \int_{SO(10)/SU(5)} \mathrm{d}(i^*(\eta_\tau)), \tag{3.87}$$

where "i" is the map $i : SO(10)/SU(5) \to PS \setminus \{0\}$ given in the subsection 3.4.1. Finally, applying the Stokes theorem

$$\int_\Gamma (\delta\tau)^{1,\dots,11} = \int_{SO(10)/SU(5)} \mathrm{d}(i^*(\eta_\tau)) = \int_{\partial(SO(10)/SU(5))} i^*(\eta_\tau) \tag{3.88}$$

and since the $SO(10)/SU(5)$ space is a compact manifold without boundary, $\partial(SO(10)/SU(5)) = \varnothing$, then we can conclude

$$Q(\mathcal{A}) = 0. \tag{3.89}$$

Thus, we have shown that the tree level scattering amplitude is Q_T closed.

Now we will show that the global (i.e $(\delta\Omega)^{IJ} = \Omega^J - \Omega^I = 0$) and Q exact (i.e $\langle\, Q(\Omega)\,\rangle$) functions are decoupled, that is, they are $Q_T = Q + \delta$ exact. A Q exact function inside to the scattering amplitude is given by

$$\langle\, Q(\Omega)\,\rangle = \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta Y_C^1 \dots Y_C^{11} Q(\Omega(\lambda,\theta,k)). \tag{3.90}$$

Only the terms with 5 θ's and 3 λ's in $Q(\Omega)$ will contribute, because there are 11 θ's coming from the 11 PCO's and the scattering amplitude must have ghost number zero. So, we focus on the global term

$$\Omega(\lambda,\theta,k,e) = \lambda^\alpha \lambda^\beta \theta^{\gamma_1} \dots \theta^{\gamma_6} f_{\alpha\beta\gamma_1\dots\gamma_6}(k_i, e_i), \tag{3.91}$$

where k_i are the momenta and e_i are the polarizations. We can write (3.90) as

$$\int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \, Y_C^1 ... Y_C^{11} Q(\Omega(\lambda,\theta,k)) = -\int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \, Q(Y_C^1 ... Y_C^{11} \Omega(\lambda,\theta,k)) + \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \, Q(Y_C^1 ... Y_C^{11}) \Omega(\lambda,\theta,k). \tag{3.92}$$

The term $Y_C^1 ... Y_C^{11} \Omega(\lambda,\theta,k)$ is identically zero because there are 17 θ's. So

$$\begin{aligned} \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \, Y_C^1 ... Y_C^{11} Q(\Omega(\lambda,\theta,k)) &= \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \, Q(Y_C^1 ... Y_C^{11}) \Omega(\lambda,\theta,k) \\ &= \frac{1}{11!} \sum_{I_1,...,I_{11}=1}^{11} \epsilon_{I_1 ... I_{11}} \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \, Q(Y_C^{I_1} ... Y_C^{I_{11}}) \Omega(\lambda,\theta,k) \\ &= \frac{1}{11!} \sum_{I_1,...,I_{11}=1}^{11} \epsilon_{I_1 ... I_{11}} \int_\Gamma (\delta\kappa)^{I_1 ... I_{11}}, \end{aligned} \tag{3.93}$$

where $\kappa^{I_1 ... I_{10}}$ is the holomorphic 11-form

$$\kappa^{I_1 ... I_{10}} = [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \frac{C^{I_1}\theta ... C^{I_{10}}\theta}{C^{I_1}\lambda ... C^{I_{10}}\lambda} \lambda^\alpha \lambda^\beta \theta^{\gamma_1} ... \theta^{\gamma_6} f_{\alpha\beta\gamma_1 ... \gamma_6}(k_i, e_i) \in C^9(\underline{U}, \Omega^{11}). \tag{3.94}$$

Note that $\delta \langle Q(\Omega) \rangle = 0$, since $(\delta(\delta\kappa))^{I_1 ... I_{12}} = 0$. Using the same procedure that allowed as go from (3.82) and to conclude in (3.89) we have

$$\langle Q(\Omega) \rangle = \frac{1}{11!} \sum_{I_1,...,I_{11}=1}^{11} \epsilon_{I_1 ... I_{11}} \int_\Gamma (\delta\kappa)^{I_1 ... I_{11}} = \int_{\partial(SO(10)/SU(5))} i^*(\eta_\kappa) = 0. \tag{3.95}$$

Therefore we have shown that every global and exact function inside to the scattering amplitude is decoupled. Nevertheless, since $QY_C^I = 1$ we could think that every global and Q-closed function $\Psi(\lambda^\alpha, \theta^\alpha)$ is decoupled from the amplitude because it could be written as $\Psi(\lambda^\alpha, \theta^\alpha) = Q(Y_C^I \Psi(\lambda^\alpha, \theta^\alpha))$. However, note that $Q(Y_C^I \Psi(\lambda^\alpha . \theta^\alpha))$ is a local function defined in U_I, therefore, when it is inside the scattering amplitude

$$\langle Q(Y_C^I \Psi) \rangle = \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta Y_C^1 \ldots Y_C^{11} Q(Y_C^I \Psi(\lambda,\theta)), \tag{3.96}$$

the integrand is not a Čech cochain and the action of the operator δ is not defined. For this reason, $Q(Y_C^I \Psi)$ does not belong to the cohomology of the BRST operator $Q + \delta$ and in conclusion, it is not allowed to write $\Psi = Q(Y_C^I \Psi)$.
For a general case we must show that the scattering amplitude decouple the states which are Q_T exact, i.e

$$\langle (Q+\delta)(\Omega) \rangle = 0 \tag{3.97}$$

for any Ω.
First of all, we know that the BRST operator is nilpotent $(Q+\delta)^2 = 0$ and we want

to show that the BRST exact terms are decoupled from the scattering amplitude. The Q_T exact terms are written as

$$\langle\,(Q+\delta)(\Omega)\,\rangle = \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \prod_{I=1}^{11} Y^I_C (Q+\delta)(\Omega). \tag{3.98}$$

However, as the 11-form $[\mathrm{d}\lambda]\prod_{I=1}^{11} Y^I_C$ is a 10-Čech cochain, $C^{10}(\underline{U},\Omega^{11})$, then it is possible that the product of the cochains $[\mathrm{d}\lambda]\prod_{I=1}^{11} Y^I_C$ and $(Q+\delta)(\Omega)$ is not well defined, because the product of two cochains is not always a cochain. For instance, let us consider the following 2 cochains

$$Y^I_C = \frac{C^I\theta}{C^I\lambda} \in C^0(\underline{U},\mathcal{O}), \qquad \Omega^J = \frac{\Lambda_{mn}(C^J\gamma^{mn}\theta)}{(C^J\lambda)} \in C^0(\underline{U},\mathcal{O}). \tag{3.99}$$

Clearly

$$\Psi^{IJ} \equiv Y^I_C\,\Omega^J = \frac{(C^I\theta)\,(C^J\gamma^{mn}\theta)\Lambda_{mn}}{(C^I\lambda)(C^J\lambda)} \neq -\frac{(C^J\theta)\,(C^I\gamma^{mn}\theta)\Lambda_{mn}}{(C^I\lambda)(C^J\lambda)} \notin C^1(\underline{U},\mathcal{O}), \tag{3.100}$$

In the particular case when Ω is a global holomorphic function in $PS \setminus \{0\}$ the product with any Čech cochain is well defined, for example the vertex operators in (3.77), or as in the computation (3.90). Note also that the Čech operator is not a derivate operator, i.e it does not satisfy the Leibniz rule. So it is not well defined acting on the product (3.100)

$$(\delta\Psi)^{IJK} \neq (\delta Y)^{IJ}\Omega^K \pm Y^I(\delta\Omega)^{JK}. \tag{3.101}$$

Therefore the expressions $(Q+\delta)\,\langle\,(Q+\delta)(\Omega)\,\rangle$ and $\langle\,(Q+\delta)(Q+\delta)(\Omega)\,\rangle$ are not equal i.e $(Q+\delta)\,\langle\,(Q+\delta)(\Omega)\,\rangle \neq \langle\,(Q+\delta)(Q+\delta)(\Omega)\,\rangle$, and in most cases the left hand side is not defined when Ω has Čech labels. Therefore the expression (3.97) does not make sense unless that Ω will be a global holomorphic function, like we assumed in (3.90).

From the analysis above we can conclude that for the tree level scattering amplitudes, the naive existence of the homotopy operator [57][52] given by

$$\xi = \left.\frac{C^I\theta}{C^I\lambda}\right|_{U_I} + \left.\frac{C^I\theta C^J\theta}{C^I\lambda C^J\lambda}\right|_{U_I\cap U_J} + ... + \left.\frac{C^1\theta C^2\theta ... C^{11}\theta}{C^1\lambda C^2\lambda ... C^{11}\lambda}\right|_{U_1\cap U_2\cap ...\cap U_{11}}, \tag{3.102}$$

which by definition satisfies

$$(Q+\delta)(\xi V_1V_2V_3U_1...U_{N-3}) = V_1V_2V_3U_1...U_{N-3} \tag{3.103}$$

for $V_1V_2V_3$ unintegrated vertex operators and $U_1...U_{N-3}$ integrated vertex operators, is not allowed because

$$\xi V_1V_2V_3U_1...U_{N-3} \tag{3.104}$$

is not a global function on $PS \setminus \{0\}$. Therefore at tree level it is sufficient to decouple the global and Q exact functions, see (3.90).

3.5.2 Lorentz and Supersymmetry Invariance

Now we show that although the new lowering picture operators are neither Lorentz nor supersymmetry invariant, the scattering amplitude is invariant under both transformations.

Lorentz Invariance

It is easy to show that the action of the Lorentz generators $M^{mn} = (1/2)\int \mathrm{d}z[(\omega\gamma^{mn}\lambda)+(p\gamma^{mn}\theta)]$ on the PCO's is Q exact:

$$M^{mn}Y_C^I = -\frac{1}{2}Q\left[\frac{(C^I\gamma^{mn}\theta)(C^I\theta)}{(C^I\lambda)^2}\right], \tag{3.105}$$

then, replacing this in the scattering amplitude we get

$$\begin{aligned} & M^{mn}(\mathcal{A}) \qquad (3.106)\\ = \;& \frac{1}{2\,11!}\sum_{I_1,\dots,I_{11}=1}^{11}\epsilon_{I_1,\dots,I_{11}}\int_\Gamma[\mathrm{d}\lambda]\int \mathrm{d}^{16}\theta\sum_{i=1}^{11}(-1)^iQ\left[\frac{(C^{I_i}\gamma^{mn}\theta)(C^{I_i}\theta)}{(C^{I_i}\lambda)^2}\right]\frac{C^{I_1}\theta...\widehat{C^{I_i}\theta}...C^{I_{11}}\theta}{C^{I_1}\lambda...\widehat{C^{I_i}\lambda}...C^{I_{11}}\lambda}\\ & \lambda^\alpha\lambda^\beta\lambda^\gamma\; f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i)\\ = \;& \frac{1}{2\,11!}\sum_{I_1,\dots,I_{11}=1}^{11}\epsilon_{I_1,\dots,I_{11}}\int_\Gamma[\mathrm{d}\lambda]\int \mathrm{d}^{16}\theta\sum_{i=1}^{11}(-1)^{i-1}\frac{(C^{I_i}\gamma^{mn}\theta)(C^{I_i}\theta)}{(C^{I_i}\lambda)^2}Q\left(\frac{C^{I_1}\theta...\widehat{C^{I_i}\theta}...C^{I_{11}}\theta}{C^{I_1}\lambda...\widehat{C^{I_i}\lambda}...C^{I_{11}}\lambda}\right)\\ & \lambda^\alpha\lambda^\beta\lambda^\gamma\; f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i)\end{aligned}$$

where the term $\frac{\widehat{C^{I_i}\theta}}{\widehat{C^{I_i}\lambda}}$ means it must be removed from the expression. Making an algebraic manipulation we find

$$\begin{aligned}\sum_{i=1}^{11}(-1)^{i-1}\frac{(C^{I_i}\gamma^{mn}\theta)(C^{I_i}\theta)}{(C^{I_i}\lambda)^2}Q\left(\frac{C^{I_1}\theta...\widehat{C^{I_i}\theta}...C^{I_{11}}\theta}{C^{I_1}\lambda...\widehat{C^{I_i}\lambda}...C^{I_{11}}\lambda}\right) &\equiv \sum_{i=1}^{11}(-1)^{i-1}\pi^{I_i}Q\left(Y_C^{I_1}...\widehat{Y_C^{I_i}}...Y_C^{I_{11}}\right)\\ &= (\delta\psi)^{I_1...I_{11}} \end{aligned} \tag{3.107}$$

where we define

$$\pi^I \equiv \frac{(C^I\gamma^{mn}\theta)(C^I\theta)}{(C^I\lambda)^2} \tag{3.108}$$

and $\psi^{I_1...I_{10}}$ is given by

$$\begin{aligned}\psi^{I_1...I_{10}} &= -\frac{1}{9!}\pi^{[I_1}Y_C^{I_2}...Y_C^{I_{10}]} \qquad (3.109)\\ &\equiv -\frac{1}{9!}\left(\pi^{I_1}Y_C^{I_2}Y_C^{I_3}...Y_C^{I_{10}} - \pi^{I_2}Y_C^{I_1}Y_C^{I_3}...Y_C^{I_{10}} + \text{all permutation}\right) \in C^9(\underline{U},\mathcal{O}).\end{aligned}$$

We define the holomorphic 11-form

$$\Psi^{I_1...I_{10}} = [\mathrm{d}\lambda]\int \mathrm{d}^{16}\theta\,\psi^{I_1...I_{10}}\lambda^\alpha\lambda^\beta\lambda^\gamma\; f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i) \in C^9(\underline{U},\Omega^{11}). \tag{3.110}$$

So we can write (3.106) in the following way

$$M^{mn}(\mathcal{A}) = \frac{1}{2\ 11!} \sum_{I_1,\ldots,I_{11}=1}^{11} \epsilon_{I_1,\ldots,I_{11}} \int_\Gamma (\delta\Psi)^{I_1\ldots I_{11}}. \tag{3.111}$$

With the same procedure used to show the BRST invariance of the amplitude, i.e following the steps from (3.82) to (3.89) we obtain

$$\sum_{I_1,\ldots,I_{11}=1}^{11} \epsilon_{I_1,\ldots,I_{11}} \int_\Gamma (\delta\Psi)^{I_1\ldots I_{11}} = \int_{\partial(SO(10)/SU(5))} i^*(\eta_\Psi) = 0 \tag{3.112}$$

since $\partial(SO(10)/SU(5)) = \varnothing$. Finally, we conclude the tree level scattering amplitude is Lorentz invariant

$$M^{mn}(\mathcal{A}) = 0. \tag{3.113}$$

Invariance under Supersymmetry

Now we show that the tree level scattering amplitude is invariant under supersymmetry transformations. We call the supersymmetry generator "q", which is given by

$$q = \varepsilon^\alpha q_\alpha \tag{3.114}$$

where ε^α is a Grassmann constant spinor spinor and

$$q_\alpha = \int \mathrm{d}z\,(p_\alpha + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta \partial x_m + \frac{1}{24}\gamma^m_{\alpha\beta}\gamma_{m\,\gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta).$$

It is easy to see that the action of q on the PCO's is

$$q(Y^I_C) = \varepsilon^\alpha q_\alpha(Y^I_C) = Q\left[\frac{(\varepsilon C^I)(C^I\theta)}{(C^I\lambda)^2}\right]. \tag{3.115}$$

Therefore in the tree level scattering amplitude we have

$$\begin{aligned} & q(\mathcal{A}) \qquad\qquad (3.116)\\ = \ & \frac{1}{11!}\sum_{I_1,\ldots,I_{11}=1}^{11} \epsilon_{I_1,\ldots,I_{11}} \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \sum_{i=1}^{11}(-1)^{i-1} Q\left[\frac{(\varepsilon C^{I_i})(C^{I_i}\theta)}{(C^{I_i}\lambda)^2}\right] \frac{C^{I_1}\theta\ldots\widehat{C^{I_i}\theta}\ldots C^{I_{11}}\theta}{C^{I_1}\lambda\ldots\widehat{C^{I_i}\lambda}\ldots C^{I_{11}}\lambda} \\ & \lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i) \\ = \ & \frac{1}{11!}\sum_{I_1,\ldots,I_{11}=1}^{11} \epsilon_{I_1,\ldots,I_{11}} \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \sum_{i=1}^{11}(-1)^{i} \frac{(\varepsilon C^{I_i})(C^{I_i}\theta)}{(C^{I_i}\lambda)^2} Q\left(\frac{C^{I_1}\theta\ldots\widehat{C^{I_i}\theta}\ldots C^{I_{11}}\theta}{C^{I_1}\lambda\ldots\widehat{C^{I_i}\lambda}\ldots C^{I_{11}}\lambda}\right) \\ & \lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i). \end{aligned}$$

Making a similar algebraic manipulation like in (3.107) we get

$$\begin{aligned} & [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \sum_{i=1}^{11}(-1)^{i} \frac{(\varepsilon C^{I_i})(C^{I_i}\theta)}{(C^{I_i}\lambda)^2} Q\left(\frac{C^{I_1}\theta\ldots\widehat{C^{I_i}\theta}\ldots C^{I_{11}}\theta}{C^{I_1}\lambda\ldots\widehat{C^{I_i}\lambda}\ldots C^{I_{11}}\lambda}\right) \\ & \lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i) \\ = \ & (\delta\Phi)^{I_1\ldots I_{11}}, \end{aligned} \tag{3.117}$$

where we have the following definitions

$$\Phi^{I_1 \ldots I_{10}} = [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \, \phi^{I_1 \ldots I_{10}} \, \lambda^\alpha \lambda^\beta \lambda^\gamma \; f_{\alpha\beta\gamma}(\theta, \, k_i, \, e_i) \in C^9(\underline{U}, \Omega^{11}), \tag{3.118}$$

$$\phi^{I_1 \ldots I_{10}} \;\; = \;\; \frac{1}{9!} \varphi^{[I_1} \, Y_C^{I_2} \ldots Y_C^{I_{10}]}, \tag{3.119}$$

$$\varphi^I = \frac{(\varepsilon C^I)(C^I\theta)}{(C^I\lambda)^2}. \tag{3.120}$$

Using the same argument as in (3.112) it is clear that

$$q(\mathcal{A}) = \frac{1}{11!} \sum_{I_1, \ldots, I_{11}=1}^{11} \epsilon_{I_1, \ldots, I_{11}} \int_\Gamma (\delta\Phi)^{I_1 \ldots I_{11}} = \int_{\partial(SO(10)/SU(5))} i^*(\eta_\Phi) = 0. \tag{3.121}$$

So the tree level scattering amplitude is invariant under supersymmetry transformations.

In conclusion, we succeeded in proving the invariance under the total BRST, Lorentz and supersymmetry transformations, where the Čech-Dolbeault language played a central role.

3.6 Relation with Twistor Space

In the subsection 3.3.3 we studied the tree-level scattering amplitude in the projective pure spinor space. In this section we will show that the result found there, given by the integral (3.42), is the same found by Berkovits and Cherkis in [22]. In that reference, the projective pure spinor space allowed to get the Green's function for a massless scalar field in $d = 10$ dimensions.

The Green's function for a massless scalar field in $d = 10$ dimensions is given by the integral

$$\Phi(x) = \int_{\tilde{\Gamma}} [\mathrm{d}\tilde{\lambda}] \; F(\tilde{\lambda}, \omega)|_{\omega = x\tilde{\lambda}}, \tag{3.122}$$

which is written covariantly [22]. In this integral, $\tilde{\lambda}^\alpha$ is a projective pure spinor, $[\mathrm{d}\tilde{\lambda}]$ is the measure of the projective pure spinor space given by (3.38), while $F(\tilde{\lambda}, \omega)|_{\omega = x\tilde{\lambda}}$ is given by

$$F(\tilde{\lambda}, \omega) = \frac{\epsilon_{\alpha_1 \ldots \alpha_{11} \beta_1 \ldots \beta_5} A_1^{\alpha_1} \ldots A_{11}^{\alpha_{11}} (\gamma^m \omega)^{\beta_1} (\gamma^n \omega)^{\beta_2} (\gamma^p \omega)^{\beta_3} (\gamma_{mnp})^{\beta_4 \beta_5}}{\prod_{r=1}^{11} (A_r^\alpha \omega_\alpha)}, \quad \omega_\alpha = (x \cdot \gamma \tilde{\lambda})_\alpha. \tag{3.123}$$

A_I^α's are constant spinors and the cycle $\tilde{\Gamma}$ is given by ten out of the eleven poles of $F(\tilde{\lambda}, \omega)$. The measure (3.38) is not suitable for obtaining the relationship between twistors and scattering amplitude, so we will modify it as follows. First note that for $|x| \neq 0$ then $(x \cdot \gamma \tilde{\lambda})_\alpha$ is a pure spinor if and only if $\tilde{\lambda}^\alpha$ is also a pure spinor. This is very easy to show:

1. In the backward direction: If $\tilde{\lambda}^\alpha$ is a pure spinor in ten dimensions, then $(x\cdot\gamma\tilde{\lambda})_\alpha$ is also a pure spinor. So, we must prove that $(x\cdot\gamma\tilde{\lambda})_\alpha(\gamma^m)^{\alpha\beta}(x\cdot\gamma\tilde{\lambda})_\beta = 0$ using the condition $\tilde{\lambda}\gamma\tilde{\lambda} = 0$. Then,

$$
\begin{aligned}
(x\cdot\gamma\tilde{\lambda})_\alpha(\gamma^m)^{\alpha\beta}(x\cdot\gamma\tilde{\lambda})_\beta &= x^n x^p\{\tilde{\lambda}^\delta(\gamma_n)_{\delta\alpha}(\gamma^m)^{\alpha\beta}(\gamma_p)_{\beta\rho}\tilde{\lambda}^\rho\} \\
&= 2\,x^m x^p(\tilde{\lambda}\gamma_p\tilde{\lambda}) - x^n x^p(\tilde{\lambda}\gamma^m\gamma_n\gamma_p\tilde{\lambda}) \\
&= -\frac{1}{2}x^n x^p(\tilde{\lambda}\gamma^m\{\gamma_n,\gamma_p\}\tilde{\lambda}) \\
&= -\frac{1}{2}x\cdot x(\tilde{\lambda}\gamma^m\tilde{\lambda}) = 0. \qquad (3.124)
\end{aligned}
$$

2. Let's now make the prove in the forward direction, i.e, assuming that $(x\cdot\gamma\tilde{\lambda})_\alpha$ is a pure spinor, then $\tilde{\lambda}^\alpha$ is also a pure spinor. We start defining the pure spinor ρ_α: $\rho_\alpha \equiv (x\cdot\gamma\tilde{\lambda})_\alpha$. Then, writing $\tilde{\lambda}^\alpha$ in terms of ρ_α we find

$$\tilde{\lambda}^\alpha = \frac{1}{x\cdot x}(\rho\gamma\cdot x)^\alpha. \qquad (3.125)$$

Since ρ_α is a pure spinor, then performing a similar computation as in the proof in the backward direction, it is trivial to show that $\tilde{\lambda}^\alpha$ is a pure spinor: $\tilde{\lambda}\gamma^m\tilde{\lambda} = 0$.

Using the previous property we redefine the measure $[\mathrm{d}\tilde{\lambda}]$ given in (3.38) by

$$
\begin{aligned}
&[\mathrm{d}\tilde{\lambda}]'(\tilde{\lambda}\gamma\cdot x\gamma^m)^{\alpha_1}(\tilde{\lambda}\gamma\cdot x\gamma^n)^{\alpha_2}(\tilde{\lambda}\gamma\cdot x\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5} = \\
&\frac{2^3}{10!|x|^8}\epsilon^{\alpha_1\ldots\alpha_5\beta_1\ldots\beta_{11}}\mathrm{d}(\tilde{\lambda}\gamma\cdot x)_{\beta_1}\wedge\ldots\wedge\mathrm{d}(\tilde{\lambda}\gamma\cdot x)_{\beta_{10}}(\tilde{\lambda}\gamma\cdot x)_{\beta_{11}}. \qquad (3.126)
\end{aligned}
$$

Performing a simple computation as in (B.35) we can show that $[\mathrm{d}\tilde{\lambda}] = e^{i\phi}[\mathrm{d}\tilde{\lambda}]'$, where $\phi\in\mathbb{R}$ is a constant. Then, up to a phase factor, we write

$$\Phi(x) = \int_{\tilde{\Gamma}}[\mathrm{d}\tilde{\lambda}]'\;F(\tilde{\lambda},\omega)|_{\omega=x\tilde{\lambda}}. \qquad (3.127)$$

Replacing $F(\tilde{\lambda},\omega)|_{\omega=x\tilde{\lambda}}$ and using the measure (3.126) we get

$$\Phi(x) = \frac{2^3 5!}{|x|^8}\int_{\tilde{\Gamma}}\frac{\mathrm{d}(A_1 x\cdot\gamma\tilde{\lambda})\wedge\ldots\wedge\mathrm{d}(A_{10}x\cdot\gamma\tilde{\lambda})}{(A_1 x\cdot\gamma\tilde{\lambda})\ldots(A_{10}x\cdot\gamma\tilde{\lambda})}, \qquad (3.128)$$

where without loss of generality, we chose $\tilde{\Gamma} = \{\lambda^\alpha\in SO(10)/U(5) : |(A_I x\cdot\gamma\tilde{\lambda})| = \epsilon_I,\ I = 1,\ldots,10\}$. From (3.42) and (4.45) the relationship between the tree level scattering amplitude and the Green's function for the massless scalar field is clear

$$
\begin{aligned}
C^I_\alpha &\rightarrow (A_I\gamma\cdot x)_\alpha, \\
K &\rightarrow \frac{1}{(2\pi i)|x|^8}. \qquad (3.129)
\end{aligned}
$$

This result was not known using the old PCO's. Although the construction for the scattering amplitudes at the genus g is in progress we think that it is likely to have a relationship between loops scattering amplitudes and massless solutions for higher-spin [22].

3.7 Comments About the Loop-Level

Now we give a glance about the scattering amplitude at the loop-level. The loop level in the minimal pure spinor formalism has two fundamental ingredients: the picture raising operators and the b-ghost. The picture raising operators are needed to absorb the zero modes of the field ω_α and some of the zero modes of the field p_α, given in the action of the minimal formalism (1.42).

Because at the loop level the complex structure of the Riemann surfaces have deformations, known as the moduli space, in order to fix these deformations it is necessary to introduce the b-ghost. In the pure spinor formalism the b-ghost is not a fundamental field [48][57] and therefore its construction in terms of the others fields is such that satisfies

$$\{Q_T, b(z)\} = T(z), \tag{3.130}$$

where Q_T is the BRST charge and $T(z)$ is the stress-energy tensor.

In [52] was given the b-ghost for the minimal pure spinor formalism. Nevertheless it has not been used to compute scattering amplitudes. One reason for not using it is the difficulty for dealing with the Čech indices inside the scattering amplitude. In this section we will give some directions for computing scattering amplitudes with b-ghost in the minimal pure spinor formalism.

However before that, we give a brief review of the Berkovits's prescription [48] for computing multiloop scattering amplitudes using the pure spinor formalism.

3.7.1 Review of the Multiloop Prescription in The Minimal Pure Spinor Formalism

The prescription to compute multiloop amplitudes in the minimal pure spinor formalism was spelled out in [48], which we now briefly review.

The multiloop prescription in the pure spinor formalism was made possible by the construction of the analogous operators of the picture changing operators in the RNS formalism, which are necessary to absorb the zero-modes of the various variables.

So the analysis of zero modes will play a crucial rôle in the multiloop prescription. Remember that a conformal weight one variable Φ_1 has g zero-modes in a genus g Riemann surface, while a conformal weight zero variable Φ_0 always has one zero mode in every genus.

In the pure spinor formalism the zero modes of λ^α, ω_α will require insertions to absorb these zero modes. The Berkovits proposal is given in a gauge invariant way as follows

$$Y_C^I = C_\alpha^I \theta^\alpha \delta(C^I \lambda), \quad Z_B^P = \frac{1}{2} B_{mn}^P (\lambda \gamma^{mn} d) \delta(\frac{1}{2} B_{rs}^P \omega \gamma^{rs} \lambda), \quad Z_J = (\lambda^\alpha d_\alpha) \delta(\omega \lambda), \tag{3.131}$$

where C_α^I is a constant spinor and B_{mn}^P is constant tensors. They will be responsible for killing the eleven zero modes of λ^α and $11g$ zero modes of ω_α. So, the zero modes will be absorbed by the insertions of the operators (3.131), but one will need explicit measures to integrate what is left.

The measure for integration over the eleven λ zero-modes was given in (3.132) while the measure for the ω_α zero modes, which has ghost number -8, is [67][32]

$$[\mathrm{d}\omega] = \frac{1}{11!5!} (\lambda \gamma^m)_{\alpha_1} (\lambda \gamma^n)_{\alpha_2} (\lambda \gamma^p)_{\alpha_3} (\gamma_{mnp})_{\alpha_4 \alpha_5} \epsilon^{\alpha_1 \ldots \alpha_5 \beta_1 \ldots \beta_{11}} \mathrm{d}\omega_{\beta_1} \wedge \ldots \wedge \mathrm{d}\omega_{\beta_{11}}. \tag{3.132}$$

The measure of the variable d_α is the flat grassmann measure

$$[\mathrm{d}d] = \frac{1}{16!} \epsilon_{\alpha_1, \ldots, \alpha_{16}} \frac{\partial}{\partial d_{\alpha_1}} \cdots \frac{\partial}{\partial d_{\alpha_{16}}}. \tag{3.133}$$

In [67] it was shown that the measure $[\mathrm{d}\omega]$ is gauge invariant, however it is easier to check that the whole measure (for a Riemann surface of genus g)

$$[\mathrm{d}\theta][\mathrm{d}\lambda] \bigwedge_{I=1}^{g} [\mathrm{d}\omega^I] \prod_{J=1}^{g} [\mathrm{d}d^J]$$

is BRST and gauge invariant.

To compute multiloop amplitudes over a g-genus Riemann surface one needs to have a measure for the integration over the moduli space of Riemann surfaces. The standard way to achieve this is through the insertion of $3g-3$ factors containing the b-ghost and the Beltrami differential, which is a conformal weight $(-1, 1)$ differential defined by

$$\mu_z^{\ \bar{z}} = g^{z\bar{z}} \frac{\partial g_{zz}}{\partial \tau}.$$

Explicitly the b-ghost insertion reads

$$(b, \mu) = \int \mathrm{d}^2 z \, b_{zz} \mu^z_{\ \bar{z}}$$

However the b-ghost must satisfy the property of $\{Q, b(z)\} = T(z)$ because (b, μ) must be BRST-invariant after the integration over moduli space. Nevertheless, in the minimal Berkovits approach [48] there is no such object, because there is no gauge invariant operator with ghost number -1 (with respect to $J = (\omega\lambda)$).

The idea to overcome this difficulty was to construct an operator $b(u, z)$ such that

$$\{Q, b_B(u, z)\} = T(u) Z_B(z)$$

because whenever one needs to insert the $3g-3$ b-ghosts in the scattering amplitude prescription one also needs to insert $10g$ of Z_B and $1g$ of Z_J to deal with the zero modes of ω_α. Then the idea was to borrow $3g-3$ Z_B's into the factor containing the measure for the moduli space. Therefore the insertion of (b_B, μ) in the pure spinor amplitude prescription will respect its BRST-closedness property up to a total derivative in moduli space.

The multiloop amplitude prescription for genus higher than one is given by

$$A_{g>1} = \int \mathrm{d}^2\tau_1 \ldots \mathrm{d}^2\tau_{3g-3} \langle \prod_{P=1}^{3g-3} \int \mathrm{d}^2 u_P \mu_P(u_P) \tilde{b}_{B_P}(u_P, z_P)$$

$$\prod_{P=3g-2}^{10g} Z_B^P(z_P) \prod_{R=1}^{g} Z_J(v_R) \prod_{I=1}^{11} Y_C^I(y_I) \mid^2 \prod_{T=1}^{N} \int \mathrm{d}^2 t_T U_T(t_T) \rangle,$$

where the b_B-ghost is a complicated operator whose expression can be looked in [48] (see also detailed computations in [60][61]). For the genus one surface the prescription is given by

$$A_{\text{one-loop}} = \int \mathrm{d}^2\tau \langle \mid \int \mathrm{d}^2 u \mu(u) \tilde{b}_{B_1}(u, z_1)$$

$$\prod_{P=2}^{10} Z_B^P(z_P) Z_J(v) \prod_{I=1}^{11} Y_C^I(y_I) \mid^2 V_1(t_1) \prod_{T=2}^{N} \int \mathrm{d}^2 t_T U_T(t_T) \rangle,$$

where due to translational invariance of the torus one can fix the position of one unintegrated vertex operator V_1.

The $\langle \quad \rangle$ brackets means the integration over the zero modes of the various variables using the measures described above together with the Berezin integrals over $\int \mathrm{d}^{16}\theta$ and $\int \mathrm{d}^{16} d$.

In a similar way as in the tree level scattering amplitude, the Berkovits' multiloop prescription is not independent of the constant spinors C_α^I's and the constant tensors B_{mn}^P, as it was shown by Skenderis and Hoogeveen in [23]. They showed also that it is necessary to integrate by the constants C^I's and B_{mn}^P's in order to decouple the non physical states.
Note also that this prescription includes the anomalous point $\lambda^\alpha = 0$. Moreover, there is no relationship between the minimal and non minimal multiloop scattering amplitudes via Čech-Dolbeault isomorphism, as it is suggested from the pure spinor partition function [26]. So, all these arguments lead us to conclude that the Berkovits' multiloop prescription must be modified to get a well defined minimal pures spinor formalism.

3.7.2 Product of Čech Cochains

In this subsection we define a product between the Čech cochains such that the result will be also a Čech cochain. The aim is to obtain a well defined scattering amplitude, i.e since the loop level scattering amplitude includes the b-ghost and the lowering and raising picture changing operators, which are mathematical objects define locally, then it is necessary that the product of all these objects will be a Čech cochain, such that the BRST operator $Q_T = Q+\delta$ is also well defined allowing in this manner to establish a relationship between the minimal and non-minimal formalism.

As we have shown in the example (3.100), the product of two Čech cochains is not in general a Čech cochain. This implies that the Čech operator δ is not defined acting on this product, because it does not satisfy the Leibniz rule, i.e it is not a derivative operator, see the example (3.101). So we are going to define a product between the Čech cochains and show how the Čech operator acts on them. This is a small step towards the definition of loop-level scattering amplitudes.

As we showed previously with the example (3.100), considering two general cochains ψ^I and τ^I in the Abelian group of holomorphic function, the product in most cases is not a Čech cochain

$$\psi^I \tau^J \notin C^1(\underline{U}, \mathcal{O}) \tag{3.134}$$

because $\psi^I\tau^J \neq -\psi^J\tau^I$. So, we define the following antisymmetric product

$$\psi^I * \tau^J \equiv \frac{1}{2}\left(\psi^I\tau^J - \psi^J\tau^I\right)\Big|_{U_I\cap U_J} \equiv \frac{1}{2}\psi^{[I}\tau^{J]} = \pi^{IJ} \in C^1(\underline{U}, \mathcal{O}), \tag{3.135}$$

which looks like an exterior product. Obviously this product is antisymmetric in the index I, J, i.e $\psi^I * \tau^J = -\psi^J * \tau^I$, however the exchange of the ψ^I and τ^J depends if they are grassmann or bosonic variables, this means

$$\psi^I * \tau^J = (-)^{(deg(\psi^I)\cdot deg(\tau^J))}\tau^J * \psi^I, \tag{3.136}$$

where $deg(\psi^I) = 0$ or 1 if ψ^I is a bosonic or grassmann variable respectively. Note that if the product of the Čech cochains is well defined, as in the case of the product of the picture lowering operators $Y_C^I Y_C^J$, then it is in agreement with (3.135):

$$Y_C^I * Y_C^J = \frac{1}{2}Y_C^{[I}Y_C^{J]} = Y_C^I Y_C^J. \tag{3.137}$$

Now, acting with the Čech operator on (3.135) we get

$$\begin{aligned}(\delta\pi)^{IJK} &= [\delta(\psi*\tau)]^{IJK} = \frac{1}{2^2}(\delta\psi)^{[IJ}\tau^{K]} = -\frac{1}{2^2}\psi^{[I}(\delta\tau)^{JK]} \\ &= \frac{3}{4}\frac{1}{3!}\left((\delta\psi)^{[IJ}\tau^{K]} - \psi^{[I}(\delta\tau)^{JK]}\right) \\ &= \frac{3}{4}\left((\delta\psi)^{IJ} * \tau^K - \psi^I * (\delta\tau)^{JK}\right) \in C^2(\underline{U}, \mathcal{O}).\end{aligned} \tag{3.138}$$

Note that δ acts like the exterior derivative over elements with star product, but with a coefficient in the front. This is a beautiful property. If we have three Čech cochains $\psi^I, \tau^J, \rho^K \in C^0(\underline{U}, \mathcal{O})$ then we define

$$\psi^I * \tau^J * \rho^K = \frac{1}{3!}\psi^{[I}\tau^J\rho^{K]} = \chi^{IJK} \in C^2(\underline{U}, \mathcal{O}). \tag{3.139}$$

It is simple to see that this product is associative. Acting with the Čech operator we have

$$\begin{aligned}(\delta\chi)^{IJKL} &= \frac{1}{3!\,2}(\delta\psi)^{[IJ}\tau^K\rho^{L]} = -\frac{1}{3!\,2}\psi^{[I}(\delta\tau)^{JK}\rho^{L]} = \frac{1}{3!\,2}\psi^{[I}\tau^J(\delta\rho)^{KL]} \\ &= \frac{4}{3!}\frac{1}{4!}\left((\delta\psi)^{[IJ}\tau^K\rho^{L]} - \psi^{[I}(\delta\tau)^{JK}\rho^{L]} + \psi^{[I}\tau^J(\delta\rho)^{KL]}\right) \\ &= \frac{4}{3!}((\delta\psi)^{IJ} * \tau^K * \rho^L - \psi^I * (\delta\tau)^{JK} * \rho^L + \psi^I * \tau^J * (\delta\rho)^{KL}) \in C^3(\underline{U}, \mathcal{O}).\end{aligned} \tag{3.140}$$

Again, the Čech operator acts like the exterior derivative operator over elements with the star product, nevertheless it has a coefficient in the front. It is straightforward to generalize this procedure for higher cochains with values on any Abelian group. Notice that the expressions (3.109) and (3.119) are just the $*$ product.
If we use this product between the homotopy operator (3.102) and the PCO's inside the tree level scattering amplitude the result vanishes because there are 11 patches to cover the pure spinor space.

3.7.3 The b-ghost

Unlike the tree-level scattering amplitude, in higher orders of the genus expansion the cover $\underline{U} = \{U_I\}$ given by the eleven patches $U_I = PS \setminus D_I$, $I = 1, .., 11$ (see 3.55) is not enough. The explanation is simple, since the b-ghost is a linear combination of 0,1,2 and 3-Čech cochains in the pure spinor space [52] and the product of the 11 picture lowering operators $\prod_{I=1}^{11} Y_C^I$ is a 10-Čech cochain then, with the antisymmetric $*$ product it is clear that the scattering amplitude will vanish if the number of patches is less than $11 + 4(3g-3)$, $g > 1$, where g is the genus of the Riemann surface. In the particular case when $g = 1$ this number is $11 + 4$. One can think to add more patches to the tree level scattering amplitude (see the appendix B.1.2) and so to apply the $*$ product with the naive homotopy operator, however this product needs to be better understood, since actually the operator δ is not a derivate operator strictly speaking because of those coefficients in the front of (3.138) and (3.140) *. Note also that the tree level scattering amplitude must be δ closed in contrast to the genus-g, as we discuss later.

*Actually, we wish that the operator $Q_T = Q + \delta$ acts like the exterior derivate, however we do not succeed yet.

So, in this approach the b-ghost is given by

$$b = b_{(0)} + b_{(1)} + b_{(2)} + b_{(3)} \tag{3.141}$$

with

$$b^{\mu}_{(0)} = \frac{A^{\mu}_{\alpha} G^{\alpha}}{(A^{\mu}\lambda)}, \quad b^{\mu\nu}_{(1)} = \frac{A^{\mu}_{\alpha} A^{\nu}_{\beta} H^{[\alpha\beta]}}{(A^{\mu}\lambda)(A^{\nu}\lambda)}, \tag{3.142}$$

$$b^{\mu\nu\rho}_{(2)} = \frac{A^{\mu}_{\alpha} A^{\nu}_{\beta} A^{\rho}_{\gamma} K^{[\alpha\beta\gamma]}}{(A^{\mu}\lambda)(A^{\nu}\lambda)(A^{\rho}\lambda)}, \quad b^{\mu\nu\rho\kappa}_{(3)} = \frac{A^{\mu}_{\alpha} A^{\nu}_{\beta} A^{\rho}_{\gamma} A^{\kappa}_{\delta} L^{[\alpha\beta\gamma\delta]}}{(A^{\mu}\lambda)(A^{\nu}\lambda)(A^{\rho}\lambda)(A^{\kappa}\lambda)},$$

where the specific form of the numerators (G, H, K, L) is [57]

$$G^{\alpha} = \frac{1}{4}\left(2\Pi^{m}(\gamma_m d)^{\alpha} - N_{mn}(\gamma^{mn}\partial\theta)^{\alpha} - J_{\lambda}\partial\theta^{\alpha} - \partial^2\theta^{\alpha}\right), \tag{3.143}$$

$$H^{[\alpha\beta]} = \frac{(\gamma^{mnp})^{[\alpha\beta]}}{192}\left[(d\gamma_{mnp}d) + 24 N_{mn}\Pi_p\right],$$

$$K^{[\alpha\beta\gamma]} = \frac{(\gamma^{m}d)^{[\alpha}(\gamma_{mnp})^{\beta\gamma]}}{16} N^{np}, \quad L^{[\alpha\beta\gamma\delta]} = \frac{(\gamma^{pqr})^{[\alpha\beta}(\gamma_{mnp})^{\gamma\delta]}}{2^7} N^{mn} N_{qr},$$

and the A^{μ}_{α}'s belong to a bigger set of constant spinors $A^{\mu} \in \{C^I, V^i\} \equiv \{C^1, ..., C^{11}, V^1, ..., V^{4(3g-3)}\}$, i.e $\mu \in \{I, i\}$, where the V^i's are linearly independent vectors in $\mathbb{C}^{16}$ such that the hypersurfaces $P^i \equiv \{\lambda^{\alpha} \in PS : V^i_{\alpha}\lambda^{\alpha} = 0\}$ satisfy $D_1 \cap .. \cap D_{11} \cap P^1 \cap ... \cap P^{4(3g-3)} = \{0\}$. We define the patches $U_I = PS \setminus D_I$, $U^i = PS \setminus P^i$ and get the cover $\mathfrak{U} = \{U_I, U^i\}$ where $PS \setminus \{0\} = \bigcup_{I=1}^{11} U_I \bigcup_{i=1}^{4(3g-3)} U^i$. Note that the C^I's are the same as in the tree level case, so the cycle Γ given in (3.20) is a good definition to compute the scattering amplitude. Using the commutators and anticommutators given in [52]

$$\{Q, G^{\alpha}(z)\} = \lambda^{\alpha} T(z), \quad \left[Q, H^{[\alpha\beta]}\right] = \lambda^{[\alpha} G^{\beta]}, \quad \{Q, K^{[\alpha\beta\gamma]}\} = \lambda^{[\alpha} H^{\beta\gamma]}, \tag{3.144}$$

$$\left[Q, L^{[\alpha\beta\gamma\delta]}\right] = \lambda^{[\alpha} K^{\beta\gamma\delta]}, \quad \lambda^{[\eta} L^{\alpha\beta\gamma\delta]} = 0,$$

where $T(z)$ is the stress-energy tensor

$$T(z) = \frac{1}{2}\partial x^m \partial x_m + p_{\alpha}\partial\theta^{\alpha} - \omega_{\alpha}\partial\lambda^{\alpha},$$

it is easy to verify that the b-ghost (3.141) satisfies

$$\{Q + \delta, b(z)\} = T(z), \tag{3.145}$$

where, for instance

$$(\delta\, b_{(0)})^{\mu\nu} = \frac{A^{\nu}_{\alpha} G^{\alpha}}{(A^{\nu}\lambda)} - \frac{A^{\mu}_{\alpha} G^{\alpha}}{(A^{\mu}\lambda)} = \frac{A^{\mu}_{\alpha} A^{\nu}_{\beta} \lambda^{[\alpha} G^{\beta]}}{(A^{\mu}\lambda)(A^{\nu}\lambda)} = \frac{A^{\mu}_{\alpha} A^{\nu}_{\beta}\left[Q, H^{[\alpha\beta]}\right]}{(A^{\mu}\lambda)(A^{\nu}\lambda)}.$$

3.7.4 The Ghost Number Bidegree

As in bosonic string theory, the b-ghost must have ghost number -1 and the BRST charge must increase the ghost number by one unit. Note that $b_{(0)}$ has ghost number -1 but $b_{(1)}, b_{(2)}, b_{(3)}$ have ghost number $-2, -3, -4$ respectively (the numerators have ghost number zero, see [52]), where the ghost current is given by $J_\lambda = \lambda^\alpha \omega_\alpha$. However, since the total BRST charge is $Q_T = Q + \delta$ and the Čech operator increases the number of patches in one, then the number of patches is also a ghost number. So we define the total ghost number by

$$J_T = \int \mathrm{d}z J_\lambda + J_\delta,$$

$$J_\delta \equiv (\sum_\mu \mu \partial_\mu - 1),$$

where the operator J_δ acts on the Čech cochains in the Čech labels, for example

$$(\sum_\eta \eta \partial_\eta - 1)\; b_{(3)}^{\mu\nu\rho\kappa} = 3\, b_{(3)}^{\mu\nu\rho\kappa}.$$

Therefore the b-ghost (3.141) has J_T ghost number -1, as expected. In the tree level scattering amplitude the J_δ ghost number is not relevant because this amplitude is δ closed, see the subsection 3.5.1. Nevertheless, at loop level this ghost number become very important since the relation (3.145) means that the scattering amplitude is $Q_T = Q + \delta$ closed up to boundary terms in the moduli space [69], i.e if we consider a loop level scattering amplitude where the $*$ product is used to insert the b-ghost, then we expect to get

$$\begin{aligned}(Q+\delta)\left\langle \ldots\ldots \int \mathrm{d}z \mu_{\bar{z}}^{z}(z) b(z) \right\rangle &= \left\langle \ldots\ldots \int \mathrm{d}z \mu_{\bar{z}}^{z}(z)(Q+\delta)(b(z)) \right\rangle \\ &= \left\langle \ldots\ldots \int \mathrm{d}z \mu_{\bar{z}}^{z}(z) T(z) \right\rangle = \int_{\mathcal{M}} \frac{\partial}{\partial \tau^i} \langle \ldots\ldots \rangle ,\end{aligned} \tag{3.146}$$

where .... means the global insertions, $\mu_{\bar{z}}^{z}(z)$ is the Beltrami differential, τ^i's are the Teichmüller parameters and $\mathcal{M}$ is the Moduli space. Now, with the aim to see the importance of the J_δ ghost number at the loop level we can regard the 1-loop scattering amplitude. In this amplitude we have 11 zero modes of the pure spinor λ^α and 11 zero modes of the spinor ω_α [48], so at 1 loop the zero modes of λ^α and ω_α form the pure spinor phase space. Integrating somehow the zero modes of the fields ω_α, d_α and θ^α then the scattering amplitude shall behave as

$$\int_\Gamma [\mathrm{d}\lambda] \frac{A^{I_1} \ldots A^{I_{11}}}{(A^{I_1}\lambda) \ldots (A^{I_{11}}\lambda)} \frac{A^{I_{12}} A^{I_{13}} \lambda^4}{(A^{I_{12}}\lambda)(A^{I_{13}}\lambda)}, \tag{3.147}$$

where we are not being careful with the spinorial indices. Note that this amplitude is a 12-Čech cochain and have J_λ ghost number -1. However this ghost number must always be zero, therefore it should be compensated with one J_δ ghost number[†] and so we will get an 11-Čech cochain. As the tree level scattering amplitude is a 10-Čech cochain and it is related to the Green's function for the massless scalar field then the 1 loop amplitude, which should be a 11-Čech cochain, suggests that it should be related to the Green's function for the massless higher-spin field [22]. This was only a simple and crude analysis about the 1-loop scattering amplitude. Actually the full analysis must be over the whole pure spinor phase space. This is because, for instance, at two loops we should get a Čech cochain in the pure spinor space bigger than 11, so its corresponding Dolbeault cochain will be identically zero and the amplitude will vanish. Therefore it is necessary to regard the whole pure spinor phase space, i.e the space of the λ^α's and ω_α's.

One could think that the scattering amplitude must have $J_T = J_\lambda + J_\delta$ ghost number zero, but that is not true. For example, in the tree level amplitude it is impossible to construct the picture lowering operator such that the amplitude has J_T ghost number zero and the origin is removed from the pure spinor space. So the conditions that the scattering amplitude has J_λ ghost number zero is necessary in order to get a physical amplitude, i.e a $(Q + \delta)$ closed scattering amplitude.

This was just a glance about the loop level scattering amplitudes, which is a work in progress [38].

[†]Perhaps because (3.147) is δ exact.

Chapter 4

Scattering Amplitudes and Unitarity in the Non-minimal Pure Spinor Formalism

In this chapter we obtain the coefficient of the type IIB (and IIA [102][103]) one and two-loop massless four-point amplitude from a straightforward computation and for the whole supermultiplet. To achieve that we use pure spinor measures which present the feature of having simple forms for all genera, in deep contrast with the complicated superstring measure for the RNS formalism (it was gave a short review in the introduction, for details one can see [97],[98]). As mentioned in [66], it is still an unsolved problem to find the precise normalizations for the chiral bosonization formulæ of [96]. Therefore the two-loop coefficient can not be obtained from a direct calculation in the RNS formalism. In fact, computing the amplitude up to the overall coefficient already required several years of effort which resulted in an impressive series of papers [95],[64], so the strategy adopted in [66] was to *fix* the two-loop coefficient indirectly by using factorization (unitarity) . So in this respect the calculations of this thesis make it very clear how the pure spinor formalism can surpass the RNS limitations. But to present our results we have chosen to adopt the clear conventions of [66], which also eases the detection of any mismatches.

In the following section we define our space time dimensions and give a short review about the prescription to compute scattering amplitude in the nonminimal pure spinor formalism [57]. In the section 2, we will give a review to the $x^m(z,\bar{z})$ fields contribution and we justify the normalization of the path integral measures. We compute the non-zero modes contributions for every fields and show that in the non-minimal pure spinor formalism the scattering amplitudes are independent of the functional determinants. We also normalize the zero modes, which need special treatment as we explain. In the section 3, we will compute the integral on the pure spinors space. This is the key to check the unitarity of the nonminimal pure spinor formalism at two loop, i.e to check the relationship (see the appendix C.1)

$$C_1^2 = 8\pi^2 C_0 C_2$$

where C_0, C_1 and C_2 are the overall factors of the tree-level, one and two loop scattering amplitude respectively. In this Section we arrive to the following result

$$\boxed{\int_{\mathcal{O}(-1)}[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]\, e^{-a\,\lambda\overline{\lambda}} = (2\pi)^{11}(a^8\cdot 12\cdot 5)^{-1}\,, \qquad a\in\mathbb{R}^+}$$

where $\mathcal{O}(-1)$ is the line bundle blow-up at the origin with base space $SO(10)/U(5)$, i.e the pure spinors space.

The computations of the three- and four-point amplitudes at tree-level are performed in section 4 to show that the conventions of section 1 match the RNS ones of (1.7) [66], i.e. $\mathcal{A}_0^{\mathrm{PS}} = \mathcal{A}_0^{\mathrm{RNS}}$, where

$$\mathcal{A}_0^{\mathrm{PS}} = (2\pi)^{10}\delta^{(10)}(k)\,\kappa^4 e^{-2\lambda}\left(\frac{\sqrt{2}}{2^{12}\pi^6\alpha'^5}\right)\left(\frac{\alpha'}{2}\right)^8 K\bar{K}\,C(s,t,u)\,.$$

Then we use the very same machinery of the tree-level computation to obtain also the full supersymmetric one- and two-loop amplitudes – including their precise coefficients – in sections 5 and 6,

$$\mathcal{A}_1^{\mathrm{PS}} = (2\pi)^{10}\delta^{(10)}(k)\frac{\kappa^4 K\bar{K}}{2^9\pi^2\alpha'^5}\left(\frac{\alpha'}{2}\right)^8\int_{\mathcal{M}_1}\frac{d^2\tau}{\tau_2^5}\prod_{i=2}^{4}\int d^2z_i\prod_{i<j}^{4}F_1(z_i,z_j)^{\alpha' k^i\cdot k^j}, \tag{4.1}$$

$$\mathcal{A}_2^{\mathrm{PS}} = (2\pi)^{10}\delta^{(10)}(k)\kappa^4 e^{2\lambda}\frac{\sqrt{2}K\bar{K}}{2^{10}\alpha'^5}\left(\frac{\alpha'}{2}\right)^{10}\int_{\mathcal{M}_2}\frac{d^2\Omega_{IJ}}{(\det\mathrm{Im}\Omega_{IJ})^5}\int_{\Sigma_4}|\mathcal{Y}_s|^2\prod_{i<j}F_2(z_i,z_j)^{\alpha' k^i\cdot k^j} \tag{4.2}$$

which explicitly shows that with the pure spinor formalism those coefficients follow directly from a first principles computation. However, we find disagreement with the RNS results reported by [66], namely

$$\mathcal{A}_1^{\mathrm{PS}} = \frac{1}{2^2}\mathcal{A}_1^{\mathrm{RNS}},\quad \mathcal{A}_2^{\mathrm{PS}} = \frac{1}{2^4}\mathcal{A}_2^{\mathrm{RNS}}. \tag{4.3}$$

As we argue in section 5 that [66] forgot the two factors of 1/2 from the GSO projection in the left- and right-moving sectors in their measure. This observation will also explain the $1/2^4$ mismatch at two-loops of section 6, as [66] fixed the two-loop coefficient using a factorization constraint which depends quadratically on the one-loop coefficient*.

*For a compact Riemann surface S of genus g the correct factor is $1/2^{2g}$, which is the number of spin structures over S and is in agreement with factorization.

In the appendix C.3 we present the detailed covariant computation of the two-loop kinematic factor needed in section 6. This appendix can be regarded as a fully $SO(10)$-covariant proof of the 2-loop equivalence† between the non-minimal and minimal pure spinor formalisms, and is analogous to the covariant proof of [67] for the 1-loop case. The appendix C.3 is devoted to proving a formula mentioned *en passant* in [49] which is used to rewrite the two-loop amplitude in terms of integrals in the period matrix instead of in the Teichmüller parameters.

4.1 Review on the scattering amplitude prescription in the non-minimal pure spinor formalism

In order to obtain the overall coefficients of the scattering amplitudes it is necessary to define our space-time dimensions.

We will give a brief review of the non-minimal pure spinor formalism. The idea is to introduce our own conventions and to normalize the massless vertex operator in the same way as in the D'Hoker, Phong and Gutperle's paper [66]. the superstring theory action on the left sector of the non-minimal pure spinor formalism was proposed by Berkovits [57] and it is given by

$$S = \frac{1}{2\pi\alpha'}\int_{\Sigma_g} \mathrm{d}^2 z\ \left(\partial x^m \bar{\partial} x_m + \alpha' p_\alpha \bar{\partial}\theta^\alpha - \alpha'\omega_\alpha\bar{\partial}\lambda^\alpha - \alpha'\bar{\omega}^\alpha\bar{\partial}\bar{\lambda}_\alpha + \alpha' s^\alpha \bar{\partial} r_\alpha\right) \quad (4.4)$$

where we defined the space time dimensions of the variables and coupling constant α' as follows

$$[x^m] = 1, \quad [\alpha'] = 2, \quad [p_\alpha] = [\omega_\alpha] = \left[\bar{\lambda}_\alpha\right] = [r_\alpha] = -1/2, \quad (4.5)$$
$$[\theta^\alpha] = [\lambda^\alpha] = [\bar{\omega}^\alpha] = [s^\alpha] = 1/2. \quad (4.6)$$

The OPE's for the matter variables are easily computed

$$x^m(z)x_n(w) \sim -\frac{\alpha'}{2}\delta^m_n \ln|z-w|^2, \quad p_\alpha(z)\theta^\beta(w) \sim \frac{\delta^\beta_\alpha}{z-w}. \quad (4.7)$$

The complex bosonic spinors λ^α and $\bar{\lambda}_\alpha$ the pure spinor constraint

$$\lambda\gamma^m\lambda = \bar{\lambda}\gamma^m\bar{\lambda} = 0, \qquad m = 0,\ 1,\ 2, ..., 9 \quad (4.8)$$

†As will be mentioned in appendix C.3, there is a loophole in the 2-loop equivalence proof of [62]. Some terms in the non-minimal pure spinor kinematic factor were argued to vanish using a $U(5)$ decomposition but, as will be shown explicitly using the identities of [63], are in fact proportional to the kinematic factor of the minimal pure spinor formalism. As this loophole only affects the proportionality constant, it does not alter the conclusions of [62] but had to be taken into account here.

and the fermionic spinor r_α satisfies the constraint

$$\bar{\lambda}\gamma^m r = 0. \tag{4.9}$$

Because of the constraints on λ^α, $\bar{\lambda}_\alpha$ and r_α, their conjugate momenta ω_α, $\bar{\omega}^\alpha$ and s^α are defined up to a gauge transformation,

$$\delta\omega_\alpha = \Lambda_m(\gamma^m\lambda)_\alpha \tag{4.10}$$

$$\delta\bar{\omega}^\alpha = \bar{\Lambda}_m(\gamma^m\bar{\lambda})^\alpha - \phi_m(\gamma^m r)^\alpha, \quad \delta s^\alpha = \phi_m(\gamma^m\bar{\lambda})^\alpha \ , \tag{4.11}$$

for arbitrary Λ_m, $\bar{\Lambda}_m$ and ϕ_m.
Analogously to the minimal pure spinor formalism variables (given in the introduction) where

$$N_{mn} = \frac{1}{2}(\omega\gamma_{mn}\lambda), \quad J_\lambda = \omega_\alpha\lambda^\alpha, \quad T_\lambda = \omega_\alpha\partial\lambda^\alpha,$$

the new variables also have their associated Lorentz and ghost currents, which we give in a covariant and gauge invariant combination [57]

$$\bar{N}_{mn} = \frac{1}{2}(\bar{\omega}\gamma_{mn}\overline{\lambda} - s\gamma_{mn}r), \quad \bar{J}_{\overline{\lambda}} = \bar{\omega}^\alpha\overline{\lambda}_\alpha - s^\alpha r_\alpha, \quad T_{\overline{\lambda}} = \bar{\omega}^\alpha\partial\overline{\lambda}_\alpha - s^\alpha\partial r_\alpha.$$

Furthermore one also defines

$$S_{mn} = \frac{1}{2}(s\gamma_{mn}\overline{\lambda}), \quad S = s^\alpha\overline{\lambda}_\alpha, \quad J_r = (rs),$$

and the total ghost current to be

$$J_b = J_\lambda - \bar{J}_{\overline{\lambda}} + J_r = \omega_\alpha\lambda^\alpha - \bar{\omega}^\alpha\overline{\lambda}_\alpha.$$

The non-minimal BRST operator is defined by

$$Q_{NM} = \int \mathrm{d}z\lambda^\alpha d_\alpha + \int \mathrm{d}z\,\bar{\omega}^\alpha r_\alpha \equiv Q + \Delta, \tag{4.12}$$

where Q is the same as in (1.22).
Using the Kugo-Ojima (KO) quartet mechanism [58][59] one can show that the cohomology of the non-minimal BRST operator (4.12) doesn't depend on the "quartet" of non-minimal variables (r_α, s^α), $(\overline{\lambda}_\alpha, \overline{\omega}^\alpha)$.
The physical states are eigenvalues of the non-minimal ghost number $\bar{J}_{\overline{\lambda}}\Psi = n\Psi$. Now, since $\bar{J}_{\overline{\lambda}}$ is BRST exact

$$\bar{J}_{\overline{\lambda}} = [Q_{NM}, s^\alpha\overline{\lambda}_\alpha] \quad \Rightarrow \quad [Q_{NM}, \bar{J}_\beta] = 0,$$

then all states with non-zero non-minimal ghost charge are Q-exact

$$\Psi = [Q_{NM}, s^\alpha\overline{\lambda}_\alpha\Psi]/n.$$

So the physical states have $n = 0$ and the vertices of the minimal pure spinor formalism can be used.

Since $\overline{\omega}^{\alpha}$ is the conjugate momenta of the field $\overline{\lambda}_{\alpha}$ and r_{α} can be interpreted as the differential form $\mathrm{d}\overline{\lambda}_{\alpha}$ then we can represent the operator $\int \mathrm{d}z\, \bar{\omega}^{\alpha} r_{\alpha}$ as the Dolbeault operator in the pure spinor space

$$\int \mathrm{d}z\ \bar{\omega}^{\alpha} r_{\alpha} \quad \longrightarrow \quad \mathrm{d}\overline{\lambda}_{\alpha} \frac{\partial}{\partial \overline{\lambda}_{\alpha}} \wedge \equiv \overline{\partial}.$$

So the BRST charge can be written as

$$Q_{NM} = \overline{\partial} + Q. \tag{4.13}$$

This representation is very useful to get a rigorous relationship between the minimal and non minimal pure spinor formalism, we will talk about this in the chapter 5.

Note that if we would interpret the operator Q as the contraction operator [52], i.e if Q comes from a gauge fixing (which is a mystery yet [40][41]), then the non minimal BRST charge would be a holomorphic equivariant operator. This fact may be interesting to related the pure spinor approach with some topological non-linear sigma model with supersymmetry worldsheet (0,2). It would be very useful to obtain the origin of the pure spinor formalism [42].

In this subsection it is import to rewrite the massless vertex operators given in the introduction. The idea is to define the normalization of these vertex in order to get the correct overall factors in the scattering amplitudes.
From the introduction we have the massless vertex operators are given by

- Unintegrated vertex operator

$$V = \lambda^{\alpha} A_{\alpha}(x, \theta) \tag{4.14}$$

- Integrated vertex operator

$$U = \partial\theta^{\alpha} A_{\alpha} + \Pi^{m} A_{m} + \frac{\alpha'}{2} d_{\alpha} W^{\alpha} + \frac{\alpha'}{4} N_{mn} \mathcal{F}_{mn} \tag{4.15}$$

where $[Q, U] = \partial V$ and the super-Yang Mills fields A_{α}, A_m, W^{α} and $\mathcal{F}_{mn}$ were given in (1.46).
The dimensions of the superfields are

$$[A_{\alpha}] = 1/2, \quad [A_m] = 0, \quad [W^{\alpha}] = -1/2, \quad [\mathcal{F}_{mn}] = -1,$$

hence the massless vertex operators have the following dimensions

$$[V] = [\lambda^{\alpha} A_{\alpha}] = 1,\ [U] = [\partial\theta^{\alpha} A_{\alpha} + A_m \Pi^m + \frac{\alpha'}{2} d_{\alpha} W^{\alpha} + \frac{\alpha'}{4} N_{mn} \mathcal{F}_{mn}] = 1\ . \tag{4.16}$$

These vertex operators have the same normalization as the massless vertex operator of the RNS formalism given in [66]. For example, the closed superstring massless operator in the NS-NS sector is [66]

$$V = e_m \bar{e}_n \int \mathrm{d}^2 z \, (\partial x^m + ik \cdot \psi_+ \psi_+^m)(\bar{\partial} x^n + ik \cdot \psi_- \psi_-^n) e^{ik \cdot x} \; , \tag{4.17}$$

where the dimension of V is two if the dimension of the polarization vectors is zero.

4.1.1 Topological String and Scattering Amplitude Prescription

In this subsection we will give the prescription for computing scattering amplitudes in the nonminimal pure spinor formalism. This short review is based on the Berkovits' paper [57].

The scattering amplitude prescription in the nonminimal pure spinor formalism is based on the idea to interpret this formalism as a critical topological string, i.e we must get the algebra of a $N = (2, 2)$ topological string

$$\begin{aligned} T(z)T(w) &\to \frac{c/2}{(z-w)^4} + \frac{2T}{(z-w)^2} + \frac{\partial T}{(z-w)} \\ T(z)G^{\pm} &\to \frac{3}{2} \frac{G^{\pm}}{(z-w)^2} + \frac{\partial G^{\pm}}{(z-w)} \\ G^{+}(z)G^{-}(w) &\to \frac{2c/3}{(z-w)^3} + \frac{2J}{(z-w)^2} + \frac{T}{(z-w)} \\ T(z)J(w) &\to \frac{\hat{c}}{(z-w)^3} + \frac{J}{(z-w)^2} + \frac{\partial J}{(z-w)} \\ J(z)G^{\pm}(w) &\to \pm \frac{G^{\pm}}{(z-w)} \\ J(z)J(w) &\to \frac{\hat{c}}{(z-w)^2}. \end{aligned} \tag{4.18}$$

where (4.18) is the left sector of $N = (2, 2)$ topological algebra, c is the central charge of the stress tensor and $\hat{c}$ is the ghost anomaly.

Now, from the gauge symmetry (4.10) some amounts gauge invariant that we can construct are

$$J_\lambda = \omega_\alpha \lambda^\alpha, \quad \bar{J_{\bar{\lambda}}} = \overline{\omega}^\alpha \overline{\lambda}_\alpha - s^\alpha r_\alpha, \quad J_r = r_\alpha s^\alpha. \tag{4.19}$$

These currents generate a $U(1)$ global symmetry. In particular we have the following curent

$$J \equiv J_\lambda - \bar{J_{\bar{\lambda}}} + J_r = \omega_\alpha \lambda^\alpha - \overline{\omega}^\alpha \overline{\lambda}_\alpha, \tag{4.20}$$

which has the same ghost anomaly like the bosonic string, i.e

$$T(z)J(y) = \frac{\hat{c}}{(z-y)^3} + \frac{J(y)}{(z-y)^2} + \frac{\partial J(y)}{(z-y)} + \text{reg} \tag{4.21}$$

where $\hat{c} = -3$. Note that although in the bosonic string theory the algebra generates by the set $\{T(z),\, j_{BRST}(z),\, b(z),\, J(z) = bc\}$ [72] is not exactly the same as (4.18), from the identification $G^+(z) = 2j_{BRST}$ and $G^-(z) = b(z)$, the topological strings can be viewed as a special type of "bosonic string" since that their scattering amplitude prescriptions are the same.

In order to obtain the algebra (4.18) in the nonminimal pure spinor formalism, it is necessary to find a composite field $b(z)$, which is known as the b-ghost, such that it satisfies

$$\{Q, b(z)\} = T(z). \tag{4.22}$$

where the stress tensor of the non-minimal pure spinor formalism is given by

$$T(z) = -\frac{1}{\alpha'}\partial x^m \partial x_m - p_\alpha \partial\theta^\alpha + \omega_\alpha \partial\lambda^\alpha + \bar{\omega}^\alpha \partial\bar{\lambda}_\alpha - s^\alpha \partial r_\alpha.$$

This composite $b-$field was given in [57] and it has the form

$$\begin{aligned} b(z) &= s^\alpha \partial\bar{\lambda}_\alpha + \frac{\bar{\lambda}_\alpha G^\alpha}{(\lambda\bar{\lambda})} + \frac{\bar{\lambda}_\alpha r_\beta H^{[\alpha\beta]}}{(\lambda\bar{\lambda})^2} - \frac{\bar{\lambda}_\alpha r_\beta r_\gamma K^{[\alpha\beta\gamma]}}{(\lambda\bar{\lambda})^3} - \frac{\bar{\lambda}_\alpha r_\beta r_\gamma r_\delta L^{[\alpha\beta\gamma\delta]}}{(\lambda\bar{\lambda})^4} \\ &= s^\alpha \partial\bar{\lambda}_\alpha + \frac{\bar{\lambda}_\alpha(2\Pi^m(\gamma_m d)^\alpha - N_{mn}(\gamma^{mn}\partial\theta)^\alpha - J_\lambda \partial\theta^\alpha - \frac{1}{4}\partial^2\theta^\alpha)}{4(\lambda\bar{\lambda})} \\ &\quad + \frac{(\bar{\lambda}\gamma^{mnp}r)(\frac{\alpha'}{2}d\gamma_{mnp}d + 24N_{mn}\Pi_p)}{192(\lambda\bar{\lambda})^2} - \frac{\frac{\alpha'}{2}(r\gamma_{mnp}r)(\bar{\lambda}\gamma^m d)N^{np}}{16(\lambda\bar{\lambda})^3} \\ &\quad + \frac{\frac{\alpha'}{2}(r\gamma_{mnp}r)(\bar{\lambda}\gamma^{pqr}r)N^{mn}N_{qr}}{128(\lambda\bar{\lambda})^4}. \end{aligned} \tag{4.23}$$

where the numerators (G, H, K, L) are the same as in (3.143). Since the b-ghost is not defined on the tip of the pure spinor cone, i.e in $\lambda = 0$, then this point must be removed from the theory, such as was claimed in the chapter 2. So the nonminimal pure spinor formalism is free of anomalies.
However the set $\{j_{BRST}, T(z), J(z), b(z)\}$ still does not satisfy the algebra (4.18), because the coefficient of the double pole between $J(z)$ with itself is not equal to ghost anomaly $\hat{c} = -3$. It can be seen from the OPE's [57]

$$\begin{aligned} &J_\lambda(z)J_\lambda(y) \to \frac{-4}{(z-y)^2}, \quad J_{\bar{\lambda}}(z)J_{\bar{\lambda}}(y) \to \text{reg}, \\ &J_r(z)J_r(y) \to \frac{11}{(z-y)^2}, \quad J_{\bar{\lambda}}(z)J_r(y) \to \frac{8}{(z-y)^2}, \end{aligned} \tag{4.24}$$

thus we have

$$J(z)J(y) \to \frac{-9}{(z-y)^2}. \tag{4.25}$$

Nevertheless, we can modify $J(z)$ by an exact $BRST$ term

$$\begin{aligned} J_{nonmin} &= J + \{Q, s^\alpha\overline{\lambda}_\alpha - 2\frac{\overline{\lambda}_\alpha\partial\theta^\alpha}{(\lambda\overline{\lambda})}\} \\ &= \omega_\alpha\lambda^\alpha - s^\alpha r_\alpha - 2\frac{\overline{\lambda}_\alpha\partial\lambda^\alpha + r_\alpha\partial\theta^\alpha}{(\lambda\overline{\lambda})} + 2\frac{(\lambda^\alpha r_\alpha)(\overline{\lambda}_\beta\partial\theta^\beta)}{(\lambda\overline{\lambda})^2} \end{aligned} \tag{4.26}$$

where J_{nonmin} is well defined since the point $\lambda^\alpha = 0$ must be removed from the pure spinor space. Therefore the new set $\{j_{BRST}, T(z), J_{nonmin}(z), b(z)\}$ satisfy the algebra (4.18) and so we have that the nonminimal pure spinor formalism can be interpreted as a critical topological string theory $N = 2$, $\hat{c} = -3$. In this way the scattering amplitude prescription in the nonminimal pure spinor formalism is given by the topological string prescription, i.e in the same way as the bosonic string theory

- Tree Level

$$\mathcal{A}_{tree} = e^{-2\mu}\left\langle \left| V^1(z_1)V^2(z_2)V^3(z_3)\int U^4(z_4)\ldots\int U^N(z_N)\right|^2\right\rangle \tag{4.27}$$

where $e^{-2\mu}$ is the string coupling constant, z_1, z_2 and z_3 are fix-points on the worldsheet, and the square $|\cdot|^2$ is because we are interesting just in the closed string.

- 1-Loop

$$\mathcal{A}_{1-loop} = \frac{1}{2}\int_{\mathcal{M}_1} \mathrm{d}^2\tau \left\langle \left| V^1(z_1)(b,\mu)\int U^2(z_2)\ldots\int U^N(z_N)\right|^2\right\rangle \tag{4.28}$$

where $\mathcal{M}_1$ is the moduli space of the torus, τ is the Teichmuller parameter, z_1 is a fix-point and (b,μ) is the inner product between the b-ghost and Beltrami differential $\mu_{\bar{z}}^{z}$.

- 2-loop

$$\mathcal{A}_{2-loop} = \frac{1}{2}e^{2\mu}\int_{\mathcal{M}_2} \mathrm{d}^6\tau \left\langle \left| \prod_{i=1}^{3}(b,\mu_i)\int U^1(z_1)\ldots\int U^N(z_N)\right|^2\right\rangle \tag{4.29}$$

where $\mathcal{M}_2$ is the Moduli space of the Riemann surface with genus $g = 2$.

- Multiloop

$$\mathcal{A}_{g>2} = \int_{\mathcal{M}_g} \mathrm{d}^{6g-6}\tau \left\langle \left| \prod_{i=1}^{3g-3}(b,\mu_i)\int U^1(z_1)\ldots\int U^N(z_N)\right|^2\right\rangle \tag{4.30}$$

where $\mathcal{M}_g$ is the Moduli space of the Riemann surface with genus $g > 2$.

In the previous prescription (4.27), (4.28) and (4.30) the simbol $\langle \cdot \rangle$ means integration by the fundamental fields $x^m, p_\alpha, \theta^\alpha, \omega_\alpha, \lambda^\alpha, \overline{\omega}^\alpha, \overline{\lambda}_\alpha, s^\alpha, r_\alpha$.
The 1/2 factor in the prescription at 1-loop (2-loop) comes from any Riemann surface of genus $g = 1$ ($g = 2$) can be written like an algebraic curve of the form $y^2 = z(z-1)(z-\tau)$ ($y^2 = z(z-1)(z-\tau_1)(z-\tau_2)(z-\tau_3)$) *, so by the $\mathbb{Z}_2$ symmetry $y \to -y$ it is necessary to introduce the 1/2 factor.

The b-ghost insertion is [72][73]

$$(b, \mu_j) = \frac{1}{2\pi}\int d^2y_j b_{zz}\mu_{j\,\bar{z}}^{z}, \qquad j = 1, \ldots, 3g-3. \tag{4.31}$$

where the normalization $1/2\pi$ comes from bosonic string theory [72] because the topological prescription is based on it.

However since the pure spinor space is a non-compact space then it is necessary to introduce a regulator to get the convergence on the pure spinor zero-modes. Obviously this regulator can not affect the scattering amplitude prescription therefore it must be $BRST$ exact, i.e it must have the form $\mathcal{N} = 1 + \{Q, \text{something}\}$ since the $BRST$ exact functions are decoupled of the theory. In [57][67] were proposed the regulators for tree-level, one and two-loop, which have the simple form

$$\mathcal{N}_{tree} = \exp(\{Q, \chi_T\}) = \exp(\{Q, -(\overline{\lambda}\theta)_0\}) = \exp[-(\lambda\overline{\lambda})_0 - (r\theta)_0], \tag{4.32}$$

$$\mathcal{N}_{1-loop} = \exp(\{Q, \chi_1\}) = \exp(\{Q, -(\overline{\lambda}\theta)_0 - (\omega s)_0\}) = \exp[-(\lambda\overline{\lambda})_0 - (\omega\overline{\omega})_0 - (r\theta)_0 + (sd)_0],$$

$$\begin{aligned}\mathcal{N}_{2-loop} &= \exp(\{Q, \chi_2\}) = \exp(\{Q, -(\overline{\lambda}\theta)_0 - \sum_{I=1}^{2}(\omega^I s^I)_0\}) \\ &= \exp[-(\lambda\overline{\lambda})_0 - \sum_{I=1}^{2}(\omega^I\overline{\omega}^I)_0 - (r\theta)_0 + \sum_{I=1}^{2}(s^I d^I)_0]\end{aligned}$$

where the label "0" means the zero modes and the index I count the number of zero modes. We will talk about the zero modes in the subsection 4.2.3. These regulators can not generalizated for more loops and number of points since the existence of the naive homotopy operator

$$\xi = \frac{(\overline{\lambda}\theta)}{(\lambda\overline{\lambda}) + (r\theta)} \qquad Q(\xi) = 1. \tag{4.33}$$

To remove this operator from the pure spinor formalism it is necessary to study the behavior of scattering amplitude when $\lambda \to 0$. This study will imply the need to define new regulators, however we do not go into detail about these topics, we suggest to see the papers [57][52][55] [53].

*Where τ and (τ_1, τ_2, τ_3) are the Teichmuller parameters for $g = 1$ and $g = 2$ Riemann surfaces respectively.

4.2 Integration Measures and Normalizations

In this subsection, in order to obtain the right overall factors of the scattering amplitudes to check the unitariry condition (C.22) we define and normalize the integration meausures from first principles, i.e we use the same method as the path integral in quantum mechanics. Note also that since the nonminimal pure spinor formalism is a $N = 2$ $\hat{c} = -3$ topological theory then this ghost anomaly must be reflected on the integration measure of the zero modes. This fact is very important to construct the integration meausures, however because the form of J_{nonmin} is complicated then it is enough to consider J, such as it was argued in [57].

4.2.1 Superflat space

First, we normalize the measures of the superflat space (x^m, θ^α) and the conjugate momentum p_α † and after find the non-zero modes contributions.

In order to normalize the integration measure of the x^m from its phase space, as the path integral in quantum mechanics, then it is useful to consider the first order action

$$S = \frac{1}{\pi\alpha'} \int \mathrm{d}^2 z\, (g^{i\bar{j}} p_i p_{\bar{j}} + p_i \bar{\partial} x^i + p_{\bar{i}} \partial x^{\bar{i}}) \tag{4.34}$$

where the index $i, \bar{i} = 1, ..., 5$, p_i and $p_{\bar{i}}$ are (1,0) and (0,1) forms with conformal weight (1,0) and (0,1) respectively and $g^{i\bar{j}} = \delta^{i\bar{j}}$.

In this first order action we can easily see that the conjugate momentum of the x^i and $x^{\bar{i}}$ fields are $P_i := p_i/\pi\alpha'$ and $P_{\bar{i}} := p_{\bar{i}}/\pi\alpha'$ respectively, so the Dirac brackets (DB) are

$$\begin{aligned}
\left[P_i(\sigma), x^j(\sigma')\right]_{DB} &= \left[\frac{p_i(\sigma)}{\pi\alpha'}, x^j(\sigma')\right]_{DB} = i\delta_i^j \delta(\sigma - \sigma'),\\
\left[P_{\bar{i}}(\sigma), x^{\bar{j}}(\sigma')\right]_{DB} &= \left[\frac{p_{\bar{i}}(\sigma)}{\pi\alpha'}, x^{\bar{j}}(\sigma')\right]_{DB} = i\delta_{\bar{i}}^{\bar{j}} \delta(\sigma - \sigma').
\end{aligned}$$

In quantum mechanics, because of the commutator relation $[p, x] = i$ one has the identity

$$\int \frac{\mathrm{d}x}{\sqrt{2\pi}} \frac{\mathrm{d}p}{\sqrt{2\pi}} e^{-ipx} = 1, \tag{4.35}$$

and the integration measure on the phase space in the path integral is [72]

$$\frac{\mathrm{d}x}{\sqrt{2\pi}} \frac{\mathrm{d}p}{\sqrt{2\pi}}. \tag{4.36}$$

† Let us remember that x^m parametrizes the flat space $\mathbb{R}^{10}$ (after the Wick rotation)

In the same way, the measure on the phase space in the path integral for the action (4.34) is

$$\begin{aligned}\prod_{z\bar{z}}\prod_{i,\bar{i},j,\bar{j}}\frac{\mathrm{d}P_i}{\sqrt{2\pi}}\frac{\mathrm{d}P_{\bar{i}}}{\sqrt{2\pi}}\frac{\mathrm{d}x^j}{\sqrt{2\pi}}\frac{\mathrm{d}x^{\bar{j}}}{\sqrt{2\pi}} &= \prod_{z\bar{z}}\prod_{i,\bar{i},j,\bar{j}}\frac{\mathrm{d}p_i}{\pi\alpha'\sqrt{2\pi}}\frac{\mathrm{d}p_{\bar{i}}}{\pi\alpha'\sqrt{2\pi}}\frac{\mathrm{d}x^j}{\sqrt{2\pi}}\frac{\mathrm{d}x^{\bar{j}}}{\sqrt{2\pi}}\\ &= \prod_{z\bar{z}}\prod_{i,\bar{i},j,\bar{j}}\frac{\mathrm{d}p_i}{\sqrt{2\pi^2\alpha'}}\frac{\mathrm{d}p_{\bar{i}}}{\sqrt{2\pi^2\alpha'}}\frac{\mathrm{d}x^j}{\sqrt{2\pi^2\alpha'}}\frac{\mathrm{d}x^{\bar{j}}}{\sqrt{2\pi^2\alpha'}},\end{aligned}$$

hence if we compute the integral by p_i, $p_{\bar{i}}$ fields we get

$$\langle \dots \rangle = \prod_{M\geq 0, m=1}^{10}\int \frac{\mathrm{d}\,x_M^m}{\sqrt{2\pi^2\alpha'}}\; e^{\left[-\frac{1}{2\pi\alpha'}\sum_{N\neq 0}(\lambda_N^2 x_N\cdot x_N+\dots)\right]}\dots \tag{4.37}$$

where we have expanded the field $x^m(z,\bar{z})$ in terms of a complete set $X_I(z,\bar{z})$ of eigenfunctions of the worldsheet Laplacian operator

$$\begin{aligned}x^m(z,\bar{z}) &= \sum_{M=0}^{\infty} x_M^m X_M(z,\bar{z}),\\ \partial\bar{\partial}X_M(z,\bar{z}) &= -\lambda_M^2 X_M(z,\bar{z})\\ \int_{\Sigma_g}\mathrm{d}^2z\, X_M(z,\bar{z})X_N(z,\bar{z}) &= \delta_{MN},\end{aligned} \tag{4.38}$$

and "..." means another contribution and insertions.

Note that the p^m fields have $10g$ zero modes in a Riemann surface of genus g and their normalizations do not affect the answer at one and two loops, because the term Π^m in the vertex operator (4.15) will not have contribution, as we will show later. However at three-loop these modes can become to be important (it is a work in progress).

Exponential Vertex

In order to compute scattering amplitudes to 4 points at tree-level, one and two-loop it is useful to find the contributions of the operators $:\exp(ik_j\cdot x):$ in (4.37), so we have

$$\begin{aligned}\left\langle \prod_{j=1}^{4} :e^{ik_j\cdot x}: \right\rangle &= \prod_{Im}\int\frac{\mathrm{d}x_I^m}{\sqrt{2\pi^2\alpha'}}\exp\left[-\frac{1}{2\pi\alpha'}\sum_{I\neq 0}\left(\lambda_I^2 x_I\cdot x_I - 2\pi\alpha' i x_I\cdot J_I\right) + ix_0\cdot J_0\right]\\ &= (2\pi)^{10}\delta^{(10)}(J_0)\left(2\pi^2\alpha'\det{}'\partial\bar{\partial}\right)^{-5}\exp\left[-\sum_{I\neq 0}\frac{\pi\alpha'}{2\lambda_I^2}J_I\cdot J_I\right]\end{aligned} \tag{4.39}$$

where

$$J^m(z,\bar{z}) = \sum_{i=1}^{4} k_i^m \delta^{(2)}(z,\bar{z}) = \sum_I J_I^m X_I(z,\bar{z}) \tag{4.40}$$

$$J_I^m = \int_{\Sigma_g} \mathrm{d}^2 z\, J^m(z,\bar{z}) X_I(z,\bar{z}). \tag{4.41}$$

In particular

$$J_0^m = X_0 \int_{\Sigma_g} \mathrm{d}^2 z\, J^m(z,\bar{z}) = X_0 \sum_{i=1}^{4} k_i^m,$$

thus, we obtain

$$\left\langle \prod_{j=1}^{4} : e^{ik_j\cdot x} : \right\rangle = \frac{(2\pi)^{10}\delta^{(10)}(X_0 k)}{\left(2\pi^2\alpha'\det{}'\partial\bar{\partial}\right)^5} \exp\left[-\frac{1}{2}\sum_{i\neq j} k_i\cdot k_j \sum_{I\neq 0}\frac{\pi\alpha'}{\lambda_I^2} X_I(z_i,\bar{z}_i) X_I(z_j,\bar{z}_j)\right]$$

where $k = \sum_{j=1}^4 k_j^m$. The term

$$\sum_{I\neq 0}\frac{\pi\alpha'}{\lambda_I^2} X_I(z_i,\bar{z}_i) X_I(z_j,\bar{z}_j)$$

is the Green's function and it satisfies [69]

$$-\frac{1}{\pi\alpha'}\partial\bar{\partial}G(z,w) = \sum_{I\neq 0} X_I(z,\bar{z}) X_I(w,\bar{w}) \tag{4.42}$$

$$= \delta^{(2)}(z-w) - X_0^2,$$

$$-\frac{1}{\pi\alpha'}\partial_z\partial_{\bar{w}}G(z,w) = -\delta^{(2)}(z-w) + \frac{1}{\alpha'}\sum_{I,J=1}^{g} w_I(z)(\mathrm{Im}\Omega)^{-1}_{IJ}\bar{w}_J(w) \tag{4.43}$$

where $w_I(z)\mathrm{d}z$ is a global holomorphic 1-form over the Riemann surface Σ_g of genus g and Ω_{IJ} is the period matrix. The normalization of the X_0 mode to be

$$X_0^2 = (A_g)^{-1}, \tag{4.44}$$

where A_g. is the area of Σ_g.
So the Green's function for the surface Σ_g is [69]

$$G(z,w) = -\frac{\alpha'}{2}\ln F^2(z,w), \tag{4.45}$$

$$F(z,w) = |E(z,w)| \exp\left[-\pi(\mathrm{Im}\Omega)^{-1}_{IJ}(\mathrm{Im}\int_w^z w_I)(\mathrm{Im}\int_w^z w_J)\right], \tag{4.46}$$

and therefore the final expression for the bosonic contribution is [69][66]

$$\langle\prod_{j=1}^{4} : e^{ik_j\cdot x} :\rangle_g = (2\pi)^{10}\delta^{(10)}(k)\frac{A_g^5}{(2\pi^2\alpha'\det{}'\partial\bar{\partial})^5}\prod_{i,j=1}^{4} F_g(z_i,z_j)^{\alpha' k^i\cdot k^j} \tag{4.47}$$

Normalization of the $[\mathrm{d}\theta^\alpha], [\mathrm{d}p_\beta]$ measures and non-zero modes contribution

Now we normalize the integration measures of the (θ^α, p_β) variables and compute the non-zero modes contribution up to OPE's. As we will see later, the only contribution, in this thesis, from OPE's is given for 4-points at tree level.

From the action of the p_α, θ^α grassmann fields

$$S_{p\theta} = \frac{1}{2\pi}\int \mathrm{d}^2 z \; p_\alpha \bar{\partial}\theta^\alpha \,, \tag{4.48}$$

we get the anticommutation relation

$$\left\{P_\alpha(\sigma), \theta^\beta(\sigma')\right\}_{DB} := \left\{\frac{p_\alpha(\sigma)}{2\pi}, \theta^\beta(\sigma')\right\}_{DB} = \delta_\alpha^\beta \delta(\sigma - \sigma'). \tag{4.49}$$

Therefore, the measure of the phase space in the path integral is [72]

$$\prod_{z,\bar{z}}\prod_{\alpha\beta} \mathrm{d}P_\alpha \, \mathrm{d}\theta^\beta = \prod_{z,\bar{z}}\prod_{\alpha\beta}(2\pi \mathrm{d}p_\alpha)\, \mathrm{d}\theta^\beta = \prod_{z,\bar{z}}\prod_{\alpha\beta}(\sqrt{2\pi}\mathrm{d}p_\alpha)\, (\sqrt{2\pi}\mathrm{d}\theta^\beta), \tag{4.50}$$

and the contribution of the non-zero modes of p_α and θ^α fields is given by

$$\begin{aligned}
&\prod_{\alpha\,\beta}\int [\mathrm{d}P_\alpha]'[\mathrm{d}\theta^\beta]'\exp\left(-\frac{1}{2\pi}\int_{\Sigma_g} \mathrm{d}^2 z \, p_\alpha \bar{\partial}\theta^\alpha\right) \\
&= \prod_{\alpha\,\beta\, M\neq 0}\int (\sqrt{2\pi}\mathrm{d}p_{\alpha\, M})(\sqrt{2\pi}\mathrm{d}\theta^\beta_M)\exp\left(-\frac{1}{2\pi}\sum_{N\neq 0}\lambda_N p_{\alpha\, N}\theta^\alpha_N\right) \\
&= \left[\det{}'\left(\bar{\partial}\right)\right]^{16},
\end{aligned}$$

where $(p_{a\,M},\, \theta^\alpha_M)$ are Grassmann numbers. The contribution of the fields of opposite worldsheet chirality, i.r the right sector $(\hat{\theta}^\alpha, \hat{p}_\beta)$, is $(\det{}'\partial)^{16}$. Thus the total non-zero modes contribution of the fermions p_α and θ^β is

$$[\det{}'(\partial\bar{\partial})]^{16}.$$

4.2.2 Normalization of the Pure Spinor measures

Covariant measures

Before to normalize the measures we define them.
As it was argued in [57] we can work with the ghost current J instead of J_{nonmin}. Now, let us remember that since the fields $(\lambda^\alpha, \overline{\lambda}_\alpha, r_\alpha, \theta^\alpha)$, with J ghost number $(1, -1, 0, 0)$, are worldsheet scalars then for any compact Riemann surface these fields have $(11, 11, 11, 16)$ zero-modes respectively. On the other hand, the fields

$(\omega_\alpha, \overline{\omega}^\alpha, s^\alpha, p_\alpha)$, with J ghost number $(-1,1,0,0)$, have conformal weight $(1,0)$, therefore the number of zero-modes on the Riemann surface of genus g of these fields is $(11g, 11g, 11g, 16g)$ respectively [69].
Note that at tree level the only zero-modes contribution comes from the fields $(\lambda^\alpha, \overline{\lambda}_\alpha, r_\alpha, \theta^\alpha)$ and so the integration measure $[\mathrm{d}\lambda][\mathrm{d}\overline{\lambda}][\mathrm{d}r][\mathrm{d}\theta]$ must have ghost number -3 since the anomaly of the ghost current J is $\hat{c} = -3$.
The measure $[\mathrm{d}\theta]$ was given in the previous subsection, $[\mathrm{d}\theta] \propto \mathrm{d}^{16}\theta$, and clearly it has ghost number zero, therefore the ghost anomaly -3 comes from $[\mathrm{d}\lambda][\mathrm{d}\overline{\lambda}][\mathrm{d}r]$. The global holomorphic top form $[\mathrm{d}\lambda]$ was found in (3.132), which has ghost number 8. The measure $[\mathrm{d}\overline{\lambda}]$ is the complex conjugate of $[\mathrm{d}\lambda]$, see (2.75), and it is easy to see that this has ghost number -8. Then the measure $[\mathrm{d}\lambda][\mathrm{d}\overline{\lambda}]$ has ghost number zero, which implies that the measure $[\mathrm{d}r]$ must have ghost number -3. So it was written in a covariant way in [57]

$$[\mathrm{d}r] = \frac{1}{11!5!}(\overline{\lambda}\gamma^m)^{\alpha_1}(\overline{\lambda}\gamma^n)^{\alpha_2}(\overline{\lambda}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5}\epsilon_{\alpha_1\ldots\alpha_5\delta_1\ldots\delta_{11}}\partial_r^{\delta_1}\ldots\partial_r^{\delta_{11}} \tag{4.51}$$

where $\partial_r^\delta = \frac{\partial}{\partial r_\delta}$.

In the same way as the minimal pure spinor formalism, the ghost anomaly, the geometrical interpretation of $\hat{c} = -3$ is the first Chern class of the projective superspace $\mathcal{Q}_{10} \times \Pi\mathcal{T}\overline{\mathcal{Q}}_{10}$ ‡, where $\mathcal{Q}_{10}$ is the projective pure spinor (section 2.8) and $\Pi\mathcal{T}\overline{\mathcal{Q}}_{10}$ is the superspace given by $\overline{\lambda}\gamma^m\overline{\lambda} = 0$, $\overline{\lambda}\gamma^m r = 0$. So, as in the minimal pure spinor formalism, the ghost number anomaly on the Riemann surface Σ_g of genus g is [18][52][80]

$$\frac{1}{2}c_1(\Sigma_g)c_1(\mathcal{Q}_{10} \times \Pi\mathcal{T}\overline{\mathcal{Q}}_{10}) = \frac{1}{2}(2-2g)(-3) = 3g - 3. \tag{4.52}$$

As we saw previously the measure $[\mathrm{d}\lambda][\mathrm{d}\overline{\lambda}][\mathrm{d}r][\mathrm{d}\theta]$ contributes with ghost number -3 then the measure $\Pi_{I=1}^g[\mathrm{d}\omega^I][\mathrm{d}\overline{\omega}^I][\mathrm{d}s^I][\mathrm{d}p^I]$ must contribute with $3g$, where the index I counts the number of zero-modes. Since the measure $[\mathrm{d}p^I] \propto \mathrm{d}^{16}p^I$ does not have ghost number, then it implies that the measure $[\mathrm{d}\omega^I][\mathrm{d}\overline{\omega}^I][\mathrm{d}s^I]$ has ghost number 3. The measure $[\mathrm{d}\omega^I]$ was given in a covariant way in the previous chapter

$$[\mathrm{d}\omega^I] = \frac{1}{11!5!}(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\epsilon^{\alpha_1\ldots\alpha_5\delta_1\ldots\delta_{11}}\mathrm{d}\omega^I_{\delta_1}\wedge\ldots\wedge\mathrm{d}\omega^I_{\delta_{11}}, \tag{4.53}$$

which is gauge invariant [67] and has ghost number -8. The measure $[\mathrm{d}\overline{\omega}^I]$ is the conjugate complex of $[\mathrm{d}\omega^I]$, so

$$[\mathrm{d}\overline{\omega}_I] = \frac{1}{5!11!}(\overline{\lambda}\gamma^m)^{\alpha_1}(\overline{\lambda}\gamma^n)^{\alpha_2}(\overline{\lambda}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5}\epsilon_{\alpha_1\ldots\alpha_5\delta_1\ldots\delta_{11}}\mathrm{d}\overline{\omega}_I^{\delta_1}\wedge\ldots\wedge\mathrm{d}\overline{\omega}_I^{\delta_{11}}, \tag{4.54}$$

‡We are using the notation of [18][20]

where it is clear that $[\mathrm{d}\overline{\omega}^I]$ has ghost number 8. Thus the measure $[\mathrm{d}s^I]$ has ghost number 3 and we write this in a covariant way as

$$[\mathrm{d}s_I](\overline{\lambda}\gamma^m)^{\alpha_1}(\overline{\lambda}\gamma^n)^{\alpha_2}(\overline{\lambda}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5} = \frac{1}{11!}\epsilon^{\alpha_1\ldots\alpha_5\rho_1\ldots\rho_{11}}\partial^{s_I}_{\rho_1}\ldots\partial^{s_I}_{\rho_{11}} \tag{4.55}$$

where $\partial^{s_I}_\rho \equiv \frac{\partial}{\partial s^\rho_I}$.
Now we normalize these measure using the same method as the previous section.

Normalization of the pure spinor measures and contribution of the non-zero modes

The action of the pure spinor variables on a chart $\lambda^+ \neq 0$ and gauge $\omega_a = 0$ (see subsection 1.2) is given by [47]

$$S = -\frac{1}{2\pi}\int_{\Sigma_g} \mathrm{d}^2 z\,(\beta\bar{\partial}\gamma + \frac{1}{2}v^{ab}\bar{\partial}u_{ab} + \bar{\beta}\bar{\partial}\bar{\gamma} + \frac{1}{2}\bar{v}_{ab}\bar{\partial}\bar{u}^{ab}).$$

The contribution of the non-zero modes is given by

$$\prod_{M\neq 0}\int \frac{[\mathrm{d}\beta_M]}{\sqrt{4\pi^2}} \wedge \frac{[\mathrm{d}\gamma_M]}{\sqrt{4\pi^2}} \bigwedge_{a<b,\,c<d} \frac{[\mathrm{d}v^{ab}_M]}{\sqrt{4\pi^2}}\frac{[\mathrm{d}u_{cd\,M}]}{\sqrt{4\pi^2}} \wedge \frac{[\mathrm{d}\bar{\beta}_M]}{\sqrt{4\pi^2}} \wedge \frac{[\mathrm{d}\bar{\gamma}_M]}{\sqrt{4\pi^2}} \bigwedge_{e<f,\,g<h} \frac{[\mathrm{d}\bar{v}_{ef\,M}]}{\sqrt{4\pi^2}}\frac{[\mathrm{d}\bar{u}^{gh}_M]}{\sqrt{4\pi^2}}$$
$$\exp\left(-\frac{1}{2\pi}\sum_{N\neq 0}\lambda_N(\beta_N\gamma_N + \frac{1}{2}v^{ab}_N u_{ab\,N} + \bar{\beta}_N\bar{\gamma}_N + \frac{1}{2}\bar{v}_{ab\,N}\bar{u}^{ab}_N)\right) \tag{4.56}$$

where the $\{\lambda_N\}$ are the eigenvalues of the $\bar{\partial}$ operator and we write the measure in the same way as in the previous Section [§]. We can write the argument of the exponential function in the following form (for example for the (γ,β) fields)

$$\exp\left(-\frac{1}{2\pi}\sum_{N\neq 0}\lambda_N(\beta_N\gamma_N + \bar{\beta}_N\bar{\gamma}_N)\right) = \exp\left(\frac{1}{2\pi}V^\dagger A\, V\right)\ , \tag{4.57}$$

where $V^T = (\gamma_N, \bar{\beta}_N)$, A is the matrix

$$A := \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}\ , \tag{4.58}$$

and B is the matrix $A := \mathrm{diag}(\lambda_N)$. The same happens for the (v^{ab}, u_{cd}) fields. Therefore the non-zero modes contribution of the pure spinors is $(\det\,\overline{\partial})^{-22}$.

Now, from the normalization of the measures in (4.56) we can normalize the covariant measures $[\mathrm{d}\lambda][\mathrm{d}\overline{\lambda}][\mathrm{d}\omega][\mathrm{d}\overline{\omega}]$. The idea is to show the following

$$[\mathrm{d}\lambda_M]\wedge[\mathrm{d}\omega^M] = (4\pi^2)^{-11}\bigwedge_{a<b,\,c<d}\mathrm{d}\gamma_M\mathrm{d}u^M_{ab}\mathrm{d}\beta_M\mathrm{d}v^{cd}_M \tag{4.59}$$
$$\left[\mathrm{d}\overline{\lambda}_M\right]\wedge[\mathrm{d}\overline{\omega}^M] = (4\pi^2)^{-11}\bigwedge_{a<b,\,c<d}\mathrm{d}\bar{\gamma}_M\mathrm{d}\bar{u}^{ab}_M\mathrm{d}\bar{\beta}_M\mathrm{d}\bar{v}^M_{cd}$$

[§] Note that when the mode is different to zero, $N \neq 0$, the number of degree of freedom of the variables $\{\gamma, u_{ab}\}$ and $\{\beta, v^{ab}\}$ is the same [72] where M is a mode index.

Following (4.56) we define the normalization of the integration measures $[d\lambda][d\omega]$ for the M^{th} mode in a covariant as

$$[d\lambda_M](\lambda_M\gamma^m)_{\alpha_1}(\lambda_M\gamma n)_{\alpha_2}(\lambda_M\gamma p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} = \tag{4.60}$$
$$= \frac{2^3(4\pi^2)^{-11/2}}{11!}\epsilon_{\alpha_1\ldots\alpha_5\rho_1\ldots\rho_{11}} d\lambda_M^{\rho_1} \wedge \ldots \wedge d\lambda_M^{\rho_{11}},$$

$$[d\omega^M] = \frac{(4\pi^2)^{-11/2}}{2^3 11!5!}(\lambda_M\gamma^m)_{\alpha_1}(\lambda_M\gamma^n)_{\alpha_2}(\lambda_M\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\epsilon^{\alpha_1\ldots\alpha_5\delta_1\ldots\delta_{11}} d\omega^M_{\delta_1} \wedge \ldots \wedge d\omega^M_{\delta_{11}}$$

where the term 2^3 was explained in the subsection 3.3.4. Taking the wedge product we get

$$[d\lambda_I] \wedge [d\omega^I] = \frac{(4\pi^2)^{-11}}{11!} d\lambda_I^{\alpha_1} \wedge d\omega^I_{\alpha_1} \wedge \ldots \wedge d\lambda_I^{\alpha_{11}} \wedge d\omega^I_{\alpha_{11}}. \tag{4.61}$$

On the chart $U_{+++++} = \{\lambda^+ \neq 0\}$

$$\begin{aligned}\lambda^\alpha &= (\lambda^+, \lambda^{ab}, \lambda^a)\,,\\ \lambda^+ &= \gamma\,, \quad \lambda_{ab} = \gamma u_{ab}\,, \quad \lambda^a = -\frac{1}{8}\gamma\epsilon^{abcde}u_{bc}u_{de}\end{aligned}$$

and in the gauge $\omega_a = 0$

$$\begin{aligned}\omega_\alpha &= (\omega_+, \omega^{ab}, \omega_a) \qquad a,\, b = 1,\, 2, \ldots, 5\\ \omega_+ &= \beta - \frac{1}{2\gamma}v^{ab}u_{ab}\,, \quad \omega^{ab} = \frac{1}{\gamma}v^{ab}\,, \quad \omega_a = 0,\end{aligned}$$

we obtain the measure in the form desired

$$[d\lambda_M] \wedge [d\omega^M] = (4\pi^2)^{-11} \bigwedge_{a<b,\, c<d} d\gamma_M du^M_{ab} d\beta_M dv^{cd}_M. \tag{4.62}$$

For the $\overline{\lambda}_\alpha$ and $\bar{\omega}^\alpha$ fields we can think as the conjugate complex of (4.60), however for future computations it is useful change the measures $[d\overline{\lambda}][d\overline{\omega}]$. In the definition of the measure $[d\overline{\lambda}]$ we can contract it with the tensor $(\lambda^M\gamma^m)_{\alpha_1}(\lambda^M\gamma^n)_{\alpha_2}(\lambda^M\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}$, so for the M^{th} mode we get

$$\begin{aligned}[d\overline{\lambda}^M] &= \frac{(4\pi^2)^{-11/2}}{2^3 11!5!(\lambda_M\overline{\lambda}^M)^3}(\lambda_M\gamma^m)_{\alpha_1}(\lambda_M\gamma^n)_{\alpha_2}(\lambda_M\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\\ &\quad \epsilon^{\alpha_1\ldots\alpha_5\rho_1\ldots\rho_{11}} d\overline{\lambda}^M_{\rho_1} \wedge \ldots \wedge d\overline{\lambda}^M_{\rho_{11}}.\end{aligned} \tag{4.63}$$

Then the measure $[d\overline{\omega}]$ can be written as

$$[d\overline{\omega}_M](\lambda_M\gamma^m)_{\alpha_1}(\lambda_M\gamma^n)_{\alpha_2}(\lambda_M\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} = (4\pi^2)^{-11/2}\frac{2^3(\lambda_M\overline{\lambda}^M)^3}{11!} \tag{4.64}$$

$$\epsilon_{\alpha_1...\alpha_5\delta_1...\delta_{11}} \mathrm{d}\overline{\omega}^{\delta_1}_M \wedge ... \wedge \mathrm{d}\overline{\omega}^{\delta_{11}}_M ,$$

and therefore

$$[\mathrm{d}\overline{\lambda}^M] \wedge [\mathrm{d}\overline{\omega}_M] = \frac{(4\pi^2)^{-11}}{11!} \mathrm{d}\overline{\lambda}^M_{\alpha_1} \wedge \mathrm{d}\overline{\omega}^{\alpha_1}_M \wedge ... \wedge \mathrm{d}\overline{\lambda}^M_{\alpha_{11}} \wedge \mathrm{d}\overline{\omega}^{\alpha_{11}}_M , \tag{4.65}$$

as we were expecting.

The contribution of the fields of opposite worldsheet chirality is $(\det'\partial)^{-22}$. So, the contribution of the non zero modes of the pure spinors is $(\det'\partial\bar{\partial})^{-22}$.

Using the same method for the r_α and s^α grassmann field we can normalize the covariant measure in the path integral for the M^{th} mode as [67]

$$[\mathrm{d}r^M] = \frac{(2\pi)^{11/2}}{11!5!} (\overline{\lambda}^M \gamma^m)^{\alpha_1} (\overline{\lambda}^M \gamma^n)^{\alpha_2} (\overline{\lambda}^M \gamma^p)^{\alpha_3} (\gamma_{mnp})^{\alpha_4\alpha_5} \epsilon_{\alpha_1...\alpha_5\delta_1...\delta_{11}} \partial^{\delta_1}_{r^M} ... \partial^{\delta_{11}}_{r^M}$$

$$[\mathrm{d}s_M]\, (\overline{\lambda}^M \gamma^m)^{\alpha_1} (\overline{\lambda}^M \gamma^n)^{\alpha_2} (\overline{\lambda}^M \gamma^p)^{\alpha_3} (\gamma_{mnp})^{\alpha_4\alpha_5} = \tag{4.66}$$

$$= \frac{(2\pi)^{11/2}}{11!} \epsilon^{\alpha_1...\alpha_5\rho_1...\rho_{11}} \partial^{s_M}_{\rho_1} ... \partial^{s_M}_{\rho_{11}} .$$

Thus

$$[\mathrm{d}r^M][\mathrm{d}s_M] = (2\pi)^{11} \partial^1_{r^M} \partial^{s_M}_1 ... \partial^{11}_{r^M} \partial^{s_M}_{11} . \tag{4.67}$$

In an analogous way as the previous case we get the contribution from the non-zero modes

$$\left(\det'\partial\bar{\partial}\right)^{11} . \tag{4.68}$$

Finally, the total contribution of the non-zero modes of the (λ^α, ω_α, $\overline{\lambda}_\alpha$, $\overline{\omega}^\alpha$, r_α, s^α) fields is

$$\left(\det'\partial\bar{\partial}\right)^{-11} \left(\det'\partial\bar{\partial}\right)^{-11} \left(\det'\partial\bar{\partial}\right)^{16} \left(\det'\partial\bar{\partial}\right)^{11} = \left(\det'\partial\bar{\partial}\right)^{5} . \tag{4.69}$$

Although we have computed the path integral of the pure spinors in a particular chart and gauge, the answer is correct because the $\{\gamma = 0\} = SO(10)/U(5)$ space has measure zero with respect to the pure spinors space. Note also that the determinant factors cancels out from (4.47).

In this section we have obtained the normalization of all integration measures for any M-mode. Nevertheless the zero modes requires a special treatment since there is a different number of these modes for the fields and their conjugate momenta is different, as it was explained at the beginning of this subsection. This fact implies that the integration measures are not dimensionless, eq. (4.5), therefore powers of α' are necessary to get dimensionless measures. Note also that to have the same convention as in [66] the zero modes of the fields $(\omega_\alpha, \overline{\omega}^\alpha, p_\alpha, s^\alpha)$ should be expanded around of the global holomorphic (1,0)-forms w_I, $I = 1, .., g$ over Riemann surface Σ_g of genus g. As these forms are not orthonormal it is necessary to introduce a Jacobian factor. Due to these reasons we devote a subsection on the normalization of zero modes.

4.2.3 The normalization of zero-modes

Since the dimension of the zero Čech cohomology group $H^0(\Sigma_g, \Omega^1)$, where $\Omega^1(\Sigma_g)$ is the sheaf of holomorphic 1-forms over Σ_g, is equal to the genus g of the Riemann surface we expand a generic conformal weight (1,0) field as [57]

$$\phi(z) = \hat{\phi}(z) + \sum_{I=1}^{g} w_I(z)\phi^I \tag{4.70}$$

where ϕ represents each component of the fields $(\omega_\alpha, \overline{w}^\alpha, s^\alpha, d_\alpha)$, ϕ^I are the zero modes and $\{w_I(z)\mathrm{d}z\}$ is a basis of the $H^0(\Sigma_g, \Omega^1)$ group such that

$$\int_{a_I} w_J(z)\mathrm{d}z = \delta_{IJ}, \qquad \int_{b_I} w_J(z)\mathrm{d}z = \Omega_{IJ} \qquad I, J = 1, 2, \ldots, g$$

$$(w_I, w_J) \equiv \int_{\Sigma_g} w_I\,\bar{w}_J\;\mathrm{d}z \wedge \mathrm{d}\bar{z} = 2\,\mathrm{Im}\Omega_{IJ} \quad \text{and} \quad \int_{a_I} \mathrm{d}z\,\hat{\phi}(z) = 0, \tag{4.71}$$

where a_I and b_J are the generators of the $H^1(\Sigma_g, \mathbb{Z}) = \mathbb{Z}^{2g}$ homology group and Ω_{IJ} is the period matrix [94]. If we expand ϕ over another basis $\{\alpha_J\}$ related by $w_I = B_I^J \alpha_J$ then [68],

$$\det\left(\int_{\Sigma_g} w_I\,\bar{w}_J\,\mathrm{d}z \wedge d\bar{z}\right) = \det|B|^2 \det\left(\int_{\Sigma_g} \alpha_I\,\bar{\alpha}_J\;\mathrm{d}z \wedge \mathrm{d}\bar{z}\right)$$

so that for

$$|\det B| = \sqrt{\det(2\,\mathrm{Im}\Omega_{IJ})} = Z_g^{-1} \tag{4.72}$$

the basis $\{\alpha_J\}$ is orthonormal, $(\alpha_I, \alpha_J) = \delta_{IJ}$. Expanding the fields over the new basis as $\phi = \sum_{J=1}^{g} \phi'^J \alpha_J$ one can show that the measure satisfies

$$\mathrm{d}\phi'^1 \cdots \mathrm{d}\phi'^g = \det(\mathrm{B})^\epsilon \mathrm{d}\phi^1 \cdots \mathrm{d}\phi^g, \tag{4.73}$$

where $\epsilon = +1(-1)$ for bosonic (fermionic) fields. In the non-minimal formalism the integration measures for conformal weight-one fields is defined in terms of the ϕ' components, but it is more convenient to use the $\{w_I\}$ basis in explicit computations. To account for this we absorb the Jacobian (4.72) equally into each of the $[\mathrm{d}\phi^I]$ measures as $(\det(\mathrm{B})^{\epsilon/g}\mathrm{d}\phi^1)\cdots(\det(\mathrm{B})^{\epsilon/g}\mathrm{d}\phi^g)$.

Similarly, for the conformal weight-zero variables $(\lambda^\alpha, \overline{\lambda}_\alpha, r_\alpha, \theta^\alpha)$ we have the expansion in a complete set of eigenfunctions for the Laplacian of the worldsheet [71]

$$\lambda^\alpha(z) = \lambda_0^\alpha \Lambda_0 + \sum_M \lambda_M^\alpha \Lambda_M(z, \bar{z}) \tag{4.74}$$

and $\Lambda_0 = 1$ is the generator of the cohomology group $H^0(\Sigma_g, \mathcal{O}) = \mathbb{C}$, where $\mathcal{O}$ is the sheaf of holomorphic functions over Σ_g. Because the norm of Λ_0 is $||\Lambda_0||^2 = A_g$ the

measures of the scalars must have the Jacobian $A_g^{\epsilon/2}$ (where $\epsilon = +1(-1)$ for bosonic (fermionic) fields).
Finally we have the following normalizations for the zero modes¶ ($I = 1, ..., g$)

$$[\mathrm{d}\lambda]_0 T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = c_\lambda \epsilon_{\alpha_1...\alpha_5\rho_1...\rho_{11}} \mathrm{d}\lambda^{\rho_1}...\mathrm{d}\lambda^{\rho_{11}} \tag{4.75}$$

$$[\mathrm{d}\bar{\lambda}]_0 \bar{T}^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = c_{\overline{\lambda}} \epsilon^{\alpha_1...\alpha_5\rho_1...\rho_{11}} \mathrm{d}\bar{\lambda}_{\rho_1}...\mathrm{d}\bar{\lambda}_{\rho_{11}} \tag{4.76}$$

$$[\mathrm{d}\omega^I]_0 = c_\omega T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \epsilon^{\alpha_1...\alpha_5\rho_1...\rho_{11}} \mathrm{d}\omega^I_{\rho_1}...\mathrm{d}\omega^I_{\rho_{11}} \tag{4.77}$$

$$[\mathrm{d}\overline{\omega}^I]_0 T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = c_{\overline{\omega}} \epsilon_{\alpha_1...\alpha_5\rho_1...\rho_{11}} \mathrm{d}\overline{\omega}^{\rho_1}_I...\mathrm{d}\overline{\omega}^{\rho_{11}}_I \tag{4.78}$$

$$[\mathrm{d}r]_0 = c_r \bar{T}^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \epsilon_{\alpha_1...\alpha_5\delta_1...\delta_{11}} \partial_r^{\delta_1}...\partial_r^{\delta_{11}} \tag{4.79}$$

$$[\mathrm{d}s^I]_0 = c_s T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \epsilon^{\alpha_1...\alpha_5\rho_1...\rho_{11}} \partial^{s^I}_{\rho_1} ... \partial^{s^I}_{\rho_{11}} \tag{4.80}$$

$$[\mathrm{d}\theta]_0 = c_\theta \mathrm{d}^{16}\theta, \quad [\mathrm{d}d^I] = c_d \mathrm{d}^{16} d^I \tag{4.81}$$

with the constants

$$c_\lambda = \left(\frac{\alpha'}{2}\right)^{-2} \frac{2^3}{11!} \left(\frac{A_g}{4\pi^2}\right)^{11/2} \qquad c_\omega = \left(\frac{\alpha'}{2}\right)^{2} \frac{(4\pi^2)^{-11/2}}{2^3\, 11!5!\, Z_g^{11/g}} \tag{4.82}$$

$$c_{\overline{\lambda}} = \left(\frac{\alpha'}{2}\right)^{2} \frac{2^3}{11!} \left(\frac{A_g}{4\pi^2}\right)^{11/2} \qquad c_{\overline{\omega}} = \left(\frac{\alpha'}{2}\right)^{-2} \frac{2^3(4\pi^2)^{-11/2}(\lambda\bar{\lambda})^3}{11!\, Z_g^{11/g}} \tag{4.83}$$

$$c_r = \left(\frac{\alpha'}{2}\right)^{-2} \frac{R}{11!5!} \left(\frac{2\pi}{A_g}\right)^{11/2} \qquad c_s = \left(\frac{\alpha'}{2}\right)^{2} \frac{(2\pi)^{11/2}R^{-1}}{2^6 11!5!(\lambda\bar{\lambda})^3} Z_g^{11/g} \tag{4.84}$$

$$c_\theta = \left(\frac{\alpha'}{2}\right)^{4} \left(\frac{2\pi}{A_g}\right)^{16/2} \qquad c_d = \left(\frac{\alpha'}{2}\right)^{-4} (2\pi)^{16/2} Z_g^{16/g} \tag{4.85}$$

where A_g is the area of the Riemann surface Σ_g,

$$Z_g = \frac{1}{\sqrt{\det(2\mathrm{Im}(\Omega_{IJ}))}}, \quad g \geq 1, \tag{4.86}$$

and the tensors $T_{\alpha_1...\alpha_5}$, $\bar{T}^{\alpha_1...\alpha_5}$ are defined as

$$T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = (\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} \tag{4.87}$$

$$\bar{T}^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = (\bar{\lambda}\gamma^m)^{\alpha_1}(\bar{\lambda}\gamma^n)^{\alpha_2}(\bar{\lambda}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5} \tag{4.88}$$

and satisfy

$$T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}\bar{T}^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = 5!\, 2^6 (\lambda\bar{\lambda})^3. \tag{4.89}$$

Note that we have added the $(\alpha'/2)$ factors to get dimensionless measures.
We have also put the constant R and R^{-1} on the measures $[\mathrm{d}r]$ and $[\mathrm{d}s^I]$ respectively. This constant is arbitrary and parametrizes the freedom in choosing the normalization of the tree-level amplitude. Note that since $C_0 \propto R$ and $C_2 \propto R^{-1}$ then it does not affect the condition $C_1 = 8\pi^2 C_0 C_2$, therefore we can set it such that our constant at tree level (C_0) has the same value as one given in [66] and so to compare in a straightforward way our results with the obtaining in [66].

¶The index "0" in the integration measures means the zero modes.

On the normalization of the holomorphic 1-forms

The result of scattering amplitudes in the pure spinor formalism does not depend on the normalization of the holomorphic 1-forms $w_I(z)$. To see this one notes that in closed string amplitudes* at genus g the difference between the number of independent fermionic and bosonic conformal weight-one left-moving variables is always $16g + 11g - 11g - 11g = 5g$, corresponding to $d^I_\alpha, s^{\alpha\, I}, \omega^I_\alpha$ and $\overline{\omega}^{\alpha\, I}$. As Z_g appear in the conformal weight-one measures as $Z_g^{1/g}$, their total contribution to closed string amplitudes is always $|Z_g^5|^2 = Z_g^{10}$. Furthermore, when saturating the $11g$ $s^{\alpha\, I}$ zero modes the regulator factor $\mathcal{N}$ provides $11g$ d^I_α zero-modes as well – because they appear in the combination $(s^I d^I)$ in $\mathcal{N}$ and there is nowhere else to get $s^{I\,\alpha}$ zero-modes from. So to complete the saturation of d^I_α the b-ghosts and external vertices will always provide $5g$ factors of $|d^I_\alpha w_I(z)|^2$, which scales as x^{10g} under $w_I(z) \to x w_I(z)$. To finish the proof it suffices to note from (4.71) and (4.72) that Z_g scales as $Z_g \to x^{-g} Z_g$ and therefore $|Z_g^5|^2$ offsets the scaling of the $|w_I^{5g}|^2$ factors from the b-ghosts and external vertices.

4.3 Integration over pure spinor zero-modes

This section is the key to check the unitarity of the formalism, where we will show that this integral has the result

$$\int_{\mathcal{O}(-1)} [\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}]\, e^{-a\,\lambda\overline{\lambda}} = (2\pi)^{11}(a^8 \cdot 12 \cdot 5)^{-1}\,, \qquad a \in \mathbb{R}^+. \tag{4.90}$$

Since the pure spinor is a non-compact space then we have introduced a regulator, which has a Gaussian form in the pure spinors variables, to obtain a convergent integral on the pure spinor zero-modes (subsection 4.1.1). In this section we show how to solve these integrals over pure spinor space.

We are interested in the integrals of the form

$$\int [\mathrm{d}\lambda]_0 \wedge [\mathrm{d}\overline{\lambda}]_0 \wedge \prod_{I=1}^{g} [\mathrm{d}\omega^I]_0 \wedge [\mathrm{d}\overline{\omega}^I]_0 \; e^{-a(\lambda\overline{\lambda}) - b\sum_{I=1}^{g}(\omega^I\overline{\omega}^I)}\,, \tag{4.91}$$

where a, b are positive constants, λ^α is a pure spinor and the measures are the same as in previous subsection. For convenience to remove the normalization factors we

*The analysis can be trivially modified to the open string.

define the measures without the label "0" as

$$[\mathrm{d}\lambda]_0 = \left(\frac{\alpha'}{2}\right)^{-2}\left(\frac{A_g}{4\pi^2}\right)^{11/2}[\mathrm{d}\lambda] \tag{4.92}$$

$$\left[\mathrm{d}\omega^I\right]_0 = \left(\frac{\alpha'}{2}\right)^{2}\frac{1}{(Z_g)^{11/g}(4\pi^2)^{11/2}}\left[\mathrm{d}\omega^I\right] \tag{4.93}$$

$$\left[\mathrm{d}\overline{\lambda}\right]_0 = \left(\frac{\alpha'}{2}\right)^{2}\left(\frac{A_g}{4\pi^2}\right)^{11/2}\left[\mathrm{d}\overline{\lambda}\right] \tag{4.94}$$

$$\left[\mathrm{d}\overline{\omega}^I\right]_0 = \left(\frac{\alpha'}{2}\right)^{-2}\frac{1}{(Z_g)^{11/g}(4\pi^2)^{11/2}}\left[\mathrm{d}\overline{\omega}^I\right]. \tag{4.95}$$

So (4.91) becomes

$$\left(\frac{A_g}{Z_g}\right)^{11}\frac{1}{(4\pi^2)^{11(g+1)}}\int[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]\wedge\prod_{I=1}^{g}[\mathrm{d}\omega^I]\wedge[\mathrm{d}\overline{\omega}^I]\; e^{-a(\lambda\overline{\lambda})-b\sum_{I=1}^{g}(\omega^I\overline{\omega}^I)}, \tag{4.96}$$

and we only focus on the integral. The integration by the variables ω_α and $\overline{\omega}^\alpha$ is relatively simple, replacing the measures

$$\left[\mathrm{d}\omega^I\right]\wedge\left[\mathrm{d}\overline{\omega}^I\right] = \frac{(\lambda\overline{\lambda})^3}{11!}\mathrm{d}\omega^I_{\alpha_1}\wedge\mathrm{d}\overline{\omega}^{\alpha_1}_I\wedge\ldots\wedge\mathrm{d}\omega^I_{\alpha_1}\wedge\mathrm{d}\overline{\omega}^{\alpha_1}_I \tag{4.97}$$

and fix the gauge $\omega_a = \overline{\omega}^a = 0$ on (4.96) we have

$$(\lambda\overline{\lambda})^{3g}\int\prod_{I=1}^{g}\mathrm{d}\omega^I_+\wedge\mathrm{d}\overline{\omega}^+_I\bigwedge_{a<b,\,c<d}\mathrm{d}\omega^{ab}_I\mathrm{d}\overline{\omega}^I_{cd}\; e^{-b\sum_{I=1}^{g}(\omega^I_+\overline{\omega}^+_I-\frac{1}{2}\omega^{ab}_I\overline{\omega}^I_{ab})}. \tag{4.98}$$

This is a simple Gaussian integral and its result is $(2\pi b^{-1})^{11g}$. Therefore we get

$$\int[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]\wedge\prod_{I=1}^{g}[\mathrm{d}\omega^I]\wedge[\mathrm{d}\overline{\omega}^I]e^{-a(\lambda\overline{\lambda})-b\sum_{I=1}^{g}(\omega^I\overline{\omega}^I)} = \left(\frac{2\pi}{b}\right)^{11g}\int[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]e^{-a(\lambda\overline{\lambda})}(\lambda\overline{\lambda})^{3g}$$

where it is useful to remember that g is the genus of the Riemann surface Σ_g, i.e $g\in\mathbb{N}$. Note that

$$\int[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]e^{-a(\lambda\overline{\lambda})}(\lambda\overline{\lambda})^{3g} = (-1)^{3g}\frac{\partial^{3g}}{\partial a^{3g}}\int[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]e^{-a(\lambda\overline{\lambda})},$$

thus the integral of our interest is simply

$$\int[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]\,e^{-a\lambda\overline{\lambda}}. \tag{4.99}$$

As it was shown in the subsection 2.6.1 we can easily see that the measure $[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]$ is

$$\begin{aligned}[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}] &= \frac{1}{11!(\lambda\overline{\lambda})^3}\mathrm{d}\lambda^{\alpha_1}\wedge\mathrm{d}\overline{\lambda}_{\alpha_1}\wedge\ldots\wedge\mathrm{d}\lambda^{\alpha_{11}}\wedge\mathrm{d}\overline{\lambda}_{\alpha_{11}}\\ &= \frac{1}{11!(\lambda\overline{\lambda})^3}\partial\overline{\partial}(\lambda\overline{\lambda})\wedge\ldots\wedge\partial\overline{\partial}(\lambda\overline{\lambda})\\ &= \frac{\Omega^{11}}{11!},\end{aligned}$$

where

$$\Omega = \frac{1}{(\lambda\overline{\lambda})^{3/11}}\partial\overline{\partial}(\lambda\overline{\lambda})$$

is the Kähler form‖ on the pure spinors space in $D = 2n = 10$ dimension.
The Kähler form of the pure spinors space in any dimension is given by

$$\Omega_{D=2n} = \frac{1}{(\lambda\overline{\lambda})^{\frac{\dim_{\mathbb{C}} PS - c_1}{\dim_{\mathbb{C}} PS}}}\partial\overline{\partial}(\lambda\overline{\lambda}), \tag{4.100}$$

where $c_1 = 2n - 2$ is the first Chern class of the tangent bundle over $SO(2n)/U(n)$ [54] and $\dim_{\mathbb{C}} PS = \frac{n(n-1)}{2} + 1$ is the complex dimension of the pure spinors space. In the parametrization (on the chart $\lambda^+ \neq 0$)

$$\lambda^+ = \gamma, \quad \lambda_{ab} = \gamma u_{ab}, \quad \lambda^a = \frac{\gamma}{8}\epsilon^{abcde}u_{bc}u_{de}, \tag{4.101}$$

where $u_{ab} = -u_{ba}$, the integration measure on pure spinors space is

$$\frac{\Omega^{11}}{11!} = \gamma^7 \mathrm{d}\gamma \bigwedge_{a<b} \mathrm{d}u_{ab} \wedge \bar{\gamma}^7 \mathrm{d}\bar{\gamma} \bigwedge_{c<d} \mathrm{d}\bar{u}^{cd}. \tag{4.102}$$

Writing (4.99) in the coordinates (4.101) we get

$$\int [\mathrm{d}\lambda][\mathrm{d}\overline{\lambda}]\, e^{-a\lambda\overline{\lambda}} = \int (\gamma\bar{\gamma})^7 \mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma} \bigwedge_{a<b,\, c<d} \mathrm{d}u_{ab}\mathrm{d}\bar{u}^{cd}\, e^{-a\gamma\bar{\gamma}(1+\frac{1}{2}u_{ab}\bar{u}^{ab}+\frac{1}{8^2}\epsilon^{abcde}\epsilon_{afghi}u_{bc}u_{de}\bar{u}^{fg}\bar{u}^{hi})}\ .$$

The γ, $\bar{\gamma}$ variables can be integrated easily

$$\int (\gamma\bar{\gamma})^7 \mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma}\; e^{-b\gamma\bar{\gamma}} = -\frac{\partial^7}{\partial b^7}\int \mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma}\; e^{-b\gamma\bar{\gamma}} = (2\pi)\cdot 7! \cdot \frac{1}{b^8}\ , \tag{4.103}$$

where

$$b := a(1 + \frac{1}{2}u_{ab}\bar{u}^{ab} + \frac{1}{8^2}\epsilon^{abcde}\epsilon_{afghi}u_{bc}u_{de}\bar{u}^{fg}\bar{u}^{hi}). \tag{4.104}$$

So (4.99) has now the form

$$\int [\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}]\, e^{-a\lambda\overline{\lambda}} = \frac{(2\pi)\cdot 7!}{a^8}\int_{SO(10)/U(5)} \alpha\ , \tag{4.105}$$

where

$$\alpha := \frac{\bigwedge_{a<b,\, c<d} \mathrm{d}u_{ab}\mathrm{d}\bar{u}^{cd}}{(1 + \frac{1}{2}u_{ab}\bar{u}^{ab} + \frac{1}{8^2}\epsilon^{abcde}\epsilon_{afghi}u_{bc}u_{de}\bar{u}^{fg}\bar{u}^{hi})^8} \tag{4.106}$$

is a global form on $SO(10)/U(5)$, therefore it belongs to the $H^{20}_{DR}(SO(10)/U(5))$ de-Rham cohomology group [94][29]. Note that the number 8 is the first Chern class

‖easily we can see that $(\lambda\overline{\lambda})$ is a scalar function (global) on the pure spinors space.

of the tangent bundle over $SO(10)/U(5)$.

The α-form can be written as

$$\alpha = \frac{\omega^{10}}{10!}, \tag{4.107}$$

where

$$\omega = -\partial\bar{\partial}\ \ln(\tilde{\lambda}\tilde{\bar{\lambda}}) \tag{4.108}$$

and $\tilde{\lambda}$ and $\tilde{\bar{\lambda}}$ are projective pure spinors, in others words

$$\tilde{\lambda}\tilde{\bar{\lambda}} = 1 + \frac{1}{2}u_{ab}\bar{u}^{ab} + \frac{1}{8^2}\epsilon^{abcde}\epsilon_{afghi}u_{bc}u_{de}\bar{u}^{fg}\bar{u}^{hi}, \tag{4.109}$$

where $\{u_{ab}\}$ is a complex parametrization on $SO(10)/U(5)$. The 2-form ω is the Kähler form, so $\ln(\tilde{\lambda}\tilde{\bar{\lambda}})$ is the Kähler potential [94]. From the identity

$$\partial\overline{\partial} = \frac{1}{2}\mathrm{d}(\partial - \overline{\partial})$$

we can see that ω is closed, i.e $\mathrm{d}\omega = 0$, therefore $SO(10)/U(5)$ is a Kähler manifold.

From the algebraic geometry point of view, the projective pure spinors space in $d = 2n = 10$ is a variety (manifold) on the projective space $\mathbb{C}P^{15}$, then its Kälher form is the pullback of the Kähler form of $\mathbb{C}P^{15}$ given by [94][30]

$$\omega = f^{*}\Omega, \tag{4.110}$$

where Ω is the Fubini-Study [94] metric of $\mathbb{C}P^{15}$ and

$$f : SO(10)/U(5) \to \mathbb{C}P^{15} \tag{4.111}$$

is the map given by the pure spinor constraint $(\lambda\gamma^{m}\lambda) = 0$ and the equivalent relation $\lambda \sim c\lambda$, $c \in \mathbb{C}^{*}$. As $SO(10)/U(5)$ is a closed manifold on $\mathbb{C}P^{15}$, then it belongs to the $H_{20}(\mathbb{C}P^{15}) = \mathbb{Z}$ homology group [31], so the projective pure spinors space is proportional to the $[\mathbb{C}P^{10}]$ homology class because $\mathbb{C}P^{10}$ is the generator of the $H_{20}(\mathbb{C}P^{15})$ homology group [31][37]. The proportionality factor is called the " degree" of a variety and it is a integer number since $H_{20}(\mathbb{C}P^{15}) = \mathbb{Z}$. The concept of degree was introduced in the subsection 3.3.4 and let us remember that it is given by (3.46)

$$\deg(SO(10)/U(5)) = {}^{\#}(SO(10)/U(5) \cdot \mathbb{C}P^{5}), \tag{4.112}$$

where ${}^{\#}(SO(10)/U(5) \cdot \mathbb{C}P^{5})$ are the intersection numbers between $SO(10)/U(5)$ and $\mathbb{C}P^{5}$ inside $\mathbb{C}P^{15}$. Hence the integral (4.105) can be written as

$$\int_{SO(10)/U(5)} \frac{\omega^{10}}{10!} = \deg(SO(10)/U(5)) \int_{\mathbb{C}P^{10}} \frac{\Omega^{10}}{10!}\bigg|_{\mathbb{C}P^{10}}. \tag{4.113}$$

In the above chapter we found the degree of the projective pure spinor space using a geometrical method and we showed that this number is $\deg(SO(10)/U(5)) = 12$. Nevertheless, in order to get a relationship between the scattering amplitudes of the minimal and non-minimal formalism (the next chapter) it is useful to obtain an analytical expression in terms of the topological invariants of the projective pure spinor degree.
Since that the pure spinors space is identified with the total space of the line bundle $\mathcal{O}(-1)$; which is the inverse of the line bundle $\mathcal{L} = \mathcal{O}(1)$ [94][18]. The first Chern class $c_1(\mathcal{L})$ of $\mathcal{L}$ is simply the pullback of the hyperplane class H [94][29]

$$c_1(\mathcal{L}) = f^* H \tag{4.114}$$

and the degree of the projective pure spinors space is given by

$$\begin{aligned}
\int_{SO(10)/U(5)} c_1(\mathcal{L})^{10} &= \deg(SO(10)/U(5)) \int_{\mathbb{C}P^{10}} H^{10}\Big|_{\mathbb{C}P^{10}} \\
&= \deg(SO(10)/U(5)) \int_{\mathbb{C}P^{10}} \frac{c_{10}(T\mathbb{C}P^{10})}{11} \\
&= \deg(SO(10)/U(5)),
\end{aligned} \tag{4.115}$$

where $\int_{\mathbb{C}P^{10}} c_{10}(T\mathbb{C}P^{10})$ is the Euler characteristic of $\mathbb{C}P^{10}$. So we have the following important result

$$\deg(SO(10)/U(5)) = \int_{SO(10)/U(5)} c_1(\mathcal{L})^{10} = 12. \tag{4.116}$$

As we said previously, this number was found using a geometrical method in the above chapter, however we can also get it in a simple way from the pure spinor partition function, as we describe below for pure spinor in dimension $D = 10$ and in the appendix C.2 for pure spinor in lower dimensions.

The Riemann-Roch formula gives us an expression for the pure spinors character at level zero [54]

$$Z_{10}(t) = \int_{SO(10)/U(5)} \frac{1}{1 - te^{-c_1(\mathcal{L})}} Td(T(SO(10)/U(5))), \tag{4.117}$$

where $Td(T(SO(10)/U(5)))$ is the Todd genus

$$Td(T(SO(10)/U(5))) = 1 + \frac{1}{2} c_1(T(SO(10)/U(5)) + \ldots \ . \tag{4.118}$$

Expanding $Z_{10}(t)$ near to $t = 1$ or near to $\epsilon = 1 - t = 0$, the most singular term is [18]

$$\frac{1}{\epsilon^{11}} \int_{SO(10)/U(5)} c_1(\mathcal{L})^{10}. \tag{4.119}$$

The pure spinors character can also be computed with the reducibility method, in this case the result is [54][26]

$$Z_{10}(t) = \frac{1+5t+5t^2+t^3}{(1-t)^{11}}. \tag{4.120}$$

Again, expanding near $\epsilon = 0$ we get that the most singular term is

$$\frac{12}{\epsilon^{11}}. \tag{4.121}$$

Comparing both results we conclude that the projective pure spinors degree is

$$\deg(SO(10)/U(5)) = \int_{SO(10)/U(5)} c_1(\mathcal{L})^{10} = 12. \tag{4.122}$$

Finally, the integral (4.99) has been solved in a easy way

$$\begin{aligned}
\int_{\mathcal{O}(-1)} [\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}]\ e^{-a\lambda\overline{\lambda}} &= \frac{(2\pi)\cdot 7!}{a^8} \int_{SO(10)/U(5)} \frac{(f^*\Omega)^{10}}{10!} \\
&= \frac{(2\pi)\cdot 7!\cdot 12}{a^8\cdot 10!} \cdot \int_{\mathbb{C}P^{10}} \Omega^{10}\big|_{\mathbb{C}P^{10}} \\
&= \frac{(2\pi)^{11}\cdot 7!\cdot 12}{a^8\cdot 10!} \\
&= \frac{(2\pi)^{11}}{a^8\cdot 60}.
\end{aligned} \tag{4.123}$$

Actually, we can compute (4.99) for any dimension using the Kähler form (4.100) (see appendix C.2)

$$\int_{\mathcal{O}(-1)} [\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}]\ e^{-a\lambda\overline{\lambda}} = \frac{(2\pi)^{c_1(T\mathbb{C}P^{n(n-1)/2})}}{a^{c_1(T\mathcal{Q}_{2n})}} \cdot \frac{c_1(T\mathcal{Q}_{2n})!}{c_1(T\mathbb{C}P^{n(n-1)/2})!} \cdot \frac{c_1(T\mathbb{C}P^{n(n-1)/2})}{c_1(T\mathcal{Q}_{2n})} \cdot \deg(\mathcal{Q}_{2n})$$

where $c_1(T\mathcal{Q}_{2n}) = 2n-2$ is the first Chern class of the tangent bundle over projective pure spinors space $\mathcal{Q}_{2n} \equiv SO(2n)/U(n)$, $c_1(T\mathbb{C}P^{n(n-1)/2}) = (n(n-1)+2)/2$ is the first Chern class of the tangent bundle over projective space $\mathbb{C}P^{n(n-1)/2}$ and $\deg(\mathcal{Q}_{2n})$ is the degree of the projective pure spinors space

$$[\mathcal{Q}_{2n}] = \deg(\mathcal{Q}_{2n})[\mathbb{C}P^{n(n-1)/2}]. \tag{4.124}$$

So the result for the general integral (4.91) is

$$\begin{aligned}
&\int [\mathrm{d}\lambda]_0 \wedge [\mathrm{d}\overline{\lambda}]_0 \wedge \prod_{I=1}^{g} [\mathrm{d}\omega^I]_0 \wedge [\mathrm{d}\overline{\omega}^I]_0\ e^{-a(\lambda\overline{\lambda}) - b\sum_{I=1}^{g}(\omega^I\overline{\omega}^I)} \\
&= \left(\frac{A_g}{Z_g}\right)^{11} \frac{1}{(4\pi^2)^{11(g+1)}} \left(\frac{2\pi}{b}\right)^{11g} \frac{(2\pi)^{11}}{7!\,60} \frac{(7+3g)!}{a^{8+3g}}.
\end{aligned} \tag{4.125}$$

To avoid cluttering in the formulæ we define the genus g bracket $\langle\,\rangle_{(n,g)}$ as

$$\langle M(\lambda,\overline{\lambda},\theta,r)\rangle_{(n,g)} \equiv \int [d\theta]_0[dr]_0[d\lambda]_0[d\overline{\lambda}]_0 \frac{e^{-(\lambda\overline{\lambda})-(r\theta)}}{(\lambda\overline{\lambda})^{3-n}} M(\lambda,\overline{\lambda},\theta,r) \tag{4.126}$$

for an arbitrary pure spinor superfield $M(\lambda,\overline{\lambda},\theta,r)$. Which together with (4.89) imply that

$$N_{(n,g)} \equiv \langle \lambda^3\theta^5\rangle_{(n,g)} = 2^7 R \left(\frac{2\pi}{A_g}\right)^{5/2} \left(\frac{\alpha'}{2}\right)^2 \frac{(7+n)!}{7!}, \quad n \geq 0, \tag{4.127}$$

where we used the abbreviated notation $(\lambda^3\theta^5) = (\lambda\gamma^r\theta)(\lambda\gamma^s\theta)(\lambda\gamma^t\theta)(\theta\gamma_{rst}\theta)$. Due to the identities of [63] it is required for 4 external points at the tree-level, one- and two-loop amplitudes

$$\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(n,g)} = -\frac{K}{2^9\,3^2\,5}\langle(\lambda^3\theta^5)\rangle_{(n,g)} \tag{4.128}$$

where K denotes the kinematic factor. When the contributions come from only the external bosons then this K is the same as in [66], which is

$$K = (e^1\cdot e^2)\big[2tu(e^3\cdot e^4) - 4t(k^1\cdot e^3)(k^2\cdot e^4)\big] + \text{perm.} \tag{4.129}$$

So, from (4.127) and (4.47) it follows that in amplitudes of closed string states the factors of A_g cancel in the always-present product of,

$$|N_{(n,g)}|^2\langle\prod_{i=1}^{N} e^{ik\cdot x}\rangle_g = (2\pi)^{10}\delta^{(10)}(k)\frac{\sqrt{2}}{2^2\pi^6\alpha'^5}\left(\frac{\alpha'}{2}\right)^4\left(\frac{(7+n)!}{7!}\right)^2\prod_{i<j}F_g(z_i,z_j)^{\alpha' k^i\cdot k^j}. \tag{4.130}$$

Now we can compute in an exact way the scattering amplitudes using the pure spinor formalism and so to check unitariry.

4.4 Tree Level Scattering Amplitude

To fix the normalizations at tree-level to match those of [66] we need two conditions [81], therefore we also evaluate the three-point amplitude, which is given by

$$\mathcal{A}_{\rm t} = \tilde{\kappa}^3 e^{-2\mu}\langle|\mathcal{N}_{tree}V(0)V(1)V(\infty)|^2\rangle, \tag{4.131}$$

where $\tilde{\kappa}$ is the normalization constant of the massless vertex operators. Using (4.47), the component expansion found in [67] and the fact that $(k^i\cdot k^j)=0$

$$\mathcal{A}_{\rm t} = (2\pi)^{10}\delta^{(10)}(k)\tilde{\kappa}^3 e^{-2\mu}\frac{A_0^5}{(2\pi^2\alpha')^5}|K_t|^2$$

hence,

$$\mathcal{A}_{\rm t}=(2\pi)^{10}\delta^{(10)}(k)\tilde{\kappa}^3e^{-2\mu}\frac{R^2\,2^{10}}{\pi^5\alpha'^5}\left(\frac{\alpha'}{2}\right)^4W_3\bar{W}_3 \tag{4.132}$$

where we used that

$$|K_t|^2=|\langle(\lambda A^1)(\lambda A^2)(\lambda A^3)\rangle_{(3,0)}|^2=\frac{|N_{(3,0)}|^2}{2880^2}W_3\bar{W}_3=R^2\,2^{10}\left(\frac{2\pi}{A_0}\right)^5\left(\frac{\alpha'}{2}\right)^4W_3\bar{W}_3$$

and $W_3=(e^1\cdot e^2)(k^2\cdot e^3)+(e^1\cdot e^3)(k^1\cdot e^2)+(e^2\cdot e^3)(k^3\cdot e^1)$ is the 3-pt kinematic factor in the RNS computation of [66].

Now we must compute the four points scattering amplitude at tree level in order to get the normalization of R.
The massless four-point amplitude at tree-level is given by 4.27,

$$\mathcal{A}_0=\tilde{\kappa}^4e^{-2\mu}\int d^2z_4\langle|\mathcal{N}_{tree}\,V^1(0)V^2(1)V^3(\infty)U^4(z_4)|^2\rangle. \tag{4.133}$$

The amplitude (4.133) was computed in components by [99] and later expressed in pure spinor superspace up to an overall normalization in [63], where it was used that $\langle\prod_{j=1}^4 e^{ik_jx(z_i,\bar{z}_i)}\rangle=|z_4|^{-\frac{1}{2}\alpha' t}|1-z_4|^{-\frac{1}{2}\alpha' u}$. The normalization of the tree-level amplitude of [63] can be determined *a posteriori* by using the precise value for the expectation value of the exponentials,

$$\langle\prod_{j=1}^4 e^{ik_jx(z_i,\bar{z}_i)}\rangle_0=(2\pi)^{10}\delta^{(10)}(k)\left(\frac{A_0}{2\pi^2\alpha'}\right)^5|z_4|^{-\frac{1}{2}\alpha' t}|1-z_4|^{-\frac{1}{2}\alpha' u}, \tag{4.134}$$

Doing that in the computations of [63] we obtain,

$$\mathcal{A}_0=(2\pi)^{10}\delta^{(10)}(k)\tilde{\kappa}^4e^{-2\mu}\left(\frac{4\pi}{2\pi^2\alpha'}\right)^5\left(\frac{\alpha'}{2}\right)^4K_0\bar{K}_0C(s,t,u), \tag{4.135}$$

where

$$C(s,t,u)=2\pi\frac{\Gamma(-\frac{\alpha' s}{4})\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})}, \tag{4.136}$$

the parameters $s=-2(k^1\cdot k^2)$, $t=-2(k^2\cdot k^3)$, $u=-2(k^1\cdot k^3)$ are the Mandelstam variables satisfying $s+t+u=0$ and the kinematic factor K_0 is given by the pure spinor superspace expression [63]

$$K_0=\langle(\lambda A^1)(\lambda\gamma^mW^2)(\lambda\gamma^nW^3)\mathcal{F}^4_{mn}\rangle_{(3,0)}=-\frac{K}{2^9\,3^2\,5}\langle(\lambda^3\theta^5)\rangle_{(3,0)} \tag{4.137}$$

where the last equality follows from (4.128). Using (4.127) we get

$$K_0=K\frac{N^{(3,0)}}{(2^9\,3^2\,5)}=\frac{R}{\sqrt{2}}\left(\frac{\alpha'}{2}\right)^2K, \tag{4.138}$$

and therefore

$$\mathcal{A}_0 = (2\pi)^{10}\delta^{(10)}(k)\tilde{\kappa}^4 e^{-2\mu}\frac{R^2}{2}\left(\frac{2}{\pi\alpha'}\right)^5\left(\frac{\alpha'}{2}\right)^8 K\bar{K}C(s,t,u) \tag{4.139}$$

In the normalization conventions of [66] the tree-level three- and four-point amplitudes were shown to be given by**

$$\mathcal{A}_{\rm t}^{\rm RNS} = (2\pi)^{10}\delta^{(10)}(k)\kappa^3 e^{-2\lambda}\left(\frac{\sqrt{2}}{2^6\pi^6\alpha'^5}\right)\left(\frac{\alpha'}{2}\right)^4 W_3\bar{W}_3, \tag{4.140}$$

$$\mathcal{A}_0^{\rm RNS} = (2\pi)^{10}\delta^{(10)}(k)\kappa^4 e^{-2\lambda}\left(\frac{\sqrt{2}}{2^{12}\pi^6\alpha'^5}\right)\left(\frac{\alpha'}{2}\right)^8 K\bar{K}C(s,t,u). \tag{4.141}$$

Comparing the RNS results of (4.140) and (4.141) with the corresponding PS amplitudes of (4.132) and (4.139) it follows that

$$\tilde{\kappa} = \kappa, \qquad R^2\, e^{-2\mu} = \frac{\sqrt{2}}{2^{16}\pi}\, e^{-2\lambda}, \tag{4.142}$$

so setting $R^2 = \frac{\sqrt{2}}{2^{16}\pi}$ the PS and RNS tree-level normalization conventions are the same, i.e $e^{-2\mu} = e^{-2\lambda}$. After this tree-level matching is done there remains no more freedom to adjust conventions.

4.5 One-loop

The one-loop massless four-point amplitude is given by (4.28),

$$\mathcal{A}_1 = \frac{1}{2}\kappa^4\int_{\mathcal{M}_1} \mathrm{d}^2\tau_1 \prod_{i=2}^{4}\int \mathrm{d}^2 z_i \langle|\mathcal{N}_{1-loop}\,(b,\mu_1)V^1(0)U^i(z_i)|^2\rangle. \tag{4.143}$$

where following [66] we use $d^2\tau = d\tau\wedge d\bar{\tau}$, $d^2z = dz\wedge d\bar{z}$ (in particular $\int_{\Sigma_1} d^2z = 2\tau_2$). The regulator in (4.32) is $\mathcal{N}_{1-loop} = e^{-(\lambda\bar{\lambda})-(\omega^1\bar{\omega}^1)-(r\theta)+(s^1d^1)}$ and the b-ghost insertion written in (4.31) reads

$$(b,\mu_1) = \frac{1}{2\pi}\int \mathrm{d}^2 z b_{zz}\mu_{\bar{z}}^{z}. \tag{4.144}$$

As discussed in [57], there is an unique way to saturate the zero-modes of all variables. The b-ghost must provide two d^1_α zero-modes with $\frac{1}{2^6 3}(\frac{\alpha'}{2})(\bar{\lambda}\gamma^{mnp}r)(d^1\gamma_{mnp}d^1)w_1w_1$, where $w_1 = 1$ is the holomorphic 1-form in the torus. Therefore the integral (4.144) is easily computed to give

$$(b,\mu_1) = \frac{1}{2^7 3\pi}\left(\frac{\alpha'}{2}\right)\frac{(\bar{\lambda}\gamma^{mnp}r)(d^1\gamma_{mnp}d^1)}{(\lambda\bar{\lambda})^2},$$

**Note that $[\mathcal{A}_t] = 6$ and $[\mathcal{A}_0] = 8$, so in [66] the factors of $(\alpha'/2)$ were forgotten.

because $\int \mathrm{d}^2 z w_1 w_1 \mu_1 = 1$ [72]. The integrated vertices contribute three d^1_α zero-modes via $(\frac{\alpha'}{2})^3(d^1W^2)(d^1W^3)(d^1W^4)$, so (4.143) becomes

$$\mathcal{A}_1 = \frac{1}{2^{15}3^2\,\pi^2}\kappa^4\left(\frac{\alpha'}{2}\right)^8\int_{\mathcal{M}_1}\mathrm{d}^2\tau\prod_{i=2}^4\int d^2z_i|\mathcal{K}_1|^2\langle\prod_{j=1}^4 :e^{ik_j\cdot X(z_i)}:\rangle_1, \tag{4.145}$$

where the computation of the zero-mode integrations in

$$\mathcal{K}_1 = \int[\mathrm{d}d^1]_0[\mathrm{d}s^1]_0[\mathrm{d}\omega^1]_0\wedge[d\overline{\omega}^1]_0\, e^{-(\omega^1\overline{\omega}^1)+(s^1d^1)}$$

$$\times\langle(\overline{\lambda}\gamma^{mnp}r)(d^1\gamma_{mnp}d^1)(\lambda A^1)(d^1W^2)(d^1W^3)(d^1W^4)\rangle_{(1,1)} \tag{4.146}$$

is straightforward and goes as follows. Using the measures (4.77) and (4.78) and the results of (4.98) and (4.125) one gets

$$\int[\mathrm{d}\omega]_0\wedge[\mathrm{d}\overline{\omega}]_0\, e^{-(\omega\overline{\omega})} = \frac{(\lambda\overline{\lambda})^3}{(2\pi)^{11}Z_1^{22}}. \tag{4.147}$$

Hence,

$$\mathcal{K}_1 = \frac{1}{(2\pi)^{11}Z_1^{22}}\int[\mathrm{d}d^1]_0[\mathrm{d}s^1]_0\, e^{(s^1d^1)}\langle(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d)(\lambda A^1)(dW^2)(dW^3)(dW^4)\rangle_{(4,1)}. \tag{4.148}$$

The integration over $[\mathrm{d}s]_0$ using the measure (4.80) leads to

$$\mathcal{K}_1 = \frac{(2\pi)^{-11/2}}{2^6(11!5!)Z_1^{11}R}\left(\frac{\alpha'}{2}\right)^2\int[\mathrm{d}d^1]_0 T_{\alpha_1\ldots\alpha_5}\epsilon^{\alpha_1\ldots\alpha_5\delta_1\ldots\delta_{11}}d_{\delta_1}\ldots d_{\delta_{11}}$$

$$\langle(\overline{\lambda}\gamma^{mnp}r)(d^1\gamma_{mnp}d^1)(\lambda A^1)(d^1W^2)(d^1W^3)(d^1W^4)\rangle_{(1,1)}. \tag{4.149}$$

Using the identities

$$\int \mathrm{d}^{16}d\, d_{\rho_1}\ldots d_{\rho_{16}} = \epsilon_{\rho_1\ldots\rho_{16}}, \qquad \epsilon_{\rho_1\ldots\rho_{16}}\epsilon^{\alpha_1\ldots\alpha_5\rho_1\ldots\rho_{11}} = 11!5!\delta^{\alpha_1\ldots\alpha_5}_{\rho_{12}\ldots\rho_{16}}, \tag{4.150}$$

$$(\gamma^{abc})^{\rho_{12}\rho_{13}}(\gamma_{m_1n_1p_1})_{\rho_{12}\rho_{13}} = -2^5 3\,\delta^{abc}_{m_1n_1p_1}, \tag{4.151}$$

$$(\lambda\gamma^{m_1})_{[\alpha_1}(\lambda\gamma^{n_1})_{\alpha_2}(\lambda\gamma^{p_1})_{\alpha_3}(\gamma_{m_1n_1p_1})_{\alpha_4\alpha_5]} = (\lambda\gamma^{m_1})_{\alpha_1}(\lambda\gamma^{n_1})_{\alpha_2}(\lambda\gamma^{p_1})_{\alpha_3}(\gamma_{m_1n_1p_1})_{\alpha_4\alpha_5} \tag{4.152}$$

the integration over $[\mathrm{d}d^1]_0$ is easily performed and (4.149) becomes

$$\mathcal{K}_1 = \frac{3(2\pi)^{5/2}Z_1^5}{2R}\left(\frac{\alpha'}{2}\right)^{-2}\langle(\overline{\lambda}\gamma^{mnp}D)(\lambda A^1)(\lambda\gamma_m W^2)(\lambda\gamma_n W^3)(\lambda\gamma_p W^4)\rangle_{(1,1)} \tag{4.153}$$

where we also used that [62] $\int e^{-(r\theta)}r_\alpha(\ldots) = \int D_\alpha e^{-(r\theta)}(\ldots)$. Using the identity [67]

$$\langle(\overline{\lambda}\gamma^{mnp}D)(\lambda A^1)(\lambda\gamma_m W^2)(\lambda\gamma_n W^3)(\lambda\gamma_p W^4)\rangle_{(1,1)} = 40\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(2,1)}$$

$$= \frac{K}{2^6\,3^2}\langle(\lambda^3\theta^5)\rangle_{(2,1)}$$

where in the last line we used (4.128), the kinematic factor (4.153) can be written as

$$\mathcal{K}_1 = \frac{(2\pi)^{5/2} Z_1^5 K}{3R\,2^7}\left(\frac{\alpha'}{2}\right)^{-2}\langle(\lambda^3\theta^5)\rangle_{(2,1)}. \tag{4.154}$$

Using the definition (4.127) one concludes from (4.154) that

$$|\langle\mathcal{K}_1\rangle|^2 = \frac{(2\pi)^5 Z_1^{10}}{2^{14}3^2R^2}K\bar{K}|N_{(2,1)}|^2\left(\frac{\alpha'}{2}\right)^{-4}. \tag{4.155}$$

The amplitude (4.145) therefore is given by

$$\mathcal{A}_1 = \frac{(2\pi)^5}{2^{29}3^4R^2\pi^2}K\bar{K}\kappa^4\left(\frac{\alpha'}{2}\right)^4\int_{\mathcal{M}_1}\mathrm{d}^2\tau Z_1^{10}\prod_{i=2}^{4}\int \mathrm{d}^2 z_i|N_{(2,1)}|^2\langle\prod_{j=1}^{4}:e^{ik_j\cdot X(z_i)}:\rangle_1$$

which upon using (4.130),

$$|N_{(2,1)}|^2\langle\prod_{j=1}^{4}:e^{ik_jX(z_i)}:\rangle = (2\pi)^{10}\delta^{(10)}(k)\frac{2^{25}3^4R^2}{(2\pi)^5\alpha'^5}\left(\frac{\alpha'}{2}\right)^4\prod_{i<j}F_1(z_i,z_j)^{\alpha k^i\cdot k^j}$$

and $Z_1^{10} = (2\tau_2)^{-5}$ finally becomes

$$\mathcal{A}_1 = (2\pi)^{10}\delta^{(10)}(k)\frac{\kappa^4K\bar{K}}{2^9\pi^2\alpha'^5}\left(\frac{\alpha'}{2}\right)^8\int_{\mathcal{M}_1}\frac{d^2\tau}{\tau_2^5}\prod_{i=2}^{4}\int d^2z_i\prod_{i<j}F_1(z_i,z_j)^{\alpha k^i\cdot k^j}. \tag{4.156}$$

It should be pointed out that the previous computation in [32] claimed that the 1-loop computation in the pure spinor formalism agreed with the RNS result of [66], but it was incorrectly used that $\int \mathrm{d}^2 z w_1 w_1 \mu_{\bar{z}}^{z} = 2$ instead of $= 1$. And to compare with the result of [66] one takes into account the translation invariance of the torus to integrate the "extra" $\int \frac{d^2z_1}{\tau_2} = 2$ integral in their equation (2.22) to conclude that (4.156) differs[††] by $\frac{1}{4}$ from the RNS result reported in [66]. We argue that the one-loop result of [66] is missing the two factors of 1/2 from the GSO projection for both the left- and right-moving sectors, explaining the $1/2^2$ discrepancy[‡‡].

4.6 Two-loop

The two-loop massless four-point amplitude in the non-minimal pure spinor formalism is given by

$$\mathcal{A}_2 = \frac{1}{2}\kappa^4 e^{2\lambda}\prod_{i=1}^{4}\prod_{j=1}^{3}\int_{\mathcal{M}_2}\mathrm{d}^2\tau_j\int \mathrm{d}^2 z_i\langle|\mathcal{N}_{2-loop}(b,\mu_j)U^i(z_i)|^2\rangle \tag{4.157}$$

[††] There is a missing factor of $(\alpha'/2)^8$ in [66].

[‡‡] We thank Eric D'Hoker for kindly confirming to us their missing 1/4 factor [78].

where

$$(b,\mu_j)=\frac{1}{2\pi}\int \mathrm{d}^2y_j b_{zz}\mu_{j\,\bar z}^{z}. \tag{4.158}$$

The 32 (22) zero-modes of d_α (s^α) are denoted by d^I_α (s^α_I) for $I=1,2$. As it is simple to see [57], they are saturated by the different factors of (4.157) as

$$\mathcal{N}_{2-loop}\to(s^1d^1)^{11}(s^2d^2)^{11}\quad \prod_{j=1}^{3}(b,\mu_j)\to(d^1)^3(d^2)^3\quad U^1U^2U^3U^4\to(d^1)^2(d^2)^2, \tag{4.159}$$

so that each b-ghost contributes only zero-modes with the term $(\frac{\alpha'}{2})\frac{(\overline{\lambda}\gamma^{mnp}r)}{192(\lambda\overline{\lambda})^2}(d\gamma_{mnp}d)$. The expansion $d_\alpha(y_i)=\hat{d}_\alpha(z)+d^1_\alpha w_1(y_i)+d^2_\alpha w_2(y_i)$ implies a zero-mode contribution of

$$(d\gamma_{mnp}d)(y)=(d^1\gamma_{mnp}d^1)f_{11}(y)+2(d^1\gamma_{mnp}d^2)f_{12}(y)+(d^2\gamma_{mnp}d^2)f_{22}(y)$$

where $f_{ij}(y)\equiv w_i(y)w_j(y)$, $i,j=1,2$ is the basis of holomorphic quadratic differentials for the genus-2 Riemann surface [74]. It follows from a short computation that,

$$\prod_{j=1}^{3}(b,\mu_j)=c_b\prod_{j=1}^{3}\int \mathrm{d}^2y_j\mu_j(y_j)\Delta(y_1,y_2)\Delta(y_2,y_3)\Delta(y_3,y_1)$$
$$\frac{1}{(\lambda\overline{\lambda})^6}(\overline{\lambda}\gamma_{abc}r)(\overline{\lambda}\gamma_{def}r)(\overline{\lambda}\gamma_{ghi}r)(d^1\gamma^{abc}d^1)(d^1\gamma^{def}d^2)(d^2\gamma^{ghi}d^2) \tag{4.160}$$

where $c_b=\frac{2}{(384\pi)^3}(\frac{\alpha'}{2})^3$ and $\Delta(y,z)=w_1(y)w_2(z)-w_2(y)w_1(z)$. In the computation of (4.160) one can check that combinations containing a different number of d^1_α and d^2_α zero modes e.g.,

$$(\overline{\lambda}\gamma_{abc}r)(\overline{\lambda}\gamma_{def}r)(\overline{\lambda}\gamma_{ghi}r)(d^1\gamma^{abc}d^2)(d^1\gamma^{def}d^2)(d^2\gamma^{ghi}d^2)$$

vanish trivially due to the index symmetries, confirming the zero mode counting of (4.159). Using the period matrix parametrization of moduli space the b-ghost insertions become

$$\int_{\mathcal{M}_2}\mathrm{d}^2\tau_1\mathrm{d}^2\tau_2\mathrm{d}^2\tau_3|\prod_{j=1}^{3}(b,\mu_j)|^2=$$
$$=c_b^2\int_{\mathcal{M}_2}\mathrm{d}^2\Omega_{IJ}|\frac{1}{(\lambda\overline{\lambda})^6}(\overline{\lambda}\gamma_{abc}r)(\overline{\lambda}\gamma_{def}r)(\overline{\lambda}\gamma_{ghi}r)(d^1\gamma^{abc}d^1)(d^1\gamma^{def}d^2)(d^2\gamma^{ghi}d^2)|^2$$

where $\int\mathrm{d}^2\Omega_{IJ}=\int\mathrm{d}^2\Omega_{11}\mathrm{d}^2\Omega_{12}\mathrm{d}^2\Omega_{22}$ and we used the identity of the appendix C.4. The integration over $[\mathrm{d}\omega^I]_0\wedge[\mathrm{d}\overline{\omega}^I]_0$ can be done using the results of (4.98)(4.125) [32] taking into account the different normalizations for the measures (4.77) and (4.78),

$$\int[\mathrm{d}\omega^1]_0\wedge[\mathrm{d}\overline{\omega}^1]_0\wedge[\mathrm{d}\omega^2]_0\wedge[\mathrm{d}\overline{\omega}^2]_0\,e^{-(\omega^1\overline{\omega}^1)-(\omega^2\overline{\omega}^2)}=\frac{(\lambda\overline{\lambda})^6}{(2\pi)^{22}}Z_2^{-22} \tag{4.161}$$

It is straightforward to use the measure (4.80) to integrate over $[\mathrm{d}s^1]_0[\mathrm{d}s^2]_0$, and the amplitude (4.157) becomes

$$
\begin{aligned}
\mathcal{A}_2 = \frac{\kappa^4 e^{2\lambda}}{2^{56}\pi^{26}3^6(11!5!)^4}\left(\frac{\alpha'}{2}\right)^8 \int_{\mathcal{M}_2} \mathrm{d}^2\Omega_{IJ}|Z_2^{-11}\int [\mathrm{d}\theta]_0[\mathrm{d}d^1]_0[\mathrm{d}d^2]_0[\mathrm{d}r]_0[\mathrm{d}\lambda]_0\wedge[\mathrm{d}\overline{\lambda}]_0 \\
\frac{e^{-(\lambda\overline{\lambda})-(r\theta)}}{(\lambda\overline{\lambda})^6}(\overline{\lambda}\gamma_{abc}r)(\overline{\lambda}\gamma_{def}r)(\overline{\lambda}\gamma_{ghi}r)(d^1\gamma^{abc}d^1)(d^1\gamma^{def}d^2)(d^2\gamma^{ghi}d^2) \\
(\lambda\gamma^{m_1})_{\alpha_1}(\lambda\gamma^{n_1})_{\alpha_2}(\lambda\gamma^{p_1})_{\alpha_3}(\gamma_{m_1n_1p_1})_{\alpha_4\alpha_5}(\lambda\gamma^{m_2})_{\beta_1}(\lambda\gamma^{n_2})_{\beta_2}(\lambda\gamma^{p_2})_{\beta_3}(\gamma_{m_2n_2p_2})_{\beta_4\beta_5} \\
\epsilon^{\alpha_1\ldots\alpha_5\rho_1\ldots\rho_{11}}\epsilon^{\beta_1\ldots\beta_5\delta_1\ldots\delta_{11}} d^1_{\rho_1}\ldots d^1_{\rho_{11}} d^2_{\delta_1}\ldots d^2_{\delta_{11}} \\
\big[(d^1W^1)(d^1W^2)(d^2W^3)(d^2W^4)w_1(z_1)w_1(z_2)w_2(z_3)w_2(z_4) \\
+(d^1W^1)(d^2W^2)(d^1W^3)(d^2W^4)w_1(z_1)w_2(z_2)w_1(z_3)w_2(z_4) \\
+(d^1W^1)(d^2W^2)(d^2W^3)(d^1W^4)w_1(z_1)w_2(z_2)w_2(z_3)w_1(z_4) \\
+(d^2W^1)(d^2W^2)(d^1W^3)(d^1W^4)w_2(z_1)w_2(z_2)w_1(z_3)w_1(z_4) \\
+(d^2W^1)(d^1W^2)(d^1W^3)(d^2W^4)w_2(z_1)w_1(z_2)w_1(z_3)w_2(z_4) \\
+(d^2W^1)(d^1W^2)(d^2W^3)(d^1W^4)w_2(z_1)w_1(z_2)w_2(z_3)w_1(z_4)\big]\big|^2 \times \langle\prod_{j=1}^4 : e^{ik_j\cdot X} :\rangle_2
\end{aligned}
\tag{4.162}
$$

where the only non-vanishing contribution from the external vertices contains two d^1 and two d^2 zero-modes coming from $(\alpha'/2)^4(dW)^4$. Integrating the d_α zero-modes in (4.162) using (4.81) and (4.150) — (4.152) one gets

$$
\mathcal{A}_2 = \frac{\pi^6}{2^4 3^2}\left(\frac{\alpha'}{2}\right)^6 \int_{\mathcal{M}_2} \mathrm{d}^2\Omega_{IJ} Z_2^{10}\big|\mathcal{K}_2\big|^2 \times \langle\prod_{j=1}^4 : e^{ik_j\cdot X} :\rangle_2 \tag{4.163}
$$

where the non-minimal kinematic factor $\mathcal{K}$ is given by

$$
\begin{aligned}
\mathcal{K}_2 = \langle(\overline{\lambda}\gamma_{m_1n_1p_1}r)(\overline{\lambda}\gamma_{def}r)(\overline{\lambda}\gamma_{m_2n_2p_2}r)(\lambda\gamma^{m_1defm_2}\lambda)\big[\\
+(\lambda\gamma^{n_1}W^1)(\lambda\gamma^{p_1}W^2)(\lambda\gamma^{n_2}W^3)(\lambda\gamma^{p_2}W^4)\,(H_{1234}+H_{3412}) \\
+(\lambda\gamma^{n_1}W^1)(\lambda\gamma^{p_1}W^3)(\lambda\gamma^{n_2}W^2)(\lambda\gamma^{p_2}W^4)\,(H_{1324}+H_{2413}) \\
+(\lambda\gamma^{n_1}W^1)(\lambda\gamma^{p_1}W^4)(\lambda\gamma^{n_2}W^2)(\lambda\gamma^{p_2}W^3)\,(H_{1423}+H_{2314})\,\big]\rangle_{(-3,2)}
\end{aligned}
\tag{4.164}
$$

and we defined

$$
H_{ijkl} = w_1(z_i)w_1(z_j)w_2(z_k)w_2(z_l). \tag{4.165}
$$

In the Appendix C.3 we will show that

$$
\mathcal{K}_2 = 2^{12}\,3^3\,5\,\mathcal{Y}_s\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(0,2)} = 2^3\,3\,\mathcal{Y}_s\,K\langle(\lambda^3\theta^5)\rangle_{(0,2)} \tag{4.166}
$$

where the second equality follows from (4.128) and $\mathcal{Y}_s$, which has space-time dimension -2, is given by

$$\mathcal{Y}_s = -s\Delta(1,4)\Delta(2,3) + t\Delta(1,2)\Delta(3,4), \tag{4.167}$$

where $\Delta(i,j) \equiv w_1(z_i)w_2(z_j) - w_1(z_j)w_2(z_i)$ and $w_I(z)$ is the basis of holomorphic 1-forms discussed previously. Hence (4.163) is given by

$$\mathcal{A}_2 = \kappa^4 e^{2\lambda} 2^2 \pi^6 K\bar{K} \left(\frac{\alpha'}{2}\right)^6 \int_{\mathcal{M}_2} \mathrm{d}^2\Omega_{IJ} Z_2^{10} |\mathcal{Y}_s|^2 |N_{(0,2)}|^2 \langle \prod_{j=1}^{4} : e^{ik_j\cdot X} :\rangle_2. \tag{4.168}$$

From the formula (4.130) we get

$$|N_{(0,2)}|^2 \langle \prod_{j=1}^{4} : e^{ik_j\cdot X} :\rangle_2 = (2\pi)^{10}\delta^{(10)}(k) \frac{\sqrt{2}}{2^2\pi^6\alpha'^5} \left(\frac{\alpha'}{2}\right)^4 \prod_{i<j} F_2(z_i,z_j)^{\alpha k^i\cdot k^j} \tag{4.169}$$

which together with $Z_2^{10} = 2^{-10}\mathrm{det}(\mathrm{Im}\Omega_{IJ})^{-5}$ implies that

$$\mathcal{A}_2 = (2\pi)^{10}\delta^{(10)}(k)\kappa^4 e^{2\lambda} \frac{\sqrt{2}K\bar{K}}{2^{10}\alpha'^5} \left(\frac{\alpha'}{2}\right)^{10} \int_{\mathcal{M}_2} \frac{\mathrm{d}^2\Omega_{IJ}}{(\mathrm{detIm}\Omega_{IJ})^5} \int_{\Sigma_4} |\mathcal{Y}_s|^2 \prod_{i<j} F_2(z_i,z_j)^{\alpha k^i\cdot k^j} \tag{4.170}$$

which is the final result for the 2-loop amplitude*. And we have shown that the computation of the whole supersymmetric amplitude including its coefficient is straightforward using the non-minimal pure spinor formalism.

We used the genus-g measures in the non-minimal pure spinor formalism to find the overall coefficient of the two-loop amplitude and have shown that there are no major differences in carrying out the computations when compared against the analogous calculations for the tree-level and one-loop amplitudes. In fact, this task is significantly simplified by the pure spinor superspace identities of [63] linking the four-point kinematic factors. These observations must be compared against the unsolved difficulties in the RNS formalism, which besides having no explicit computations for the whole supermultiplet has to rely on a factorization procedure to find the two-loop coefficient. Furthermore, we argued that the mismatch of 1/16 found in the two-loop amplitude compared with the result of [66] is due to a missing factor of 1/4 from the GSO projection in their one-loop amplitude.

*The coefficient obtained here is 1/16 times the result reported by [66]. This difference can be accounted for by the missing factor of 1/4 in their 1-loop result which is used as input in their fixing of the 2-loop coefficient through factorization.

Chapter 5

Equivalence Between the Minimal and Non-Minimal Formalism at Tree Level

As it was argued in [26] there is a formal relationship between the minimal and non minimal pure spinor formalism. For instance, using the Čech-Dolbeault isomorphism, explained in the subsection 3.4.1, we can map the minimal $b-$ghost (3.141) in (4.23) as follows [52]:
Setting the cover for the $PS \setminus \{0\}$ space as (see appendix B.1.2)

$$\underline{U} = \{U_\alpha\}, \qquad \text{where} \quad U_\alpha = \{\lambda \in PS \,:\, \lambda^\alpha \neq 0\},\ \alpha = 1, ..., 16, \tag{5.1}$$

note that $U_1 \cap ... \cap U_{16} = \{0\}$ and therefore $PS \setminus \{0\} = U_1 \cup ... \cup U_{16}$, we can choose the following partition of the unity

$$\rho_\alpha = \frac{\lambda^\alpha \overline{\lambda}_\alpha}{(\lambda\overline{\lambda})}. \tag{5.2}$$

From the representation of the Dolbeault operator given in the introduction

$$\overline{\partial} \;\rightarrow\; \int \mathrm{d}z\, (\overline{\omega} r),$$

it is simple to see [52]

$$\overline{\partial}\rho_\alpha = \frac{(\lambda\overline{\lambda}) r_\alpha - (r\lambda)\overline{\lambda}_\alpha}{(\lambda\overline{\lambda})} \lambda^\alpha.$$

So applying the method described in the subsection 3.4.1 we get the following map

$$b_{min} = b^{\alpha}_{(0)} + b^{\alpha\beta}_{(1)} + b^{\alpha\beta\gamma}_{(2)} + b^{\alpha\beta\gamma\delta}_{(3)} \xrightarrow{\text{Čech-Dol}} b_{nonmin} - s\partial\overline{\lambda}, \tag{5.3}$$

where b_{min} is (3.141) and b_{nonmin} is (4.23). The term $s\partial\overline{\lambda}$ comes from the non minimal variables and it can not be to obtain from the Čech-Dolbeault isomorphism.
Other example is the naive homotopy operator. Using the same cover (5.1) and the partition of unity (5.2) we can map (3.102) to (4.33).

One purpose of this chapter is to show the equivalence between the minimal and non minimal scattering amplitudes at tree-level.

We must find the Dolbeault cocycle corresponding to the scattering amplitude (3.28) using the isomorphism $H^{10}(PS\setminus\{0\},\Omega^{11}) \approx H^{(11,10)}_{\bar\partial}(PS\setminus\{0\})$, which was explained in the section 3.4.
Since the elements of the group $H^{(11,10)}_{\bar\partial}(PS\setminus\{0\})$ are (11,10)-forms, they can not be evaluated in the whole space of the pure spinor minus the origin. However, $PS\setminus\{0\}$ can be contracted to the space $SO(10)/SU(5)$, which can be thought as the boundary in the infinite of the $PS\setminus\{0\}$ space. Then, by the isomorphism (3.70), the elements of $H^{(11,10)}_{\bar\partial}(PS\setminus\{0\})$ can be evaluated in the $SO(10)/SU(5)$ space. As will be explained in this section, this fact means that the picture lowering operators are not related to any particular regulator.
Now we show how to get the Dolbeault cocycle corresponding to (3.28).
The scattering amplitude (3.28) can be written as

$$\begin{aligned}\mathcal{A} &= \int_\Gamma [\mathrm{d}\lambda]\frac{\epsilon^{\alpha_1\ldots\alpha_5\beta_1\ldots\beta_{11}}C^1_{\beta_1}\ldots C^{11}_{\beta_{11}}}{C^1\lambda\ldots C^{11}\lambda}(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}K \qquad (5.4)\\ &= \frac{1}{11!}\sum_{I_1\ldots I_{11}}\epsilon_{I_1\ldots I_{11}}\int_\Gamma[\mathrm{d}\lambda]\frac{\epsilon^{\alpha_1\ldots\alpha_5\beta_1\ldots\beta_{11}}C^{I_1}_{\beta_1}\ldots C^{I_{11}}_{\beta_{11}}}{C^{I_1}\lambda\ldots C^{I_{11}}\lambda}(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}K\\ &\equiv \frac{1}{11!}\sum_{I_1\ldots I_{11}}\epsilon_{I_1\ldots I_{11}}\int_\Gamma\beta^{I_1\ldots I_{11}} = \int_\Gamma\beta^{1,\ldots,11}\end{aligned}$$

where the θ^α's have been integrated. Clearly $\beta^{I_1\ldots I_{11}}$ is a Čech cochain*

$$\beta^{I_1\ldots I_{11}} \in C^{10}(\underline{U},\Omega^{11}) \qquad (5.5)$$

where $\underline{U}$ is the cover of the $PS\setminus\{0\}$ space, which was defined in the section 3.4, i.e $\underline{U}=\{U_I\}$, $I=1,\ldots,11$, and the patches U_I's are given by $U_I = PS\setminus D_I$, where D_I is the hypersurface $D_I=\{\lambda^\alpha\in PS : C^I_\alpha\lambda^\alpha = 0\}$. Remember that $PS\setminus\{0\} = U_1\cup\ldots\cup U_{11}$. Since there are 11 patches to cover $PS\setminus\{0\}$ then $\beta^{I_1\ldots I_{11}}$ is in the Čech cohomology because $C^{11}(\underline{U},\Omega^{11})=\{0\}$ and $(\delta\beta)^{I_1\ldots I_{12}}\in C^{11}(\underline{U},\Omega^{11})$, so $(\delta\beta)^{I_1\ldots I_{12}}=0$, so we can write

$$\beta^{I_1\ldots I_{11}} \in H^{10}(PS\setminus\{0\},\Omega^{11}). \qquad (5.6)$$

Now, using the partition of unity (3.63) we can find the Dolbeault cocycle, η_β, given by (3.65) and (3.68)

$$\eta_\beta = \frac{1}{10!}\sum_{I_1\ldots I_{11}=1}^{11}\beta^{I_1\ldots I_{11}}\rho_{I_1}\wedge\bar\partial\rho_{I_2}\wedge\ldots\wedge\bar\partial\rho_{I_{11}}. \qquad (5.7)$$

Note that, since $\beta^{I_1\ldots I_{11}}$ is an element of $H^{10}(PS\setminus\{0\},\Omega^{11})$, then $\eta_\beta\in H^{(11,10)}_{\bar\partial}(PS\setminus\{0\})$, as was explained in the subsection 3.4.1. The Dolbeault cohomology group

*In [18] was shown that the measure $[\mathrm{d}\lambda]$ is defined globally on $PS\setminus\{0\}$, so, the Čech indices come only from the PCO's.

$H_{\bar{\partial}}^{(11,10)}(PS \setminus \{0\})$ was computed in [18], $H_{\bar{\partial}}^{(11,10)}(PS \setminus \{0\}) = \mathbb{C}$, so it only has one generator which is η_β.

The computation (5.7) is not straightforward because it is needed to make a non-trivial global transformation from $SO(10)/U(5)$ to itself in order to find an expression for the Dolbeault cocycle η_β independent of the constants C^I's. See the simple example given in the appendix B.1.3. To avoid this difficulty, we use the concept of the degree of the projective pure spinor space, which was introduced in the subsection 3.3.4 and computed in an analytical way in the section 4.3, in order to obtain η_β in a simpler way.

5.1 The Dolbeault Cocycle

Using the degree of the projective pure spinor space we can compute easily the Dolbeault cocycle corresponding to the form $\beta^{I_1 \dots I_{11}}$. The equation (4.115) means

$$\deg(SO(10)/U(5)) = \int_{SO(10)/U(5)} \frac{\omega^{10}}{(2\pi i)^{10}}, \tag{5.8}$$

where ω is (4.108)

$$\omega = -\partial\bar{\partial} \ln(\tilde{\lambda}\tilde{\bar{\lambda}}). \tag{5.9}$$

So, from the subsection 3.3.4 we have that the scattering amplitude (5.4) is

$$\mathcal{A} = \int_\Gamma \beta^{1,\dots,11} = (2\pi i)\ 2^3\ 5! \int_{SO(10)/U(5)} \omega^{10}\ K. \tag{5.10}$$

Writing ω^{10} in coordinates as in (4.106), we have

$$\begin{aligned} \int_\Gamma \beta^{1,\dots,11} &= (2\pi i)\, 2^3\, 5! \int_{SO(10)/U(5)} \omega^{10}\, K \\ &= (2\pi i)\ 2^3\ 5! \int_{\mathbb{C}^{20}} \frac{(10!)\ \bigwedge_{a<b,\, c<d} \mathrm{d}u_{ab}\mathrm{d}\bar{u}^{cd}}{(1 + \frac{1}{2}u_{ab}\bar{u}^{ab} + \frac{1}{8^2}\epsilon^{abcde}\epsilon_{afghi}u_{bc}u_{de}\bar{u}^{fg}\bar{u}^{hi})^8} K \\ &= 2^3\ 5! \int_{\mathbb{C}^{20}} \int_0^{2\pi} \frac{i(10!)\ \mathrm{d}\phi \bigwedge_{a<b,\, c<d} \mathrm{d}u_{ab}\mathrm{d}\bar{u}^{cd}}{(1 + \frac{1}{2}u_{ab}\bar{u}^{ab} + \frac{1}{8^2}\epsilon^{abcde}\epsilon_{afghi}u_{bc}u_{de}\bar{u}^{fg}\bar{u}^{hi})^8} K. \end{aligned} \tag{5.11}$$

So (5.11) is a 21-form evaluated locally on the $SO(10)/SU(5)$ space given by (3.71). This can be seen in the following simple way: the variables u_{ab} parametrize the projective pure spinor space in the patch $\lambda^+ \neq 0$, i.e $\tilde{\lambda}^\alpha = (1, u_{ab}, \frac{1}{8}\epsilon^{abcde}u_{bc}u_{de})$, and ϕ parametrizes the circle $\gamma = e^{i\phi}$. So we locally have the space $SO(10)/U(5)\Big|_{\lambda^+\neq 0} \times U(1)$. Since $U(5) = U(1) \times SU(5)$ then we get the space $SO(10)/U(5)\Big|_{\lambda^+\neq 0} \times U(1) = SO(10)/SU(5)\Big|_{\lambda^+\neq 0}$. Note that we have done just a local analysis. Actually, it is impossible to write globally the space $SO(10)/SU(5)$ as the product between the projective pure spinor space and the circle, $SO(10)/SU(5) \neq SO(10)/U(5) \times U(1)$.

The expression (5.11) means that we found the Dolbeault cocycle η_β evaluated in the space $SO(10)/SU(5)$ locally, i.e we got $(i^*\eta_\beta)\Big|_{\lambda^+\neq 0}$, where i is the embedding $i : SO(10)/SU(5) \to PS \setminus \{0\}$, explained in the sub-subsection 3.4.1. In the following, we are going to obtain η_β in a covariant way in the $PS \setminus \{0\}$ space.

Remember that the holomorphic pure spinor measure $[\mathrm{d}\lambda]$ was given in (3.47). We define a new antiholomorphic 10-form in the $PS \setminus \{0\}$ space as

$$\left[\mathrm{d}\overline{\lambda}\right]' (\overline{\lambda}\gamma^m)^{\alpha_1}(\overline{\lambda}\gamma^n)^{\alpha_2}(\overline{\lambda}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5} = \frac{2^3}{10!}\epsilon^{\alpha_1\ldots\alpha_5\beta_1\ldots\beta_{11}}\mathrm{d}\overline{\lambda}_{\beta_1}\wedge\ldots\wedge\mathrm{d}\overline{\lambda}_{\beta_{10}}\overline{\lambda}_{\beta_{11}}, \tag{5.12}$$

where $\overline{\lambda}_\alpha$ is a pure spinor, $\overline{\lambda}_\alpha(\gamma^m)^{\alpha\beta}\overline{\lambda}_\beta = 0$. Note that (5.12) has the same algebraic expression as in (B.33), with the difference that in this case $\overline{\lambda}_\alpha$ belongs to the $PS \setminus \{0\}$ space while the one in (B.33) it is a projective pure spinor. It is easy to see that in the parametrization on the patch $\lambda^+ \neq 0$

$$\lambda^\alpha = \gamma(1, u_{ab}, \epsilon^{abcde}u_{bc}u_{de}/8), \qquad \bar{\lambda}_\alpha = \bar{\gamma}(1, \bar{u}^{ab}, \epsilon_{abcde}\bar{u}^{bc}\bar{u}^{de}/8), \tag{5.13}$$

the (11,10)-form $[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]'$ becomes

$$[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]' = \gamma^7\bar{\gamma}^8\mathrm{d}\gamma\wedge\mathrm{d}u_{12}\wedge\ldots\wedge\mathrm{d}u_{45}\wedge\mathrm{d}\bar{u}^{12}\wedge\ldots\wedge\mathrm{d}\bar{u}^{45}. \tag{5.14}$$

The $SO(10)/SU(5)$ space given in (3.71) is parametrized on the patch $\lambda^+ \neq 0$ in the following way

$$\lambda^\alpha = r\, e^{i\phi}(1, u_{ab}, \epsilon^{abcde}u_{bc}u_{de}/8), \qquad \text{where } r \text{ is positive constant.} \tag{5.15}$$

So, we can write the 21-form of (5.11) as

$$\frac{[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]'}{(\lambda\overline{\lambda})^8}\Bigg|_{SO(10)/SU(5)}\Bigg|_{\lambda^+\neq 0} = \frac{i\ \mathrm{d}\phi\bigwedge_{a<b,\,c<d}\mathrm{d}u_{ab}\mathrm{d}\bar{u}^{cd}}{(1+\frac{1}{2}u_{ab}\bar{u}^{ab}+\frac{1}{8^2}\epsilon^{abcde}\epsilon_{afghi}u_{bc}u_{de}\bar{u}^{fg}\bar{u}^{hi})^8}. \tag{5.16}$$

Using the pure spinor constraint it is not hard to verify that the (11,10)- form $[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]'/(\lambda\overline{\lambda})^8$ is $\bar{\partial}$ closed on $PS \setminus \{0\}$:

$$\bar{\partial}\left(\frac{[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]'}{(\lambda\overline{\lambda})^8}\right) = 0. \tag{5.17}$$

Therefore, the (11,10)-form $[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]'/(\lambda\overline{\lambda})^8$ belongs to cohomology group $H^{(11,10)}_{\bar{\partial}}(PS\setminus\{0\})$ and the pull back i^* is just the restriction

$$i^*\left(\frac{[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]'}{(\lambda\overline{\lambda})^8}\right) = \frac{[\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]'}{(\lambda\overline{\lambda})^8}\Bigg|_{SO(10)/SU(5)}, \tag{5.18}$$

which is an element of the de-Rham cohomology group $H^{21}_{DR}(SO(10)/SU(5))$. Finally, we found the Dolbeault cocycle η_β corresponding to $\beta^{1,\dots,11}$

$$\beta^{1,\dots,11} \xrightarrow{\text{Čech-Dol}} \eta_\beta \equiv 2^3\, 5!\, (10!) \frac{[d\lambda] \wedge [d\overline{\lambda}]'}{(\lambda\overline{\lambda})^8} K, \tag{5.19}$$

and (5.11) in a covariant way is given by

$$\int_\Gamma \beta^{1,\dots,11} = \int_{SO(10)/SU(5)} \eta_\beta\Big|_{SO(10)/SU(5)} \equiv 2^3\, 5! \int_{SO(10)/SU(5)} (10!) \frac{[d\lambda] \wedge [d\overline{\lambda}]'}{(\lambda\overline{\lambda})^8}\Bigg|_{SO(10)/SU(5)} K. \tag{5.20}$$

Using the Čech-Dolbeault isomorphism we have gone from a theory in an 11-cycle Γ to a theory in the whole $SO(10)/SU(5)$ space. Furthermore, notice that since the non-minimal pure spinor formalism is defined in the whole pure spinor space $PS \setminus \{0\}$, which is a non-compact space, then there are an infinite number of global functions on it such that the amplitude does not change. These functions are called regulators. This is in contrast with the $SO(10)/SU(5)$ space, which is a compact manifold whose unique generator is given by (5.18).

Note that integrating the non compact direction of the $PS \setminus \{0\}$ space we get the space $SO(10)/SU(5)$. This means that for any regulator in the non-minimal formalism after integrating the non compact direction of the $PS \setminus \{0\}$ space, one must get the expression (5.20). We will be more explicit by using coordinates in the following. If λ^α is a pure spinor, then it can be written as $\lambda^\alpha = \gamma\tilde{\lambda}^\alpha$, where $\gamma \in \mathbb{C}^* = U(1) \times \mathbb{R}^+$ and $\tilde{\lambda}^\alpha$ is a projective pure spinor. So, setting $\gamma = \rho\, e^{i\phi}$, where $e^{i\phi} \in U(1)$ and $\rho \in \mathbb{R}^+$ and integrating by ρ in the non-minimal formalism we must get (5.20) for any regulator. In this manner we obtained a formalism in the compact manifold $SO(10)/SU(5)$ instead of the whole pure spinor space, which is not surprising since removing the point $\lambda^\alpha = 0$ this space can be deformed (homotopically) to $SO(10)/SU(5)$.

5.1.1 A Particular Regulator

In this subsection we would like to illustrate what we said in the last paragraph with a particular regulator.

The most useful regulator in the non-minimal pure spinor formalism for computing the tree level scattering amplitude is

$$\mathcal{N} = \exp(-\overline{\lambda}_\alpha \lambda^\alpha - r_\alpha \theta^\alpha), \tag{5.21}$$

as given in [57], where r_α is a spinor such that $r_\alpha(\gamma^m)^{\alpha\beta}\overline{\lambda}_\beta = 0$. After integrating the variables r_α and θ^α we get [33]

$$\mathcal{A} = 2^3\, 5! \int_{PS} [\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}]\, e^{-(\lambda\overline{\lambda})}\, (\lambda\overline{\lambda})^3\, K \tag{5.22}$$

where the measure $[\mathrm{d}r]$ was given in (4.79) [32][57]

$$[\mathrm{d}r] = \frac{1}{2^3\, 5!\, 11!} (\overline{\lambda}\gamma^m)^{\alpha_1} (\overline{\lambda}\gamma^n)^{\alpha_2} (\overline{\lambda}\gamma^p)^{\alpha_3} (\gamma_{mnp})^{\alpha_4\alpha_5} \epsilon_{\alpha_1,\ldots,\alpha_5\beta_1\ldots\beta_{11}} \partial_r^{\beta_1} \ldots \partial_r^{\beta_{11}}, \tag{5.23}$$

where we have introduced the factor 2^3 for convenience. We replaced the vertex operators in the amplitude by $(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\, K$, where K is the kinematic factor. Using the coordinates $\lambda^\alpha = \gamma\tilde{\lambda}^\alpha = \rho\, e^{i\phi}\tilde{\lambda}^\alpha$, which were explained previously, the integration measure is (see (4.102))

$$\begin{aligned} [\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}] = (\gamma\overline{\gamma})^7\, \mathrm{d}\gamma \wedge \mathrm{d}\overline{\gamma} \wedge [\mathrm{d}\tilde{\lambda}] \wedge [\mathrm{d}\overline{\tilde{\lambda}}] &= -2\, i\, (\rho^2)^7\, \rho\, \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge [\mathrm{d}\tilde{\lambda}] \wedge [\mathrm{d}\overline{\tilde{\lambda}}] \\ &= -2\, i\, (\rho^2)^7\, \rho\, \mathrm{d}\rho \wedge [\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}]' \Big|_{SO(10)/SU(5)}\Big|_{r=1}, \end{aligned} \tag{5.24}$$

where $SO(10)/SU(5)|_{r=1}$ means that the space $SO(10)/SU(5)$ has size $r = 1$ (see (3.71)). So, integrating the non-compact variable ρ from r_0 to r we get

$$\begin{aligned} & 2^3\, 5! \int_{PS} [\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}]\, e^{-(\lambda\overline{\lambda})}\, (\lambda\overline{\lambda})^3\, K \\ = \; & 2^3\, 5! \int_{SO(10)/SU(5)} (10!) [\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}]' \Bigg(\frac{e^{-\rho^2(\tilde{\lambda}\overline{\tilde{\lambda}})}}{(\tilde{\lambda}\overline{\tilde{\lambda}})^8} + \frac{\rho^2 e^{-\rho^2(\tilde{\lambda}\overline{\tilde{\lambda}})}}{(\tilde{\lambda}\overline{\tilde{\lambda}})^7} + \frac{\rho^4 e^{-\rho^2(\tilde{\lambda}\overline{\tilde{\lambda}})}}{2(\tilde{\lambda}\overline{\tilde{\lambda}})^6} + \\ & + \frac{\rho^6 e^{-\rho^2(\tilde{\lambda}\overline{\tilde{\lambda}})}}{3!(\tilde{\lambda}\overline{\tilde{\lambda}})^5} + \frac{\rho^8 e^{-\rho^2(\tilde{\lambda}\overline{\tilde{\lambda}})}}{4!(\tilde{\lambda}\overline{\tilde{\lambda}})^4} + \frac{\rho^{10} e^{-\rho^2(\tilde{\lambda}\overline{\tilde{\lambda}})}}{5!(\tilde{\lambda}\overline{\tilde{\lambda}})^3} \frac{\rho^{12} e^{-\rho^2(\tilde{\lambda}\overline{\tilde{\lambda}})}}{6!(\tilde{\lambda}\overline{\tilde{\lambda}})^2} + \frac{\rho^{14} e^{-\rho^2(\tilde{\lambda}\overline{\tilde{\lambda}})}}{7!(\tilde{\lambda}\overline{\tilde{\lambda}})} + \\ & + \frac{\rho^{16} e^{-\rho^2(\tilde{\lambda}\overline{\tilde{\lambda}})}}{8!} + \frac{\rho^{18} (\tilde{\lambda}\overline{\tilde{\lambda}}) e^{-\rho^2(\tilde{\lambda}\overline{\tilde{\lambda}})}}{9!} + \frac{\rho^{20} (\tilde{\lambda}\overline{\tilde{\lambda}})^2 e^{-\rho^2(\tilde{\lambda}\overline{\tilde{\lambda}})}}{10!} \Bigg) \Bigg|_{SO(10)/SU(5)|_{\rho=r_0}}^{SO(10)/SU(5)|_{\rho=r}} K, \end{aligned} \tag{5.25}$$

Note that $SO(10)/SU(5)|_{\rho=r} - SO(10)/SU(5)|_{\rho=r_0}$ is the boundary of the finite pure spinor space, i.e

$$PS_{r_0,r} \equiv \{\lambda^\alpha \in \mathbb{C}^{16} : \lambda^\alpha (\gamma^m)_{\alpha\beta} \lambda^\beta = 0 \text{ and } r_0^2 \leq \lambda^\alpha \overline{\lambda}_\alpha \leq r^2\}, \tag{5.26}$$

where r_0, r are positive constants. In order to obtain the whole pure spinor space, $PS \setminus \{0\}$, we must take the limits $r_0 \to 0$ and $r \to \infty$, so we get the equivalence

$$\begin{aligned} 2^3\, 5! \int_{PS} [\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}]\, e^{-(\lambda\overline{\lambda})}\, (\lambda\overline{\lambda})^3 K &= 2^3\, 5! \int_{SO(10)/SU(5)} (10!) \frac{[\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}]'}{(\lambda\overline{\lambda})^8} \Bigg|_{SO(10)/SU(5)} K \\ &= \int_\Gamma \beta^{1,\ldots,11}. \end{aligned} \tag{5.27}$$

This is the reason why we say that $SO(10)/SU(5)$ is the "boundary" of the $PS\setminus\{0\}$ space, although it is a non-compact space. The equivalence (5.27) holds for any regulator because the Čech-Dolbeault isomorphism relates the minimal formalism with a formalism in $SO(10)/SU(5)$, which only has one cohomology generator given by $([\mathrm{d}\lambda]\wedge[\mathrm{d}\overline{\lambda}]'/(\lambda\overline{\lambda})^8)|_{SO(10)/SU(5)}$.

Although the equivalence between the minimal and non-minimal formalisms is somewhat premature because in tree level we can absorb any number in the coupling constant $e^{-2\mu}$ [33], the previous result is beautiful and it will be very import for computing loop amplitudes [38].

5.2 Independence of the Constant Spinors C^I_α's

Using the results found in the previous section we show that the scattering amplitude at tree level with the new proposal for the PCO, given in the chapter 3, is independent of the choice of the constant spinors C^I's. This implies that they do not need to be integrated, in contrast with the analysis presented in [48][23], where it did was necessary.

We will present an example of pure spinors in four dimensions, where the conditions of linear independence for the C^I's and the intersection of the hyperplanes D_I's in the origin are equivalent. However, in ten dimensions it is not sufficient that the C^I's are linearly independent, so, based on the assumption that the hypersurfaces $D_I = \{C^I\lambda = 0\}, I = 1, ..., 11$, meet just in the origin, we will show that the scattering amplitude is independent of the C^I's choice.

5.2.1 Pure Spinors in $d = 4$: A Simple Example

Before we show the independence of the C^I's for pure spinors in ten dimensions, we give a simple example in four dimensions in order to understand how this can be achieved.

Consider the pure spinor space in $d = 4$ dimensions, i.e the flat space $\mathbb{C}^2$. In this case the integral corresponding to (5.4) is given by

$$\int_\Gamma \vartheta = \int_\Gamma [\mathrm{d}\lambda]\frac{\epsilon^{cd}C^1_c\,C^2_d}{(C^1\lambda)(C^2\lambda)}, \qquad c, d = 1, 2 \tag{5.28}$$

where $\lambda^a = (\lambda^1, \lambda^2)$ are the coordinates of $\mathbb{C}^2$ and the measure is simply $[\mathrm{d}\lambda] = 2^{-1}\epsilon_{ab}\,\mathrm{d}\lambda^a\wedge\mathrm{d}\lambda^b = \mathrm{d}\lambda^1\wedge\mathrm{d}\lambda^2$. Clearly, the vectors $C^j, j = 1, 2$ must be linearly independent in order to obtain an integral different from zero, i.e the determinant $\det(C^j_a)\neq 0$. This implies that the intersection of the hyperplanes $C^j\lambda = 0$, $j = 1, 2$

is just the origin. To compute (5.28), firstly we consider the simple case when $C^j_a = \delta^j_a$. Then the integral is

$$\int_\Gamma \vartheta = \int_\Gamma \frac{\mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2}{\lambda^1 \lambda^2}. \tag{5.29}$$

Secondly we define in a natural way the 2-cycle Γ as the torus $\Gamma = \{\lambda^a \in \mathbb{C}^2 : |\lambda^1| = \epsilon^1 \text{ and } |\lambda^2| = \epsilon^2\}$, where ϵ^1, ϵ^2 are positive arbitrary constants. So (5.29) is a trivial integral and its answer is

$$\int_\Gamma \vartheta = (2\pi i)^2. \tag{5.30}$$

Once the answer is known, we must know what happens if we choose two arbitrary vectors C^i_a but keep the same 2-cycle $\Gamma = \{\lambda^a \in \mathbb{C}^2 : |\lambda^1| = \epsilon^1 \text{ and } |\lambda^2| = \epsilon^2\}$. In other words, we want to know the answer to the question: is (5.28) independent of the constants C^j's?. We will show that the answer is affirmative and its result is the same as in (5.30).

From (5.28) we have

$$\int_{|\lambda^2|=\epsilon^2} \int_{|\lambda^1|=\epsilon^1} \frac{(a_1 b_2 - a_2 b_1)\mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2}{(a_1\lambda^1 + a_2\lambda^2)(b_1\lambda^1 + b_2\lambda^2)}, \tag{5.31}$$

where $C^1 = (a_1, a_2)$ and $C^2 = (b_1, b_2)$. Without loss of generality, we can set $a_2, b_1 \neq 0$. To solve (5.31), first note that since ϵ^1 is an arbitrary constant, it can be set to a very large value such that the pole $-(b_2/b_1)\lambda^2$ is inside of the cycle $|\lambda^1| = \epsilon^1$, so integrating λ^1 we have

$$\int_{|\lambda^2|=\epsilon^2} \int_{|\lambda^1|=\epsilon^1} \frac{(a_1 b_2 - a_2 b_1)\mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2}{(a_1\lambda^1 + a_2\lambda^2)(b_1\lambda^1 + b_2\lambda^2)} = (2\pi i) \int_{|\lambda^2|=\epsilon^2} \frac{(a_1 b_2 - a_2 b_1)\mathrm{d}\lambda^2}{(a_1 b_2 - a_2 b_1)\lambda^2}, \tag{5.32}$$

getting the same answer as in (5.30). However, since the integral depends on a very large value of ϵ^1, this is not a satisfactory way for computing, so we must look for a better analysis.

As $\det(C^j_a) = (a_1 b_2 - a_2 b_1) \neq 0$, then we can make the following change of variables

$$\begin{aligned} \lambda^1 &= M^{-1}(b_2 z^1 - a_2 z^2) \\ \lambda^2 &= M^{-1}(-b_1 z^1 + a_1 z^2) \end{aligned} \tag{5.33}$$

where $M = (a_1 b_2 - a_2 b_1)$. Using these new coordinates (5.32) becomes

$$\int_\Gamma \frac{\mathrm{d}z^1 \wedge \mathrm{d}z^2}{z^1 z^2}. \tag{5.34}$$

where Γ is the 2-cycle given by $|b_2 z^1 - a_2 z^2| = \epsilon^1 |M|$ and $|a_1 z^2 - b_1 z^1| = \epsilon^2 |M|$. Since ϵ^1 and ϵ^2 are positive arbitrary constants then $|z_1| > 0$, $|z_2| > 0$ and applying the triangle inequality we get

$$\begin{aligned} 0 < |z^1| &\leq (\epsilon^1 |a_1| + \epsilon^2 |a_2|) \\ 0 < |z^2| &\leq (\epsilon^1 |b_1| + \epsilon^2 |b_2|) \end{aligned} \tag{5.35}$$

where without loss of generality, we set $a_2, b_1 \neq 0$. Therefore the torus $\Gamma = \{(z^1, z^2) \in \mathbb{C}^2 : |b_2 z^1 - a_2 z^2| = \epsilon^1|M|, |a_1 z^2 - b_1 z^1| = \epsilon^2|M|\}$ can be deformed to the torus $\Gamma' = \{(z^1, z^2) \in \mathbb{C}^2 : |z^1| = (\epsilon^1|a_1| + \epsilon^2|a_2|)/2, |z^2| = (\epsilon^1|b_1| + \epsilon^2|b_2|)/2\}$. So, we have shown that the integral (5.28) is independent of the constants C^j's when we fix the integration cycle, because it can always be deformed to a cycle of the type $|C^i\lambda| = \varepsilon^j$, $j = 1, 2$, for some ε^j, $j = 1, 2$. Formally we are saying the following: remember that the integral (5.28) just depends of the classes of the homology cycle $[\Gamma]$ and the cocycle of cohomology $[\vartheta]$ (see subsection 3.3.1). So, if $\det(C^j_a) \neq 0$ then all the holomorphic 2-forms ϑ are in the same cohomology class $[\vartheta]$ and all the 2-cycle $\Gamma = \{(\lambda^1, \lambda^2) \in \mathbb{C}^2 : |C^j\lambda| = \epsilon^j, j = 1, 2\}$ are in the same homology class $[\Gamma]$. Now we must show the same for pure spinors in $d = 10$.

5.2.2 Pure Spinors in $d = 10$

In the previous example the conditions $\det(C^j_a) \neq 0$ and $\{C^1\lambda = 0\} \cap \{C^2\lambda = 0\} = \{0\}$ were equivalent. However, in the pure spinor space in $d = 10$ the condition $\det(C^I_\alpha) \neq 0$ does not make sense because $I = 1, .., 11$ and $\alpha = 1, ..., 16$, but remember that we have always claimed that $D_1 \cap ... \cap D_{11} = \{0\}$, where $D_I = \{\lambda^\alpha \in PS : C^I_\alpha \lambda^\alpha = 0\}$. In this case is not easy to follow the same analysis of the previous example because PS is not a flat space. Therefore we will make use of the ideas presented previously in this chapter to prove that the tree level scattering amplitude (3.28) is independent of the C^I's.
From (5.4) we have that the amplitude is given by

$$\mathcal{A} = \int_\Gamma \beta^{1,...,11} \tag{5.36}$$

where the 11-cycle Γ was defined as $\Gamma = \{\lambda^\alpha \in PS : |C^I\lambda| = \epsilon^I, I = 1, ..., 11\}$, $\epsilon^I \in \mathbb{R}^+$. In the subsection 5.1 we found the Dolbeault cocycle

$$\eta_\beta = 2^3\, 5!\, \frac{[\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]'}{(\lambda\bar{\lambda})^8}\Bigg|_{SO(10)/SU(5)} K \tag{5.37}$$

corresponding to $\beta^{1,...,11}$, thanks to the isomorphism from the group $H^{10}(PS \smallsetminus \{0\}, \Omega^{11})$ to $H^{21}_{DR}(SO(10)/SU(5))$ (see subsection 3.4.1). As the Dolbeault cocycle η_β is independent of the constants C^I's, then, choosing another set of constant spinors C'^I, $I = 1, .., 11$, such that they satisfy the same condition $D'_1 \cap ... \cap D'_{11} = \{0\}$, its Čech cocycle $\beta'^{1,...,11}$ is in the same cohomology class as $\beta^{1,...,11}$, because $\beta'^{1,...,11}$ and $\beta^{1,...,11}$ have the same corresponding Dolbeault cocycle η_β and the groups $H^{10}(PS \smallsetminus \{0\}, \Omega^{11})$ and $H^{21}_{DR}(SO(10)/SU(5))$ are isomorphic. It means that

$$\int_\Gamma \beta^{1,...,11} = \int_\Gamma \beta'^{1,...,11}, \tag{5.38}$$

because the cohomology classes $[\beta^{1,\ldots,11}]$ and $[\beta'^{1,\ldots,11}]$ are the same.
So we have shown that the tree level scattering amplitude (5.4) is independent of the constant spinors C^I's and therefore it is not needed to integrate over them.

Chapter 6

Conclusions

In the chapter 2, we have reproduced some of the ideas of Nekrasov [18] to show in an explicit way that the point $\lambda = 0$ must be removed from the pure spinor formalism to get a anomaly free theory.
So, in the chapter 3 we proposed a new "picture lowering" operator and computed the scattering amplitude at tree level in such a way that we eliminated the singular point of the pure spinor space, getting in this way a theory free of anomalies [18]. Since the new picture operators are defined just on each patch of the pure spinor space, it is necessary to introduce the Čech operator as part of the BRST charge in order to have a well defined formalism [34]. Therefore, we have introduced the Čech formalism for the scattering amplitudes computation, which seems to be the correct formulation [52][26].

Note that in this thesis the tree level scattering amplitude in the minimal formalism was always computed using three unintegrated vertex operators and the rest were integrated vertex operators. In contrast with the minimal formalism, in the non-minimal formulation it is possible to compute tree level amplitude with all the vertex operators unintegrated [57]. The difficulty in the minimal formalism is the b-ghost. Although we gave a glance about how to treat this issue, in order to continue with the loop-level this subject must be further developed [34][38].

Using the Čech-Dolbeault isomorphism, we also showed in an elegant manner that the tree-level scattering amplitude is BRST, Lorentz and supersymmetric invariant.

We obtained also a relationship between the tree-level scattering amplitude in the pure spinor formalism and the Green's function for the massless scalar field in the twistor formalism [34][22]. We believe that perhaps there is a relationship between the loop-level scattering amplitudes in the pure spinor formalism and the Green's function for the higher-spin massless fields [22], which we would like to explore in the future.

In the chapter 4, after to normalize the integration measures from first principles we showed that the integration over the pure spinor space is given in terms of the topological invariants [32]. This result can be generalized to any complex manifold M embedded in some $\mathbb{C}P^n$ space, except when M is a Calabi-Yau manifold because in this case the first Chern class vanishes $c_1(M) = 0$. So we used the genus-g measures in the non-minimal pure spinor formalism to find the overall coefficient of the two-loop amplitude and have shown that there are no major differences in carrying out the computations when compared against the analogous calculations for the tree-level and one-loop amplitudes [32][33]. In fact, this task is significantly simplified by the pure spinor superspace identities of [63] linking the four-point kinematic factors.
These observations must be compared against the unsolved difficulties in the RNS formalism, which besides having no explicit computations for the whole supermultiplet has to rely on a factorization procedure to find the two-loop coefficient. Furthermore, we argued that the mismatch of 1/16 found in the two-loop amplitude compared with the result of [66] is due to a missing factor of 1/4 from the GSO projection in their one-loop amplitude.

In the last chapter, using the Čech-Dolbeault isomorphism, we found the corresponding Dolbeault cocycle for the tree level scattering amplitude (3.28). What is interesting here is that the Dolbeault cocycle must not be evaluated in whole pure spinor space, but in the $SO(10)/SU(5)$, which can be thought like a sphere in the pure spinor space. This confirms that the singular point was removed from the pure spinor space [34]. Moreover, since the de-Rham cohomology group of this manifold has just one generator given by

$$\left.\frac{[\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]'}{(\lambda\bar{\lambda})^8}\right|_{SO(10)/SU(5)},$$

i.e $H^{21}_{DR}(SO(10)/SU(5)) = \mathbb{C}$ [18], then these picture operates do not correspond to any particular regulator of the non-minimal formalism since they directly involve cohomology generators.
In contrast with the PCO's proposed in [48], with the new PCO's proposed the tree level scattering amplitude is independent of the choice of the constants spinors C^I's. That is because the cohomology class of the scattering amplitude is the same when the constants C^I's satisfy the constraint
$\{C^1\lambda = 0\} \cap \{C^2\lambda = 0\} \cap ... \cap \{C^{11}\lambda = 0\} = \{0\}$, for λ^α satisfying the pure spinor condition.

6.1 Some Perspectives

One of our perspectives is to extend the new picture changing operators at loop level.
We believe that using PCO's there are not restriction about the number of points or genus of the Riemann surfaces. So this opens a new possibility to compute scattering amplitudes.
Nevertheless it is not an easy work and new problems arise, like the geometry structure of the pure spinor phase space.

Let us remember that the tree-level scattering amplitude using the new PCO is equivalent to the theory on the compact space $SO(10)/SU(5)$, as it was shown in chapter 5. Given that $SO(10)/SU(5)$ up to a global constants has only one global volume form given by (5.18), then the question that arises is how to introduce the b-ghost on this space and then be able to compute scattering amplitudes at the loop level. Actually, at the loop level it is necessary a cover over the space $(\lambda^\alpha, \omega^I_\alpha)$, $I = 1 \ldots g$, on which the operator δ would act. In particular, at one loop this space would be the pure spinor phase space $(\lambda^\alpha, \omega_\alpha)$. Therefore, applying the Čech-Dolbeault isomorphism as explained in the subsection 3.4, it would not be found the space $SO(10)/SU(5)$ and in fact, at this time we do not know which space it would be. However, this is nice since it opens up the possibility to have non-constant global functions. In order to understand better this kind of amplitudes, it will be useful to know first the geometry of the space $(\lambda^\alpha, \omega^I_\alpha)$.

With the collaboration of C.R. Mafra, we are computing 4 point at three loop. This is a complex work, which is almost impossible in the RNS formalism. We want to find the coefficient of the D^6R^4 contribution, which was conjectured by Green and Vanhove in [45].
We have obtained some results. In [46] using modular invariance, they generalized the higher genus 4-point superstring amplitude. However from our preliminary results it seems that this generalization should be corrected, this is very important in the development of high-genus amplitudes.

Appendix A

Appendix

A.1 Connection and Curvature on Holomorphic Vector Bundles

The goal of this appendix is that the reader remembers the simple concepts of connection and curvature on complex manifolds, which are very useful to show the anomalies of the beta-gamma system.

This appendix is based on the section 0.5 of [94].

Let M be a complex manifold of complex dimension $dim_{\mathbb{C}}(M) = d$, and $\pi : E \to M$ a holomorphic vector bundle with fiber $\mathbb{C}^k$. A hermitian metric on E is a hermitian inner product on each fiber E_z of E, varying smoothly with $z \in M$, i.e if $s = \{s_1, .., s_k\}$ is a holomorphic frame for E, then the functions

$$h_{ij}(z) = (s_i(z), s_j(z)) \ , \tag{A.1}$$

are C^∞.
A frame s for E is called unitary if $s_1, ..., s_k$ is an orthonormal basis for E_z for each $z \in M$. It is clear that unitary frames always exist locally.
Definition. A connection on E is a map

$$D : \Gamma(E) \to \Gamma(TM^* \otimes E), \tag{A.2}$$

where $\Gamma(E)$ is the space of holomorphic section on E and TM^* is the cotangent bundle, satisfying Leibnitz's rule

$$D(\gamma_1 + \gamma_2) = D\gamma_1 + D\gamma_2 \, , \quad \gamma_1, \gamma_2 \in \Gamma(E) \tag{A.3}$$

$$D(f\gamma) = \mathrm{d}f \otimes \gamma + fD\gamma \, , \quad \gamma \in \Gamma(E) \tag{A.4}$$

where f is a function on M.

Now, let $U \cap M$ and $s_1(z), ..., s_k(z)$ be a holomorphic frame for E, where $z \in U$, then we can write

$$Ds_i = \sum_j \omega_{ij} s_j \; , \tag{A.5}$$

where (ω_{ij}) is a matrix of 1-forms. In matrix way we have

$$Ds = \omega s \; . \tag{A.6}$$

Note that the matrix ω completely determines the connection.
So, for a general holomorphic section $\xi \in \Gamma(U)$

$$\xi = \sum_i \xi_i s_i \tag{A.7}$$

D acts

$$D\xi = \sum_i (\mathrm{d}\xi_i + \sum_j \xi_j \omega_{ji}) s_i \; . \tag{A.8}$$

It is simple see the connection matrix ω at a point $z_0 \in U$ depends on the choice of frame in a neighborhood of z_0, for instance, if $s'_1, ..., s'_k$ is another holomorphic frame then

$$s'_i(z) = \sum_j g_{ij}(z) s_j(z) \; , \tag{A.9}$$

where $g_{ij}(z)$ is holomorphic and non-singular. So

$$Ds'_i = \sum_j \mathrm{d}g_{ij} s_j + \sum_{k,j} g_{ik} \omega_{kj} s_j \; , \tag{A.10}$$

and therefore

$$\omega' = \mathrm{d}g \cdot g^{-1} + g \cdot \omega \cdot g^{-1} \, , \;\; (g = (g_{ij})) \, . \tag{A.11}$$

We can also note that the hermitian metric has the following transformation

$$h(s'_i, s'_j) = h'_{ij} = g_{ik} h_{kl} g^*_{jl} \quad \longrightarrow \quad h' = g\, h\, g^\dagger . \tag{A.12}$$

There is in general no natural connection on a vector bundle E. However as E is holomorphic and hermitian we can make two requirements that dictate a canonical choice of connection

(1) From the decomposition $TM^* = TM^{+*} \oplus TM^{-*}$, we can write $D = D' + D''$ with $D' : \Gamma(E) \to \Gamma(TM^{+*} \otimes E)$ and $D'' : \Gamma(E) \to \Gamma(TM^{-*} \otimes E)$. Now we say that a connection D on E is compatible with the complex structure if $D'' = \overline{\partial}$.

(2) D is said to be compatible with the hermitian metric if

$$d(\xi,\eta) = (D\xi,\eta) + (\xi, D\eta)\ . \tag{A.13}$$

Lemma. If E is a hermitian vector bundle, there is a unique connection D on E compatible with both the metric and the complex structure.
Proof. Let $s_1, ..., s_k$ be a holomorphic frame for E, and $h_{ij} = (s_i, s_j)$ the hermitian metric. So

$$\begin{aligned} \mathrm{d}h_{ij} &= \mathrm{d}(s_i, s_j) \\ &= \sum_k \omega_{ik} h_{kj} + \sum_k \overline{\omega}_{jk} h_{ik} \\ &= (1,0) + (0,1)\ . \end{aligned}$$

Since $\mathrm{d} = \partial + \bar{\partial}$ then we have

$$\begin{aligned} \partial h_{ij} &= \sum_k \omega_{ik} h_{kj}\,, \quad \partial h = \omega h\,, \\ \overline{\partial} h_{ij} &= \sum_k \overline{\omega}_{jk} h_{ik}\,. \quad \overline{\partial} h = h \overline{\omega}^T\ . \end{aligned}$$

It is simple to see that the solution for these equations is $\omega = (\partial h)h^{-1}$. Clearly ω is a (1,0)-form.

Now, in a natural way we can force the Leibnitz's rule

$$D(\psi \wedge \xi) = \mathrm{d}\psi \otimes \xi + (-1)^p \psi \wedge D\xi, \tag{A.14}$$

where ψ is a p-form over M and $\xi \in \Gamma(E)$, and so to compute the map D^2

$$D^2 : \Gamma^E \to \Gamma(\bigwedge^2 TM^* \otimes E)\ .$$

For instance, if f is a C^∞ function over M then

$$\begin{aligned} D^2(f\xi) &= D(\mathrm{d}f \otimes \xi + fD\xi) \\ &= -\mathrm{d}f \wedge D\xi + \mathrm{d}f \wedge D\xi + fD^2\xi \\ &= fD^2\xi\ . \end{aligned}$$

In the same way as we represented the map D as a matrix of 1-forms we can represent the map D^2 as a matrix of 2-forms (called the curvature matrix of D)

$$D^2 s_i = \sum_j F_{ij} s_j\ . \tag{A.15}$$

If s and s' are two frames at the point $z \in U$, $(s' = gs)$, then

$$\begin{aligned} D^2 s'_i &= D^2(\sum_j g_{ij} s_j) \\ &= \sum_{j,k} g_{ij} F_{jk} s_k \\ &= \sum g_{ij} F_{jk} g^{-1}_{kl} e'_l \ , \end{aligned}$$

so

$$F' = gFg^{-1} \ . \tag{A.16}$$

Note that the curvature matrix can be given in terms of the connection matrix (using the definition)

$$D^2 s_i = D(\sum \omega_{ij} s_j) \tag{A.17}$$

$$= \sum (\mathrm{d}\omega_{ij} - \sum \omega_{ik} \wedge \omega_{kj}) s_j \ . \tag{A.18}$$

In matrix notation we have

$$F = \mathrm{d}\omega - \omega \wedge \omega \ . \tag{A.19}$$

Replacing the connection compatible with both the metric and the complex structure we get

$$F = -(\partial\bar{\partial}h)h^{-1} + (\partial h)h^{-1} \wedge (\bar{\partial}h)h^{-1} \ . \tag{A.20}$$

and therefore F is a (1,1)-form.

A.2 Introduction to the Spectral Sequences

Since spectral sequences is not a easy subject of the algebraic geometry then we only give some basic tools to understand the subsection 2.9. The references for this appendix are [29] and [94].

Definition. A spectral sequence is a sequence $\{E_r, d_r\}$ $(r \geq 0)$ of bigraded groups

$$E_r = \bigoplus_{p,q \geq 0} E_r^{p,q} \tag{A.21}$$

together with differentials

$$d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}, \qquad d_r^2 = 0 \tag{A.22}$$

such that

$$H^*(E_r) = E_{r+1}. \tag{A.23}$$

In practice we will always have $E_r = E_{r+1} = ...$ for $r \geq r_0$; we call this limit group E_∞ and say that the spectral sequence $\{E_r\}$ converges to E_∞.

Since the pure spinor space is the total space of a line bundle, subsection 2.7.1, then we define the spectral sequence of a fiber bundle.

A.2.1 The Spectral Sequence of a Fiber Bundle

Let $\pi : E \to M$ be a fiber bundle with fiber F over a manifold M. Indeed, given a good cover $\underline{U} = \{U_\alpha\}$ of M, i.e every intersection of the patches U_α is always homeomorphic to any $\mathbb{R}^n$, then $\pi^{-1}\underline{U}$ is a cover on E and we can from the double complex

$$K^{p,q} = C^p(\pi^{-1}\underline{U}, \Omega^q) = \bigoplus_{\alpha_0 < \ldots < \alpha_p} \Omega^q(\pi^{-1}U_{\alpha_0<\ldots<\alpha_p}) \ . \tag{A.24}$$

where $\Omega^q(\pi^{-1}U_{\alpha_0<\ldots<\alpha_p})$ is the abelian group of q-form (real forms instead of holomorphic forms, see section 2.4).
We define the term E_1 as the de-Rham cohomology of $\Omega^*(\pi^{-1}U_{\alpha_0<\ldots<\alpha_p})$, i.e

$$E_1^{p,q} = H_{\mathrm{d}}^{p,q}(E) = \bigoplus_{\alpha_0<\ldots<\alpha_p} H^q(\pi^{-1}U_{\alpha_0<\ldots<\alpha_p}) = C^p(\underline{U}, \mathcal{H}^q), \tag{A.25}$$

where d means exterior derivative and $\mathcal{H}^q$ is the abelian group $\mathcal{H}^q(U) = H^q(\pi^{-1}U)$. Since $\underline{U}$ is a good cover then the abelian group $\mathcal{H}^q$ is simply $\mathcal{H}^q(F) = H^q(F)$.

From the previous definition d_1 is a nilpotent operator such that (see (A.22)) $d_1 : E_1^{p,q} \to E_1^{p+1,q}$, so in this case we define d_1 as the Čech operator $d_1 = \delta$ on $E_1^{p,q}$. Therefore the E_2 term is

$$E_2^{p,q} = H_\delta H_{\mathrm{d}}^{p,q}(E) = H_\delta^p(\underline{U}, \mathcal{H}^q) = H^p(M, H^q(F)) \tag{A.26}$$

Note that if $\eta \in E_2^{p,q}$ then we have

$$\mathrm{d}\eta_{\alpha_0\ldots\alpha_p} = 0, \qquad (\delta\eta)_{\alpha_0\ldots\alpha_{p+1}} = \mathrm{d}\, c_{\alpha_0\ldots\alpha_{p+1}}$$

where $c_{\alpha_0\ldots\alpha_{p+1}}$ is some $(q-1)$-form in $\Omega^q(\pi^{-1}U_{\alpha_0\ldots\alpha_{p+1}})$. Since $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ then we define d_2 as the Čech operator applied to $c_{\alpha_0\ldots\alpha_{p+1}}$, so

$$E_3^{p,q} = H_\delta H_\delta H_{\mathrm{d}}^{p,q}(E). \tag{A.27}$$

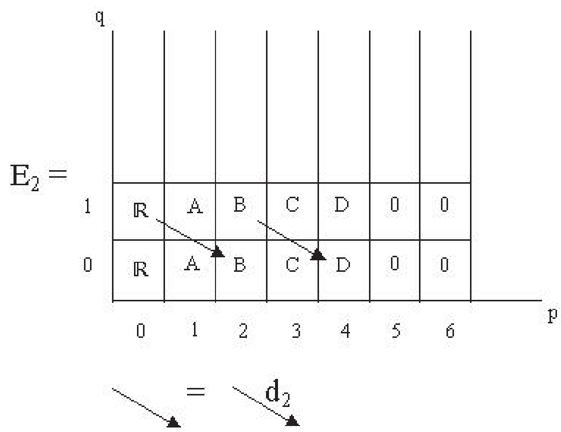
In the cases of our interest the operator $d_3 = d_4 = ... = 0$, therefore $E_3 = E_4 = ...$ and

$$E_3 = H^*_{DR}(E) \ , \qquad H^i(E) = \bigoplus_{p+q=i} E_3^{p,q}. \tag{A.28}$$

Now we give a simple example.
Example. To compute the de-Rham cohomology of $\mathbb{C}P^2$ from the fibration

$$\begin{array}{ccc} S^1 & \longrightarrow & S^5 \\ & & \Big\downarrow \pi \\ & & \mathbb{C}P^2 \end{array} \tag{A.29}$$

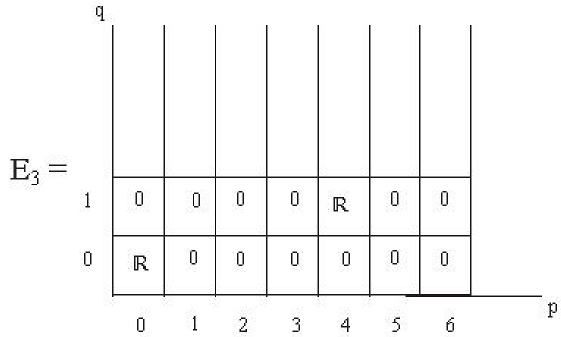
As the de-Rham cohomology of S^1 is $H^0(S^1) = H^1(S^1) = \mathbb{R}$, then the term E_2 is given by the table $E_2^{p,q} = H^p(\mathbb{C}P^2, H^q(S^1))$



Since d_3 moves down two steps, then $d_3 = 0$. Similarly

$$d_4 = d_5 = ... = 0$$

So the spectral sequence degenerates at the E_3 term and $E_3 = E_4 = ... = E_\infty = H^*(S^5)$. As the de-Rham cohomology of S^5 is well known, $H^0(S^5) = H^5(S^5) = \mathbb{R}$, then the E_3 table is

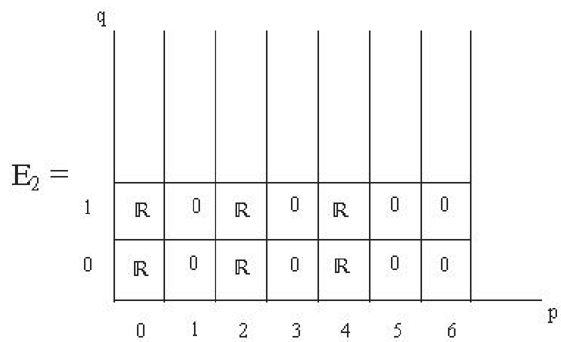


This means

$$d_2 : \quad \mathbb{R} \to B, \quad B \to D, \tag{A.30}$$

$$0 \to A, \quad A \to C, \quad C \to 0\,, \tag{A.31}$$

must all be isomorphisms. It follows that



Therefore the de-Rham cohomology of $\mathbb{C}P^2$ is
$H^0(\mathbb{C}P^2) = H^2(\mathbb{C}P^2) = H^4(\mathbb{C}P^2) = \mathbb{R}$.

Appendix B

Appendix

B.1 Some Simple Examples

B.1.1 The Pure Spinor Condition in the $U(5)$ decomposition

We will give an example of a point $p \in PS$, for which in the $U(5)$ decomposition, is necessary to consider both conditions $\chi^a = 0$ and $\zeta_a = 0$ in order to have a well defined tangent space at p.

Consider for instance the point $p = (\lambda^+ = 0, \lambda_{ab} = 0, \lambda^a = \delta^{1a})$ in the pure spinor space. Then, the gradient vectors $V^a = (\lambda^a, -\frac{1}{4}\epsilon^{abcde}\lambda_{bc}, \lambda^+\delta^{ab})$, which generate the holomorphic tangent space to the cone given by

$$\chi^a \equiv \lambda^+\lambda^a - \frac{1}{8}\epsilon^{abcde}\lambda_{bc}\lambda_{de} = 0, \quad a, b, c, d, e = 1, ..., 5,$$

do not generate a tangent space of complex dimension 11 at the point p. This is because $V^i = (0, ..., 0)$ for $i = 2, .., 5$, i.e only V^1 is different from zero at p, which means that p is a singular point* of the space $\chi^a = 0$, $a = 1, ..., 5$. So $\chi^a = 0$ does not describe completely the pure spinor space since actually PS only has one singular point: $\lambda^\alpha = 0$. For that reason, we must consider the rest of the pure spinor equations

$$\zeta_a = \lambda^b\lambda_{ba} = 0, \qquad a, b = 1, ..., 5. \tag{B.1}$$

Note that p is actually a point in the pure spinor space since it satisfies both $\chi^a = 0$ and $\zeta_a = 0$. In contrast, there exists points which do not satisfy simultaneously both set of equations. To the five equations $\zeta_a = 0$ corresponds five gradient vectors $A_a = (0, \lambda^b\delta^c_a - \lambda^c\delta^b_a, \lambda_{ba}) = (0, \lambda^{[b}\delta^{c]}_a, \lambda_{ba})$. Therefore at the point p we have in addition to V^1, four linearly independent vectors

$$A_i = (0, \lambda^{[1}\delta^{j]}_i, 0), \qquad i, j = 2, 3, .., 5, \tag{B.2}$$

*We say that p is a singular point of PS (or any space) if and only if it is not possible to define an unique tangent space in p with the same dimension of PS.

where $[1j]$ are the components $([12],[13],...,[15])$ of such vectors and we have a well defined tangent space at p. Summarizing, with this particular example for the point p, we argued the necessity of considering the conditions $\zeta_a = 0$ and now we have at every point of the pure spinor space, except for the origin, a tangent space of complex dimension 11. In other words, the pure spinor space without the origin is a smooth manifold embedded in $\mathbb{C}^{16}$. Note however that for $\lambda^+ \neq 0$, the five vectors V^a's are linearly independent and the solutions of the equations $\chi^a = 0$ satisfy trivially the equations $\zeta_a = 0$. Therefore, when $\lambda^+ \neq 0$ the five equations $\chi^a = 0$ are enough to describe the pure spinor space.

B.1.2 Another Cover For The Pure Spinor Space

We argued that the constant spinors C^I's given in (3.12) are not a good choice because the intersection of the hypersurfaces $\{C^I\lambda = 0\}, I = 1,...,11$ is the non compact space $\mathbb{C}^5$. This means that the union of the patches $U_I = PS \setminus \{C^I\lambda = 0\}$, where the scattering amplitude is supported, is not the whole pure spinor space, i.e

$$U_1 \cup ... \cup U_{11} = PS \setminus \mathbb{C}^5. \tag{B.3}$$

Then one question arises: Is it possible to complete the patches U_I in such a way that they form a cover of the $PS \setminus \{0\}$ space? Obviously the answer is positive. Here we show what is the difficulty for completing the patches for the tree level scattering amplitude.

In [52] it was proposed the cover for the pure spinor space $\mathcal{U} = \{U_\alpha\}$, $\alpha = 1,...,16$, with the patches U_α's given by

$$U_\alpha = PS \setminus \mathcal{D}_\alpha, \qquad \mathcal{D}_\alpha \equiv \{\lambda^\alpha \in PS : \lambda^\alpha = 0\}. \tag{B.4}$$

Clearly $\mathcal{U}$ is a cover of the pure spinor space without the origin

$$\bigcup_{\alpha=1}^{16} U_\alpha = U_1 \cup ... \cup U_{16} = PS \setminus \{0\}. \tag{B.5}$$

So, we can define the following picture operators

$$Y^\alpha = \frac{\theta^\alpha}{\lambda^\alpha}. \tag{B.6}$$

The first difficulty here is that there are 16 PCO's instead of 11, however this is not really a problem. Note that in the $U(5)$ decomposition, i.e $\lambda^\alpha = (\lambda^+, \lambda_{ab}, \lambda^a)$, $a, b = 1,...,5$ and $\lambda_{ab} = -\lambda_{ba}$, and choosing the picture operators

$$Y^+ = \frac{\theta^+}{\lambda^+} \quad \text{and} \quad Y_{ab} = \frac{\theta_{ab}}{\lambda_{ab}}, \tag{B.7}$$

we fall in the first example of the subsection 3.2.2, with the difference that now we have a cover for $PS \setminus \{0\}$. So the tree level scattering amplitude is given by

$$\mathcal{A} = \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \prod_{i=1}^{11} \frac{\theta^i}{\lambda^i} \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \tag{B.8}$$

where

$$\frac{\theta^1}{\lambda^1} \equiv \frac{\theta^+}{\lambda^+}, \quad \frac{\theta^2}{\lambda^2} \equiv \frac{\theta_{12}}{\lambda_{12}}, \cdots, \frac{\theta^{11}}{\lambda^{11}} \equiv \frac{\theta_{45}}{\lambda_{45}} \tag{B.9}$$

and the cycle Γ is given by $\Gamma = \{\lambda^\alpha \in PS : |\lambda^i| = \varepsilon^i, i = 1, ..., 11\}$, $\varepsilon^i \in \mathbb{R}^+$. Now we must verify if (B.8) is a physical amplitude, i.e if it is $Q_T = Q + \delta$ closed.

In the same way as in (3.82) it is not hard to show that (B.8) is Q closed. Then now we must show that the amplitude (B.8) is δ closed. Since in this case there are 16 patches then the analysis can not be similar to the one presented in the subsection 5. Acting with the δ operator in (B.8) we get

$$(\delta\mathcal{A})^{1,\ldots,11,j} = \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \left(\prod_{i=2}^{11} \frac{\theta^i}{\lambda^i} \frac{\theta^j}{\lambda^j} - \prod_{i=3}^{11} \frac{\theta^1}{\lambda^1} \frac{\theta^j}{\lambda^j} \frac{\theta^i}{\lambda^i} + \ldots + \prod_{i=1}^{11} \frac{\theta^i}{\lambda^i} \right) \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta), \tag{B.10}$$

where j is any number from 12 to 16. Naively (B.10) can be written as

$$(\delta\mathcal{A})^{1,\ldots,11,j} = \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \, Q \left(\prod_{i=1}^{11} \frac{\theta^i}{\lambda^i} \frac{\theta^j}{\lambda^j} \right) \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta), \tag{B.11}$$

nevertheless that is not true. In the subsection 3.3.1 we said that the scattering amplitude also depends of the homology class of the cycle Γ. Since the computation (B.11) has 12 Čech labels and Γ is a 11-cycle we need to be careful. From (B.10) we can see that just the term

$$\int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \prod_{i=1}^{11} \frac{\theta^i}{\lambda^i} \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \tag{B.12}$$

contributes, since the other terms vanish because the cycle $|\lambda^j| = \varepsilon^j$ is not in Γ. Therefore the scattering amplitude (B.8) is not physical.

Actually, we have shown that the cycle Γ is a trivial element of the homology group $H_{11}(PS \setminus \mathcal{D})$, where $\mathcal{D} = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_{16}$. This is because the intersection $\mathcal{D}_1 \cap \ldots \cap \mathcal{D}_{11}$ is $\mathbb{C}^5$, so the difficulty of using the cover $\mathcal{U}$ and the PCO's (B.6) is to get a well defined cycle Γ such that we can write $(\delta\mathcal{A})^{1,..,11,j}$ like in (B.11), i.e a non trivial element of the homology group $H_{11}(PS \setminus \mathcal{D})$. Note that if we add to the cover $\underline{U} = \{U_I\}, I = 1, .., 11$, where the patches U_I's are given in (3.55), more patches, then there is not problem. The reason is simple, since the condition $D_1 \cap \ldots \cap D_{11} = \{0\}$, for the D_I's given in (3.55), then the cycle Γ (3.20) will always be a non trivial element of the homology class, so applying the δ operator to the amplitude we get something equal to (B.11).

In conclusion, for tree level scattering amplitude with 3 unintegrated vertex operators and the remaining integrated, it is sufficient to work with 11 patches such that they cover the pure spinor space without the origin $PS \setminus \{0\}$.

B.1.3 The Čech-Dolbeault Correspondence for Pure Spinor in $d = 4$

Our next simple example is the pure spinor space in $d = 4$ dimensions, i.e $PS = \mathbb{C}^2$. We choose the coordinates $\lambda^a = (\lambda^1, \lambda^2)$ and consider the integral

$$I = \int_\Gamma \frac{\mathrm{d}(C^1\lambda) \wedge \mathrm{d}(C^2\lambda)}{(C^1\lambda)(C^2\lambda)} = \int_\Gamma \psi, \tag{B.13}$$

where $C^i\lambda = C^i_a\lambda^a$, $\det(C^i_a) \neq 0$ and Γ is given by $\Gamma = \{\lambda^a \in \mathbb{C}^2 : |C^i\lambda| = \varepsilon^i\}$, $\varepsilon^i \in \mathbb{R}^+$. We can write (B.13) as

$$I = \int_\Gamma [\mathrm{d}\lambda] \frac{\epsilon^{ab}C^1_aC^2_b}{(C^1\lambda)(C^2\lambda)}, \tag{B.14}$$

where $[\mathrm{d}\lambda] = (1/2)\epsilon_{ab}\mathrm{d}\lambda^a \wedge \mathrm{d}\lambda^b = \mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2$.
Note that $\mathbb{C}^2$ can be seen as the total space of the universal line bundle $\mathcal{O}(-1)$ over $\mathbb{C}P^1$, i.e $\lambda^a = \gamma\tilde{\lambda}^a$ where γ is the fiber and $\tilde{\lambda}^a$ are the coordinates of $\mathbb{C}P^1$. So, without loss of generality we choose $\Gamma = \{\lambda^a \in \mathcal{O}(-1) : |\gamma| = \varepsilon, |C^1\tilde{\lambda}| = \varepsilon^1$, where $\tilde{\lambda}^a \in \mathbb{C}P^1\}\, \varepsilon, \varepsilon^1 \in \mathbb{R}^+$ (as in the sub-subsection 3.3.2) and the measure $[\mathrm{d}\lambda]$ is given like in reference [18] by $[\mathrm{d}\lambda] = \gamma\mathrm{d}\gamma \wedge [\mathrm{d}\tilde{\lambda}]$, where $[\mathrm{d}\tilde{\lambda}] = \epsilon_{ab}\mathrm{d}\tilde{\lambda}^a\tilde{\lambda}^b$ is the measure for the twistor space in $d = 4$ [22]. Then, integrating γ we get

$$\int_\Gamma [\mathrm{d}\lambda] \frac{\epsilon^{ab}C^1_aC^2_b}{(C^1\lambda)(C^2\lambda)} = \int_\Gamma \frac{\mathrm{d}\gamma}{\gamma} \wedge [\mathrm{d}\tilde{\lambda}] \frac{\epsilon^{ab}C^1_aC^2_b}{(C^1\tilde{\lambda})(C^2\tilde{\lambda})} = (2\pi i) \int_{|C^1\tilde{\lambda}|=\varepsilon_1} [\mathrm{d}\tilde{\lambda}] \frac{\epsilon^{ab}C^1_aC^2_b}{(C^1\tilde{\lambda})(C^2\tilde{\lambda})}, \tag{B.15}$$

where the right hand side has the same form as the Green's function for the massless scalar field in $d = 4$ using the twistor language [22]. This result in $d = 4$ is analogous to the one obtained in $d = 10$, see (3.42).
Now, using the partition of unity

$$\rho_i = \frac{|C^i\lambda|^2}{(|C^1\lambda|^2 + |C^2\lambda|^2)}, \qquad i = 1, 2 \tag{B.16}$$

subordinated to the cover $\mathcal{U} = \{U_1, U_2\}$, where

$$U_i = \mathbb{C}^2 \setminus \{C^i\lambda = 0\}, \qquad i = 1, 2,$$

we find the Dolbeault cocycle corresponding to ψ. Note that from the condition $\det(C^i_a) \neq 0$ then $\{C^1\lambda = 0\} \cap \{C^2\lambda = 0\} = \{0\}$, so we get $\mathbb{C}^2 \setminus \{0\} = U_1 \cup U_2$. Since ψ is a (2,0) holomorphic form over $U_1 \cap U_2$ then ψ is 1-Čech cochain, i.e

$\psi \in C^1(\mathfrak{U}, \Omega^2)$, where $\Omega^2(\mathbb{C}^2 \setminus \{0\})$ is the abelian group of the (2,0) holomorphic forms over $\mathbb{C}^2 \setminus \{0\}$. So, using (3.65), the Dolbeault cocycle corresponding to $\psi \equiv \psi_{12} = -\psi_{21}$ is given by

$$\eta_\psi = \sum_{\alpha,\beta=1}^{2} \psi_{\alpha\beta}\rho_\alpha \wedge \bar{\partial}\rho_\beta = \psi_{12}\rho_1 \wedge \bar{\partial}\rho_2 + \psi_{21}\rho_2 \wedge \bar{\partial}\rho_1 = \psi_{12} \wedge \bar{\partial}\rho_2 \tag{B.17}$$

where $1, 2$ are the Čech labels. Replacing ψ_{12} and ρ_2 in η_ψ we get

$$\eta_\psi = \frac{\mathrm{d}(C^1\lambda) \wedge \mathrm{d}(C^2\lambda) \wedge \left[(\bar{C}^1\overline{\lambda})\mathrm{d}(\bar{C}^2\overline{\lambda}) - (\bar{C}^2\overline{\lambda})\mathrm{d}(\bar{C}^1\overline{\lambda})\right]}{(|C^1\lambda|^2 + |C^2\lambda|^2)^2}. \tag{B.18}$$

Note that this (2,1)-form is global on $\mathbb{C}^2 \setminus \{0\}$.
Therefore from the Čech-Dolbeault correspondence we have

$$\int_\Gamma \psi_{12} = \int_{S^3} \eta_\psi|_{S^3}, \tag{B.19}$$

where S^3 is the sphere $|\lambda^1|^2 + |\lambda^2|^2 = r^2$, $r \in \mathbb{R}^+$. Since S^3 is a $U(1)$-line bundle over $\mathbb{C}P^1$ space then we can write η_ψ in the S^3 coordinates

$$\lambda^a = r\, e^{i\theta}(1, u),$$

where $e^{i\theta}$ parametrizes the fiber $U(1)$, u parametrizes the $\mathbb{C}P^1$ space and r is the size of S^3. So,

$$\eta_\psi|_{S^3} = i\frac{|\epsilon^{ab}C_a^1 C_b^2|^2}{(|C_1^1 + C_2^1\, u|^2 + |C_1^2 + C_2^2\, u|^2)^2}\mathrm{d}\theta \wedge \mathrm{d}u \wedge \mathrm{d}\bar{u}. \tag{B.20}$$

Note that the constant r does not appear and the $U(1)$ part is decoupled. Therefore we can perform a global transformation from $\mathbb{C}P^1 \to \mathbb{C}P^1$ to eliminate the C^I's constants. This transformation is known as the Möbius transformation

$$v = \frac{C_1^1 + C_2^1\, u}{C_1^2 + C_2^2\, u}, \quad \text{where} \quad \begin{pmatrix} C_1^1 & C_2^1 \\ C_1^2 & C_2^2 \end{pmatrix} \in GL(2, \mathbb{C}). \tag{B.21}$$

With this transformation we obtain

$$\eta_\psi|_{S^3} = i\frac{1}{(1 + v\bar{v})^2}\mathrm{d}\theta \wedge \mathrm{d}v \wedge \mathrm{d}\bar{v}. \tag{B.22}$$

(B.22) is the $d = 4$ equivalent to (5.11) for pure spinors in $d = 10$ and $\eta_\psi|_{S^3}$ is a generator of the de-Rham cohomology group $H^3_{DR}(S^3) = \mathbb{C}$ in coordinates. Integrating by $\mathrm{d}\theta$ we have the following equality

$$\int_{|C^1\tilde{\lambda}|=\varepsilon_1} [\mathrm{d}\tilde{\lambda}]\frac{\epsilon^{ab}C_a^1 C_b^2}{(C^1\tilde{\lambda})(C^2\tilde{\lambda})} = \int_{\mathbb{C}^2} \frac{1}{(1 + v\bar{v})^2}\mathrm{d}v \wedge \mathrm{d}\bar{v} = (2\pi i)\int_{\mathbb{C}P^1} H, \tag{B.23}$$

where the hyperplane class H is written locally as $H = (1/(2\pi i))(1 + v\bar{v})^{-2}\mathrm{d}v \wedge \mathrm{d}\bar{v}$ [32]. So (B.23) is just $(2\pi i)$ times the degree of the projective complex space $\mathbb{C}P^1$, which is one.

B.1.4 Global Integrals

Now we want to give a simple example with the aim to explore the global definition of the degree of a hypersurface. Let us consider the following cone in $\mathbb{C}^4$

$$\chi \equiv z_1 z_2 - z_3 z_4 = 0 \tag{B.24}$$

and the integral

$$I = \int_\Gamma \frac{\mathrm{d}f^1 \wedge \mathrm{d}f^2 \wedge \mathrm{d}f^3}{f^1 f^2 f^3}, \tag{B.25}$$

where $f^i = C^i Z = C^i_1 z_1 + C^i_2 z_2 + C^i_3 z_3 + C^i_4 z_4$ and $\Gamma = \{Z \in \mathbb{C}^4 : \chi = 0 \text{ and } ||f^i|| = \varepsilon_i\}$, $\varepsilon_i \in \mathbb{R}^+$. We choose the C^i's in a similar way to (3.12), i.e, $f^1 = z_1$, $f^2 = z_2$, $f^3 = z_3$.

Note that the intersection

$$\{\tilde{f}^1 = 0\} \cap \{\tilde{f}^2 = 0\} \cap \{\tilde{\chi} = 0\}\Big|_{\mathbb{C}P^3} = \{[0,0,1,0],[0,0,0,1]\}$$

where $\{\tilde{f}^i = 0\} \equiv \{f^i = 0\}/\sim$ and the equivalence relation is given by $Z \sim cZ$, $c \in \mathbb{C}^*$. The same is true for $\tilde{\chi}$. This means that the degree of the smooth manifold $\tilde{\chi} = 0$ embedded in $\mathbb{C}P^3$ is deg($\tilde{\chi} = 0$)=2. So we would expect that (B.25) will be $(2\pi i)^3 2$ from the discussion of the sub-subsection 3.3.4.

Now, it is important to note that the intersection

$$\{f^1 = 0\} \cap \{f^2 = 0\} \cap \{f^3 = 0\} \cap \{\chi = 0\}\Big|_{\mathbb{C}^4} = \mathbb{C},$$

which, as we will explain, implies that the integral (B.25) is not well defined. Replacing the f^i's in (B.25) we have an integral like in $\mathbb{C}^3$

$$\int_{|z_i|=\varepsilon_i} \frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3}{z_1 z_2 z_3} = (2\pi i)^3, \tag{B.26}$$

where we have lost all the information about the cone, in fact we are in one chart. If we want to obtain global information, we must write the integral in the following way

$$I = \frac{1}{(2\pi i)} \int_R \frac{\mathrm{d}f^1 \wedge \mathrm{d}f^2 \wedge \mathrm{d}f^3 \wedge \mathrm{d}\chi}{f^1 f^2 f^3 \chi} = \frac{1}{(2\pi i)} \int_R \frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}(z_1 z_2 - z_3 z_4)}{z_1 z_2 z_3 (z_1 z_2 - z_3 z_4)}, \tag{B.27}$$

where R is given by $R = \{Z \in \mathbb{C}^4 : ||f^i|| = \varepsilon_i, |\chi| = \varepsilon\}$. Integrating first z_1 and then z_2, z_3 and z_4, we would obtain as a result $(2\pi i)^3$. Nevertheless, note that the pole f^3 is eliminated and it should be recovered from χ. This implies that the cycles $|f^3| = \varepsilon_3$ and $|\chi| = \varepsilon$ were mixed. To understand this better, let us first integrate over the cycle $|f^3| = \varepsilon_3$ in (B.27). We will obtain

$$\begin{aligned} \frac{1}{(2\pi i)} \int_R \frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}(z_1 z_2 - z_3 z_4)}{z_1 z_2 z_3 (z_1 z_2 - z_3 z_4)} &= \frac{1}{(2\pi i)} \int_R \frac{\mathrm{d}z_3 \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge (-z_3)\mathrm{d}z_4}{z_1 z_2 z_3 (z_1 z_2 - z_3 z_4)} \\ &= \frac{-1}{(2\pi i)} \int_R \frac{\mathrm{d}z_3 \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_4}{z_1 z_2 z_3 (\frac{z_1 z_2}{z_3} - z_4)} \end{aligned} \tag{B.28}$$

so we get an infinite in the denominator and the integral is zero. Therefore we have a contraction and (B.25) is not well defined for $f^i = z_i$, $i = 1, 2, 3$. This contraction comes from the fact that the integral is not well defined globally for those f^i's, i.e changing the order in which we compute the integral (B.27) is equivalent to a change of chart in the cone.

In the pure spinor formalism the f^I's, given by the constant spinors C^I's (3.12), have the same problem. Although we can not do the same trick as (B.27), because the constraints (3.11) $\chi_a = 0$ do not describe the whole pure spinor space, it can be useful to understand this more complicated problem. For the constrains $\chi_a = 0$ we have

$$I \;=\; \int_R \frac{(\mathrm{d}f^1)\wedge ... \wedge(\mathrm{d}f^{11})\wedge \mathrm{d}(\chi_1)\wedge ... \wedge \mathrm{d}(\chi_5)}{f^1 ... f^{11}(\chi_1)...(\chi_5)} \tag{B.29}$$

where R goes around every pole. Integrating first by the cycle $|\lambda^+| = \varepsilon$ we get an infinite in the denominator, just as in the previous example.

Now, changing $f^3 = z_3$ by $f^3 = z_3 - z_4$ in the example of the cone $\chi = z_1 z_2 - z_3 z_4 = 0$, we get the intersection

$$\{f^1 = 0\} \cap \{f^2 = 0\} \cap \{f^3 = 0\} \cap \{\chi = 0\}\Big|_{\mathbb{C}^4} = \{0\}$$

with multiplicity $m_{\{0\}} = 2$, which comes from the equation $z_4^2 = 0$. So, we have the integral

$$I = \frac{1}{(2\pi i)}\int_R \frac{\mathrm{d}f^1\wedge \mathrm{d}f^2 \wedge \mathrm{d}f^3 \wedge \mathrm{d}\chi}{f^1 f^2 f^3 \chi} = \frac{1}{(2\pi i)}\int_R \frac{\mathrm{d}z_1\wedge \mathrm{d}z_2\wedge \mathrm{d}(z_3 - z_4)\wedge \mathrm{d}(z_1 z_2 - z_3 z_4)}{z_1 z_2 (z_3 - z_4)(z_1 z_2 - z_3 z_4)}. \tag{B.30}$$

This integral does not have any problems and its result is the expected $(2\pi i)^3 2$ (which was explained in the sub-subsection 3.3.4 and matches with the Bezout theorem [94]).

B.2 Proof of the Identity $[\mathbf{d}\tilde{\lambda}] = \mathbf{d}u_{12}\wedge ... \wedge \mathbf{d}u_{45}$.

Let us give again the statement that we want to proof.

If $\tilde{\lambda}^\alpha$ is an element of the projective pure spinors space in 10 dimensions, i.e. if $\tilde{\lambda}^\alpha \in SO(10)/U(5)$, then the integration measure $[\mathrm{d}\tilde{\lambda}]$ defined by [22]

$$[\mathrm{d}\tilde{\lambda}](\tilde{\lambda}\gamma^m)_{\alpha_1}(\tilde{\lambda}\gamma^n)_{\alpha_2}(\tilde{\lambda}\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} = \frac{2^3}{10!}\epsilon_{\alpha_1...\alpha_5\beta_1...\beta_{11}}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge...\wedge\mathrm{d}\tilde{\lambda}^{\beta_{10}}\tilde{\lambda}^{\beta_{11}}, \tag{B.31}$$

written in the parametrization $\tilde{\lambda}^\alpha = (\tilde{\lambda}^+, \tilde{\lambda}_{ab}, \tilde{\lambda}^a) = (1, u_{ab}, \frac{1}{8}\epsilon^{abcde}u_{bc}u_{de})$ is

$$[\mathrm{d}\tilde{\lambda}] = \mathrm{d}u_{12}\wedge ... \wedge \mathrm{d}u_{45}. \tag{B.32}$$

Proof

Since $SO(10)/U(5)$ is a complex manifold we can write an anti-holomorphic measure as [32]

$$[\mathrm{d}\bar{\tilde{\lambda}}](\bar{\tilde{\lambda}}\gamma^m)^{\alpha_1}(\bar{\tilde{\lambda}}\gamma^n)^{\alpha_2}(\bar{\tilde{\lambda}}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5} = \frac{2^3}{10!}\epsilon^{\alpha_1...\alpha_5\beta_1...\beta_{11}}\mathrm{d}\bar{\tilde{\lambda}}_{\beta_1}\wedge...\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\beta_{10}}\bar{\tilde{\lambda}}_{\beta_{11}}, \quad \text{(B.33)}$$

or in a more appropriate way as

$$[\mathrm{d}\bar{\tilde{\lambda}}] = \frac{1}{2^3 5!10!}\frac{1}{(\tilde{\lambda}\bar{\tilde{\lambda}})^3}(\tilde{\lambda}\gamma^m)_{\alpha_1}(\tilde{\lambda}\gamma^n)_{\alpha_2}(\tilde{\lambda}\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\epsilon^{\alpha_1...\alpha_5\delta_1...\delta_{11}}\mathrm{d}\bar{\tilde{\lambda}}_{\delta_1}\wedge...\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\delta_{10}}\bar{\tilde{\lambda}}_{\delta_{11}}, \quad \text{(B.34)}$$

where $\bar{\tilde{\lambda}}_\alpha = (\bar{\tilde{\lambda}}_+, \bar{\tilde{\lambda}}^{ab}, \bar{\tilde{\lambda}}_a) = (1, \bar{u}^{ab}, \frac{1}{8}\epsilon_{abcde}\bar{u}^{bc}\bar{u}^{de})$. From (3.38) and (B.34) it is simple to see that

$$\begin{aligned}
[\mathrm{d}\tilde{\lambda}]\wedge[\mathrm{d}\bar{\tilde{\lambda}}] &= \frac{1}{5!(10!)^2}\frac{1}{(\tilde{\lambda}\bar{\tilde{\lambda}})^3}\epsilon_{\alpha_1...\alpha_5\beta_1...\beta_{11}}\epsilon^{\alpha_1...\alpha_5\delta_1...\delta_{11}}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge...\wedge\mathrm{d}\tilde{\lambda}^{\beta_{10}}\tilde{\lambda}^{\beta_{11}}\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\delta_1}\wedge...\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\delta_{10}}\bar{\tilde{\lambda}}_{\delta_{11}} \\
&= \frac{1}{(10!)^2}\frac{1}{(\tilde{\lambda}\bar{\tilde{\lambda}})^3}\delta^{\delta_1}_{[\beta_1}\delta^{\delta_2}_{\beta_2}...\delta^{\delta_{11}}_{\beta_{11}]}\tilde{\lambda}^{\beta_{11}}\bar{\tilde{\lambda}}_{\delta_{11}}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge...\wedge\mathrm{d}\tilde{\lambda}^{\beta_{10}}\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\delta_1}\wedge...\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\delta_{10}} \\
&= \frac{1}{10!}\frac{1}{(\tilde{\lambda}\bar{\tilde{\lambda}})^2}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge...\wedge\mathrm{d}\tilde{\lambda}^{\beta_{10}}\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\beta_1}\wedge...\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\beta_{10}} \\
&\quad -\frac{10}{10!}\frac{1}{(\tilde{\lambda}\bar{\tilde{\lambda}})^3}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge...\wedge\mathrm{d}\tilde{\lambda}^{\beta_9}\wedge\bar{\tilde{\lambda}}_{\alpha_1}\mathrm{d}\tilde{\lambda}^{\alpha_1}\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\beta_1}\wedge...\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\beta_9}\wedge\tilde{\lambda}^{\alpha_2}\mathrm{d}\bar{\tilde{\lambda}}_{\alpha_2} \\
&= -\frac{1}{10!}\left(\frac{1}{(\tilde{\lambda}\bar{\tilde{\lambda}})^2}\partial\bar{\partial}(\tilde{\lambda}\bar{\tilde{\lambda}})\wedge...\wedge\partial\bar{\partial}(\tilde{\lambda}\bar{\tilde{\lambda}})\right. \\
&\quad \left. -\frac{10}{(\tilde{\lambda}\bar{\tilde{\lambda}})^3}\partial(\tilde{\lambda}\bar{\tilde{\lambda}})\wedge\bar{\partial}(\tilde{\lambda}\bar{\tilde{\lambda}})\wedge\partial\bar{\partial}(\tilde{\lambda}\bar{\tilde{\lambda}})\wedge...\wedge\partial\bar{\partial}(\tilde{\lambda}\bar{\tilde{\lambda}})\right) \\
&= \frac{1}{10!}\left(i(\tilde{\lambda}\bar{\tilde{\lambda}})^{8/10}\partial\bar{\partial}\ln(\tilde{\lambda}\bar{\tilde{\lambda}})\right)^{10}. \qquad \text{(B.35)}
\end{aligned}$$

In the section 4.3, see [32], it was shown that

$$\frac{\omega^{10}}{10!} = \frac{\mathrm{d}u_{12}\wedge...\wedge\mathrm{d}u_{45}\wedge\mathrm{d}\bar{u}^{12}\wedge...\wedge\mathrm{d}\bar{u}^{45}}{(\tilde{\lambda}\bar{\tilde{\lambda}})^8}, \quad \text{(B.36)}$$

where

$$\omega = -\partial\bar{\partial}\ln(\tilde{\lambda}\bar{\tilde{\lambda}}) \quad \text{(B.37)}$$

and

$$(\tilde{\lambda}\bar{\tilde{\lambda}}) = (1 + \frac{1}{2}u_{ab}\bar{u}^{ab} + \frac{1}{8^2}\epsilon^{a_1b_1c_1d_1e_1}\epsilon_{a_1b_2c_2d_2e_2}u_{b_1c_1}u_{d_1e_1}\bar{u}^{b_2c_2}\bar{u}^{d_2e_2}).$$

So, we have shown that

$$[\mathrm{d}\tilde{\lambda}] = \exp(i\phi)\mathrm{d}u_{12}\wedge...\wedge\mathrm{d}u_{45} \quad \text{(B.38)}$$

where $\phi\in\mathbb{R}$ is a constant. Since this phase factor does not affect the amplitude we can set $\phi = 0$ and thus the identity was proven ■

Appendix C

Appendix

C.1 S-Duality: A short review

In this appendex, we collect the predictions of S-duality for the R^4 and D^4R^4 terms in the low energy effective action of Type IIB, and express them in a notation that will facilitate their comparison with the results from perturbation theory.

Our idea is to show which is the relationship between the overall factors of the tree-level, one and two-loop 4 point scattering amplitude.

C.1.1 S-duality prediction for R^4 and D^4R^4 terms

The massless spectrum of Type IIB string theory contains two scalar fields, the NS-NS dilaton ϕ and the R-R axion χ. The theory has a remarkable non-perturbative $SL(2,\mathbb{Z})$ S-duality symmetry under which a complex combination $\tau = \chi + ie^{-\phi} = \tau_1 + i\tau_2$ transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \tag{C.1}$$

It was conjectured in [76] that the R^4 terms, with $SL(2,\mathbb{Z})$ invariance, in the Type IIB effective action take the following form, (in the string frame),

$$S_{R^4} = \kappa_{10}^2 \int \mathrm{d}^{10}x\, \sqrt{-G}\, \mathcal{R}^4\, e^{-\frac{1}{2}\phi}\, 2\zeta(3) E_{3/2}(\tau, \bar{\tau}), \tag{C.2}$$

where $\mathcal{R}^4 = t_8 t_8 R^4$, R is the Riemann tensor and t_8 is the well-known kinematic tensor which enters both the tree-level and one-loop superstring 4-point functions and is given by [5][6][67]

$$\begin{aligned} t_8^{m_1 n_1 m_2 n_2 m_3 n_3 m_4 n_4} = -\frac{1}{2}\Big[& \left(\delta^{m_1 m_2}\delta^{n_1 n_2} - \delta^{m_1 n_2}\delta^{n_1 m_2}\right)\left(\delta^{m_3 m_4}\delta^{n_3 n_4} - \delta^{m_3 n_4}\delta^{n_3 m_4}\right) \\ & + \left(\delta^{m_2 m_3}\delta^{n_2 n_3} - \delta^{m_2 n_3}\delta^{n_2 m_3}\right)\left(\delta^{m_4 m_1}\delta^{n_4 n_1} - \delta^{m_4 n_1}\delta^{n_4 m_1}\right) \end{aligned} \tag{C.3}$$

$$+\left(\delta^{m_1m_3}\delta^{n_1n_3}-\delta^{m_1n_3}\delta^{n_1m_3}\right)\left(\delta^{m_2m_4}\delta^{n_2n_4}-\delta^{m_2n_4}\delta^{n_2m_4}\right)\Big]$$

$$+\frac{1}{2}\Big[\delta^{n_1m_2}\delta^{n_2m_3}\delta^{n_3m_4}\mathrm{d}^{n_4m_1}+\delta^{n_1m_3}\delta^{n_3m_2}\delta^{n_2m_4}\delta^{n_4m_1}+\delta^{n_1m_3}\delta^{n_3m_4}\delta^{n_4m_2}\delta^{n_2m_1}$$

$$+45\,\text{terms obtained by antisymmetrizing on each pair of indices}\Big].$$

Furthermore, $\zeta(s)$ is the Riemann zeta function, and $E_{3/2}(\tau,\bar{\tau})$ is the non-holomorphic Eisenstein series of weight $s=3/2$. For general s, the non-holomorphic Eisenstein series E_s is defined by

$$2\zeta(2s)E_s(\tau,\bar{\tau})=\sum_{(m,n)\neq(0,0)}\frac{\tau_2^s}{|m+n\tau|^{2s}} \tag{C.4}$$

and satisfies the following differential equation

$$4\tau_2^2\partial_\tau\partial_{\bar{\tau}}E_s(\tau,\bar{\tau})=s(s-1)E_s(\tau,\bar{\tau}) \tag{C.5}$$

In [7] it was shown that (C.5) is a consequence of supersymmetry. The series expression (C.4) is the unique S-duality invariant solution with these perturbative terms. In [8] the form of the R^4 term (C.2) was derived from a one-loop calculation in eleven dimensional supergravity. The expansion of the non-holomorphic Eisenstein series $2\zeta(3)E_{3/2}(\tau,\bar{\tau})$ is given in [9]

$$2\zeta(3)E_{3/2}(\tau,\bar{\tau})=2\zeta(3)e^{-\frac{3}{2}\phi}+\frac{2\pi^2}{3}e^{\frac{1}{2}\phi}+\text{non}-\text{perturbative} \tag{C.6}$$

It follows that the R^4 term in the effective action (C.2) gets perturbative contributions only from tree-level and one-loop. The vanishing of two-loop contributions was proven in [10].

In [11] a two-loop calculation in eleven dimensional supergravity was used to calculate the D^4R^4 terms explicitly in the Type IIB effective action. It was found that it has a S-duality invariant form (still expressed in the string frame),

$$S_{D^4R^4}=\left(\frac{\alpha'}{2}\right)^2\frac{\kappa_{10}^2}{2^2}\int \mathrm{d}^{10}x\sqrt{-G}\,D^4\mathcal{R}^4\,e^{\frac{1}{2}\phi}\,\zeta(5)\,E_{5/2}(\tau,\bar{\tau}) \tag{C.7}$$

For $E_{5/2}$ the expansion in large negative ϕ results in two perturbative terms, given by

$$2\zeta(5)E_{5/2}=2\zeta(5)e^{-\frac{5}{2}\phi}+\frac{\pi^4}{90}\times\frac{8}{3}\,e^{\frac{3}{2}\phi}+\text{non}-\text{perturbative} \tag{C.8}$$

From (C.7) it is clear that the two terms come from a tree-level and a two-loop contribution, but that the one-loop contribution is absent*. This is in accord with [12], where the one-loop contribution to this term in the action was shown to be zero.

*We can see this since the coefficient in e^{ϕ} is the Euler number of the Riemann surface.

C.1.2 Matching S-Duality and Perturbative predictions

From the actions (C.2), (C.7) and the expansions (C.6), (C.8) we obtain the contributions from tree-level, one and two-loop 4-point functions

$$A_0^{(R^4)} = \mathcal{R}^4\kappa_{10}^2 2\zeta(3)e^{-2\phi},$$

$$A_1^{(R^4)} = \mathcal{R}^4\kappa_{10}^2\frac{2\pi^2}{3}, \tag{C.9}$$

and

$$A_0^{(D^4R^4)} = \mathcal{R}^4(s^2+t^2+u^2)\kappa_{10}^2\frac{(\alpha')^2}{2^4}\zeta(5)e^{-2\phi},$$

$$A_2^{(D^4R^4)} = \mathcal{R}^4\kappa_{10}^2\frac{(\alpha')^2}{2^4}\frac{4\pi^4}{270}e^{2\phi}(s^2+t^2+u^2). \tag{C.10}$$

In the previous section we gave the superstring contributions at tree-level, one and two-loo 4-point. In the low energy limit the exponential function $\exp(-\alpha'\sum k_i\cdot k_j G(z_i,z_j))\approx 1$ (in order to get the R^4 and D^4R^4 contributions from one and two-loop), and the combination of the Gamma functions in the tree-level amplitude is given by[†]

$$\frac{\Gamma(-s\alpha'/4)\Gamma(-t\alpha'/4)\Gamma(-u\alpha'/4)}{\Gamma(1+s\alpha'/4)\Gamma(1+t\alpha'/4)\Gamma(1+u\alpha'/4)} = \frac{2^6}{(\alpha')^3 stu}+2\zeta(3)+\frac{\zeta(5)}{16}(\alpha')^2(s^2+t^2+u^2)+... \tag{C.11}$$

The first term arises through 1-particle reducible Feynman diagrams from the Einstein-Hilbert action. The second term in (C.11) gives the following tree-level contribution to the R^4 terms in the effective action

$$\mathcal{A}_0^{(R^4)}(\epsilon_i,k_i) = 2\pi\left(\frac{\alpha'}{2}\right)^8 C_0\, K\bar{K}\, 2\zeta(3)e^{-2\lambda}, \tag{C.12}$$

and the third term gives the following tree-level contribution to the D^4R^4 terms

$$\mathcal{A}_0^{(D^4R^4)}(\epsilon_i,k_i) = 2\pi\left(\frac{\alpha'}{2}\right)^8 C_0\, K\bar{K}\,(s^2+t^2+u^2)\frac{(\alpha')^2}{2^4}\zeta(5)e^{-2\lambda}. \tag{C.13}$$

Using the fact that the integration of the moduli space of the Torus is [66]

$$\int_{\mathcal{M}_1} \mathrm{d}\tau\wedge\mathrm{d}\bar{\tau}(\mathrm{Im}\tau)^{-2} = \frac{2\pi}{3}, \tag{C.14}$$

so the R^4 contributions from one loop superstring theory is

$$\mathcal{A}_1^{(R^4)}(\epsilon_i,k_i) = \frac{32\pi}{3}C_1\left(\frac{\alpha'}{2}\right)^8 K\bar{K}. \tag{C.15}$$

[†]Throughout, we shall omit the overall momentum conservation factor $(2\pi)^{10}\delta^{(10)}(k_1+k_2+k_3+k_4)$ when expressing the scattering amplitudes, and use the Mandelstam variables, $s=-(k_1+k_2)^2$, $t=-(k_2+k_3)^2$, and $u=-(k_1+k_3)^2$.

Now, since the property of the holomorphic $(1,0)$-forms on the Riemann surface of genus $g = 2$ (Σ_2)

$$\int_{\Sigma_2} \mathrm{d}^2 z\, \omega_I \omega_J^* = 2\mathrm{Im}\Omega_{IJ}, \tag{C.16}$$

we get

$$\prod_{i=1}^{4} \int \mathrm{d}^2 z_i |\mathcal{Y}_s|^2 = 32(s^2 + t^2 + u^2)(\det \mathrm{Im}\Omega)^2. \tag{C.17}$$

The volume of the genus two moduli space is given by [66]

$$\int_{\mathcal{M}_2} \frac{\mathrm{d}^2 \Omega_{IJ}}{(\det \mathrm{Im}\Omega)^3} = \frac{8\pi^3}{270}. \tag{C.18}$$

Therefore the D^4R^4 contribution from the two-loop is

$$\mathcal{A}_2^{(D^4R^4)}(\epsilon_i, k_i) = \frac{2^6\pi^3}{270} C_2\, e^{2\lambda} \left(\frac{\alpha'}{2}\right)^{10} K\bar{K}\, (\alpha')^2 (s^2 + t^2 + u^2). \tag{C.19}$$

Finally the superstring contributions for the R^4 and D^4R^4 terms are

$$\mathcal{A}_0^{(R^4)}(\epsilon_i, k_i) = 2\pi \left(\frac{\alpha'}{2}\right)^{8} C_0\, K\bar{K}\, 2\zeta(3) e^{-2\lambda},$$

$$\mathcal{A}_1^{(R^4)}(\epsilon_i, k_i) = \frac{32\pi}{3} C_1 \left(\frac{\alpha'}{2}\right)^{8} K\bar{K}, \tag{C.20}$$

and

$$\mathcal{A}_0^{(D^4R^4)}(\epsilon_i, k_i) = 2\pi \left(\frac{\alpha'}{2}\right)^{8} C_0\, K\bar{K}\, (s^2 + t^2 + u^2) \frac{(\alpha')^2}{2^4} \zeta(5) e^{-2\lambda},$$

$$\mathcal{A}_2^{(D^4R^4)}(\epsilon_i, k_i) = \frac{2^6\pi^3}{270} C_2\, e^{2\lambda} \left(\frac{\alpha'}{2}\right)^{10} K\bar{K}\, (\alpha')^2 (s^2 + t^2 + u^2). \tag{C.21}$$

The predictions of S-duality and superstring perturbation theory require the matching of (C.9) with (C.20), and (C.10) with (C.21). Using the conversion relation $K\bar{K} = 2^6 \mathcal{R}^4$, these matching conditions are equivalent to the following relations,

$$C_1^2 = 8\pi^2 C_0 C_2 \tag{C.22}$$

This relation is precisely the factorization condition on the two-loop 4-point function [66].

C.2 Pure spinors in lower dimensions and partition function

The aim of studying pure spinors in lower dimensions $(D = 2n < 10)$ is to have a better feeling of some algebraic properties of the pure spinors space. At the end of the appendix we make some remarks and give a nice geometric interpretation of the character of pure spinors.

We know that in $D = 4, 6, 8$ the projective pure spinors space are $\mathbb{C}P^1$, $\mathbb{C}P^3$ and a quadric variety embedded in $\mathbb{C}P^7$, respectively.

$\mathbb{C}P^1$ and $\mathbb{C}P^3$ are the trivial cases because in $D = 4, 6$ the pure spinors don't have any constraints and the pure spinors space is the simple blow-up of the origin [30] (the pure spinors space is the total space of the line bundle $\mathcal{O}(-1)$). In these cases the Kälher form of the pure spinors space is simply

$$\Omega = \partial\overline{\partial}(\lambda\overline{\lambda}), \tag{C.23}$$

where we have used the general formula (4.100)

$$\boxed{\Omega_{D=2n} = (\lambda\overline{\lambda})^{-\frac{\dim_{\mathbb{C}} PS - c_1}{\dim_{\mathbb{C}} PS}} \partial\overline{\partial}(\lambda\overline{\lambda})}$$

and the notation

$$\lambda\overline{\lambda} = \gamma\bar{\gamma}(1 + z\bar{z}), \qquad \text{for } D = 4\,, \tag{C.24}$$

$$\lambda\overline{\lambda} = \gamma\bar{\gamma}(1 + z\bar{z} + u\bar{u} + v\bar{v}), \qquad \text{for } D = 6\,, \tag{C.25}$$

where $\{z\}$ parametrize $\mathbb{C}P^1$, $\{z,\, u,\, v\}$ parametrize $\mathbb{C}P^3$, $\{\gamma\}$ is the fiber and c_1 is the first Chern class of projective pure spinors space. From [54] we can see that in $D = 4, 6$ the first Chern class of the tangent bundle over the projective pure spinors space is

$$\begin{aligned} c_1(T\mathbb{C}P^1) &= 2, \\ c_1(T\mathbb{C}P^3) &= 4 \end{aligned} \tag{C.26}$$

and it has the same value of the complex dimension of the pure spinors space ($\dim_{\mathbb{C}} PS$).

The integration measures for the pure spinors space in $D = 4, 6$ are given by

$$\frac{\Omega^2}{2!} = \omega \wedge \bar{\omega} \qquad \text{for } D = 4\,, \tag{C.27}$$

$$\frac{\Omega^4}{4!} = \omega \wedge \bar{\omega} \qquad \text{for } D = 6\,, \tag{C.28}$$

where

$$\omega = \gamma\, \mathrm{d}\gamma \wedge \mathrm{d}z \qquad \text{for } D = 4 \tag{C.29}$$

$$\omega = \gamma^3\, \mathrm{d}\gamma \wedge \mathrm{d}z \wedge \mathrm{d}u \wedge \mathrm{d}v \qquad \text{for } D = 6 \tag{C.30}$$

are the holomorphic top forms, which agree with the ones of [18]. To compute (C.26) is very easy from the following exact sequence of bundles (the Euler sequence)[94][30]

$$0 \longrightarrow \mathbb{C} \longrightarrow H^{\oplus(n+1)} \longrightarrow T\mathbb{C}P^n \longrightarrow 0\ , \tag{C.31}$$

where $\mathbb{C}$ is a trivial bundle, H is the hyperplane class and $T\mathbb{C}P^n$ is the tangent bundle on $\mathbb{C}P^n$. This sequence implies that

$$H^{\oplus(n+1)} = T\mathbb{C}P^n \oplus \mathbb{C}. \tag{C.32}$$

Therefore, the total Chern class of the tangent bundle on $\mathbb{C}P^n$ is

$$c(T\mathbb{C}P^n) = (1+H)^{n+1} \tag{C.33}$$

where we have denoted the first Chern class of the hyperplane bundle H with the same letter H. Now it is clear that $c_1(T\mathbb{C}P^n) = (n+1)H$ and that $c_n(T\mathbb{C}P^n) = (n+1)H^n$. As the Euler characteristic of a complex manifold M of complex dimension n is [94]

$$\chi(M) = \int_M c_n(TM)\ , \tag{C.34}$$

then we have that

$$\int_{\mathbb{C}P^n} H^n = 1, \tag{C.35}$$

which was used in (4.115) and (4.123). Let's apply the previous results to the pure spinors space in $D = 4$.
We know that the integration measure on the pure spinors space in $D = 4$ is

$$\frac{\Omega^2}{2!} = -\gamma\bar{\gamma}\,\mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma} \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{C.36}$$

Let's integrate the function $\exp\{-a\lambda\overline{\lambda}\}$, with $a \in \mathbb{R}^+$,

$$\begin{aligned}\int_{\mathcal{O}(-1)} [\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}]\ e^{-a\lambda\overline{\lambda}} &= -\int_{\mathbb{C}^2} \gamma\bar{\gamma}\,\mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma} \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z}\, e^{-a\gamma\bar{\gamma}(1+z\bar{z})} \\ &= \frac{\pi}{a^2 i}\int_{\mathbb{C}} \frac{2}{(1+z\bar{z})^2}\mathrm{d}z \wedge \mathrm{d}\bar{z}. \end{aligned} \tag{C.37}$$

We can see that $g_{z\bar{z}} = 2/(1+z\bar{z})^2$ is the metric of S^2 with radius 1 on a chart homeomorphic to $\mathbb{C}$. The area of a sphere with radius R is $4\pi R^2$, so the integral (C.37) is $4\pi^2/a^2$. Nevertheless we want to show how to compute the integral (C.37) using simple topological properties of the projective pure spinors space (S^2). Let's remember that the first Chern class of a complex manifold $\mathcal{M}$ is given by the expression

$$c_1(T\mathcal{M}) = \frac{i}{2\pi}\partial\overline{\partial}\ln\det(g_{i\bar{j}}), \tag{C.38}$$

so, in our example we have

$$c_1(TS^2) = \frac{2}{2\pi i}\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+z\bar{z})^2}. \tag{C.39}$$

Note that the number 2 on the numerator, which comes of the exponent of $(1+z\bar{z})^2$, is simply the first Chern class of the tangent bundle with respect to the hyperplane bundle H ($c_1(TS^2) = 2H$)[‡], hence

$$H = \frac{1}{2\pi i} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+z\bar{z})^2} \tag{C.40}$$

on the chart. Now, using (C.35) we can easily compute (C.37)

$$\int_{\mathcal{O}(-1)} [\mathrm{d}\lambda] \wedge [\mathrm{d}\overline{\lambda}]\ e^{-a\lambda\overline{\lambda}} = \frac{2\pi}{a^2} 2\pi \int_{\mathbb{C}} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{2\pi i(1+z\bar{z})^2} = \frac{4\pi^2}{a^2} \int_{\mathbb{C}P^1} H = \frac{4\pi^2}{a^2}, \tag{C.41}$$

as expected.

We can get the same result (C.35) from the partition function, for example, computing the partition function for $\mathcal{O}(-1)$ over $\mathbb{C}P^n$ in the zero level with the reducibility method [27] we have

$$Z_{\mathcal{O}(-1)}(t) = \frac{1}{(1-t)^{n+1}}. \tag{C.42}$$

Expanding around to $\epsilon = 1 - t = 0$ the most singular term is

$$\frac{1}{\epsilon^{n+1}}, \tag{C.43}$$

and by comparing with the Riemann-Roch formula (4.117) we get (C.35).

Now we discuss some aspects of intersection theory. It is clear that in $\mathbb{C}P^n$ we have a set $\{\mathbb{C}P^m\}$ with $m \leq n$ which is embedded it. It is easy to see that these $\mathbb{C}P^m$'s intersect transversally of a point [94], i.e

$$^{\#}(\mathbb{C}P^m \cdot \mathbb{C}P^{n-m}) = 1\,, \qquad m \leq n. \tag{C.44}$$

As the homology groups of $\mathbb{C}P^n$ are [31]

$$H_{2i}(\mathbb{C}P^n) = \mathbb{Z}\,, \qquad i = 1, 2, ..., n \tag{C.45}$$

then by (C.44) we can take the homology generators to be the $[\mathbb{C}P^i]$ classes. With this, we define the degree of a closed variety V of complex dimension m by

$$\deg(V) = {}^{\#}(V \cdot \mathbb{C}P^{n-m}). \tag{C.46}$$

This is a topological number because it depends only on the homology class.

Now we compute the degree for projective pure spinors in $D = 8$.The projective pure spinors space in $D = 8$ ($\mathcal{Q}_8$) is a hypersurface in $\mathbb{C}P^7$. It is given in terms of

[‡]This is the same argument by which the number 8 is in the 20-form (4.106).

homogeneous coordinates $\{\lambda^+, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{23},$ $\lambda_{24}, \lambda_{34}, \lambda_{1234}\}$ on $\mathbb{C}P^7$ as the zero locus of [14]

$$\lambda^+\lambda_{1234} - \lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{24} - \lambda_{23}\lambda_{14} = 0. \tag{C.47}$$

Since $\deg(\mathcal{Q}_8)$ is the number of points where $\mathcal{Q}_8$ and $\mathbb{C}P^1$ are intersected, if we take $\mathbb{C}P^1$ as the locus $\lambda_{12} = \lambda_{13} = \lambda_{14} = \lambda_{23} = \lambda_{24} = \lambda_{34} = 0$, the $\deg(\mathcal{Q}_8)$ will be the number of solutions of the homogeneous polynomial

$$\lambda^+\lambda_{1234} = 0. \tag{C.48}$$

The solutions of this polynomial are the points $[1, 0, 0, 0, 0, 0, 0, 0]$ and $[0, 0, 0, 0, 0, 0, 0, 1]$, therefore

$$\deg(\mathcal{Q}_8) = 2. \tag{C.49}$$

Using the partition function we get the same answer, i.e, the partition function for $\mathcal{O}(-1)$ over $\mathcal{Q}_8$ is given by [27]

$$Z_{\mathcal{Q}_8}(t) = \frac{1+t}{(1-t)^7}\,. \tag{C.50}$$

Expanding near to $\epsilon = 1 - t = 0$, the most singular term of $Z_{\mathcal{Q}_8}(t)$ is

$$\frac{2}{\epsilon^7}, \tag{C.51}$$

so, by comparing with the Riemann-Roch formula (4.117) we get

$$\int_{\mathcal{Q}_8} c_1(\mathcal{L})^6 = 2. \tag{C.52}$$

Actually this result was expected, since $\mathcal{Q}_8$ is a hypersurface given by a homogeneous polynomial of degree 2, then the first Chern class of the divisor $[\mathcal{Q}_8]$ is

$$c_1([\mathcal{Q}_8]) = 2H, \tag{C.53}$$

which is Poincaré dual to $\mathcal{Q}_8$ [94][29]. So

$$\int_{\mathcal{Q}_8} c_1(\mathcal{L})^6 = \int_{\mathcal{Q}_8} (f^*H)^6 = \int_{\mathbb{C}P^7} H^6 \wedge c_1([\mathcal{Q}_8]) = 2\int_{\mathbb{C}P^7} H^7 = 2. \tag{C.54}$$

where $f : \mathcal{Q}_8 \to \mathbb{C}P^7$ is the embeding.

We now have a geometric interpretation to the result found in [54]. In [54] it was shown that the partition function of pure spinors can be written as a rational function[§]

$$Z_{\mathcal{O}(-1)}(t) = \frac{P(t)}{Q(t)}\,, \tag{C.55}$$

[§] we are only interested in the zero level.

where $P(t)$ and $Q(t)$ are polynomials. In $D = 2n$ the $Q(t)$ polynomial has the form [54][26]

$$Q(t) = (1-t)^{\dim_{\mathbb{C}} PS}. \tag{C.56}$$

In [54] it was also shown that $Z_{\mathcal{O}(-1)}(t)$ can be written as an infinite product (*ghost−ghost*)

$$Z_{\mathcal{O}(-1)}(t) = \prod_{n=1}^{\infty}(1-t^n)^{-N_n}. \tag{C.57}$$

The N_n coefficients contain the information about the Virasoro central charge, ghost number anomaly, etc

$$\frac{1}{2}c_{\text{vir}} = \sum_n N_n\,, \tag{C.58}$$

$$a_{\text{ghost}} = \sum_n nN_n. \tag{C.59}$$

From (C.55) and (C.57) we have

$$\ln(-x)\sum_n N_n + \sum_n \ln(n)N_n + \frac{x}{2}\sum_n nN_n + \sum_{g=1}^{\infty}\frac{B_{2g}}{2g(2g)!}x^{2g}\sum_n n^{2g}N_n$$
$$= -\ln P(e^x) + \ln Q(e^x), \tag{C.60}$$

where $\{B_g\}$ are the Bernoulli numbers. Replacing (C.56) in the previous expression we get

$$\ln(1-e^x)^{\dim_{\mathbb{C}}PS} = (\dim_{\mathbb{C}}PS)\ln(-x) + \frac{\dim_{\mathbb{C}}PS}{2}x + \frac{\dim_{\mathbb{C}}PS}{24}x^2 + \ldots \quad . \tag{C.61}$$

Without loss of generality we can suppose that

$$P(e^x) = y + a\,e^x + b\,e^{2x} + c\,e^{3x} + \ldots \quad , \tag{C.62}$$

so

$$-\ln P(e^x) = -\ln P(1) - \partial_x \ln P(x)|_{x=1}\,x + \ldots.. \tag{C.63}$$

$$= -\ln P(1) - \frac{\partial_x P(x)|_{x=1}}{P(1)}x + \ldots.. \tag{C.64}$$

$$= -\ln(y+a+b+c+\ldots) - \frac{a+2\,b+3\,c+\ldots}{y+a+b+c+..}x + \ldots.. \tag{C.65}$$

$$\tag{C.66}$$

and therefore we have

$$\frac{1}{2}c_{\text{vir}} = \sum_n N_n = \dim_{\mathbb{C}}PS\,, \tag{C.67}$$

$$a_{\text{ghost}} = \sum_n nN_n = \dim_{\mathbb{C}}PS - 2\frac{\partial_x P(x)|_{x=1}}{P(1)}\,, \tag{C.68}$$

$$\ln P(1) = -\sum_n \ln(n)N_n = \ln(\deg\mathcal{Q}_{2n}), \qquad \mathcal{Q}_{2n} := SO(2n)/U(n). \tag{C.69}$$

From the Riemann-Roch formula (4.117) and by expanding (C.55) with (C.56) near to $\epsilon = 1 - t = 0$ it is clear than $\deg(\mathcal{Q}_{2n}) = P(1)$.
We know that a_{ghost} is the first Chern class of $T\mathcal{Q}_{2n}$ and that the $\deg(\mathcal{Q}_{2n})$ gives the homology class

$$[\mathcal{Q}_{2n}] = \deg(\mathcal{Q}_{2n})[\mathbb{C}P^{n(n-1)/2}]\,, \tag{C.70}$$

in others words, the $\deg(\mathcal{Q}_{2n})$ gives us the Poincaré dual class of $\mathcal{Q}_{2n}$. Noting that the homology class of $\mathcal{Q}_{2n}$ is an integer number times the homology class of $\mathbb{C}P^{n(n-1)/2}$, we can interpret $\dim_{\mathbb{C}} PS = 1 + n(n-1)/2$ as the first Chern class of $T\mathbb{C}P^{n(n-1)/2}$. Thus we have

$$c_1(T\mathbb{C}P^{n(n-1)/2}) = \sum_n N_n\,, \tag{C.71}$$

$$c_1(T\mathcal{Q}_{2n}) = \sum_n nN_n\,, \tag{C.72}$$

$$\deg(\mathcal{Q}_{2n}) = \exp(-\sum_n \ln(n)N_n) = \left(\prod_n n^{N_n}\right)^{-1}. \tag{C.73}$$

With these geometric interpretation we get a geometric constraint on the coefficients of the $P(t)$ polynomial

$$\deg(\mathcal{Q}_{2n})\left\{c_1(T\mathbb{C}P^{n(n-1)/2}) - c_1(T\mathcal{Q}_{2n})\right\} = 2\,\partial_x P(x)|_{x=1}. \tag{C.74}$$

We can also rewrite the integration measure of the pure spinors space (4.100) as

$$\Omega_{D=2n} = (\lambda\overline{\lambda})^{-\frac{c_1(T\mathbb{C}P^{n(n-1)/2}) - c_1(T\mathcal{Q}_{2n})}{c_1(T\mathbb{C}P^{n(n-1)/2})}}\partial\overline{\partial}(\lambda\overline{\lambda}) = (\lambda\overline{\lambda})^{\frac{-2\,\partial_x P(x)|_{x=1}}{\deg(\mathcal{Q}_{2n})c_1(T\mathbb{C}P^{n(n-1)/2})}}\partial\overline{\partial}(\lambda\overline{\lambda})\,, \tag{C.75}$$

where we interpret the term $\{c_1(T\mathbb{C}P^{n(n-1)/2}) - c_1(T\mathcal{Q}_{2n})\}$ as a topological deviation and find a relationship between the integration measure and the character of the pure spinors space.

C.3 Non-minimal two-loop kinematic factor

The non-minimal two-loop computation of section 5 leads to the kinematic factor

$$K = \langle(\overline{\lambda}\gamma^{abc}D)(\overline{\lambda}\gamma^{ghi}D)(\overline{\lambda}\gamma^{def}D)(\lambda\gamma_{adefg}\lambda)\big[(\lambda\gamma^b W^1)(\lambda\gamma^c W^2)(\lambda\gamma^h W^3)(\lambda\gamma^i W^4)\big]\rangle_{(-3,2)}. \tag{C.76}$$

In [62] it was shown¶ that (C.76) is proportional to $\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^s W)\mathcal{F}_{mn}\mathcal{F}_{pq}\mathcal{F}_{rs}\rangle_{(0,2)}$, the kinematic factor obtained in the minimal pure spinor formalism [49], whose

¶There is a loophole in the proof of [62] though. In that proof the terms in (C.76) which are of the form $kWWWF$ where argued to vanish after summing ovstraightforward wayer the permutations. However we show here that by using the identities of [63] those terms are actually proportional to $W\mathcal{F}\mathcal{F}\mathcal{F}$, so the conclusions of [62] still hold true.

equivalence with the RNS result of [64] was established in [50][63]. We will now evaluate all the terms in (C.76) to find the exact coefficient announced in (4.166).

To simplify the covariant computation of (C.76) we use $(\overline{\lambda}\gamma^{def}D)(\lambda\gamma_{adefg}\lambda)$ $= 48(\lambda\overline{\lambda})(\lambda\gamma^{ag}D) - 48(\lambda\gamma^{ag}\overline{\lambda})(\lambda D)$ and drop the last term because $(\lambda\gamma^m W^I)$ is BRST-closed. And for the same reason we can use $(\lambda\gamma^a\gamma^g D)$ instead of $(\lambda\gamma^{ag}D)$ in the first term. Therefore (C.76) becomes

$$K = 48\langle(\overline{\lambda}\gamma^{ghi}D)(\lambda\gamma^a\gamma^g D)(\overline{\lambda}\gamma^{abc}D)\big[(\lambda\gamma^b W^1)(\lambda\gamma^c W^2)(\lambda\gamma^h W^3)(\lambda\gamma^i W^4)\big]\rangle_{(-2,2)}. \tag{C.77}$$

The strategy to evaluate and simplify[‖] (C.77) is straightforward due to the identities obeyed by the pure spinor λ^α. One uses the SYM equation of motion for W^α in the form of

$$(\overline{\lambda}\gamma^{abc}D)(\lambda\gamma^m W^1) = \frac{1}{4}(\lambda\gamma^m\gamma^{m_1n_1}\gamma^{abc}\overline{\lambda})\mathcal{F}^1_{m_1n_1} \tag{C.78}$$

$$(\lambda\gamma^a\gamma^g D)(\lambda\gamma^m W^2) = \frac{1}{4}(\lambda\gamma^{agm_2n_2m}\lambda)\mathcal{F}^2_{m_2n_2} \tag{C.79}$$

and uses gamma matrix identities[**] in such a way as to get factors which vanish by the pure spinor property of $(\lambda\gamma^m)_\alpha(\lambda\gamma_m)_\beta = 0$. For example, one gets identities like

$$(\lambda\gamma^b\gamma^{m_1n_1}\gamma^{abc}\overline{\lambda})(\lambda\gamma^a\gamma^g D)\big[\mathcal{F}^1_{m_1n_1}(\lambda\gamma^c W^2)\big] = 48(\lambda\overline{\lambda})(\lambda\gamma^a\gamma^g D)\big[\mathcal{F}^1_{ac}(\lambda\gamma^c W^2)\big] \tag{C.80}$$

and

$$\mathcal{F}^3_{rs}(\lambda\gamma^h\gamma^{rs}\gamma^{abc}\overline{\lambda})(\lambda\gamma^a)_\alpha(\lambda\gamma^b)_\beta(\lambda\gamma^c)_\gamma =$$
$$16(\lambda\overline{\lambda})(\delta^h_b\mathcal{F}^3_{ac} - \delta^h_c\mathcal{F}^3_{ab} - \delta^h_a\mathcal{F}^3_{bc})(\lambda\gamma^a)_\alpha(\lambda\gamma^b)_\beta(\lambda\gamma^c)_\gamma. \tag{C.81}$$

Following the above steps (C.77) becomes

$$K = 576\langle(\overline{\lambda}\gamma^{ghi}D)(\lambda\gamma^a\gamma^g D)\big[\mathcal{F}^1_{ab}(\lambda\gamma^b W^2)(\lambda\gamma^h W^3)(\lambda\gamma^i W^4)$$
$$-\frac{1}{3}\mathcal{F}^3_{ab}(\lambda\gamma^b W^1)(\lambda\gamma^h W^2)(\lambda\gamma^i W^4) - \frac{1}{3}\mathcal{F}^4_{ab}(\lambda\gamma^b W^1)(\lambda\gamma^h W^2)(\lambda\gamma^i W^3) + (1\leftrightarrow 2)\big]\rangle_{(-1,2)}$$
$$-192\langle(\overline{\lambda}\gamma^{gai}D)(\lambda\gamma^a\gamma^g D)\big[\mathcal{F}^3_{bc}(\lambda\gamma^i W^4)(\lambda\gamma^b W^1)(\lambda\gamma^c W^2) + (3\leftrightarrow 4)\big]\rangle_{(-1,2)}. \tag{C.82}$$

The last line of (C.82) vanishes. To see this note that the factor inside brackets is BRST-closed, so that we can replace $(\lambda\gamma^a\gamma^g D)$ by $(\lambda\gamma^{ag}D)$. Furthermore $(\overline{\lambda}\gamma^{gai}D)(\lambda\gamma^{ga}D) = -(\overline{\lambda}\gamma^{ga}\gamma^i D)(\lambda\gamma^{ga}D) - 2(\overline{\lambda}\gamma^a D)(\lambda\gamma^{ia}D)$ and the last term vanishes when acting on $\mathcal{F}^3_{bc}(\lambda\gamma^i W^4)(\lambda\gamma^b W^1)(\lambda\gamma^c W^2)$ because $(\lambda\gamma^{ia}D) = (\lambda\gamma^i\gamma^a D) - \delta^i_a(\lambda D)$ and $(\lambda\gamma^i)_\alpha(\lambda\gamma_i)_\beta = 0$ due to the pure spinor property. Therefore by using the gamma matrix identity of

$$(\gamma^{mn})_\alpha{}^\delta(\gamma_{mn})_\beta{}^\sigma = -8\delta^\sigma_\alpha\delta^\delta_\beta - 2\delta^\delta_\alpha\delta^\sigma_\beta + 4\gamma^m_{\alpha\beta}\gamma^{\delta\sigma}_m \tag{C.83}$$

‖These kind of computations confirm the observations made long ago that pure spinors simplify the description of super-Yang-Mills theory [65].

**The package GAMMA [92] is often very useful for these manipulations.

and dropping the term proportional to the BRST charge and using momentum conservation (so that D_α and D_β effectively anti-commute) we get

$$(\overline{\lambda}\gamma^{ga}\gamma^i D)(\lambda\gamma^{ga}D) = 8(\lambda\overline{\lambda})(D\gamma^i D) + 4(\overline{\lambda}\gamma^m D)(\lambda\gamma^m\gamma^i D). \qquad \text{(C.84)}$$

The first term in the RHS of (C.84) is proportional to k^i and vanishes by momentum conservation, while the last term vanishes when acting on $\mathcal{F}^3_{bc}(\lambda\gamma^i W^4)(\lambda\gamma^b W^1)(\lambda\gamma^c W^2)$ for the same reason as explained above.

For convenience we write (C.82) as

$$K = 576K_{a_1} - 192K_{a_2} - 192K_{a_3} + (1 \leftrightarrow 2) \qquad \text{(C.85)}$$

where

$$K_{a_1} \equiv \langle(\overline{\lambda}\gamma^{ghi}D)(\lambda\gamma^a\gamma^g D)\big[\mathcal{F}^1_{ab}(\lambda\gamma^b W^2)(\lambda\gamma^h W^3)(\lambda\gamma^i W^4)\big]\rangle_{(-1,2)}$$

while K_{a_2} and K_{a_3} can be obtained by permuting the labels in K_{a_1}. Using the SYM equations of motion and a few gamma matrix identities we get

$$K_{a_1} = +\langle(\overline{\lambda}\gamma^{ghi}D)\Big[6k^1_c(\lambda\gamma^g W^1)(\lambda\gamma^c W^2)(\lambda\gamma^h W^3)(\lambda\gamma^i W^4)$$
$$-\frac{1}{4}(\lambda\gamma^{mnpqg}\lambda)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}(\lambda\gamma^h W^3)(\lambda\gamma^i W^4) - \frac{1}{4}(\lambda\gamma^{agmnh}\lambda)\mathcal{F}^1_{ac}\mathcal{F}^3_{mn}(\lambda\gamma^c W^2)(\lambda\gamma^i W^4)$$
$$-\frac{1}{4}(\lambda\gamma^{agmnh}\lambda)\mathcal{F}^1_{ac}\mathcal{F}^4_{mn}(\lambda\gamma^c W^2)(\lambda\gamma^i W^3)\Big]\rangle_{(-1,2)}. \qquad \text{(C.86)}$$

After a long and tedious computation using straightforward manipulations and identities like $(\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^I_{mn}\mathcal{F}^J_{pq} = (\lambda\gamma^{mnpqr}\lambda)\mathcal{F}^J_{mn}\mathcal{F}^I_{pq}$ and [49]

$$(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^s W^4)\big[\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs} + \mathcal{F}^3_{mn}\mathcal{F}^1_{pq}\mathcal{F}^2_{rs} + \mathcal{F}^2_{mn}\mathcal{F}^3_{pq}\mathcal{F}^1_{rs}\big] = 0 \qquad \text{(C.87)}$$

one gets

$$K_{a_1} = -\frac{1}{2}\langle k^1_m(\overline{\lambda}\gamma^{ghi}\gamma^n W^1)\mathcal{F}^2_{pq}(\lambda\gamma^{mnpqg}\lambda)(\lambda\gamma^h W^3)(\lambda\gamma^i W^4)\rangle_{(-1,2)} + (1 \leftrightarrow 2)$$
$$-\frac{1}{4}\langle\big(2\mathcal{F}^3_{rs}k^1_{[a}(\overline{\lambda}\gamma^{ghi}\gamma_{c]}W^1) + 2k^3_r(\overline{\lambda}\gamma^{ghi}\gamma^s W^3)\mathcal{F}^1_{ac}\big)(\lambda\gamma^{agrsh}\lambda)(\lambda\gamma^c W^2)(\lambda\gamma^i W^4)\rangle_{(-1,2)}$$
$$-\frac{1}{4}\langle\big(2\mathcal{F}^4_{rs}k^1_{[a}(\overline{\lambda}\gamma^{ghi}\gamma_{c]}W^1) + 2k^4_r(\overline{\lambda}\gamma^{ghi}\gamma^s W^4)\mathcal{F}^1_{ac}\big)(\lambda\gamma^{agrsh}\lambda)(\lambda\gamma^c W^2)(\lambda\gamma^i W^3)\rangle_{(-1,2)}$$
$$+\langle(\lambda\gamma^{mnpqr}\lambda)\big[\big(\mathcal{F}^1_{mn}\mathcal{F}^3_{pq}\mathcal{F}^2_{rs} - 4\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}\big)(\lambda\gamma^s W^4) - 3\mathcal{F}^3_{mn}\mathcal{F}^4_{pq}\mathcal{F}^1_{rs}(\lambda\gamma^s W^2) + (3 \leftrightarrow 4)\big]$$
$$-72k^1_m(\lambda\gamma^m W^2)\big[\mathcal{F}^1_{hi}(\lambda\gamma^h W^3)(\lambda\gamma^i W^4) + \mathcal{F}^3_{hi}(\lambda\gamma^h W^1)(\lambda\gamma^i W^4) + \mathcal{F}^4_{hi}(\lambda\gamma^h W^1)(\lambda\gamma^i W^3)\big]$$
$$+24k^1_m(\lambda\gamma^m W^4)\mathcal{F}^2_{hi}(\lambda\gamma^h W^1)(\lambda\gamma^i W^3) + 24k^1_m(\lambda\gamma^m W^3)\mathcal{F}^2_{hi}(\lambda\gamma^h W^1)(\lambda\gamma^i W^4)\rangle_{(0,2)}$$
$$\text{(C.88)}$$

To simplify the $\langle\,\rangle_{(-1,2)}$ terms in (C.88) it is convenient to have $\overline{\lambda}_\alpha$ in the combination $(\lambda\overline{\lambda})$ by using the identities,

$$(\overline{\lambda}\gamma^{ghi}\gamma^n W^1)(\lambda\gamma^{mnpqg}\lambda)(\lambda\gamma^h W^3)(\lambda\gamma^i W^4) = 2(\lambda\overline{\lambda})(W^3\gamma^{gi}\gamma_n W^1)(\lambda\gamma^{mnpqg}\lambda)(\lambda\gamma^i W^4) \tag{C.89}$$

and similarly

$$(\overline{\lambda}\gamma^{ghi}\gamma^a W^1)(\lambda\gamma^{agrsh}\lambda)(\lambda\gamma^c W^2)(\lambda\gamma^i W^4) = 2(\lambda\overline{\lambda})(W^4\gamma^{ahi}W^1)(\lambda\gamma^{ahirs}\lambda)(\lambda\gamma^c W^2) \tag{C.90}$$

$$(\overline{\lambda}\gamma^{ghi}\gamma^c W^1)(\lambda\gamma^{agrsh}\lambda)(\lambda\gamma^c W^2)(\lambda\gamma^i W^4) = 2(\lambda\overline{\lambda})(W^4\gamma^{ghc}W^1)(\lambda\gamma^{agrsh}\lambda)(\lambda\gamma^c W^2). \tag{C.91}$$

In [63] it was proved that

$$\langle(\lambda\gamma^{mnpqr}\lambda)(\lambda\gamma^s W^4)\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}\rangle_{(n,g)} = -16(k^1\cdot k^2)\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(n,g)} \tag{C.92}$$

and that $\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(n,g)}$ is completely symmetric in the particle labels, hence

$$\begin{aligned}\langle(\lambda\gamma^{mnpqr}\lambda)\big[\left(\mathcal{F}^1_{mn}\mathcal{F}^3_{pq}\mathcal{F}^2_{rs} - 4\mathcal{F}^1_{mn}\mathcal{F}^2_{pq}\mathcal{F}^3_{rs}\right)(\lambda\gamma^s W^4) - 3\mathcal{F}^3_{mn}\mathcal{F}^4_{pq}\mathcal{F}^1_{rs}(\lambda\gamma^s W^2)\big]\rangle_{(0,2)}\\ +(3\leftrightarrow 4) = +240(k^1\cdot k^2)\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(0,2)},\end{aligned}$$

where we also used the momentum conservation relation of $(k^1\cdot k^3)+(k^1\cdot k^4) = -(k^1\cdot k^2)$. The last two lines of (C.88) can be simplified by using $(\lambda\gamma^m W) = QA^m - k^m(\lambda A)$ and by noticing that the terms of the form $Q(A^m)\mathcal{F}_{pq}(\lambda\gamma^p W)(\lambda\gamma^q W)$ are BRST exact and therefore vanish. Doing that one gets

$$\begin{aligned}-72\langle k^1_m(\lambda\gamma^m W^2)\big[\mathcal{F}^1_{hi}(\lambda\gamma^h W^3)(\lambda\gamma^i W^4)+\mathcal{F}^3_{hi}(\lambda\gamma^h W^1)(\lambda\gamma^i W^4)+\mathcal{F}^4_{hi}(\lambda\gamma^h W^1)(\lambda\gamma^i W^3)\big]\\ +24k^1_m(\lambda\gamma^m W^4)\mathcal{F}^2_{hi}(\lambda\gamma^h W^1)(\lambda\gamma^i W^3) + 24k^1_m(\lambda\gamma^m W^3)\mathcal{F}^2_{hi}(\lambda\gamma^h W^1)(\lambda\gamma^i W^4)\rangle_{(0,2)}\\ = +240(k^1\cdot k^2)\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(0,2)}.\end{aligned} \tag{C.93}$$

Feeding the results above into the expression for K_{a_1} in (C.88) one can write it as $K_{a_1} = K_{a_{11}} + K_{a_{12}}$, where

$$\begin{aligned}K_{a_{11}} = -\langle k^1_r(\lambda\gamma^{mnpqr}\lambda)(W^3\gamma_{mns}W^1)(\lambda\gamma^s W^4)\mathcal{F}^2_{pq}\rangle_{(0,2)} + (1\leftrightarrow 2)\\ -\big[\langle(\mathcal{F}^3_{rs}k^1_{[a}(W^4\gamma_{gh|c]}W^1) + k^3_r(W^4\gamma_{ghs}W^3)\mathcal{F}^1_{ac})(\lambda\gamma^{agrsh}\lambda)(\lambda\gamma^c W^2)\rangle_{(0,2)} + (3\leftrightarrow 4)\big]\end{aligned} \tag{C.94}$$

and

$$K_{a_{12}} = +480(k^1\cdot k^2)\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(0,2)} \tag{C.95}$$

Furthermore, by using the gamma matrix identities $\gamma^{mnp} = \gamma^{mn}\gamma^p - \eta^{mn}\gamma^p + \eta^{am}\gamma^n$ and

$$(\gamma^{mn})_\alpha{}^\delta(\gamma_{mn})_\beta{}^\sigma = -8\delta_\alpha^\sigma\delta_\beta^\delta + 4\gamma^m_{\alpha\beta}\gamma_m^{\delta\sigma} - 2\delta_\alpha^\delta\delta_\beta^\sigma,$$

the pure spinor identities $(\lambda\gamma^{amnpq}\lambda)(\lambda\gamma_a)_\beta = (\lambda\gamma^m)_\alpha(\lambda\gamma_m)_\beta = 0$, the equation of motion $k^I_m(\lambda\gamma^m W^I) = 0$ and the results above, $K_{a_{11}}$ (and its permutations $K_{a_{21}}$ and $K_{a_{31}}$) can be further simplified. In fact, one can show that

$$-\langle k^1_r(\lambda\gamma^{mnpqr}\lambda)(W^3\gamma_{mns}W^1)(\lambda\gamma^s W^4)\mathcal{F}^2_{pq}\rangle_{(0,2)}$$

$$= 32\langle k^1_m(\lambda\gamma^m W^4)(\lambda\gamma^p W^3)(\lambda\gamma^q W^1)\mathcal{F}^2_{pq}\rangle_{(0,2)} + (3 \leftrightarrow 4)$$

$$= -32\left((k^1\cdot k^3) + (k^1\cdot k^4)\right)\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(0,2)}. \tag{C.96}$$

From $\gamma^{mnp}_{\alpha\beta}\gamma^{\gamma\delta}_{mnp} = 48(\delta^\gamma_\alpha\delta^\delta_\beta - \delta^\delta_\alpha\delta^\gamma_\beta)$ and the equation of motion for W^α_3 it follows that,

$$-k^3_r(\lambda\gamma^{agrsh}\lambda)(W^4\gamma_{ghs}W^3)\mathcal{F}^1_{ac}(\lambda\gamma^c W^2) = 48(k^3\cdot k^4)(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}$$

and

$$\frac{1}{2}\mathcal{F}^3_{rs}k^1_c(W^4\gamma_{gha}W^1)(\lambda\gamma^{agrsh}\lambda)(\lambda\gamma^c W^2) = 48(k^1\cdot k^2)(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}.$$

From (C.96) one also gets

$$-\frac{1}{2}\mathcal{F}^3_{rs}k^1_a(W^4\gamma_{ghc}W^1)(\lambda\gamma^{agrsh}\lambda)(\lambda\gamma^c W^2) = 16(k^1\cdot k^3)(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}. \tag{C.97}$$

Plugging the identities (C.96) – (C.97) in (C.94) and summing over the indicated permutations leads to

$$K_{a_{11}} = 240(k^1\cdot k^2)\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(0,2)} \tag{C.98}$$

hence

$$K_{a_1} = K_{a_{11}} + K_{a_{12}} = 720(k^1\cdot k^2)\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(0,2)}. \tag{C.99}$$

From (C.85) and (C.99) and their permutations one arrives at the final result[††] for (C.76),

$$K = +720\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(0,2)}\times$$

$$\times\left[576(k^1\cdot k^2) - 192(k^3\cdot k^2) - 192(k^4\cdot k^1) + 576(k^2\cdot k^1) - 192(k^3\cdot k^1) - 192(k^4\cdot k^2)\right]$$

$$= 3\cdot 2^7\cdot 2880(k^1\cdot k^2)\langle(\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(0,2)}. \tag{C.100}$$

The complete kinematic factor (4.164) is obtained using the result (C.100) and

[††]To check results we performed explicit component expansion computations with especially-crafted programs using FORM [79].

permuting its labels. The first line of (4.164) is given by (C.100) while the second and third are obtained by replacing $s \to u$ and $s \to t$ respectively. The final result is therefore

$$\mathcal{K}_2 = -3 \cdot 2^6 \cdot 2880 \langle (\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(0,2)} \big[$$

$$s(H_{1234} + H_{3412}) + u(H_{1324} + H_{2413}) + t(H_{1423} + H_{2314})\big]$$

$$= 2^{12}\, 3^3\, 5\, \mathcal{Y}_s \langle (\lambda A^1)(\lambda\gamma^m W^2)(\lambda\gamma^n W^3)\mathcal{F}^4_{mn}\rangle_{(0,2)} \tag{C.101}$$

where we used the Mandelstam variables and $u = -t - s$ together with

$$H_{1234} + H_{3412} - H_{1324} - H_{2413} = \Delta(1,4)\Delta(2,3)$$

$$H_{1423} + H_{2314} - H_{1324} - H_{2413} = -\Delta(1,2)\Delta(3,4).$$

and the definition (4.167). With (C.101) the expression for the kinematic factor (4.164) is finally demonstrated.

C.4 Period matrix parametrization of genus-two moduli space

Let $\mu_i{}^z_{\bar{z}}$ $(i = 1,2,3)$ be the Beltrami differentials, τ_i $(i = 1,2,3)$ the Teichmüller parameters and $w_I(z)$ $(I = 1,2)$ the holomorphic 1- forms over Σ_2, then [49]

$$\int d^2\tau_1 d^2\tau_2 d^2\tau_3 \Big| \prod_{i=1}^{3} \int d^2 z_i \mu_i(z_i)\Delta(1,2)\Delta(2,3)\Delta(3,1)\Big|^2 = \int d^2\Omega_{11} d^2\Omega_{12} d^2\Omega_{22} \tag{C.102}$$

where $\Delta(i,j) = w_1(z_i)w_2(z_j) - w_1(z_j)w_2(z_i)$. To prove this one uses the identity [101][69]

$$\int d^2 z\ w_I\,(z) w_J(z)\,\mu_i(z) = \frac{\delta\Omega_{IJ}}{\delta\tau_i} \tag{C.103}$$

and expands $\Delta(1,2)\Delta(2,3)\Delta(3,1)$ to get

$$\prod_{i=1}^{3} \int d^2 z_i \mu_i(z_i)\Delta(1,2)\Delta(2,3)\Delta(3,1) = -\frac{\delta\Omega_{11}}{\delta\tau_i}\frac{\delta\Omega_{12}}{\delta\tau_j}\frac{\delta\Omega_{22}}{\delta\tau_k}\epsilon^{ijk}. \tag{C.104}$$

So

$$d\tau_1 \wedge d\tau_2 \wedge d\tau_3 \prod_{i=1}^{3} \int d^2 z_i \mu_i(z_i)\Delta(1,2)\Delta(2,3)\Delta(3,1) = -\frac{\delta\Omega_{11}}{\delta\tau_i}\frac{\delta\Omega_{12}}{\delta\tau_j}\frac{\delta\Omega_{22}}{\delta\tau_k}\epsilon^{ijk} d\tau_1 \wedge d\tau_2 \wedge d\tau_3$$

$$= -\frac{\delta\Omega_{11}}{\delta\tau_i}\frac{\delta\Omega_{12}}{\delta\tau_j}\frac{\delta\Omega_{22}}{\delta\tau_k} d\tau_i \wedge d\tau_j \wedge d\tau_k$$

$$= -\delta\Omega_{11} \wedge \delta\Omega_{12} \wedge \delta\Omega_{22}.$$

Multiplying the last expression by its complex conjugate we get (C.102).

References

[1] M. B. Green, J. H. Schwarz and L. Brink, "N=4 Yang-Mills And N=8 Supergravity As Limits Of String Theories," Nucl. Phys. B **198**, 474 (1982).
S. Sannan, "Gravity as the limit of the type II superstring theory," Phys. Rev. D **34**, 1749 (1986).

[2] J. J. Atick, G. W. Moore and A. Sen, "Catoptric Tadpoles," Nucl. Phys. B **307**, 221 (1988).
E. J. Martinec, "Nonrenormalization Theorems And Fermionic String Finiteness," Phys. Lett. B **171**, 189 (1986).
S. Mandelstam, "The n loop string amplitude: Explicit formulas, finiteness and absence of ambiguities," Phys. Lett. B **277**, 82 (1992).

[3] P. Ramond, "Dual Theory for Free Fermions," Phys. Rev. D **3**, 2415 (1971).
A. Neveu and J. H. Schwarz, "Factorizable dual model of pions," Nucl. Phys. B **31** (1971) 86.
A. Neveu and J. H. Schwarz, "Quark Model of Dual Pions," Phys. Rev. D **4**, 1109 (1971).

[4] M. B. Green and J. H. Schwarz, "Covariant Description Of Superstrings," Phys. Lett. B **136**, 367 (1984).
M. B. Green and J. H. Schwarz, "Properties Of The Covariant Formulation Of Superstring Theories," Nucl. Phys. B **243**, 285 (1984).

[5] M. B. Green and J. H. Schwarz, "Supersymmetrical String Theories," Phys. Lett. B **109**, 444 (1982).

[6] M.B. Green, J.H. Schwarz, and E. Witten, "Superstring Theory", Cambridge University Press (1987), Vols I and II.

[7] M. B. Green and S. Sethi, "Supersymmetry constraints on type IIB supergravity," Phys. Rev. D **59**, 046006 (1999) [arXiv:hep-th/9808061].

[8] M. B. Green, M. Gutperle and P. Vanhove, "One loop in eleven dimensions," Phys. Lett. B **409**, 177 (1997) [arXiv:hep-th/9706175].

[9] A. Terras, *Harmonic analysis on symmetric spaces and applications I*, Springer, New York-Berlin 1985

[10] E. D'Hoker and D. H. Phong, "Two-loop superstrings. VI: Non-renormalization theorems and the 4-point function," arXiv:hep-th/0501197.

[11] M. B. Green, H. h. Kwon and P. Vanhove, "Two loops in eleven dimensions," Phys. Rev. D **61**, 104010 (2000) [arXiv:hep-th/9910055].

[12] M. B. Green and P. Vanhove, "The low energy expansion of the one-loop type II superstring amplitude," Phys. Rev. D **61**, 104011 (2000) [arXiv:hep-th/9910056].

[13] Gualtieri, M., "Generalized complex geometry", ArXiv Mathematics e-prints, arXiv:math/0401221,2004,

[14] E. Cartan, "Lecons sur la Theorie des Spineurs", Hermann, Paris, 1937

[15] P.S. Howe, "Pure spinor lines in superspace and ten-dimensional supersymmetric theories," Phys. Lett. B258: 141 (1991).

[16] N. Berkovits, "Relating the RNS and pure spinor formalisms for the superstring," JHEP **0108**, 026 (2001) [arXiv:hep-th/0104247].

[17] C.R. Mafra, "Four-point one-loop amplitude computation in the pure spinor formalism ", JHEP 0601:075 (2006), [arXiv:hep-th/0512052]

[18] N. A. Nekrasov, "Lectures on curved beta-gamma systems, pure spinors, and anomalies," arXiv:hep-th/0511008.

[19] A. S. Losev, A. Marshakov, A. M. Zeitlin, "On first order formalism in string theory," Phys. Lett. **B633**, 375-381 (2006). [hep-th/0510065].

[20] C. Saemann, "Aspects of twistor geometry and supersymmetric field theories within superstring theory," arXiv:hep-th/0603098.

[21] L.P. Hughston, "Applications of $SO(8)$ Spinors", pp.253-287 in Gravitation and Geometry: a volume in honor of Ivor Robinson (eds. W. Rindler and A. Trautman), Bibliopolis, Naples (1987).

[22] N. Berkovits and S. A. Cherkis, "Pure spinors are higher-dimensional twistors," JHEP **0412**, 049 (2004) [arXiv:hep-th/0409243].

[23] J. Hoogeveen and K. Skenderis, "Decoupling of unphysical states in the minimal pure spinor formalism I," JHEP **1001**, 041 (2010) [arXiv:0906.3368 [hep-th]].

[24] N. Berkovits, J. Hoogeveen and K. Skenderis, "Decoupling of unphysical states in the minimal pure spinor formalism II," JHEP **0909**, 035 (209) [arXiv:0906.3371 [hep-th]].

[25] J. Hoogeveen and K. Skenderis, "BRST quantization of the pure spinor superstring," JHEP **0711**, 081 (2007) [arXiv:0710.2598 [hep-th]].

[26] Y. Aisaka, E. A. Arroyo, N. Berkovits and N. Nekrasov, "Pure Spinor Partition Function and the Massive Superstring Spectrum," JHEP **0808**, 050 (2008) [arXiv:0806.0584 [hep-th]].

[27] Y. Aisaka and E. A. Arroyo, "Hilbert space of curved $\beta\gamma$ systems on quadric cones," JHEP **0808**, 052 (2008) [arXiv:0806.0586 [hep-th]].

[28] Griffiths and Harris, "Principles of Algebraic Geometry", [Wiley Classics Library Edition Published 1994]

[29] Raoul Bott and Loring W.Tu "Differential Forms in Algebraic Topology", [Springer-Verlag published 1982]

[30] H.B. Lawson, Jr. and M.L. Michelsohn, "Spin geometry", Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989. MR 91g:5300

[31] W.S. Massey, "A Basic Course in Algebraic Topology", [Springer 1991]

[32] H. Gomez, "One-loop Superstring Amplitude From Integrals on Pure Spinors Space," JHEP **0912**, 034 (2009) [arXiv:0910.3405 [hep-th]].

[33] H. Gomez and C. R. Mafra, "The Overall Coefficient of the Two-loop Superstring Amplitude Using Pure Spinors," JHEP **1005**, 017 (2010) [arXiv:1003.0678 [hep-th]].

[34] O. A. Bedoya and H. Gomez, "A New Proposal for the Picture Changing Operators in the Minimal Pure Spinor Formalism," JHEP **1108**, 025 (2011) [arXiv:1106.1253 [hep-th]].

[35] J. H. Schwarz, "Superstring Theory," Phys. Rept. **89**, 223 (1982).

[36] W. Siegel, "Classical Superstring Mechanics," Nucl. Phys. B **263**, 93 (1986).

[37] William Fulton, "Intersection theory", [Springer 1998]

[38] Work in progress

[39] N. Berkovits and B. Carlini Vallilo, "Consistency of superPoincare covariant superstring tree amplitudes." JHEP 0007:015,2000, [arXiv:hep-th;0004171]

[40] N. Berkovits, "Pure spinors, twistors, and emergent supersymmetry," [arXiv:1105.1147 [hep-th]].

[41] I. Oda, "Emergence of Superstring from Pure Spinor," [arXiv:1106.3548 [hep-th]].

[42] M. Matone, L. Mazzucato, I. Oda, D. Sorokin and M. Tonin, "The superembedding origin of the Berkovits pure spinor covariant quantization of superstrings," Nucl. Phys. B **639**, 182 (2002) [arXiv:hep-th/0206104].

[43] N. Berkovits, O. Chandia, "Lorentz invariance of the pure spinor BRST cohomology for the superstring," Phys. Lett. **B514**, 394-400 (2001). [hep-th/0105149].

[44] N. Berkovits and O. Chandia, "Massive superstring vertex operator in D = 10 superspace," JHEP **0208**, 040 (2002) [arXiv:hep-th/0204121].

[45] M. B. Green, P. Vanhove, "Duality and higher derivative terms in M theory," JHEP **0601**, 093 (2006). [hep-th/0510027].

[46] M. Matone, R. Volpato, "Higher genus superstring amplitudes from the geometry of moduli space," Nucl. Phys. **B732**, 321-340 (2006). [hep-th/0506231].

[47] N. Berkovits, "Super-Poincare covariant quantization of the superstring," JHEP **0004**, 018 (2000) [arXiv:hep-th/0001035].

[48] N. Berkovits, "Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring," JHEP **0409**, 047 (2004) [arXiv:hep-th/0406055].

[49] N. Berkovits, "Super-Poincare covariant two-loop superstring amplitudes," JHEP **0601**, 005 (2006) [arXiv:hep-th/0503197].

[50] N. Berkovits and C. R. Mafra, "Equivalence of two-loop superstring amplitudes in the pure spinor and RNS formalisms," Phys. Rev. Lett. **96**, 011602 (2006) [arXiv:hep-th/0509234].

[51] J. Milnor, *"Morse Theory"*, Princeton University Press, (1963).

[52] N. Berkovits and N. Nekrasov, "Multiloop superstring amplitudes from non-minimal pure spinor formalism," JHEP **0612**, 029 (2006) [arXiv:hep-th/0609012].

[53] Y. Aisaka and N. Berkovits, "Pure Spinor Vertex Operators in Siegel Gauge and Loop Amplitude Regularization," JHEP **0907**, 062 (2009) [arXiv:0903.3443 [hep-th]].

[54] N. Berkovits and N. Nekrasov, "The character of pure spinors," Lett. Math. Phys. **74**, 75 (2005) [arXiv:hep-th/0503075].

[55] P. A. Grassi and P. Vanhove, "Higher-loop amplitudes in the non-minimal pure spinor formalism," JHEP **0905**, 089 (2009) [arXiv:0903.3903 [hep-th]].

[56] N. Berkovits, "Explaining pure spinor superspace," arXiv:hep-th/0612021.

[57] N. Berkovits, "Pure spinor formalism as an N = 2 topological string," JHEP **0510**, 089 (2005) [arXiv:hep-th/0509120].

[58] T. Kugo and I. Ojima, "Local Covariant Operator Formalism Of Nonabelian Gauge Theories And Quark Confinement Problem," Prog. Theor. Phys. Suppl. **66**, 1 (1979).

[59] G. N. Rybkin, "State Space in BRST Quantization and Kugo-Ojima Quartets," Int. J. Mod. Phys. A **6**, 1675 (1991).

[60] I. Oda and M. Tonin, "Y-formalism in pure spinor quantization of superstrings," Nucl. Phys. B **727**, 176 (2005) [arXiv:hep-th/0505277].

[61] I. Oda and M. Tonin, "On the b-antighost in the pure spinor quantization of superstrings," Phys. Lett. B **606**, 218 (2005) [arXiv:hep-th/0409052].

[62] N. Berkovits and C. R. Mafra, "Some superstring amplitude computations with the non-minimal pure spinor formalism," JHEP **0611**, 079 (2006) [arXiv:hep-th/0607187].

[63] C. R. Mafra, "Pure Spinor Superspace Identities for Massless Four-point Kinematic Factors," JHEP **0804**, 093 (2008) [arXiv:0801.0580 [hep-th]].

[64] E. D'Hoker and D. H. Phong, "Two-Loop Superstrings VI: Non-Renormalization Theorems and the 4-Point Function," Nucl. Phys. B **715**, 3 (2005) [arXiv:hep-th/0501197].

[65] P. S. Howe, "Pure Spinors Lines In Superspace And Ten-Dimensional Supersymmetric Theories," Phys. Lett. B **258**, 141 (1991) [Addendum-ibid. B **259**, 511 (1991)]. P. S. Howe, "Pure Spinors, Function Superspaces And Supergravity Theories In Ten-Dimensions And Eleven-Dimensions," Phys. Lett. B **273**, 90 (1991).

[66] E. D'Hoker, M. Gutperle and D. H. Phong, "Two-loop superstrings and S-duality," Nucl. Phys. B **722**, 81 (2005) [arXiv:hep-th/0503180].

[67] C. R. Mafra, "Superstring Scattering Amplitudes with the Pure Spinor Formalism," arXiv:0902.1552 [hep-th].

[68] A. A. Beilinson and Yu. I. Manin, "The Mumford Form and the Polyakov Measure in String Theory," Commun. Math. Phys. **107**, 359 (1986).

[69] E. D'Hoker and D. H. Phong, "The Geometry of String Perturbation Theory," Rev. Mod. Phys. **60**, 917 (1988).

[70] E. D'Hoker and D. H. Phong, "Multiloop Amplitudes For The Bosonic Polyakov String," Nucl. Phys. B **269**, 205 (1986).

[71] W. I. Weisberger, "Normalization of the Path Integral Measure and the Coupling Constants for Bosonic Strings," Nucl. Phys. B **284**, 171 (1987).

[72] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string," *Cambridge, UK: Univ. Pr. (1998) 402 p*

[73] J. Polchinski, "Factorization of Bosonic String Amplitudes," Nucl. Phys. B **307**, 61 (1988).

[74] Hershel M. Farkas and Irwin Kra, "Riemann Surfaces", Second Edition, Springer-Verlag, New York, 1992.

[75] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," Commun. Math. Phys. **165**, 311 (1994) [arXiv:hep-th/9309140].

[76] M. B. Green and M. Gutperle, "Effects of D-instantons," Nucl. Phys. B **498**, 195 (1997) [arXiv:hep-th/9701093].

[77] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond," *Cambridge, UK: Univ. Pr. (1998) 531 p*

[78] Eric D'Hoker, private communication.

[79] J. A. M. Vermaseren, "New features of FORM," arXiv:math-ph/0010025. M. Tentyukov and J. A. M. Vermaseren, "The multithreaded version of FORM," arXiv:hep-ph/0702279.

[80] E. Witten, "Two-dimensional models with (0,2) supersymmetry: Perturbative aspects," arXiv:hep-th/0504078.

[81] S. Weinberg, "Coupling Constants And Vertex Functions In String Theories," Phys. Lett. B **156**, 309 (1985). S. Weinberg, "Covariant Path Integral Approach To String Theory,"

[82] I. Oda and M. Tonin, "Y-formalism and b ghost in the Non-minimal Pure Spinor Formalism of Superstrings," Nucl. Phys. B **779**, 63 (2007) [arXiv:0704.1219 [hep-th]].

[83] R. Medina, F. T. Brandt and F. R. Machado, "The open superstring 5-point amplitude revisited," JHEP **0207**, 071 (2002) [arXiv:hep-th/0208121]. L. A. Barreiro and R. Medina, "5-field terms in the open superstring effective action," JHEP **0503**, 055 (2005) [arXiv:hep-th/0503182].

[84] S. Stieberger, "Open & Closed vs Pure Open String Disk Amplitudes," arXiv:0907.2211 [hep-th].

[85] D. Oprisa and S. Stieberger, "Six gluon open superstring disk amplitude, multiple hypergeometric series and Euler-Zagier sums," arXiv:hep-th/0509042.

[86] D. M. Richards, "The One-Loop Five-Graviton Amplitude and the Effective Action," JHEP **0810**, 042 (2008) [arXiv:0807.2421 [hep-th]]. D. M. Richards, "The One-Loop H^2R^3 and $H^2(DH)^2R$ Terms in the Effective Action," JHEP **0810**, 043 (2008) [arXiv:0807.3453 [hep-th]].

[87] O. A. Bedoya and N. Berkovits, "GGI Lectures on the Pure Spinor Formalism of the Superstring," arXiv:0910.2254 [hep-th].

[88] N. Berkovits, "Explaining the Pure Spinor Formalism for the Superstring," JHEP **0801**, 065 (2008) [arXiv:0712.0324 [hep-th]].

[89] C. R. Mafra and C. Stahn, "The One-loop Open Superstring Massless Five-point Amplitude with the Non-Minimal Pure Spinor Formalism," JHEP **0903**, 126 (2009) [arXiv:0902.1539 [hep-th]].

[90] C. R. Mafra, "Simplifying the Tree-level Superstring Massless Five-point Amplitude," JHEP **1001**, 007 (2010) [arXiv:0909.5206 [hep-th]].

[91] M. B. Green, J. G. Russo and P. Vanhove, "Low energy expansion of the four-particle genus-one amplitude in type II superstring theory," JHEP **0802**, 020 (2008) [arXiv:0801.0322 [hep-th]].

[92] U. Gran, "GAMMA: A Mathematica package for performing Gamma-matrix algebra and Fierz transformations in arbitrary dimensions," arXiv:hep-th/0105086.

[93] A. Basu, "The D**10 R**4 term in type IIB string theory," Phys. Lett. B **648**, 378 (2007) [arXiv:hep-th/0610335].

[94] Griffiths and Harris, "Principles of Algebraic Geometry", [Wiley Classics Library Edition Published 1994]

[95] E. D'Hoker and D. H. Phong, "Two-Loop Superstrings V: Gauge Slice Independence of the N-Point Function," Nucl. Phys. B **715**, 91 (2005) [arXiv:hep-th/0501196]. E. D'Hoker and D. H. Phong, "Two-Loop Superstrings IV, The Cosmological Constant and Modular Forms," Nucl. Phys. B **639**, 129 (2002) [arXiv:hep-th/0111040]. E. D'Hoker and D. H. Phong, "Two-Loop Superstrings III, Slice Independence and Absence of Ambiguities," Nucl. Phys. B **636**, 61 (2002) [arXiv:hep-th/0111016]. E. D'Hoker and D. H. Phong, "Two-Loop Superstrings II, The Chiral Measure on Moduli Space," Nucl. Phys. B **636**, 3 (2002) [arXiv:hep-th/0110283]. E. D'Hoker and D. H. Phong, "Two-Loop Superstrings I, Main Formulas," Phys. Lett. B **529**, 241 (2002) [arXiv:hep-th/0110247].

[96] E. Verlinde and H. Verlinde, "Chiral Bosonization, determinants and the string partition function", Nucl. Phys. **B288** (1987) 357.

[97] E. D'Hoker and D. H. Phong, "Lectures on Two-Loop Superstrings," arXiv:hep-th/0211111.

[98] A. Morozov, "NSR Superstring Measures Revisited," JHEP **0805**, 086 (2008) [arXiv:0804.3167 [hep-th]].

[99] G. Policastro and D. Tsimpis, "R**4, purified," arXiv:hep-th/0603165.

[100] N. Berkovits, "ICTP lectures on covariant quantization of the superstring," arXiv:hep-th/0209059.

[101] A. A. Belavin, V. Knizhnik, A. Morozov and A. Perelomov, "Two- and Three-loop Amplitudes in the Bosonic String Theory," JETP Lett. **43**, 411 (1986) [Phys. Lett. B **177**, 324 (1986)].

[102] M. B. Green, H. h. Kwon and P. Vanhove, "Two loops in eleven dimensions," Phys. Rev. D **61**, 104010 (2000) [arXiv:hep-th/9910055].

[103] N. Berkovits, "New higher-derivative R**4 theorems," Phys. Rev. Lett. **98**, 211601 (2007) [arXiv:hep-th/0609006].