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LINEARIZATION OF THE HIGHER ANALOGUE OF COURANT ALGEBROIDS

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Abstract. In this paper, we show that the spaces of sections of the n-th differential operator bundle $\mathfrak{D}^n E$ and the n-th skew-symmetric jet bundle $\mathfrak{J}_n E$ of a vector bundle E are isomorphic to the spaces of linear n-vector fields and linear n-forms on E^* respectively. Consequently, the n-omni-Lie algebroid $\mathfrak{D}E \oplus \mathfrak{J}_n E$ introduced by Bi-Vitagliano-Zhang can be explained as certain linearization, which we call pseudo-linearization of the higher analogue of Courant algebroids $TE^* \oplus \wedge^n T^*E^*$. On the other hand, we show that the omni n-Lie algebroid $\mathfrak{D}E \oplus \wedge^n \mathfrak{J}E$ can also be explained as certain linearization, which we call Weinstein-linearization of the higher analogue of Courant algebroids $TE^* \oplus \wedge^n T^*E^*$. We also show that n-Lie algebroids, local n-Lie algebras and Nambu-Jacobi structures can be characterized as integrable subbundles of omni n-Lie algebroids.

1. **Introduction.** This paper aims to study linearization of the higher analogue of Courant algebroids $TE^* \oplus \wedge^n T^*E^*$.

1.1. **Omni-Lie algebras and omni-Lie algebroids.** Courant algebroids were introduced by Liu, Weinstein and Xu in [32] and have been found many applications in mathematical physics. See the survey article [28] for more details. The notion of an omni-Lie algebra was introduced by Weinstein in [41] to study the linearization of the standard Courant algebroid $TM \oplus T^*M$. Then it was further studied in [27, 36, 37]. An **omni-Lie algebra** associated to a vector space V is a triple $(\mathrm{gl}(V) \oplus V, (\cdot,\cdot), \{\cdot,\cdot\})$, where $(\cdot,\cdot)$ is the V-valued pairing given by

$$(A+u, B+v) = Av + Bu, \quad \forall\ A+u, B+v \in \mathrm{gl}(V) \oplus V,$$

and $\{\cdot,\cdot\}$ is the bilinear bracket operation given by

$$\{A+u, B+v\} = [A,B] + Av.$$

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Note that $(\mathrm{gl}(V)\oplus V,\{\cdot,\cdot\})$ is not a Lie algebra, but a Leibniz algebra, which provides a natural example of Leibniz algebras. Moreover, Dirac structures of the omni-Lie algebra $\mathrm{gl}(V)\oplus V$ characterize all Lie algebra structures on V, and this is one of the most important properties of an omni-Lie algebra. Let M be the vector space V^* in the standard Courant algebroid $TM\oplus T^*M$, and consider linear vector fields, which are in one-to-one correspondence with $\mathrm{gl}(V)$, and constant 1-forms on V^*, which are in one-to-one correspondence with V. Then the Dorfman bracket in the standard Courant algebroid $TM\oplus T^*M$ reduces to the bracket in the omni-Lie algebra $\mathrm{gl}(V)\oplus V$ given above. We use the terminology "base-linearization" to indicate this process. Different generalizations of an omni-Lie algebra have been given recently with applications in different aspects.

The notion of an omni-Lie algebroid was introduced in [7], which can be viewed as the geometric generalization of an omni-Lie algebra from a vector space to a vector bundle. Lie algebroid structures on a vector bundle E (or local Lie algebra structures when E is a line bundle) can be characterized as Dirac structures of the omni-Lie algebroid $\mathfrak{D}E\oplus\mathfrak{J}E$, where $\mathfrak{D}E$ is the covariant differential operator bundle and $\mathfrak{J}E$ is the first jet bundle of E. Omni-Lie algebroids provide a general framework to study Jacobi structures, contact structures and odd dimensional analogues of generalized complex structures [9, 26, 30, 38, 39, 40]. Omni-Lie algebroids are also natural examples of E-Courant algebroids introduced in [8]. Similar to the fact that $\mathrm{gl}(V)$ can be understood as linear vector fields on V^*, it is well-known that $\mathfrak{D}E$ corresponds to linear vector fields on the dual bundle E^*. But there are different explanations of $\mathfrak{J}E$:

- Sections of $\mathfrak{J}E$ can be understood as constant 1-forms on E^*. This explanation is supported by the fact that the pairing between $\mathfrak{D}E$ and $\mathfrak{J}E$ takes values in E, whose sections are linear functions on E^*. On the other hand, when E reduces to a vector space V, we have $\mathfrak{J}V=V$, which is understood as the space of constant 1-forms on V^*. Therefore, this point of view is consistent with Weinstein's original idea in the study of linearization of the standard Courant algebroid $TM\oplus T^*M$.
- Sections of $\mathfrak{J}E$ can also be understood as linear 1-forms on E^*. This explanation is supported by the fact that $\mathfrak{J}E$ corresponds to linear sections of the double vector bundle $(T^*E^*;E,E^*;M)$.

When $\mathfrak{J}E$ is understood as fiberwise constant 1-forms on E^*, we will say that the omni-Lie algebroid $\mathfrak{D}E\oplus\mathfrak{J}E$ is the Weinstein-linearization of the standard Courant algebroid $TE^*\oplus T^*E^*$; when $\mathfrak{J}E$ is understood as linear 1-forms on E^*, we will say that the omni-Lie algebroid $\mathfrak{D}E\oplus\mathfrak{J}E$ is the pseudo-linearization of the standard Courant algebroid $TE^*\oplus T^*E^*$. Even though Weinstein-linearization and pseudo-linearization are the same in this situation, namely we both obtain the omni-Lie algebroid $\mathfrak{D}E\oplus\mathfrak{J}E$, in the sequel we will see that different geometric objects can be obtained using different explanations of $\mathfrak{J}E$.

1.2. **Omni n-Lie algebras and n-omni-Lie algebroids.** Recently, the higher analogues of the standard Courant algebroid $TM\oplus\wedge^nT^*M$ are widely studied due to applications in Nambu-Poisson structures, multisymplectic structures, L_∞-algebra theory and topological field theory [1, 4, 15, 19, 22]. In particular, Dirac structures of the higher analogues of the standard Courant algebroid $TM\oplus\wedge^nT^*M$ are deeply studied in [2, 6, 20, 42]. In [31], the authors introduced the notion of an omni n-Lie algebra $\mathrm{gl}(V)\oplus\wedge^nV$ and proved that it is the base-linearization of the higher analogue of the standard Courant algebroid $TM \oplus \wedge^n T^*M$. Moreover, the $(n+1)$-Lie algebra structures on V can be characterized as integrable subspaces of the omni n-Lie algebra $\mathrm{gl}(V) \oplus \wedge^n V$. n-Lie algebras (also called Filippov algebras) are the underlying algebraic structures of Nambu-Poisson structures, and have many applications in mathematical physics. See the review article [13] for more details.

To study the higher analogue of the omni-Lie algebroid, the notion of an n-omni-Lie algebroid was introduced in [3], which is the direct sum of the covariant differential operator bundle $\mathfrak{D}E$ and the n-th skew-symmetric jet bundle $\mathfrak{J}_n E$ together with a pairing and a bracket operation. Multicontact structures can be characterized as integrable subbundles of n-omni-Lie algebroids. Note that when the vector bundle E reduces to a vector space V, one can not obtain the aforementioned omni n-Lie algebra since $\mathfrak{J}_n V = 0$. On the other hand, $(n+1)$-Lie algebroid structures on E can not be characterized by integrable subbundles of the n-omni-Lie algebroid $\mathfrak{D}E \oplus \mathfrak{J}_n E$.

1.3. **Main results.** In this paper, we give an alternative explanation of the n-omni-Lie algebroid introduced in [3]. We find that linear n-vector fields and linear n-forms on a vector bundle E are sections of the n-th differential operator bundle $\mathfrak{D}^n E$ and the n-th skew-symmetric jet bundle $\mathfrak{J}_n E$ respectively. As a consequence, the n-omni-Lie algebroid $\mathfrak{D}E \oplus \mathfrak{J}_n E$ can be viewed as certain linearization, called pseudo-linearization, of the higher analogue of Courant algebroids $TE^* \oplus \wedge^n T^*E^*$. On the other hand, if we understand $\mathfrak{J}E$ as constant 1-forms on E^*, it is natural to use constant n-forms, that is $\wedge^n \mathfrak{J}E$, to replace $\mathfrak{J}_n E$ that was used in [3]. Based on this observation, we introduce the notion of an omni n-Lie algebroid. More precisely, an omni n-Lie algebroid associated to a vector bundle E is the direct sum bundle $\mathfrak{D}E \oplus \wedge^n \mathfrak{J}E$ together with an anchor, a pairing and a bracket operation (see Definition 3.2). When E reduces to a vector space V, we obtain the omni n-Lie algebra $\mathrm{gl}(V) \oplus \wedge^n V$ naturally. Moreover, we show that the omni n-Lie algebroid $\mathfrak{D}E \oplus \wedge^n \mathfrak{J}E$ can also be viewed as certain linearization, called Weinstein-linearization, of $TE^* \oplus \wedge^n T^*E^*$. When $\mathrm{rank}E \geq 2$ (resp. $\mathrm{rank}E = 1$), $(n+1)$-Lie algebroid structures (local $(n+1)$-Lie algebra structures) on E can be characterized as integrable subbundles of the omni n-Lie algebroid $\mathfrak{D}E \oplus \wedge^n \mathfrak{J}E$.

We summarize n-omni-Lie algebroids and omni n-Lie algebroids by the following table:

omni n-Lie algebroids	n-omni-Lie algebroids
$\mathfrak{D}E \oplus \wedge^n \mathfrak{J}E$	$\mathfrak{D}E \oplus \mathfrak{J}_n E$
Weinstein-linearization	pseudo-linearization
$(n+1)$-Lie algebroid structures on E	higher Dirac-Jacobi structures
Nambu-Jacobi structures on M	exact multisymplectic structures
Leibniz algebroid structures on $\wedge^n \mathfrak{J}E$	-
omni n-Lie algebra $\mathrm{gl}(V) \oplus \wedge^n V$	-

2. **Pseudo-linearization of $TE^* \oplus \wedge^n T^*E^*$ and n-omni-Lie algebroids.** The goal of this section is to give a geometric explanation of the n-omni-Lie algebroid $\mathfrak{D}E \oplus \mathfrak{J}_n E$ introduced in [3]. We understand it as a linearization of the higher analogue of Courant algebroids $TE^* \oplus \wedge^n T^*E^*$ in the sense that $\Gamma(\mathfrak{D}E)$ and $\Gamma(\mathfrak{J}_n E)$ are spaces of linear vector fields and linear n-forms ([5]) on the vector bundle E^*.

We first recall the n-th differential operator bundle $\mathfrak{D}^n E$ and the n-th skew-symmetric jet bundle $\mathfrak{J}_n E$ of a vector bundle E.

A covariant differential operator for a vector bundle $E \to M$ is a smooth map $\mathfrak{d} : \Gamma(E) \to \Gamma(E)$, such that there is an element $X_{\mathfrak{d}} \in \mathfrak{X}^1(M)$, called the symbol, satisfying

$$\mathfrak{d}(fu) = f\mathfrak{d}(u) + X_{\mathfrak{d}}(f)u, \qquad \forall\ f \in C^\infty(M), u \in \Gamma(E).$$

The covariant differential operator bundle $\mathfrak{D}E$ of a vector bundle E with the commutator bracket $[\cdot,\cdot]$ is a Lie algebroid, which is indeed the gauge Lie algebroid of the frame bundle $\mathcal{F}(E)$. The corresponding Atiyah sequence is as follows:

$$0 \to \mathrm{End}(E) \xrightarrow{\mathbb{i}} \mathfrak{D}E \xrightarrow{\mathbb{j}} TM \to 0. \tag{1}$$

The first jet bundle $\mathfrak{J}E$ of a vector bundle E is the bundle whose fiber over a point $m \in M$ is the space of equivalence classes of sections of E, where $[u]_m = [v]_m$ for $u, v \in \Gamma(E)$ if $u(m) = v(m)$ and $d_m\langle u, \xi\rangle = d_m\langle v, \xi\rangle$ for any $\xi \in \Gamma(E^*)$. In [7], the authors proved that the first jet bundle $\mathfrak{J}E$ may be considered as an E-dual bundle of $\mathfrak{D}E$, i.e.

$$\mathfrak{J}E \cong \{\nu \in \mathrm{Hom}(\mathfrak{D}E, E) \,|\, \nu(\Phi) = \Phi \circ \nu(\mathrm{Id}_E), \quad \forall\ \Phi \in \mathrm{End}(E)\}.$$

Associated to the jet bundle $\mathfrak{J}E$, there is a jet sequence of E given by:

$$0 \to \mathrm{Hom}(TM, E) \xrightarrow{\mathbb{e}} \mathfrak{J}E \xrightarrow{\mathbb{p}} E \to 0. \tag{2}$$

This sequence does not necessarily split, but on the section level, it does:

$$\mathbb{d} : \Gamma(E) \to \Gamma(\mathfrak{J}E), \qquad \mathbb{d}u(\mathfrak{d}) := \mathfrak{d}(u), \qquad \forall\ u \in \Gamma(E), \mathfrak{d} \in \Gamma(\mathfrak{D}E). \tag{3}$$

A useful formula is

$$\mathbb{d}(fu) = df \otimes u + f\mathbb{d}u, \quad \forall\ u \in \Gamma(E),\ f \in C^\infty(M).$$

There is an E-valued pairing between $\mathfrak{J}E$ and $\mathfrak{D}E$ defined by

$$\langle \mu, \mathfrak{d}\rangle := \mathfrak{d}(u), \qquad \mu \in (\mathfrak{J}E)_m, \mathfrak{d} \in (\mathfrak{D}E)_m,$$

where $u \in \Gamma(E)$ satisfies $\mu = [u]_m$. In particular, one has

$$\begin{aligned} \langle \mu, \Phi\rangle &= \Phi \circ \mathbb{p}(\mu), \quad \forall\ \Phi \in \mathrm{End}(E),\ \mu \in \mathfrak{J}E; \\ \langle \mathfrak{y}, \mathfrak{d}\rangle &= \mathfrak{y} \circ \mathbb{j}(\mathfrak{d}), \quad \forall\ \mathfrak{y} \in \mathrm{Hom}(TM, E),\ \mathfrak{d} \in \mathfrak{D}E. \end{aligned}$$

The **n-th differential operator bundle** $\mathfrak{D}^n E$ is introduced in [10, 35] as

$$\mathfrak{D}^n E := \mathrm{Hom}(\wedge^n \mathfrak{J}E, E)_{\mathfrak{D}E} = \{\mathfrak{d} \in \mathrm{Hom}(\wedge^n \mathfrak{J}E, E) | \mathrm{Im}(\mathfrak{d}_\sharp) \subset \mathfrak{D}E\}, \qquad n \geq 2,$$

where $\mathfrak{d}_\sharp : \wedge^{n-1}\mathfrak{J}E \to \mathrm{Hom}(\mathfrak{J}E, E)$ is defined by

$$\mathfrak{d}_\sharp(\mu_1, \cdots, \mu_{n-1})(\mu_n) = \mathfrak{d}(\mu_1, \cdots, \mu_n), \qquad \forall\ \mu_1, \cdots \mu_n \in \Gamma(\mathfrak{J}E).$$

When $\mathrm{rank}E \geq 2$, it fits into the following exact sequence:

$$0 \to \mathrm{Hom}(\wedge^n E, E) \xrightarrow{\mathbb{f}} \mathfrak{D}^n E \xrightarrow{\mathbb{q}} \mathrm{Hom}(\wedge^{n-1} E, TM) \to 0. \tag{4}$$

There is a graded Lie algebra structure on sections of $\mathfrak{D}^\bullet E$ ([10]) given as follows:

$$[\mathfrak{d}_1, \mathfrak{d}_2] = (-1)^{(k+1)(l+1)}\mathfrak{d}_1 \circ \mathfrak{d}_2 - \mathfrak{d}_2 \circ \mathfrak{d}_1 \in \Gamma(\mathfrak{D}^{k+l-1}E), \tag{5}$$

for $\mathfrak{d}_1 \in \Gamma(\mathfrak{D}^k E)$ and $\mathfrak{d}_2 \in \Gamma(\mathfrak{D}^l E)$, where

$$\begin{aligned} &\mathfrak{d}_2 \circ \mathfrak{d}_1(\mathbb{d}u_1, \cdots, \mathbb{d}u_{k+l-1}) \\ = &\sum_{\tau \in Sh(k,l-1)} (-1)^\tau \mathfrak{d}_2(\mathbb{d}(\mathfrak{d}_1(\mathbb{d}u_{\tau_1}, \cdots, \mathbb{d}u_{\tau_k})), \mathbb{d}u_{\tau_{k+1}}, \cdots, \mathbb{d}u_{\tau_{k+l-1}}), \end{aligned}$$

for $u_i \in \Gamma(E)$ and τ is taken over all $(k, l-1)$-shuffles.

In [8], the authors introduced the **n-th skew-symmetric jet bundle**

$$\mathfrak{J}_n E := \mathrm{Hom}(\wedge^n \mathfrak{D}E, E)_{\mathfrak{J}E} = \{\mu \in \mathrm{Hom}(\wedge^n \mathfrak{D}E, E) | \mathrm{Im}(\mu_\sharp) \subset \mathfrak{J}E\}, \qquad n \geq 2,$$

where $\mu_\sharp : \wedge^{n-1}\mathfrak{D}E \to \mathrm{Hom}(\mathfrak{D}E, E)$ is the induced bundle map

$$\mu_\sharp(\mathfrak{d}_1, \cdots \mathfrak{d}_{n-1})(\mathfrak{d}_n) = \mu(\mathfrak{d}_1, \cdots \mathfrak{d}_{n-1}, \mathfrak{d}_n), \qquad \forall\ \mathfrak{d}_1, \cdots \mathfrak{d}_n \in \Gamma(\mathfrak{D}E).$$

Moreover, the n-th skew-symmetric jet bundle also fits into an exact sequence

$$0 \to \mathrm{Hom}(\wedge^n TM, E) \xrightarrow{\mathbb{e}} \mathfrak{J}_n E \xrightarrow{\mathbb{p}} \mathrm{Hom}(\wedge^{n-1} TM, E) \to 0. \tag{6}$$

There is a complex $\mathbb{d} : \mathfrak{J}_\bullet E \to \mathfrak{J}_{\bullet+1} E$. It is given as a subcomplex of the Chavelley-Eilenberg complex of the Lie algebroid $\mathfrak{D}E$ with the natural representation on the vector bundle E, whose differential $d_{\mathrm{CE}} : \mathrm{Hom}(\wedge^n \mathfrak{D}E, E) \to \mathrm{Hom}(\wedge^{n+1}\mathfrak{D}E, E)$ is defined by

$$\begin{aligned} d_{\mathrm{CE}}\mu(\mathfrak{d}_1, \cdots, \mathfrak{d}_{n+1}) = & \sum_{i=1}^{n+1} (-1)^{i+1} \mathfrak{d}_i \big(\mu(\mathfrak{d}_1, \cdots, \hat{\mathfrak{d}_i}, \cdots, \mathfrak{d}_{n+1})\big) \\ & + \sum_{i<j} (-1)^{i+j} \mu([\mathfrak{d}_i, \mathfrak{d}_j], \mathfrak{d}_1, \cdots, \hat{\mathfrak{d}_i}, \cdots, \hat{\mathfrak{d}_j}, \cdots, \mathfrak{d}_{n+1}), \end{aligned} \tag{7}$$

for $\mathfrak{d}_i \in \Gamma(\mathfrak{D}E)$. See [8, Lemma 3.6] for details.

2.1. **Linear n-vector fields on a vector bundle.** For a vector bundle $p_E : E \to M$, identify $\Gamma(E^*)$ with the space of functions on E which are linear along each fiber. A section of the first differential operator bundle $\mathfrak{D}E^*$ maps a section of E^* to a section of E^*, which can be viewed as a linear vector field on E. Denote by $\mathfrak{X}^1_{lin}(E)$ the space of linear vector fields on E. We have $\Gamma(\mathfrak{D}E^*) \cong \mathfrak{X}^1_{lin}(E)$. We shall generalize this result to linear n-vector fields. Actually, linear n-vector fields studied in [5, 25] are isomorphic to sections of the n-th differential operator bundle $\mathfrak{D}^n E$ introduced in [10, 35].

An n-vector field $\pi \in \mathfrak{X}^n(E)$ is called **linear** if $\pi(d\phi_1, \cdots, d\phi_n) \in \Gamma(E^*)$ when $\phi_1, \cdots, \phi_n \in \Gamma(E^*)$ ([25]).

Linear multivector fields on a vector bundle can be described alternatively by the homogeneity structure; see the details in [18, 29]. Let $E \to M$ be a vector bundle. The monoid $\mathbb{R}_{\geq 0}$ of nonnegative real numbers acts on E by (fiberwise scalar multiplication)

$$h : \mathbb{R}_{\geq 0} \times E \to E, \quad (\lambda, e) \mapsto h_\lambda(e) := \lambda e.$$

This action is called the **homogeneity structure** on E. The homogeneity structure fully characterizes the vector bundle structure. In particular, a function on E is linear if it satisfies $h^*_\lambda f = \lambda f$, i.e. $f \in \Gamma(E^*)$.

Proposition 1. ([5, 29]) Let $\pi \in \mathfrak{X}^n(E)$. The following statements are equivalent:

(1) π is linear;
(2) $h^*_\lambda \pi = \lambda^{1-n}\pi$;
(3) Let $\{x^i\}$ be local coordinates on M and $\{u^j\}$ be local coordinates on the fiber of E. Then π has the local expression

$$\begin{aligned} \pi \quad = \quad & \frac{1}{n!}\pi_j^{i_1\cdots i_n}(x) u^j \frac{\partial}{\partial u^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial u^{i_n}} \\ & + \frac{1}{(n-1)!}\pi^{i_1\cdots i_{n-1},j}(x) \frac{\partial}{\partial u^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial u^{i_{n-1}}} \wedge \frac{\partial}{\partial x^j}. \end{aligned}$$

Denote by $\mathfrak{X}^n_{lin}(E)$ the space of linear n-vector fields on E. As explained in [25], a linear n-vector field π for $n \geq 2$ has the properties that

$$\pi(d\phi_1, \cdots, d\phi_{n-1}, dp_E^* f) = p_E^* g_{\phi_1,\cdots,\phi_{n-1},f} \tag{8}$$

for some $g_{\phi_1,\cdots,\phi_{n-1},f} \in C^\infty(M)$ and

$$\iota_{dp_E^* f_1}\iota_{dp_E^* f_2}\pi = 0, \qquad \forall\ \phi_1, \cdots \phi_{n-1} \in \Gamma(E^*), f, f_1, f_2 \in C^\infty(M).$$

As a consequence, from a linear n-vector field π, we obtain $\delta_0 : C^\infty(M) \to \Gamma(\wedge^{n-1}E)$ and $\delta_1 : \Gamma(E) \to \Gamma(\wedge^n E)$ given by

$$\delta_0(f)(\phi_1, \cdots, \phi_{n-1}) := g_{\phi_1,\cdots,\phi_{n-1},f},$$

and

$$\delta_1(u)(\phi_1, \cdots, \phi_n) := \sum_{i=1}^n (-1)^{i+n} g_{\phi_1,\cdots,\hat{\phi_i},\cdots,\phi_n,\phi_i(u)} - \langle \pi(d\phi_1, \cdots, d\phi_n), u\rangle,$$

for $u \in \Gamma(E)$. This correspondence is actually one-to-one; see [25].

We are now at the position to state our main result in this section.

Theorem 2.1. *For a vector bundle E, the space of linear multivector fields $\mathfrak{X}^\bullet_{lin}(E)$ with the Schouten bracket $[\cdot,\cdot]_S$ is a graded Lie algebra. Moreover, we have the isomorphism*

$$(\Gamma(\mathfrak{D}^\bullet E^*), [\cdot,\cdot]) \cong (\mathfrak{X}^\bullet_{lin}(E), [\cdot,\cdot]_S), \qquad \mathfrak{d} \mapsto \hat{\mathfrak{d}},$$

of graded Lie algebras, where $\hat{\mathfrak{d}}$ is determined by

$$\hat{\mathfrak{d}}(d\phi_1, \cdots, d\phi_n) := (-1)^n \mathfrak{d}(\mathbb{d}\phi_1, \cdots, \mathbb{d}\phi_n), \qquad \phi_i \in \Gamma(E^*), \tag{9}$$

where $\mathbb{d} : \Gamma(E) \to \Gamma(\mathfrak{J}E)$ is the natural map given by (3).

Proof. First, we prove that linear multivector fields on E are closed under the Schouten bracket, namely,

$$[\mathfrak{X}^l_{lin}(E), \mathfrak{X}^k_{lin}(E)]_S \subset \mathfrak{X}^{k+l-1}_{lin}(E).$$

In fact, for $X \in \mathfrak{X}^l_{lin}(E)$ and $Y \in \mathfrak{X}^k_{lin}(E)$, by the relation that

$$h_\lambda^*[X,Y]_S = [h_\lambda^* X, h_\lambda^* Y]_S = [\lambda^{1-l}X, \lambda^{1-k}Y]_S = \lambda^{1-(k+l-1)}[X,Y]_S,$$

we have $[X,Y] \in \mathfrak{X}^{k+l-1}_{lin}(E)$.

Secondly, we check that $\mathfrak{d} \mapsto \hat{\mathfrak{d}}$ is an isomorphism of graded vector spaces. We shall find its inverse. For $X \in \mathfrak{X}^n_{lin}(E)$, define $\check{X} \in \mathrm{Hom}(\wedge^n \mathfrak{J}E^*, E^*)$ by

$$\check{X}(\mathbb{d}\phi_1, \cdots, \mathbb{d}\phi_n) := (-1)^n X(d\phi_1, \cdots, d\phi_n) \in \Gamma(E^*), \qquad \phi_i \in \Gamma(E^*).$$

The function linear property of $\check{X}$ requires that

$$\begin{aligned} \check{X}(\mathbb{d}\phi_1, \cdots, \mathbb{d}\phi_{n-1}, df \otimes \phi_n) &= (-1)^n X(d\phi_1, \cdots, d\phi_{n-1}, dp_{E^*}^* f)\phi_n, \\ \check{X}(\mathbb{d}\phi_1, \cdots, \mathbb{d}\phi_{n-2}, dg \otimes \phi_{n-1}, df \otimes \phi_n) &= 0. \end{aligned}$$

By (8), there exists a vector field $\mathbb{j}(\check{X}_\sharp(d\phi_1, \cdots, d\phi_{n-1})) \in \mathfrak{X}^1(M)$ such that

$$\begin{aligned} \check{X}(\mathbb{d}\phi_1, \cdots, \mathbb{d}\phi_{n-1}, df \otimes \phi_n) &= \mathbb{j}(\check{X}_\sharp(d\phi_1, \cdots, d\phi_{n-1}))(f)\phi_n \\ &= (df \otimes \phi_n) \circ \mathbb{j}(\check{X}_\sharp(\mathbb{d}\phi_1, \cdots, \mathbb{d}\phi_{n-1})). \end{aligned}$$

Thus $\check{X} \in \Gamma(\mathfrak{D}^n E^*)$. So we get a map $\mathfrak{X}^n_{lin}(E) \to \mathfrak{D}^n(E^*), X \to \check{X}$, which is actually the inverse of the map $\mathfrak{d} \mapsto \hat{\mathfrak{d}}$.

At last, we show that

$$\widehat{[\mathfrak{d}, \mathfrak{t}]} = [\hat{\mathfrak{d}}, \hat{\mathfrak{t}}]_S, \qquad \mathfrak{d} \in \Gamma(\mathfrak{D}^l E^*), \mathfrak{t} \in \Gamma(\mathfrak{D}^k E^*).$$

Actually, by (5), we have

$$\begin{aligned}
& \widehat{[\mathfrak{d},\mathfrak{t}]}(d\phi_1,\cdots,d\phi_{k+l-1}) \\
= & (-1)^{k+l-1}[\mathfrak{d},\mathfrak{t}](\mathbb{d}\phi_1,\cdots,\mathbb{d}\phi_{k+l-1}) \\
= & (-1)^{kl}\sum_{\sigma\in Sh(k,l-1)}(-1)^{\sigma}\mathfrak{d}(\mathbb{d}(\mathfrak{t}(\mathbb{d}\phi_{\sigma_1},\cdots,\mathbb{d}\phi_{\sigma_k})),\mathbb{d}\phi_{\sigma_{k+1}},\cdots,\mathbb{d}\phi_{\sigma_{k+l-1}}) \\
& -(-1)^{k+l-1}\sum_{\tau\in Sh(l,k-1)}(-1)^{\tau}\mathfrak{t}(\mathbb{d}(\mathfrak{d}(\mathbb{d}\phi_{\tau_1},\cdots,\mathbb{d}\phi_{\tau_l})),\mathbb{d}\phi_{\tau_{l+1}},\cdots,\mathbb{d}\phi_{\tau_{k+l-1}}) \\
= & [\hat{\mathfrak{d}},\hat{\mathfrak{t}}]_S(d\phi_1,\cdots,d\phi_{k+l-1}).
\end{aligned}$$

We thus proved that $\mathfrak{d}\mapsto\hat{\mathfrak{d}}$ defines an isomorphism of graded Lie algebras. □

By [10, 35], the exact sequence (4) always splits when $\mathrm{rank}E\geq 2$.

Corollary 1. *If* $\mathrm{rank}E\geq 2$*, then we have*

$$\mathfrak{X}^n_{lin}(E)\cong\Gamma(\mathfrak{D}^nE^*)\cong\Gamma(\wedge^nE\otimes E^*)\oplus\Gamma(\wedge^{n-1}E\otimes TM).$$

Example 1. *When* $E=TM$ *for a manifold* M*, we have*

$$\mathfrak{X}^n_{lin}(TM)\cong\Gamma(\mathfrak{D}^nT^*M)\cong\mathfrak{X}^n(M)\otimes\Omega^1(M)\oplus(\mathfrak{X}^{n-1}(M)\otimes\mathfrak{X}^1(M)).$$

Example 2. *When* $E=T^*M$ *for a manifold* M*, we have*

$$\mathfrak{X}^n_{lin}(T^*M)\cong\Gamma(\mathfrak{D}^nTM)\cong\Omega^n(M)\otimes\mathfrak{X}^1(M)\oplus(\Omega^{n-1}(M)\otimes\mathfrak{X}^1(M)).$$

Example 3. *When* $E=V^*$ *is a vector space, we have* $\mathfrak{D}V=\mathrm{gl}(V)$ *and* $\mathfrak{J}V=V$*. In this case,*

$$\mathfrak{X}^n_{lin}(V^*)\cong\Gamma(\mathfrak{D}^nV)=\mathrm{Hom}(\wedge^nV,V).$$

When $E=M\times V^*$*, we have* $\mathfrak{D}(M\times V)=TM\oplus\mathrm{gl}(V)$ *and* $\mathfrak{J}(M\times V)=(T^*M\otimes V)\oplus(M\times V)$*. Furthermore, the space of linear* n*-vector fields on* $M\times V^*$ *is*

$$\mathfrak{X}^n_{lin}(M\times V^*)\cong\Gamma(\mathfrak{D}^n(M\times V))\cong\mathrm{Hom}(\wedge^nV,V)\oplus(\mathfrak{X}^1(M)\otimes\wedge^{n-1}V^*).$$

Example 4. *Consider the case* $E=M\times\mathbb{R}$*. Then we have* $\mathfrak{D}E=TM\times\mathbb{R}$ *and* $\mathfrak{J}E=T^*M\times\mathbb{R}$*. By definitions, we obtain*

$$\mathfrak{X}^n_{lin}(M\times\mathbb{R})\cong\Gamma(\mathfrak{D}^n(M\times\mathbb{R}))\cong\mathfrak{X}^n(M)\oplus\mathfrak{X}^{n-1}(M),\qquad n\geq 1.$$

2.2. **Linear n-forms on a vector bundle.** It is well known that linear 1-forms on a vector bundle E can be viewed as sections of the first jet bundle $\mathfrak{J}E^*$. Here we find that linear n-forms studied in [5] are sections of the n-th jet bundle $\mathfrak{J}_nE$ introduced in [8].

Definition 2.2. ([5]) An n-form Λ on a vector bundle E is called **linear** if the induced map $\Lambda^\sharp:\oplus^{n-1}_ETE\to T^*E$:

$$\begin{array}{ccc}
\oplus^{n-1}_ETE & \xrightarrow{\Lambda^\sharp} & T^*E \\
\downarrow & & \downarrow \\
\oplus^{n-1}TM & \xrightarrow{\lambda} & E^*
\end{array},$$

is a morphism of vector bundles, where $\lambda:\oplus^{n-1}TM\to E^*$ is the covering map on the base manifolds. The space of linear n-forms on E is denoted by $\Omega^n_{lin}(E)$.

As the map λ is skew-symmetric, it is a bundle map $\wedge^{n-1}TM \to E^*$. In particular, a linear 1-form is a section of $T^*E \to E$ which induces a bundle map from $E \to M$ to $T^*E \to E^*$. In other words, it is a linear section of $T^*E \to E$ in the double vector bundle $(T^*E; E, E^*; M)$, which is a section of $\mathfrak{J}E^*$.

A simpler description of linear n-forms is by using the homogeneity structure. Similar to Proposition 1, we have the following results for linear n-forms, which are based on results from [5, 18, 29].

Proposition 2. *Let $\Lambda \in \Omega^n(E)$. Then the following statements are equivalent:*

(1) *Λ is a linear n-form;*
(2) *$h_\lambda^*\Lambda = \lambda\Lambda$;*
(3) *Choose a local coordinate $\{x^i, u^j\}$ on E, where $\{x^i\}$ and $\{u^j\}$ are the coordinate functions on M and the fiber respectively. Then Λ is locally of the form*

$$\begin{aligned}\Lambda = &\frac{1}{n!}\Lambda_{i_1\cdots i_n,j}(x)u^j dx^{i_1}\wedge\cdots\wedge dx^{i_n}\\ &+\frac{1}{(n-1)!}\lambda_{i_1\cdots i_{n-1},j}(x)dx^{i_1}\wedge\cdots dx^{i_{n-1}}\wedge du^j.\end{aligned} \tag{10}$$

It is obvious from (2) in the above proposition that linear forms are closed under the de Rham differential.

We give another equivalent description of linear n-forms on E by use of the linear vector fields on E.

Lemma 2.3. *An n-form $\Lambda \in \Omega^n(E)$ is linear if and only if there exists a bundle map $\lambda : \wedge^{n-1}TM \to E^*$, such that*

$$\Lambda(X_1,\cdots,X_n)\in\Gamma(E^*),\qquad \Lambda(X_1,\cdots,X_{n-1},\Phi) = \Phi\circ\lambda(\underline{X_1},\cdots,\underline{X_{n-1}}),$$

where $X_i \in \mathfrak{X}^1_{lin}(E)$ which determines $\underline{X_i} \in \mathfrak{X}^1(M)$ and $\Phi \in \mathfrak{X}^1_{lin}(E)$ satisfying $\Phi(dp_E^* f) = 0$ for any $f \in C^\infty(M)$.

Proof. Taking a local coordinate (x^i, u^j) for E, a linear vector field $X \in \mathfrak{X}^1_{lin}(E)$ has the form

$$X = f_j^k(x)u^j\frac{\partial}{\partial u^k} + f^i(x)\frac{\partial}{\partial x^i}.$$

If Λ is linear, by the local formula (10), it is direct to see that $\Lambda(X_1,\cdots,X_n)\in\Gamma(E^*)$ for $X_i \in \mathfrak{X}^1_{lin}(E)$. Then suppose

$$X_l = f_{j_l}^{k_l}(x)u^{j_l}\frac{\partial}{\partial u^{k_l}} + f^{i_l}(x)\frac{\partial}{\partial x^{i_l}},$$

we have $\underline{X_l} = f^{i_l}(x)\frac{\partial}{\partial x^{i_l}} \in \mathfrak{X}^1(M)$. Assume $\Phi = f_j^k(x)u^j\frac{\partial}{\partial u^k}$, we have

$$\begin{aligned}\Lambda(X_1,\cdots,X_{n-1},\Phi) &= \Lambda(f^{i_1}(x)\frac{\partial}{\partial x^{i_1}},\cdots,f^{i_{n-1}}(x)\frac{\partial}{\partial x^{i_{n-1}}}, f_j^k(x)u^j\frac{\partial}{\partial u^k})\\ &= \Phi\circ\lambda(\underline{X_1},\cdots,\underline{X_{n-1}}).\end{aligned}$$

It is similar for the converse. □

The following theorem is a dual of Theorem 2.1.

Theorem 2.4. *For a vector bundle E, we have $d\Omega^n_{lin}(E) \subset \Omega^{n+1}_{lin}(E)$. Moreover, we have an isomorphism of cochain complexes:*

$$(\Gamma(\mathfrak{J}_\bullet E^*), \mathbb{d}) \cong (\Omega^\bullet_{lin}(E), d), \qquad \mu \mapsto \hat{\mu},$$

where $\hat{\mu}$ *is defined by*

$$\hat{\mu}(X_1, \cdots, X_n) = \mu(\check{X}_1, \cdots, \check{X}_n), \qquad X_i \in \mathfrak{X}^1_{lin}(E)$$

and $\check{X}_i \in \Gamma(\mathfrak{D}E^*)$ *is defined by* $\check{X}_i(\mathbb{d}\phi) = X_i(d\phi)$ *for* $\phi \in \Gamma(E^*)$.

Proof. By using the homogeneity, an n-form $\Lambda \in \Omega^n(E)$ is linear if $h^*_\lambda \Lambda = \lambda\Lambda$. It is obvious that the de Rham differential $d\Lambda$ is a linear $(n+1)$-form. This fact can also be seen from the local expression (10) of linear forms. We get that $d\Omega^n_{lin}(E) \subset \Omega^{n+1}_{lin}(E)$.

Then we check that $\mu \mapsto \hat{\mu}$ is well-defined. It is obvious that $\hat{\mu}(X_1, \cdots, X_n) \in \Gamma(E^*)$, and

$$\hat{\mu}(X_1, \cdots, X_{n-1}, \Phi) = \mu(\check{X}_1, \cdots, \check{X}_{n-1}, \Phi) = \Phi \circ \mathbb{p}(\mu)(\underline{X}_1, \cdots, \underline{X_{n-1}}),$$

where $\mathbb{p} : \mathfrak{J}_n E^* \to \mathrm{Hom}(\wedge^{n-1}TM, E^*)$ is the map in (6). So $\hat{\mu} \in \Omega^n_{lin}(E)$ and the associated bundle map $\wedge^{n-1}TM \to E^*$ is $\mathbb{p}(\mu)$.

Then we define an inverse map of $\mu \mapsto \hat{\mu}$. For $\Lambda \in \Omega^n_{lin}(E)$, define

$$\check{\Lambda}(\mathfrak{d}_1, \cdots, \mathfrak{d}_n) = \Lambda(\hat{\mathfrak{d}_1}, \cdots, \hat{\mathfrak{d}_n}), \qquad \mathfrak{d}_i \in \Gamma(\mathfrak{D}E^*),$$

where $\hat{\mathfrak{d}}_i \in \mathfrak{X}^1_{lin}(E)$ is defined by $\hat{\mathfrak{d}}_i(d\phi) = \mathfrak{d}_i(\mathbb{d}\phi)$ for $\phi \in \Gamma(E^*)$. By Lemma 2.3, we can check that $\check{\Lambda} \in \Gamma(\mathfrak{J}_n E^*)$. We get a map

$$\Omega^n_{lin}(E) \to \Gamma(\mathfrak{J}_n E^*), \qquad \Lambda \mapsto \check{\Lambda},$$

and it is the inverse of the map $\mu \mapsto \hat{\mu}$. We get an isomorphism.

Now it is left to check $\mu \mapsto \hat{\mu}$ actually gives an isomorphism of cochain complexes, namely

$$\widehat{\mathbb{d}\mu} = d\hat{\mu}, \qquad \mu \in \Gamma(\mathfrak{J}_n E^*).$$

We have the following diagram:

$$\begin{array}{ccc} (\Gamma(\mathfrak{J}_\bullet E^*), \mathbb{d}) & \xrightarrow{\subset} & (\mathrm{Hom}(\wedge^n \mathfrak{D}E^*, E^*), d_{\mathrm{CE}}) \\ \downarrow & & \downarrow \\ (\Omega^\bullet_{lin}(E), d) & \xrightarrow{\subset} & (\mathrm{Hom}(\wedge^n \mathfrak{X}^1_{lin}(E), \Gamma(E^*)), d), \end{array}$$

where the left two complexes are subcomplexes of the right two complexes. We note that the right vertical side is actually an isomorphism of cochain complexes. As $\Gamma(\mathfrak{D}E) \cong \mathfrak{X}^1_{lin}(E)$, comparing the Chavelley-Eilenberg differential (7) and the de Rham differential, we have $\widehat{\mathbb{d}\mu} = \widehat{d_{\mathrm{CE}}\mu} = d\hat{\mu}$ for $\mu \in \Gamma(\mathfrak{J}_n E^*)$. ☐

The exact sequence (6) of $\mathfrak{J}_n E$ splits at the level of sections ([8]).

Corollary 2. *We have*

$$\Omega^n_{lin}(E) \cong \Gamma(\mathfrak{J}_n E^*) \cong \Gamma(\wedge^n T^*M \otimes E^*) \oplus \Gamma(\wedge^{n-1} T^*M \otimes E^*).$$

Example 5. *When* $E = TM$ *for a manifold* M*, we have*

$$\Omega^n_{lin}(TM) \cong \Gamma(\mathfrak{J}_n T^*M) \cong \Omega^n(M) \otimes \Omega^1(M) \oplus (\Omega^{n-1}(M) \otimes \Omega^1(M)).$$

Example 6. *When* $E = T^*M$ *for a manifold* M*, we have*

$$\Omega^n_{lin}(T^*M) \cong \Gamma(\mathfrak{J}_n TM) \cong \Omega^n(M) \otimes \mathfrak{X}^1(M) \oplus (\Omega^{n-1}(M) \otimes \mathfrak{X}^1(M)).$$

Comparing with Example 2, we see $\mathfrak{X}^n_{lin}(T^*M) \cong \Omega^n_{lin}(T^*M)$.

Example 7. *When $E = V^*$ is a vector space, we have*

$$\Omega^n_{lin}(V^*) \cong \Gamma(\mathfrak{J}_n V) = 0, \qquad n \geq 2.$$

Actually, for $\mu \in \Gamma(\mathfrak{J}_2 V)$, as $\mu(A \wedge B) = B\mu(A \wedge \mathrm{Id}_V) = -BA\mu(\mathrm{Id}_V \wedge \mathrm{Id}_V) = 0$ for $A, B \in \mathrm{gl}(V)$, we see $\mu = 0$.

When $E = M \times V^*$, we get

$$\Omega^n_{lin}(M \times V^*) \cong \Gamma(\mathfrak{J}_n(M \times V)) \cong \Omega^n(M) \otimes V \oplus \Omega^{n-1}(M) \otimes V.$$

Example 8. *Consider the case $E = M \times \mathbb{R}$, the trivial line bundle. We have $\mathfrak{D}E = TM \times \mathbb{R}$ and $\mathfrak{J}E = T^*M \times \mathbb{R}$. By definition, we obtain*

$$\Omega^n_{lin}(M \times \mathbb{R}) \cong \Gamma(\mathfrak{J}_n E^*) \cong \Omega^n(M) \oplus \Omega^{n-1}(M).$$

2.3. Pseudo-linearization of higher analogues of Courant algebroids $TE^* \oplus \wedge^n T^* E^*$.

In this section, as consequences of Theorems 2.1 and 2.4, we show that the n-omni-Lie algebroid $\mathfrak{D}E \oplus \mathfrak{J}_n E$ introduced in [3] is certain linearization of the higher analogue of Courant algebroids $TE^* \oplus \wedge^n T^* E^*$ ([1, 42]).

For a manifold M, on the vector bundle $\mathcal{T}^n := TM \oplus \wedge^n T^*M$, there exists a natural non-degenerate symmetric pairing with values in $\wedge^{n-1}T^*M$:

$$(X + \alpha, Y + \beta) = \iota_X \beta + \iota_Y \alpha, \qquad \forall\ X, Y \in \mathfrak{X}^1(M), \alpha, \beta \in \Omega^n(M),$$

an anchor map

$$\rho : \mathcal{T}^n \to TM, \qquad \rho(X + \alpha) = X,$$

and a higher Dorfman bracket on $\Gamma(\mathcal{T}^n)$:

$$\{X + \alpha, Y + \beta\} = [X, Y] + L_X \beta - \iota_Y d\alpha.$$

They satisfy some properties similar to that for a Courant algebroid. The quadruple $(\mathcal{T}^n, (\cdot,\cdot), \{\cdot,\cdot\}, \rho)$ is called **a higher analogue of Courant algebroids** in [1, 42]. See [11] for a similar structure on $A \oplus \wedge^n A^*$ for any Lie algebroid A and relation to the shifted cotangent bundle $T^*[n]A[1]$.

The **n-omni-Lie algebroid** of a vector bundle E ([3]) is the quadruple $(\mathfrak{D}E \oplus \mathfrak{J}_n E, (\cdot,\cdot), \{\cdot,\cdot\}, \rho)$, where $\rho : \mathfrak{D}E \oplus \mathfrak{J}_n E \to \mathfrak{D}E$ is the projection to the first summand, $(\cdot,\cdot)$ is the $\mathfrak{J}_{n-1}E$-valued pairing

$$(\mathfrak{d} + \mu, \mathfrak{t} + \nu) = \iota_{\mathfrak{d}} \nu + \iota_{\mathfrak{t}} \mu, \qquad \forall\ \mathfrak{d}, \mathfrak{t} \in \Gamma(\mathfrak{D}E), \mu, \nu \in \Gamma(\mathfrak{J}_n E),$$

and the bracket $\{\cdot,\cdot\}$ is

$$\{\mathfrak{d} + \mu, \mathfrak{t} + \nu\} = [\mathfrak{d}, \mathfrak{t}] + L_{\mathfrak{d}} \nu - \iota_{\mathfrak{t}} \mathrm{d}\mu.$$

Here $L_{\mathfrak{d}} : \Gamma(\mathfrak{J}_n E) \to \Gamma(\mathfrak{J}_n E)$ is defined in the following way: for $\nu \in \Gamma(\mathfrak{J}_n E) \subset \mathrm{Hom}(\wedge^n \mathfrak{D}E, E)$, suppose $\nu = \omega \otimes u$ for $\omega \in \Gamma(\wedge^n(\mathfrak{D}E)^*)$ and $u \in \Gamma(E)$. Define

$$L_{\mathfrak{d}} \nu = (L_{\mathfrak{d}} \omega) \otimes u + \omega \otimes \mathfrak{d}(u).$$

It is proved in [8, Proposition 3.2] that $L_{\mathfrak{d}} \nu \in \Gamma(\mathfrak{J}_n E)$.

By Theorems 2.1 and 2.4, linear vector fields and linear n-forms on E^* can be seen as sections of $\mathfrak{D}E$ and $\mathfrak{J}_n E$ respectively. So linear sections of $TE^* \oplus \wedge^n T^* E^*$ are sections of the vector bundle $\mathfrak{D}E \oplus \mathfrak{J}_n E$. Also, linear multivector fields and linear forms on a vector bundle are closed under the Schouten bracket and the de Rham differential respectively. The following lemma states that the linearity is also preserved by the Lie derivative and the contraction.

Lemma 2.5. *We have*

$$\iota_{\mathfrak{X}^1_{lin}(E)} \Omega^n_{lin}(E) \subset \Omega^{n-1}_{lin}(E), \qquad L_{\mathfrak{X}^1_{lin}(E)} \Omega^n_{lin}(E) \subset \Omega^n_{lin}(E).$$

Proof. For $X \in \mathfrak{X}^1_{lin}(E)$ and $\Lambda \in \Omega^n_{lin}(E)$, since $h^*_\lambda X = X$ and $h^*_\lambda \Lambda = \lambda\Lambda$, it is obvious that $h^*_\lambda(\iota_X\Lambda) = \lambda\iota_X\Lambda$ and $h^*_\lambda(L_X\Lambda) = \lambda L_X\Lambda$. The conclusion follows immediately. □

Recall from Theorems 2.1 and 2.4 that we have the isomorphisms $\Gamma(\mathfrak{D}^\bullet E) \cong \mathfrak{X}^\bullet_{lin}(E^*), \mathfrak{d} \mapsto \hat{\mathfrak{d}}$ and $\Gamma(\mathfrak{J}_\bullet E) \cong \Omega^\bullet_{lin}(E^*), \mu \mapsto \hat{\mu}$.

Theorem 2.6. *For a vector bundle E, the n-omni-Lie algebroid $\mathfrak{D}E \oplus \mathfrak{J}_n E$ is induced from the higher analogue of Courant algebroids $(TE^* \oplus \wedge^n T^*E^*, (\cdot,\cdot), \{\cdot,\cdot\}, \rho)$ by restricting to $\mathfrak{X}^1_{lin}(E^*) \oplus \Omega^n_{lin}(E^*)$. Precisely, we have*

$$\begin{aligned} \widehat{(\mathfrak{d},\mu)} &= (\hat{\mathfrak{d}},\hat{\mu}); \\ \widehat{[\mathfrak{d},\mathfrak{t}]} &= [\hat{\mathfrak{d}},\hat{\mathfrak{t}}]_S; \\ \widehat{L_\mathfrak{d}\mu} &= L_{\hat{\mathfrak{d}}}\hat{\mu}; \\ \widehat{\iota_\mathfrak{d}\mathbb{d}\mu} &= \iota_{\hat{\mathfrak{d}}}d\hat{\mu}, \end{aligned}$$

for $\mathfrak{d}, \mathfrak{t} \in \Gamma(\mathfrak{D}E)$ and $\mu \in \Gamma(\mathfrak{J}_n E)$.

Proof. By Theorems 2.1 and 2.4, we know that $\Gamma(\mathfrak{D}E) \cong \mathfrak{X}^1_{lin}(E^*)$, $\Gamma(\mathfrak{J}_n E) \cong \Omega^n_{lin}(E^*)$, and we have $\widehat{[\mathfrak{d},\mathfrak{t}]} = [\hat{\mathfrak{d}},\hat{\mathfrak{t}}]_S$. We claim that

$$\iota_{\hat{\mathfrak{d}}}\hat{\mu} = \widehat{\iota_\mathfrak{d}\mu}, \qquad \mathfrak{d} \in \Gamma(\mathfrak{D}E), \mu \in \Gamma(\mathfrak{J}_n E). \tag{11}$$

Indeed, for $X_1, \cdots, X_{n-1} \in \mathfrak{X}^1_{lin}(E^*)$, we have

$$\begin{aligned} \iota_{\hat{\mathfrak{d}}}\hat{\mu}(X_1,\cdots,X_{n-1}) &= \hat{\mu}(\hat{\mathfrak{d}}, X_1,\cdots,X_{n-1}) \\ &= \mu(\mathfrak{d}, \check{X}_1,\cdots,\check{X}_{n-1}) = \widehat{\iota_\mathfrak{d}\mu}(X_1,\cdots,X_{n-1}), \end{aligned}$$

where $\check{X}_i \in \Gamma(\mathfrak{D}E)$ is defined by $\check{X}_i(\mathbb{d}u) = X_i(du)$ for $u \in \Gamma(E)$. We thus have $\widehat{(\mathfrak{d},\mu)} = \widehat{\iota_\mathfrak{d}\mu} = \iota_{\hat{\mathfrak{d}}}\hat{\mu} = (\hat{\mathfrak{d}},\hat{\mu})$. By Theorem 2.4, we have

$$d\hat{\mu} = \widehat{\mathbb{d}\mu}. \tag{12}$$

By use of (11) and (12), we have

$$\widehat{\iota_\mathfrak{d}\mathbb{d}\mu} = \iota_{\hat{\mathfrak{d}}}\widehat{\mathbb{d}\mu} = \iota_{\hat{\mathfrak{d}}}d\hat{\mu},$$

and

$$\widehat{L_\mathfrak{d}\mu} = \widehat{\iota_\mathfrak{d}\mathbb{d}\mu} + \widehat{\mathbb{d}\iota_\mathfrak{d}\mu} = \iota_{\hat{\mathfrak{d}}}\widehat{\mathbb{d}\mu} + d\widehat{\iota_\mathfrak{d}\mu} = \iota_{\hat{\mathfrak{d}}}d\hat{\mu} + d\iota_{\hat{\mathfrak{d}}}\hat{\mu} = L_{\hat{\mathfrak{d}}}\hat{\mu}.$$

This completes the proof. □

As a consequence, we call n-omni-Lie algebroids the **pseudo-linearization** of higher analogues of Courant algebroids. As the linearization, n-omni-Lie algebroids inherit many properties of the higher analogues of Courant algebroids. By [1, Theorem 2.2, 2.5] and Theorem 2.6, we recover the following result in [3], where they proved it by direct calculation. In preparation, we first recall the definition of Leibniz algebroids.

A **Leibniz algebroid** ([16, 17, 23]) is a vector bundle E with a bracket $\{\cdot,\cdot\}$ on $\Gamma(E)$ and a bundle map $\rho : E \to TM$, called the anchor map, satisfying that

$$\{u,\{v,w\}\} = \{\{u,v\},w\} + \{v,\{u,w\}\}, \qquad \{u, fv\} = f\{u,v\} + \rho(u)(f)v,$$

for all $u, v, w \in \Gamma(E)$ and $f \in C^\infty(M)$.

Corollary 3. *Let $(\mathfrak{D}E \oplus \mathfrak{J}_n E, (\cdot,\cdot), \{\cdot,\cdot\}, \rho)$ be an n-omni-Lie algebroid. Then*

(i) *$(\mathfrak{D}E \oplus \mathfrak{J}_n E, \{\cdot,\cdot\}, \mathbb{j} \circ \rho)$ is a Leibniz algebroid, where $\mathbb{j} : \mathfrak{D}E \to TM$ is the map in (1);*

(ii) $\{e, e\} = \frac{1}{2}\mathbb{d}(e, e)$;
(iii) $\rho(e_1)(e_2, e_3) = (\{e_1, e_2\}, e_3) + (e_2, \{e_1, e_3\})$,
for all $e, e_i \in \Gamma(\mathfrak{D}E \oplus \mathfrak{J}_n E)$.

Remark 1. When $E = V$, a vector space, the n-omni-Lie algebroid for $n \geq 2$ is just $\mathrm{gl}(V)$. So n-omni-Lie algebroids do not include omni n-Lie algebras studied in [31] as special cases.

3. **Weinstein-linearization of $TE^* \oplus \wedge^n T^*E^*$ and omni n-Lie algebroids.** On the vector bundle $\mathfrak{D}E \oplus \wedge^n \mathfrak{J}E$, we introduce an $E \otimes \wedge^{n-1}\mathfrak{J}E$-valued pairing

$$(\mathfrak{d} + \alpha, \mathfrak{t} + \beta) = \iota_{\mathfrak{d}}\beta + \iota_{\mathfrak{t}}\alpha, \qquad \forall\ \mathfrak{d}, \mathfrak{t} \in \Gamma(\mathfrak{D}E), \alpha, \beta \in \Gamma(\wedge^n \mathfrak{J}E), \tag{13}$$

and a bracket

$$\{\mathfrak{d} + \alpha, \mathfrak{t} + \beta\} := [X, Y] + L_{\mathfrak{d}}\beta - \iota_{\mathfrak{t}}\mathbb{d}\alpha, \tag{14}$$

where $L_{\mathfrak{d}} : \Gamma(\wedge^n \mathfrak{J}E) \to \Gamma(\wedge^n \mathfrak{J}E)$ is defined by

$$L_{\mathfrak{d}}(\alpha_1 \wedge \cdots \alpha_n) = \sum_{i=1}^{n} \alpha_1 \wedge \cdots \wedge L_{\mathfrak{d}}\alpha_i \wedge \alpha_{i+1} \wedge \cdots \wedge \alpha_n, \qquad \alpha_i \in \Gamma(\mathfrak{J}E),$$

and $\mathbb{d} : \Gamma(\wedge^n \mathfrak{J}E) \to \Gamma(\mathfrak{J}_2 E \otimes \wedge^{n-1}\mathfrak{J}E)$ is defined by

$$\mathbb{d}(\alpha_1 \wedge \cdots \wedge \alpha_n) = \sum_{i=1}^{n} (-1)^{i-1} (\mathbb{d}\alpha_i) \otimes \alpha_1 \wedge \cdots \wedge \hat{\alpha_i} \wedge \cdots \wedge \alpha_n,$$

where $\hat{\alpha_i}$ means taking α_i out. The contraction

$$\iota_{\mathfrak{t}} : \Gamma(\mathfrak{J}_2 E \otimes \wedge^{n-1}\mathfrak{J}E) \longrightarrow \Gamma(\wedge^n \mathfrak{J}E)$$

is defined by

$$\begin{aligned}\iota_{\mathfrak{t}}(\omega \otimes \alpha_1 \wedge \cdots \wedge \alpha_{n-1}) &:= \omega(\mathfrak{t}) \wedge \alpha_1 \wedge \cdots \alpha_{n-1} \\ &\quad + \sum_{i=1}^{n} (-1)^i \iota_{\mathfrak{t}}\alpha_i \otimes \omega \wedge \alpha_1 \wedge \cdots \wedge \hat{\alpha_i} \wedge \cdots \wedge \alpha_{n-1},\end{aligned} \tag{15}$$

for $\omega \in \Gamma(\mathfrak{J}_2 E)$ and $\alpha_i \in \Gamma(\mathfrak{J}E)$.

The following lemma makes sure that $\iota_{\mathfrak{t}}$ is well-defined.

Lemma 3.1. *For* $\mathfrak{t} \in \Gamma(\mathfrak{D}E)$ *and* $\omega \otimes \alpha_1 \wedge \cdots \wedge \alpha_{n-1} \in \Gamma(\mathfrak{J}_2 E \otimes \wedge^{n-1}\mathfrak{J}E)$*, we have*

$$\iota_{\mathfrak{t}}(\omega \otimes \alpha_1 \wedge \cdots \wedge \alpha_{n-1}) \in \Gamma(\wedge^n \mathfrak{J}E).$$

Proof. Let us prove this result by using Theorem 2.4. With respect to the relation between $\Omega^2_{lin}(E^*)$ and $\wedge^2\Omega^1_{lin}(E^*)$, we have the following assertion:

$$\Gamma(E) \otimes \Omega^2_{lin}(E^*) \hookrightarrow \wedge^2\Omega^1_{lin}(E^*). \tag{16}$$

It is easily seen from the homogeneity structure on E^*. In fact, for $f \in \Gamma(E)$ and $\Lambda \in \Omega^2_{lin}(E^*)$, we have $h^*_\lambda f = \lambda f$ and $h^*_\lambda \Lambda = \lambda\Lambda$. Then, $h^*_\lambda(f\Lambda) = h^*_\lambda(f)h^*_\lambda(\Lambda) = \lambda^2 f\Lambda$, which implies that $f\Lambda \in \wedge^2\Omega^1_{lin}(E^*)$. Thus we proved (16). By Theorem 2.4, we further get

$$\Gamma(E \otimes \mathfrak{J}_2 E) \hookrightarrow \Gamma(\wedge^2 \mathfrak{J}E). \tag{17}$$

This implies that $\iota_{\mathfrak{t}}\alpha_i \otimes \omega$ in (15) belongs to $\Gamma(\wedge^2 \mathfrak{J}E)$. So both of the two terms in the right hand side of (15) belong to $\Gamma(\wedge^n \mathfrak{J}E)$. We obtain $\iota_{\mathfrak{t}}(\omega \otimes \alpha_1 \wedge \cdots \wedge \alpha_{n-1}) \in \Gamma(\wedge^n \mathfrak{J}E)$. □

Based on this lemma, for $\mathfrak{t} \in \Gamma(\mathfrak{D}E)$ and $\alpha \in \Gamma(\wedge^n \mathfrak{J}E)$, we have $\iota_{\mathfrak{t}} \mathbb{d}\alpha \in \Gamma(\wedge^n \mathfrak{J}E)$. Since $L_{\mathfrak{t}}\alpha \in \Gamma(\wedge^n \mathfrak{J}E)$ by definition, we further get $\mathbb{d}\iota_{\mathfrak{t}}\alpha \in \Gamma(\wedge^n \mathfrak{J}E)$.

Definition 3.2. The **omni n-Lie algebroid**[1] associated to a vector bundle E is a quadruple $(\mathfrak{D}E \oplus \wedge^n \mathfrak{J}E, (\cdot,\cdot), \{\cdot,\cdot\}, \rho)$, where ρ is the anchor map

$$\rho : \mathfrak{D}E \oplus \wedge^n \mathfrak{J}E \to \mathfrak{D}E, \qquad \rho(X + \alpha) = X,$$

the $E \otimes \wedge^{n-1}\mathfrak{J}E$-valued pairing $(\cdot,\cdot)$ and the bracket $\{\cdot,\cdot\}$ are given by (13) and (14) respectively.

3.1. **Weinstein-linearization of higher analogues of Courant algebroids $TE^* \oplus \wedge^n T^*E^*$.** In this subsection, we show that the omni n-Lie algebroid given in the last subsection is certain linearization of the higher analogue of Courant algebroids $TE^* \oplus \wedge^n T^*E^*$.

By Theorem 2.4, we have the isomorphism $\Gamma(\wedge^n \mathfrak{J}E) \cong \wedge^n \Omega^1_{lin}(E^*), \alpha \mapsto \hat{\alpha}$, where

$$\hat{\alpha} = \hat{\alpha_1} \wedge \cdots \wedge \hat{\alpha_n} \tag{18}$$

if $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_n$ for $\alpha_i \in \Gamma(\mathfrak{J}E)$. Here $\hat{\alpha_i} \in \Omega^1_{lin}(E^*)$.

Theorem 3.3. *The embedding of sections of the omni n-Lie algebroid $\mathfrak{D}E \oplus \wedge^n \mathfrak{J}E$ into the higher analogue of Courant algebroids $TE^* \oplus \wedge^n T^*E^*$ given by (9) and (18) defines a sub-Leibniz algebroid satisfying*

$$\begin{aligned}
\widehat{(\mathfrak{d}, \alpha)} &= (\hat{\mathfrak{d}}, \hat{\alpha}); \\
\widehat{[\mathfrak{d}, \mathfrak{t}]} &= [\hat{\mathfrak{d}}, \hat{\mathfrak{t}}]_S; \\
\widehat{L_{\mathfrak{d}}\alpha} &= L_{\hat{\mathfrak{d}}}\hat{\alpha}; \\
\widehat{\iota_{\mathfrak{d}}\mathbb{d}\alpha} &= \iota_{\hat{\mathfrak{d}}} d\hat{\alpha},
\end{aligned}$$

for $\mathfrak{d}, \mathfrak{t} \in \Gamma(\mathfrak{D}E)$ and $\alpha \in \Gamma(\wedge^n \mathfrak{J}E)$.

Proof. Based on Theorem 2.1 and 2.4, following the same manner as in Theorem 2.6, we could get this result. We omit the details here. □

As a consequence, we call omni n-Lie algebroids the **Weinstein-linearization** of higher analogues of Courant algebroids.

By Theorem 3.3 and the properties of higher analogues of Courant algebroids $TE^* \oplus \wedge^n T^*E^*$, we get the following relations.

Proposition 3. *The omni n-Lie algebroid $(\mathfrak{D}E \oplus \wedge^n \mathfrak{J}E, (\cdot,\cdot), \{\cdot,\cdot\}, \rho)$ has the properties:*

- $\{e_1, \{e_2, e_3\}\} = \{\{e_1, e_2\}, e_3\} + \{e_2, \{e_1, e_3\}\}$;
- $\rho(\{e_1, e_2\}) = [\rho(e_1), \rho(e_2)]$;
- $\{e_1, f e_2\} = f\{e_1, e_2\} + (\mathbb{j} \circ \rho)(e_1) f e_2$;
- $\{e, e\} = \frac{1}{2}\mathbb{d}(e, e)$;
- $\rho(e_1)(e_2, e_3) = (\{e_1, e_2\}, e_3) + (e_2, \{e_1, e_3\})$,

for all $e_1, e_2, e_3 \in \Gamma(\mathfrak{D}E \oplus \wedge^n \mathfrak{J}E)$, where $\mathbb{j} : \mathfrak{D}E \to TM$ is the map in (1) and $\mathbb{d} : \Gamma(E \otimes \wedge^{n-1}\mathfrak{J}E) \to \Gamma(\wedge^n \mathfrak{J}E)$ is defined by the map

$$\mathbb{d}(u \otimes \alpha_1 \wedge \cdots \wedge \alpha_{n-1}) = \mathbb{d}u \wedge \alpha_1 \wedge \cdots \wedge \alpha_{n-1} + \sum_{i=1}^{n-1} (-1)^{i-1} u \otimes \mathbb{d}\alpha_i \wedge \alpha_1 \wedge \cdots \wedge \hat{\alpha_i} \wedge \cdots \wedge \alpha_{n-1},$$

[1]By personal communication with Luca Vitagliano, we learned that this definition was also given in their original version of [3], but not appeared in the published version.

for $u \in \Gamma(E)$ and $\alpha_i \in \Gamma(\mathfrak{J}E)$, which is well-defined by (17).

Corollary 4. *We have that $(\mathfrak{D}E \oplus \wedge^n \mathfrak{J}E, \{\cdot,\cdot\}, \mathbb{j}\circ\rho)$ is a Leibniz algebroid, where $\mathbb{j} : \mathfrak{D}E \to TM$ is the map in (1).*

When the vector bundle E is a vector space, denoted by V, we have $\mathfrak{D}E = \mathrm{gl}(V)$ and $\mathfrak{J}E = V$. In this case, the Lie derivative of $\mathfrak{D}E$ on $\wedge^n\mathfrak{J}E$ and the contraction of $\wedge^n\mathfrak{J}E$ by $\mathfrak{D}E$ are:

$$L_X : \wedge^n V \to \wedge^n V, \qquad L_X(\alpha_1 \wedge \cdots \wedge \alpha_n) = \sum_{i=1}^{n} \alpha_1 \wedge \cdots \wedge X\alpha_i \wedge \cdots \wedge \alpha_n, \tag{19}$$

and $\iota_X : \wedge^n V \to V \otimes \wedge^{n-1} V$,

$$\iota_X(\alpha_1 \wedge \cdots \wedge \alpha_n) = \sum_{i=1}^{n} (-1)^{i-1} X\alpha_i \otimes \alpha_1 \wedge \cdots \wedge \hat{\alpha_i} \wedge \cdots \wedge \alpha_n, \tag{20}$$

for $X \in \mathrm{gl}(V)$ and $\alpha_i \in V$.

The omni n-Lie algebroid for a vector space V is

$$(\mathrm{gl}(V) \oplus \wedge^n V, (\cdot,\cdot), \{\cdot,\cdot\}),$$

where the pairing $(\cdot,\cdot)$ takes values in $V \otimes \wedge^{n-1} V$ and is given by

$$(X + \alpha, Y + \beta) = \iota_X\beta + \iota_Y\alpha, \qquad \forall\ X, Y \in \mathrm{gl}(V), \alpha, \beta \in \wedge^n V,$$

and the bracket $\{\cdot,\cdot\}$ is defined by

$$\{X + \alpha, Y + \beta\} = [X, Y] + L_X\beta.$$

Here $\iota_X\beta$ and $L_X\beta$ are defined by (20) and (19). This is the omni n-Lie algebra introduced in [31], which is the base-linearization of the higher analogue of the standard Courant algebroid $TM \oplus \wedge^n T^*M$.

3.2. Integrable subbundles of omni n-Lie algebroids and n-Lie algebroids.

The notion of n-Lie algebroids, also called Filippov algebroids, was introduced in [17]. In this subsection, we show that the graph of $\Pi^\sharp : \wedge^n\mathfrak{J}E \to \mathfrak{D}E$ for $\Pi \in \Gamma(\mathfrak{D}^{n+1}E)$ is an integrable subbundle of the omni n-Lie algebroid $\mathfrak{D}E \oplus \wedge^n\mathfrak{J}E$ if and only if it defines an $(n+1)$-Lie algebroid structure on E.

Definition 3.4. ([17]) An n-Lie algebroid is a vector bundle E with a skew-symmetric n-bracket on its sections

$$[\cdot, \cdots, \cdot] : \Gamma(E) \times \cdots \times \Gamma(E) \to \Gamma(E)$$

satisfying the fundamental identity:

$$[u_1, \cdots, u_{n-1}, [v_1, \cdots, v_n]] = \sum_{i=1}^{n} [v_1, \cdots, v_{i-1}, [u_1, \cdots, u_{n-1}, v_i], \cdots, v_n], \tag{21}$$

for all $u_i, v_i \in \Gamma(E)$, and a bundle map $\rho : \wedge^{n-1}E \to TM$, called the anchor map, such that the Leibniz rule holds:

$$[u_1, \cdots, fu_n] = f[u_1, \cdots, u_n] + \rho(u_1, \cdots, u_{n-1})(f)u_n, \quad \forall\ u_i \in \Gamma(E), f \in C^\infty(M).$$

When E is a vector space, this recovers the notion of an ***n*-Lie algebra** ([14]). The section space $\Gamma(E)$ of an n-Lie algebroid with the n-bracket $[\cdot, \cdots, \cdot]$ is an n-Lie algebra.

Remark 2. An n-Lie algebra structure on V gives rise to a Leibniz algebra structure on $\wedge^{n-1}V$ ([12]). Applying this to the section space $\Gamma(E)$ of an n-Lie algebroid E, we obtain a Leibniz algebra structure on $\Gamma(\wedge^{n-1}E)$ given by

$$\mathfrak{u}\circ\mathfrak{v}=\sum_{i=1}^{n-1}v_1\wedge\cdots v_{i-1}\wedge[u_1,\cdots,u_{n-1},v_i]\wedge v_{i+1}\wedge\cdots\wedge v_{n-1},$$

for $\mathfrak{u}=u_1\wedge\cdots\wedge u_{n-1}$ and $\mathfrak{v}=v_1\wedge\cdots\wedge v_{n-1}$. Then we deduce that the anchor map

$$\rho:(\Gamma(\wedge^{n-1}E),\circ)\to(\mathfrak{X}^1(M),[\cdot,\cdot])$$

in the definition of n-Lie algebroids is a Leibniz algebra morphism. This can be proved by replacing v_n by fv_n in the fundamental identity (21) and then using the Leibniz rule. This condition was listed in the definition of Filippov algebroids in [17], which is redundant.

Recall from [7, 9] that a Lie algebroid structure on a vector bundle E gives rise to a bundle map from $\mathfrak{J}E$ to $\mathfrak{D}E$ whose graph is an integrable subbundle of the omni-Lie algebroid $\mathfrak{D}E\oplus\mathfrak{J}E$. Moreover, Lie algebroid structures on a vector bundle E correspond to linear Poisson structures on E^*. We shall generalize these results to n-Lie algebroids and linear Nambu-Poisson structures.

Let $\Pi\in\Gamma(\mathfrak{D}^{n+1}E)$. By definition, it gives a bundle map $\Pi^\sharp:\wedge^n\mathfrak{J}E\to\mathfrak{D}E$. Conversely, a bundle map $\wedge^n\mathfrak{J}E\to\mathfrak{D}E$ gives a skew-symmetric map $\wedge^{n+1}\mathfrak{J}E\to E$, if and only if it is a section of $\mathfrak{D}^{n+1}E$.

Define the graph of $\Pi^\sharp:\wedge^n\mathfrak{J}E\to\mathfrak{D}E$ by

$$\mathcal{G}_{\Pi^\sharp}=\{\Pi^\sharp(\alpha)+\alpha;\alpha\in\wedge^n\mathfrak{J}E\}\subset\mathfrak{D}E\oplus\wedge^n\mathfrak{J}E. \tag{22}$$

The following result needs the assumption $\mathrm{rank}E\geq 2$ to guarantee that the anchor map is function linear.

Theorem 3.5. *If E is a vector bundle with $\mathrm{rank}E\geq 2$, then there is a one-one correspondence between $(n+1)$-Lie algebroid structures $(E,[\cdot,\cdots,\cdot],\rho)$ and integrable subbundles $\mathcal{G}_{\Pi^\sharp}$ defined by (22) of the omni n-Lie algebroid $\mathfrak{D}E\oplus\wedge^n\mathfrak{J}E$ coming from $\Pi\in\Gamma(\mathfrak{D}^{n+1}E)$, which is given by*

$$[u_1,\cdots,u_{n+1}]\;=\;\Pi(\mathbb{d}u_1,\mathbb{d}u_2,\cdots,\mathbb{d}u_{n+1}), \tag{23}$$

$$\rho(u_1,\cdots,u_n)\;=\;\mathfrak{j}(\Pi^\sharp(\mathbb{d}u_1,\cdots,\mathbb{d}u_n)), \tag{24}$$

where $u_i\in\Gamma(E)$ and $\mathfrak{j}:\mathfrak{D}E\to TM$ is the map in (1).

Proof. By Theorem 2.1, for $\Pi\in\Gamma(\mathfrak{D}^{n+1}E)$, we have $\hat\Pi\in\mathfrak{X}^{n+1}_{lin}(E^*)$. Recall that a Nambu-Poisson structure of order n on a manifold M is an $(n+1)$-vector field $\Pi\in\mathfrak{X}^{n+1}(M)$ such that

$$L_{\Pi^\sharp(df_1\wedge\cdots\wedge df_n)}\Pi=0,\quad\forall f_1,\cdots,f_n\in C^\infty(M).$$

It is from [1, Theorem 1] that $\hat\Pi$ is a Nambu-Poisson structure on E^* if and only if the graph of $\hat\Pi^\sharp:\wedge^nT^*E^*\to TE^*$ is an integrable subbundle of the higher analogue of Courant algebroids $TE^*\oplus\wedge^nT^*E^*$.

By Theorem 3.3, if the graph $\mathcal{G}_{\Pi^\sharp}\subset\mathfrak{D}E\oplus\wedge^n\mathfrak{J}E$ is integrable, then the graph $\mathcal{G}_{(-1)^{n+1}\hat\Pi^\sharp}\subset TE^*\oplus\wedge^nT^*E^*$ is also integrable, which is equivalent to that $(-1)^{n+1}\hat\Pi$ is a Nambu-Poisson structure on E^*. In particular, this Nambu-Poisson structure is linear, meaning that

$$[u_1,\cdots,u_{n+1}]=(-1)^{n+1}\hat\Pi(du_1,\cdots,du_{n+1})=\Pi(\mathbb{d}u_1,\cdots,\mathbb{d}u_{n+1})\in\Gamma(E).$$

Therefore, we obtain an $(n+1)$-Lie algebra structure defined by (23) on $\Gamma(E)$. Also, we have

$$\begin{aligned}[u_1,\cdots,u_n,fu_{n+1}] &= \Pi(\mathbb{d}u_1,\cdots,f\mathbb{d}u_{n+1}+df\otimes u_{n+1})\\ &= f[u_1,\cdots,u_{n+1}]+\mathbb{j}(\Pi^\sharp(\mathbb{d}u_1,\cdots,\mathbb{d}u_n))(f)u_{n+1}\\ &= f[u_1,\cdots,u_{n+1}]+\rho(u_1,\cdots,u_n)(f)u_{n+1}.\end{aligned}$$

We claim that $(E,[\cdot,\cdots,\cdot],\rho)$ is an $(n+1)$-Lie algebroid. It suffices to check that $\rho:\wedge^n\Gamma(E)\to\mathfrak{X}^1(M)$ defined by (24) induces a bundle map from $\wedge^n E$ to TM. This is equivalent to

$$\mathbb{j}(\Pi^\sharp(df\otimes u_1,\mathbb{d}u_2,\cdots,\mathbb{d}u_n))=0,\qquad\forall\ f\in C^\infty(M),u_i\in\Gamma(E),\tag{25}$$

as

$$\rho(fu_1,\cdots,u_n)=\mathbb{j}(\Pi^\sharp(df\otimes u_1,\mathbb{d}u_2,\cdots,\mathbb{d}u_n))+f\rho(u_1,\cdots,u_n).$$

In fact, as Π is skew-symmetric, for $df\otimes u_1,df'\otimes u_1'\in\Gamma(T^*M\otimes E)$, we have

$$\begin{aligned}(\Pi^\sharp(df\otimes u_1,\mathbb{d}u_2,\cdots,\mathbb{d}u_n),df'\otimes u_1') &= \mathbb{j}(\Pi^\sharp(df\otimes u_1,\mathbb{d}u_2,\cdots,\mathbb{d}u_n))(f')u_1'\\ &= -\mathbb{j}(\Pi^\sharp(df'\otimes u_1',\mathbb{d}u_2,\cdots,\mathbb{d}u_n))(f)u_1.\end{aligned}$$

Since rank$E\geq 2$, we choose u_1 and u_1' to be independent. So the coefficients of u_1 and u_1' in the above formulas must be zero. Thus we proved (25). Hence $(E,[\cdot,\cdots,\cdot],\rho)$ is an $(n+1)$-Lie algebroid.

From the above verification, the converse direction also holds. □

Recall that for a bivector field $\Pi\in\mathfrak{X}^2(M)$ on a manifold M, the graph of $\Pi^\sharp:T^*M\to TM$ in $TM\oplus T^*M$ is integrable with respect to the standard Courant bracket if and only if $\Pi^\sharp[\alpha,\beta]_\Pi=[\Pi^\sharp(\alpha),\Pi^\sharp(\beta)]$ for $\alpha,\beta\in\Omega^1(M)$, where $[\alpha,\beta]_\Pi=L_{\Pi^\sharp(\alpha)}\beta-L_{\Pi^\sharp(\beta)}\alpha+d(\Pi^\sharp(\beta),\alpha)$. That is, Π is a Poisson structure. Moreover, $(T^*M,[\cdot,\cdot]_\Pi,\Pi^\sharp)$ is a Lie algebroid. Analogously, we have

Proposition 4. *For $\Pi\in\Gamma(\mathfrak{D}^{n+1}E)$, its graph $\mathcal{G}_{\Pi^\sharp}$ given by (22) is an integrable subbundle of the omni n-Lie algebroid $\mathfrak{D}E\oplus\wedge^n\mathfrak{J}E$ if and only if*

$$\Pi^\sharp[\alpha,\beta]_\Pi=[\Pi^\sharp(\alpha),\Pi^\sharp(\beta)],\qquad\forall\ \alpha,\beta\in\Gamma(\wedge^n\mathfrak{J}E),$$

where the bracket $[\cdot,\cdot]_\Pi$ on $\wedge^n\mathfrak{J}E$ is defined as

$$[\alpha,\beta]_\Pi=L_{\Pi^\sharp(\alpha)}\beta-L_{\Pi^\sharp(\beta)}\alpha+\mathbb{d}(\Pi^\sharp(\beta),\alpha).$$

Moreover, such an integrable subbundle induces a Leibniz algebroid $(\wedge^n\mathfrak{J}E,[\cdot,\cdot]_\Pi,\mathbb{j}\circ\Pi^\sharp)$, where $\mathbb{j}:\mathfrak{D}E\to TM$ is the bundle map in (1).

Proof. By the calculation

$$\{\Pi^\sharp(\alpha)+\alpha,\Pi^\sharp(\beta)+\beta\}=[\Pi^\sharp(\alpha),\Pi^\sharp(\beta)]+L_{\Pi^\sharp(\alpha)}\beta-\iota_{\Pi^\sharp(\beta)}\mathbb{d}\alpha,\qquad\forall\ \alpha,\beta\in\Gamma(\wedge^n\mathfrak{J}E),$$

we see that $\mathcal{G}_{\Pi^\sharp}$ is closed with respect to the Dorfman bracket (14) if and only if $\Pi^\sharp[\alpha,\beta]_\Pi=[\Pi^\sharp(\alpha),\Pi^\sharp(\beta)]$. This is equivalent to that $[\cdot,\cdot]_\Pi$ satisfies the Jacobi identity. It is left to check the Leibniz rule. For $f\in C^\infty(M)$, we have

$$[\alpha,f\beta]_\Pi=f[\alpha,\beta]_\Pi+(L_{\Pi^\sharp(\alpha)}f)\beta=f[\alpha,\beta]_\Pi+\mathbb{j}(\Pi^\sharp(\alpha))(f)\beta.$$

So $(\wedge^n\mathfrak{J}E,[\cdot,\cdot]_\Pi,\mathbb{j}\circ\Pi^\sharp)$ is a Leibniz algebroid. □

For a Lie algebroid E, the first jet bundle $\mathfrak{J}E$ has a natural Lie algebroid structure with the bracket such that $[\mathbb{d}u_1,\mathbb{d}u_2]=\mathbb{d}[u_1,u_2]_E$. Similar to this, we have the following result.

Proposition 5. *Let $(E, [\cdot, \cdots, \cdot]_E, \rho_E)$ be an $(n+1)$-Lie algebroid. Then*

(1) *there exists a unique $(n+1)$-Lie algebroid structure $(\mathfrak{J}E, [\cdot, \cdots, \cdot], \rho_{\mathfrak{J}E})$ on $\mathfrak{J}E$ such that*

$$[\mathbb{d}u_1, \cdots, \mathbb{d}u_{n+1}] = \mathbb{d}[u_1, \cdots, u_n]_E, \qquad \rho_{\mathfrak{J}E}(\mathbb{d}u_1, \cdots, \mathbb{d}u_n) = \rho_E(u_1, \cdots, u_n);$$

(2) *there exists a unique Leibniz algebroid structure $(\wedge^n \mathfrak{J}E, [\cdot, \cdot], \rho)$ on $\wedge^n \mathfrak{J}E$ such that*

$$[\mathbb{d}u_1 \wedge \cdots \wedge \mathbb{d}u_n, \mathbb{d}v_1 \wedge \cdots \wedge \mathbb{d}v_n] = \sum_{i=1}^{n} \mathbb{d}v_1 \wedge \cdots \mathbb{d}[u_1, \cdots, u_n, v_i]_E \wedge \cdots \wedge \mathbb{d}v_n,$$

and $\rho(\mathbb{d}u_1 \wedge \cdots \wedge \mathbb{d}u_n) = \rho_E(u_1, \cdots, u_n)$, for $u_i, v_i \in \Gamma(E)$.

Proof. The proof is standard. We omit the details. □

Applying Theorem 3.5 and Proposition 4 to the case when E is a vector space V, we get a Leibniz algebra $\mathrm{gl}(V) \oplus \wedge^n V$. Let $\Pi : \wedge^{n+1} V \to V$ be a skew-symmetric linear map. It induces a linear map $\Pi^\sharp : \wedge^n V \to \mathrm{gl}(V)$:

$$\Pi^\sharp(\alpha)(\alpha_{n+1}) = \Pi(\alpha, \alpha_{n+1}), \qquad \forall\ \alpha \in \wedge^n V, \alpha_{n+1} \in V.$$

Corollary 5. *Let $\Pi : \wedge^{n+1} V \to V$ be a linear map. Then the following statements are equivalent:*

(1) *the graph $\mathcal{G}_{\Pi^\sharp} \subset \mathrm{gl}(V) \oplus \wedge^n V$ is a sub-Leibniz algebra;*
(2) *V with the bracket $\{\alpha_1, \cdots, \alpha_{n+1}\} = \Pi(\alpha_1, \cdots, \alpha_{n+1})$ is an $(n+1)$-Lie algebra;*
(3) *$\wedge^n V$ with the bracket $[\alpha, \beta]_\Pi = L_{\Pi^\sharp(\alpha)} \beta$ is a Leibniz algebra;*
(4) *$\Pi^\sharp [\alpha, \beta]_\Pi = [\Pi^\sharp(\alpha), \Pi^\sharp(\beta)]$, where $[\alpha, \beta]_\Pi := L_{\Pi^\sharp(\alpha)} \beta$ for $\alpha, \beta \in \wedge^n V$.*

The equivalence of (1) and (2) is exactly [31, Theorem 3.4].

3.3. **Integrable subbundles of omni n-Lie algebroids and Nambu-Jacobi structures.** We explored the integrable subbundles of omni n-Lie algebroids when $\mathrm{rank}(E) \geq 2$ and found Theorem 3.5. Now we study the case when $\mathrm{rank}(E) = 1$ and particularly when E is a trivial line bundle.

A **local n-Lie algebra** is a vector bundle E such that $\Gamma(E)$ has an n-Lie algebra structure with the property $\mathrm{supp}[u_1, \cdots, u_n] \subset \mathrm{supp} u_1 \cap \cdots \cap \mathrm{supp} u_n$ for $u_i \in \Gamma(E)$. When E is the trivial line bundle $M \times \mathbb{R}$, it recovers the definition of Nambu-Jacobi structures on a manifold.

Nambu-Jacobi structures are the generalization of both Jacobi structures and Nambu-Poisson structures; see [20, 21, 33, 34].

Definition 3.6. A **Nambu-Jacobi structure** of order n on a manifold M ($2 \leq n \leq \dim M$) is a linear skew-symmetric n-bracket $[\cdot, \cdots, \cdot] : C^\infty(M) \times \cdots \times C^\infty(M) \to C^\infty(M)$, which is a first order differential operator, i.e.

$$[g_1 g_2, f_1, \cdots, f_{n-1}] = g_1[g_2, f_1, \cdots, f_{n-1}] + g_2[g_1, f_1, \cdots, f_{n-1}] - g_1 g_2 [1, f_1, \cdots, f_{n-1}],$$

and satisfies the fundamental identity (21), i.e.

$$[f_1, \cdots, f_{n-1}, [g_1, \cdots, g_n]] = \sum_{i=1}^{n} [g_1, \cdots, g_{i-1}, [f_1, \cdots, f_{n-1}, g_i], g_{i+1}, \cdots, g_n],$$

for $f_i, g_i \in C^\infty(M)$.

A manifold with such a bracket on its function space is called a **Nambu-Jacobi manifold of order** n. A **Jacobi manifold** is a Nambu-Jacobi manifold of order 2 and a **Nambu-Poisson manifold** is a Nambu-Jacobi manifold whose bracket vanishes if one of the functions is constant. When $n > 2$, a Nambu-Jacobi structure of order n is equivalently determined by a compatible pair of Nambu-Poisson structures $\Lambda \in \mathfrak{X}^n(M)$ and $\Gamma \in \mathfrak{X}^{n-1}(M)$; see [24, 33] for details.

Theorem 3.7. *Assume that E is a line bundle. There is a one-one correspondence between local $(n+1)$-Lie algebra structures $[\cdot, \cdots, \cdot]$ on E and integrable subbundles $\mathcal{G}_{\Pi^\sharp} \subset \mathfrak{D}E \oplus \wedge^n \mathfrak{J}E$ for $\Pi \in \Gamma(\mathfrak{D}^{n+1}E)$ satisfying that*

$$[u_1, \cdots, u_{n+1}] = \Pi(\mathbb{d}u_1, \cdots, \mathbb{d}u_{n+1}), \qquad \forall\ u_i \in \Gamma(E).$$

Moreover, Π induces an $(n+1)$-Lie algebroid structure on E if and only if $\mathbb{j} \circ \Pi^\sharp \circ \mathbb{e} = 0$, where $\mathbb{j} : \mathfrak{D}E \to TM$ and $\mathbb{e} : T^*M \otimes E \to \mathfrak{J}E$ are given in (1) and (2) respectively.

Proof. The first part of the proof of Theorem 3.5 still holds until the assumption $\mathrm{rank}E \geq 2$ is used. So we also get an $(n+1)$-Lie bracket on $\Gamma(E)$ if the graph $\mathcal{G}_{\Pi^\sharp}$ is integrable. This bracket is local. When $\mathrm{rank}E = 1$, the map $\rho : \wedge^n \Gamma(E) \to \mathfrak{X}^1(M)$ is not necessarily a bundle map. So in general, we obtain a local $(n+1)$-Lie algebra on E, which is an $(n+1)$-Lie algebroid if and only if (25) holds, namely, $\mathbb{j} \circ \Pi^\sharp \circ \mathbb{e} = 0$. The converse is easy to get. □

Now we study in detail the case of $E = M \times \mathbb{R}$, the trivial line bundle over M and build the relation between omni n-Lie algebroids and Nambu-Jacobi structures. For this, we first make some preparations. In this case, $\mathfrak{D}E = TM \times \mathbb{R}$ and $\mathfrak{J}E = T^*M \times \mathbb{R}$. Here $\mathfrak{D}E$ is a Lie algebroid with the Lie bracket and the anchor given by

$$[X+f, Y+g] = [X, Y] + Xg - Yf, \quad \rho(X+f) = X, \quad \forall\, X, Y \in \mathfrak{X}^1(M), f, g \in C^\infty(M).$$

The pairing between $\mathfrak{D}E$ and $\mathfrak{J}E$ is given by

$$(X+f, \xi+g) = \iota_X \xi + fg, \qquad \xi \in \Omega^1(M). \tag{26}$$

And the action of $\Gamma(\mathfrak{D}E)$ on $\Gamma(E)$ is

$$(X+g)f = Xf + gf. \tag{27}$$

Before writing down the structures of the omni n-Lie algebroid $\mathfrak{D}E \oplus \wedge^n \mathfrak{J}E$ when $E = M \times \mathbb{R}$ for a general n, we first make clear of the differential $\mathbb{d}$ on $\mathfrak{J}_\bullet E$ in this case.

Lemma 3.8. *When $E = M \times \mathbb{R}$, we have $\mathfrak{J}_n E = \Omega^n(M) \oplus \Omega^{n-1}(M)$ and the differential $\mathbb{d} : \mathfrak{J}_n E \to \mathfrak{J}_{n+1} E$ is given by*

$$\begin{aligned} \mathbb{d}(f) &= df + f, \qquad \forall\ f \in C^\infty(M); \\ \mathbb{d}(\alpha_n + \alpha_{n-1}) &= d\alpha_n + \alpha_n - d\alpha_{n-1}, \quad \forall\ \alpha_n \in \Omega^n(M), \alpha_{n-1} \in \Omega^{n-1}(M), n \geq 1. \end{aligned}$$

Proof. By the pairing (26) and the action (27) for $E = M \times \mathbb{R}$, we have

$$\mathbb{d}f(X+g) = (X+g)f = Xf + gf = (df + f, X+g), \quad \forall f, g \in C^\infty(M), X \in \mathfrak{X}^1(M).$$

Therefore, $\mathbb{d} : \Gamma(E) \to \Gamma(\mathfrak{J}E)$ is given by $\mathbb{d}f = df + f$. To see the action of $\mathbb{d}$ on $\Gamma(\mathfrak{J}E)$, for $\alpha \in \Omega^1(M)$, we have

$$\begin{aligned} \mathbb{d}\alpha(X+f, Y+h) &= (X+f)\iota_Y \alpha - (Y+h)\iota_X \alpha - \iota_{[X,Y]}\alpha \\ &= d\alpha(X, Y) + f\iota_Y \alpha - h\iota_X \alpha. \end{aligned}$$

Hence we obtain $\mathbb{d}\alpha = d\alpha + \alpha$. Here $\alpha \in \Omega^1(M)$ is viewed as an element in $\mathfrak{J}_2 E$ by $\alpha(f, Y) = f\iota_Y\alpha$ and $\alpha(Y, f) = -f\iota_Y\alpha$.

Then, for $g \in C^\infty(M)$ as a section of $\mathfrak{J}E$, we have

$$\begin{aligned} \mathbb{d}g(X+f, Y+h) &= (X+f)(gh) - (Y+h)(gf) - g(Xh - Yf) \\ &= h(Xg) - f(Yg) \\ &= dg(h, X) - dg(f, Y), \end{aligned}$$

thus we get $\mathbb{d}g = -dg \in \Omega^1(M)$, which is seen as a section of $\mathfrak{J}_2 E$. The higher degrees are similar to get. □

Lemma 3.9. *The Lie derivative of $\mathfrak{D}E = TM \times \mathbb{R}$ on $\mathfrak{J}E = T^*M \times \mathbb{R}$ and the contraction of $\mathfrak{J}E$ by $\mathfrak{D}E$ for $E = M \times \mathbb{R}$ are given by*

$$L_{X+f}(\xi + g) = L_X\xi + f\xi + gdf + Xg + fg; \tag{28}$$

$$\iota_{X+f}\mathbb{d}(\xi + g) = \iota_X d\xi + f\xi - fdg - \iota_X\xi + Xg, \tag{29}$$

where $X \in \mathfrak{X}^1(M), \xi \in \Omega^1(M), f, g \in C^\infty(M)$.

Proof. By the general formula for the Lie derivative and the contraction of an omni-Lie algebroid, using (26), we have

$$\begin{aligned} (L_{X+f}(\xi+g), Y+h) &= (X+f)(\iota_Y\xi + gh) - (\xi + g, [X,Y] + Xh - Yf) \\ &= X(\iota_Y\xi + gh) + f\iota_Y\xi + fgh - \xi([X,Y]) - g(Xh - Yf) \\ &= (L_X\xi + f\xi + gdf + Xg + fg, Y + h). \end{aligned}$$

So we get that

$$L_{X+f}(\xi + g) = L_X\xi + f\xi + gdf + Xg + fg.$$

By the fact that $\mathbb{d}(g) = dg + g$ for any $g \in C^\infty(M) = \Gamma(E)$, we obtain

$$\begin{aligned} \iota_{X+f}\mathbb{d}(\xi+g) &= L_{X+f}(\xi+g) - \mathbb{d}(X+f, \xi+g) \\ &= L_X\xi + f\xi + gdf + Xg + fg - d(\iota_X\xi + fg) - \iota_X\xi - fg \\ &= \iota_X d\xi + f\xi - fdg - \iota_X\xi + Xg. \end{aligned}$$

This is (29). We finishes the proof. □

As a consequence, by using the Leibniz rule of the Lie derivative, the Lie derivative of $\mathfrak{D}E$ on $\wedge^n\mathfrak{J}E$ is also clear.

Proposition 6. *For $E = M \times \mathbb{R}$, the omni n-Lie algebroid $(\mathfrak{D}E \oplus \wedge^n\mathfrak{J}E, (\cdot,\cdot), \{\cdot,\cdot\}, \rho)$ is defined as follows:*

- $\mathfrak{D}E \oplus \wedge^n\mathfrak{J}E = TM \times \mathbb{R} \oplus \big(\wedge^n T^*M \oplus \wedge^{n-1}T^*M\big)$,
- *the $(\wedge^{n-1}T^*M \oplus \wedge^{n-2}T^*M)$-valued pairing is given by*

$$(X + f, \alpha_n + \alpha_{n-1}) = \iota_X\alpha_n + f\alpha_{n-1} - \iota_X\alpha_{n-1}, \tag{30}$$

- *the bracket is given by*

$$\begin{aligned} \{X+f, Y+g\} &= [X,Y] + Xg - Yf; &(31)\\ \{X+f, \alpha_n + \alpha_{n-1}\} &= L_{X+f}(\alpha_n + \alpha_{n-1}) \\ &= L_X\alpha_n + f\alpha_n + df \wedge \alpha_{n-1} + L_X\alpha_{n-1} + f\alpha_{n-1}; &(32)\\ \{\alpha_n + \alpha_{n-1}, X+f\} &= -\iota_{X+f}\mathbb{d}(\alpha_n + \alpha_{n-1}) \\ &= -\iota_X d\alpha_n - f\alpha_n + fd\alpha_{n-1} + \iota_X\alpha_n - \iota_X d\alpha_{n-1}, &(33) \end{aligned}$$

where $X, Y \in \mathfrak{X}^1(M), f, g \in C^\infty(M), \alpha_n \in \Omega^n(M), \alpha_{n-1} \in \Omega^{n-1}(M)$.

Proof. By (26), we have

$$(X+f,\alpha_2)=\iota_X\alpha_2,\qquad (X+f,\alpha_1)=(X+f,1\wedge\alpha_1)=-\iota_X\alpha_1+f\alpha_1,$$

for $\alpha_2\in\Omega^2(M)$ and $\alpha_1\in\Omega^1(M)$, treating as sections of $\wedge^2\mathfrak{J}E$. Based on this idea, we get (30). (31) is clear. By (28) and the Leibniz rule, we get

$$\{X+f,\alpha_n+\alpha_{n-1}\}=L_{X+f}(\alpha_n+\alpha_{n-1})=L_X\alpha_n+L_X\alpha_{n-1}+f\alpha_n+df\wedge\alpha_{n-1}+f\alpha_{n-1}.$$

This is (32). Similarly we have

$$\begin{aligned}\{\alpha_n+\alpha_{n-1},X+f\} &= -L_{X+f}(\alpha_n+\alpha_{n-1})+\mathbb{d}(\alpha_n+\alpha_{n-1},X+f)\\ &= -L_{X+f}(\alpha_n+\alpha_{n-1})+\mathbb{d}(\iota_X\alpha_n+f\alpha_{n-1}-\iota_X\alpha_{n-1})\\ &= -L_X\alpha_n-f\alpha_n-df\wedge\alpha_{n-1}-L_X\alpha_{n-1}-f\alpha_{n-1}\\ &\quad +d(\iota_X\alpha_n+f\alpha_{n-1})+\iota_X\alpha_n+f\alpha_{n-1}+d\iota_X\alpha_{n-1}\\ &= -\iota_Xd\alpha_n-f\alpha_n+fd\alpha_{n-1}+\iota_X\alpha_n-\iota_Xd\alpha_{n-1},\end{aligned}$$

where the third identity follows from Lemma 3.8. Hence we get (33). □

Remark 3. When $n=1$, we obtain the structure of the omni-Lie algebroid for $E=M\times\mathbb{R}$:

$$\mathfrak{D}E\oplus\mathfrak{J}E=(TM\times\mathbb{R})\oplus(T^*M\times\mathbb{R}),$$

where the Dorfman bracket is

$$\begin{aligned}\{X+f,Y+g\} &= [X,Y]+Xg-Yf;\\ \{X+f,\xi+g\} &= L_X\xi+f\xi+gdf+Xg+fg;\\ \{\xi+g,X+f\} &= -\iota_Xd\xi-f\xi+fdg+\iota_X\xi-Xg.\end{aligned}$$

for all $X,Y\in\mathfrak{X}^1(M)$, $f,g\in C^\infty(M)$ and $\xi\in\Omega^1(M)$.

The skew-symmetrization of this bracket is

$$\begin{aligned}[X+f,Y+g] &= [X,Y]+Xg-Yf;\\ [X+f,\xi+g] &= L_X\xi-\frac{1}{2}d\iota_X\xi+f\xi+\frac{1}{2}(gdf-fdg)+Xg-\frac{1}{2}\iota_X\xi+\frac{1}{2}fg.\end{aligned}$$

This is exactly the bracket given in [40] by Wade in the study of conformal Dirac structures.

Also, the Lie derivative and contraction in Lemma 3.9 coincide with that in [26], where they defined it directly.

Let $E=M\times\mathbb{R}$ and $\Pi\in\Gamma(\mathfrak{D}^{n+1}E)$. The bundle map

$$\Pi:\wedge^{n+1}\mathfrak{J}E=\wedge^{n+1}T^*M\oplus\wedge^nT^*M\to E=M\times\mathbb{R}$$

has two components

$$\Pi=\Lambda+\Gamma\in\mathfrak{X}^{n+1}(M)\oplus\mathfrak{X}^n(M).$$

As a consequence of Theorem 3.7, we have

Proposition 7. *With the above notations, the graph of $\Pi^\sharp=\Lambda^\sharp+\Gamma^\sharp$ defines an integrable subbundle of the omni n-Lie algebroid $TM\times\mathbb{R}\oplus(\wedge^nT^*M\times\mathbb{R})$ if and only if it defines a Nambu-Jacobi structure of order $n+1$ on M whose Lie bracket is*

$$[f_1,\cdots,f_{n+1}] = \Lambda(df_1,\cdots,df_{n+1})+\sum_{i=1}^{n+1}(-1)^{i-1}f_i\Gamma(df_1,\cdots,\hat{df_i},\cdots df_{n+1}),$$

for $f_i\in C^\infty(M)$.

Proof. By definition, Nambu-Jacobi structures on M of order $n+1$ are local $(n+1)$-Lie algebra structures on the trivial line bundle $M \times \mathbb{R}$. So by Theorem 3.7 and Lemma 3.8, we obtain a Nambu-Jacobi structure on M with the bracket

$$\begin{aligned}[f_1, \cdots, f_{n+1}] &= (\Lambda + \Gamma)(\mathbb{d} f_1, \cdots, \mathbb{d} f_n) \\ &= (\Lambda + 1 \wedge \Gamma)(df_1 + f_1, \cdots, df_n + f_n) \\ &= \Lambda(df_1, \cdots, df_{n+1}) + \sum_{i=1}^{n+1} (-1)^{i-1} f_i \Gamma(df_1, \cdots, \hat{df_i}, \cdots df_{n+1}),\end{aligned}$$

which finishes the proof. □

This Nambu-Jacobi structure also appeared in [21, 34] with a different sign convention.

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