Alejandro Gil García

Curvature, symmetries and hypersurfaces of supergravity c-map spaces

Dissertation zur Erlangung des Doktorgrades an der

Universität Hamburg
Fakultät für Mathematik, Informatik und Naturwissenschaften
Fachbereich Mathematik

Hamburg, 2024

1. Gutachter: Prof. Dr. Vicente Cortés
2. Gutachter: Prof. Dr. Jens Heber
3. Gutachter: Prof. Dr. Andriy Haydys

Datum der Disputation: 26.06.2024
Vorsitzende der Prüfungskommission: Prof. Dr. Jörg Teschner

Abstract

This thesis deals with a particular class of quaternionic Kähler manifolds of negative scalar curvature, namely supergravity c-map spaces. In particular, we study curvature and symmetry properties and, for a particular case, the induced geometry on some hypersurfaces.

Supergravity c-map spaces are quaternionic Kähler manifolds in the image of the supergravity c-map. This construction produces a quaternionic Kähler manifold from a projective special Kähler manifold. Despite its physical origins, which give this construction its name, the supergravity c-map is completely understood within the framework of differential geometry. Moreover, the supergravity c-map admits a one-parameter deformation giving rise to a one-parameter family of quaternionic Kähler manifolds, known as the one-loop deformed supergravity c-map spaces. These are the main object of study of this thesis.

Mathematically, the (deformed) supergravity c-map has been explained in a two-step process. In this process one starts with a conical affine special Kähler (CASK) manifold M, which is a $\mathbb{C}^*$-bundle over a projective special Kähler (PSK) manifold $\bar{M}$. Then, one constructs a pseudo-hyperkähler structure on the cotangent bundle $N = T^*M$ of the CASK manifold M, the so-called rigid c-map structure. Finally, one applies the HK/QK correspondence to obtain a quaternionic Kähler manifold $\bar{N}$ of negative scalar curvature from the pseudo-hyperkähler manifold N. In this setting, the obtained quaternionic Kähler metric is precisely the supergravity c-map metric. The key point of this construction is the HK/QK correspondence. This correspondence was interpreted as an instance of a more general construction known as the twist construction. This will be the formalism used in this thesis to obtain the main results.

The first goal of this thesis is to show that any deformed supergravity c-map is not locally homogeneous. To obtain this result it is enough to show, using the twist formalism, that the (squared) norm of an abstract curvature tensor related to the curvature tensor of the pseudo-hyperkähler rigid c-map metric on $N = T^*M$ is not constant. In this process we also obtain an explicit formula for the Riemann curvature tensor of the rigid c-map metric in terms of tensors of the CASK manifold M. As an important corollary, we show that the deformed supergravity c-map applied to a homogeneous PSK manifold produces a cohomogeneity one quaternionic Kähler manifold.

The second goal of this thesis is to study the isometry group of a deformed supergravity c-map space. It is known how to describe the Killing vector fields of the supergravity c-map metric and, for the particular case of the deformation of the symmetric space $\mathrm{SU}(n,2)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))$, it is even known how to integrate these vector fields to obtain a group which acts effectively and isometrically on the quaternionic Kähler manifold. In this thesis we obtain such group of isometries, without integrating the Killing vector fields, for a large subclass of supergravity c-map spaces, namely supergravity q-map spaces.

The last goal of this thesis is to study the induced geometry on the hypersurface orbit of the cohomogeneity one quaternionic Kähler manifold obtained by applying the deformed supergravity c-map to $\mathbb{C}\mathrm{H}^{n-1}$. This corresponds to the one-parameter deformation of the symmetric space $\mathrm{SU}(n,2)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))$. We mainly study its Ricci tensor and deduce that the hypersurface equipped with the induced metric is not a Ricci soliton, in contrast with the undeformed case.

Zusammenfassung

Diese Arbeit befasst sich mit einer bestimmten Klasse quaternionischer Kähler-Mannigfaltigkeiten negativer Skalarkrümmung, den sogenannten Supergravitations-c-Abbildungsräumen. Insbesondere untersuchen wir Krümmungs- und Symmetrieeigenschaften dieser Räume sowie für einen Spezialfall die induzierte Geometrie auf einigen Hyperebenen.

Supergravitations-c-Abbildungsräume sind quaternionische Kähler-Mannigfaltigkeiten im Bild der Supergravitations-c-Abbildung, einer Konstruktion, die eine quaternionische Kähler-Mannigfaltigkeit aus deiner projektiven speziellen Kählermannigfaltigkeit erzeugt. Trotz ihres physikalischen Ursprungs, der die Supergravitations-c-Abbildung auch ihren Namen verdankt, ist sie vollständig im Rahmen der Differentialgeometrie verstanden. Desweiteren erlaubt die Supergravitations-c-Abbildung eine Deformation, die eine Ein-Parameter-Familie quaternionischer Kählermannigfaltigkeiten hervorbringt, genannt Supergravitations-c-Abbildungsräume mit Schleifenverformung. Diese sind Hauptgegenstand der Arbeit.

Mathematisch wurde die (deformierte) Supergravitations-c-Abbildung in einem zweistufigen Prozess erklärt. In diesem Prozess beginnt man mit einer konischen affinen speziellen Kähler-Mannigfaltigkeit (KASK) M, die eine $\mathbb{C}^*$-Faserung über einer projektiven speziellen Kähler-Mannigfaltigkeit (PSK) $\bar{M}$ ist. Dann konstruiert man eine pseudo-hyperkählersche Struktur auf dem Kotangentialbündel $N = T^*M$ der KASK-Mannigfaltigkeit M, die sogenannte rigide c-Abbildung. Schließlich verwendet man die HK/QK-Korrespondenz, um aus der pseudo-hyperkählerschen Mannigfaltigkeit N eine quaternionische Kähler-Mannigfaltigkeit $\bar{N}$ negativer Skalarkrümmung zu erhalten. In diesem Zusammenhang ist die resultierende quaternionische Kähler-Metrik gerade die Supergravitations-c-Abbildungsmetrik. Der entscheidende Punkt dieser Konstruktion ist die HK/QK-Korrespondenz, welche als Spezialfall einer allgemeineren Konstruktion interpretiert werden kann, der Twist-Konstruktion. Dieser Formalismus wird für die Hauptresultate dieser Arbeit relevant sein.

Das erste Ziel dieser Arbeit ist zu zeigen, dass deformierte Supergravitations-c-Abbildungsräume niemals lokal homogen sind. Um dieses Ergebnis zu erhalten, genügt es, unter Verwendung des Twist-Formalismus zu zeigen, dass die (quadratische) Norm eines abstrakten Krümmungstensors, der mit dem Krümmungstensor der pseudo-hyperkählerschen rigiden c-Abbildungsmetrik auf $N = T^*M$ zusammenhängt, nicht konstant ist. In diesem Prozess erhalten wir auch eine explizite Formel für den Riemann-

Krümmungstensor der rigiden c-Abbildungsmetrik in Bezug auf Tensoren der KASK-Mannigfaltigkeit M. Als wichtiges Korollar zeigen wir, dass die deformierte Supergravitations-c-Abbildung homogene PSK-Mannigfaltigkeiten auf quaternionische Kählermannigfaltigkeiten mit Kohomogenität eins abbildet.

Das zweite Ziel dieser Arbeit ist es, die Isometriegruppe eines deformierten Supergravitations-c-Abbildungsraumes zu untersuchen. Es ist bekannt, wie man die Killing-Vektorfelder der Supergravitations-c-Abbildungsmetrik beschreibt, und für den speziellen Fall der Deformation des symmetrischen Raums $\mathrm{SU}(n,2)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))$ ist sogar bekannt, wie man diese Vektorfelder integriert, um eine Gruppe zu erhalten, die effektiv und isometrisch auf der quaternionischen Kähler-Mannigfaltigkeit wirkt. In dieser Arbeit erhalten wir eine solche Isometriegruppe für eine große Unterklasse von Supergravitations-c-Abbildungsräumen, den Supergravitations-q-Abbildungsräumen, ohne die Killing-Vektorfelder zu integrieren.

Das letzte Ziel dieser Arbeit ist es, die induzierte Geometrie auf der Hyperebene der quaternionischen Kähler-Mannigfaltigkeit der Kohomogenität eins zu untersuchen, die durch Anwendung der deformierten Supergravitations-c-Abbildung auf $\mathbb{C}\mathrm{H}^{n-1}$ erhalten wird. Dies entspricht der Ein-Parameter-Deformation des symmetrischen Raums $\mathrm{SU}(n,2)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))$. Wir untersuchen hauptsächlich ihren Ricci-Tensor und schlussfolgern, dass die Hyperebene mit der induzierten Metrik kein Ricci-Soliton ist, im Gegensatz zum undeformierten Fall.

Acknowledgments

First and foremost, I am genuinely indebted to my supervisor Vicente Cortés, for his patient mentorship during the past three years. I am deeply grateful to him for always having time to discuss my work, helping me with the questions I had, even the smallest or simplest ones, and guiding me to become a professional mathematician. I am also grateful to him for teaching me not only mathematics, but also for sharing his valuable insights into academic life as a whole.

I would also like to acknowledge the postdocs who accompanied me during this academic journey: Arpan Saha, Danu Thung and Iván Tulli. Your willingness to spend countless hours discussing and sharing your experiences, knowledge and diverse perspectives has been immensely valuable to me. My path as a mathematician would have been significantly different without each of you. Thank you sincerely.

I also want to express my gratitude to my other collaborators, Giovanni Bazzoni, Diego Conti, Adela Latorre, Markus Röser and Carlos Shahbazi, whose different points of view and unique ways of doing mathematics have deeply enriched my training. A big thanks goes out to my colleagues at Geomatikum as well, Andreas, Leander, Marco, Mateo, Oskar and Paula, for so many pleasant conversations and for creating a fantastic working atmosphere. My special thanks also go to Stefanie Risse for all her help with paperwork and bureaucratic issues and for her exceptional care and support throughout my time there.

Heartfelt thanks to my love Deniz and my friends in Hamburg, Andrea, Andrés, Cósima, Daniel, David, Isabela, Lina and Óscar, for making me feel at home. Finally, to my family and friends in Madrid, Álvaro, Javi, Nevado and Sergio, for your affection through time and space.

Thank you very much to all of you who have been part of this journey!

Contents

Chapter 1

Introduction

Quaternionic Kähler manifolds are Riemannian manifolds of dimension $4n > 4$ such that their holonomy group is contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$ [Ber55]. In terms of more concrete objects, quaternionic Kähler manifolds are Riemannian manifolds (M, g) of non-zero scalar curvature equipped with a parallel skew-symmetric almost quaternionic structure $\mathscr{Q} \subset \mathrm{End}(TM)$. The metric g and the subbundle $\mathscr{Q}$ can be used to construct a global 4-form Ω which is in some sense the analogous of the fundamental 2-form of a Kähler manifold. Note that, despite the name, quaternionic Kähler manifolds are not Kähler in general.

One of the main differences of quaternionic Kähler geometry with respect to the other special geometries appearing in Berger list of Riemannian holonomy groups is that quaternionic Kähler manifolds are Einstein but not Ricci-flat, since we are assuming non-zero scalar curvature [Ber66]. This has two important consequences: the theory depends on the sign of the scalar curvature and there exist non-flat homogeneous examples.

We note now that a 4-dimensional quaternionic Kähler manifold has holonomy contained in $\mathrm{Sp}(1)\mathrm{Sp}(1) \cong \mathrm{SO}(4)$, that is, generic holonomy. Therefore, a 4-dimensional quaternionic Kähler manifold may be defined as an oriented Riemannian manifold which is Einstein and self-dual.

It turns out that the theory of (complete) quaternionic Kähler manifolds of positive or negative scalar curvature is very different. Quaternionic Kähler manifolds of positive scalar curvature have very strong topological restrictions, in particular they must be compact and simply connected [Sal82]. The only known examples of such manifolds are the so-called Wolf spaces [Wol65], which are symmetric quaternionic Kähler manifolds of compact type. In fact, LeBrun and Salamon proved that, up to homothety, there are only finitely many $4n$-dimensional complete quaternionic Kähler manifolds of positive scalar curvature for any $n \in \mathbb{N}$ [LS94]. In the same paper, the authors proved strong conditions on the second homotopy group as well as on the Betti numbers. These results can all be viewed as a strong supporting evidence for their famous conjecture, which states that the symmetric spaces are the only examples of complete quaternionic Kähler manifolds of positive scalar curvature. This conjecture has been proved up to dimension 16 [Hit81, FK82, PS91, BWW22].

On the other hand, quaternionic Kähler geometry of negative scalar curvature has shown to be very rich. The non-compact duals of the Wolf spaces are symmetric examples. These moreover admit cocompact lattices [Bor63], yielding examples of locally symmetric compact quaternionic Kähler manifolds of negative scalar curvature. Alekseevsky found the first examples of non-symmetric homogeneous quaternionic Kähler manifolds of negative scalar curvature [Ale75]. These are homogeneous under a completely solvable group of isometries, and are known as Alekseevsky spaces. He claimed their classification, although some examples were missing [dWVP92]. The list was completed by Cortés [Cor96a]. Very recently, it has been shown by Böhm and Lafuente that all homogeneous quaternionic Kähler manifolds of negative scalar curvature are Alekseevsky spaces [BL23]. It was shown by LeBrun that complete non-locally homogeneous quaternionic Kähler manifolds exist in abundance [LeB91]. However, his proofs are not constructive.

Quaternionic Kähler manifolds also play an important role in physics, since they appear in supergravity and string theories. Physicists discovered that one can construct a quaternionic Kähler manifold of negative scalar curvature starting from a projective special Kähler manifold [FS90]. This construction is known as the supergravity c-map and manifolds in their image are known as supergravity c-map spaces. The quaternionic Kähler metric of the supergravity c-map space is sometimes referred to as the Ferrara-Sabharwal metric. An alternative proof of this construction was given by Hitchin in [Hit09]. Furthermore, it was also shown by physicists that supergravity c-map spaces admit a one-parameter deformation by quaternionic Kähler metrics [RSV06]. These are known as the (one-loop) deformed supergravity c-map metrics.

The mathematical proof that the (one-loop) deformed supergravity c-map metric is indeed quaternionic Kähler of negative scalar curvature is described in several steps, see [ACDM15].

First, given a projective special Kähler manifold $\bar{M}$, one can consider a conical affine special Kähler manifold M on a $\mathbb{C}^*$-bundle over $\bar{M}$. Then one can equip the cotangent bundle $N = T^*M$ of M with a (semi-flat) pseudo-hyperkähler structure, known as the rigid c-map structure [CFG89, Fre99b, ACD02], which is moreover equipped with a rotating circle symmetry [ACM13]. It was first shown by Haydys [Hay08] that given a positive-definite hyperkähler manifold with a rotating circle symmetry one can obtain a quaternionic Kähler manifold also equipped with a circle symmetry. This construction is known as the HK/QK correspondence. However, the quaternionic Kähler metrics obtained by this method have positive scalar curvature (and are incomplete). This correspondence was then generalized in [ACM13] to obtain quaternionic Kähler manifolds of negative scalar curvature by allowing indefinite hyperkähler metrics on N. We note that the HK/QK correspondence was interpreted in [MS15] as an instance of a more general construction known as the twist construction [Swa10]. Finally, one checks that the metric obtained by this method is the deformed supergravity c-map metric [ACDM15]. Summarizing we have the following diagram:

$$\begin{tikzcd} M^{2n} \arrow[r, "\text{rigid c-map}"] & N^{4n} \arrow[d, "\text{HK/QK}"] \\ \bar{M}^{2n-2} \arrow[u, "\mathbb{C}^*"] \arrow[r, "\text{supergravity c-map}"] & \bar{N}^{4n} \end{tikzcd}$$

We then conclude that the deformed supergravity c-map produces a one-parameter family of quaternionic Kähler metrics (depending on a real parameter $c \in \mathbb{R}$) from a projective special Kähler manifold. Note that the case $c = 0$ is the Ferrara-Sabharwal metric. For a fixed projective special Kähler manifold, the metrics in the image of the deformed supergravity c-map are locally isometric for different values of $c > 0$ [CDS17].

There is a special class of supergravity c-map spaces known as supergravity q-map spaces. These arise as the composition of the supergravity r-map and the supergravity c-map. The supergravity r-map produces a projective special Kähler manifold $\bar{M}$ of (real) dimension $2n-2$ from a projective special real manifold $\mathcal{H}$ of dimension $n-2$. This construction was introduced in [dWVP92]. Therefore, applying the supergravity q-map to a projective special real manifold $\mathcal{H}$ of dimension $n-2$, we obtain a quaternionic Kähler manifold of negative scalar curvature of dimension $4n$. Since the supergravity c-map admits a one-parameter deformation, so does the supergravity q-map. It is known that for any $c \geq 0$, a supergravity q-map space is complete provided that the projective special real manifold is complete [CDS17]. It is also known that all homogeneous quaternionic Kähler manifolds of negative scalar curvature, except $\mathbb{H}\mathrm{H}^n$ and $\mathrm{SU}(n,2)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))$, are in the image of the supergravity q-map [dWVP92]. However, the latter is still in the image of the supergravity c-map.

Finally, examples in all dimensions of complete non-locally homogeneous quaternionic Kähler manifolds of negative scalar curvature with two ends, one of finite volume and the other one of infinite volume, have been constructed [CRT21, Cor23]. Nevertheless, the problem of finding complete non-locally symmetric quaternionic Kähler manifolds of finite volume is still open, in contrast with all the other holonomy groups in Berger list, for which even compact non-locally symmetric examples are known [Yau78, Bea83, Joy96a, Joy96b, Joy96c].

1.1 Outline and summary of the results

This thesis is framed withing the study of (deformed) supergravity c-map spaces, which was partially carried out in previous PhD thesis [Dyc15, Sah20, Thu20]. In this work we extend and generalize some of these results and we deepen on the geometric comprehension of the supergravity c-map.

As the name suggests, the supergravity c-map has a physical origin. In particular, it arises from string theory. Briefly speaking, in the context of type II (super)string theory, when compactifying the 10-dimensional theory on a Calabi-Yau 3-fold, one obtains a 4-dimensional effective field theory with a moduli space which splits as a product of the vector multiplet moduli space the and hypermultiplet moduli space. These are a projective special Kähler and a quaternionic Kähler manifold, respectively (see Section 1.2 for details). The quaternionic Kähler metric, sometimes called tree-level metric, admits several quantum corrections (in the string coupling g_s). The deformed supergravity c-map corresponds to the perturbative corrections of the theory (in some sense, the easiest to understand). This case was first done in the physics literature [RSV06, Ale07] and nowadays it is completely understood in differential-geometric terms [ACDM15]. Nevertheless, the tree-level metric also receives non-perturbative corrections. These have been studied extensively in the physics literature (see e.g. [Ale13] and references therein) although understanding them from the pure mathematical point of view is still a work in progress and it is far from being completely understood [CT22a, CT22b, CT24].

The thesis is structured as follows:

- In Chapter 2 we present a detailed exposition about quaternionic Kähler manifolds, which are the main object of interest of this thesis. We introduce them first from the point of view of holonomy theory and then as particular cases of almost quaternionic manifolds. We give an overview of their general properties and then we distinguish them depending on the sign of the scalar curvature (since they are Einstein). We further study the positive and negative case giving examples and stating the major open problem of the field. This chapter does not contain original results.
- In Chapter 3 we introduce hyperkähler manifolds also as particular cases of almost quaternionic manifolds. We give some examples and explain in detail the Swann bundle, which is a conical (pseudo-)hyperkähler manifold canonically associated to a quaternionic Kähler manifold, see Theorem 3.1.14. Next we introduce the HK/QK correspondence, which is a way to construct quaternionic Kähler manifolds equipped with a circle action from (pseudo-)hyperkähler manifolds equipped with a rotating circle action, see Theorem 3.2.4. Finally, we explain how this correspondence can be interpreted as an instance of a more general construction known as the twist construction. We explain it in detail, introducing the concept of $\mathscr{H}$-relatedness and giving the tensors which are $\mathscr{H}$-related with the quaternionic Kähler metric and its Riemann curvature tensor, see Theorem 3.3.9 and Theorem 3.3.11. This chapter does not contain original results.
- In Chapter 4 we introduce supergravity c-map spaces, which is the class of quaternionic Kähler manifolds we will work with. As a first step, we introduce (affine and projective) special Kähler geometry. We give some examples and explain a general way to construct them. We also give an explicit formula for

the Riemann curvature tensor of an affine special Kähler manifold in Proposition 4.1.29. Although Subsection 4.1.2 does not contain original results, we prove some results again giving also formulas in local coordinates, which will be useful in the following chapter. Next we introduce the rigid c-map, which equips the total space of $N = T^*M$, where M is an affine special Kähler manifold, with a (semi-flat) (pseudo-)hyperkähler structure, see Theorem 4.2.1. In the case where M is furthermore conical, the rigid c-map admits a rotating Killing vector field generating a rotating circle action, see Proposition 4.2.4, hence we can apply the HK/QK correspondence. The composition of the rigid c-map and the HK/QK correspondence is the supergravity c-map, see Theorem 4.3.2. Next we express the supergravity c-map metric in local coordinates. This allows us to determine large groups of isometries and to study the completeness of the metric. Finally we introduce a subclass of supergravity c-map spaces, namely supergravity q-map spaces. These arise as the result of applying to a projective special real manifold $\mathcal{H}$ the composition of the supergravity r-map and the supergravity c-map. This chapter does not contain original results.

- Chapter 5 is the first original chapter of this thesis and contains the results of [CGS23, CGT24]. In Section 5.1 we compute the Riemann curvature tensor Rm_N of the rigid c-map metric g_N on $N = T^*M$ for any affine special Kähler manifold M, see Theorem 5.1.4. If M is furthermore conical, then we get that the Riemann curvature tensor Rm_N is a section of the subbundle

$$\mathrm{Sym}^2(\Lambda^2\mathcal{Z}^\perp)\oplus\left(\Lambda^2\mathcal{Z}^\perp\vee(\mathcal{Z}^\perp\wedge\mathcal{Z})\right)\subset\mathrm{Sym}^2(\Lambda^2T^*N),$$

where $\mathcal{Z} := (\mathbb{H}Z)^*$, $\mathcal{Z}^\perp := ((\mathbb{H}Z)^\perp)^*$ and $\mathbb{H}Z := \mathrm{span}\{Z, I_1Z, I_2Z, I_3Z\}$, see Proposition 5.1.6. In particular, Rm_N vanishes if at least two of the entries belong to $\mathbb{H}Z$. In Section 5.2 we obtain the first main result of this thesis: a quaternionic Kähler manifold in the image of the deformed supergravity c-map is not locally homogeneous, see Theorem 5.2.6. To prove this it is enough to show that $\|\mathrm{Rm}^c_{\mathrm{H}}\|^2_{g^c_{\mathrm{H}}}$, which is $\mathscr{H}$-related with $\|\mathrm{Rm}^c_{\bar{N}}\|^2_{g^c_{\bar{N}}}$, is not constant on $N = T^*M$ for $c > 0$. This is done in Proposition 5.2.4 by showing that the derivative of $\|\mathrm{Rm}^c_{\mathrm{H}}\|^2_{g^c_{\mathrm{H}}}$ in the direction of Ξ, given by (22), is not zero. For that we compute $\mathscr{L}_\Xi\|\mathrm{Rm}^c_{\mathrm{H}}\|^2_{g^c_{\mathrm{H}}}$ explicitly in Proposition 5.2.3. As a corollary of this first main result we obtain that the deformed supergravity c-map applied to a simply connected homogeneous projective special Kähler manifold gives us a complete cohomogeneity one quaternionic Kähler manifold, see Corollary 5.2.7. This generalizes the results of [CST21] and answers positively to a conjecture stated in [Thu20]. In Section 5.3 we explain how to construct isometries for a rigid c-map space and how these interact, giving rise to an isometric action of the group $\mathrm{Aut}(M)\ltimes\mathbb{R}^{2n}$, $n = \dim_{\mathbb{C}} M$, on $N = T^*M$, see Proposition 5.3.4. Finally, in Section 5.4 we describe how to lift the action of $\mathrm{Aut}(M)\ltimes\mathbb{R}^{2n}$ on N to an action of $\mathrm{Aut}(M)\ltimes\mathrm{Heis}_{2n+1}$, given by (31), on $P = N\times\mathbb{S}^1$. In the case of supergravity q-map spaces, i.e. when the quaternionic Kähler manifold $\bar{N}$ is determined by a projective special real manifold $\mathcal{H} \subset \mathbb{R}^{n-1}$, we show that the group

$$\left((\mathbb{R}_{>0} \times \mathrm{Aut}(\mathcal{H})) \ltimes \mathbb{R}^{n-1}\right) \ltimes (\mathrm{Heis}_{2n+1}/\mathcal{F}),$$

where $\mathcal{F}$ is an infinite cyclic subgroup of the Heisenberg center, acts isometrically and effectively on the quaternionic Kähler manifold $\bar{N} \subset P$ (for $c \geq 0$) viewed as a hypersurface of the circle bundle P, see Theorem 5.4.4. This gives us the second main result of this thesis, which generalizes the results of [CDJL21], where such group was described only for the undeformed case $c = 0$.

- Chapter 6 is the second original chapter of this thesis and contains part of an unpublished work in progress with Vicente Cortés and Markus Röser. In Section 6.1 we consider a manifold $\bar{N} := (0,\infty) \times K$ equipped with an Einstein metric $g = f(\rho)\mathrm{d}\rho^2 + g_\rho$, where g_ρ is a metric on the hypersurface $\bar{N}_\rho := \{\rho\} \times K$. In Lemma 6.1.5 we compute the Ricci curvature tensor of $(\bar{N}_\rho, g_\rho)$:

$$\mathrm{Ric}_{\bar{N}_\rho} = \lambda g_\rho + \left(\frac{1}{4f}\mathrm{tr}\left(\frac{\partial}{\partial\rho}g_\rho\right) - \frac{f'}{4f^2}\right)\frac{\partial}{\partial\rho}g_\rho - \frac{2}{f}h_\rho^2 + \frac{1}{2f}\frac{\partial^2}{\partial\rho^2}g_\rho.$$

Then we apply this formula to the one-loop deformation of the non-compact symmetric space $\bar{N} = \mathrm{SU}(n,2)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))$, which is a cohomogeneity one quaternionic Kähler manifold, and obtain explicit expressions for the eigenvalues of the Ricci endomorphism of $(\bar{N}_\rho, g_\rho^c)$, see Proposition 6.1.8. We devote Section 6.2 to express $(\bar{N}_\rho, g_\rho^c)$ as a Riemannian solvmanifold. For that we first identify $\bar{N}_\rho$ with a simply connected solvable Lie group L and compute the structure constants of the corresponding Lie algebra $\mathfrak{l}$ in Lemma 6.2.1, Lemma 6.2.2 and Lemma 6.2.3. Then we express the metric g_ρ^c as a left-invariant metric on L and compute its Ricci endomorphism expressed on a basis of $\mathfrak{l}$, see Proposition 6.2.6. With this information we deduce that $(\bar{N}_\rho, g_\rho^c)$ is a solvsoliton for $c = 0$ (which agrees with the general result of [DST21]) and is not a solvsoliton for $c > 0$, yielding the third main result of this thesis, see Theorem 6.2.15 and Theorem 6.2.17.

1.2 Physical background and motivation

From the pure mathematical point of view, the supergravity c-map construction may be seen rather unnatural. Thus, before starting with the mathematical formulation, we would like to explain briefly (and not exhaustively) the motivation for such construction and the physical ideas that give rise to it. Furthermore, this gives us the opportunity to explain some concepts and notation used in physics to try to make them more accessible to a mathematical audience (a very good reference for this, in the author's opinion, is [Ham17]). Some useful references for this section are [Fre99a, FVP12, Tan14, Cec15].

Note first of all that in the physics literature the Einstein summation convention is widely used. It establishes that when an index variable appears twice in a single term and is not otherwise defined, then there is summation of that term over all the values of the index. We will use this convention throughout the whole subsection.

Typically, when in physics we talk about a (classical) **field theory** we refer to:

- A base smooth manifold X equipped with a pseudo-Riemannian metric η. The pair (X,η) is usually called a **spacetime**. In several interesting cases, we just take $X=\mathbb{R}^4$ equipped with the Minkowski metric $\eta=\mathrm{diag}(1,-1,-1,-1)$. We will denote the pair $(\mathbb{R}^4,\eta)$ just by $\mathbb{R}^{1,3}$.
- Smooth sections of some bundles over X. These are the **fields** of the theory. The theory itself depend on which kind of fields do we have. For instance, we can consider sections on a trivial bundle $X\times M$. In this case these correspond just to functions $\phi : X \longrightarrow M$, where M is usually known as the **scalar** or **target manifold**. Some other examples of fields are connection 1-forms of a principal G-bundle or spinors.

The theories can be described using a **Lagrangian** $\mathcal{L}$, that is, an algebraic expression in terms of the fields and derivatives of the fields (more precisely, $\mathcal{L}$ is a section of a jet bundle, see e.g. [CH17]). The simplest example is the **Klein-Gordon** Lagrangian of a free scalar field $\phi : \mathbb{R}^{1,3} \longrightarrow \mathbb{R}$ with mass $m\in\mathbb{R}$:

$$\mathcal{L}_{\mathrm{KG}}[\phi]=\frac{1}{2}\partial^\mu\phi\partial_\mu\phi-\frac{1}{2}m^2\phi^2. \tag{1}$$

Let us unravel this a bit, as this compact expression may seem a bit mysterious at first sight. Usually for the Minkowski spacetime the coordinates are labeled from 0 to 3. The coordinate $x^0=t$ is the temporal direction (we set the speed of light to be equal to 1) while the coordinates $x^1=x$, $x^2=y$, $x^3=z$ are the spatial directions. The Greek indices run from 0 to 3 while the Latin indices run from 1 to 3. Thus $\partial_\mu := \frac{\partial}{\partial x^\mu}$ for $\mu=0,1,2,3$. We also have

$$\partial^0=\partial_\nu\eta^{0\nu}=\partial_0 \quad \text{and} \quad \partial^i=\partial_\nu\eta^{i\nu}=-\partial_i \quad \text{for } i=1,2,3,$$

where $\eta^{\mu\nu}$ denotes the entries of the inverse metric $\eta^{-1}=\mathrm{diag}(1,-1,-1,-1)$. Hence the expression (1) can be written as

$$\mathcal{L}_{\mathrm{KG}}[\phi]=\frac{1}{2}\left(\left(\frac{\partial\phi}{\partial x^0}\right)^2-\sum_{i=1}^{3}\left(\frac{\partial\phi}{\partial x^i}\right)^2\right)-\frac{1}{2}m^2\phi^2.$$

Some other common Lagrangians are:

- The **Yang-Mills** Lagrangian of a connection 1-form A on a principal G-bundle over $X=\mathbb{R}^{1,3}$ for G a compact Lie group:

$$\mathcal{L}_{\mathrm{YM}}[A]=-\frac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu},$$

where $a = 1, \dots, \dim_{\mathbb{R}} G$, and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a{}_{bc} A^b_\mu A^c_\nu$$

is the curvature 2-form of the connection 1-form A, $f^a{}_{bc}$ are the structure constants of the Lie algebra $\mathfrak{g}$ and $g \in \mathbb{R}$ is the so-called coupling constant.

- The **Dirac** Lagrangian of a free Dirac spinor $\psi : \mathbb{R}^{1,3} \longrightarrow \mathbb{C}^4$ of mass $m \in \mathbb{R}$:

$$\mathcal{L}_{\mathrm{D}}[\psi] = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi,$$

where $\bar{\psi} := \psi^\dagger \gamma^0$ and γ^μ are the so-called gamma matrices (we also have set here $\hbar = 1$). Here $\psi^\dagger$ denotes the conjugate transpose of ψ.

- The **Einstein-Hilbert** Lagrangian is a slightly different case since we are in a gravity theory. Here we fix a smooth manifold X but we do not fix a metric. Instead we take a (pseudo-Riemannian) metric g as a field itself. Then the Lagrangian has the form

$$\mathcal{L}_{\mathrm{EH}}[g] = \frac{1}{16\pi G}\sqrt{|g|}R,$$

where G is the Newtonian constant of gravitation, $|g|$ is the absolute value of the determinant of the matrix representation of the metric tensor on the manifold X and R is the Ricci scalar of the metric g (also called scalar curvature and denoted scal).

The above Lagrangians are simple in the sense that each of them only depend on a type of field. The situation is more complicated when there are several fields **coupled** in the Lagrangian which interact between them. An example of such situation is the Lagrangian of the **Standard Model** of particle physics $\mathcal{L}_{\mathrm{SM}}[\phi, A, \psi]$, which depend on scalar fields, connection 1-forms and spinors (for a detailed discussion see [Ham17]).

Given a Lagrangian, the dynamics of the theory are determined by a functional on the space of all possible configurations, which is known as the **action** of the theory. This is the integral of the Lagrangian over the whole spacetime:

$$\mathcal{S} = \int \mathrm{d}^n x \mathcal{L}.$$

Here n denotes the dimension of the spacetime X and $\mathrm{d}^n x := \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$ is the usual way of denoting in the physical literature a volume form. Physical theories are governed by the **Hamilton's principle of least action**. This says that the variation of the action must be zero:

$$\delta \mathcal{S} = 0.$$

From this statement one can deduce the **equations of motion** of the theory. These correspond to the Euler-Lagrange equations of the Lagrangian $\mathcal{L}$. Very well-known equations are precisely the equations of motion of a field theory:

- The equations of motion of the Klein-Gordon Lagrangian give the **wave equation**
$$(\Box + m^2)\phi = 0,$$
where $\Box = \partial^\mu \partial_\mu$ is the d'Alembert operator.
- The equations of motion of the Dirac Lagrangian give the **Dirac equation**
$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$
- The equations of motion of the Einstein-Hilbert Lagrangian give the vacuum **Einstein field equations**
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0,$$
where $R_{\mu\nu}$ denotes the Ricci curvature tensor of g. This equation is equivalent to $R_{\mu\nu} = 0$. Hence, a solution to these equations is a Ricci-flat metric.

Now let us consider a field theory on $\mathbb{R}^{1,3}$. To have a meaningful physical theory, its corresponding action $\mathcal{S} = \int \mathrm{d}^4 x \mathcal{L}$ must be invariant under the **Poincaré group** (or its algebra). The Poincaré group is the isometry group of $\mathbb{R}^{1,3}$. It consists of translations in spacetime and Lorentz transformations:
$$\mathrm{Isom}(\mathbb{R}^{1,3}) = \mathbb{R}^{1,3} \rtimes \mathrm{O}(1,3).$$
The Lie algebra relations of the Poincaré group are given by
$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[M_{\mu\nu}, P_\rho] &= i(\eta_{\rho\nu} P_\mu - \eta_{\rho\mu} P_\nu), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma}),
\end{aligned}$$
where P_μ are the 4 generators of the translations and $M_{\mu\nu}$ are the 6 generators of the Lorentz transformations (note that $M_{\mu\nu} = -M_{\nu\mu}$).

However, the theories we are interested in have more symmetries that just Poincaré. These are **supersymmetric** theories. To describe them, we first need to enhance the Poincaré Lie algebra to a Lie superalgebra. A **Lie superalgebra** is a $\mathbb{Z}_2$-graded algebra, i.e. a vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, together with a bilinear product satisfying the following properties for all $X_i \in \mathfrak{g}_i$, $X_j \in \mathfrak{g}_j$:

- Grading: $X_i X_j \in \mathfrak{g}_{i+j \bmod 2}$.
- Supersymmetry: $X_i X_j = -(-1)^{ij} X_j X_i$.
- Super Jacobi identity:
$$X_k(X_\ell X_m)(-1)^{km} + X_\ell(X_m X_k)(-1)^{\ell k} + X_m(X_k X_\ell)(-1)^{m\ell} = 0.$$

The vector subspace $\mathfrak{g}_0$ is called the even part and $\mathfrak{g}_1$ the odd part. It follows that the bilinear product is:

- Anti-symmetric on $\mathfrak{g}_0 \times \mathfrak{g}_0$ and maps to $\mathfrak{g}_0$ (written as $[\cdot,\cdot]$).
- Symmetric on $\mathfrak{g}_1 \times \mathfrak{g}_1$ and maps to $\mathfrak{g}_0$ (written as $\{\cdot,\cdot\}$).
- Anti-symmetric on $\mathfrak{g}_0 \times \mathfrak{g}_1$ and $\mathfrak{g}_1 \times \mathfrak{g}_0$ and maps to $\mathfrak{g}_1$ (written as $[\cdot,\cdot]$).

The notation $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ is just a different notation for the bilinear product on the algebra $\mathfrak{g}$. On two general elements in $\mathfrak{g}$, which have components in both $\mathfrak{g}_0$ and $\mathfrak{g}_1$, the product is neither symmetric nor anti-symmetric. It is not difficult to show that:

- The vector subspace $\mathfrak{g}_0$ with the product $[\cdot,\cdot]$ is a Lie algebra.
- The map $\phi : \mathfrak{g}_0 \longrightarrow \mathrm{End}(\mathfrak{g}_1)$ with $\phi(X)V = [X,V]$ is a representation of the Lie algebra $\mathfrak{g}_0$ on the vector space $\mathfrak{g}_1$.
- The map $\{\cdot,\cdot\} : \mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$ is a vector space-valued symmetric bilinear form.

The **Poincaré superalgebra** is then the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ is the Poincaré algebra and

$$\mathfrak{g}_1 = \mathrm{span}_{\mathbb{R}}\{Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}} \mid I = 1,\ldots,\mathcal{N}\},$$

where $Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}$, $\alpha, \dot{\alpha} = 1,2$, are $2\mathcal{N}$ Weyl spinors. In dimension 4, these constitute a set of $\mathcal{N}$ Majorana spinors. The elements of $\mathfrak{g}_1$ are called (infinitesimal) supersymmetries. When $\mathcal{N} = 1$ we have **minimal supersymmetry** whereas for $\mathcal{N} > 1$ we have **extended supersymmetry**.

The commutation relations in $\mathfrak{g}_1$ are given as follows:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta^{IJ}, \quad \{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta} Z^{IJ}, \quad \{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \varepsilon_{\dot{\alpha}\dot{\beta}} (Z^\dagger)^{IJ},$$

where σ_μ are the Pauli matrices defined as

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\sigma^\mu = (\sigma^0, \sigma^i) = (\sigma_0, -\sigma_i)$. The matrix ε is given by

$$\varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $Z^{IJ} = -Z^{JI}$ are generators called **central charges**. They commute with all the generators of the algebra.

The commutation relations between $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are given as follows:

$$[P_\mu, Q^I_\alpha] = 0, \quad [P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0,$$

$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta, \quad [M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}},$$

where

$$(\sigma^{\mu\nu})_{\alpha}{}^{\beta} = \frac{1}{4}\big((\sigma^{\mu})_{\alpha\dot{\gamma}}(\bar{\sigma}^{\nu})^{\dot{\gamma}\beta} - (\sigma^{\nu})_{\alpha\dot{\gamma}}(\bar{\sigma}^{\mu})^{\dot{\gamma}\beta}\big),$$
$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}\big((\bar{\sigma}^{\mu})^{\dot{\alpha}\gamma}(\sigma^{\nu})_{\gamma\dot{\beta}} - (\bar{\sigma}^{\nu})^{\dot{\alpha}\gamma}(\sigma^{\mu})_{\gamma\dot{\beta}}\big)$$

and $\bar{\sigma}^{\mu} = (\sigma_0, \sigma_i)$.

Remark 1.2.1. The Poincaré superalgebra can be defined in any dimension. We take $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathbb{R}^{1,n-1} \oplus \mathfrak{so}(1,n-1)$ is the Poincaré algebra in dimension n and $\mathfrak{g}_1$ again contains the supersymmetric generators. Note that since these are spinors, they depend on the dimension n and are not necessarily as in dimension 4. The commutation relations also depend on the specific dimension. This implies that, whenever we talk about a supersymmetric theory, we must specify the dimension of the spacetime. We point out that Poincaré superalgebras in any signature, any dimension and any number of odd generators have been classified in [AC97a].

If the generators of the supersymmetry transformation depend on the coordinates of the spacetime, then the supersymmetric theory is called **local**. Since the anti-commutator of two supersymmetries is a translation (basically a vector), theories which are invariant under spacetime-dependent supersymmetries are invariant under the action of all vector fields and hence under all infinitesimal diffeomorphisms. Therefore, local supersymmetric theories are diffeomorphism invariant, i.e. theories of gravity. Such theories are also called **supergravity theories**. In contrast, theories in which the generators of the supersymmetry do not depend on the coordinates of the spacetime are called **global** or **rigid supersymmetric theories**.

In Poincaré invariant theories we study the irreducible representations, also called **multiplets**, of the Poincaré algebra, since they are used to construct the fields of the theory. Similarly, in supersymmetric theories we study the irreducible representations of the Poincaré superalgebra. They are called **supermultiplets** and depend on the number $\mathcal{N}$. These are constructed from the multiplets of the Poincaré algebra and hence they contain fields of different types, that is, they can contain scalar fields, gauge fields (1-forms), fermions (spinors) and/or gravitational fields (metrics).

We are interested in $\mathcal{N} = 2$ (local and global) supersymmetric theories in dimension 4. In this case, there exist several (massless) supermultiplets relevant for us:

- The **vector multiplet** consists of a complex scalar field, a gauge field and 2 Weyl fermions. The scalar manifold of vector multiplets of $\mathcal{N} = 2$ rigid supersymmetry is an affine special Kähler manifold and the scalar manifold of vector multiplets of $\mathcal{N} = 2$ supergravity is a projective special Kähler manifold [dWVP84].
- The **hypermultiplet** consists of 4 real scalar fields and 2 Weyl fermions. The scalar manifold of hypermultiplets of $\mathcal{N} = 2$ rigid supersymmetry is a hyperkähler manifold [AF81] and the scalar manifold of hypermultiplets of $\mathcal{N} = 2$ supergravity is a quaternionic Kähler manifold of negative scalar curvature [BW83].

	Vector multiplet	Hypermultiplet
Rigid supersymmetry	Affine special Kähler	Hyperkähler
Local supersymmetry	Projective special Kähler	Quaternionic Kähler

Table 1: Scalar manifolds of $\mathcal{N} = 2$ supersymmetric theories

A way to relate these geometries was first obtained from superstring theory (for the following discussion see e.g. [vdA08, Ale13] and references therein). Briefly speaking (for our purposes), a **superstring theory** is a supersymmetric field theory in which the spacetime is a manifold of dimension 10. There are five different superstring theories but we are only interested in the so-called **type II superstring theories**: type IIA and IIB. Since the spacetime around us appears to be 4-dimensional and the theory is 10-dimensional, physicists developed a method known as **compactification** on an **internal manifold**. That is, the 10-dimensional spacetime is divided in the usual 4-dimensional Minkowski space $\mathbb{R}^{1,3}$ and the other six dimensions are very small and constitute the internal manifold X, which is assumed to be compact. Therefore the spacetime is of the form

$$\mathbb{R}^{1,3} \times X.$$

The presence of supersymmetry in the theory implies that the internal manifold X is a Calabi-Yau manifold [CHSW85].

The 10-dimensional theory, when compactified on a Calabi-Yau manifold X, gives us a 4-dimensional effective field theory on $\mathbb{R}^{1,3}$. This is, roughly speaking, an approximation of the theory. It turns out that the 4-dimensional effective field theory is a $\mathcal{N} = 2$ supergravity theory.

The scalar fields of the 4-dimensional effective field theory parameterize the so-called **moduli space** of the type II theory. This is a Riemannian manifold

$$\mathscr{M} = \mathscr{M}_{\mathrm{VM}} \times \mathscr{M}_{\mathrm{HM}},$$

where the subscripts refer to vector multiplet and hypermultiplet, respectively. The space $\mathscr{M}_{\mathrm{VM}}$ is a projective special Kähler manifold and the space $\mathscr{M}_{\mathrm{HM}}$ is a quaternionic Kähler manifold. In type IIA, the scalar fields of the vector multiplet moduli space are the (complexified) Kähler moduli of the Calabi-Yau manifold X while the hypermultiplet moduli space contains the complex structure moduli space of X. In type IIB the role of the moduli spaces is interchanged. By further compactifying to $\mathbb{R}^{1,2} \times \mathbb{S}^1_R \times X$ one can use **T-duality** to relate the moduli space of the 4-dimensional effective field theory of type IIA with the one from type IIB, namely the vector multiplet moduli space in type IIA/B to the the hypermultiplet moduli space in type IIB/A (on the same Calabi-Yau manifold X). This relation is called the **local** or **supergravity c-map** [CFG89]. Therefore, the local c-map explains how to construct a quaternionic Kähler manifold of negative scalar curvature from a projective special Kähler manifold. In the situation where there is no gravity, the c-map is called the **rigid c-map**, which explains how to construct a hyperkähler manifold from an affine special Kähler one. The explicit construction of the local c-map metric was carried out in [FS90]. Hence, this metric is sometimes referred to as the **Ferrara-Sabharwal metric**.

Although the supergravity c-map has its origin in string theory, it can be also formulated purely in terms of supergravity using dimensional reduction. Briefly speaking, **dimensional reduction** from a theory in dimension $n+1$ to a theory in dimension n consists of compactifying the spacetime $\tilde{X}^{n+1}$ on a circle $\mathbb{S}^1_R$ of radius R, that is

$$\tilde{X}^{n+1} = X^n \times \mathbb{S}^1_R.$$

Then we can expand the fields on $\tilde{X}^{n+1}$ in Fourier series and the Fourier coefficients are fields on X^n. If we just take into account the zeroth terms of the expansion we talk about dimensional reduction, otherwise we talk about compactification. Starting from vector multiplets on a $\mathcal{N}=2$ supergravity theory in four dimensions, dimensional reduction to three dimensions yields hypermultiplets on a $\mathcal{N}=4$ supergravity theory. The scalar manifold in this theory is also known to be quaternionic Kähler, so we have obtained a quaternionic Kähler manifold from a projective special Kähler manifold again. It turns out that this construction precisely reproduces the supergravity c-map [FS90, dWVP92]. Similarly, dimensional reduction of vector multiplets on a $\mathcal{N}=2$ supergravity theory in five dimensions (whose scalar manifold is known to be projective special real) to vector multiplets on a $\mathcal{N}=2$ supergravity theory in four dimensions leads to the **supergravity r-map**, which assigns a projective special Kähler manifold to each projective special real manifold [dWVP92]. The composition of the supergravity r-map and the supergravity c-map is called the **supergravity q-map**. For more details see e.g. [LM20] and references therein.

Chapter 2

Quaternionic Kähler geometry

In this first preliminary chapter we introduce and describe in detail the main object of interest of this thesis: quaternionic Kähler manifolds. In Section 2.1 we recall the basics of holonomy theory and introduce quaternionic Kähler manifolds using this language. In Section 2.2 we describe quaternionic Kähler manifolds in terms of some tensor fields and explain some of their general properties, such as their curvature or some spaces canonically associated to them. In Section 2.3 we focus on complete quaternionic Kähler manifolds with positive scalar curvature, describing the additional properties that these manifolds have. Finally, in Section 2.4 we do the same with complete quaternionic Kähler manifolds of negative scalar curvature. The latter will be the focus of attention of this thesis, so their study will be extended throughout the text. None of the results mentioned in this chapter are original to this thesis and the references will be properly cited.

2.1 Holonomy and Berger theorem

The history of quaternionic Kähler manifolds starts with the celebrated Berger theorem about the classification of the possible holonomy groups of a simply connected, irreducible and non-symmetric Riemannian manifold (M,g). In this list appears the Lie group

$$\mathrm{Sp}(n)\mathrm{Sp}(1) := (\mathrm{Sp}(n) \times \mathrm{Sp}(1))/\mathbb{Z}_2,$$

so this points out that may exist examples of Riemannian manifolds with that holonomy group. The search of manifolds with holonomy in Berger list has become one of the greatest problems in modern differential geometry, leading to plenty of new interesting theories, techniques and results. We briefly recall here the well-known theory of holonomy groups to understand the statement of Berger theorem and its consequences. Basic references for this are [Bes87, Joy00].

Throughout the text we will assume that all the manifolds are connected, unless stated otherwise.

2.1.1 Definition and properties of the holonomy group

Let (M,g) be a Riemannian manifold and let $x \in M$. Given a loop $\gamma : [0,1] \longrightarrow M$ based at x, the Levi-Civita connection ∇ of the metric g defines a parallel transport map

$$P_\gamma : T_xM \longrightarrow T_xM$$

as $P_\gamma(v) := V_{\gamma(1)}$ for $v \in T_xM$, where V is the unique vector field such that $\nabla_{\gamma'} V = 0$ and $V_{\gamma(0)} = v$. Given a loop γ based on x, we define the inverse loop by $\gamma^{-1}(t) := \gamma(1-t)$. For two paths γ_1, γ_2 on M such that $\gamma_1(0) = x = \gamma_2(1)$ and $\gamma_1(1) = \gamma_2(0)$, we define the loop $\gamma_2 \circ \gamma_1$ based at x by

$$\gamma_2 \circ \gamma_1(t) := \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

We have that $P_{\gamma^{-1}} = P_\gamma^{-1}$ and $P_{\gamma_2 \circ \gamma_1} = P_{\gamma_2} \circ P_{\gamma_1}$. Moreover, P_γ is an orthogonal transformation of T_xM. Indeed

$$g(P_\gamma(v), P_\gamma(w)) = g(V_{\gamma(1)}, W_{\gamma(1)}) = g(V_{\gamma(0)}, W_{\gamma(0)}) = g(v,w),$$

where the second equality follows from the fact that $g(V,W)$ is constant along γ since ∇ is metric.

We then define the **holonomy group** of g at $x \in M$ as

$$\mathrm{Hol}(g)_x := \{P_\gamma \mid \gamma : [0,1] \longrightarrow M, \gamma(0) = \gamma(1) = x\} \subseteq \mathrm{O}(T_xM).$$

The holonomy group depends on the basepoint x only up to conjugation, i.e. if γ is a path from x to y in M, then

$$\mathrm{Hol}(g)_y = P_\gamma \circ \mathrm{Hol}(g)_x \circ P_\gamma^{-1}.$$

As a consequence, the holonomy groups at various points of M are in fact all isomorphic. Then we simply talk about the holonomy group of (M,g), denoted by $\mathrm{Hol}(g)$, since it is defined up to conjugation.

We denote by $\mathrm{Hol}^0(g)$ the **restricted holonomy group**, which is the subgroup of $\mathrm{Hol}(g)$ consisting of maps P_γ coming from contractible loops γ. Next we state some important properties of $\mathrm{Hol}^0(g)$.

Proposition 2.1.1. *Let (M,g) be a Riemannian manifold. Then*

(a) $\mathrm{Hol}^0(g)$ *is connected.*

(b) $\mathrm{Hol}^0(g)$ *is the identity component of* $\mathrm{Hol}(g)$.

(c) $\mathrm{Hol}^0(g)$ *is a normal subgroup of* $\mathrm{Hol}(g)$.

(d) *There is a natural, surjective group homomorphism* $\pi_1(M) \longrightarrow \mathrm{Hol}(g)/\mathrm{Hol}^0(g)$. *Thus, if M is simply connected, then* $\mathrm{Hol}(g) = \mathrm{Hol}^0(g)$.

Moreover, the following non-trivial property also holds.

Theorem 2.1.2 ([BL52])**.** *Let* (M,g) *be a Riemannian manifold of dimension* n*. Then the restricted holonomy group* $\mathrm{Hol}^0(g)$ *is a closed subgroup of* $\mathrm{O}(n)$*. In particular,* $\mathrm{Hol}^0(g)$ *is compact Lie group.*

Since $\mathrm{Hol}^0(g)_x \subseteq \mathrm{SO}(T_xM)$ is a Lie group, we can consider its Lie algebra. We denote by

$$\mathfrak{hol}(g)_x \subseteq \mathfrak{so}(T_xM) \cong \Lambda^2 T_x^* M$$

the **holonomy algebra** of (M,g).

Remark 2.1.3. The holonomy group $\mathrm{Hol}(g)$ is not just an abstract group, it comes equipped with a natural representation on T_xM, or equivalently, $\mathrm{Hol}(g)$ is embedded as a subgroup of $\mathrm{SO}(T_xM)$. Therefore we can think of the holonomy group as a representation, and we will refer to it as the **holonomy representation**.

There is a fundamental relationship between the holonomy group (or its Lie algebra) and the curvature of (M,g). The holonomy algebra constraints the curvature.

Proposition 2.1.4. *Let* (M,g) *be a Riemannian manifold. Then for every* $x \in M$

$$\mathrm{Rm}(g)_x \in \mathrm{Sym}^2(\mathfrak{hol}(g)_x) \subseteq \mathrm{Sym}^2(\Lambda^2 T_x^* M),$$

where $\mathrm{Rm}(g)$ *is the* $(0,4)$*-curvature tensor of* g*.*

There is a kind of converse to this result, known as the Ambrose-Singer theorem. It says that the holonomy algebra is generated by the curvature.

Theorem 2.1.5 ([AS53, Theorem 2])**.** *Let* (M,g) *be a Riemannian manifold. For* $x \in M$*, the Lie algebra* $\mathfrak{hol}(g)_x$ *is the subspace spanned by the elements*

$$P_\gamma^{-1} \circ R(v,w)_y \circ P_\gamma,$$

where $\gamma : [0,1] \longrightarrow M$ *is a path with* $\gamma(0) = x$ *and* $\gamma(1) = y$*,* $P_\gamma : T_xM \longrightarrow T_yM$ *is the parallel transport map and* $v,w \in T_yM$*.*

This shows that the curvature completely determines $\mathfrak{hol}(g)$, and hence $\mathrm{Hol}^0(g)$ (up to coverings). For instance, if the manifold is flat, so that $\mathrm{Rm}(g) = 0$, then $\mathfrak{hol}(g) = 0$, and therefore $\mathrm{Hol}^0(g)$ is trivial.

Let S be a tensor field on M. We say that S is a **constant tensor** or **parallel** (with respect to the Levi-Civita connection ∇) if

$$\nabla S = 0.$$

The next result, usually known as the **holonomy principle**, shows that the constant tensors on M are determined entirely by the holonomy group $\mathrm{Hol}(g)$.

Theorem 2.1.6. *Let (M,g) be a Riemannian manifold and let $x \in M$. If S is a parallel tensor field, then S_x is fixed by the action of $\mathrm{Hol}(g)_x$. Conversely, if S_0 is fixed by the action of $\mathrm{Hol}(g)_x$, then there exists a unique parallel tensor field S such that $S_x = S_0$.*

Thus, given a Riemannian manifold (M,g), the holonomy group $\mathrm{Hol}(g)$ determines the constant tensors on M, and the constant tensors on M determine the holonomy group $\mathrm{Hol}(g)$. Therefore, studying the holonomy group and studying the constant tensors come down to the same thing.

2.1.2 Holonomy group and products

Given a Riemannian manifold (M,g), we may now ask which is its holonomy group. More precisely, one can ask which subgroups of $\mathrm{O}(n)$ can occur as the holonomy group of a Riemannian manifold of dimension n. We will see that this question reduces to the case where the holonomy group acts irreducibly on the tangent space.

Definition 2.1.7. Let (M,g) be a Riemannian manifold.

- We say that (M,g) is **reducible** if it is isometric to $(M_1 \times M_2, g_1 \times g_2)$ for (M_i, g_i) Riemannian manifolds with $\dim M_i > 0$.
- We say that (M,g) is **locally reducible** if every point has a reducible open neighborhood.
- We say that (M,g) is **irreducible** if it is not locally reducible.

For a product metric we have the following result.

Proposition 2.1.8. *Let (M_1,g_1), (M_2,g_2) be Riemannian manifolds. Then the product metric $g_1 \times g_2$ has holonomy $\mathrm{Hol}(g_1 \times g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2)$.*

If g is a Riemannian metric and the holonomy representation of g is reducible, then the metric itself is at least locally reducible, and its holonomy group is a product. Therefore we have the following.

Proposition 2.1.9. *Let (M,g) be an irreducible Riemannian manifold. Then the representations of $\mathrm{Hol}(g)$ and $\mathrm{Hol}^0(g)$ on T_xM are irreducible for all $x \in M$.*

By assuming that (M,g) is a Riemannian manifold with M simply connected and g complete we obtain a sort of converse to Proposition 2.1.8 due to the following celebrated theorem by de Rham.

Theorem 2.1.10 ([dR52, Théorème III])**.** *Let (M,g) be a complete, simply connected Riemannian manifold. Then there exist complete, simply connected Riemannian manifolds (M_j, g_j) for $j = 1,\dots,k$, such that the holonomy representation of each $\mathrm{Hol}(g_j)$ is irreducible, (M,g) is isometric to the product $(M_1 \times \cdots \times M_k, g_1 \times \cdots \times g_k)$, and*

$$\mathrm{Hol}(g) = \mathrm{Hol}(g_1) \times \cdots \times \mathrm{Hol}(g_k).$$

2.1.3 Holonomy and symmetric spaces

We now briefly describe the theory of Riemannian symmetric spaces. These were introduced by Cartan, who also classified them completely [Car26, Car27] (see e.g. [Hel78] for more details).

Definition 2.1.11. Let (M,g) be a Riemannian manifold.

- We say that (M,g) is a **symmetric space** if for every $x \in M$ there exists an isometry $s_x : M \longrightarrow M$ such that $s_x(x) = x$ and $\mathrm{d}(s_x)_x = -\mathrm{Id}$.
- We say that (M,g) is a **locally symmetric space** if every point $x \in M$ admits an open neighborhood U_x in M and an isometry $s_x : U_x \longrightarrow U_x$ such that $s_x(x) = x$ and $\mathrm{d}(s_x)_x = -\mathrm{Id}$.

Cartan proved the following characterization of locally symmetric spaces.

Theorem 2.1.12. *Let (M,g) be a Riemannian manifold, ∇ the Levi-Civita connection of g and R the $(1,3)$-curvature tensor. Then (M,g) is a locally symmetric space if and only if $\nabla R = 0$.*

Clearly, any symmetric space is locally symmetric. The following result tells us when the converse is true.

Proposition 2.1.13. *Let (M,g) be a complete, simply connected locally symmetric space. Then (M,g) is a symmetric space.*

Here are some properties of symmetric spaces.

Proposition 2.1.14. *Let (M,g) be a symmetric space. Then (M,g) is complete and M is homogeneous under the action of the identity component of the isometry group, i.e. $M \cong G/K$ where $G = \mathrm{Isom}^0(M,g)$ and K is the (compact) stabilizer of G at any point $x \in M$.*

Remark 2.1.15. Any Riemannian homogeneous space is complete.

We say that (M,g) is **irreducible symmetric** if it is symmetric and its holonomy $\mathrm{Hol}^0(g)$ is irreducible. As a consequence of Theorem 2.1.10, a simply connected symmetric space is decomposed in a unique manner into a Riemannian product of irreducible symmetric spaces. This explains why we restrict ourselves to simply connected irreducible symmetric spaces. In this case we have the following.

Proposition 2.1.16. *Let $(M \cong G/K, g)$ be an irreducible simply connected symmetric space. Then the holonomy group $\mathrm{Hol}(g)$ is equal to K.*

Thus symmetric spaces are Riemannian manifolds for which the holonomy group is known.

Let us mention further properties of symmetric spaces which will be relevant later.

Proposition 2.1.17. *Let (M,g) be an irreducible symmetric space. Then (M,g) is Einstein, i.e.* $\mathrm{Ric}(g) = \lambda g$ *for some* $\lambda \in \mathbb{R}$*, where* $\mathrm{Ric}(g)$ *is the Ricci curvature of* g*.*

We say that an irreducible symmetric space (M,g) is of **compact type** if (M,g) has non-negative sectional curvature, and of **non-compact type** is (M,g) has non-positive sectional curvature.

Lemma 2.1.18. *Let (M,g) be a non-flat irreducible symmetric space. If (M,g) is of compact type, then M is a compact manifold. If (M,g) is of non-compact type, then M is a non-compact manifold.*

Remark 2.1.19. It is well-known that the Ricci and scalar curvature of (M,g) at any point $x \in M$ can be expressed as the sum of the sectional curvatures evaluated on an orthonormal basis of T_xM. Therefore, a Riemannian manifold with constant sectional curvature is Einstein and has constant scalar curvature.

2.1.4 Berger classification theorem

Now we are ready to state the Berger classification theorem of the possible holonomy groups for a Riemannian manifold (under the appropriate assumptions).

Theorem 2.1.20 ([Ber55, Théorème III.3])**.** *Let (M,g) be a simply connected, irreducible and non-symmetric Riemannian manifold. Then the holonomy group* $\mathrm{Hol}(g)$ *is one of the following:*

$\mathrm{Hol}(g)$	$\dim_{\mathbb{R}} M$
$\mathrm{SO}(n)$	n
$\mathrm{U}(m)$	$n = 2m\,(m \geq 2)$
$\mathrm{SU}(m)$	$n = 2m\,(m \geq 2)$
$\mathrm{Sp}(m)$	$n = 4m\,(m \geq 2)$
$\mathrm{Sp}(m)\mathrm{Sp}(1)$	$n = 4m\,(m \geq 2)$
G_2	$n = 7$
$\mathrm{Spin}(7)$	$n = 8$

Table 2: Berger list of holonomy groups.

In fact, Berger also included the case $n = 16$ and $\mathrm{Hol}(g) = \mathrm{Spin}(9)$, but it was shown by Alekseevsky [Ale68b] and independently by Brown and Gray [BG72] that any Riemannian metric with holonomy group $\mathrm{Spin}(9)$ is symmetric.

Berger proved that the groups on Table 2 are the only possibilities, but he did not show whether these groups actually occur as holonomy groups. Nowadays it is known that all the groups on Berger list occur as the holonomy groups of irreducible and non-locally symmetric Riemannian manifolds, although this has taken a considerable amount of effort during the last decades.

Riemannian manifolds with holonomy contained in $\mathrm{U}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ are Kähler, Calabi-Yau and hyperkähler, respectively. Similarly, we then may define quaternionic Kähler manifolds as follows.

Definition 2.1.21. Let (M,g) be a Riemannian manifold of dimension $4n > 4$. We say that it is **quaternionic Kähler** if $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(n)\mathrm{Sp}(1)$.

Remark 2.1.22. In the case $n = 1$ it happens that $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4)$, the generic case, so any simply connected 4-manifold has holonomy contained in $\mathrm{Sp}(1)\mathrm{Sp}(1)$. Later we will provide an alternative definition that extends to $n = 1$.

Note that $\mathrm{Sp}(n) \subset \mathrm{Sp}(n)\mathrm{Sp}(1)$, so a quaternionic Kähler manifold may have holonomy contained in $\mathrm{Sp}(n)$, and then it would be hyperkähler. We will exclude that case here (see Theorem 2.2.14). Then, in this work, quaternionic Kähler manifolds will be those whose holonomy is contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$ but not in $\mathrm{Sp}(n)$.

As a final remark, despite the name, quaternionic Kähler manifolds are in general not Kähler, although there are some examples that can have both structures (see Remark 2.3.9).

2.2 Quaternionic Kähler manifolds

We have defined quaternionic Kähler manifolds in terms of holonomy, but we would like to have an equivalent definition that allows us to work with more concrete geometric objects, rather than an abstract condition on the holonomy.

Some references for this section are the book of Besse [Bes87], the book of Boyer and Galicki [BG08] and the articles of Salamon [Sal82, Sal99].

Definition 2.2.1. Let M be a smooth manifold of dimension $4n$. We say that M is **almost quaternionic** if there exists a rank three subbundle $\mathcal{Q} \subset \mathrm{End}(TM)$ such that for every point $x \in M$ we have $\mathcal{Q}_x = \mathrm{span}_{\mathbb{R}}\{I_1, I_2, I_3\} \subset \mathrm{End}(T_xM)$ where

$$I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -\mathrm{Id}\,.$$

In other words, $\mathbb{R}\,\mathrm{Id} \oplus \mathcal{Q}$ is, at each point, a subalgebra isomorphic to the algebra of quaternions $\mathbb{H}$, hence the name quaternionic. Notice that although $\mathcal{Q}$ admits local frames of almost complex structures, these are not necessarily global. In particular, almost quaternionic manifolds are not necessarily almost complex manifolds. Nevertheless, locally we can perform similar constructions as in the almost complex case.

Definition 2.2.2. Let (M,g) be a Riemannian manifold and let $\mathcal{Q}$ be an almost quaternionic structure on M. We say that the metric g is **adapted** or **compatible** with the almost quaternionic structure if

$$g(I_k v, I_k w) = g(v,w)$$

for $I_k \in \mathcal{Q}_x$, $k = 1,2,3$, and all $v, w \in T_xM$ at all points $x \in M$. We call $(M,g,\mathcal{Q})$ an **almost quaternionic Hermitian manifold**.

Any almost quaternionic manifold admits an almost quaternionic Hermitian structure. Indeed, let $\tilde{g}$ be an arbitrary metric on M, then the metric g defined as

$$g(v,w) := \frac{1}{4}\Big(\tilde{g}(v,w) + \sum_{k=1}^{3} \tilde{g}(I_k v, I_k w)\Big)$$

is a metric compatible with the almost quaternionic structure.

Let (M,g) be a Riemannian manifold. We can define a bundle metric on $\mathrm{End}(TM)$ such that Id has unit norm by

$$\langle A,B\rangle := \frac{1}{\dim M}\operatorname{tr}(AB^*), \tag{2}$$

where B^* denotes the adjoint of B with respect to g. Then, the local almost complex structures $\{I_1,I_2,I_3\}$ form an orthonormal frame of $\mathscr{Q}$ with respect to the metric (2).

Given an adapted metric g on the almost quaternionic manifold $(M,\mathscr{Q})$, we have an isometric bundle embedding $\mathscr{Q} \subset \Lambda^2 T^*M$ which associates to each $I \in \mathscr{Q}_x$ the non-degenerate 2-form ω_I defined by

$$\omega_I(v,w) := g(Iv,w)$$

for $v,w \in T_xM$. The 2-forms associated to the orthonormal frame $\{I_1,I_2,I_3\}$ are denoted by $\{\omega_1,\omega_2,\omega_3\}$, where $\omega_k := \omega_{I_k}$. Using these 2-forms, we can construct a 4-form

$$\Omega := \sum_{k=1}^{3} \omega_k \wedge \omega_k, \tag{3}$$

which turns out to be globally defined, since two frames are related by an $\mathrm{SO}(3)$ transformation, and non-degenerate (i.e. $\Omega^n \neq 0$). The 4-form Ω is usually called the **fundamental 4-form** of the almost quaternionic Hermitian manifold $(M,g,\mathscr{Q})$. This form plays a similar role to the fundamental 2-form in the case of Hermitian manifolds.

Now we define quaternionic Kähler manifolds as a special class of almost quaternionic Hermitian manifolds.

Definition 2.2.3. Let $(M,g,\mathscr{Q})$ be an almost quaternionic Hermitian manifold of dimension $4n > 4$. We say that it is a **quaternionic Kähler manifold** if the subbundle $\mathscr{Q}$ is preserved by the Levi-Civita connection ∇ of g, i.e. $\nabla\Gamma(\mathscr{Q}) \subset \Gamma(T^*M \otimes \mathscr{Q})$.

Since ∇ preserves $\mathscr{Q}$, by the holonomy principle, the holonomy group of g must be the subgroup of $\mathrm{SO}(4n)$ that preserves $\mathscr{Q}$. That is, the subgroup which at each tangent space T_xM preserves the linear subspace $\mathrm{span}_{\mathbb{R}}\{I_1,I_2,I_3\} \subset \mathrm{End}(T_xM)$. This subgroup has to be contained precisely in $\mathrm{Sp}(n)\mathrm{Sp}(1)$.

Conversely, let (M,g) be a Riemannian manifold with holonomy contained in the group $\mathrm{Sp}(n)\mathrm{Sp}(1)$. The tangent space T_xM admits three endomorphisms I_1, I_2 and I_3

satisfying the quaternionic relations, i.e. $I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -\mathrm{Id}$. These endomorphisms are preserved by $\mathrm{Sp}(n)$, but not by $\mathrm{Sp}(1)$, which permutes them. Nevertheless, the linear subspace $\mathrm{span}_{\mathbb{R}}\{I_1, I_2, I_3\} \subset \mathrm{End}(T_x M)$ is preserved by $\mathrm{Sp}(1)$ and then by $\mathrm{Sp}(n)\mathrm{Sp}(1)$. Declaring the basis $\{e_j, I_1 e_j, I_2 e_j, I_3 e_j\}$, $j = 1, \dots, n$, being orthonormal implies that the almost quaternionic structure $\mathcal{Q}$ is compatible with the metric g. By the holonomy principle (see Theorem 2.1.6) the subbundle $\mathcal{Q}$ is parallel in the sense that $\nabla\Gamma(\mathcal{Q}) \subset \Gamma(T^*M \otimes \mathcal{Q})$.

As we pointed out in Remark 2.1.22, the definition of quaternionic Kähler manifolds for dimension 4 should be adapted. The following definition includes a further requirement which is automatically satisfied when the dimension of M is greater than 4 (see Proposition 2.2.5).

Definition 2.2.4. Let $(M, g, \mathcal{Q})$ be an almost quaternionic Hermitian manifold of dimension 4. We say that it is a **quaternionic Kähler manifold** if the subbundle $\mathcal{Q}$ is preserved by the Levi-Civita connection ∇ of g, i.e. $\nabla\Gamma(\mathcal{Q}) \subset \Gamma(T^*M \otimes \mathcal{Q})$, and every $J \in \Gamma(\mathcal{Q})$ satisfies

$$g(R(JX,Y)Z,W) + g(R(X,JY)Z,W) + g(R(X,Y)JZ,W) + g(R(X,Y)Z,JW) = 0 \quad (4)$$

for every $X, Y, Z, W \in \Gamma(TM)$, where R is the curvature tensor.

Proposition 2.2.5. *Let $(M, g, \mathcal{Q})$ be a quaternionic Kähler manifold of dimension $4n > 4$. Then the equation (4) holds for every $J \in \Gamma(\mathcal{Q})$ and all $X, Y, Z, W \in \Gamma(TM)$.*

Proof. Let $\{I_1, I_2, I_3\}$ be a local frame of $\mathcal{Q}$. Since all the objects involved are tensors, it is enough to show the result for a basis. Following the proof of [Bes87, Theorem 14.39], we can write

$$[R(X,Y), I_\alpha] = \eta_\gamma(X,Y) I_\beta - \eta_\beta(X,Y) I_\gamma,$$

where (α, β, γ) is a cyclic permutation of (1,2,3), and η_α are locally defined 2-forms. By [Bes87, Lemma 14.40], for $4n > 4$, the 2-forms η_α can be written as

$$\eta_\alpha(X,Y) = \frac{2}{n+2} \mathrm{Ric}(I_\alpha X, Y),$$

where Ric is the Ricci curvature of g. Then we have

$$\begin{aligned} g(R(I_\alpha X,Y)Z,W) + g(R(X,I_\alpha Y)Z,W) &= \frac{2}{n+2}\mathrm{Ric}(I_\gamma Z,W) g(I_\beta X,Y) \\ &\quad - \frac{2}{n+2}\mathrm{Ric}(I_\beta Z,W) g(I_\gamma X,Y), \\ g(R(X,Y)I_\alpha Z,W) + g(R(X,Y)Z,I_\alpha W) &= \frac{2}{n+2}\mathrm{Ric}(I_\gamma X,Y) g(I_\beta Z,W) \\ &\quad - \frac{2}{n+2}\mathrm{Ric}(I_\beta X,Y) g(I_\gamma Z,W). \end{aligned}$$

Summing both terms, the result follows from the fact that quaternionic Kähler manifolds are Einstein (see Theorem 2.2.13), i.e. $\mathrm{Ric} = \lambda g$ for some $\lambda \in \mathbb{R}$. $\square$

Let us now briefly describe an equivalent way to define quaternionic Kähler manifolds of dimension 4.

Let $(M,g,\mathscr{Q})$ be a quaternionic Kähler manifold of dimension $4n$. If we regard the curvature tensor R of M as a symmetric endomorphism of $\Lambda^2 T^*M$, then

$$R|_{\mathscr{Q}} = \lambda \,\mathrm{Id}_{\mathscr{Q}},$$

where λ is a positive multiple of the scalar curvature of M (see e.g. [BG08]).

If (M,g) is an oriented Riemannian 4-manifold, the Hodge star operator $\star$ satisfies $\star^2 = \mathrm{Id}$ for 2-forms. Then we can decompose

$$\Lambda^2 T^*M = \Lambda^2_+ \oplus \Lambda^2_-,$$

where $\Lambda^2_\pm$ are the ± 1-eigenspaces of $\star$. These are the bundles of **self-dual** and **anti-self-dual** 2-forms, respectively. With respect to this decomposition of $\Lambda^2 T^*M$, the curvature tensor R has the following form

$$R = \begin{pmatrix} W_+ + \frac{\mathrm{scal}}{12}\mathrm{Id} & \mathrm{Ric}_0 \\ \mathrm{Ric}_0 & W_- + \frac{\mathrm{scal}}{12}\mathrm{Id} \end{pmatrix},$$

where $W_\pm$ are the self-dual and anti-self-dual parts of the Weyl tensor, respectively, $\mathrm{Ric}_0 := \mathrm{Ric} - \frac{\mathrm{scal}}{4} g$ is the trace-free Ricci tensor and scal is the scalar curvature. If the 4-manifold is Einstein (i.e. $\mathrm{Ric}_0 = 0$) and self-dual (i.e. $W_- = 0$) we have

$$R|_{\mathscr{Q}} = \frac{\mathrm{scal}}{12}\mathrm{Id}_{\mathscr{Q}},$$

where $\mathscr{Q} = \Lambda^2_-$. Therefore, we obtain the following characterization of quaternionic Kähler 4-manifolds.

Proposition 2.2.6. *Let (M,g) be an oriented Riemannian 4-manifold. Then it is quaternionic Kähler if and only if (M,g) is Einstein and self-dual.*

As an example of this equivalence, in [Sah20, Example 2.1.17] it is written in detail how to explicitly describe the quaternionic Kähler structure of the (Einstein and self-dual) complex hyperbolic plane $\mathbb{C}\mathrm{H}^2$.

Remark 2.2.7. In the literature, Proposition 2.2.6 often appears as the definition of 4-dimensional quaternionic Kähler manifolds. Further motivation for this definition comes from the following fact. Let M be a quaternionic Kähler manifold. We say that a submanifold $N \subset M$ is **quaternionic** if T_xN is an $\mathbb{H}$-submodule of T_xM for all $x \in N$. It was shown in [Mar90, Proposizione 9.1] that a 4-dimensional quaternionic submanifold of a quaternionic Kähler manifold is Einstein and self-dual with respect to the induced metric.

As we have already said, the fundamental 4-form Ω, given by (3), plays a similar role on quaternionic Hermitian geometry to the fundamental 2-form on Hermitian geometry. Thus quaternionic Kähler manifolds are analogues of Kähler manifolds in some sense.

Proposition 2.2.8. *Let $(M,g,\mathscr{Q})$ be a quaternionic Kähler manifold. Then the fundamental 4-form is parallel, i.e. $\nabla\Omega = 0$. In particular, Ω is closed.*

Proof. This proof follows [Thu20, Lemma 2.6]. It is enough to prove that $\nabla\Omega^{\otimes} = 0$ for $\Omega^{\otimes} := \sum_{k=1}^{3} \omega_k \otimes \omega_k$, since this implies $\nabla\Omega = 0$. Let $(M,g,\mathscr{Q})$ be a quaternionic Kähler manifold and consider an orthonormal frame $\{\omega_1,\omega_2,\omega_3\}$ of $\mathscr{Q} \subset \Lambda^2 T^*M$. For $\omega_i \otimes \omega_j \in \Gamma(\mathscr{Q}\otimes\mathscr{Q})$ and $X \in \Gamma(TM)$ we have

$$\begin{aligned}
\langle \nabla_X \Omega^{\otimes}, \omega_i \otimes \omega_j \rangle &= \sum_{k=1}^{3} \langle \nabla_X \omega_k \otimes \omega_k + \omega_k \otimes \nabla_X \omega_k, \omega_i \otimes \omega_j \rangle \\
&= \sum_{k=1}^{3} \left(\langle \nabla_X \omega_k, \omega_i \rangle \langle \omega_k, \omega_j \rangle + \langle \omega_k, \omega_i \rangle \langle \nabla_X \omega_k, \omega_j \rangle \right) \\
&= \langle \nabla_X \omega_j, \omega_i \rangle + \langle \nabla_X \omega_i, \omega_j \rangle = X \langle \omega_j, \omega_i \rangle = 0,
\end{aligned}$$

where in the last equation we have used that the Levi-Civita connection is compatible with the bundle metric (2). This means that $\nabla_X\Omega^{\otimes}$ is orthogonal to $\omega_i \otimes \omega_j$ for every $i,j \in \{1,2,3\}$, so $\nabla_X\Omega^{\otimes} \in \Gamma((\mathscr{Q}\otimes\mathscr{Q})^{\perp})$ for every $X \in \Gamma(TM)$. We conclude that $\nabla\Omega^{\otimes} = 0$ since $\nabla_X\Omega^{\otimes} \in \Gamma(\mathscr{Q}\otimes\mathscr{Q})$ for $\mathscr{Q}$ being preserved by ∇.

We can give an alternative proof without using the tensor $\Omega^{\otimes}$. It was noticed in [Ish74] that given a local orthonormal frame $\{\omega_1,\omega_2,\omega_3\}$ of $\mathscr{Q} \subset \Lambda^2 T^*M$, we have

$$\nabla_X \omega_k = \sum_{\ell=1}^{3} \theta_{k\ell}(X)\omega_\ell,$$

where $\theta_{k\ell}$ are 1-forms satisfying $\theta_{k\ell} = -\theta_{\ell k}$ for all $k,\ell = 1,2,3$. This implies that $\nabla_X\Omega = 2\sum_{k=1}^{3} \nabla_X \omega_k \wedge \omega_k = 0$ for all $X \in \Gamma(TM)$, hence Ω is parallel. ◻

This gives us the following immediate corollary.

Corollary 2.2.9. *Let $(M,g,\mathscr{Q})$ be a compact quaternionic Kähler manifold of dimension $4n$. Then $b_{4k}(M) > 0$ for $k = 1,\dots,n$.*

Note that in dimension 4 we always have $\nabla\Omega = 0$ since Ω is a multiple of the Riemannian volume form, which is always parallel. For dimension $4n > 4$, $\nabla\Omega = 0$ characterizes the quaternionic Kähler manifolds among the almost quaternionic Hermitian ones.

Proposition 2.2.10. *Let $(M,g,\mathscr{Q})$ be an almost quaternionic Hermitian manifold of dimension $4n > 4$. If the fundamental 4-form Ω is parallel, then $(M,g,\mathscr{Q})$ is quaternionic Kähler.*

Proof. Given a 4-form Ω on a manifold M, we can define the endomorphism field $B_\Omega \in \Gamma(\mathrm{End}(\Lambda^2 T^*M))$ by

$$B_\Omega \sigma := \star(\star\Omega \wedge \sigma),$$

where $\star$ is the Hodge star operator and $\sigma \in \Gamma(\Lambda^2 T^*M)$. The Levi-Civita connection commutes with the Hodge star operator. Thus, if Ω is parallel, so is B_Ω.

Now let $x \in M$ be a point. We have the decomposition

$$\Lambda^2 T_x^* M \cong \mathfrak{so}(4n) = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \oplus \mathfrak{k},$$

where $\mathfrak{k}$ is the orthogonal complement of $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ in $\mathfrak{so}(4n)$. The endomorphism $B_\Omega : \Lambda^2 T_x^* M \longrightarrow \Lambda^2 T_x^* M$ has precisely $\mathfrak{sp}(n)$, $\mathfrak{sp}(1)$ and $\mathfrak{k}$ as eigenspaces with distinct eigenvalues [ACD03]. Suppose that $B_\Omega \omega = \lambda \omega$ for some $\omega \in \mathcal{Q}_x \cong \mathfrak{sp}(1)$ and $\lambda \in \mathbb{R}$. Then $B_\Omega(\nabla_X \omega) = \nabla_X(B_\Omega \omega) = \lambda \nabla_X \omega$. This implies that $\nabla_X \omega \in \mathcal{Q}_x$. Hence the almost quaternionic Hermitian manifold is quaternionic Kähler. □

In the Hermitian case, if the 2-form ω is closed, then the manifold is Kähler. For almost quaternionic Hermitian manifolds the analogue is true for dimension $4n > 8$ and for dimension 8 we require an additional property.

Theorem 2.2.11 ([Swa91, Theorem 2.2])**.** *Let $(M, g, \mathcal{Q})$ be an almost quaternionic Hermitian manifold. If $4n > 8$, then $\mathrm{d}\Omega = 0$ implies $\nabla\Omega = 0$ and M is quaternionic Kähler. In dimension 8, an almost quaternionic Hermitian manifold is quaternionic Kähler if and only if $\mathrm{d}\Omega = 0$ and the algebraic ideal generated by $\mathcal{Q} \subset \Lambda^2 T^*M$ is a differential ideal (i.e. closed under exterior differentiation).*

Remark 2.2.12. In [Sal01] Salamon gave the first example of a compact quaternionic Hermitian manifold of dimension 8 with $\mathrm{d}\Omega = 0$ but whose algebraic ideal generated by $\mathcal{Q} \subset \Lambda^2 T^*M$ is not differential. This confirms that indeed the closedness of Ω is not sufficient for $n = 2$. More examples were constructed in [CM15]. However, none of these examples is Einstein.

2.2.1 Curvature of quaternionic Kähler manifolds

Manifolds with holonomy group appearing in Berger list (Table 2) have strongly restricted curvature tensors, except for $\mathrm{SO}(n)$ (the generic case) and $\mathrm{U}(n)$ (the Kähler case). It turns out that for manifolds with holonomy contained in $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, G_2 or $\mathrm{Spin}(7)$, the Ricci tensor vanishes identically, i.e. these manifolds are Ricci-flat. For manifolds with holonomy contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$ we have the following fundamental result.

Theorem 2.2.13 ([Ber66])**.** *Let $(M, g, \mathcal{Q})$ be a quaternionic Kähler manifold. Then (M, g) is Einstein, i.e.* $\mathrm{Ric} = \lambda g$ *for* $\lambda \in \mathbb{R}$.

We can ask when the case $\lambda = 0$ occurs. The following theorem answers this question.

Theorem 2.2.14 ([Ber66])**.** *Let $(M, g, \mathcal{Q})$ be a quaternionic Kähler manifold. Then (M, g) is Ricci-flat if and only if it is locally hyperkähler, i.e.* $\mathrm{Hol}^0(g) \subseteq \mathrm{Sp}(n)$*. Moreover, if (M, g) is not Ricci-flat, then it is irreducible.*

Remark 2.2.15. Note that Theorem 2.2.14 implies, in particular, that the product of two quaternionic Kähler manifolds is not quaternionic Kähler.

In this work we are excluding the possibility of $\mathrm{Hol}^0(g) \subseteq \mathrm{Sp}(n)$. Therefore, we will only consider quaternionic Kähler manifolds which are Einstein with $\lambda \neq 0$, which implies that the scalar curvature is non-zero. This fact naturally divides the theory into the quaternionic Kähler manifolds with positive and negative scalar curvature. We will study them in more detail in the following sections.

For the curvature tensor of a quaternionic Kähler manifold we have the following decomposition due to Alekseevsky.

Theorem 2.2.16 ([Ale68b])**.** *Let $(M, g, \mathcal{Q})$ be a quaternionic Kähler manifold of dimension $4n$. Then its Riemann curvature tensor R is of the form*

$$R = \nu R_{\mathbb{HP}^n} + R_{\mathrm{HK}},$$

where $\nu := \frac{\mathrm{scal}}{4n(n+2)}$ is the reduced scalar curvature of M, $R_{\mathbb{HP}^n}$ is formally the curvature tensor of the quaternionic projective space $\mathbb{HP}^n$ and R_{HK} is an algebraic curvature tensor of hyperkähler type, this means that it is trace-free and commutes with every section of $\mathcal{Q}$, i.e. for every $X, Y \in \Gamma(TM)$ and every $J \in \Gamma(\mathcal{Q})$ we have

$$[R_{\mathrm{HK}}(X,Y), J] = 0.$$

Remark 2.2.17. The algebraic curvature tensor R_{HK} is sometimes called the quaternionic Weyl curvature of M.

Note that, as a corollary of Theorem 2.2.16, the Ricci tensor of any quaternionic Kähler manifold is completely determined by the Ricci tensor of $\mathbb{HP}^n$, since R_{HK} is trace-free. The projective quaternionic space is Einstein by Proposition 2.1.17 (since it is an irreducible symmetric space), and this gives an easy proof that any quaternionic Kähler manifold is Einstein.

2.2.2 Twistor space

To finish this section, we will see that to each quaternionic Kähler manifold we can associate a particular type of complex manifold, called the twistor space. This allows us to use the powerful tools of complex algebraic geometry to study the properties of quaternionic Kähler manifolds. The original idea is due to Penrose, who outlined how the metric properties of an Einstein self-dual Lorentzian 4-manifold can be encoded in the complex geometry of a bundle of this space (see [Pen76]). This idea was then developed and formulated for Riemannian manifolds.

The starting point of the twistor construction is an oriented Riemannian 4-manifold (M,g). We consider a bundle $\pi : Z \longrightarrow M$ where $Z := \mathbb{S}(\Lambda^2_-)$ is the unit sphere bundle over the rank three bundle of anti-self-dual 2-forms. Then Z is a manifold of real dimension 6 whose fibers are 2-spheres diffeomorphic to $\mathbb{C}\mathrm{P}^1$. Using the metric to identify 2-forms and skew-adjoint endomorphisms of TM, an anti-self-dual 2-form at $x \in M$ becomes an endomorphism J_x which defines an almost complex structure on T_xM, i.e. $J_x^2 = -\mathrm{Id}$. Using the Levi-Civita connection we may split the tangent bundle of Z as

$$TZ \cong \pi^*(TM) \oplus T^{\mathrm{V}}Z.$$

At a point $z \in Z$, we have $T_z^{\mathrm{V}}Z = T_z(Z_{\pi(z)}) = T_z\mathbb{C}\mathrm{P}^1$ since the fiber of Z is $\mathbb{C}\mathrm{P}^1$. Then we define an almost complex structure on $T_zZ = T_{\pi(z)}M \oplus T_z\mathbb{C}\mathrm{P}^1$ by

$$J_z := J_{\pi(z)} \oplus J_{\mathbb{C}\mathrm{P}^1},$$

where $J_{\pi(z)}$ is the almost complex structure on $T_{\pi(z)}M$ and $J_{\mathbb{C}\mathrm{P}^1}$ is the natural complex structure on the fiber $\mathbb{C}\mathrm{P}^1$. Therefore, the twistor space Z is always an almost complex manifold. It is natural to ask whether this almost complex structure is integrable. The following theorem answers this question.

Theorem 2.2.18 ([AHS78, Theorem 4.1])**.** *Let (M,g) be a 4-dimensional oriented Riemannian manifold and let Z be its twistor space. Then Z is a complex manifold if and only if (M,g) is self-dual.*

In particular, thanks to the equivalent characterization of 4-dimensional quaternionic Kähler manifolds given by Proposition 2.2.6, the twistor space of a 4-dimensional quaternionic Kähler manifold is a complex manifold.

It is important to note that this construction can be inverted, so we can construct self-dual manifolds starting with a twistor space (see [Pen76] and [Bes87] for details).

We now generalize this construction to the higher-dimensional case.

Let $(M,g,\mathcal{Q})$ be a quaternionic Kähler manifold of dimension $4n$. At any point $x \in M$, there exist many almost complex structures on T_xM given by any linear combination

$$I_a := a_1I_1 + a_2I_2 + a_3I_3,$$

where $(a_1,a_2,a_3) \in \mathbb{S}^2$ and $I_1,I_2,I_3 \in \mathcal{Q}_x$. Similarly as before, we can consider a bundle $\pi : Z \longrightarrow M$ where $Z := \mathbb{S}(\mathcal{Q})$ is the unit sphere bundle over $\mathcal{Q}$. Then Z is a manifold of real dimension $4n+2$ whose fibers are 2-spheres. We can always construct (pointwise) an almost complex structure on Z by considering the sum $I_a \oplus J_{\mathbb{C}\mathrm{P}^1}$ of an almost complex structure I_a defined on T_xM and the natural complex structure $J_{\mathbb{C}\mathrm{P}^1}$ on $\mathbb{C}\mathrm{P}^1$.

Definition 2.2.19. Let $(M,g,\mathcal{Q})$ be a quaternionic Kähler manifold. Then we say that the almost complex manifold $Z = \mathbb{S}(\mathcal{Q})$ is its **twistor space**.

As before, the fundamental property of Z is that the almost complex structure defined above is integrable.

Theorem 2.2.20 ([Sal82, Theorem 4.1])**.** *Let $(M, g, \mathscr{Q})$ be a quaternionic Kähler manifold of dimension $4n$. Then its twistor space Z is a complex manifold.*

An arbitrary complex manifold cannot be realized as a twistor space, since Z carries more structure.

Theorem 2.2.21 ([Sal82])**.** *Let $(M, g, \mathscr{Q})$ be a quaternionic Kähler manifold of dimension $4n$ and Z be its twistor space. Then:*

(a) *Z is a complex manifold of complex dimension $2n+1$ such that for each point $x \in M$ the fiber $\pi^{-1}(x)$ of $\pi : Z \longrightarrow M$ is a complex curve $\mathbb{C}\mathrm{P}^1$ in Z with normal bundle $2n\mathscr{O}(1) = \mathscr{O}(1) \otimes \mathbb{C}^{2n}$.*

(b) *Z carries a **real structure**, i.e. an anti-holomorphic involution $\sigma : Z \longrightarrow Z$ with $\pi \circ \sigma = \pi$, acting as $\sigma : J \longmapsto -J$ under the identification of points $z \in Z$ with almost complex structures J on $T_{\pi(z)}M$.*

(c) *If g has non-zero scalar curvature, then Z carries a **complex contact structure**, i.e. a complex line bundle L together with a holomorphic 1-form θ taking values in L such that $\theta \wedge (\mathrm{d}\theta)^n$ is nowhere zero.*

As in the 4-dimensional case, the twistor construction can always be inverted to recover uniquely (up to homothety) the quaternionic Kähler structure of M, although this description is highly non-explicit (see [LeB89]).

We will see in the following section that twistor spaces are a fundamental tool for studying the properties of positive quaternionic Kähler manifolds.

2.3 Positive quaternionic Kähler manifolds

We have seen in Theorem 2.2.13 that quaternionic Kähler manifolds are Einstein with non-zero scalar curvature. In this section we focus on the case when the scalar curvature is positive. For the definition of positive quaternionic Kähler manifolds we will also require the property of being (geodesically) complete. For a more complete and detailed exposition, see [Sal99, BG08, Ama09] and references therein.

Definition 2.3.1. Let $(M, g, \mathscr{Q})$ be a quaternionic Kähler manifold. We say that it is **positive** if it is complete and its scalar curvature is positive.

Myers theorem states that if (M, g) is a complete Einstein manifold with positive scalar curvature, then M is compact with finite fundamental group. Therefore we obtain the following as an immediate application.

Proposition 2.3.2. *Let $(M, g, \mathscr{Q})$ be a positive quaternionic Kähler manifold. Then M is compact with finite fundamental group.*

By Corollary 2.2.9, we know that $b_{4k}(M) > 0$ for positive quaternionic Kähler manifolds. We can say more about their topology. For that we need to use the properties of the twistor space. The next theorem is fundamental in the theory of positive quaternionic Kähler manifolds.

Theorem 2.3.3 ([Sal82, Theorem 6.1]). *Let $(M, g, \mathcal{Q})$ be a positive quaternionic Kähler manifold. Then its twistor space Z is a compact complex manifold that admits a Kähler-Einstein metric of positive scalar curvature.*

We can say even more since $\pi : (Z, \hat{g}) \longrightarrow (M, g)$ is actually a Riemannian submersion with totally geodesic fibers, where $\hat{g}$ denotes the Kähler-Einstein metric given by Theorem 2.3.3.

Remark 2.3.4. The first Chern class of the twistor space Z of a positive quaternionic Kähler manifold is positive (see [Sal82]). In other words, Z is a Fano manifold. Moreover, since Z is also a complex contact manifold (see Theorem 2.2.21), the first Chern class $c_1(Z)$ is divisible by $n+1$ [Kob59], where $2n+1$ is the complex dimension of Z. Such Z are very special objects in complex algebraic geometry, and a lot is known about them.

Salamon also proved that Z is simply connected and has only (p, p)-cohomology. Using these results he obtained further restrictions on the topology of a positive quaternionic Kähler manifold.

Theorem 2.3.5 ([Sal82, Theorem 6.6]). *Let $(M, g, \mathcal{Q})$ be a positive quaternionic Kähler manifold. Then M is simply connected and has odd Betti numbers equal to zero.*

Much more can be said about the topology of positive quaternionic Kähler manifolds. We refer the interested reader to the references cited so far.

Now it is time to give examples of positive quaternionic Kähler manifolds. The archetypal example is the quaternionic projective space $\mathbb{H}\mathrm{P}^n = (\mathbb{H}^{n+1} \setminus \{0\})/\mathbb{H}^*$, which can be viewed as a symmetric space

$$\mathbb{H}\mathrm{P}^n \cong \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n) \times \mathrm{Sp}(1)}.$$

When it is equipped with its symmetric metric, $\mathbb{H}\mathrm{P}^n$ is a quaternionic Kähler manifold since, by Proposition 2.1.16, the holonomy group of a symmetric space is given by its isotropy group. Note that the isotropy group is really $\mathrm{Sp}(n)\mathrm{Sp}(1)$, not $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. Indeed, since $\mathrm{Sp}(n+1)$ only acts nearly effectively on $\mathbb{H}\mathrm{P}^n$, one could more precisely write $\mathbb{H}\mathrm{P}^n \cong (\mathrm{Sp}(n+1)/\mathbb{Z}_2)/\mathrm{Sp}(n)\mathrm{Sp}(1)$.

The quaternionic projective space $\mathbb{H}\mathrm{P}^n$ is compact, is Einstein as every irreducible symmetric space, and using the quaternionic Hopf fibration $\mathrm{Sp}(1) \hookrightarrow \mathbb{S}^{4n+3} \to \mathbb{H}\mathrm{P}^n$ and the theory of Riemannian submersions, one can show that the sectional curvature is positive, and therefore, the scalar curvature too. For $n = 1$, we have that $\mathbb{H}\mathrm{P}^1 \cong \mathbb{S}^4$ is Einstein and self-dual.

We have already said that quaternionic Kähler manifolds are in general not Kähler. Most of the positive ones are not even almost complex manifolds. In the case of quaternionic projective spaces this result is due to [Mas62]. For an arbitrary positive quaternionic Kähler manifolds we have the following results.

Theorem 2.3.6 ([AMP98, Theorem 3.8])**.** *Let $(M, g, \mathcal{Q})$ be a positive quaternionic Kähler manifold. Then it does not admit a compatible almost complex structure.*

An almost complex structure J on $(M, g, \mathcal{Q})$ is **compatible** with the quaternionic structure $\mathcal{Q}$ if $J \in \Gamma(\mathcal{Q})$. With this result in mind we may ask if at least there exist positive quaternionic Kähler manifolds admitting a non-compatible almost complex structure. The answer is the following theorem (see also Remark 2.3.9).

Theorem 2.3.7 ([GMS11, Theorem 1.1])**.** *Let $(M, g, \mathcal{Q})$ be a positive quaternionic Kähler manifold which is not the complex Grassmannian* $\mathrm{Gr}_2(\mathbb{C}^{n+2})$*. Then it does not admit an almost complex structure.*

There are further examples of quaternionic Kähler symmetric spaces. Its holonomy group coincides with the isotropy group, and for a quaternionic Kähler manifold this has to be isomorphic to $K\mathrm{Sp}(1)$ for some compact Lie group $K \subseteq \mathrm{Sp}(n)$. From the Cartan list of irreducible symmetric spaces it is not difficult to detect these holonomy groups. This was made by Wolf, who classified the quaternionic Kähler symmetric spaces (both positive and negative scalar curvature). So these spaces are usually known as **Wolf spaces**. Moreover, Wolf not only identified them, but he observed that this classification matched the classification of simply connected homogeneous complex contact manifolds obtained by Boothby [Boo62], who proved that there is a one-to-one correspondence between them and compact simple Lie groups (excluding $\mathrm{SU}(2)$).

Note that a positive quaternionic Kähler symmetric space is always of compact type.

Theorem 2.3.8 ([Wol65, Theorem 6.1])**.** *Let $(M, g, \mathcal{Q})$ be a positive quaternionic Kähler symmetric space. Then $M \cong G/K$ is one of the following table:*

G	K	$\dim_{\mathbb{R}} M$
$\mathrm{Sp}(n+1)$	$\mathrm{Sp}(n)\times\mathrm{Sp}(1)$	$4n$
$\mathrm{SU}(n+2)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))$	$4n$
$\mathrm{SO}(n+4)$	$\mathrm{SO}(n)\times\mathrm{SO}(4)$	$4n$
G_2	$\mathrm{SO}(4)$	8
F_4	$\mathrm{Sp}(3)\mathrm{Sp}(1)$	28
E_6	$\mathrm{SU}(6)\mathrm{Sp}(1)$	40
E_7	$\mathrm{Spin}(12)\mathrm{Sp}(1)$	64
E_8	$\mathrm{E}_7\mathrm{Sp}(1)$	112

Table 3: Wolf spaces of compact type.

Moreover, there is a one-to-one correspondence between positive quaternionic Kähler symmetric spaces and compact simply connected homogeneous complex contact manifolds.

The three families of Wolf spaces of compact type in Table 3 can be realized as the following manifolds:

$$\begin{aligned}\mathbb{H}\mathrm{P}^n &\cong \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\times\mathrm{Sp}(1)},\\ \mathrm{Gr}_2(\mathbb{C}^{n+2}) &\cong \frac{\mathrm{SU}(n+2)}{\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))},\\ \widetilde{\mathrm{Gr}}_4(\mathbb{R}^{n+4}) &\cong \frac{\mathrm{SO}(n+4)}{\mathrm{SO}(n)\times\mathrm{SO}(4)}.\end{aligned}$$

Note that there are coincidences $\mathbb{H}\mathrm{P}^1 \cong \mathbb{S}^4 \cong \widetilde{\mathrm{Gr}}_4(\mathbb{R}^5)$ and $\mathrm{Gr}_2(\mathbb{C}^3) \cong \mathbb{C}\mathrm{P}^2$ for $n = 1$ and $\mathrm{Gr}_2(\mathbb{C}^4) \cong \widetilde{\mathrm{Gr}}_4(\mathbb{R}^6)$ for $n = 2$. Up to this coincidences, there is exactly one Wolf space of compact type for each compact simple Lie group except $\mathrm{SU}(2)$.

Remark 2.3.9. It is well-known that the complex Grassmannian can be embedded as a complex submanifold into a complex projective space of appropriate dimension (via the Plücker embedding). Therefore it inherits the Fubini-Study metric, so it is Kähler. This does not contradict Theorem 2.3.6 since this complex structure is not compatible with the quaternionic Kähler structure. That is, $(\mathrm{Gr}_2(\mathbb{C}^{n+2}), g, \mathcal{Q})$ is quaternionic Kähler and $(\mathrm{Gr}_2(\mathbb{C}^{n+2}), g, J)$ is Kähler but $J \notin \Gamma(\mathcal{Q})$. In fact, it is also known that $\mathrm{Gr}_2(\mathbb{C}^{n+2})$ is the only positive quaternionic Kähler manifold admitting an almost complex structure (see Theorem 2.3.7).

One can try to look for more examples of positive quaternionic Kähler manifolds in the broader class of homogeneous spaces, but it turns out that Wolf spaces are the only homogeneous positive quaternionic Kähler manifolds.

Theorem 2.3.10 ([AC97b, Theorem 1.1])**.** *Let $(M, g, \mathcal{Q})$ be a quaternionic Kähler manifold which is homogeneous under an unimodular group. Then (M, g) is a Wolf space, i.e. is a symmetric space.*

Since the isometry group of a compact manifold is compact, and therefore unimodular, we obtain the following well-known result.

Theorem 2.3.11 ([Ale68a, Theorem 1])**.** *Let $(M, g, \mathcal{Q})$ be a compact homogeneous quaternionic Kähler manifold. Then (M, g) is a Wolf space.*

In particular, any homogeneous positive quaternionic Kähler manifold is a Wolf space. Wolf spaces are the only known examples of positive quaternionic Kähler manifolds, and it is conjectured that they are the only ones. LeBrun and Salamon proved the following rigidity result.

Theorem 2.3.12 ([LS94, Theorem 0.1])**.** *There are only finitely many positive quaternionic Kähler manifolds up to homothety of dimension* $4n$ *for any* $n \in \mathbb{N}$.

In the same paper they also proved constraints on the second homotopy group as well as the Betti numbers. These results (together with Theorem 2.3.14 and Theorem 2.3.15 below) can be viewed as strong supporting evidence for their famous conjecture, stated in the same paper.

Conjecture 2.3.13 (LeBrun-Salamon)**.** *Let* $(M, g, \mathcal{Q})$ *be a positive quaternionic Kähler manifold. Then* (M, g) *is homothetic to a Wolf space.*

The theory of twistor spaces for positive quaternionic Kähler manifolds has played a crucial role to show that the conjecture is true in low dimensions. Recall that in the positive case, twistor spaces are Kähler-Einstein manifolds of positive scalar curvature (i.e. Fano manifolds) admitting a complex contact structure. These spaces are very constrained and, since their complex geometric properties determine the quaternionic Kähler metric completely, positive quaternionic Kähler manifolds are also very constrained. Indeed, by a result of LeBrun [LeB95], every compact complex contact manifold admitting a Kähler-Einstein metric is the twistor space of a quaternionic Kähler manifold.

What we know so far about the conjecture is the following.

Theorem 2.3.14 ([Hit81, Theorem 6.1],[FK82])**.** *Let* $(M, g, \mathcal{Q})$ *be a positive quaternionic Kähler manifold of dimension* 4*, i.e. an Einstein and self-dual manifold of positive scalar curvature. Then* (M, g) *is homothetic to* $\mathbb{H}\mathrm{P}^1$ *or* $\mathbb{C}\mathrm{P}^2$*, i.e. is a Wolf space.*

Theorem 2.3.15 ([PS91, Theorem 1.1])**.** *Let* $(M, g, \mathcal{Q})$ *be a positive quaternionic Kähler manifold of dimension* 8*. Then* (M, g) *is homothetic to* $\mathbb{H}\mathrm{P}^2$*,* $\mathrm{Gr}_2(\mathbb{C}^4)$ *or* $\mathrm{G}_2/\mathrm{SO}(4)$*, i.e. is a Wolf space.*

In [HH02] it was claimed that the conjecture is also true in dimension 12, but some errors were found in their proof (see [Ama09] for details). Nevertheless, the conjecture has been recently shown to be true in dimension 12 and 16.

Theorem 2.3.16 ([BWW22, Theorem 1.1])**.** *Let* $(M, g, \mathcal{Q})$ *be a positive quaternionic Kähler manifold. Then:*

- (M, g) *is homothetic to* $\mathbb{H}\mathrm{P}^3$*,* $\mathrm{Gr}_2(\mathbb{C}^5)$ *or* $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$ *if* $\dim_{\mathbb{R}} M = 12$.
- (M, g) *is homothetic to* $\mathbb{H}\mathrm{P}^4$*,* $\mathrm{Gr}_2(\mathbb{C}^6)$ *or* $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^8)$ *if* $\dim_{\mathbb{R}} M = 16$.

There exist more partial results that prove the LeBrun-Salamon conjecture under additional hypothesis (see e.g. [ORSW21, PW22]) and it is widely believed that it is true, but proving it remains one of the major open problems in quaternionic Kähler geometry.

2.4 Negative quaternionic Kähler manifolds

We finish this chapter studying quaternionic Kähler manifolds of negative scalar curvature. We will see that the theory in this case is much more different and flexible, in terms of existence of examples, that the positive case.

Definition 2.4.1. Let $(M, g, \mathscr{Q})$ be a quaternionic Kähler manifold. We say that it is **negative** if it is complete and its scalar curvature is negative.

First of all, compactness is not guaranteed, as the Myers theorem does not apply for negatively curved Einstein manifolds. About the topology, to the author's knowledge, not too much can be said specific to negative scalar curvature, in contrast with positive quaternionic Kähler manifolds. In the compact case, apart from Corollary 2.2.9, there is the following result (compare with Theorem 2.3.5).

Theorem 2.4.2 ([SW02, Theorem 6.6])**.** *Let $(M, g, \mathscr{Q})$ be a compact negative quaternionic Kähler manifold of dimension $4n$. Then $b_{2k+1}(M) = 0$ for $2k+1 < n$.*

In the negative case, the twistor space is not so useful as in the positive case to determine the geometry of the underlying quaternionic Kähler manifold, since Z does not admit a Kähler-Einstein metric, but a pseudo-Kähler-Einstein metric (see [Bes87]). However, in this case the twistor space can be used to prove the "opposite result" of Theorem 2.3.12 (see Theorem 2.4.8 below). But recall that, even if the quaternionic Kähler metric is determined by the complex geometry of the twistor space Z, typically it is not possible to recover the metric explicitly from Z.

The archetypal example of a negative quaternionic Kähler manifold is the quaternionic hyperbolic space $\mathbb{H}\mathrm{H}^n$, which is the dual symmetric space of $\mathbb{H}\mathrm{P}^n$, then

$$\mathbb{H}\mathrm{H}^n \cong \frac{\mathrm{Sp}(n,1)}{\mathrm{Sp}(n)\times \mathrm{Sp}(1)}.$$

As for $\mathbb{H}\mathrm{H}^n$, we can consider the dual symmetric spaces of all Wolf spaces of compact type, which are always non-compact and with negative scalar curvature by the general theory of symmetric spaces. Then we obtain the classification of the Wolf spaces of non-compact type.

Theorem 2.4.3 ([Wol65, Theorem 6.7])**.** *Let $(M, g, \mathscr{Q})$ be a negative quaternionic Kähler symmetric space. Then $M \cong G/K$ is one of the following table:*

G	K	$\dim_{\mathbb{R}} M$
$\mathrm{Sp}(n,1)$	$\mathrm{Sp}(n)\times\mathrm{Sp}(1)$	$4n$
$\mathrm{SU}(n,2)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))$	$4n$
$\mathrm{SO}(n,4)$	$\mathrm{SO}(n)\times\mathrm{SO}(4)$	$4n$

G_2^2	$\mathrm{SO}(4)$	8
F_4^4	$\mathrm{Sp}(3)\mathrm{Sp}(1)$	28
E_6^2	$\mathrm{SU}(6)\mathrm{Sp}(1)$	40
E_7^{-5}	$\mathrm{Spin}(12)\mathrm{Sp}(1)$	64
E_8^{-24}	$\mathrm{E}_7\mathrm{Sp}(1)$	112

Table 4: Wolf spaces of non-compact type.

Moreover, there is a one-to-one correspondence between negative quaternionic Kähler symmetric spaces and non-compact duals of the compact simply connected homogeneous complex contact manifolds.

Remark 2.4.4. The superindices for the exceptional Lie groups denote the signature of the corresponding Killing form B on the Lie algebra, where the signature is defined here as the number of positive values minus the number of negative values when B is expressed in diagonal form.

By a result of Borel [Bor63], every Riemannian symmetric space of non-compact type admits smooth quotients by discrete cocompact groups of isometries (i.e. with compact quotients). Therefore, applying this result to the Wolf spaces of non-compact type of Table 4, we obtain compact locally symmetric examples of negative quaternionic Kähler manifolds. Note that positive quaternionic Kähler manifolds do not have any smooth quotients since they are simply connected by Theorem 2.3.5.

We now ask whether there exist examples of non-locally symmetric negative quaternionic Kähler manifolds. It turns out that this case is not as restrictive as the positive case. The first non-locally symmetric examples were found by Alekseevsky in [Ale75] in the classification of quaternionic Kähler manifolds homogeneous under a simply transitive completely solvable group of isometries. Such manifolds are called **Alekseevsky spaces**. It was pointed out by the physicists de Wit and Van Proeyen in [dWVP92] that his classification was already incomplete. This was fixed by Cortés in [Cor96a], who proved it rigorously via Lie-theoretical methods.

Theorem 2.4.5 ([Cor96a, Theorem 2.28])**.** *Let $(M,g,\mathcal{Q})$ be a negative quaternionic Kähler manifold homogeneous under a simply transitive completely solvable group of isometries, i.e. an Alekseevsky space. If (M,g) is symmetric, then is one of Table 4. If (M,g) is non-symmetric, then belongs to one of the following discrete infinite series:*

$$\mathscr{T}(p) \text{ for } p \geq 1,$$
$$\mathscr{W}(p,q) \text{ for } 1 \leq p \leq q,$$
$$\mathscr{V}(\ell,k) \text{ for } k \not\equiv 0 \,(\mathrm{mod}\, 4) \text{ and } (\ell,k) \notin \{(1,1),(1,2)\},$$
$$\mathscr{V}(p,q;k) \text{ for } k \equiv 0 \,(\mathrm{mod}\, 4) \text{ and } (p+q,k) \notin \{(1,4),(1,8)\}.$$

Alekseevsky conjectured in [Ale75] that Alekseevsky spaces are the only possible homogeneous negative quaternionic Kähler manifolds. This was shown to be true recently by Böhm and Lafuente. In fact, they proved a more general result known as the

Alekseevsky conjecture [BL23, Theorem A], which states that a connected homogeneous Einstein manifold of dimension n with negative scalar curvature is diffeomorphic to $\mathbb{R}^n$.

Let us explain succinctly how this implies the result. Let $(M, g, \mathcal{Q})$ be a homogeneous negative quaternionic Kähler manifold. In virtue of the resolution of the Alekseevsky conjecture, (M, g) admits a simply transitive solvable Lie group of isometries. Then, by the result of Lauret [Lau10, Theorem 3.1], it is a standard Einstein solvmanifold in the sense of Heber [Heb98]. Finally, by [Heb98, Theorem B], such a manifold admits a simply transitive completely solvable group of isometries. Then we can conclude:

Theorem 2.4.6 ([BL23, Corollary C])**.** *Let $(M, g, \mathcal{Q})$ be a homogeneous negative quaternionic Kähler manifold. Then it is an Alekseevsky space.*

In particular, now we know that a homogeneous quaternionic Kähler manifold is a Wolf space (if it is positive) or an Alekseevsky space (if it is negative).

One may ask if the non-symmetric Alekseevsky spaces admit compact quotients. But in this direction we have the following result.

Theorem 2.4.7 ([AC99, Corollary 1.28])**.** *Let $(M, g, \mathcal{Q})$ be an Alekseevsky space. Then it admits quotients of finite volume if and only if it is symmetric.*

The existence of a compact (or even of finite volume) non-locally symmetric quaternionic Kähler manifold (of positive or negative scalar curvature) is still open. In the negative case, LeBrun proved that there is an infinite-dimensional moduli space of complete quaternionic Kähler metrics on $\mathbb{R}^{4n}$ for every $n \in \mathbb{N}$.

Theorem 2.4.8 ([LeB91])**.** *There exists an infinite-dimensional family of pairwise distinct deformations of the standard quaternionic Kähler metric on $\mathbb{H}\mathrm{H}^n$, each member of which is a (complete) negative quaternionic Kähler manifold.*

This result says that negative quaternionic Kähler manifolds occur in abundance and suggests that it may be possible to construct many more examples, including examples which are not homogeneous.

Meanwhile, physicists discovered a way to produce new examples of quaternionic Kähler manifolds known as the supergravity c-map. Briefly speaking, this construction takes as input a projective special Kähler manifold (see Section 4.1) and gives us a quaternionic Kähler metric of negative scalar curvature [FS90]. Moreover, these metrics admit a deformation giving rise to a one-parameter family of quaternionic Kähler manifolds [RSV06]. Supergravity c-map spaces will be the main focus of this work although we do not explain them in detail now (see Section 4.3). For the moment, it is important to say that the mathematics inspired by these physical constructions have led to many further examples of (complete) negative quaternionic Kähler manifolds that are not (even locally) homogeneous. Particular examples of such manifolds were found in [CHM12, CDJL21, CST22]. One of the main goals of this thesis is to prove that any deformed supergravity c-map space is not locally homogeneous (see Theorem 5.2.6).

Using the properties of supergravity c-map spaces, it has been shown in [CRT21] that there exist non-locally homogeneous (complete) negative quaternionic Kähler manifolds with two ends, one of finite volume and the other one of infinite volume. These examples has been found only in dimension 4 and 8. Nevertheless, using an alternative method based on the quaternionic Kähler metrics constructed on bundles over hyperkähler manifolds by [Fow23], Cortés has proved [Cor23] that non-locally homogeneous (complete) negative quaternionic Kähler manifolds with two ends, one of finite volume and the other one of infinite volume, exist in all dimensions $4n \geq 4$.

Chapter 3

The HK/QK correspondence and the twist construction

In this second preliminary chapter we introduce hyperkähler geometry and describe how it is related with quaternionic Kähler geometry. In particular, we explain how to obtain quaternionic Kähler manifolds from hyperkähler ones. In Section 3.1 we define hyperkähler manifolds. We briefly recall some of their properties, pointing out similarities and differences with the quaternionic Kähler geometry, and mentioning some examples. In this section we also explain in more detail how to construct a canonical hyperkähler manifold associated to any quaternionic Kähler manifold, the so-called Swann bundle. In Section 3.2 we introduce the HK/QK correspondence, which is a way to obtain (a one-parameter family of) quaternionic Kähler metrics from hyperkähler manifolds equipped with (rotating) circle actions. Finally, in Section 3.3 we introduce the twist construction and explain how the HK/QK correspondence can be recovered from this general method. None of the results mentioned in this chapter are original to this thesis and the references will be properly cited.

3.1 Hyperkähler manifolds

As we have already said, hyperkähler manifolds are those Riemannian manifolds (M,g) whose holonomy group is contained in the compact symplectic group $\mathrm{Sp}(n)$. Similarly as the quaternionic Kähler case, we are interested in a definition in terms of more concrete objects which we can manipulate better, i.e. in terms of tensors. Some references for this section are, as before, the books [Bes87, Joy00] and the survey [Dan99].

There are several ways to state the definition of hyperkähler manifolds. We will define them as a particular case of almost quaternionic Hermitian manifolds.

Definition 3.1.1. Let $(M,g,\mathscr{Q})$ be an almost quaternionic Hermitian manifold. We say that it is **hyperkähler** if $\mathscr{Q}$ admits a global trivialization $\{I_1,I_2,I_3\}$ of endomorphisms satisfying the quaternionic relations and which are covariantly constant, i.e.

$\nabla I_k = 0$ for $k = 1,2,3$, where ∇ is the Levi-Civita connection of g. We will denote a hyperkähler manifold as (M, g, I_1, I_2, I_3).

These three endomorphisms are global almost complex structures and, since they are covariantly constant, they are integrable, as follows from

$$N_{I_k}(X,Y) = (\nabla_Y I_k) I_k X - (\nabla_X I_k) I_k Y + (\nabla_{I_k Y} I_k) X - (\nabla_{I_k X} I_k) Y.$$

For each I_k we define a 2-form $\omega_k := g(I_k \cdot, \cdot)$ which is covariantly constant, hence closed, hence Kähler. This implies that a hyperkähler manifold is Kähler with respect to each complex structure I_k. Moreover, note that on a hyperkähler manifold (M, g, I_1, I_2, I_3), the endomorphism $I_a := a_1 I_1 + a_2 I_2 + a_3 I_3$, where $a = (a_1, a_2, a_3) \in \mathbb{S}^2$, is also a complex structure and (M, g, I_a) is again a Kähler manifold. Therefore the metric g is Kähler in lots of different ways, with respect to a whole 2-sphere of complex structures. Because of this, we call g hyperkähler.

As in the case of quaternionic Kähler manifolds, we can construct the corresponding twistor space $Z = M \times \mathbb{C}\mathrm{P}^1$ where we are considering

$$\mathbb{C}\mathrm{P}^1 \cong \mathbb{S}^2 \cong \{a_1 I_1 + a_2 I_2 + a_3 I_3 \mid (a_1, a_2, a_3) \in \mathbb{S}^2\}$$

as the natural 2-sphere of complex structures on M. We can also define an almost complex structure J on Z which turns out to be integrable. Hence, if (M, g, I_1, I_2, I_3) is a hyperkähler manifold of real dimension $4n$, then (Z, J) is a complex manifold of complex dimension $2n+1$ called the **twistor space** of M. The twistor space comes equipped with additional holomorphic data. In fact, this holomorphic data is sufficient to reconstruct the hyperkähler structure (see [HKLR87]). The moral of this is that hyperkähler manifolds can be written solely in terms of holomorphic data and so they can be studied and some explicit examples can be found using complex algebraic geometry.

Notice that an almost Hermitian manifold (M, g, J) (i.e. J is not integrable) is Kähler if and only if J is integrable and $\omega = g(J\cdot, \cdot)$ is closed. This can be seen by using the well-known formula

$$2g((\nabla_X J)Y, Z) = 3\mathrm{d}\omega(X,Y,Z) - 3\mathrm{d}\omega(X, JY, JZ) - g(JX, N_J(Y,Z)).$$

For a manifold (M, g, I_1, I_2, I_3) equipped with three almost complex structures compatible with the metric g and satisfying the quaternionic relations, it is enough to require that the corresponding 2-forms ω_k are closed to obtain a hyperkähler manifold. This result is usually known as the Hitchin lemma [Hit87, Lemma 6.8]. More specifically, what Hitchin lemma says is that if $\mathrm{d}\omega_k = 0$, then this already implies that the almost complex structure I_k is integrable and then (M, g, I_1, I_2, I_3) is hyperkähler.

The fact that hyperkähler manifolds have holonomy contained in $\mathrm{Sp}(n) \subseteq \mathrm{SU}(2n)$ implies immediately that the metric g in Ricci-flat. As a corollary, if M is compact, the first Chern class $c_1(M)$ vanishes.

Hyperkähler manifolds can be studied also from the point of view of algebraic geometry. To see this we need to introduce an extra structure.

Definition 3.1.2. A **holomorphic symplectic manifold** (M,J,Ω) is a complex manifold (M,J) where $\Omega \in \Omega^2(M,\mathbb{C})$ is a closed and non-degenerate holomorphic 2-form.

Proposition 3.1.3. *Let (M,g,I_1,I_2,I_3) be a hyperkähler manifold. Then the complex 2-form*

$$\Omega := \omega_2 + i\omega_3 \in \Omega^2(M,\mathbb{C})$$

is a parallel holomorphic symplectic structure with respect to I_1.

Proof. The 2-form Ω is clearly closed, non-degenerate and parallel. To show that it is holomorphic, let $X \in \Gamma(T^{0,1}M)$, i.e. $I_1X = -iX$. Then we have

$$\begin{aligned}
\Omega(X,Y) &= \omega_2(X,Y) + i\omega_3(X,Y) = g(I_2X,Y) + ig(I_3X,Y) \\
&= g(I_3I_1X,Y) - ig(I_2I_1X,Y) = -ig(I_3X,Y) - g(I_2X,Y) \\
&= -i\omega_3(X,Y) - \omega_2(X,Y) = -\Omega(X,Y)
\end{aligned}$$

for all vector fields Y. Hence $\iota_X\Omega = 0$ and since X was arbitrary, the 2-form Ω belongs to $\Omega^{2,0}(M)$. Since Ω is closed, $\bar{\partial}\Omega = (\mathrm{d}\Omega)^{2,1} = 0$ and then Ω is holomorphic. □

Thus every hyperkähler manifold can be viewed as a Kähler manifold which is also holomorphic symplectic with parallel holomorphic symplectic form. Conversely, a Kähler manifold with parallel holomorphic symplectic structure is hyperkähler. Indeed, since the metric is Kähler and the holomorphic symplectic structure is parallel, then the holonomy group must be contained in $\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{Sp}(2n,\mathbb{C})$.

The next theorem shows that for compact manifolds we do not need to assume that the holomorphic symplectic form is parallel.

Theorem 3.1.4 ([Bea83, Proposition 4])**.** *Let (M,J) be a compact complex manifold admitting a Kähler metric and a holomorphic symplectic structure. Then M admits a unique hyperkähler metric in every Kähler class.*

Before giving some examples of hyperkähler manifolds, we can say something about their topology, at least in the compact case. Similarly as for positive quaternionic Kähler manifolds (see Theorem 2.3.5), we have the following result.

Proposition 3.1.5 ([Bea83, Proposition 4])**.** *Let (M,g) be a compact manifold of dimension $4n$ with $\mathrm{Hol}(g) = \mathrm{Sp}(n)$. Then M is simply connected.*

Much is known about the cohomology of compact hyperkähler manifolds. For the interested reader we refer to [GJH03] and references therein.

3.1.1 Examples of hyperkähler manifolds

The first example of hyperkähler manifold is $\mathbb{R}^{4n} \cong \mathbb{H}^n$ equipped with the flat hyperkähler structure. Identifying $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ and writing $q \in \mathbb{H}^n$ as $q = z + wj$ for

$z, w \in \mathbb{C}^n$, the hyperkähler structure is determined by the following tensors

$$\begin{aligned} g &= \sum_{k=1}^{n} (\mathrm{d}z_k \mathrm{d}\bar{z}_k + \mathrm{d}w_k \mathrm{d}\bar{w}_k)\,, \\ \omega_1 &= \frac{i}{2} \sum_{k=1}^{n} (\mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k + \mathrm{d}w_k \wedge \mathrm{d}\bar{w}_k)\,, \\ \omega_2 &= \frac{1}{2} \sum_{k=1}^{n} (\mathrm{d}z_k \wedge \mathrm{d}w_k + \mathrm{d}\bar{z}_k \wedge \mathrm{d}\bar{w}_k)\,, \\ \omega_3 &= \frac{1}{2i} \sum_{k=1}^{n} (\mathrm{d}z_k \wedge \mathrm{d}w_k - \mathrm{d}\bar{z}_k \wedge \mathrm{d}\bar{w}_k)\,. \end{aligned}$$

The holomorphic symplectic form with respect to the complex structure I_1 is

$$\Omega = \omega_2 + i\omega_3 = \sum_{k=1}^{n} \mathrm{d}z_k \wedge \mathrm{d}w_k.$$

This structure corresponds to consider $\mathbb{H}^n$ as the cotangent bundle of $\mathbb{C}^n$. Note that the above structure is invariant under translations, so we can also define a flat hyperkähler structure on the torus $\mathbb{T}^{4n}$.

In contrast with the quaternionic Kähler case, where we have several symmetric and homogeneous examples, it turns out that the only homogeneous hyperkähler manifolds are the Euclidean space $\mathbb{R}^{4n}$ and the torus $\mathbb{T}^{4n}$ (and products of them). This follows from a more general result.

Theorem 3.1.6 ([AK75, Theorem 1])**.** *Let (M, g) be a Ricci-flat homogeneous space. Then (M, g) is flat, i.e. is the product of a Euclidean space and a flat torus.*

The product of two hyperkähler manifolds is again hyperkähler. Conversely, we can look at the de Rham decomposition of hyperkähler manifolds. The following theorem is a consequence of a well-known result about the decomposition of a complete Ricci-flat Kähler manifold (see e.g. [Bea83, Bes87]).

Theorem 3.1.7. *Let (M, g) be a complete hyperkähler manifold. Then*

$$\tilde{M} \cong \mathbb{R}^{4k} \times M_1 \times \cdots \times M_r,$$

where $\tilde{M}$ is the universal covering of M and each M_j is a complete simply connected $4m_j$-dimensional manifold with holonomy $\mathrm{Sp}(m_j)$. If M is furthermore compact, then the factors M_j above are also compact and

$$M \cong (\mathbb{T}^{4k} \times M_1 \times \cdots \times M_r)/\Gamma,$$

where Γ is a finite group of holomorphic transformations.

There are only a few examples of compact hyperkähler manifolds known in each dimension. Note that $\mathrm{Sp}(1) = \mathrm{SU}(2)$, so hyperkähler 4-manifolds are precisely Calabi-Yau. The only compact 4-manifolds carrying metrics with holonomy exactly $\mathrm{Sp}(1)$ are **K3 surfaces**, which are simply connected compact complex surfaces with trivial canonical bundle. An important example of a K3 surface is

$$S = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{CP}^3 .$$

It is important in the sense that it is some kind of model for K3 surfaces. In particular, Kodaira [Kod64] showed that every K3 surface is a deformation of a non-singular quartic surface in $\mathbb{CP}^3$. Thus al K3 surfaces are diffeomorphic to S. The so-called **Enriques surfaces** are $\mathbb{Z}_2$-quotients of K3 surfaces. They are examples of locally hyperkähler manifolds which are not hyperkähler, since their restricted holonomy group is $\mathrm{Sp}(1)$ but they are not simply connected. Note also that the flat torus $\mathbb{T}^4$ is hyperkähler but its holonomy is trivial.

In dimension greater than 4, in contrast to Calabi-Yau manifolds, examples of compact hyperkähler manifolds are difficult to find, and only a few are known in each dimension. The first examples were two series of manifolds due to Beauville [Bea83], which generalize an example of Fujiki [Fuj83] in real dimension 8. Two further examples of compact hyperkähler manifolds have been constructed by O'Grady in real dimension 20 [O'G99] and 12 [O'G03]. In [Bea11] it is conjectured that there are only finitely many compact hyperkähler manifolds (up to deformation) in every dimension.

In the non-compact case, many examples of complete hyperkähler manifolds are known. Some examples can be constructed via the so-called hyperkähler quotient construction, introduced in [HKLR87], which generalized the well-known symplectic quotient construction to hyperkähler manifolds. A representative collection of examples can be found in [Hit92].

Another source of examples particularly interesting for us is the cotangent bundle of some complex manifolds. It is natural to look for hyperkähler structures on these spaces since, for any complex manifold M, the cotangent bundle T^*M admits a holomorphic symplectic structure. The first examples of complete metrics with holonomy precisely $\mathrm{Sp}(n)$ are due to Calabi, who constructed a hyperkähler metric on $T^*\mathbb{CP}^n$, usually known as the Calabi metric.

Theorem 3.1.8 ([Cal79, Théorème 5.3])**.** *The cotangent bundle of the complex projective space $\mathbb{CP}^n$ admits a complete hyperkähler metric for any $n \in \mathbb{N}$.*

The Calabi construction was generalized by Biquard and Gauduchon to the cotangent bundle of compact Hermitian symmetric spaces G/K in [BG97]. Here the authors construct a complete hyperkähler metric which is invariant under the action of G and restricts to the symmetric Kähler metric on G/K when it is restricted to the zero section. They also considered the non-compact dual Hermitian symmetric spaces, but in this case the hyperkähler metric is incomplete and can only be defined on an open neighborhood of the zero section. This local construction was then generalized independently by Feix [Fei01] and Kaledin [Kal01], who constructed hyperkähler metrics on a neighborhood of the zero section of the cotangent bundle of any real-analytic Kähler manifold. Although the hyperkähler metric is in general incomplete and only locally defined, they recover the metrics of Calabi and Biquard-Gauduchon when the base manifold is a compact Hermitian symmetric space.

Finally, if we allow indefinite signature, there are further examples of pseudo-Kähler manifolds whose cotangent bundle admits a pseudo-hyperkähler structure (see Section 4.2). In this work we are particularly interested in these spaces.

3.1.2 Swann bundle

An important link between hyperkähler and quaternionic Kähler geometry was obtained by Swann in [Swa91], where he showed that over any quaternionic Kähler manifold (with non-zero scalar curvature) one can construct a bundle whose total space carries a natural conical (pseudo-)hyperkähler structure encoding the quaternionic Kähler geometry of the base. We briefly recall this construction here (see also [ACDM15] and [Ion19]).

Definition 3.1.9. A (pseudo-)Riemannian manifold (M,g) is called **conical** if it admits a vector field ξ (with $g(\xi,\xi)\neq 0$) satisfying $\nabla\xi = \mathrm{Id}$, where ∇ is the Levi-Civita connection of g.

We have the following characterizations of being conical.

Lemma 3.1.10. *Let (M,g) be a (pseudo-)Riemannian manifold. Then the following are equivalent:*

(a) *(M,g,ξ) is a conical (pseudo-)Riemannian manifold.*

(b) *There exists a function κ on M such that $g=\nabla^2\kappa=\nabla \mathrm{d}\kappa$, i.e. the metric is given by the Hessian of κ.*

(c) *There exists a function κ on M such that the vector field ξ, defined by $\iota_\xi g = \mathrm{d}\kappa$, is homothetic, i.e. $\mathscr{L}_\xi g = 2g$.*

Proof. Suppose (a) holds. We define $\kappa := \frac{1}{2}g(\xi,\xi)$, thus $\mathrm{d}\kappa = g(\xi,\cdot) = \iota_\xi g$. The following computation implies (b):

$$(\nabla \mathrm{d}\kappa)(X,Y) = X(Y(\kappa)) - \mathrm{d}\kappa(\nabla_X Y) = Xg(\xi,Y) - g(\xi,\nabla_X Y) = g(X,Y).$$

To obtain (c) we use $(\mathscr{L}_\xi g)(X,Y) = g(\nabla_X\xi,Y) + g(X,\nabla_Y\xi)$ and $\nabla\xi = \mathrm{Id}$.

Suppose (b) holds. We define ξ by the equation $\mathrm{d}\kappa = \iota_\xi g$. Then the expression $g(X,Y) = (\nabla^2\kappa)(X,Y) = g(\nabla_X\xi,Y)$ implies $\nabla\xi = \mathrm{Id}$ since g is non-degenerate. Thus (a) holds.

Suppose (c) holds. The equation $\mathscr{L}_\xi g = 2g$ implies

$$2g(X,Y) = (\mathscr{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi).$$

Since the Hessian $\nabla^2 \kappa$ of a function is symmetric, we get $g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X)$. Combining these expressions, and using again that the metric g is non-degenerate, we show that $\nabla \xi = \mathrm{Id}$, thus (a) holds. □

If the conical manifold is furthermore hyperkähler, then we can say much more about the interaction between these two structures.

Lemma 3.1.11. *Let $(M, g, I_1, I_2, I_3, \xi)$ be a conical hyperkähler manifold. Then:*

(a) *The function $\kappa = \frac{1}{2} g(\xi, \xi)$ is a global Kähler potential for all three Kähler forms, that is $2\omega_k = \mathrm{d}\mathrm{d}^c_k \kappa$, where $\mathrm{d}^c_k = -I_k^* \mathrm{d}$ is the d^c-operator associated with I_k, $k = 1,2,3$. Such a potential is called* ***hyperkähler potential****.*

(b) *The distribution $\mathbb{H}\xi := \mathrm{span}\{\xi, I_1\xi, I_2\xi, I_3\xi\}$ determines a Lie algebra isomorphic to $\mathbb{R} \oplus \mathfrak{sp}(1)$, that is $[\xi, I_k \xi] = 0$ and $[I_j \xi, I_k \xi] = 2\sum_{\ell=1}^3 \varepsilon_{kj\ell} I_\ell \xi$, where $\varepsilon_{jk\ell}$ is the Levi-Civita symbol.*

(c) *Each $I_k \xi$ is ω_k-Hamiltonian with Hamiltonian function κ.*

(d) *Each $I_k \xi$ is a Killing vector field.*

Proof. (a) We have $\mathrm{d}^c_k \kappa = -I_k^* \mathrm{d}\kappa = -g(\xi, I_k \cdot) = g(I_k \xi, \cdot) =: \theta_k$. Then

$$\mathrm{d}\theta_k(X,Y) = X(\theta_k(Y)) - Y(\theta_k(X)) - \theta_k([X,Y]) = g(\nabla_X I_k \xi, Y) - g(\nabla_Y I_k \xi, X).$$

Using $\nabla I_k = 0$ and $\nabla \xi = \mathrm{Id}$ we obtain the desired result.

(b) This follows from the fact that ∇ is torsion-free, $\nabla I_k = 0$, $\nabla \xi = \mathrm{Id}$ and the quaternionic relations.

(c) We have $\iota_{I_k \xi} \omega_k = \omega_k(I_k \xi, \cdot) = -g(\xi, \cdot) = -\mathrm{d}\kappa$.

(d) We have $(\mathscr{L}_{I_k \xi} g)(X,Y) = g(\nabla_X I_k \xi, Y) + g(X, \nabla_Y I_k \xi)$. Using $\nabla I_k = 0$ and $\nabla \xi = \mathrm{Id}$ we obtain $\mathscr{L}_{I_k \xi} g = 0$, i.e. $I_k \xi$ is Killing. □

By similar computations as before, we furthermore obtain that

$$\mathscr{L}_{I_j \xi} \omega_k = 2 \sum_{\ell=1}^{3} \varepsilon_{kj\ell} \omega_\ell.$$

Identifying $\mathfrak{sp}(1)$ with $\mathrm{Im}\,\mathbb{H}$, we can summarize this by saying that $q \in \mathfrak{sp}(1)$ acts on the $\mathrm{Im}\,\mathbb{H}$-valued 2-form $\omega = \omega_1 i + \omega_2 j + \omega_3 k$ by $\mathscr{L}_q \omega = [\omega, q] = \omega \cdot q - q \cdot \omega$.

Definition 3.1.12. Let (M, g, I_1, I_2, I_3) be a hyperkähler manifold. We say that an infinitesimal $\mathrm{Sp}(1)$-action by Killing vector fields **permutes complex structures** (or is **permuting**) if $\mathscr{L}_q \omega = [\omega, q]$ for every $q \in \mathfrak{sp}(1) \cong \mathrm{Im}\,\mathbb{H}$.

Therefore, due to the above discussion, we conclude that a conical hyperkähler manifold carries an infinitesimal $\mathrm{Sp}(1)$-action which permutes the complex structures. In fact, Swann [Swa91, Proposition 5.5] proved that the existence of an infinitesimal $\mathrm{Sp}(1)$-action is essentially equivalent to require that the hyperkähler manifold is conical.

We recall the following notions which will be use throughout the thesis.

Definition 3.1.13. Let (M,g,I_1,I_2,I_3) be a hyperkähler manifold. We say that a vector field Z on M is:

- **Tri-holomorphic** if $\mathscr{L}_Z I_1 = 0$, $\mathscr{L}_Z I_2 = 0$ and $\mathscr{L}_Z I_3 = 0$.
- **Rotating** if $\mathscr{L}_Z I_1 = 0$, $\mathscr{L}_Z I_2 = I_3$ and $\mathscr{L}_Z I_3 = -I_2$.

Now let $(M,g,\mathscr{Q})$ be a quaternionic Kähler manifold. We can construct over M the principal $\mathrm{SO}(3)$-bundle $\mathrm{SO}(\mathscr{Q})$ of oriented orthonormal frames of the quaternionic structure bundle $\mathscr{Q}$. For $q \in \mathrm{Sp}(1)$ and $v \in \mathrm{Im}\,\mathbb{H}$, the map $q \longmapsto (qvq^{-1})$ defines the universal covering $\mathrm{Sp}(1) \longrightarrow \mathrm{SO}(3)$ with kernel $\mathbb{Z}_2$ (recall $\mathrm{Sp}(1) \cong \mathrm{Spin}(3)$). This gives us the standard action of $\mathrm{Sp}(1)$ on $\mathrm{Im}\,\mathbb{H} \cong \mathbb{R}^3$, which extends to an action on $\mathbb{H} \cong \mathbb{R} \oplus \mathrm{Im}\,\mathbb{H}$ by acting trivially on the first factor. It induces a well-defined action of $\mathrm{SO}(3)$ on the quotient $\mathbb{H}^*/\mathbb{Z}_2 \cong \mathrm{SO}(3) \times \mathbb{R}_{>0}$. If we denote this action by ρ, then we define the following associated principal bundle:

$$\mathscr{U}(M) := \mathrm{SO}(\mathscr{Q}) \times_\rho (\mathbb{H}^*/\mathbb{Z}_2).$$

Theorem 3.1.14 ([Swa91, Theorem 3.5])**.** *Let $(M,g,\mathscr{Q})$ be a quaternionic Kähler manifold. Then $\mathscr{U}(M) = \mathrm{SO}(\mathscr{Q}) \times_\rho (\mathbb{H}^*/\mathbb{Z}_2)$ carries a conical (pseudo-)hyperkähler structure whose hyperkähler potential function is given by $|q|^2$ for $q \in \mathbb{H}^*$.*

Definition 3.1.15. Let $(M,g,\mathscr{Q})$ be a quaternionic Kähler manifold. We say that the bundle $\mathscr{U}(M)$ equipped with its conical (pseudo-)hyperkähler structure is the **Swann bundle** of $(M,g,\mathscr{Q})$.

Remark 3.1.16. For a quaternionic Kähler manifold M of dimension $4n$, the hyperkähler metric of the Swann bundle $\mathscr{U}(M)$ is positive-definite if $\nu > 0$ and of signature $(4,4n)$ if $\nu < 0$, where $\nu = \frac{\mathrm{scal}}{4n(n+2)}$ is the reduced scalar curvature of M.

The Swann bundle $\mathscr{U}(M)$ admits a homothetic $\mathbb{H}^*$-action and the subgroup $\mathrm{Sp}(1)$ of $\mathbb{H}^*$ acts isometrically, but it permutes the complex structures of $\mathscr{U}(M)$, so we cannot apply the hyperkähler quotient construction introduced in [HKLR87] (since the action has to be tri-holomorphic). However, if we fix one complex structure I_k, then there is a subgroup $\mathrm{U}(1) \subset \mathrm{Sp}(1)$ which preserves I_k and we have a moment map for this circle action. The generator of this circle action is $I_k\xi$ which, indeed, preserve I_k, since it is Killing and

$$\mathscr{L}_{I_k\xi}\omega_k = \mathrm{d}\iota_{I_k\xi}\omega_k = -\mathrm{d}\mathrm{d}\kappa = 0.$$

Up to a constant, the moment map is $\kappa = \frac{1}{2}g(\xi,\xi) : \mathscr{U}(M) \longrightarrow \mathbb{R}$, since $\mathrm{d}\kappa = -\iota_{I_k\xi}\,\omega_k$, and fiberwise the level sets correspond to spheres in $\mathbb{H}$. The level sets of κ are preserved by the $\mathrm{Sp}(1)$-action. Indeed, for $I_j\xi \in \mathfrak{sp}(1)$ we have

$$\iota_{I_j\xi}\mathrm{d}\kappa = -\omega_k(I_k\xi, I_j\xi) = \omega_j(\xi,\xi) = 0.$$

The $\mathrm{Sp}(1)$-quotient of a level set of the moment map κ is just the original quaternionic Kähler manifold M.

This discussion generalizes as follows.

Theorem 3.1.17 ([Swa91, Theorem 5.1])**.** *Let (M,g,I_1,I_2,I_3) be a hyperkähler manifold admitting an isometric $\mathrm{Sp}(1)$-action such that:*

(a) *There is a finite subgroup Γ of $\mathrm{Sp}(1)$ such that $\mathrm{Sp}(1)/\Gamma$ acts freely on M.*

(b) *$\mathrm{Sp}(1)$ induces a transitive action on the sphere of complex structures compatible with the hyperkähler structure.*

(c) *If X_k denotes a generator of the $\mathrm{U}(1)$-subgroup preserving I_k, then the (real) linear span of I_kX_k in TM is independent of the choice of complex structure.*

Let $\mathrm{U}(1) \subset \mathrm{Sp}(1)$ be a subgroup preserving a complex structure I_k and $\mu : M \longrightarrow \mathbb{R}$ be a moment map for this $\mathrm{U}(1)$ with respect to the Kähler structure ω_k. Then $\mu^{-1}(x)$ is $\mathrm{Sp}(1)$-invariant and $\mu^{-1}(x)/(\mathrm{Sp}(1)/\Gamma)$ is a quaternionic Kähler manifold.

Remark 3.1.18. This result also holds in the pseudo-Riemannian case, under the assumption that the restriction of the hyperkähler metric to the tangent spaces to the $\mathrm{Sp}(1)$-orbits is non-degenerate. This case is relevant for quaternionic Kähler manifolds of negative scalar curvature.

The next result says that Swann bundles are characterized among all hyperkähler manifolds, at least locally, by the existence of a permuting $\mathrm{Sp}(1)$-action.

Theorem 3.1.19 ([Swa91, Theorem 5.9])**.** *If (N,g,I_1,I_2,I_3) is a hyperkähler manifold satisfying the hypothesis of Theorem 3.1.17, then N is locally homothetic to the Swann bundle $\mathscr{U}(M)$ of the quaternionic Kähler manifold $M = \mu^{-1}(x)/(\mathrm{Sp}(1)/\Gamma)$.*

It is worthwhile to mention the relation between the Swann bundle and the twistor space of a quaternionic Kähler manifold. As we have seen, the twistor space Z of a quaternionic Kähler manifold $(M,g,\mathscr{Q})$ is the sphere bundle $\mathbb{S}(\mathscr{Q})$ consisting of almost complex structures compatible with the quaternionic structure on M. Recall that Z always admits an integrable almost complex structure (see Theorem 2.2.20). We can obtain the twistor space from $\mathscr{U}(M)$ as follows. By fixing one of the complex structures, say I_k, we obtain a $\mathrm{U}(1)$-action fixing I_k, and $\mathbb{C}^* = \mathrm{U}(1) \times \mathbb{R}_{>0}$ acts holomorphically on the complex manifold $(\mathscr{U}(M), I_k)$. The quotient $(\mathscr{U}(M), I_k)/\mathbb{C}^*$ then gives us Z, which is independent of the chosen complex structure (see [Swa91,Hit13]).

Moreover, the Swann bundle construction commutes with the hyperkähler quotient [HKLR87] and the quaternionic Kähler quotient [GL88] constructions. Recall that a vector field on a hyperkähler manifold is called tri-holomorphic if it preserves the three complex structures. Now, let $(M, g, \mathscr{Q})$ be a quaternionic Kähler manifold with fundamental 4-form Ω. In [GL88] the concept of quaternionic Killing field was introduced. This is a Killing field that preserves both $\mathscr{Q}$ and Ω. However, this definition was later shown to be superfluous, since any Killing vector field $X \in \Gamma(TM)$ satisfies $\mathscr{L}_X \Gamma(\mathscr{Q}) \subset \Gamma(\mathscr{Q})$ and $\mathscr{L}_X \Omega = 0$ (see [ACDP03] or [BG08]).

In [Swa91, Lemma 4.1] is shown that any Killing vector field X on M can be lifted to a tri-Hamiltonian Killing vector field $\tilde{X}$ on the Swann bundle $\mathscr{U}(M)$.

Theorem 3.1.20 ([Swa91, Theorem 4.6])**.** *Let $(M, g, \mathscr{Q})$ be a quaternionic Kähler manifold and let G be a compact connected Lie group. If G acts freely and isometrically on M, then G induces a free, tri-holomorphic, isometric action on the Swann bundle $\mathscr{U}(M)$. The (pseudo-)hyperkähler quotient of $\mathscr{U}(M)$ by this G-action is precisely the Swann bundle of the quaternionic Kähler quotient of M by G, that is*

$$\mathscr{U}(M)/\!/\!/G = \mathscr{U}(M/\!/\!/G).$$

3.1.3 Bundles over quaternionic Kähler manifolds

Here we summarize what we have presented so far about bundle constructions over quaternionic Kähler manifolds. We recap the structures that these bundles have, showing how rich is quaternionic Kähler geometry in terms on interplay with several other geometric structures.

Let $(M, g, \mathscr{Q})$ be a quaternionic Kähler manifold of dimension $4n$. Then we can define the following bundles over it:

(1) **Twistor space $\mathscr{Z}(M)$:** We have seen that over a quaternionic Kähler manifold we can take the sphere bundle $\mathbb{S}(\mathscr{Q})$ of almost complex structures compatible with $\mathscr{Q}$. The $(4n+2)$-dimensional manifold $\mathscr{Z}(M) := \mathbb{S}(\mathscr{Q})$ admits an almost complex structure which turns out to be integrable (see Theorem 2.2.20). Moreover, the twistor space carries a real structure and a complex contact structure (see Theorem 2.2.21). In the case where the quaternionic Kähler manifold is positive, $\mathscr{Z}(M)$ admits a (positive-definite) Kähler-Einstein metric (see Theorem 2.3.3). In the case where the quaternionic Kähler manifold is negative, the twistor space admits an indefinite Kähler-Einstein metric of signature $(2, 4n)$ (see [Bes87]).

(2) **Konishi bundle $\mathscr{S}(M)$:** We have seen that over a quaternionic Kähler manifold we can take the principal $\mathrm{SO}(3)$-bundle $\mathrm{SO}(\mathscr{Q})$ of oriented orthonormal frames of $\mathscr{Q}$. It was shown in [Kon75, Theorem 2] that the $(4n+3)$-dimensional manifold $\mathscr{S}(M) := \mathrm{SO}(\mathscr{Q})$ admits a (positive-definite) 3-Sasakian metric if the quaternionic Kähler manifold is positive, and an indefinite 3-Sasakian metric of signature $(3, 4n)$ if the quaternionic Kähler manifold is negative. Briefly speaking, (S, g) is 3-Sasakian if its metric cone $(\mathbb{R}_{>0} \times S, \mathrm{d}r^2 + r^2 g)$ is hyperkähler, so

3-Sasakian geometry is the odd-dimensional analogue of hyperkähler geometry. Every 3-Sasakian manifold (S,g) is Einstein and, if (S,g) is complete, then it is compact with finite fundamental group. We refer the interested reader to [BG08] for further study on this geometry.

(3) **Swann bundle $\mathscr{U}(M)$:** We have seen that over a quaternionic Kähler manifold we can take the Konishi bundle $\mathscr{S}(M)$ and then take the Cartesian product with $\mathbb{R}_{>0}$. This is a principal bundle with fiber $\mathrm{SO}(3)\times\mathbb{R}_{>0}\cong\mathbb{H}^*/\mathbb{Z}_2$. The $(4n+4)$-dimensional manifold $\mathscr{U}(M):=\mathscr{S}(M)\times\mathbb{R}_{>0}$ can be also seen as the associated bundle $\mathscr{S}(M)\times_\rho(\mathbb{H}^*/\mathbb{Z}_2)$, where ρ is the action of $\mathrm{SO}(3)$ on $\mathbb{H}^*/\mathbb{Z}_2$. The manifold $\mathscr{U}(M)$ admits a conical hyperkähler structure with positive-definite metric if M is positive, and indefinite metric with signature $(4,4n)$ if M is negative (see Theorem 3.1.14 and Remark 3.1.16).

In particular [BG08], for a quaternionic Kähler manifold $(M,g,\mathscr{Q})$ we have the following fibrations defined by $\mathbb{Z}_2\subset\mathbb{R}^*\subset\mathbb{C}^*\subset\mathbb{H}^*$:

- $\mathbb{H}^*/\mathbb{C}^*\cong\mathbb{S}^2\longrightarrow\mathscr{Z}(M)\longrightarrow M$.
- $\mathbb{H}^*/\mathbb{R}^*\cong\mathrm{SO}(3)\longrightarrow\mathscr{S}(M)\longrightarrow M$.
- $\mathbb{H}^*/\mathbb{Z}_2\longrightarrow\mathscr{U}(M)\longrightarrow M$.
- $\mathbb{C}^*/\mathbb{R}^*\cong\mathbb{S}^1\longrightarrow\mathscr{S}(M)\longrightarrow\mathscr{Z}(M)$.
- $\mathbb{C}^*/\mathbb{Z}_2\longrightarrow\mathscr{U}(M)\longrightarrow\mathscr{Z}(M)$.
- $\mathbb{R}^*/\mathbb{Z}_2\cong\mathbb{R}_{>0}\longrightarrow\mathscr{U}(M)\longrightarrow\mathscr{S}(M)$.

These six fibrations are the six arrows of the following diagram:

$$\begin{tikzcd}
& \mathscr{U}(M) \arrow[dl] \arrow[dr] \arrow[dd] & \\
\mathscr{Z}(M) \arrow[dr] & & \mathscr{S}(M) \arrow[ll] \arrow[dl] \\
& M &
\end{tikzcd}$$

Let us see how this works in a particular example. Consider the quaternionic Kähler manifold

$$M=\mathbb{H}\mathrm{P}^n\cong\mathbb{S}^{4n+3}/\mathrm{Sp}(1)\cong\mathrm{Sp}(n+1)/(\mathrm{Sp}(n)\times\mathrm{Sp}(1)).$$

The corresponding bundles over M are the following:

- $\mathscr{Z}(M)=\mathbb{C}\mathrm{P}^{2n+1}\cong\mathbb{S}^{4n+3}/\mathrm{U}(1)\cong\mathrm{Sp}(n+1)/(\mathrm{Sp}(n)\times\mathrm{U}(1))$.
- $\mathscr{S}(M)=\mathbb{R}\mathrm{P}^{4n+3}\cong\mathbb{S}^{4n+3}/\mathbb{Z}_2\cong\mathrm{Sp}(n+1)/(\mathrm{Sp}(n)\times\mathbb{Z}_2)$.
- $\mathscr{U}(M)=(\mathbb{H}^{n+1}\setminus\{0\})/\mathbb{Z}_2$.

Under the appropriate identifications, we can see that all these spaces fit in the above diagram. Since $\mathbb{H}\mathrm{P}^n$ is a positive quaternionic Kähler manifold, the spaces $\mathscr{Z}(M)$, $\mathscr{S}(M)$ and $\mathscr{U}(M)$, carry Kähler-Einstein, 3-Sasakian and conical hyperkähler structures, respectively, all of them with positive-definite metric.

3.2 The HK/QK correspondence

We have seen in the previous sections that quaternionic Kähler and hyperkähler geometries are "intrinsically" very different, in the sense that their geometric properties differ a lot, e.g. quaternionic Kähler manifolds are in general not even almost complex manifolds whereas hyperkähler manifolds have a whole 2-sphere of integrable almost complex structures. Nevertheless, their "extrinsic" geometry is very similar, in the sense of which spaces we can associate to them, e.g. both geometries have associated a twistor space which is a complex manifold (see also Remark 3.2.3).

Hence, it is natural to ask whether it is possible to relate these two geometries via an additional space which connects them. We have seen an instance of this relation. Indeed, to any quaternionic Kähler manifold M we can associate a conical hyperkähler manifold, i.e. the Swann bundle $\mathscr{U}(M)$ (see Theorem 3.1.14). Moreover, isometric group actions on M lift to isometric and tri-holomorphic group actions on $\mathscr{U}(M)$, thus we can perform the hyperkähler quotient to obtain a new hyperkähler manifold (see Theorem 3.1.20). In particular, when we have a circle action on M, with this procedure we obtain a new hyperkähler manifold of the same dimension as M. This procedure is known as the **QK/HK correspondence** (see also [APP11]).

Summarizing, if we start with a quaternionic Kähler manifold equipped with a circle action, it is possible to obtain a hyperkähler manifold of the same dimension equipped with a rotating circle action. We next show that this construction can be inverted to obtain quaternionic Kähler manifolds.

Let (M, g, I_1, I_2, I_3) be a hyperkähler manifold. Suppose that it admits **HK/QK data**, i.e. a tuple $(Z, \omega_1, \omega_{\mathrm{H}}, f_Z^c, f_{\mathrm{H}}^c)$ such that

- $\omega_1 := g(I_1\cdot,\cdot)$ is integral (i.e. ω_1 has integral periods),
- Z is a rotating Killing vector field preserving I_1 and rotating the other two into each other (we assume for simplicity that Z generates a free circle action),
- $\omega_{\mathrm{H}} := \omega_1 + \mathrm{d}\iota_Z g$,
- f_Z^c is a nowhere vanishing function such that $\iota_Z\omega_1 = -\mathrm{d}f_Z^c$,
- $f_{\mathrm{H}}^c := f_Z^c + g(Z,Z)$ is nowhere vanishing.

Lemma 3.2.1. *The function f_{H}^c is ω_{H}-Hamiltonian.*

Proof. We have that

$$\iota_Z\omega_{\mathrm{H}} = \iota_Z\omega_1 + \iota_Z\mathrm{d}\iota_Z g = -\mathrm{d}f_Z^c + \mathscr{L}_Z\iota_Z g - \mathrm{d}(g(Z,Z)),$$

but $\mathscr{L}_Z\iota_Z g = (\mathscr{L}_Z g)(Z,\cdot) + g(\mathscr{L}_Z Z,\cdot) = 0$, so $\iota_Z\omega_{\mathrm{H}} = -\mathrm{d}(f_Z^c + g(Z,Z)) = -\mathrm{d}f_{\mathrm{H}}^c$. □

Note that there is a freedom of adding a constant to the Hamiltonian functions f_Z^c and f_{H}^c, so long as the shifted Hamiltonian functions are still nowhere vanishing. This is reflected in the superscript c in f_Z^c and f_{H}^c.

Given this data, it was shown in [Hay08, Theorem 3 and Theorem 7] that we can construct a quaternionic Kähler manifold $\bar{M}$ of positive scalar curvature equipped with a circle action such that the given hyperkähler manifold M may be recovered as a hyperkähler reduction of the Swann bundle $\mathscr{U}(\bar{M})$ of $\bar{M}$ by a lift of the circle action at a non-zero level set. This construction is known as the **HK/QK correspondence**.

Remark 3.2.2. Although the quaternionic Kähler manifolds obtained by Haydys have positive scalar curvature, they are not complete. This can be seen, for example, by realizing that a quaternionic Kähler manifold in the image of the HK/QK correspondence always admits an integrable almost complex structure compatible with the quaternionic structure (see [Bat99, Proposition 3.3] or [Sal99, Section 7]), so they can not be complete by Theorem 2.3.6.

Remark 3.2.3. In [Hit13], Hitchin discusses the HK/QK correspondence from the point of view of the corresponding twistor spaces. Briefly speaking, given a hyperkähler manifold M with a circle action, we can construct a principal $\mathbb{C}^*$-bundle P over its twistor space $\mathscr{Z}(M)$. We also assume that the holomorphic vector field associated to the circle action generates a $\mathbb{C}^*$-action on P. Then, after removing the fixed points, we can define the quotient $\bar{\mathscr{Z}} := P/\mathbb{C}^*$. Hitchin showed that $\bar{\mathscr{Z}}$ is a complex contact manifold and, in particular, it is the twistor space of a quaternionic Kähler manifold $\bar{M}$, i.e. $\bar{\mathscr{Z}} = \mathscr{Z}(\bar{M})$.

The HK/QK correspondence was generalized in [ACM13] by allowing indefinite metrics on the hyperkähler manifold M. In this case we can construct a quaternionic pseudo-Kähler manifold $\bar{M}$ of non-zero scalar curvature. The signature of the resulting quaternionic pseudo-Kähler manifold and the sign of its scalar curvature depend on the signature of the pseudo-hyperkähler manifold M and the signs of the functions f_Z^c and f_{H}^c. The cases when one obtains a (positive-definite) quaternionic Kähler metric were specified in [ACM13] and include the case of quaternionic Kähler metrics of positive scalar curvature considered by Haydys [Hay08], who started with a (positive-definite) hyperkähler metric, as we have explained before.

In the following theorem we focus on the cases which yield a positive-definite metric of negative scalar curvature, of relevance to this thesis.

Theorem 3.2.4 ([ACM13, Corollary 2]). *Let (M,g,I_1,I_2,I_3) be a pseudo-hyperkähler manifold of dimension $4n+4 \geq 4$ equipped with the HK/QK data $(Z,\omega_1,\omega_{\mathrm{H}},f_Z^c,f_{\mathrm{H}}^c)$, and let P be a principal $\mathbb{S}^1$-bundle over M such that $c_1(P)\otimes\mathbb{R} = [\omega_1] = [\omega_{\mathrm{H}}]$. Then there is a lift of the circle action on M generated by Z to $P\times\mathbb{H}^*$, so that its quotient by the lifted action, $\hat{M} := (P\times\mathbb{H}^*)/\mathbb{S}^1$, carries a conical pseudo-hyperkähler structure with hyperkähler reduction (M,g,I_1,I_2,I_3). The conical pseudo-hyperkähler manifold $\hat{M}$ is the Swann bundle of a (positive-definite) quaternionic Kähler manifold $\bar{M}$ of negative scalar curvature if and only if g is positive-definite and $f_Z^c > 0$, or if the signature of g is $(4n,4)$ and $f_Z^c > 0$ while $f_{\mathrm{H}}^c < 0$.*

Note that explicit expressions for all the above data, including the quaternionic Kähler metric, are obtained in [Hay08, ACM13], [ACDM15, Theorem 2]. We have however omitted these in the statement of Theorem 3.2.4 to avoid redundancy, since we will describe the metric below in Section 3.3 using the language of Swann twist construction.

Moreover, since there is the freedom of adding a constant term to the Hamiltonian function f_Z^c, Theorem 3.2.4 gives us, if necessary after restricting to open sets, a one-parameter family of quaternionic Kähler manifolds of fixed scalar curvature associated to a pseudo-hyperkähler manifold.

3.3 The twist construction

We have seen that given a quaternionic Kähler manifold M endowed with a G-action we can construct a hyperkähler manifold $\mathscr{U}(M)/\!/\!/G$, where $\mathscr{U}(M)$ is the Swann bundle of M. This way of construct new manifolds with geometric structures by using an auxiliary bundle can be studied in a much more general context.

Let M be a manifold and let G be a Lie group acting on M. Suppose in addition that M is equipped with a geometric structure which is invariant under the G-action. Then one can ask how to construct a new manifold with a G-action and the same, or a closely related, G-invariant geometric structure. A way to perform this construction, using an intermediate bundle over the original manifold, was considered by Joyce in [Joy92]. The strategy he followed is the following:

- First, we construct a principal G-bundle P over the manifold M.
- Second, we lift the original G-action on M to a G-action on P in such a way that it commutes with the principal G-action.
- Finally, we quotient P by the lifted G-action and we study the geometry of the quotient space $\bar{M} := P/G$, which is also equipped with a G-action.

By this procedure (under certain assumptions on the G-action and its lifts), Joyce provided a new construction of hypercomplex [Joy92, Theorem 2.1] and quaternionic [Joy92, Theorem 2.2] manifolds, which are obtained by "twisting" M by the G-bundle P. In this case we need moreover a choice of a **quaternionic connection** on P, that is a connection 1-form on P whose curvature is of type $(1,1)$ with respect to each almost complex structure from the quaternionic structure.

This general construction was also considered by Swann in [Swa10], who studied it in the case where G is a torus. He constructed several examples of various types of (hyper-)complex and (hyper-)Hermitian geometries (such as strong Kähler with torsion and hyperkähler with torsion structures on compact simply connected manifolds). This construction is more general that the one described by Joyce in the sense that it can be applied to more geometric structures and we do not necessarily need a quaternionic connection. We briefly recall how this construction works in general and then we focus on the case where $G = \mathbb{S}^1$, which in the most relevant for us. All the details can be found in [Swa10].

Let M be a manifold and let $\pi : P \longrightarrow M$ be a principal $\mathbb{T}^n$-bundle with structural group T_P. We write $\mathfrak{t}_P$ for the Lie algebra of T_P. Now assume that there is an action of $T_M \cong \mathbb{T}^n$ on M and write $Z : \mathfrak{t}_M \longrightarrow \Gamma(TM)$ for the infinitesimal action, which can be regarded as an element of $\Gamma(TM) \otimes \mathfrak{t}_M^*$. Let $\eta \in \Omega^1(P, \mathfrak{t}_P)$ be a connection 1-form on P with curvature $\omega \in \Omega^2(M, \mathfrak{t}_P)$, i.e. $\pi^*\omega = \mathrm{d}\eta$. We want to determine the conditions so that the T_M-action is covered by an abelian Lie group action on P preserving η and commuting with T_P.

Proposition 3.3.1 ([Swa10, Proposition 2.1])**.** *The T_M-action on M induced by Z lifts to an $\mathbb{R}^n$-action on P preserving the connection 1-form η if and only if $\mathscr{L}_Z\omega = 0$ and $\iota_Z\omega = -\mathrm{d}f$ for some $f \in \mathscr{C}^\infty(M, \mathfrak{t}_P \otimes \mathfrak{t}_M^*)$.*

Note that the lift is not unique since it depends on the choice of f.

Now suppose that ω is a closed 2-form with values in $\mathbb{R}^n \cong \mathfrak{t}_P$.

Definition 3.3.2. We say that a T_M-action on M is **ω-Hamiltonian** if $\mathscr{L}_Z\omega = 0$ and $\iota_Z\omega = -\mathrm{d}f$ for some $f \in \mathscr{C}^\infty(M, \mathfrak{t}_P \otimes \mathfrak{t}_M^*)$.

In this construction we would like to start with a manifold M with a T_M-action and a 2-form $\omega \in \Omega^2(M, \mathfrak{t}_n)$ that is T_M-invariant. We then want to construct a principal $\mathbb{T}^n$-bundle P with a connection η whose curvature is ω in such a way that T_M lifts to a $\mathbb{T}^n$-action on P preserving η and commuting with the principal action. The following result provides us with the setting where this construction holds.

Proposition 3.3.3 ([Swa10, Proposition 2.3])**.** *Suppose that the manifold M admits an ω-Hamiltonian T_M-action for some integral closed 2-form. Then there is a principal $\mathbb{T}^n$-bundle $P \longrightarrow M$ such that:*

(a) *There exists an n-torus action on P commuting with the principal torus action and covering the T_M-action on M.*

(b) *The space P admits a principal $\mathbb{T}^n$-connection η whose curvature is ω and which is invariant under the lifted torus action.*

In fact, such a lift exists for any $\mathbb{T}^n$-bundle P with $c_1(P) \otimes \mathbb{R} = [\omega]$.

We now proceed to explain the twist construction. Suppose that M is a manifold with an effective ω-Hamiltonian T_M-action where $\omega \in \Omega^2(M, \mathfrak{t}_n)$ is integral. Let P be a principal $\mathbb{T}^n$-bundle over M with a connection η whose curvature is ω and with an $\mathring{T}_M$-action preserving η and covering the T_M-action infinitesimally. Here $\mathring{T}_M$ is some connected abelian group covering T_M. Assume that $\mathring{T}_M$ acts properly on P and that $\mathring{T}_M$ is transverse to

$$\mathscr{H} := \ker\eta \subset TP.$$

Then $\mathring{T}_M$ has discrete stabilizers and $P/\mathring{T}_M$ has the same dimension as M. This transversality is the same as requiring $f \in \mathscr{C}^\infty(M, \mathfrak{t}_P \otimes \mathfrak{t}_M^*)$ to be invertible.

Definition 3.3.4. A **twist** of M with respect to the torus action T_M, the integral closed 2-form ω and an invertible function f, is the quotient space $\bar{M} := P/\mathring{T}_M$. We say that $\bar{M}$ is a **smooth twist** if $\bar{M}$ is a smooth manifold.

Remark 3.3.5. For torus actions, a twist $\bar{M}$ will at worst be an orbifold under the assumptions above. We are interested in constructing smooth manifolds, and therefore, we only discuss geometric structures in the case of smooth twists.

We then have a double fibration structure on P with projection maps

$$M \xleftarrow{\pi} P \xrightarrow{\bar{\pi}} \bar{M}.$$

Our assumptions imply that both maps are transverse to the distribution $\mathscr{H}$. We use this to relate objects in M and $\bar{M}$ since the projection maps π and $\bar{\pi}$ induce linear isomorphisms $T_{\pi(p)}M \cong \mathscr{H}_p \cong T_{\bar{\pi}(p)}\bar{M}$ for every $p \in P$. This allows us to define the notion of $\mathscr{H}$**-relatedness** between tensors of the same type in M and $\bar{M}$.

Definition 3.3.6. Two tensors α on M and $\bar{\alpha}$ on $\bar{M}$ are said to be $\mathscr{H}$**-related**, written $\alpha \sim_{\mathscr{H}} \bar{\alpha}$, if their pullbacks to P agree on $\mathscr{H}$, that is, $\pi^*\alpha = \bar{\pi}^*\bar{\alpha}$ on $\mathscr{H}$.

Then we say that a tensor field $\bar{\alpha}$ on $\bar{M}$ is the twist of α on M if $\alpha \sim_{\mathscr{H}} \bar{\alpha}$. Moreover, the tensor $\bar{\alpha}$ is uniquely determined by α.

Not every tensor field α on M can be twisted since the T_M-invariance of α is a necessary condition (see [Swa10, Lemma 3.4]). In other words, if $\alpha \sim_{\mathscr{H}} \bar{\alpha}$, then α is T_M-invariant. Conversely, every T_M-invariant tensor field admits a twist, and explicit formulas for the twist of such tensors can be obtained.

Another feature of the twist construction is that it can be inverted in the following sense: if $\bar{M}$ is the twist of M, then M can be obtained from $\bar{M}$ via a twist, thus $\bar{M}$ can be thought as a "dual" of M. Let us see how this works. The distribution $\mathscr{H} = \ker\eta$ on P is transverse to the action of $\mathring{T}_M$. If $\mathring{T}_M$ acts freely on P then we have a principal bundle $\mathring{T}_M \longrightarrow P \longrightarrow \bar{M}$. Its connection 1-form corresponding to $\mathscr{H}$ is $\bar{\eta} = \pi^*(f^{-1})\eta$. This has curvature

$$\bar{\pi}^*\bar{\omega} = \pi^*(f^{-1}\omega) - \pi^*(f^{-1}\mathrm{d}ff^{-1}) \wedge \eta,$$

which is simply the 2-form $\bar{\omega}$ which is $\mathscr{H}$-related to $f^{-1}\omega$. Since T_P commutes with $\mathring{T}_M$, it descends to an action of a torus $T_{\bar{M}}$ on $\bar{M}$ preserving $\bar{\omega}$. Write $\bar{Z} : \mathfrak{t}_{\bar{M}} \longrightarrow \Gamma(T\bar{M})$ for the infinitesimal action of $T_{\bar{M}}$. This action is $\bar{\omega}$-Hamiltonian with $\iota_{\bar{Z}}\bar{\omega} = -\mathrm{d}(f^{-1})$. Then the original manifold M is obtained by twisting $\bar{M}$ with respect to $T_{\bar{M}}$ and $\bar{\omega}$ using f^{-1}.

We have seen so far that the twist construction is a duality between manifolds with torus actions that induces an isomorphism of the respective algebras of invariant tensor fields. Then the twist construction preserves the algebraic properties of T_M-invariant tensor fields, such as algebraic symmetries and non-degeneracy conditions. However, the twist construction does not preserve the differential conditions. In particular, the integrability conditions of geometric structures are lost when we perform the twist. Nevertheless, we can obtain explicit expressions for such differential conditions. For example, if $\alpha \sim_{\mathscr{H}} \bar{\alpha}$ are $\mathscr{H}$-related differential forms, their exterior derivatives are $\mathscr{H}$-related by (see [Swa10, Corollary 3.6])

$$\mathrm{d}\bar{\alpha} \sim_{\mathscr{H}} \mathrm{d}\alpha - f^{-1}\omega \wedge \iota_Z \alpha. \tag{5}$$

If $X \sim_{\mathscr{H}} \bar{X}$ and $Y \sim_{\mathscr{H}} \bar{Y}$ are $\mathscr{H}$-related vector fields, then their Lie brackets are $\mathscr{H}$-related by (see [Swa10, Lemma 3.7])

$$[\bar{X},\bar{Y}] \sim_{\mathscr{H}} [X,Y] + Zf^{-1}\omega(X,Y).$$

The upshot of the twist construction, and one of its great advantages, is that we can study geometric structures and their properties on the twist manifold $\bar{M}$ purely in terms of the data of the original manifold M and the twist data, without even knowing explicitly the twist manifold $\bar{M}$. To exploit this powerful machinery, we first need to find the tensor on M which is $\mathscr{H}$-related to the tensor on $\bar{M}$ that we want to study. We will see that this is the key point for the study of the quaternionic Kähler manifolds considered in this thesis.

Now, particularizing and summarizing the above construction for the case $G = \mathbb{S}^1$, we have that the twist construction takes as input a manifold M equipped with **twist data**, i.e. a triple (ω, Z, f) consisting of

- an integral closed 2-form ω,
- a vector field Z generating a circle action which is ω-Hamiltonian,
- a choice of nowhere vanishing Hamiltonian function f,

and gives as output a new manifold $\bar{M}$ with a circle action. Furthermore, it also gives a bijective correspondence called $\mathscr{H}$-relatedness between tensor fields of the same type on M and $\bar{M}$ which are invariant under the respective circle actions. In particular, if two functions $f \in \mathscr{C}^\infty(M)$ and $\bar{f} \in \mathscr{C}^\infty(\bar{M})$ are invariant under the respective circle actions and $\mathscr{H}$-related, then they are either both constant or both non-constant.

Example 3.3.7 ([MS14, Example 1])**.** A basic example of the twist construction is provided by $M = \mathbb{C}\mathrm{P}^n \times \mathbb{T}^2$. This is a Kähler manifold as a product. Suppose that Z generates one the circle actions of $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Taking ω to be the Fubini-Study 2-form on $\mathbb{C}\mathrm{P}^n$, we have that $\iota_Z\omega = 0$, so we can take $f \equiv 1$. Then $P = \mathbb{S}^{2n+1} \times \mathbb{T}^2$ and the twist is $\bar{M} = \mathbb{S}^{2n+1} \times \mathbb{S}^1$. As ω is of type $(1,1)$ we have that $\bar{M}$ is a complex manifold by [Swa10, Lemma 3.9]. However $\bar{M}$ is compact and $b_2(\bar{M}) = 0$, so $\bar{M}$ cannot be Kähler.

3.3.1 The HK/QK correspondence as a twist

As we have explained above, the twist construction is a powerful method to obtain examples of manifolds equipped with some interesting geometric structures. We can describe the tensors on the twist manifold $\bar{M}$ in terms of the tensor on the original manifold M and the twist data (ω, Z, f). We have seen that, although the algebraic properties of the tensors are preserved, the differential conditions are not.

The twist construction was applied by Macia and Swann [MS15] to the case of hyperkähler and quaternionic Kähler manifolds equipped with circle actions. In fact, they showed that the HK/QK correspondence can be seen as an instance of this procedure.

Recall that to perform the HK/QK correspondence we need a (pseudo-)hyperkähler manifold (M, g, I_1, I_2, I_3) equipped with the HK/QK data $(Z, \omega_1, \omega_{\mathrm{H}}, f_Z^c, f_{\mathrm{H}}^c)$, where the vector field Z that generates the circle action on M is ω_1-Hamiltonian, Killing and rotating, that is

$$\iota_Z\omega_1 = -\mathrm{d}f_Z^c, \quad \mathscr{L}_Z g = 0, \quad \mathscr{L}_Z\omega_1 = 0, \quad \mathscr{L}_Z\omega_2 = \omega_3, \quad \mathscr{L}_Z\omega_3 = -\omega_2.$$

Note that the HK/QK data $(Z, \omega_1, \omega_{\mathrm{H}}, f_Z^c, f_{\mathrm{H}}^c)$ induces the twist data (ω_1, Z, f_Z^c). The vector field Z does not preserve each Kähler form but it preserves the fundamental 4-form $\Omega = \sum_{k=1}^3 \omega_k \wedge \omega_k$ of the hyperkähler structure, so we can twist Ω to an $\mathscr{H}$-related 4-form $\bar{\Omega}$ on the twist manifold $\bar{M}$. This implies that $\bar{M}$ has an almost quaternionic Hermitian structure $(\bar{g}, \mathscr{Q})$. Recall that such structure is quaternionic Kähler if $\bar{\Omega}$ is parallel with respect to the Levi-Civita connection of $\bar{g}$ (see Proposition 2.2.10). If $\dim \bar{M} \geq 12$, to obtain a quaternionic Kähler structure it suffices that $\bar{\Omega}$ is closed (see Theorem 2.2.11). However, since Ω is closed and due to (5), we cannot expect that $\bar{\Omega}$ is also closed. Hence, the twist of a hyperkähler metric will not be a quaternionic Kähler metric.

This problem was solved by Macia and Swann in [MS15] by deforming the almost quaternionic Hermitian structure in a controlled way via the notion of elementary deformations [MS14]. Let Z be the rotating Killing vector field on (M, g, I_1, I_2, I_3). Then we define

$$\alpha_0 := \iota_Z g, \quad \alpha_1 := \iota_Z\omega_1, \quad \alpha_2 := \iota_Z\omega_2, \quad \alpha_3 := \iota_Z\omega_3$$

and

$$g_\alpha := \sum_{k=0}^{3} \alpha_k \otimes \alpha_k.$$

In fact, we have that $g_\alpha = g(Z,Z) g|_{\mathbb{H}Z}$, where $\mathbb{H}Z := \operatorname{span}\{Z, I_1 Z, I_2 Z, I_3 Z\}$ is the distribution generated by the quaternionic span of Z.

Definition 3.3.8. Let (M, g, I_1, I_2, I_3) be a (pseudo-)hyperkähler manifold with a rotating Killing vector field Z. An **elementary deformation** of g with respect to Z is a metric of the form

$$g_{\mathrm{H}} := \phi g + \psi g_\alpha, \tag{6}$$

where $\phi, \psi \in \mathscr{C}^\infty(M)$ are nowhere-vanishing functions.

Using the notion of elementary deformations, Macia and Swann found which are the only possible choices for the nowhere-vanishing functions ϕ and ψ in (6) and twist data (ω, Z, f) on M for which the twist manifold $\bar{M}$ is quaternionic Kähler. In fact, they proved the following.

Theorem 3.3.9 ([MS15, Theorem 4.1])**.** *Let (M, g, I_1, I_2, I_3) be a (pseudo-)hyperkähler manifold equipped with HK/QK data $(Z, \omega_1, \omega_{\mathrm{H}}, f_Z^c, f_{\mathrm{H}}^c)$. Then the quaternionic Kähler manifold $(\bar{M}, \bar{g}^c, \mathscr{Q})$ given by the HK/QK correspondence is obtained by performing the twist construction with respect to the twist data $(\omega_{\mathrm{H}}, Z, f_{\mathrm{H}}^c)$, where*

$$\omega_{\mathrm{H}} := \omega_1 + \mathrm{d}\iota_Z g \quad \textit{and} \quad f_{\mathrm{H}}^c := f_Z^c + g(Z,Z).$$

In particular, $\mathscr{Q}$ is $\mathscr{H}$-related to $\mathrm{span}\{I_1, I_2, I_3\}$ *and $\bar{g}^c$ is $\mathscr{H}$-related to the metric*

$$g_{\mathrm{H}}^c := K\left(\frac{1}{f_Z^c} g|_{(\mathbb{H}Z)^\perp} + \frac{f_{\mathrm{H}}^c}{(f_Z^c)^2} g|_{\mathbb{H}Z}\right), \tag{7}$$

where K is a non-zero constant of the same sign as f_Z^c.

Taking K to have the same sing as f_Z^c gives a quaternionic Kähler metric $\bar{g}^c$ that is positive-definite whenever the given (pseudo-)hyperkähler metric g is positive-definite when restricted to $(\mathbb{H}Z)^\perp$. The reduced scalar curvature of $\bar{g}^c$ is then given by $\nu = -\frac{1}{8K}$. Thus, the sign of f_Z^c determines the sign of the scalar curvature (they are opposite) while the choice of the constant K determines its magnitude. For our purposes, f_Z^c is taken to be positive, so we may set $K = 1$. This gives us a positive-definite quaternionic Kähler metric of reduced scalar curvature $-\frac{1}{8}$.

Note that there is a freedom of adding a constant $c \in \mathbb{R}$ to the Hamiltonian function f_Z^c. Since the metric g_{H}^c in (7) depends on f_Z^c, it also depends on the constant c and therefore also the $\mathscr{H}$-related metric $\bar{g}^c$ on $\bar{M}$. This leads to a one-parameter family of quaternionic Kähler metrics. We will see that the geometry of $\bar{M}$ has really different properties depending on the choice of the parameter c, in particular whether is zero or non-zero.

From Theorem 3.3.9 it also follows that the constructions provided in [Hay08, Hit13, ACM13] of quaternionic Kähler metrics from hyperkähler metrics with a rotating circle symmetry agree.

Remark 3.3.10. It is also possible to twist (pseudo-)hyperkähler manifolds such that the twist manifold is again (pseudo-)hyperkähler. This was done in [Swa16, Theorem 5.1 and Theorem 6.1], where in this case the vector field Z is a tri-holomorphic isometry of the original (pseudo-)hyperkähler metric.

3.3.2 Curvature under the HK/QK correspondence

We have seen above that the HK/QK correspondence can be recovered using the twist construction (see Theorem 3.3.9). This allows us to use this powerful machinery to study the properties of the quaternionic Kähler metrics we are interested in. In particular, the twist construction was used in [CST22] to obtain a tensor $\mathrm{Rm}^c_{\mathrm{H}} \in \Gamma((T^*M)^{\otimes 4})$ on the (pseudo-)hyperkähler manifold (M,g,I_1,I_2,I_3) which is $\mathscr{H}$-related to the (lowered) Riemann curvature tensor $\mathrm{Rm}^c_{\bar{M}} \in \Gamma((T^*\bar{M})^{\otimes 4})$ of the quaternionic Kähler metric $\bar{g}^c$. In order to state this result, we need to introduce some notation.

We define the Kulkarni-Nomizu map

$$\Gamma((T^*M)^{\otimes 4}) \longrightarrow \Gamma(\Lambda^2 T^*M \otimes \Lambda^2 T^*M), \quad \Phi \longmapsto \Phi^{\oslash}$$

by setting

$$\Phi^{\oslash}(A,B,C,X) := \Phi(A,C,B,X) - \Phi(A,X,B,C) + \Phi(B,X,A,C) - \Phi(B,C,A,X)$$

for arbitrary vector fields A,B,C,X on M.

We define a second map

$$\Gamma(\Lambda^2 T^*M \otimes \Lambda^2 T^*M) \longrightarrow \Gamma(\Lambda^2 T^*M \otimes \Lambda^2 T^*M), \quad \Phi \longmapsto \Phi^{\varobar}$$

by setting

$$\Phi^{\varobar}(A,B,C,X) := \Phi^{\oslash}(A,B,C,X) + 2\Phi(A,B,C,X) + 2\Phi(C,X,A,B).$$

For $(0,2)$-tensors α and β, we set

$$\alpha \oslash \beta := (\alpha \otimes \beta)^{\oslash}$$

and analogously we define $\alpha \varobar \beta$. Taking α and β symmetric one recovers the well-known Kulkarni-Nomizu product $\alpha \oslash \beta = \beta \oslash \alpha$, which is an abstract curvature tensor, i.e. a $(0,4)$-tensor with the symmetries of the (lowered) Riemann curvature tensor. Taking α and β skew-symmetric, $\alpha \varobar \beta = \beta \varobar \alpha$ is precisely six times the natural projection of the tensor $\frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha) \in \Gamma(\mathrm{Sym}^2 \Lambda^2 T^*M)$ to the subspace consisting of abstract curvature tensors.

Now we can state the result that shows which tensor is $\mathscr{H}$-related with the Riemann curvature tensor of the quaternionic Kähler metric.

Theorem 3.3.11 ([CST22, Theorem 3.4]). *Let (M,g,I_1,I_2,I_3) be a (pseudo-)hyperkähler manifold equipped with HK/QK data $(Z,\omega_1,\omega_{\mathrm{H}},f^c_Z,f^c_{\mathrm{H}})$ and let $(\bar{M},\bar{g}^c,\mathscr{Q})$ be the quaternionic Kähler manifold given by the HK/QK correspondence. Then the (lowered) Riemann curvature $\mathrm{Rm}^c_{\bar{M}}$ of the metric $\bar{g}^c$ is $\mathscr{H}$-related to the tensor*

$$\mathrm{Rm}^c_{\mathrm{H}} := \frac{1}{f^c_Z}\mathrm{Rm}_M - \frac{1}{f^c_Z f^c_{\mathrm{H}}}\mathrm{Rm}_{\mathrm{HK}} - \frac{1}{8}\mathrm{Rm}_{\mathbb{HP}}, \tag{8}$$

where $\mathrm{Rm}_{\mathrm{HK}}$ and $\mathrm{Rm}_{\mathbb{HP}}$ are defined by

$$\mathrm{Rm}_{\mathrm{HK}} := \tfrac{1}{8}\omega_{\mathrm{H}} \varobar \omega_{\mathrm{H}} + \tfrac{1}{8}\sum_{k=1}^{3} \omega_{\mathrm{H}}(I_k\cdot,\cdot) \oslash \omega_{\mathrm{H}}(I_k\cdot,\cdot),$$

$$\mathrm{Rm}_{\mathbb{HP}} := -g^c_{\mathrm{H}} \oslash g^c_{\mathrm{H}} - \sum_{k=1}^{3} g^c_{\mathrm{H}}(I_k\cdot,\cdot) \varobar g^c_{\mathrm{H}}(I_k\cdot,\cdot).$$

Note that (8) reflects a refinement of the Alekseevsky decomposition of the curvature tensor of a quaternionic Kähler metric of reduced scalar curvature $-\frac{1}{8}$ arising from the HK/QK correspondence (compare with Theorem 2.2.16). The first two terms on the right correspond to the part of hyperkähler type, while the last term corresponds to $-\frac{1}{8}$ times the (formal) curvature tensor of the quaternionic projective space of unit reduced scalar curvature. In particular, we have that both Rm_M and $\mathrm{Rm}_{\mathrm{HK}}$ are separately g^c_{H}-orthogonal to $\mathrm{Rm}_{\mathbb{HP}}$.

As an application of Theorem 3.3.11, we will proof in Section 5.2 that the norm of the curvature tensor $\mathrm{Rm}^c_{\bar{M}}$ of the metric $\bar{g}^c$ on the quaternionic Kähler side is not constant if the norm of $\mathrm{Rm}^c_{\mathrm{H}}$ on the pseudo-hyperkähler side is not constant. We indeed proceed by specializing this argument to the case of the deformed supergravity c-map.

Chapter 4

The supergravity c-map

In this last preliminary chapter we introduce the quaternionic Kähler manifolds we are interested in this thesis, the so-called supergravity c-map spaces. In Section 4.1 we introduce (affine and projective) special Kähler manifolds, which are the starting point of the supergravity c-map construction. We study basic properties of them, introduce some examples and explain a way to construct them using locally defined holomorphic functions. Moreover, we determine the curvature tensor of any affine special Kähler manifold. In Section 4.2 we explain that the total space of the cotangent bundle of any affine special Kähler manifold has a canonical hyperkähler structure. These hyperkähler manifolds are known as rigid c-map spaces and in some cases they are equipped with a rotating Killing vector field generating a rotating circle action, so we can perform the HK/QK correspondence. Finally, in Section 4.3 we explain the supergravity c-map construction. This gives us a one-parameter family of quaternionic Kähler metrics starting with a projective special Kähler manifold. We also describe the one-parameter supergravity c-map metric in local coordinates and study some properties of it. Furthermore, we explain that it is possible to obtain a subclass of supergravity c-map spaces, known as supergravity q-map spaces, starting with cubic homogeneous polynomials. As in the previous chapters, all what is presented here is well-known and not original to this thesis. The references for each result will be properly cited.

4.1 Special Kähler geometry

Our final goal is to construct (complete) quaternionic Kähler manifolds with negative scalar curvature. We have seen in the previous chapter that a way to do this is by using hyperkähler manifolds of indefinite signature equipped with a rotating circle action. Not every hyperkähler manifold admits such circle action, therefore we are interested in construct hyperkähler manifolds in which we can perform the HK/QK correspondence. A way to do this is by using special Kähler manifolds. Special Kähler geometry appeared in the physics literature in global supersymmetry and supergravity [dWVP84], and it was mathematically formulated by Freed [Fre99b]. Here we recall some definitions and well-known properties about special Kähler manifolds. Some other references are [ACD02, CM09].

Definition 4.1.1. Let (M, g, J) be a (pseudo-)Kähler manifold and let $\omega := g(J\cdot, \cdot)$ be the Kähler form. We say that it is an **affine special Kähler (ASK) manifold** if it is equipped with a flat torsion-free connection ∇ such that $\nabla\omega = 0$ and $\mathrm{d}^\nabla J = 0$.

Lemma 4.1.2. *Let (M, g, J, ∇) be an ASK manifold. The condition $\mathrm{d}^\nabla J = 0$ is equivalent to*

$$(\nabla_X J)Y = (\nabla_Y J)X$$

for all $X, Y \in \Gamma(TM)$.

Proof. Let $X, Y \in \Gamma(TM)$. Then

$$\begin{aligned}(\mathrm{d}^\nabla J)(X,Y) &= \nabla_X(JY) - \nabla_Y(JX) - J([X,Y]) \\ &= (\nabla_X J)Y + J(\nabla_X Y) - (\nabla_Y J)X - J(\nabla_Y X) - J(\nabla_X Y - \nabla_Y X) \\ &= (\nabla_X J)Y - (\nabla_Y J)X,\end{aligned}$$

where we have used that ∇ is torsion-free. □

Note that ∇ is not a metric connection, otherwise it would be the Levi-Civita connection of g, since it is torsion-free. Nevertheless, the tensor ∇g satisfies the following property.

Lemma 4.1.3. *Let (M, g, J, ∇) be an ASK manifold. Then ∇g is totally symmetric.*

Proof. It is enough to show that $(\nabla_X g)(Y, Z) = (\nabla_Z g)(Y, X)$ for $X, Y, Z \in \Gamma(TM)$. Using $g = \omega(\cdot, J\cdot)$ we get

$$(\nabla_X g)(Y,Z) = (\nabla_X \omega)(Y, JZ) + \omega(Y, (\nabla_X J)Z).$$

Now, using the properties of being affine special Kähler, we get the claimed result. □

We are interested in a more particular kind of ASK manifolds.

Definition 4.1.4. A **conical affine special Kähler (CASK) manifold** (M, g, J, ∇, ξ) is an ASK manifold (M, g, J, ∇) endowed with a vector field ξ, called the **Euler vector field**, such that:

- g is negative-definite on $\mathrm{span}\{\xi, J\xi\}$ and positive-definite on its orthogonal complement.
- $D\xi = \nabla\xi = \mathrm{Id}$, where D denotes the Levi-Civita connection of g.

Moreover, M is endowed with a principal $\mathbb{C}^*$-action generated by ξ and $J\xi$.

We collect in the following proposition some well-known facts about the interaction between the vector fields ξ and $J\xi$ and the affine special Kähler structure (see e.g. [CM09, CHM12]).

Proposition 4.1.5. *Let (M, g, J, ∇, ξ) be a CASK manifold. Then:*

(a) *The vector field ξ is holomorphic and homothetic, i.e. $\mathscr{L}_\xi g = 2g$.*

(b) *The vector field $J\xi$ is holomorphic and Killing.*

(c) *The function $f = \frac{1}{2}g(\xi, \xi)$ is a Kähler potential for g and a ω-Hamiltonian function for $J\xi$, i.e. $\mathrm{d}f = -\iota_{J\xi}\omega = -\omega(J\xi, \cdot)$.*

Proof. (a) For all $X \in \Gamma(TM)$ we have

$$(\mathscr{L}_\xi J)(X) = \mathscr{L}_\xi(JX) - J(\mathscr{L}_\xi X) = D_\xi(JX) - D_{JX}\xi - J(D_\xi X) + J(D_X\xi) = 0,$$

where we have used that D is torsion-free, $DJ = 0$ and $D\xi = \mathrm{Id}$. Hence $\mathscr{L}_\xi J = 0$, i.e. ξ is holomorphic. To see that ξ is homothetic, let $X, Y \in \Gamma(TM)$. Using also $Dg = 0$, we obtain

$$(\mathscr{L}_\xi g)(X, Y) = g(D_X\xi, Y) + g(X, D_Y\xi) = 2g(X, Y).$$

(b) The proof of $\mathscr{L}_{J\xi}J = 0$ is the same as in (a). The following shows that $J\xi$ is Killing:

$$(\mathscr{L}_{J\xi}g)(X, Y) = g(D_X J\xi, Y) + g(X, D_Y J\xi) = g(JX, Y) + g(X, JY) = 0,$$

which is zero since J is skew-symmetric.

(c) For $f = \frac{1}{2}g(\xi, \xi)$, we have $\mathrm{d}f = g(\xi, \cdot) = -\omega(J\xi, \cdot)$, which means that f is a ω-Hamiltonian function for $J\xi$. Now $\mathrm{d}\mathrm{d}^c f = -\mathrm{d}J^*\mathrm{d}f = \mathrm{d}(\omega(\xi, \cdot))$. Using that ∇ is torsion-free, $\nabla\omega = 0$ and $\nabla\xi = \mathrm{Id}$, we obtain

$$\mathrm{d}(\omega(\xi, \cdot))(X, Y) = \omega(\nabla_X\xi, Y) - \omega(\nabla_Y\xi, X) = 2\omega(X, Y).$$

Thus $\omega = \frac{1}{2}\mathrm{d}\mathrm{d}^c f$, which means that f is a Kähler potential for ω. $\square$

Corollary 4.1.6. *Let (M, g, J, ∇, ξ) be a CASK manifold. Then $D(J\xi) = \nabla(J\xi) = J$.*

Proof. From $DJ = 0$ and $D\xi = \mathrm{Id}$ follows $D(J\xi) = J$. The condition $\nabla(J\xi) = J$ is equivalent to $\nabla_\xi J = 0$ since

$$\nabla_X(J\xi) = (\nabla_X J)\xi + J(\nabla_X\xi) = (\nabla_\xi J)X + JX$$

for all $X \in \Gamma(TM)$. Moreover, ∇ is torsion-free and $\nabla\xi = \mathrm{Id}$, thus

$$(\mathscr{L}_\xi J)X = \mathscr{L}_\xi(JX) - J(\mathscr{L}_\xi X) = \nabla_\xi(JX) - \nabla_{JX}\xi - J(\nabla_\xi X) + J(\nabla_X\xi) = (\nabla_\xi J)X,$$

which is zero since ξ is holomorphic by Proposition 4.1.5 (a). $\square$

The $\mathbb{C}^*$-action on a CASK manifold casts it as the total space of a principal $\mathbb{C}^*$-bundle. The base of this bundle is what we call a projective special Kähler manifold.

Definition 4.1.7. Given a CASK manifold (M, g, J, ∇, ξ), the manifold $\bar{M} := M/\mathbb{C}^*$ inherits a (positive-definite) Kähler metric $\bar{g}$ and $(\bar{M}, \bar{g})$ is called a **projective special Kähler (PSK) manifold**.

The metric $\bar{g}$ on $\bar{M}$ is obtained by Kähler reduction exploiting the fact that $J\xi$ is a Hamiltonian Killing vector field. Indeed,

$$\bar{M} = M/\!/\mathbb{S}^1 = f^{-1}(-\tfrac{1}{2})/\mathbb{S}^1 \tag{9}$$

is the Kähler quotient of the CASK manifold M by the $\mathbb{S}^1$-action generated by $J\xi$.

Remark 4.1.8. In [Man21] an intrinsic characterization of PSK manifolds is presented. Here the projective special Kähler structure is reduced to data solely defined on the manifold itself. In particular, this characterization is obtained by means of a locally defined symmetric tensor called deviance, satisfying certain conditions: a differential one and an algebraic one. The deviance tensor emerges from the difference between two naturally occurring connections on the CASK manifold over the projective special Kähler one.

Remark 4.1.9. A PSK manifold $\bar{M}$ is in particular a **Hodge manifold**, that is, a Kähler manifold equipped with a holomorphic Hermitian line bundle $L \longrightarrow \bar{M}$ with curvature $-2\pi i\bar{\omega}$, where $\bar{\omega}$ is the Kähler form of $\bar{M}$. This implies that $[\bar{\omega}] \in H^2(\bar{M}, \mathbb{R})$ is an integral class. By removing the zero section of L we obtain a $\mathbb{C}^*$-bundle $M \longrightarrow \bar{M}$ whose total space M has the structure of a CASK manifold (see [Fre99b, Section 4]).

Following [CST21] we introduce the natural notion of symmetry in this setting.

Definition 4.1.10.

- An **automorphism** of a CASK manifold (M, g, J, ∇, ξ) is a diffeomorphism of M which preserves g, J, ∇ and ξ.
- An **automorphism** of a PSK manifold $\bar{M} = M/\!/\mathbb{S}^1$ is a diffeomorphism of $\bar{M}$ induced by an automorphism of the CASK manifold M.

The corresponding groups of automorphisms are denoted by $\mathrm{Aut}(M)$ and $\mathrm{Aut}(\bar{M})$, respectively. At the infinitesimal level we have the following notion.

Definition 4.1.11.

- An **infinitesimal automorphism** of a CASK manifold (M, g, J, ∇, ξ) is a vector field $X \in \Gamma(TM)$ such that its local flow preserves the CASK data on M. The Lie algebra of such vector fields is denoted by $\mathfrak{aut}(M)$.
- An **infinitesimal automorphism** of a PSK manifold is a vector field $\bar{X}$ induced by an infinitesimal automorphism of the corresponding CASK manifold (which always projects since it commutes with ξ and $J\xi$). The corresponding Lie algebra is denoted by $\mathfrak{aut}(\bar{M})$.

It was shown in [CST21, Proposition 2.18] that $\mathfrak{aut}(M)$ and $\mathfrak{aut}(\bar{M})$ are isomorphic when the Levi-Civita connection D and the special connection ∇ are not equal.

4.1.1 Examples and construction of special Kähler manifolds

The very first example of an ASK manifold that one can think of is the trivial one. Indeed, let (M,g,J) be a flat (pseudo-)Kähler manifold, i.e. the Levi-Civita connection D of g is flat. Then $(M,g,J,\nabla = D)$ is an ASK manifold and $\nabla J = 0$. Conversely, any ASK manifold (M,g,J,∇) such that $\nabla J = 0$ satisfies $\nabla = D$. In fact, in the case of positive-definite ASK manifolds, the only complete examples are precisely the flat ones.

Theorem 4.1.12 ([Lu99, Theorem 2])**.** *Let (M,g,J,∇) be a positive-definite ASK manifold. If the metric g is complete, then it is flat.*

Before introducing a construction of special Kähler manifolds, which yields plenty of non-flat examples, we show that these manifolds arise naturally in several interesting contexts. We now explain some of them and refer to the interested reader to [Cor02] and references therein for more details.

Example 4.1.13. Let X be a Calabi-Yau 3-fold, i.e. a compact Kähler manifold of complex dimension 3 with holonomy group $\mathrm{SU}(3)$. Then X has a holomorphic volume form $\mathrm{vol}_X \in H^{3,0}(X)$, unique up to scaling. Such a pair (X, vol_X) is called a gauged Calabi-Yau 3-fold. The **Kuranishi moduli space** S of X is a complex manifold that can be identified with a neighborhood of zero in $H^{2,1}(X)$. Denote by $\mathcal{X} \longrightarrow S$ the corresponding deformation of complex structure and consider the holomorphic line bundle $H^{3,0}(\mathcal{X}) \longrightarrow S$ with fiber $H^{3,0}(X_s)$ at $s \in S$. Denote by $H^{3,0}(\mathcal{X}) \setminus S$ the $\mathbb{C}^*$-bundle over S which is obtained from the complex line bundle $H^{3,0}(\mathcal{X})$ by removing the zero section $S \ni s \longmapsto 0 \in H^{3,0}(X_s)$. We think of it as the **moduli space of gauged Calabi-Yau 3-folds** (X_s, vol_s), $\mathrm{vol}_s \in H^{3,0}(X_s) \setminus \{0\}$, $s \in S$. We now define the period map $\mathrm{Per} : S \longrightarrow \mathbb{P}(H^3(X,\mathbb{C}))$, $s \longmapsto H^{3,0}(X_s)$. It can be shown that the cone

$$\mathscr{M}_X := \bigcup_{s \in S} \mathrm{Per}(s) \setminus \{0\} \subset H^3(X,\mathbb{C})$$

over $\mathrm{Per}(S) \cong S$ is canonically identified with the moduli space $H^{3,0}(\mathcal{X}) \setminus S$ of gauged Calabi-Yau 3-folds. In this setting we have that S is a PSK manifold and $\mathscr{M}_X$ is a CASK manifold of complex signature $(1, h^{2,1}(X))$ (see e.g. [Str90, Cor98]).

Example 4.1.14. Let X be a hyperkähler manifold. Recall that such a manifold X is automatically Kähler of even complex dimension (say $\dim_{\mathbb{C}} X = 2n$) and carries a holomorphic symplectic structure Ω (see Proposition 3.1.3). A complex submanifold $Y \subset X$ of complex dimension n is called **Lagrangian** if $\iota^*\Omega = 0$, where $\iota : Y \hookrightarrow X$ is the inclusion map. Hitchin showed in [Hit99, Theorem 3] that the moduli space of deformations of a compact complex Lagrangian submanifold Y of a hyperkähler manifold X has a naturally induced (positive-definite) ASK structure.

Example 4.1.15. Recall that a Hamiltonian system is a symplectic manifold (M^{2n}, ω) together with a smooth function $h \in \mathscr{C}^\infty(M)$ called Hamiltonian function. It gives rise to the Hamiltonian vector field X_h on M defined by $\mathrm{d}h = -\iota_{X_h}\omega$. A smooth function $f \in \mathscr{C}^\infty(M)$ on a Hamiltonian system is called a first integral if it is constant along the flow generated by X_h or, equivalently, if $\omega(X_h, X_f) = 0$. The Hamiltonian system (M^{2n}, ω, h) is called completely integrable if there is a proper map

$$F = (h = f_1, f_2, \dots, f_n) : M \longrightarrow \mathbb{R}^n$$

such that $\omega(X_{f_j}, X_{f_k}) = 0$ for all $1 \leq j,k \leq n$. An **algebraic completely integrable system** is the adaptation of this concept to holomorphic symplectic manifolds. In this case we consider Hamiltonian systems for a complex-valued Hamiltonian function h. Such a system is called algebraic completely integrable if there exists a function $F : X \longrightarrow \mathbb{C}^n$ with exactly the same properties as before, where X is the holomorphic symplectic manifold of $\dim_{\mathbb{C}} X = 2n$. The generic fibers of F are complex tori of dimension n and Lagrangian submanifolds of X. By a result of Donagi and Witten [DW96] (see also [Fre99b, Theorem 3.4]) the base manifold of an algebraic completely integrable system is an ASK manifold. Conversely, every ASK manifold M gives rise to an algebraic completely integrable system with total space $X := T^*M/\Lambda$, where Λ is a bundle of lattices. In addition, the manifold X comes equipped with a **semi-flat hyperkähler metric**. This means that, with respect to this metric, each fiber of the torus fibration $\pi : X \longrightarrow M$ is flat.

Some other examples of special Kähler manifolds, in particular of PSK manifolds, can be found in the homogeneous setting. The simplest (non-trivial) example is the complex hyperbolic space $\mathbb{C}\mathrm{H}^n$. Indeed, consider $\mathbb{C}^{n+1}$ equipped with the pseudo-Kähler structure determined by the Hermitian form $h := -\mathrm{d}z^0 \otimes \mathrm{d}\bar{z}^0 + \sum_{j=1}^n \mathrm{d}z^j \otimes \mathrm{d}\bar{z}^j$ and let $\xi := \sum_{j=0}^n (z^j \frac{\partial}{\partial z^j} + \bar{z}^j \frac{\partial}{\partial \bar{z}^j})$ be the standard Euler vector field on $\mathbb{C}^{n+1}$. Then the open subset

$$M := \{z \in \mathbb{C}^{n+1} \mid h(z,z) < 0\},$$

equipped with the above structure, is a CASK manifold. By taking the Kähler quotient by the natural $\mathbb{S}^1$-action of the unit complex numbers on $\mathbb{C}^{n+1}$ we obtain $\bar{M} = \mathbb{C}\mathrm{H}^n$, which is therefore a PSK manifold.

There are classifications of homogeneous PSK manifolds under various assumptions. For instance, homogeneous PSK manifolds associated to homogeneous real affine cubic hypersurfaces were completely classified in [dWVP92] and [Cor96b]; and [AC00] includes the classification of all homogeneous PSK manifolds of a real semisimple group.

We now present the construction of special Kähler manifolds systematically developed in [ACD02]. In fact, in this paper the authors extend the notion of special Kähler manifolds to the non-metric realm by introducing special complex and special symplectic manifolds. Nevertheless we are only interested in the special Kähler case. The following discussion in based on this work.

First of all, a flat torsion-free connection ∇ on a manifold M defines on it an affine structure, i.e. an atlas with affine transition functions. A function f on (M, ∇) is called affine if $\nabla \mathrm{d} f = 0$. A local coordinate system $(x^1, \ldots, x^n)$ on M, $n = \dim M$, is called affine if the x^j are affine functions. Any affine local coordinate system $(x^1, \ldots, x^n)$ defines a parallel local coframe $(\mathrm{d}x^1, \ldots, \mathrm{d}x^n)$. Conversely, since any parallel 1-form α is locally the differential of an affine function f, given a parallel coframe $(\alpha^1, \ldots, \alpha^n)$ defined on a simply connected domain $U \subset M$ there exist affine functions x^j on U such that $\mathrm{d}x^j = \alpha^j$. The tuple $(x^1, \ldots, x^n)$ defines an affine local coordinate system near each point in U, which is unique up to translations in $\mathbb{R}^n$.

Definition 4.1.16. Let (M, g, J, ∇) be an ASK manifold. A ∇-affine local coordinate system $\{x^j, y_j\}_{j=1}^n$ on M is called a **real special coordinate system** if

$$\omega := g(J\cdot, \cdot) = 2 \sum_{j=1}^{n} \mathrm{d}x^j \wedge \mathrm{d}y_j.$$

A **conjugate pair of special coordinates** is a pair of holomorphic local coordinates $\{z^j, w_j\}$ such that $\{x^j := \mathrm{Re}(z^j), y_j := \mathrm{Re}(w_j)\}$ is a real special coordinate system.

Theorem 4.1.17 ([ACD02, Theorem 1])**.** *Let (M, g, J, ∇) be an ASK manifold. Then M admits a real special coordinate system near any point, unique up to affine symplectic transformations. Moreover, near any point of M there exists a real special coordinate system admitting a holomorphic extension to a conjugate pair of special coordinates, i.e. there exist holomorphic functions z^j and w_j with $\mathrm{Re}(z^j) = x^j$ and $\mathrm{Re}(w_j) = y_j$.*

Hence, given an ASK manifold, we have a canonical set of real local ∇-affine coordinates, which we will denote by

$$\{q^j\}_{j=1}^{2n} := \{x^j, y_j\}_{j=1}^n.$$

The construction realizes simply connected ASK manifolds as immersed complex submanifolds of $T^*\mathbb{C}^n$, as we explain next. Let $V := T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$ be a complex symplectic vector space with canonical coordinates $\{z^j, w_j\}$ and standard complex symplectic form $\Omega = \sum_{j=1}^n \mathrm{d}z^j \wedge \mathrm{d}w_j$. Let $\tau : V \longrightarrow V$ be the standard real structure with fixed point set $V^\tau = T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$. Then $\gamma := i\Omega(\cdot, \tau\cdot)$ defines a Hermitian form on V of complex signature (n, n).

Let (M, J) be a complex manifold of complex dimension n.

Definition 4.1.18. A holomorphic immersion $\phi : M \longrightarrow V$ is called **Lagrangian** if $\phi^*\Omega = 0$ and it is called **non-degenerate** if $\phi^*\gamma$ is non-degenerate.

A Lagrangian non-degenerate immersion $\phi : M \longrightarrow V$ induces on M the following data:

- Local coordinates $x^j := \mathrm{Re}(\phi^* z^j)$ and $y_j := \mathrm{Re}(\phi^* w_j)$.
- A flat torsion-free connection ∇ defined by the condition $\nabla \mathrm{d}x^j = \nabla \mathrm{d}y_j = 0$ for $j = 1, \ldots, n$.

- A pseudo-Riemannian metric $g := \mathrm{Re}(\phi^*\gamma)$ such that (M, g, J) is a pseudo-Kähler manifold. Moreover, the Kähler form ω of (M, g, J) is given in the local ∇-affine coordinates $\{x^j, y_j\}$ by $\omega = 2\sum_{j=1}^{n} \mathrm{d}x^j \wedge \mathrm{d}y_j$.

Theorem 4.1.19 ([ACD02, Theorem 3]). *Let $\phi : M \longrightarrow V$ be a non-degenerate Lagrangian immersion with induced geometric data (g, ∇). Then (M, g, J, ∇) is an ASK manifold.*

In fact, any simply connected ASK manifold is obtained from Theorem 4.1.19.

Theorem 4.1.20 ([ACD02, Theorem 4]). *Let (M, g, J, ∇) be a simply connected ASK manifold. Then it admits a non-degenerate Lagrangian immersion $\phi : M \longrightarrow V = T^*\mathbb{C}^n$ inducing the data (g, ∇) on M. Such ϕ is unique up to affine transformations of V with linear part in $\mathrm{Sp}(\mathbb{R}^{2n})$.*

For the general case (where M is not necessarily simply connected), this gives us a local characterization of ASK manifolds. The important advantage of this characterization in terms of non-degenerate Lagrangian immersions lies in the fact that Lagrangian immersions are locally defined by a generating function. More precisely, let $U \subset \mathbb{C}^n$ be an open subset. We say that a 1-form $\sum_{j=1}^{n} F_j(z)\mathrm{d}z^j$ on U is regular if the real matrix $\mathrm{Im}(\partial F_j/\partial z^k)$ is invertible. A holomorphic function F on U is called non-degenerate if its differential $\mathrm{d}F$ is a regular holomorphic 1-form, thus a function F on U is non-degenerate if the matrix

$$\mathrm{Im}\left(\frac{\partial^2 F}{\partial z^j \partial z^k}\right)$$

in invertible. Any holomorphic 1-form ϕ on a domain $U \subset \mathbb{C}^n$ can be considered as a holomorphic immersion $\phi : U \longrightarrow V = T^*\mathbb{C}^n$, and ϕ is Lagrangian if and only if it is closed. Then we get the following.

Corollary 4.1.21 ([ACD02, Corollary 4]). *Any non-degenerate local holomorphic function on $\mathbb{C}^n$ defines an ASK manifold of complex dimension n. Conversely, any ASK manifold of complex dimension n can be locally obtained in this way.*

In view of Corollary 4.1.21, we can then define an **ASK domain** as a connected open subset $U \subset \mathbb{C}^n$ together with a non-degenerate holomorphic function F defining an ASK structure on U via the real special coordinates $x^j := \mathrm{Re}(z^j)$ and $y_j := \mathrm{Re}(\frac{\partial F}{\partial z^j})$. Such a function is called a **holomorphic prepotential**. Hence, we can rephrase the above discussion as that every ASK manifold is locally an ASK domain.

If we specialize to CASK manifolds, we further need the conjugate pair of special coordinates $\{z^j, w_j\}$ to be chosen in such a way that the $\mathbb{C}^*$-action generated by the vector fields ξ and $J\xi$ acts on them by complex multiplication [ACD02, Theorem 5]. Then these coordinates are called **conical conjugate pairs of special coordinates** and the corresponding real coordinates, $x^j := \mathrm{Re}(z^j)$ and $y_j := \mathrm{Re}(w_j)$, are simply called

conical special real coordinates. Note that in these coordinates the Euler vector field ξ takes the form

$$\xi = \sum_{j=1}^{n} \left(x^j \frac{\partial}{\partial x^j} + y_j \frac{\partial}{\partial y_j} \right) = \sum_{j=1}^{2n} q^j \frac{\partial}{\partial q^j}.$$

To get an analogous result of Theorem 4.1.19 for CASK manifolds, we need the holomorphic immersion $\phi : M \longrightarrow V = T^*\mathbb{C}^n$ being conical, that is, for every point $x \in M$ and every neighborhood U of x there exist neighborhoods U_1 of $1 \in \mathbb{C}^*$ and U_x of x such that $\lambda\phi(U_x) \subset \phi(U)$ for all $\lambda \in U_1$. Notice that we do not require the image $\phi(M)$ to be a complex cone, i.e. (globally) invariant under the $\mathbb{C}^*$-action on V.

Theorem 4.1.22 ([ACD02, Theorem 8]). *Let $\phi : M \longrightarrow V$ be a conical non-degenerate Lagrangian immersion with induced geometric data (g, ∇, ξ). Then (M, g, J, ∇, ξ) is a CASK manifold.*

We also have that locally any CASK manifold is obtained in this way.

Theorem 4.1.23 ([ACD02, Theorem 9]). *Let (M, g, J, ∇, ξ) be a simply connected CASK manifold. Then it admits a conical non-degenerate Lagrangian immersion $\phi : M \longrightarrow V = T^*\mathbb{C}^n$ inducing the data (g, ∇, ξ) on M. Such ϕ is unique up to linear transformations of V in* $\mathrm{Sp}(\mathbb{R}^{2n})$.

We observe that in this case, to get a analogous result as Corollary 4.1.21, we need the non-degenerate function F on the open subset $U \subset \mathbb{C}^n$ being conical, that is, F is locally homogeneous of degree 2, i.e. $F(\lambda z) = \lambda^2 F(z)$ for all $z \in U$ and all λ near $1 \in \mathbb{C}^*$. Then we have:

Corollary 4.1.24 ([ACD02, Corollary 6]). *Any conical non-degenerate local holomorphic function on $\mathbb{C}^n$ defines a CASK manifold of complex dimension n. Conversely, any CASK manifold of complex dimension n can be locally obtained in this way.*

Similarly as before, we now define a **CASK domain** as a $\mathbb{C}^*$-invariant connected open subset $U \subset \mathbb{C}^n$ together with a conical non-degenerate holomorphic function F, i.e. a locally homogeneous function of degree 2, defining a CASK structure on U. As before, every CASK manifold is locally a CASK domain.

As a final remark, notice that the local structure of CASK manifolds is reflected in the local structure of PSK manifolds. Indeed, let M be a simply connected CASK manifold and let $\bar{M} = M/\mathbb{C}^*$ be the corresponding PSK manifold. Then the holomorphic immersion $\phi : M \longrightarrow V = T^*\mathbb{C}^n$ of Theorem 4.1.23 induces a holomorphic immersion $\bar{\phi} : \bar{M} \longrightarrow \mathbb{P}(V) \cong \mathbb{C}\mathrm{P}^{2n-1}$. Thus PSK manifolds arise locally as open subsets of complex projective spaces. In other words, a PSK manifold is locally a **PSK domain**, that is $\bar{U} := U/\mathbb{C}^* \subset \mathbb{C}\mathrm{P}^{n-1}$, where $U \subset \mathbb{C}^n$ is a CASK domain.

4.1.2 Curvature of ASK manifolds

Further properties of special Kähler manifolds can be studied. In particular, we can determine the curvature tensor of any ASK manifold. Here we use the Einstein summation convention when working on local coordinates.

First of all, we compute the Christoffel symbols of the Levi-Civita connection D of the ASK metric g in local ∇-affine coordinates $\{q^j\}$, whose existence is guaranteed by Theorem 4.1.17.

Lemma 4.1.25. *Let (M,g,J,∇) be an ASK manifold. Then the Christoffel symbols of D in ∇-affine coordinates are given by*

$$\Gamma_{ijk} = \tfrac{1}{2}\partial_i g_{jk}.$$

Proof. Let $\{q^i\}$ be a set of local ∇-affine coordinates on M. This implies that $\nabla\partial_i = 0$ for all i, where $\partial_i := \frac{\partial}{\partial q^i}$. Then

$$(\nabla_{\partial_j} g)(\partial_i,\partial_k) = \partial_j g(\partial_i,\partial_k) - g(\nabla_{\partial_j}\partial_i,\partial_k) - g(\partial_i,\nabla_{\partial_j}\partial_k) = \partial_j g_{ik}.$$

Since $(\nabla_{\partial_j} g)(\partial_i,\partial_k) = (\nabla_{\partial_k} g)(\partial_i,\partial_j) = \partial_k g_{ij}$ by Lemma 4.1.3, the Christoffel symbols are given by

$$\Gamma_{ijk} = \tfrac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) = \tfrac{1}{2}\partial_i g_{jk}.$$

□

Let (M,g,J,∇) be an ASK manifold. We define the tensor field

$$\mathcal{S} := g^{-1}\nabla g \in \Gamma(T^*M \otimes \mathrm{End}(TM)).$$

In ∇-affine coordinates $\{q^i\}$, this is given by

$$\mathcal{S}^k_{ij} = g^{km}(\nabla g)_{ijm} = 2g^{km}\Gamma_{ijm} = 2\Gamma^k_{ij}. \tag{10}$$

The following three results are well-known in special Kähler geometry (see e.g. [Fre99b, ACD02]).

Lemma 4.1.26. *Let (M,g,J,∇) be an ASK manifold. Then the tensor $\mathcal{S}$ satisfies the following properties:*

(a) $g(\mathcal{S}_X Y,Z) = g(\mathcal{S}_X Z,Y)$,

(b) $\mathcal{S}_X Y = \mathcal{S}_Y X$,

for all vector field $X,Y,Z \in \Gamma(TM)$.

Proof. Since $\mathcal{S} = g^{-1}\nabla g$, we have that $g(\mathcal{S}_X Y,Z) = (\nabla_X g)(Y,Z)$. Since $\nabla\omega = 0$ and $g = \omega(\cdot,J\cdot)$, we furthermore have $g(\mathcal{S}_X Y,Z) = \omega(Y,(\nabla_X J)Z)$ (see also the proof of Lemma 4.1.3).

Part (a) now follows from the fact that the metric g is symmetric. Part (b) follows from part (a), the condition $(\nabla_X J)Z - (\nabla_Z J)X = (\mathrm{d}^\nabla J)(X,Z) = 0$ and the fact that g is non-degenerate. □

Proposition 4.1.27. *Let (M,g,J,∇) be an ASK manifold. Then $D-\nabla = \frac{1}{2}\mathcal{S}$, where D is the Levi-Civita connection of g.*

Proof. Let $\tilde{D} := \nabla + \frac{1}{2}\mathcal{S}$. As the Levi-Civita connection is the unique torsion-free connection preserving the metric g, the result will follow if we can show that $\tilde{D}$ is torsion-free and metric.

Since ∇ is torsion-free and $\mathcal{S}_X Y = \mathcal{S}_Y X$, it follows that $\tilde{D}$ is also torsion-free. Now let us check that $\tilde{D}$ is metric. For $X,Y,Z \in \Gamma(TM)$ we have

$$\begin{aligned}(\tilde{D}_X g)(Y,Z) &= Xg(Y,Z) - g(\tilde{D}_X Y,Z) - g(Y,\tilde{D}_X Z)\\ &= Xg(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z) - \tfrac{1}{2}g(\mathcal{S}_X Y,Z) - \tfrac{1}{2}g(Y,\mathcal{S}_X Z)\\ &= (\nabla_X g)(Y,Z) - \tfrac{1}{2}(\nabla_X g)(Y,Z) - \tfrac{1}{2}(\nabla_X g)(Z,Y) = 0,\end{aligned}$$

where in the last step we have used that ∇g is totally symmetric. □

Specializing to CASK manifolds we furthermore have:

Corollary 4.1.28. *Let (M,g,J,∇,ξ) be a CASK manifold. Then:*

(a) $\mathcal{S}_\xi X = \mathcal{S}_X \xi = 0$,

(b) $\mathcal{S}_{J\xi} X = \mathcal{S}_X J\xi = 0$,

for all $X \in \Gamma(TM)$.

Proof. By Proposition 4.1.27 we have $\frac{1}{2}\mathcal{S} = D-\nabla$. Part (a) follows from $D\xi = \nabla\xi = \mathrm{Id}$ and part (b) from Corollary 4.1.6. □

We now proceed to compute the curvature of an ASK manifold.

Proposition 4.1.29. *Let (M,g,J,∇) be an ASK manifold. Then the curvature R of the Levi-Civita connection D is*

$$R(X,Y) = -\tfrac{1}{4}[\mathcal{S}_X,\mathcal{S}_Y]. \tag{11}$$

Proof. Since $D = \nabla + \frac{1}{2}\mathcal{S}$ by Proposition 4.1.27, the curvature is

$$\begin{aligned}R(X,Y) &= [D_X,D_Y] - D_{[X,Y]}\\ &= [\nabla_X,\nabla_Y] - \nabla_{[X,Y]} + \tfrac{1}{2}[\nabla_X,\mathcal{S}_Y] - \tfrac{1}{2}[\nabla_Y,\mathcal{S}_X] - \tfrac{1}{2}\mathcal{S}_{[X,Y]} + \tfrac{1}{4}[\mathcal{S}_X,\mathcal{S}_Y]\\ &= \tfrac{1}{2}[\nabla_X,\mathcal{S}_Y] - \tfrac{1}{2}[\nabla_Y,\mathcal{S}_X] - \tfrac{1}{2}\mathcal{S}_{[X,Y]} + \tfrac{1}{4}[\mathcal{S}_X,\mathcal{S}_Y],\end{aligned}$$

where we have used that ∇ is flat. On the other hand, we have $\mathcal{S}_X = g^{-1}(\nabla_X g)$, thus

$$\begin{aligned}[\nabla_X,\mathcal{S}_Y] &= [\nabla_X, g^{-1}(\nabla_Y g)] = \nabla_X(g^{-1}(\nabla_Y g)) - g^{-1}((\nabla_Y g)\nabla_X)\\ &= (\nabla_X g^{-1})\nabla_Y g + g^{-1}(\nabla_X(\nabla_Y g)) - g^{-1}((\nabla_Y g)\nabla_X)\\ &= (\nabla_X g^{-1})\nabla_Y g + g^{-1}(\nabla_X\nabla_Y g) + g^{-1}((\nabla_Y g)\nabla_X) - g^{-1}((\nabla_Y g)\nabla_X)\\ &= -g^{-1}((\nabla_X g)g^{-1}(\nabla_Y g)) + g^{-1}(\nabla_X\nabla_Y g).\end{aligned}$$

Using again that ∇ is flat, we get

$$[\nabla_X, \mathcal{S}_Y] - [\nabla_Y, \mathcal{S}_X] - \mathcal{S}_{[X,Y]} = -g^{-1}((\nabla_X g)g^{-1}(\nabla_Y g)) + g^{-1}((\nabla_Y g)g^{-1}(\nabla_X g)),$$

which is also equal to $-[\mathcal{S}_X, \mathcal{S}_Y]$. Putting everything together we obtain

$$R(X,Y) = -\tfrac{1}{2}[\mathcal{S}_X, \mathcal{S}_Y] + \tfrac{1}{4}[\mathcal{S}_X, \mathcal{S}_Y] = -\tfrac{1}{4}[\mathcal{S}_X, \mathcal{S}_Y].$$

□

As an immediate consequence of (11) and Corollary 4.1.28, we obtain the well-known result that the Riemann curvature of any Kähler cone vanishes when applied to vector fields generating the $\mathbb{C}^*$-action.

Corollary 4.1.30. *Let (M, g, J, ∇, ξ) be a CASK manifold. Then:*

(a) $R(\xi, \cdot)\cdot = R(\cdot, \xi)\cdot = R(\cdot, \cdot)\xi = 0.$

(b) $R(J\xi, \cdot)\cdot = R(\cdot, J\xi)\cdot = R(\cdot, \cdot)J\xi = 0.$

Finally, we compute the Riemann curvature tensor of an ASK manifold in local ∇-affine coordinates.

Corollary 4.1.31. *Let (M, g, J, ∇) be an ASK manifold. Then the $(0,4)$-Riemann curvature tensor* Rm *in ∇-affine coordinates is given by*

$$\mathrm{Rm}_{ijk\ell} = g_{\ell m}(\Gamma^p_{ik}\Gamma^m_{jp} - \Gamma^p_{jk}\Gamma^m_{ip}).$$

Proof. By Proposition 4.1.29, we have $R(X,Y) = -\frac{1}{4}[\mathcal{S}_X, \mathcal{S}_Y]$ for all $X, Y \in \Gamma(TM)$. In local ∇-affine coordinates we get

$$\begin{aligned}
\mathrm{Rm}_{ijk\ell} &= \mathrm{Rm}(\partial_i, \partial_j, \partial_k, \partial_\ell) = g(R(\partial_i, \partial_j)\partial_k, \partial_\ell) = -\tfrac{1}{4}g([\mathcal{S}_{\partial_i}, \mathcal{S}_{\partial_j}]\partial_k, \partial_\ell) \\
&= -\tfrac{1}{4}g(\mathcal{S}_{\partial_i}\mathcal{S}_{\partial_j}\partial_k, \partial_\ell) + \tfrac{1}{4}g(\mathcal{S}_{\partial_j}\mathcal{S}_{\partial_i}\partial_k, \partial_\ell) = -\tfrac{1}{4}\mathcal{S}^p_{jk}\mathcal{S}^m_{ip}g_{m\ell} + \tfrac{1}{4}\mathcal{S}^p_{ik}\mathcal{S}^m_{jp}g_{m\ell} \\
&= g_{\ell m}(\Gamma^p_{ik}\Gamma^m_{jp} - \Gamma^p_{jk}\Gamma^m_{ip}),
\end{aligned}$$

where in the last equation we have used (10).

We can give an alternative proof without using (11). Indeed, the $(0,4)$-Riemann curvature tensor of a Riemannian manifold is given in local coordinates by

$$\mathrm{Rm}_{ijk\ell} = g_{\ell m}(\partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^p_{jk}\Gamma^m_{ip} - \Gamma^p_{ik}\Gamma^m_{jp}).$$

By Lemma 4.1.25, the Christoffel symbols of the Levi-Civita connection D of the metric g are $\Gamma^m_{jk} = \frac{1}{2}\partial_j g_{kr} g^{rm}$, expressed in the local ∇-affine coordinates $\{q^i\}$. Then we get

$$\begin{aligned}
\partial_i\Gamma^m_{jk} &= \tfrac{1}{2}\partial_i\partial_j g_{kr} g^{rm} + \tfrac{1}{2}\partial_j g_{rk}\partial_i g^{rm} \\
&= \tfrac{1}{2}\partial_i\partial_j g_{kr} g^{rm} - \tfrac{1}{2}\partial_j g_{kr} g^{r\alpha}\partial_i g_{\alpha\beta} g^{\beta m} \\
&= \tfrac{1}{2}\partial_i\partial_j g_{kr} g^{rm} - 2\Gamma^\alpha_{jk}\Gamma^m_{i\alpha},
\end{aligned}$$

which implies that $\partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} = -2(\Gamma^\alpha_{jk}\Gamma^m_{i\alpha} - \Gamma^\alpha_{ik}\Gamma^m_{j\alpha})$. Hence the Riemann curvature Rm is given in these coordinates by

$$\mathrm{Rm}_{ijk\ell} = -g_{\ell m}(\Gamma^p_{jk}\Gamma^m_{ip} - \Gamma^p_{ik}\Gamma^m_{jp}) = g_{\ell m}(\Gamma^p_{ik}\Gamma^m_{jp} - \Gamma^p_{jk}\Gamma^m_{ip}).$$

□

4.2 Rigid c-map spaces

As we have already mentioned, our main reason to introduce and study (affine) special Kähler manifolds is that its cotangent bundle has the structure of a pseudo-hyperkähler manifold. Moreover, in the case of a CASK manifold, its cotangent bundle even possesses a rotating circle action, so it is possible to perform the HK/QK correspondence in these spaces.

Let us consider an ASK manifold (M, g, J, ∇). As we said, we will discuss the natural geometric structure that exists on its cotangent bundle $N = T^*M$. For this, it will be useful to recall some basic facts about vector bundles (see e.g. [MS22]).

Let $\pi : E \longrightarrow M$ be a vector bundle over a manifold M. We can pull E back to a bundle over its total space: $\pi^*E \longrightarrow E$. This bundle always admits a tautological section $\Phi \in \Gamma(\pi^*E)$, which assigns to the point $e \in E$ the value e. In the case where $E = T^*M$, this is precisely (one interpretation of) the tautological 1-form λ. The tangent vectors to the fibers of E determine a canonical vertical distribution $T^{\mathrm{V}}E \subset TE$, which is moreover canonically isomorphic to π^*E. The corresponding isomorphism is denoted by

$$\mathcal{V} : \pi^*E \longrightarrow T^{\mathrm{V}}E.$$

In the case where $E = T^*M$, local coordinates $\{q^j\}$ on M induce canonical coordinates $\{q^j, p_j\}$ on T^*M and the isomorphism $\mathcal{V} : \pi^*(T^*M) \longrightarrow T^{\mathrm{V}}(T^*M)$ is implemented by mapping $\pi^*(\mathrm{d}q^j)$ to $\frac{\partial}{\partial p_j}$.

Now assume that E comes equipped with some connection ∇. Then we may use the pullback connection on π^*E to compute $(\pi^*\nabla)\Phi$. The assignment $X \longmapsto (\pi^*\nabla)_X\Phi$, where $X \in \Gamma(TE)$, then provides a left inverse to $\mathcal{V}$. Thus, a vector field X is determined by $(\pi^*\nabla)_X\Phi \in \Gamma(\pi^*E)$ and $\pi_*(X) \in \Gamma(\pi^*(TM))$. This is just another way of phrasing the fact that a connection on E induces a splitting $TE \cong \pi^*(TM) \oplus \pi^*E$. In particular, for $E = T^*M$ we obtain the splitting $T(T^*M) \cong \pi^*(TM) \oplus \pi^*(T^*M)$. More precisely, using the flat torsion-free connection ∇ on M, we can identify

$$TN = T(T^*M) = T^{\mathrm{H}}N \oplus T^{\mathrm{V}}N \cong \pi^*(TM) \oplus \pi^*(T^*M),$$

where $\pi\colon N = T^*M \longrightarrow M$ is the canonical projection, $T^{\mathrm{V}}N = \ker(\mathrm{d}\pi)$ is the vertical distribution and $T^{\mathrm{H}}N$ is the horizontal distribution defined by ∇. Then, given a vector field $X \in \Gamma(TN)$, we will think of its horizontal component X^{H} as a section of $\pi^*(TM)$

and its vertical component X^{V} as a section of $\pi^*(T^*M)$. Using these identifications, we may define the following tensor fields on $N = T^*M$:

$$g_N := \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \quad I_1 := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}, \quad I_2 := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \quad I_3 := I_1 I_2. \tag{12}$$

It was first pointed out in the physics literature [CFG89], and then mathematically described in [Fre99b, Theorem 2.1] and [ACD02, Theorem 11], that the tensors introduced in (12) define a pseudo-hyperkähler structure on the cotangent bundle on an ASK manifold.

Theorem 4.2.1. *Let (M,g,J,∇) be an ASK manifold. Then the metric g_N and the almost complex structures I_1, I_2 and I_3 given by (12) define a pseudo-hyperkähler structure on $N = T^*M$.*

Definition 4.2.2. The construction of the pseudo-hyperkähler manifold N from the ASK manifold M as explained above is known as the **rigid c-map**. A manifold in the image of the rigid c-map is called a **rigid c-map space**.

Then the rigid c-map assigns to each ASK manifold (M,g,J,∇) of real dimension $2n$ a pseudo-hyperkähler manifold $(N = T^*M, g_N, I_1, I_2, I_3)$ of real dimension $4n$. Moreover, if the pseudo-Kähler metric g has signature $(2n-2p, 2p)$, $p = 0, \dots, n$, then the pseudo-hyperkähler metric g_N has signature $(4n-4p, 4p)$.

Remark 4.2.3. In [MS15], the rigid c-map construction is obtained by using the language of principal bundles. They even conclude in [MS15, Proposition 2.4] that Theorem 4.2.1 is in fact an equivalence. Indeed, if M is a Kähler manifold such that the tensors (12) on T^*M define a hyperkähler structure, then M has to be an ASK manifold. They also point out in [MS15, Remark 2.5] that in general the hyperkähler metric obtained from the rigid c-map is different from the hyperkähler metrics on cotangent bundles constructed by Feix [Fei01] and Kaledin [Kal01].

If we star with a CASK manifold, we moreover have the vector fields ξ and $J\xi$ generating a $\mathbb{C}^*$-action. So in this case the rigid c-map space enjoys some additional properties.

Proposition 4.2.4 ([ACM13, Proposition 2])**.** *Let (M,g,J,∇,ξ) be a CASK manifold and define on the associated rigid c-map space $(N = T^*M, g_N, I_1, I_2, I_3)$ the following data:*

$$Z := -\widetilde{J\xi}, \quad \omega_1 := g_N(I_1\cdot,\cdot), \quad \omega_{\mathrm{H}} := \omega_1 + \mathrm{d}\iota_Z g_N,$$
$$f_Z := -\tfrac{1}{2} g_N(Z,Z), \quad f_{\mathrm{H}} := \tfrac{1}{2} g_N(Z,Z),$$

where $\widetilde{J\xi}$ denotes the horizontal lift of $J\xi$ with respect to ∇. Then

$$\mathscr{L}_Z g_N = 0, \quad \mathscr{L}_Z \omega_2 = \omega_3, \quad \iota_Z \omega_1 = -\mathrm{d}f_Z, \quad \iota_Z \omega_{\mathrm{H}} = -\mathrm{d}f_{\mathrm{H}}.$$

Note that Proposition 4.2.4 implies that Z is a rotating Killing vector field that generates a rotating circle action on the pseudo-hyperkähler manifold N. This implies that we are able to apply the HK/QK correspondence to N to construct quaternionic Kähler manifolds, which is our final goal. Therefore, from now on we will always assume that the rigid c-map space is also equipped with the rotating Killing vector field Z and its corresponding rotating isometric circle action.

We have then the following definition of an automorphism on a rigid c-map space.

Definition 4.2.5. A diffeomorphism $\varphi : N \longrightarrow N$ is called an **automorphism of the rigid c-map structure**, or equivalently of the pseudo-hyperkähler structure with rotating circle action, if it preserves g_N, I_1, I_2, I_3 and $f_Z^c := f_Z - \frac{1}{2}c$.

Note that an automorphism in the above sense automatically commutes with the rotating circle action. The group of all automorphisms of the rigid c-map structure is denoted by $\mathrm{Aut}_{\mathbb{S}^1}(N)$. The subgroup of ω_1-Hamiltonian automorphisms is denoted by $\mathrm{Ham}_{\mathbb{S}^1}(N)$.

In this situation, where we have a hyperkähler structure with a rotating circle action, the canonical closed 2-form $\omega_{\mathrm{H}} = \omega_1 + \mathrm{d}\iota_Z g_N$ is of type $(1,1)$ with respect to each I_k [CST22, Lemma 2.7]. Note that any vector field which is ω_1-Hamiltonian and preserves Z and g_N is automatically ω_{H}-Hamiltonian as well. This applies, in particular, to the rotating Killing field Z, whose Hamiltonian function with respect to ω_{H} we denote by $f_{\mathrm{H}}^c := f_{\mathrm{H}} - \frac{1}{2}c$ (see also Proposition 4.2.4). In the following, we will focus on Hamiltonian functions with respect to ω_{H} rather than ω_1.

There are two important sources of Hamiltonian automorphisms of the rigid c-map structure: canonical lifts of CASK automorphisms and translations in the fibers. We will study them in detail in Section 5.3 and we will see how do they transform under the HK/QK correspondence, interpreted as a twist, in Section 5.4.

For completeness of the exposition we mention that in the case of the rigid c-map the closed 2-form ω_{H} is in fact symplectic.

Proposition 4.2.6. *Let (M, g, J, ∇, ξ) be a CASK manifold and $(N = T^*M, g_N, I_1, I_2, I_3)$ the associated rigid c-map space. Then the canonical 2-form $\omega_{\mathrm{H}} = \omega_1 + \mathrm{d}\iota_Z g_N$ associated with the rotating Killing field Z is a symplectic structure.*

Proof. Since the 1-form $\iota_Z g_N$ is the pull-back of $\alpha := -\iota_{J\xi} g$ we can calculate its differential as $\pi^*\mathrm{d}\alpha$, where $\mathrm{d}\alpha$ can be expressed as twice the skew-symmetric part of the Levi-Civita covariant derivative

$$D\alpha = -g(D(J\xi), \cdot) = -g(J\cdot, \cdot) = -\omega.$$

Since this is skew-symmetric, we see that $\mathrm{d}\iota_Z g_N = \pi^*\mathrm{d}\alpha = -2\pi^*\omega$. This shows that

$$\omega_{\mathrm{H}} = \begin{pmatrix} -\omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \tag{13}$$

from where we see that ω_{H} is non-degenerate, hence symplectic. $\square$

Incidentally, the above expression for ω_{H} shows that the endomorphism

$$I_{\mathrm{H}} := g_N^{-1}\omega_{\mathrm{H}} = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

is a g_N-skew-symmetric almost complex structure, which obviously commutes with I_1, I_2 and I_3. In this way we recover the statement that ω_{H} is of type $(1,1)$ for the three complex structures.

We moreover observe the following fact.

Corollary 4.2.7. *Let (M, g, J, ∇, ξ) be a CASK manifold. Then the associated rigid c-map space $(N = T^*M, g_N, I_1, I_2, I_3)$ carries a canonical almost Kähler structure (g_N, I_{H}).*

4.3 Supergravity c-map spaces

We have explained in Section 4.2 that given a CASK manifold (M, g, J, ∇, ξ), its cotangent bundle $N = T^*M$ is a pseudo-hyperkähler manifold equipped with a rotating Killing vector field $Z = -\widetilde{J\xi}$ generating a rotating circle action (see Theorem 4.2.1 and Proposition 4.2.4). Moreover, since $g(\xi, \xi) < 0$ by definition of the CASK structure, we have

$$\begin{aligned} f_Z &:= -\tfrac{1}{2} g_N(Z, Z) = -\tfrac{1}{2}\pi^* g(\xi, \xi) > 0, \\ f_{\mathrm{H}} &:= f_Z + g_N(Z, Z) = \tfrac{1}{2} g_N(Z, Z) < 0. \end{aligned} \tag{14}$$

We therefore have a pseudo-hyperkähler manifold $N = T^*M$ equipped with HK/QK data Z, $\omega_1 = g_N(I_1\cdot, \cdot)$, $\omega_{\mathrm{H}} = \omega_1 + \mathrm{d}\iota_Z g_N$ and Hamiltonian functions $f_Z^c := f_Z - \frac{1}{2}c$ and $f_{\mathrm{H}}^c := f_{\mathrm{H}} - \frac{1}{2}c$, where $c \in \mathbb{R}$ is a constant such that the inequalities in (14) are still satisfied. Hence, if we apply the HK/QK correspondence (see Theorem 3.2.4) to the rigid c-map space N with this HK/QK data we then get a quaternionic Kähler manifold $(\bar{N}, g_{\bar{N}}^c, \mathscr{Q})$ with positive-definite metric $g_{\bar{N}}^c$ and negative scalar curvature.

In [CGS23], the composition of the rigid c-map and the HK/QK correspondence, together with the choice of the Hamiltonian functions (14), is called the supergravity c-map. Nevertheless, whenever the supergravity c-map is referred to in the literature (both mathematical and physical), the input of such construction is a PSK manifold instead of a CASK manifold. However, recall that given a CASK manifold M, the orbit space of the $\mathbb{C}^*$-action $\bar{M} = M/\mathbb{C}^*$ is a PSK manifold. Conversely, given a PSK manifold, the total space of a $\mathbb{C}^*$-bundle over it is a CASK manifold (see Remark 4.1.9), so there is a one-to-one correspondence between them. Hence we adopt the classical nomenclature for the supergravity c-map.

Definition 4.3.1. The construction of the quaternionic Kähler manifold $\bar{N}$ from the PSK manifold $\bar{M}$ as explained above is known as the **supergravity c-map**. A manifold in the image of the supergravity c-map is called a **supergravity c-map space**.

Given a positive-definite PSK manifold of real dimension $2n$, the corresponding CASK manifold has signature $(2n,2)$ and the composition of the rigid c-map and the HK/QK correspondence gives us then a positive-definite quaternionic Kähler manifold of real dimension $4n+4$ and negative scalar curvature. We can state the following result to summarize our discussion (see also [ACDM15, Theorem 5]).

Theorem 4.3.2. *The supergravity c-map assigns to each positive-definite PSK manifold a positive-definite quaternionic Kähler manifold with negative scalar curvature.*

The following diagram summarizes this construction:

$$\begin{tikzcd} M^{2n+2}\ (\mathrm{CASK}) \arrow[r, "\text{rigid c-map}"] & N^{4n+4}\ (\mathrm{HK}) \arrow[d, "\mathrm{HK/QK}"] \\ \bar{M}^{2n}\ (\mathrm{PSK}) \arrow[u, "\mathbb{C}^*"] \arrow[r, "\text{supergravity c-map}"] & \bar{N}^{4n+4}\ (\mathrm{QK}) \end{tikzcd} \tag{15}$$

We have already pointed out at the end of Section 3.2 that there is a freedom of adding a constant $c \in \mathbb{R}$ to the Hamiltonian functions f_Z^c and f_{H}^c (which explains the superscript c). This implies that the HK/QK correspondence produces a one-parameter family of quaternionic Kähler manifolds, and then so does the supergravity c-map. The case $c = 0$ is distinguished and is called the **undeformed supergravity c-map**, while the remaining cases are collectively referred to as the **deformed supergravity c-map**.

The original description of the supergravity c-map is given in term of some local coordinates [FS90]. Although we will not use this local description to prove our mains results, it is useful to deduce some important properties of supergravity c-map spaces, such as the existence of large groups of isometries or the completeness of the quaternionic Kähler metric.

4.3.1 Supergravity c-map in local coordinates

Throughout this and the following subsection we use the Einstein summation convention to work with local coordinates and the notation and conventions of [CT22b], which slightly differ from the ones used before in this work.

Let M be a CASK domain. Recall that this is an open subset $M \subset \mathbb{C}^{n+1} \setminus \{0\}$ invariant under the usual $\mathbb{C}^*$-action on $\mathbb{C}^{n+1} \setminus \{0\}$ by multiplication together with a holomorphic function $F : M \longrightarrow \mathbb{C}$ homogeneous of degree 2 with respect to the $\mathbb{C}^*$-action. With respect to the natural holomorphic coordinates $(X^0, \dots, X^n)$ of M, the matrix

$$\tau_{ij} := \left(\frac{\partial^2 F}{\partial X^i \partial X^j} \right)$$

satisfies that $\mathrm{Im}(\tau_{ij})$ has signature $(n,1)$ and $\mathrm{Im}(\tau_{ij}) X^i \bar{X}^j < 0$ for $X \in M$.

From these data we obtain a CASK manifold (M,g,J,∇,ξ) by

$$g=\mathrm{Im}(\tau_{ij})\mathrm{d}X^i\mathrm{d}\bar{X}^j,\quad \omega=g(J\cdot,\cdot)=\tfrac{i}{2}\mathrm{Im}(\tau_{ij})\mathrm{d}X^i\wedge\mathrm{d}\bar{X}^j,\quad \xi=X^i\tfrac{\partial}{\partial X^i}+\bar{X}^i\tfrac{\partial}{\partial\bar{X}^i},$$

and the flat connection ∇ is defined such that

$$\mathrm{d}x^i:=\mathrm{Re}(\mathrm{d}X^i)\quad\text{and}\quad \mathrm{d}y_i:=-\mathrm{Re}(\mathrm{d}(\tfrac{\partial F}{\partial X^i})) \tag{16}$$

is a flat frame of T^*M.

In the case of a CASK domain, ξ and $J\xi$ generate a free $\mathbb{C}^*$-action on M, so we can perform the Kähler quotient to obtain a PSK domain $(\bar{M},\bar{g})$ as explained in (9). The relation between the coordinates $\{X^i\}_{i=0}^n$ on M and the coordinates $\{z^a\}_{a=1}^n$ of $\bar{M}$ is given by $\frac{X^i}{X^0}=z^i$ and $z^0:=1$.

Now consider the manifold $\bar{N}:=\bar{M}\times\mathbb{R}_{>0}\times\mathbb{R}^{2n+3}$, where $n=\dim_{\mathbb{C}}\bar{M}$, with global coordinates on $\bar{N}$ given by $(z^a,\rho,\tilde{\zeta}_i,\zeta^i,\sigma)\in\bar{M}\times\mathbb{R}_{>0}\times\mathbb{R}^{2n+2}\times\mathbb{R}$, where $i=0,\dots,n$. Furthermore, we define

$$N_{ij}:=-2\,\mathrm{Im}(\tau_{ij}),\quad W_i:=\mathrm{d}\tilde{\zeta}_i-\tau_{ij}\mathrm{d}\zeta^j,\quad K:=N_{ij}z^i\bar{z}^j. \tag{17}$$

With respect to these coordinates we define the metric $g_{\bar{N}}$ on $\bar{N}$ by

$$g_{\bar{N}}:=\bar{g}+\frac{1}{4\rho^2}\mathrm{d}\rho^2-\frac{1}{4\rho}(N^{ij}-\frac{2}{K}z^i\bar{z}^j)W_i\bar{W}_j+\frac{1}{64\rho^2}(\mathrm{d}\sigma+\tilde{\zeta}_i\mathrm{d}\zeta^i-\zeta^i\mathrm{d}\tilde{\zeta}_i)^2, \tag{18}$$

where $\bar{g}$ is the PSK metric and N^{ij} denotes the inverse matrix of N_{ij}.

In principle, the expression for the metric $g_{\bar{N}}$ depends on the coordinates chosen on the PSK domain $\bar{M}$. Moreover, if we consider a PSK manifold covered by PSK domains $\{\bar{M}_\alpha\}$, we need to check whether all the quaternionic Kähler metrics defined on the corresponding spaces $\bar{N}_\alpha$ patch to a globally well-defined metric. This problem was solved in [CHM12, Theorem 9], where it is shown that the quaternionic Kähler manifold $\bar{N}$ constructed from all the $\{\bar{N}_\alpha\}$ does neither depend on the covering $\{\bar{M}_\alpha\}$ of $\bar{M}$ nor on the choice of coordinates on the domains $\bar{M}_\alpha$. Hence we obtain a globally defined quaternionic Kähler metric starting with a PSK manifold.

The metric $g_{\bar{N}}$ corresponds to the metric of the undeformed supergravity c-map space introduced before, i.e. the case $c=0$. This metric is sometimes called the **Ferrara-Sabharwal metric** after the physicists who first explicitly described it [FS90]. They showed that this metric is quaternionic Kähler of negative scalar curvature by direct computation. An alternative proof of this fact is due to Hitchin [Hit09].

Let $(\bar{M},\bar{g})$ be a PSK domain and $\bar{N}=\bar{M}\times G$ the corresponding quaternionic Kähler manifold with the Ferrara-Sabharwal metric $g_{\bar{N}}=\bar{g}+g_G$, where

$$g_G:=\frac{1}{4\rho^2}\mathrm{d}\rho^2-\frac{1}{4\rho}(N^{ij}-\frac{2}{K}z^i\bar{z}^j)W_i\bar{W}_j+\frac{1}{64\rho^2}(\mathrm{d}\sigma+\tilde{\zeta}_i\mathrm{d}\zeta^i-\zeta^i\mathrm{d}\tilde{\zeta}_i)^2.$$

In [CHM12] the authors show that, for a fixed $x \in \bar{M}$, $g_G(x)$ can be considered as a left-invariant Riemannian metric on a certain Lie group diffeomorphic to $\mathbb{R}^{2n+4}$. The Lie group G is defined by the following group multiplication on $\mathbb{R}^{2n+4}$ (see also [CT22b, Equation 2.13]):

$$(r, \tilde{\eta}_i, \eta^i, \kappa) \cdot (\rho, \tilde{\zeta}_i, \zeta^i, \sigma) = (r\rho, \tilde{\eta}_i + \sqrt{r}\tilde{\zeta}_i, \eta^i + \sqrt{r}\zeta^i, \kappa + r\sigma + \sqrt{r}(\eta^i\tilde{\zeta}_i - \tilde{\eta}_i\zeta^i)).$$

The Lie group G is isomorphic to the solvable Iwasawa subgroup of $\mathrm{SU}(1, n+2)$, which acts simply transitively on the complex hyperbolic space of complex dimension $n+2$. In fact, the Lie group G is a 1-dimensional solvable extension of the Heisenberg group Heis_{2n+3} of dimension $2n+3$, which is parameterized by the coordinates $(\tilde{\zeta}_i, \zeta^i, \sigma) \in \mathbb{R}^{2n+3}$. In general, when $\bar{M}$ is a PSK manifold instead of a PSK domain, we have the structure of a bundle of Lie groups $\pi : \bar{N} \longrightarrow \bar{M}$, where each fiber is isomorphic to the solvable Lie group G equipped with a left-invariant metric. Hence each fiber is homogeneous. Moreover, the Lie group G acts by isometries on $(\bar{N}, g_{\bar{N}})$. Using this description, it is possible to determine when the quaternionic Kähler metric is complete.

Theorem 4.3.3 ([CHM12, Theorem 10])**.** *Let $(\bar{M}, \bar{g})$ be a PSK manifold and $(\bar{N}, g_{\bar{N}}, \mathcal{Q})$ the associated undeformed supergravity c-map space. If $(\bar{M}, \bar{g})$ is complete, so is $(\bar{N}, g_{\bar{N}})$.*

Thus we have a very simple criterion to determine the completeness of the undeformed supergravity c-map metric.

Now we proceed to give an expression in local coordinates of the deformed supergravity c-map metric, sometimes called the **one-loop deformed supergravity c-map metric** due to its physical origins [RSV06].

Similarly as in the undeformed case, let $(\bar{M}, \bar{g})$ be a PSK domain of complex dimension n with coordinates $\{z^a\}$. We consider $\bar{N} := \bar{M} \times \mathbb{R}_{>0} \times \mathbb{R}^{2n+3}$ with global coordinates $(z^a, \rho, \tilde{\zeta}_i, \zeta^i, \sigma)$ and define the metric $g^c_{\bar{N}}$ on $\bar{N}$ by

$$\begin{aligned} g^c_{\bar{N}} := {} & \frac{\rho + c}{\rho}\bar{g} + \frac{1}{4\rho^2}\frac{\rho + 2c}{\rho + c}\mathrm{d}\rho^2 - \frac{1}{4\rho}\Big(N^{ij} - \frac{2(\rho + c)}{\rho K}z^i\bar{z}^j\Big)W_i\bar{W}_j \\ & + \frac{1}{64\rho^2}\frac{\rho + c}{\rho + 2c}(\mathrm{d}\sigma + \tilde{\zeta}_i\mathrm{d}\zeta^i - \zeta^i\mathrm{d}\tilde{\zeta}_i - 4c\mathrm{d}^c\mathcal{K})^2, \end{aligned} \tag{19}$$

where N_{ij}, W_i and K are as in (17), $\mathcal{K} := -\log K$, $\mathrm{d}^c := i(\bar{\partial} - \partial)$ and

$$\mathrm{d}^c\mathcal{K} = -i\frac{N_{ij}}{N_{k\ell}z^k\bar{z}^\ell}(z^i\mathrm{d}\bar{z}^j - \bar{z}^j\mathrm{d}z^i).$$

A first observation is that for $c = 0$ we recover the undeformed supergravity c-map (18), so $g^0_{\bar{N}} = g_{\bar{N}}$. It is known that this metric is in general incomplete for $c < 0$ [ACDM15, Remark 9]. Hence we will assume that $c \geq 0$ from now on.

Note that as in the undeformed case, we have to consider whether the metric $g^c_{\bar{N}}$ is globally well-defined in the case that $\bar{M}$ is a PSK manifold. This was solved in [CDS17, Theorem 12], where it was found that the local metrics can be patched together precisely if the coordinate σ is periodic with length $2\pi c$, $c > 0$, that is $\sigma \in \mathbb{S}^1_c := \mathbb{R}/2\pi c\mathbb{Z}$. Notice that $\mathbb{S}^1_c$ can be identified with $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ by $[x] \longmapsto [cx]$ if $c > 0$ and that $\mathbb{S}^1_0 = \mathbb{R}$.

The proof that $g^c_{\bar{N}}$, $c > 0$, is in fact a positive-definite quaternionic Kähler metric was first obtained in [ACDM15, Theorem 5]. To show by direct computation that $g^c_{\bar{N}}$ is quaternionic Kähler is considerably hard, so the problem was divided in two steps. The first one was showing that if one starts with a PSK manifold $\bar{M}$, and apply the rigid c-map to the corresponding CASK manifold M and then the HK/QK correspondence to $N = T^*M$, one gets a quaternionic Kähler manifold $\bar{N}$. The second step was showing that the metric obtained by this indirect method coincides locally with the expression of $g^c_{\bar{N}}$ in local coordinates given by (19).

One of the first remarkable features of the deformed supergravity c-map metric is that its local geometry does not depend on the particular choice of constant $c > 0$.

Proposition 4.3.4 ([CDS17, Proposition 10])**.** *Let $(\bar{M}, \bar{g})$ be a PSK manifold and $g^{c_1}_{\bar{N}}$, $g^{c_2}_{\bar{N}}$ deformed supergravity c-map metrics for $c_1, c_2 > 0$. Then $(\bar{N}, g^{c_1}_{\bar{N}})$ and $(\bar{N}, g^{c_2}_{\bar{N}})$ are locally isometric.*

Recall that in the undeformed case we have a solvable Lie group G of dimension $2n+4$ acting by isometries on $(\bar{N}, g_{\bar{N}})$. However, for $c > 0$, we only have the nilradical of G, which is the group Heis_{2n+3}, acting by isometries on $(\bar{N}, g^c_{\bar{N}})$, whose action is the same as in the undeformed case. This action preserves the fibers of $\pi : \bar{N} \longrightarrow \bar{M}$, although they are not longer homogeneous.

Although there is no general theorem asserting completeness for deformed supergravity c-map metrics arising from complete PSK manifold as in the undeformed case (see Theorem 4.3.3), there are partial results that cover the most important known examples of negative quaternionic Kähler manifolds.

The first completeness result is established under the additional assumption of regular boundary behavior for the initial PSK manifold. The precise meaning of this was introduced in [CDS17].

Definition 4.3.5. A **CASK manifold with regular boundary behavior** is a CASK manifold (M, g, J, ∇, ξ) which admits an embedding $\imath : M \longrightarrow \mathcal{M}$ into a manifold with boundary $\mathcal{M}$ such that $\imath(M) = \mathring{\mathcal{M}} := \mathcal{M} \setminus \partial\mathcal{M}$ and the tensor fields (g, J, ξ) smoothly extend to $\mathcal{M}$ such that, for all boundary points $x \in \partial\mathcal{M}$ we have $f(x) = 0$, $\mathrm{d}f_x \neq 0$, where $f = \frac{1}{2}g(\xi, \xi)$, and g_x is negative semi-definite on $T_x\partial\mathcal{M} \cap J(T_x\partial\mathcal{M})$ with kernel $\mathrm{span}_{\mathbb{R}}\{\xi_x, J\xi_x\}$.

As in the case of empty boundary, we will assume that ξ and $J\xi$ generate a principal $\mathbb{C}^*$-action on the manifold $\mathcal{M}$. Then $\bar{\mathcal{M}} = \mathcal{M}/\mathbb{C}^*$ is a manifold with boundary and its interior $\bar{M} = M/\mathbb{C}^*$ is a PSK manifold with PSK metric $\bar{g}$. If the manifold $\bar{\mathcal{M}}$ with boundary is compact, then we will call $(\bar{M}, \bar{g})$ a **PSK manifold with regular boundary behavior**.

Theorem 4.3.6 ([CDS17, Theorem 7]). *Let $(\bar{M}, \bar{g})$ be a PSK manifold with regular boundary behavior. Then $(\bar{M}, \bar{g})$ is complete.*

Now we can state the following completeness result for the deformed supergravity c-map metric.

Theorem 4.3.7 ([CDS17, Theorem 13]). *Let $(\bar{M}, \bar{g})$ be a PSK manifold with regular boundary behavior and $(\bar{N}, g^c_{\bar{N}}, \mathcal{Q})$ the associated supergravity c-map space. Then $(\bar{N}, g^c_{\bar{N}})$ is complete for all $c \geq 0$.*

Example 4.3.8 ([CDS17, Example 14]). The PSK manifold $\mathbb{C}\mathrm{H}^n$ with quadratic holomorphic prepotential $F = \frac{i}{2}((z^0)^2 - \sum_{j=1}^n (z^j)^2)$ has regular boundary behavior. Thus Theorem 4.3.6 implies the completeness of $\mathbb{C}\mathrm{H}^n$. It is well-known (see e.g. [dWVP92, Table 2]) that $(\bar{N}, g_{\bar{N}})$ is isometric to the Wolf space of non-compact type

$$\frac{\mathrm{SU}(n+1,2)}{\mathrm{S}(\mathrm{U}(n+1)\times \mathrm{U}(2))}. \tag{20}$$

Then, as a corollary of Theorem 4.3.7 (see also [CDS17, Corollary 15]), we get that the deformed supergravity c-map space $(\bar{N}, g^c_{\bar{N}})$ is complete.

In the next subsection we explain in detail the second completeness result of deformed supergravity c-map metrics. This result assumes that the PSK manifold is obtained from a cubic homogeneous polynomial, which give rise to a special class of supergravity c-map spaces.

4.3.2 Supergravity q-map spaces

Supergravity q-map spaces are a special class of supergravity c-map spaces. These arise as the composition of the supergravity r-map and the supergravity c-map. As we have seen (see Theorem 4.3.2), the supergravity c-map produces (a one-parameter family of) quaternionic Kähler metrics from a PSK manifold, while the supergravity r-map produces a PSK manifold from a projective special real manifold. We introduce these concepts and study their completeness properties. We use again the notation and conventions of [CT22b] and the Einstein summation convention. For further details we refer to the interested reader to [CDL14, CDS17, CDJL21] and references therein.

Definition 4.3.9. A **projective special real (PSR) manifold** is a Riemannian manifold $(\mathcal{H}, g_{\mathcal{H}})$ such that $\mathcal{H} \subset \mathbb{R}^n$ is a hypersurface and there is a homogeneous cubic polynomial $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying

- $\mathcal{H} \subset \{t \in \mathbb{R}^n \mid h(t) = 1\}$.

- $g_{\mathcal{H}} := -\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}}$.

We denote the coordinates of $\mathbb{R}^n$ by t^a, where $a = 1, \dots, n$. In particular we write h as follows:

$$h(t^a) = \frac{1}{6} k_{abc} t^a t^b t^c,$$

where the coefficients $k_{abc} \in \mathbb{R}$ are symmetric in the indices.

Let us consider a PSR manifold $(\mathcal{H}, g_{\mathcal{H}})$ defined by the real cubic homogeneous h and let $\mathcal{U} := \mathbb{R}_{>0} \cdot \mathcal{H} \subset \mathbb{R}^n \setminus \{0\}$. The pair $(\mathcal{U}, g_{\mathcal{U}} := -\partial^2 h)$ is sometimes called **conical affine special real (CASR) manifold** [CDM18]. We define $\bar{M} := \mathbb{R}^n + i\mathcal{U} \subset \mathbb{C}^n$ with the canonical holomorphic structure, where global holomorphic coordinates are given by $z^a := b^a + it^a \in \mathbb{R}^n + i\mathcal{U}$. On $\bar{M}$ we consider the metric

$$\bar{g} := \frac{\partial^2 \mathcal{K}}{\partial z^a \partial \bar{z}^b} \mathrm{d}z^a \mathrm{d}\bar{z}^b,$$

where $\mathcal{K} := -\log K(t)$ and $K(t) := 8h(t) = \frac{4}{3} k_{abc} t^a t^b t^c$. Then it can be shown that $(\bar{M}, \bar{g})$ is a PSK manifold [CHM12]. In fact, one can find the following explicit expression for a PSK metric coming from a PSR manifold:

$$\begin{aligned} \bar{g} &= -\frac{1}{4} \frac{\partial^2 \log h(t)}{\partial t^a \partial t^b} (\mathrm{d}b^a \mathrm{d}b^b + \mathrm{d}t^a \mathrm{d}t^b) \\ &= \left(-\frac{k_{abc} t^c}{4h(t)} + \frac{k_{acd} k_{bef} t^c t^d t^e t^f}{(4h(t))^2} \right) (\mathrm{d}b^a \mathrm{d}b^b + \mathrm{d}t^a \mathrm{d}t^b). \end{aligned}$$

Definition 4.3.10. The construction of the PSK manifold $\bar{M}$ from the PSR manifold $\mathcal{H}$ as explained above is known as the **supergravity r-map**. A manifold in the image of the supergravity r-map is called a **supergravity r-map space**.

Remark 4.3.11. A manifold in the image of the supergravity r-map is also called a projective very special Kähler manifold.

Remark 4.3.12. A similar diagram as (15) can be constructed in the case of special real geometry by introducing the rigid r-map and the ASK/PSK correspondence (see [CDM18] for details):

$$\begin{tikzcd} \mathcal{U}^n \text{ (CASR)} \arrow[r, "\text{rigid r-map}"] & M^{2n} \text{ (ASK)} \arrow[d, "\text{ASK/PSK}"] \\ \mathcal{H}^{n-1} \text{ (PSR)} \arrow[u, "\mathbb{R}_{>0}"] \arrow[r, "\text{supergravity r-map}"] & \bar{M}^{2n} \text{ (PSK)} \end{tikzcd}$$

Given a supergravity r-map space $(\bar{M}, \bar{g})$, its corresponding CASK manifold is defined via the CASK domain (M, F), where $M \subset \mathbb{C}^{n+1}$ is given by

$$M := \{(X^0, \dots, X^n) = X^0 \cdot (1, z) \in \mathbb{C}^{n+1} \mid X^0 \in \mathbb{C}^*, z \in \bar{M}\} \tag{21}$$

and

$$F(X) = -\frac{h(X^1, \dots, X^n)}{X^0} = -\frac{1}{6} k_{abc} \frac{X^a X^b X^c}{X^0}.$$

Remark 4.3.13. Note that for a CASK manifold M determined by a PSR manifold $\mathcal{H}$ as above, the functions $x^i := \mathrm{Re}(X^i)$ and $y_i := -\mathrm{Re}(\frac{\partial F}{\partial X^i})$ are globally defined and their differentials (16) give rise to a parallel frame of T^*M, hence $\mathrm{Hol}(\nabla)$ is trivial (this ensures that the action of $\mathbb{R}^{2n}$ given by Proposition 5.3.3 is global). In fact, the flat connection ∇ on any CASK domain has trivial holonomy.

Due to the following two results, projective special real geometry constitutes a powerful tool for the construction of complete PSK manifolds.

Theorem 4.3.14 ([CHM12, Theorem 4]). *Let $(\mathcal{H}, g_{\mathcal{H}})$ be a PSR manifold and $(\bar{M}, \bar{g})$ the associated supergravity r-map space. If $(\mathcal{H}, g_{\mathcal{H}})$ is complete, so is $(\bar{M}, \bar{g})$.*

The question of completeness for a PSR manifold $(\mathcal{H}, g_{\mathcal{H}})$ reduces to a simple topological question for the hypersurface $\mathcal{H} \subset \mathbb{R}^n$:

Theorem 4.3.15 ([CNS16, Theorem 2.5]). *Let $(\mathcal{H}, g_{\mathcal{H}})$ be a PSR manifold of dimension $n-1$. Then $(\mathcal{H}, g_{\mathcal{H}})$ is complete if and only if the subset $\mathcal{H} \subset \mathbb{R}^n$ is closed.*

We can now consider the composition of the supergravity r-map and the supergravity c-map to construct quaternionic Kähler manifolds from cubic homogeneous polynomials.

Definition 4.3.16. The construction of the quaternionic Kähler manifold $\bar{N}$ from the PSR manifold $\mathcal{H}$ given by the composition of the supergravity r-map and the supergravity c-map is known as the **supergravity q-map**. A manifold in the image of the supergravity q-map is called a **supergravity q-map space**.

Then the supergravity q-map assigns to each PSR manifold $(\mathcal{H}, g_{\mathcal{H}})$ of dimension $n-1$ a quaternionic Kähler manifold $(\bar{N}, g^c_{\bar{N}}, \mathcal{Q})$ of dimension $4n+4$ and negative scalar curvature.

Since the supergravity c-map depends on a parameter, so does the supergravity q-map. Thus we can also talk about undeformed and deformed supergravity q-map spaces.

The completeness of an undeformed supergravity q-map space is guaranteed by Theorem 4.3.3 and Theorem 4.3.14, as long as the starting PSR manifold is complete. For the deformed case we have the following result.

Theorem 4.3.17 ([CDS17, Theorem 27]). *Let $(\mathcal{H}, g_{\mathcal{H}})$ be a complete PSR manifold and $(\bar{N}, g^c_{\bar{N}}, \mathcal{Q})$ the associated supergravity q-map space. Then $(\bar{N}, g^c_{\bar{N}})$ is complete for all $c \geq 0$.*

Recall that all homogeneous negative quaternionic Kähler manifolds are Alekseevsky spaces (see Theorem 2.4.6) and these are precisely the Wolf spaces of non-compact type and four discrete infinite families of non-symmetric spaces (see Theorem 2.4.5). It is known (see [dWVP92] and references therein) that except for quaternionic hyperbolic spaces $\mathbb{H}\mathrm{H}^n$, all Alekseevsky spaces are in the image of the supergravity c-map. While the series (20) of Hermitian non-compact Wolf spaces can be obtained via the supergravity c-map from the PSK manifold $\mathbb{C}\mathrm{H}^n$ (see also Example 4.3.8), which is not in the image of the supergravity r-map, all the other Alekseevsky spaces are in the image of the supergravity q-map.

Summarizing what we have discussed so far, we get the following corollary.

Corollary 4.3.18. *All homogeneous negative quaternionic Kähler manifolds except the quaternionic hyperbolic spaces admit a one-parameter deformation by complete quaternionic Kähler metrics of negative scalar curvature.*

For completeness of the exposition, we explain why the quaternionic hyperbolic spaces are not supergravity c-map spaces.

Proposition 4.3.19. *The quaternionic hyperbolic space $\mathbb{H}\mathrm{H}^n$ is not a supergravity c-map space for any $n \geq 1$.*

Proof. To proof this fact we look at the totally geodesic Kähler submanifolds compatible with the quaternionic structure. Due to the work of [AM01], we know that the maximal possible dimension of a Kähler submanifold compatible with the quaternionic structure of a quaternionic Kähler manifold of dimension $4n$ is $2n$. In the case of $\mathbb{H}\mathrm{H}^n$ the only totally geodesic Kähler submanifolds of (real) dimension $2n$ compatible with the quaternionic structure are the complex hyperbolic spaces $\mathbb{C}\mathrm{H}^n$ (up to isometries of the ambient space). On the other hand, any supergravity c-map space of dimension $4n$ has a totally geodesic Kähler submanifold compatible with the quaternionic structure of the form $\mathbb{C}\mathrm{H}^1 \times \bar{M}$, where $\bar{M}$ is the underlying PSK manifold of dimension $2n-2$. In fact, the submanifold $\mathbb{C}\mathrm{H}^1 \times \bar{M} \subset \bar{N}$ is obtained as the fixed point set of the isometric involution expressed in standard fiber coordinates $(\rho, \sigma, \tilde{\zeta}_i, \zeta^i)$, $i = 1, \ldots, 2n$, by $(\rho, \sigma, \tilde{\zeta}_i, \zeta^i) \longmapsto (\rho, \sigma, -\tilde{\zeta}_i, -\zeta^i)$. Since $\mathbb{C}\mathrm{H}^n$ is irreducible, we see that $\mathbb{H}\mathrm{H}^n$ is not a supergravity c-map space if $n > 1$. The case $n = 1$ is also excluded, since the supergravity c-map space associated with a PSK manifold reduced to a point is $\mathbb{C}\mathrm{H}^2$ (belonging to the Hermitian symmetric series) and not $\mathbb{H}\mathrm{H}^1$. ☐

Example 4.3.20. As an example of the supergravity q-map construction, let

$$\mathcal{H} = \{1\} = \{t \in \mathbb{R} \mid h(t) = t^3 = 1\} \subset \mathbb{R}$$

be the only 0-dimensional PSR manifold. The associated undeformed supergravity q-map space $\bar{N}$ is isometric to the 8-dimensional Wolf space of non-compact type $\mathrm{G}_2^*/\mathrm{SO}(4)$, where G_2^* denotes the (unique) non-compact real form of G_2. The quaternionic Kähler manifold $\bar{N}$ admits a deformation by a parameter $c > 0$ to another complete quaternionic Kähler manifold which is in turn not locally homogeneous (see [CDS17, Example 28]). This result will be generalized in Theorem 5.2.6.

As a final remark, it was shown in [CDJL21, Theorem 22] that there exist complete PSR manifolds $\mathcal{H}^{n-1} \subset \mathbb{R}^n$ in each dimension $n > 1$ such that their corresponding undeformed supergravity q-map spaces are not locally homogeneous. In particular, they obtain examples of complete quaternionic Kähler manifolds of negative scalar curvature in all dimensions ≥ 12 for which the isometry group acts with cohomogeneity one [CDJL21, Theorem 3].

Chapter 5

Curvature and symmetries of supergravity c-map spaces

In this first original chapter, we present the results of the study of some curvature and symmetry properties of deformed supergravity c-map spaces. More precisely, in Section 5.1 we compute a formula for the Riemann curvature tensor of the rigid c-map metric in terms of tensors of the ASK manifold. We apply this formula to the case where the base manifold is CASK, relevant for the supergravity c-map construction. In Section 5.2 we obtain one of the main results of this thesis. Here we show that every deformed supergravity c-map space is not locally homogeneous. As a corollary, we obtain that the deformed supergravity c-map applied to a simply connected homogeneous PSK manifold gives us a complete cohomogeneity one quaternionic Kähler manifold of negative scalar curvature. Section 5.3 contains some known results on two ways to obtain Hamiltonian automorphisms of a rigid c-map space and how these automorphisms interact. Finally, in Section 5.4 we first explain how to construct a global action on the trivial circle bundle over the rigid c-map space used in the HK/QK correspondence (seen as a twist). Then we show that this action restricts to an effective an isometric action on the quaternionic Kähler manifold, considered as a submanifold of the total space of the circle bundle, for the case of supergravity q-map spaces. This gives us the second main result of the thesis.

5.1 Curvature of rigid c-map spaces

The main purpose of this section is to obtain an explicit expression for the Riemann curvature tensor of a rigid c-map space $N = T^*M$ purely in terms of the ASK structure of M. This section contains the results of [CGS23, Section 3.1]. Here we use the Einstein summation convention when working on local coordinates.

As noted in Section 4.2, the tangent space $TN = T(T^*M)$ of the total space of the cotangent bundle $N = T^*M$ of an ASK manifold M can be identified, using the flat connection ∇, with $\pi^*(TM) \oplus \pi^*(T^*M)$. This allows us to relate the Riemann curvature of g_N to pullbacks of tensor fields defined on the base M.

The computation of the Riemann curvature tensor Rm_N is long but straightforward. We will divide it in several steps. First, we compute the Levi-Civita connection D^N and the Christoffel symbols Γ^N of the rigid c-map metric g_N in terms of the Christoffel symbols Γ of the ASK metric g. After this, we compute the Riemann curvature tensor Rm_N in local coordinates. Finally, we express the curvature in a coordinate independent way.

Let (M,g,J,∇) be an ASK manifold and let $\{q^j\}$ be a set of ∇-affine local coordinates given by Theorem 4.1.17. Consider the induced coordinates $\{q^j, p_j\}$ on $N = T^*M$. Recall that we have the orthogonal decomposition $TN = T^{\mathrm{H}}N \oplus T^{\mathrm{V}}N$, so we denote

$$\partial_j := \frac{\partial}{\partial q^j} \in \Gamma(T^{\mathrm{H}}N) \quad \text{and} \quad \partial_{\tilde{k}} := \frac{\partial}{\partial p_k} \in \Gamma(T^{\mathrm{V}}N).$$

Lemma 5.1.1. *Let (M,g,J,∇) be an ASK manifold and let $\{q^j, p_j\}$ be local coordinates on the associated rigid c-map space $(N = T^*M, g_N, I_1, I_2, I_3)$. Then:*

(1) *The Levi-Civita connection D^N of g_N is given by*

$$\begin{aligned}
g_N(D^N_{\partial_i}\partial_j, \partial_k) &= \Gamma_{ijk}, & g_N(D^N_{\partial_{\tilde{i}}}\partial_j, \partial_k) &= 0,\\
g_N(D^N_{\partial_i}\partial_j, \partial_{\tilde{k}}) &= 0, & g_N(D^N_{\partial_{\tilde{i}}}\partial_j, \partial_{\tilde{k}}) &= -\Gamma^{ik}_{j},\\
g_N(D^N_{\partial_i}\partial_{\tilde{j}}, \partial_k) &= 0, & g_N(D^N_{\partial_{\tilde{i}}}\partial_{\tilde{j}}, \partial_k) &= \Gamma^{ij}_{k},\\
g_N(D^N_{\partial_i}\partial_{\tilde{j}}, \partial_{\tilde{k}}) &= -\Gamma^{jk}_{i}, & g_N(D^N_{\partial_{\tilde{i}}}\partial_{\tilde{j}}, \partial_{\tilde{k}}) &= 0.
\end{aligned}$$

(2) *The Christoffel symbols Γ^N of g_N are given by*

$$\begin{aligned}
(\Gamma^N)^s_{ij} &= \Gamma^s_{ij}, & (\Gamma^N)^s_{\tilde{i}j} &= 0,\\
(\Gamma^N)^{\tilde{t}}_{ij} &= 0, & (\Gamma^N)^{\tilde{t}}_{\tilde{i}j} &= -\Gamma^i_{jt},\\
(\Gamma^N)^s_{i\tilde{j}} &= 0, & (\Gamma^N)^s_{\tilde{i}\tilde{j}} &= \Gamma^{ijs},\\
(\Gamma^N)^{\tilde{t}}_{i\tilde{j}} &= -\Gamma^j_{it}, & (\Gamma^N)^{\tilde{t}}_{\tilde{i}\tilde{j}} &= 0.
\end{aligned}$$

Proof. (1) For the following computations recall the Koszul formula

$$\begin{aligned}
2g_N(D^N_X Y, Z) = {} & X g_N(Y,Z) + Y g_N(X,Z) - Z g_N(X,Y)\\
& - g_N([Y,Z],X) - g_N([X,Z],Y) + g_N([X,Y],Z)
\end{aligned}$$

and the fact that coordinate vector fields always commute.

The ASK manifold M is a totally geodesic submanifold of N since M is the set of fixed points of the isometry $(q,p) \longmapsto (q,-p)$. This implies that $D^N_{\partial_i}\partial_j$ is again horizontal, thus we get

$$g_N(D^N_{\partial_i}\partial_j, \partial_k) = \Gamma_{ijk} \quad \text{and} \quad g_N(D^N_{\partial_i}\partial_j, \partial_{\tilde{k}}) = 0,$$

since the decomposition $TN = T^{\mathrm{H}}N \oplus T^{\mathrm{V}}N$ is orthogonal.

Let us compute the other terms of the Levi-Civita connection D^N. We apply the Koszul formula, the fact that g_N only depends on the $\{q^j\}$ coordinates and that horizontal and vertical vector fields are perpendicular:

$$2g_N(D^N_{\partial_i}\partial_{\tilde{j}}, \partial_k) = \partial_i g_N(\partial_{\tilde{j}}, \partial_k) + \partial_{\tilde{j}} g_N(\partial_i, \partial_k) - \partial_k g_N(\partial_i, \partial_{\tilde{j}}) = \partial_{\tilde{j}} g_{ik} = 0.$$

Since the Christoffel symbols of g are given by $\Gamma_{ijk} = \frac{1}{2}\partial_i g_{jk}$ by Lemma 4.1.25, we get:

$$\begin{aligned} 2g_N(D^N_{\partial_i}\partial_{\tilde{j}}, \partial_{\tilde{k}}) &= \partial_i g_N(\partial_{\tilde{j}}, \partial_{\tilde{k}}) + \partial_{\tilde{j}} g_N(\partial_i, \partial_{\tilde{k}}) - \partial_{\tilde{k}} g_N(\partial_i, \partial_{\tilde{j}}) = \partial_i g^{jk} \\ &= -g^{js}\partial_i g_{st} g^{tk} = -2\Gamma^j_{it} g^{tk} = -2\Gamma^{jk}_i. \end{aligned}$$

Using moreover that $D^N g_N = 0$, the other terms are computed in a similar way:

$$\begin{aligned} 2g_N(D^N_{\partial_{\tilde{i}}}\partial_j, \partial_k) &= \partial_{\tilde{i}} g_N(\partial_j, \partial_k) + \partial_j g_N(\partial_{\tilde{i}}, \partial_k) - \partial_k g_N(\partial_{\tilde{i}}, \partial_j) = \partial_{\tilde{i}} g_{jk} = 0, \\ 2g_N(D^N_{\partial_{\tilde{i}}}\partial_j, \partial_{\tilde{k}}) &= \partial_{\tilde{i}} g_N(\partial_j, \partial_{\tilde{k}}) + \partial_j g_N(\partial_{\tilde{i}}, \partial_{\tilde{k}}) - \partial_{\tilde{k}} g_N(\partial_{\tilde{i}}, \partial_j) = \partial_j g^{ik} \\ &= -g^{is}\partial_j g_{st} g^{tk} = -2\Gamma^i_{jt} g^{tk} = -2\Gamma^{ik}_j, \\ g_N(D^N_{\partial_{\tilde{i}}}\partial_{\tilde{j}}, \partial_k) &= \partial_{\tilde{i}} g_N(\partial_{\tilde{j}}, \partial_k) - g_N(\partial_{\tilde{j}}, D^N_{\partial_{\tilde{i}}}\partial_k) = \Gamma^{ij}_k, \\ 2g_N(D^N_{\partial_{\tilde{i}}}\partial_{\tilde{j}}, \partial_{\tilde{k}}) &= \partial_{\tilde{i}} g_N(\partial_{\tilde{j}}, \partial_{\tilde{k}}) + \partial_{\tilde{j}} g_N(\partial_{\tilde{i}}, \partial_{\tilde{k}}) - \partial_{\tilde{k}} g_N(\partial_{\tilde{i}}, \partial_{\tilde{j}}) \\ &= \partial_{\tilde{i}} g^{jk} + \partial_{\tilde{j}} g^{ik} - \partial_{\tilde{k}} g^{ij} = 0. \end{aligned}$$

(2) We only prove the first formula since the other ones are proved in a similar way. Using (1) we have that

$$(\Gamma^N)^s_{ij} g_{sr} = g_N((\Gamma^N)^s_{ij}\partial_s, \partial_r) = g_N(D^N_{\partial_i}\partial_j, \partial_r) = \Gamma_{ijr}.$$

Therefore we obtain

$$(\Gamma^N)^u_{ij} = (\Gamma^N)^s_{ij} g_{sr} g^{ru} = \Gamma_{ijr} g^{ru} = \Gamma^u_{ij}.$$

In particular, we note that the Christoffel symbols Γ^N of g_N only depend on the coordinates $\{q^j\}$ of M. □

Using the information obtained in Lemma 5.1.1, we are able to compute the Riemann curvature tensor Rm_N in local coordinates.

Proposition 5.1.2. *Let (M,g,J,∇) be an ASK manifold and let $\{q^j,p_j\}$ be local coordinates on the associated rigid c-map space $(N=T^*M,g_N,I_1,I_2,I_3)$. Then:*

$$\begin{aligned}
&(\mathrm{Rm}_N)_{ijk\ell} = \Gamma^s_{ik}\Gamma_{js\ell} - \Gamma^s_{jk}\Gamma_{is\ell},\\
&(\mathrm{Rm}_N)_{ijk\tilde{\ell}} = 0,\\
&(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}k\ell} = \Gamma^i_{ks}\Gamma^{js}_{\ell} - \Gamma^j_{ks}\Gamma^{is}_{\ell},\\
&(\mathrm{Rm}_N)_{i\tilde{j}k\tilde{\ell}} = -\tfrac{1}{2}\partial_i\partial_k g_{sr}g^{rj}g^{s\ell} + 2\Gamma^{s\ell}_{k}\Gamma^j_{is} + \Gamma^j_{ks}\Gamma^{s\ell}_{i} + \Gamma^s_{ik}\Gamma^{j\ell}_{s},\\
&(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}\tilde{k}\ell} = 0,\\
&(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}\tilde{k}\tilde{\ell}} = \Gamma^{iks}\Gamma^{j\ell}_{s} - \Gamma^{jks}\Gamma^{i\ell}_{s}.
\end{aligned}$$

Proof. We compute each term although all computations will be similar. First of all, recall that the Christoffel symbols Γ^N of g_N only depend on the base coordinates $\{q^j\}$. For the first term we have

$$\begin{aligned}
g_N(D^N_{\partial_i}D^N_{\partial_j}\partial_k,\partial_\ell) &= g_N(D^N_{\partial_i}((\Gamma^N)^s_{jk}\partial_s + (\Gamma^N)^{\tilde{t}}_{jk}\partial_{\tilde{t}}),\partial_\ell) = g_N(D^N_{\partial_i}(\Gamma^s_{jk}\partial_s),\partial_\ell)\\
&= \partial_i\Gamma^s_{jk}g_N(\partial_s,\partial_\ell) + \Gamma^s_{jk}g_N(D^N_{\partial_i}\partial_s,\partial_\ell)\\
&= \partial_i\Gamma^s_{jk}g_{s\ell} + \Gamma^s_{jk}\Gamma_{is\ell},
\end{aligned}$$

thus

$$(\mathrm{Rm}_N)_{ijk\ell} = (\partial_i\Gamma^s_{jk} - \partial_j\Gamma^s_{ik})g_{s\ell} + \Gamma^s_{jk}\Gamma_{is\ell} - \Gamma^s_{ik}\Gamma_{js\ell}.$$

The first summand is equal to (see proof of Corollary 4.1.31)

$$(\partial_i\Gamma^s_{jk} - \partial_j\Gamma^s_{ik})g_{s\ell} = -2(\Gamma^\alpha_{jk}\Gamma^s_{i\alpha} - \Gamma^\alpha_{ik}\Gamma^s_{j\alpha})g_{s\ell} = -2(\Gamma^\alpha_{jk}\Gamma_{i\alpha\ell} - \Gamma^\alpha_{ik}\Gamma_{j\alpha\ell}).$$

Then we obtain

$$(\mathrm{Rm}_N)_{ijk\ell} = \Gamma^s_{ik}\Gamma_{js\ell} - \Gamma^s_{jk}\Gamma_{is\ell}.$$

For the second term we have

$$\begin{aligned}
g_N(D^N_{\partial_i}D^N_{\partial_j}\partial_k,\partial_{\tilde{\ell}}) &= g_N(D^N_{\partial_i}((\Gamma^N)^s_{jk}\partial_s + (\Gamma^N)^{\tilde{t}}_{jk}\partial_{\tilde{t}}),\partial_{\tilde{\ell}}) = g_N(D^N_{\partial_i}(\Gamma^s_{jk}\partial_s),\partial_{\tilde{\ell}})\\
&= \partial_i\Gamma^s_{jk}g_N(\partial_s,\partial_{\tilde{\ell}}) + \Gamma^s_{jk}g_N(D^N_{\partial_i}\partial_s,\partial_{\tilde{\ell}}) = 0,
\end{aligned}$$

hence $(\mathrm{Rm}_N)_{ijk\tilde{\ell}} = 0$. For the third term we have

$$\begin{aligned}
g_N(D^N_{\partial_{\tilde{i}}}D^N_{\partial_{\tilde{j}}}\partial_k,\partial_\ell) &= g_N(D^N_{\partial_{\tilde{i}}}((\Gamma^N)^s_{\tilde{j}k}\partial_s + (\Gamma^N)^{\tilde{t}}_{\tilde{j}k}\partial_{\tilde{t}}),\partial_\ell)\\
&= (\Gamma^N)^{\tilde{t}}_{\tilde{j}k}g_N(D^N_{\partial_{\tilde{i}}}\partial_{\tilde{t}},\partial_\ell) = -\Gamma^j_{kt}\Gamma^{it}_{\ell},
\end{aligned}$$

hence

$$(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}k\ell} = \Gamma^i_{kt}\Gamma^{jt}_{\ell} - \Gamma^j_{kt}\Gamma^{it}_{\ell}.$$

For the fourth term we have

$$\begin{aligned}
g_N(D^N_{\partial_i}D^N_{\partial_{\tilde{j}}}\partial_k,\partial_{\tilde{\ell}}) &= g_N(D^N_{\partial_i}((\Gamma^N)^s_{\tilde{j}k}\partial_s+(\Gamma^N)^{\tilde{t}}_{\tilde{j}k}\partial_{\tilde{t}}),\partial_{\tilde{\ell}}) = -g_N(D^N_{\partial_i}(\Gamma^j_{kt}\partial_{\tilde{t}}),\partial_{\tilde{\ell}})\\
&= -\partial_i\Gamma^j_{kt}g_N(\partial_{\tilde{t}},\partial_{\tilde{\ell}})-\Gamma^j_{kt}g_N(D^N_{\partial_i}\partial_{\tilde{t}},\partial_{\tilde{\ell}})\\
&= -\partial_i\Gamma^j_{kt}g^{t\ell}+\Gamma^j_{kt}\Gamma^{t\ell}_i,\\
g_N(D^N_{\partial_{\tilde{j}}}D^N_{\partial_i}\partial_k,\partial_{\tilde{\ell}}) &= g_N(D^N_{\partial_{\tilde{j}}}((\Gamma^N)^s_{ik}\partial_s+(\Gamma^N)^{\tilde{t}}_{ik}\partial_{\tilde{t}}),\partial_{\tilde{\ell}})\\
&= (\Gamma^N)^s_{ik}g_N(D^N_{\partial_{\tilde{j}}}\partial_s,\partial_{\tilde{\ell}}) = -\Gamma^s_{ik}\Gamma^{j\ell}_s.
\end{aligned}$$

The derivative of the Christoffel symbol Γ of g is given by

$$\partial_i\Gamma^j_{kt} = \tfrac{1}{2}\partial_i\partial_k g_{tr}g^{rj}-2\Gamma^\alpha_{kt}\Gamma^j_{i\alpha},$$

so we have

$$\partial_i\Gamma^j_{kt}g^{t\ell} = \tfrac{1}{2}\partial_i\partial_k g_{tr}g^{rj}g^{t\ell}-2\Gamma^{\alpha\ell}_k\Gamma^j_{i\alpha}.$$

Therefore

$$(\mathrm{Rm}_N)_{i\tilde{j}k\tilde{\ell}} = -\tfrac{1}{2}\partial_i\partial_k g_{sr}g^{rj}g^{s\ell}+2\Gamma^{s\ell}_k\Gamma^j_{is}+\Gamma^j_{ks}\Gamma^{s\ell}_i+\Gamma^s_{ik}\Gamma^{j\ell}_s.$$

For the fifth term we have

$$g_N(D^N_{\partial_{\tilde{i}}}D^N_{\partial_{\tilde{j}}}\partial_{\tilde{k}},\partial_\ell) = g_N(D^N_{\partial_{\tilde{i}}}((\Gamma^N)^s_{\tilde{j}\tilde{k}}\partial_s+(\Gamma^N)^{\tilde{t}}_{\tilde{j}\tilde{k}}\partial_{\tilde{t}}),\partial_\ell) = (\Gamma^N)^s_{\tilde{j}\tilde{k}}g_N(D^N_{\partial_{\tilde{i}}}\partial_s,\partial_\ell) = 0,$$

so $(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}\tilde{k}\ell} = 0$. For the last term we have

$$\begin{aligned}
g_N(D^N_{\partial_{\tilde{i}}}D^N_{\partial_{\tilde{j}}}\partial_{\tilde{k}},\partial_{\tilde{\ell}}) &= g_N(D^N_{\partial_{\tilde{i}}}((\Gamma^N)^s_{\tilde{j}\tilde{k}}\partial_s+(\Gamma^N)^{\tilde{t}}_{\tilde{j}\tilde{k}}\partial_{\tilde{t}}),\partial_{\tilde{\ell}})\\
&= (\Gamma^N)^s_{\tilde{j}\tilde{k}}g_N(D^N_{\partial_{\tilde{i}}}\partial_s,\partial_{\tilde{\ell}}) = -\Gamma^{jks}\Gamma^{i\ell}_s.
\end{aligned}$$

Therefore $(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}\tilde{k}\tilde{\ell}} = \Gamma^{iks}\Gamma^{j\ell}_s-\Gamma^{jks}\Gamma^{i\ell}_s$. □

The last step is to express the formulas obtained in Proposition 5.1.2 in a coordinate independent way in terms of tensors defined on the base manifold M.

Lemma 5.1.3. *Let (M,g,J,∇) be an ASK manifold and let $\{q^j,p_j\}$ be local coordinates on the associated rigid c-map space $(N=T^*M,g_N,I_1,I_2,I_3)$. Then the curvature* *tensor* Rm_N *of* N *obtained in Proposition 5.1.2 can be expressed as follows:*

$$
\begin{aligned}
(\mathrm{Rm}_N)_{ijk\ell} &= -\tfrac{1}{4}g([\mathcal{S}_{\partial_i},\mathcal{S}_{\partial_j}]\partial_k,\partial_\ell),\\
(\mathrm{Rm}_N)_{ijk\tilde{\ell}} &= \ \ 0,\\
(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}k\ell} &= -\tfrac{1}{4}g([\mathcal{S}_{(\mathrm{d}q^i)^\sharp},\mathcal{S}_{(\mathrm{d}q^j)^\sharp}]\partial_k,\partial_\ell),\\
(\mathrm{Rm}_N)_{i\tilde{j}k\tilde{\ell}} &= \ \ \tfrac{1}{2}g(\mathcal{S}_{\partial_i}\mathcal{S}_{\partial_k}(\mathrm{d}q^\ell)^\sharp,(\mathrm{d}q^j)^\sharp)\\
&\quad+\tfrac{1}{4}g(\mathcal{S}_{\partial_k}\mathcal{S}_{\partial_i}(\mathrm{d}q^\ell)^\sharp,(\mathrm{d}q^j)^\sharp)\\
&\quad+\tfrac{1}{4}g(\mathcal{S}_{\partial_i}\partial_k,\mathcal{S}_{(\mathrm{d}q^j)^\sharp}(\mathrm{d}q^\ell)^\sharp)\\
&\quad-\tfrac{1}{2}(\nabla^2_{\partial_i,(\mathrm{d}q^j)^\sharp}g)(\partial_k,(\mathrm{d}q^\ell)^\sharp),\\
(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}\tilde{k}\ell} &= \ \ 0,\\
(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}\tilde{k}\tilde{\ell}} &= -\tfrac{1}{4}g([\mathcal{S}_{(\mathrm{d}q^i)^\sharp},\mathcal{S}_{(\mathrm{d}q^j)^\sharp}](\mathrm{d}q^k)^\sharp,(\mathrm{d}q^\ell)^\sharp).
\end{aligned}
$$

Proof. For the first term, by looking at the proof of Corollary 4.1.31, we have

$$(\mathrm{Rm}_N)_{ijk\ell} = -\tfrac{1}{4}g([\mathcal{S}_{\partial_i},\mathcal{S}_{\partial_j}]\partial_k,\partial_\ell) = g_{\ell m}(\Gamma^p_{ik}\Gamma^m_{jp}-\Gamma^p_{jk}\Gamma^m_{ip}) = \Gamma^p_{ik}\Gamma_{jp\ell}-\Gamma^p_{jk}\Gamma_{ip\ell}.$$

Notice that $\partial_{\tilde{i}}\in\Gamma(T^{\mathrm{V}}N)$ corresponds to the 1-form $\mathrm{d}q^i$ on M and $(\mathrm{d}q^i)^\sharp = g^{iu}\partial_u$. Then, for the third term we have

$$
\begin{aligned}
(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}k\ell} &= -\tfrac{1}{4}g([\mathcal{S}_{(\mathrm{d}q^i)^\sharp},\mathcal{S}_{(\mathrm{d}q^j)^\sharp}]\partial_k,\partial_\ell) = -\tfrac{1}{4}g^{iu}g^{jv}g([\mathcal{S}_{\partial_u},\mathcal{S}_{\partial_v}]\partial_k,\partial_\ell)\\
&= g^{iu}g^{jv}(\Gamma^p_{uk}\Gamma_{vp\ell}-\Gamma^p_{vk}\Gamma_{up\ell}) = \Gamma^{ip}_k\Gamma^j_{p\ell}-\Gamma^{jp}_k\Gamma^i_{p\ell}\\
&= \Gamma^i_{kp}\Gamma^{jp}_{\ell}-\Gamma^j_{kp}\Gamma^{ip}_{\ell},
\end{aligned}
$$

where in the last equality we have used that $(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}k\ell} = -(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}\ell k}$.

For the fourth term, first recall that $\mathcal{S}_{\partial_i}\partial_j = \mathcal{S}^k_{ij}\partial_k$ and $\mathcal{S}^k_{ij} = 2\Gamma^k_{ij}$ by (10). We compute each of the summands:

$$
\begin{aligned}
\tfrac{1}{2}g(\mathcal{S}_{\partial_i}\mathcal{S}_{\partial_k}(\mathrm{d}q^\ell)^\sharp,(\mathrm{d}q^j)^\sharp) &= \tfrac{1}{2}g^{jv}g^{\ell x}g(\mathcal{S}_{\partial_i}\mathcal{S}_{\partial_k}\partial_x,\partial_v) = \tfrac{1}{2}\mathcal{S}^\alpha_{kx}\mathcal{S}^\beta_{i\alpha}g^{jv}g^{\ell x}g_{\beta v}\\
&= 2\Gamma^\alpha_{kx}\Gamma^\beta_{i\alpha}g^{jv}g^{\ell x}g_{\beta v} = 2\Gamma^{\alpha\ell}_k\Gamma^j_{i\alpha},\\
\tfrac{1}{4}g(\mathcal{S}_{\partial_k}\mathcal{S}_{\partial_i}(\mathrm{d}q^\ell)^\sharp,(\mathrm{d}q^j)^\sharp) &= \Gamma^{\alpha\ell}_i\Gamma^j_{k\alpha},\\
\tfrac{1}{4}g(\mathcal{S}_{\partial_i}\partial_k,\mathcal{S}_{(\mathrm{d}q^j)^\sharp}(\mathrm{d}q^\ell)^\sharp) &= \tfrac{1}{4}g^{jv}g^{\ell x}g(\mathcal{S}_{\partial_i}\partial_k,\mathcal{S}_{\partial_v}\partial_x) = \tfrac{1}{4}\mathcal{S}^\alpha_{ik}\mathcal{S}^\beta_{vx}g^{jv}g^{\ell x}g_{\alpha\beta}\\
&= \Gamma^\alpha_{ik}\Gamma^\beta_{vx}g^{jv}g^{\ell x}g_{\alpha\beta} = \Gamma^\alpha_{ik}\Gamma^{j\ell}_\alpha,\\
-\tfrac{1}{2}(\nabla^2_{\partial_i,(\mathrm{d}q^j)^\sharp}g)(\partial_k,(\mathrm{d}q^\ell)^\sharp) &= -\tfrac{1}{2}g^{jv}g^{\ell x}(\nabla^2_{\partial_i,\partial_v}g)(\partial_k,\partial_x) = -\tfrac{1}{2}g^{jv}g^{\ell x}(\nabla^2 g)_{ivkx}\\
&= -\tfrac{1}{2}g^{jv}g^{\ell x}\partial_i\partial_v g_{kx} = -\tfrac{1}{2}g^{jv}g^{\ell x}\partial_i\partial_k g_{vx},
\end{aligned}
$$

where in the last equality we have used that $\partial_v g_{kx} = \partial_k g_{vx}$ since $(\nabla g)_{ijk} = \partial_i g_{jk}$ is totally symmetric by Lemma 4.1.3. Adding all the summands we get the fourth term.

For the last term we have

$$\begin{aligned}(\mathrm{Rm}_N)_{\tilde{i}\tilde{j}\tilde{k}\tilde{\ell}} &= -\tfrac{1}{4}g([\mathcal{S}_{(\mathrm{d}q^i)^\sharp},\mathcal{S}_{(\mathrm{d}q^j)^\sharp}](\mathrm{d}q^k)^\sharp,(\mathrm{d}q^\ell)^\sharp)\\ &= -\tfrac{1}{4}g^{iu}g^{jv}g^{kw}g^{\ell x}g([\mathcal{S}_{\partial_u},\mathcal{S}_{\partial_v}]\partial_w,\partial_x)\\ &= g^{iu}g^{jv}g^{kw}g^{\ell x}(\Gamma^p_{uw}\Gamma_{vpx}-\Gamma^p_{vw}\Gamma_{upx})\\ &= \Gamma^{ikp}\Gamma^{j\ell}_{p}-\Gamma^{jkp}\Gamma^{i\ell}_{p}.\end{aligned}$$

□

After these computations, we can state the formula for the curvature tensor of any rigid c-map space.

Theorem 5.1.4. *Let (M,g,J,∇) be an ASK manifold and $(N=T^*M,g_N,I_1,I_2,I_3)$ the associated rigid c-map space. Then the curvature tensor Rm_N of N is given by*

$$\begin{aligned}\mathrm{Rm}_N(A^{\mathrm{H}},B^{\mathrm{H}},C^{\mathrm{H}},X^{\mathrm{H}}) &= -\tfrac{1}{4}g\big([\mathcal{S}_{A^{\mathrm{H}}},\mathcal{S}_{B^{\mathrm{H}}}]C^{\mathrm{H}},X^{\mathrm{H}}\big),\\ \mathrm{Rm}_N(A^{\mathrm{H}},B^{\mathrm{H}},C^{\mathrm{H}},X^{\mathrm{V}}) &= \quad 0,\\ \mathrm{Rm}_N(A^{\mathrm{H}},B^{\mathrm{H}},C^{\mathrm{V}},X^{\mathrm{V}}) &= -\tfrac{1}{4}g\big([\mathcal{S}_{A^{\mathrm{H}}},\mathcal{S}_{B^{\mathrm{H}}}](C^{\mathrm{V}})^\sharp,(X^{\mathrm{V}})^\sharp\big),\\ \mathrm{Rm}_N(A^{\mathrm{H}},B^{\mathrm{V}},C^{\mathrm{H}},X^{\mathrm{V}}) &= \quad \tfrac{1}{2}g\big(\mathcal{S}_{A^{\mathrm{H}}}\mathcal{S}_{C^{\mathrm{H}}}(X^{\mathrm{V}})^\sharp,(B^{\mathrm{V}})^\sharp\big)\\ &\quad +\tfrac{1}{4}g\big(\mathcal{S}_{C^{\mathrm{H}}}\mathcal{S}_{A^{\mathrm{H}}}(X^{\mathrm{V}})^\sharp,(B^{\mathrm{V}})^\sharp\big)\\ &\quad +\tfrac{1}{4}g\big(\mathcal{S}_{A^{\mathrm{H}}}C^{\mathrm{H}},\mathcal{S}_{(B^{\mathrm{V}})^\sharp}(X^{\mathrm{V}})^\sharp\big)\\ &\quad -\tfrac{1}{2}\big(\nabla^2_{A^{\mathrm{H}},(B^{\mathrm{V}})^\sharp}g\big)\big(C^{\mathrm{H}},(X^{\mathrm{V}})^\sharp\big),\\ \mathrm{Rm}_N(A^{\mathrm{H}},B^{\mathrm{V}},C^{\mathrm{V}},X^{\mathrm{V}}) &= \quad 0,\\ \mathrm{Rm}_N(A^{\mathrm{V}},B^{\mathrm{V}},C^{\mathrm{V}},X^{\mathrm{V}}) &= -\tfrac{1}{4}g\big([\mathcal{S}_{(A^{\mathrm{V}})^\sharp},\mathcal{S}_{(B^{\mathrm{V}})^\sharp}](C^{\mathrm{V}})^\sharp,(X^{\mathrm{V}})^\sharp\big),\end{aligned}$$

where $A,B,C,X\in T_pN$, $p\in N$, and $X^{\mathrm{H}}\in T_p^{\mathrm{H}}N$, $X^{\mathrm{V}}\in T_p^{\mathrm{V}}N$ are, respectively, horizontal and vertical components. Moreover, on the right-hand side of these formulas, horizontal and vertical vectors are identified with elements of $T_{\pi(p)}M$ and $T^*_{\pi(p)}M$, respectively, and $\alpha^\sharp\in T_{\pi(p)}M$ denotes the metric dual of $\alpha\in T^*_{\pi(p)}M$.

Proof. As we have explained, this result is obtained from a long but straightforward computation in local coordinates $\{q^j,p_j\}$ on N induced by local ∇-affine coordinates $\{q^j\}$ on M. First, in Lemma 5.1.1 we compute the Christoffel symbols of (N,g_N) in terms of the Christoffel symbols of (M,g) given in (10). Then, in Proposition 5.1.2 we compute the curvature tensor of (N,g_N) in terms of the tensor $\mathcal{S}$ and the curvature tensor of (M,g), given in Proposition 4.1.29 also in terms of $\mathcal{S}$. We conclude in Lemma 5.1.3 by expressing the final result in a coordinate independent way using only the above intrinsic identifications and basic properties of ASK manifolds (such as the complete symmetry of ∇g). □

Note that the remaining components of the Riemann curvature follow from the above by symmetries of the curvature tensor and that $\nabla^2 g$ coincides with ∇S, where S is the totally symmetric $(0,3)$-tensor which corresponds to the $(1,2)$-tensor $\mathcal{S}$.

Corollary 5.1.5. *Let (M,g,J,∇) be an ASK manifold and $(N=T^*M,g_N,I_1,I_2,I_3)$ the associated rigid c-map space. If $\nabla = D$, where D is the Levi-Civita connection of g, then* $\mathrm{Rm}_N = 0$.

Proof. If $\nabla = D$, then $\nabla g = 0$ since the Levi-Civita connection is metric. This implies that $\mathcal{S} = g^{-1}\nabla g = 0$ and then $\mathrm{Rm}_N = 0$ by Theorem 5.1.4. □

In the case where the ASK manifold M is furthermore CASK, we can say something additional.

Proposition 5.1.6. *Let (M,g,J,∇,ξ) be a CASK manifold and $(N=T^*M,g_N,I_1,I_2,I_3)$ the associated rigid c-map space. Denote $\mathcal{Z} := (\mathbb{H}Z)^*$ and $\mathcal{Z}^\perp := ((\mathbb{H}Z)^\perp)^*$. Then the curvature tensor* Rm_N *of N is a section of the subbundle*

$$\mathrm{Sym}^2(\Lambda^2\mathcal{Z}^\perp)\oplus\left(\Lambda^2\mathcal{Z}^\perp\vee(\mathcal{Z}^\perp\wedge\mathcal{Z})\right)\subset\mathrm{Sym}^2(\Lambda^2T^*N),$$

where we are using the isomorphism $T^*N\cong(\mathbb{H}Z)^*\oplus((\mathbb{H}Z)^\perp)^*$ corresponding to the decomposition $TN=\mathbb{H}Z\oplus(\mathbb{H}Z)^\perp$ and $\vee$ denotes the symmetric tensor product. In particular, $\mathrm{Rm}_N(A,B,C,X)=0$ *if at least two of the vectors A,B,C,X belong to $\mathbb{H}Z$.*

Proof. We have seen that the curvature of N is completely determined by tensors on the base M. Under the identifications $T_p^{\mathrm{H}}N\cong T_{\pi(p)}M$ and $T_p^{\mathrm{V}}N\cong T^*_{\pi(p)}M$ the horizontal vector fields Z, I_1Z on N are identified with the vector fields $-J\xi,\xi$ on M, and the vertical vector fields I_2Z, I_3Z with the 1-forms $\xi^\flat,(-J\xi)^\flat$ (with the convention $\omega = g(J\cdot,\cdot)$). Every term in Theorem 5.1.4 can be expressed in terms of the tensor $\mathcal{S}$. From Corollary 4.1.28 we know that $\mathcal{S}$ vanishes on ξ and $J\xi$, therefore all the curvature elements are zero taking into account that S and $\nabla^2 g$ are totally symmetric. In fact, the total symmetry of $S=\nabla g$ was stated in Lemma 4.1.26 and implies that of $\nabla S = \nabla^2 g$ using that $\nabla^2_{A,B}g = \nabla^2_{B,A}g$ since ∇ is flat. □

5.2 Norm of the curvature tensor

The purpose of this section is to show that any deformed supergravity c-map is not locally homogeneous. This section contains the results of [CGS23, Section 3.2].

Let us start by defining what is a locally homogeneous manifold.

Definition 5.2.1. Let (M,g) be a Riemannian manifold of dimension n. We say that it is **locally homogeneous** if for all $x\in M$ there exist n Killing vector fields defined in a neighborhood of x which are linearly independent at x.

Note that a function on a connected locally homogeneous Riemannian manifold which is invariant under any locally defined isometry is necessarily constant. Thus, to show that a Riemannian manifold is not locally homogeneous, it is enough to show that a scalar curvature invariant is not constant. Since quaternionic Kähler manifolds are Einstein (see Theorem 2.2.13), the scalar curvature is constant, so we have to use another scalar curvature invariant, namely the (square of the) norm of the curvature tensor.

We now show that the norm of the curvature of the deformed supergravity c-map metric $g^c_{\bar{N}}$ associated to the PSK manifold $\bar{M}$ is not constant on the manifold $\bar{N}$ for $c > 0$. Since

$$g^c_{\bar{N}} \sim_{\mathscr{H}} g^c_{\mathrm{H}} \quad \text{and} \quad \mathrm{Rm}^c_{\bar{N}} \sim_{\mathscr{H}} \mathrm{Rm}^c_{\mathrm{H}}$$

by Theorem 3.3.9 and Theorem 3.3.11, respectively, this is equivalent to show that the function $\|\mathrm{Rm}^c_{\mathrm{H}}\|^2_{g^c_{\mathrm{H}}}$ is not constant on the rigid c-map space $N = T^*M$, where M is the CASK manifold associated to $\bar{M}$.

In order to compute this norm, we work in a g_N-orthonormal frame $\{e_i, \varepsilon_\mu\}$ of TN that is adapted to the quaternionic distribution $\mathbb{H}Z = \mathrm{span}\{Z, I_1 Z, I_2 Z, I_3 Z\}$. This means that $\{e_i\}$ span the distribution $\mathbb{H}Z$ and $\{\varepsilon_\mu\}$ span the orthogonal complement $(\mathbb{H}Z)^\perp$.

In terms of this frame, the norm of an abstract $(0,4)$-curvature tensor $\mathcal{C}$ with respect to the metric g^c_{H} is given by

$$\begin{aligned}
\|\mathcal{C}\|^2_{g^c_{\mathrm{H}}} = \hat{g}^c_{\mathrm{H}}(\mathcal{C},\mathcal{C}) = \frac{(f^c_Z)^8}{(f^c_{\mathrm{H}})^4} \sum \mathcal{C}(e_i, e_j, e_k, e_\ell)^2 - 4\frac{(f^c_Z)^7}{(f^c_{\mathrm{H}})^3} \sum \mathcal{C}(\varepsilon_\mu, e_j, e_k, e_\ell)^2 \\
+ 2\frac{(f^c_Z)^6}{(f^c_{\mathrm{H}})^2} \sum \mathcal{C}(\varepsilon_\mu, \varepsilon_\nu, e_k, e_\ell)^2 + 4\frac{(f^c_Z)^6}{(f^c_{\mathrm{H}})^2} \sum \mathcal{C}(\varepsilon_\mu, e_j, \varepsilon_\lambda, e_\ell)^2 \\
- 4\frac{(f^c_Z)^5}{f^c_{\mathrm{H}}} \sum \mathcal{C}(\varepsilon_\mu, \varepsilon_\nu, \varepsilon_\lambda, e_\ell)^2 + (f^c_Z)^4 \sum \mathcal{C}(\varepsilon_\mu, \varepsilon_\nu, \varepsilon_\lambda, \varepsilon_\sigma)^2,
\end{aligned}$$

where $\hat{g}^c_{\mathrm{H}} := \big((g^c_{\mathrm{H}})^{-1}\big)^{\otimes 4}$ denotes the metric on the bundle $(T^*N)^{\otimes 4}$ induced by g^c_{H}.

Let us now specialize Theorem 3.3.11 to the case of the deformed supergravity c-map. Since the decomposition between the hyperkähler part and the projective quaternionic space part is orthogonal, we have

$$\begin{aligned}
\|\mathrm{Rm}^c_{\mathrm{H}}\|^2_{g^c_{\mathrm{H}}} = \frac{1}{(f^c_Z)^2} \|\mathrm{Rm}_N\|^2_{g^c_{\mathrm{H}}} + \frac{1}{(f^c_Z)^2 (f^c_{\mathrm{H}})^2} \|\mathrm{Rm}_{\mathrm{HK}}\|^2_{g^c_{\mathrm{H}}} \\
+ \frac{2}{(f^c_Z)^2 f^c_{\mathrm{H}}} \hat{g}^c_{\mathrm{H}}(\mathrm{Rm}_N, \mathrm{Rm}_{\mathrm{HK}}) + \frac{1}{64} \|\mathrm{Rm}_{\mathbb{HP}}\|^2_{g^c_{\mathrm{H}}}.
\end{aligned}$$

The final term $\frac{1}{64}\|\mathrm{Rm}_{\mathbb{HP}}\|^2_{g^c_{\mathrm{H}}}$ is just a constant depending only on the dimension of N.

Meanwhile the remaining terms can be computed to be

$$\begin{aligned}
\frac{1}{(f_Z^c)^2}\|\mathrm{Rm}_N\|^2_{g^c_{\mathrm{H}}} &= \frac{(f_Z^c)^6}{(f_{\mathrm{H}}^c)^4}R_0^N - 4\frac{(f_Z^c)^5}{(f_{\mathrm{H}}^c)^3}R_1^N + 2\frac{(f_Z^c)^4}{(f_{\mathrm{H}}^c)^2}R_{2\mathrm{a}}^N \\
&\quad + 4\frac{(f_Z^c)^4}{(f_{\mathrm{H}}^c)^2}R_{2\mathrm{b}}^N - 4\frac{(f_Z^c)^3}{f_{\mathrm{H}}^c}R_3^N + (f_Z^c)^2 R_4^N, \\
\frac{1}{(f_Z^c)^2(f_{\mathrm{H}}^c)^2}\|\mathrm{Rm}_{\mathrm{HK}}\|^2_{g^c_{\mathrm{H}}} &= \frac{(f_Z^c)^6}{(f_{\mathrm{H}}^c)^6}R_0^{\mathrm{HK}} - 4\frac{(f_Z^c)^5}{(f_{\mathrm{H}}^c)^5}R_1^{\mathrm{HK}} + 2\frac{(f_Z^c)^4}{(f_{\mathrm{H}}^c)^4}R_{2\mathrm{a}}^{\mathrm{HK}} \\
&\quad + 4\frac{(f_Z^c)^4}{(f_{\mathrm{H}}^c)^4}R_{2\mathrm{b}}^{\mathrm{HK}} - 4\frac{(f_Z^c)^3}{(f_{\mathrm{H}}^c)^3}R_3^{\mathrm{HK}} + \frac{(f_Z^c)^2}{(f_{\mathrm{H}}^c)^2}R_4^{\mathrm{HK}}, \\
\frac{1}{(f_Z^c)^2 f_{\mathrm{H}}^c}\hat{g}^c_{\mathrm{H}}(\mathrm{Rm}_N, \mathrm{Rm}_{\mathrm{HK}}) &= \frac{(f_Z^c)^6}{(f_{\mathrm{H}}^c)^5}R_0^{\mathrm{C}} - 4\frac{(f_Z^c)^5}{(f_{\mathrm{H}}^c)^4}R_1^{\mathrm{C}} + 2\frac{(f_Z^c)^4}{(f_{\mathrm{H}}^c)^3}R_{2\mathrm{a}}^{\mathrm{C}} \\
&\quad + 4\frac{(f_Z^c)^4}{(f_{\mathrm{H}}^c)^3}R_{2\mathrm{b}}^{\mathrm{C}} - 4\frac{(f_Z^c)^3}{(f_{\mathrm{H}}^c)^2}R_3^{\mathrm{C}} + \frac{(f_Z^c)^2}{f_{\mathrm{H}}^c}R_4^{\mathrm{C}}.
\end{aligned}$$

In the above, we have introduced the notation

$$\begin{aligned}
R_0^N &:= \sum \mathrm{Rm}_N(e_i, e_j, e_k, e_\ell)^2, & R_1^N &:= \sum \mathrm{Rm}_N(\varepsilon_\mu, e_j, e_k, e_\ell)^2, \\
R_{2\mathrm{a}}^N &:= \sum \mathrm{Rm}_N(\varepsilon_\mu, \varepsilon_\nu, e_k, e_\ell)^2, & R_{2\mathrm{b}}^N &:= \sum \mathrm{Rm}_N(\varepsilon_\mu, e_j, \varepsilon_\lambda, e_\ell)^2, \\
R_3^N &:= \sum \mathrm{Rm}_N(\varepsilon_\mu, \varepsilon_\nu, \varepsilon_\lambda, e_\ell)^2, & R_4^N &:= \sum \mathrm{Rm}_N(\varepsilon_\mu, \varepsilon_\nu, \varepsilon_\lambda, \varepsilon_\sigma)^2.
\end{aligned}$$

The terms of the form R^{HK} and R^{C} (where C stands for "cross-terms") are defined in a similar way, for example, $R_0^{\mathrm{C}} := \sum \mathrm{Rm}_N(e_i, e_j, e_k, e_\ell)\,\mathrm{Rm}_{\mathrm{HK}}(e_i, e_j, e_k, e_\ell)$. In particular, all the terms R^N and R^{HK} are non-negative functions since they are sums of squares, and by virtue of Proposition 5.1.6, $R_I^N = R_I^{\mathrm{C}} = 0$ for $I = 0, 1, 2\mathrm{a}, 2\mathrm{b}$.

To show that a function on a manifold is not constant, it is enough to find a direction such that the derivative of the function in that direction is not zero. The function we are interested in differentiate is $\|\mathrm{Rm}^c_{\mathrm{H}}\|^2_{g^c_{\mathrm{H}}}$ on the rigid c-map space $N = T^*M$, and as a direction we will take $\Xi \in \Gamma(TN)$, the natural lift of the Euler vector field $\xi \in \Gamma(TM)$ to N, which is given in local ∇-affine coordinates $\{q^j, p_j\}$ by

$$\Xi := \sum \left(q^j \frac{\partial}{\partial q^j} + p_j \frac{\partial}{\partial p_j} \right). \tag{22}$$

The vector field Ξ has the following properties.

Lemma 5.2.2. *Let (M, g, J, ∇, ξ) be a CASK manifold and $(N = T^*M, g_N, I_1, I_2, I_3)$ the associated rigid c-map space. The vector field Ξ given by (22) is tri-holomorphic and moreover satisfies*

$$\mathscr{L}_\Xi g_N = 2 g_N, \quad \mathscr{L}_\Xi \omega_{\mathrm{H}} = 2\omega_{\mathrm{H}}, \quad \mathscr{L}_\Xi f_Z = 2 f_Z, \quad \mathscr{L}_\Xi f_{\mathrm{H}} = 2 f_{\mathrm{H}}.$$

Proof. We use the Einstein summation convention in this proof. With respect to the local ∇-affine coordinates $\{q^j, p_j\}$, the pseudo-hyperkähler structure (12) of the rigid c-map space $N = T^*M$ is given by

$$\begin{aligned} g_N &= g_{ij}\mathrm{d}q^i\mathrm{d}q^j + g^{ij}\mathrm{d}p_i\mathrm{d}p_j, \\ \omega_1 &= \tfrac{1}{2}\omega_{ij}\mathrm{d}q^i \wedge \mathrm{d}q^j + \tfrac{1}{2}\omega^{ij}\mathrm{d}p_i \wedge \mathrm{d}p_j, \\ \omega_2 &= J^i_j\mathrm{d}q^j \wedge \mathrm{d}p_i, \\ \omega_3 &= \mathrm{d}q^i \wedge \mathrm{d}p_i. \end{aligned}$$

First note the following:

$$\mathscr{L}_\Xi \frac{\partial}{\partial q^k} = -\frac{\partial}{\partial q^k}, \quad \mathscr{L}_\Xi \frac{\partial}{\partial p_k} = -\frac{\partial}{\partial p_k}, \quad \mathscr{L}_\Xi \mathrm{d}q^k = \mathrm{d}q^k, \quad \mathscr{L}_\Xi \mathrm{d}p_k = \mathrm{d}p_k.$$

We have that $\Xi|_{TM} = \xi$ and that g_{ij} is a function on the CASK manifold M, then $\mathscr{L}_\Xi g_{ij} = \mathscr{L}_\xi g_{ij}$ and

$$\mathscr{L}_\xi g_{ij} = \mathscr{L}_\xi g(\tfrac{\partial}{\partial q^i}, \tfrac{\partial}{\partial q^j}) = (\mathscr{L}_\xi g)(\tfrac{\partial}{\partial q^i}, \tfrac{\partial}{\partial q^j}) + g(\mathscr{L}_\xi \tfrac{\partial}{\partial q^i}, \tfrac{\partial}{\partial q^j}) + g(\tfrac{\partial}{\partial q^j}, \mathscr{L}_\xi \tfrac{\partial}{\partial q^j}) = 0,$$

since $\mathscr{L}_\xi g = 2g$ by Proposition 4.1.5 (1). This also implies $\mathscr{L}_\Xi g^{ij} = 0$. Therefore, using the Leibniz rule, we get

$$\mathscr{L}_\Xi g_N = \mathscr{L}_\Xi (g_{ij}\mathrm{d}q^i\mathrm{d}q^j + g^{ij}\mathrm{d}p_i\mathrm{d}p_j) = 2g_N.$$

Using $\mathscr{L}_\xi g = 2g$ and $\mathscr{L}_\xi J = 0$ we obtain $\mathscr{L}_\xi \omega = 2\omega$, and this implies that $\mathscr{L}_\xi \omega_{ij} = 0$. Indeed

$$\mathscr{L}_\xi \omega_{ij} = \mathscr{L}_\xi \omega(\tfrac{\partial}{\partial q^i}, \tfrac{\partial}{\partial q^j}) = (\mathscr{L}_\xi \omega)(\tfrac{\partial}{\partial q^i}, \tfrac{\partial}{\partial q^j}) + \omega(\mathscr{L}_\xi \tfrac{\partial}{\partial q^i}, \tfrac{\partial}{\partial q^j}) + \omega(\tfrac{\partial}{\partial q^j}, \mathscr{L}_\xi \tfrac{\partial}{\partial q^j}) = 0.$$

This also implies that $\mathscr{L}_\xi \omega^{ij} = 0$. As before, since ω_{ij} is just a function on M we have $\mathscr{L}_\Xi \omega_{ij} = \mathscr{L}_\xi \omega_{ij}$. Then

$$\mathscr{L}_\Xi \omega_1 = \mathscr{L}_\Xi (\tfrac{1}{2}\omega_{ij}\mathrm{d}q^i \wedge \mathrm{d}q^j + \tfrac{1}{2}\omega^{ij}\mathrm{d}p_i \wedge \mathrm{d}p_j) = 2\omega_1.$$

From here we also get that $\mathscr{L}_\Xi I_1 = 0$ since $\mathscr{L}_\Xi g_N = 2g_N$ and $\omega_k = g_N(I_k\cdot, \cdot)$. A similar computation gives us $\mathscr{L}_\Xi I_2 = 0$ and $\mathscr{L}_\Xi I_3 = 0$. So far we have proved that Ξ is homothetic and tri-holomorphic.

The 2-form ω_{H} is given in local coordinates by

$$\omega_{\mathrm{H}} = -\tfrac{1}{2}\omega_{ij}\mathrm{d}q^i \wedge \mathrm{d}q^j + \tfrac{1}{2}\omega^{ij}\mathrm{d}p_i \wedge \mathrm{d}p_j,$$

as can be seen, for instance, from (13). Proceeding as before, we get $\mathscr{L}_\Xi \omega_{\mathrm{H}} = 2\omega_{\mathrm{H}}$. To conclude, note that $f_Z = -\frac{1}{2}g_N(Z,Z) = -\frac{1}{2}\pi^* g(\xi,\xi)$, which implies that $\mathscr{L}_\Xi f_Z = 2f_Z$ since $\mathscr{L}_\xi g = 2g$. Similarly we get $\mathscr{L}_\Xi f_{\mathrm{H}} = 2f_{\mathrm{H}}$. □

We now express the derivative of the function $\|\mathrm{Rm}^c_{\mathrm{H}}\|^2_{g^c_{\mathrm{H}}}$ along the direction Ξ in terms of the "curvature functions" R^N, R^{HK} and R^{C} introduced above.

Proposition 5.2.3. *Let (M,g,J,∇,ξ) be a CASK manifold and $(N=T^*M,g_N,I_1,I_2,I_3)$ the associated rigid c-map space. Let Ξ given by (22). Then*

$$\mathscr{L}_\Xi \|\mathrm{Rm}^c_{\mathrm{H}}\|^2_{g^c_{\mathrm{H}}} = \frac{1}{(f^c_{\mathrm{H}})^7}\left(\sum_{k=1}^{9} \tilde{\Omega}_k c^k\right), \tag{23}$$

where the (c-independent) functions $\tilde{\Omega}_k$ are given in terms of the functions R^N, R^{HK} and R^{C} defined above by

$$\tilde{\Omega}_9 := \frac{1}{128}\big(36R^N_3 + R^N_4\big), \tag{24a}$$
$$\tilde{\Omega}_8 := -\frac{1}{64}\big(f_Z(260R^N_3 - 6R^N_4) - 4R^{\mathrm{C}}_3 + R^{\mathrm{C}}_4\big), \tag{24b}$$
$$\tilde{\Omega}_7 := \frac{1}{32}\big((f_Z)^2(572R^N_3 + 14R^N_4) + f_Z(28R^{\mathrm{C}}_3 - 7R^{\mathrm{C}}_4)\big), \tag{24c}$$
$$\begin{aligned}\tilde{\Omega}_6 := -\frac{1}{16}\big(&(f_Z)^3(788R^N_3 - 14R^N_4) + (f_Z)^2(-36R^{\mathrm{C}}_3 + 17R^{\mathrm{C}}_4)\\ &+ f_Z(-6R^{\mathrm{HK}}_0 + 20R^{\mathrm{HK}}_1 - 8R^{\mathrm{HK}}_{2a} - 16R^{\mathrm{HK}}_{2b} + 12R^{\mathrm{HK}}_3 - 2R^{\mathrm{HK}}_4)\big),\end{aligned} \tag{24d}$$
$$\begin{aligned}\tilde{\Omega}_5 := \frac{1}{8}\big(&(f_Z)^4(660R^N_3) + (f_Z)^3(-36R^{\mathrm{C}}_3 - 15R^{\mathrm{C}}_4)\\ &+ (f_Z)^2(-30R^{\mathrm{HK}}_0 + 60R^{\mathrm{HK}}_1 - 8R^{\mathrm{HK}}_{2a} - 16R^{\mathrm{HK}}_{2b} - 12R^{\mathrm{HK}}_3 + 6R^{\mathrm{HK}}_4)\big),\end{aligned} \tag{24e}$$
$$\begin{aligned}\tilde{\Omega}_4 := -\frac{1}{4}\big(&(f_Z)^5(204R^N_3 + 14R^N_4) + (f_Z)^4(84R^{\mathrm{C}}_3 - 5R^{\mathrm{C}}_4)\\ &+ (f_Z)^3(-60R^{\mathrm{HK}}_0 + 40R^{\mathrm{HK}}_1 + 16R^{\mathrm{HK}}_{2a} + 32R^{\mathrm{HK}}_{2b} - 24R^{\mathrm{HK}}_3 - 4R^{\mathrm{HK}}_4)\big),\end{aligned} \tag{24f}$$
$$\begin{aligned}\tilde{\Omega}_3 := \frac{1}{2}\big(&(f_Z)^6(20R^N_3 - 14R^N_4) + (f_Z)^5(-12R^{\mathrm{C}}_3 + 19R^{\mathrm{C}}_4)\\ &+ (f_Z)^4(-60R^{\mathrm{HK}}_0 - 40R^{\mathrm{HK}}_1 + 16R^{\mathrm{HK}}_{2a} + 32R^{\mathrm{HK}}_{2b} + 24R^{\mathrm{HK}}_3 - 4R^{\mathrm{HK}}_4)\big),\end{aligned} \tag{24g}$$
$$\begin{aligned}\tilde{\Omega}_2 := -\big(&(f_Z)^7(28R^N_3 + 6R^N_4) + (f_Z)^6(-44R^{\mathrm{C}}_3 - 13R^{\mathrm{C}}_4)\\ &+ (f_Z)^5(-30R^{\mathrm{HK}}_0 - 60R^{\mathrm{HK}}_1 - 8R^{\mathrm{HK}}_{2a} - 16R^{\mathrm{HK}}_{2b} + 12R^{\mathrm{HK}}_3 + 6R^{\mathrm{HK}}_4)\big),\end{aligned} \tag{24h}$$
$$\begin{aligned}\tilde{\Omega}_1 := -2\big(&(f_Z)^8(8R^N_3 + R^N_4) + (f_Z)^7(-20R^{\mathrm{C}}_3 - 3R^{\mathrm{C}}_4)\\ &+ (f_Z)^6(6R^{\mathrm{HK}}_0 + 20R^{\mathrm{HK}}_1 + 8R^{\mathrm{HK}}_{2a} + 16R^{\mathrm{HK}}_{2b} + 12R^{\mathrm{HK}}_3 + 2R^{\mathrm{HK}}_4)\big).\end{aligned} \tag{24i}$$

Proof. By Lemma 5.2.2, the vector field Ξ satisfies

$$\mathscr{L}_\Xi g_N = 2g_N, \quad \mathscr{L}_\Xi \omega_{\mathrm{H}} = 2\omega_{\mathrm{H}}, \quad \mathscr{L}_\Xi f_Z = 2f_Z, \quad \mathscr{L}_\Xi f_{\mathrm{H}} = 2f_{\mathrm{H}}.$$

Since we have $f^c_Z = f_Z - \frac{1}{2}c$ and $f^c_{\mathrm{H}} = f_{\mathrm{H}} - \frac{1}{2}c = -f^c_Z - c$, it follows that

$$\mathscr{L}_\Xi f^c_Z = 2f^c_Z + c \quad \text{and} \quad \mathscr{L}_\Xi f^c_{\mathrm{H}} = -2f^c_Z - c.$$

Note in particular that Ξ generates homotheties with respect to the metric g_N. Using the general fact that any homothety of a pseudo-Riemannian manifold is affine with respect to the Levi-Civita connection and hence preserves its curvature, we have that $\mathscr{L}_\Xi \mathrm{Rm}_N = 2\,\mathrm{Rm}_N$. Moreover we have that $\mathscr{L}_\Xi \mathrm{Rm}_{\mathrm{HK}} = 4\,\mathrm{Rm}_{\mathrm{HK}}$. A long but straightforward computation using these observations and the Leibniz rule then yields the desired result. $\square$

Now that we have the explicit form of the derivative of the norm of the curvature tensor along Ξ, we show that this derivative is not zero when $c > 0$.

Proposition 5.2.4. *Let (M,g,J,∇,ξ) be a CASK manifold and $(N = T^*M, g_N, I_1, I_2, I_3)$ the associated rigid c-map space. Then $\|\mathrm{Rm}^c_{\mathrm{H}}\|^2_{g^c_{\mathrm{H}}}$ is not constant on N when $c > 0$.*

Proof. We prove it by contradiction. Let

$$F^c := \|\mathrm{Rm}^c_{\mathrm{H}}\|^2_{g^c_{\mathrm{H}}} \in \mathscr{C}^\infty(N) \quad \text{and} \quad \bar{F}^c := \|\mathrm{Rm}^c_{\bar{N}}\|^2_{g^c_{\bar{N}}} \in \mathscr{C}^\infty(\bar{N}).$$

Suppose that F^c is constant for some $c > 0$. Since F^c and $\bar{F}^c$ are $\mathscr{H}$-related, F^c is constant if and only if $\bar{F}^c$ is constant. We know that for $c, c' > 0$, the quaternionic Kähler manifolds $(\bar{N}, g^c_{\bar{N}})$ and $(\bar{N}, g^{c'}_{\bar{N}})$ are locally isometric (see Proposition 4.3.4).

Since there exists a (local) diffeomorphism $\varphi : \bar{N} \longrightarrow \bar{N}$ such that $\varphi^*\bar{F}^c = \bar{F}^{c'}$, it follows that $\bar{F}^c$ is constant if and only if $\bar{F}^{c'} = \varphi^*\bar{F}^c$ is constant. This implies that $\bar{F}^c$ is constant for all $c > 0$. By $\mathscr{H}$-relatedness, F^c is also constant for all $c > 0$. Then $\mathscr{L}_\Xi F^c = 0$ for all $c > 0$ and, by (23), this implies that $\tilde{\Omega}_k \equiv 0$ for $k = 1, \dots, 9$.

By (24a), $\tilde{\Omega}_9 \equiv 0$ implies that

$$36R_3^N + R_4^N \equiv 0,$$

but both functions are non-negative, so this means that $R_3^N \equiv R_4^N \equiv 0$. Recall that R_4^N is a sum of squares, so each of the individual terms must vanish separately, i.e. $\mathrm{Rm}_N(\varepsilon_\mu, \varepsilon_\nu, \varepsilon_\lambda, \varepsilon_\sigma) \equiv 0$. This shows that $\mathrm{Rm}_N \equiv 0$, which implies $R_3^{\mathrm{C}} \equiv 0$ and $R_4^{\mathrm{C}} \equiv 0$.

Now, by (24i), $\tilde{\Omega}_1 \equiv 0$ implies that

$$6R_0^{\mathrm{HK}} + 20R_1^{\mathrm{HK}} + 8R_{2a}^{\mathrm{HK}} + 16R_{2b}^{\mathrm{HK}} + 12R_3^{\mathrm{HK}} + 2R_4^{\mathrm{HK}} \equiv 0,$$

but, as before, all these functions are non-negative, so all of them vanish identically. Thus, we find that $\mathrm{Rm}_{\mathrm{HK}} \equiv 0$, but this is a contradiction, since for a rigid c-map space we have $\mathrm{Rm}_{\mathrm{HK}}(Z, I_1Z, Z, I_1Z) = g_N(Z,Z)^2 > 0$ by Lemma 5.2.5. Hence we can conclude that $\mathscr{L}_\Xi \|\mathrm{Rm}^c_{\mathrm{H}}\|^2_{g^c_{\mathrm{H}}} \not\equiv 0$ and therefore $\|\mathrm{Rm}^c_{\mathrm{H}}\|^2_{g^c_{\mathrm{H}}}$ is not a constant function. $\square$

Lemma 5.2.5. *Let (M,g,J,∇,ξ) be a CASK manifold and $(N = T^*M, g_N, I_1, I_2, I_3)$ the associated rigid c-map space. Then* $\mathrm{Rm}_{\mathrm{HK}}(Z, I_1Z, Z, I_1Z) = g_N(Z,Z)^2$.

Proof. In the case of a rigid c-map space with $f_Z = -\frac{1}{2}g_N(Z,Z) = -f_{\mathrm{H}}$, we have

$$\omega_{\mathrm{H}}(Z,X) = -\mathrm{d}f_{\mathrm{H}}^c(X) = \mathrm{d}f_Z^c(X) = -\omega_1(Z,X) = -g_N(I_1Z,X)$$

for all $X \in \Gamma(TN)$, where $f_Z^c = f_Z - \frac{1}{2}c$ and $f_{\mathrm{H}}^c = f_{\mathrm{H}} - \frac{1}{2}c$. Hence we get

$$\begin{aligned}(\omega_{\mathrm{H}} \varobar \omega_{\mathrm{H}})(Z,I_1Z,Z,I_1Z) &= 2\omega_{\mathrm{H}}(Z,Z)\omega_{\mathrm{H}}(I_1Z,I_1Z) + 6\omega_{\mathrm{H}}(Z,I_1Z)\omega_{\mathrm{H}}(Z,I_1Z)\\ &= 6g_N(I_1Z,I_1Z)^2 = 6g_N(Z,Z)^2,\end{aligned}$$

where the notation $\varobar$ was introduced before Theorem 3.3.11. Next we compute

$$\begin{aligned}\psi_k &:= (\omega_{\mathrm{H}}(I_k\cdot,\cdot) \owedge \omega_{\mathrm{H}}(I_k\cdot,\cdot))(Z,I_1Z,Z,I_1Z)\\ &= 2\omega_{\mathrm{H}}(I_kZ,Z)\omega_{\mathrm{H}}(I_kI_1Z,I_1Z) - 2\omega_{\mathrm{H}}(I_kZ,I_1Z)\omega_{\mathrm{H}}(I_kI_1Z,Z) \in \mathscr{C}^\infty(N)\end{aligned}$$

for $k = 1,2,3$. Using that the 2-form ω_{H} is of type $(1,1)$ with respect to each I_k [CST22, Lemma 2.7] and the quaternionic relations of I_1, I_2 and I_3, we get:

$$\psi_1 = 2g_N(Z,Z)^2 \quad \text{and} \quad \psi_2 = \psi_3 = -2g_N(I_1Z,I_2Z)^2 - 2g_N(I_1Z,I_3Z)^2.$$

Note that in the orthogonal decomposition $TN = T^{\mathrm{H}}N \oplus T^{\mathrm{V}}N$, the vector fields Z, I_1Z are horizontal and I_2Z, I_3Z are vertical, thus $\psi_2 = \psi_3 = 0$. Summing all the terms we obtain the claimed result. □

As a consequence of Proposition 5.2.4 we obtain one of the main results of this thesis.

Theorem 5.2.6. *Let $(\bar{M},\bar{g})$ be a PSK manifold and $(\bar{N}, g_{\bar{N}}^c, \mathscr{Q})$ the associated deformed supergravity c-map space. Then $(\bar{N}, g_{\bar{N}}^c)$ is not locally homogeneous for any $c > 0$.*

By the results of [CST21, CRT21, MS22], given a CASK manifold (M,g,J,∇,ξ) of real dimension $2n$ with automorphism group $\mathrm{Aut}(M)$ (recall that this is the subgroup of isometries of (M,g) preserving the full CASK structure), the associated deformed supergravity c-map space $(\bar{N}, g_{\bar{N}}^c, \mathscr{Q})$, $c > 0$, is isometrically acted on by the group $\mathrm{Aut}(M) \ltimes \mathrm{Heis}_{2n+1}$, provided that the underlying PSK manifold $(\bar{M},\bar{g})$ is simply connected (or M is a CASK domain). In particular, when $\mathrm{Aut}(M)$ acts transitively on $\bar{M}$, we obtain an action of $\mathrm{Aut}(M) \ltimes \mathrm{Heis}_{2n+1}$ that is transitive on the level sets of the norm of the quaternionic moment map associated to the circle action on $\bar{N}$. Thus, as a corollary of Theorem 5.2.6, we have the following generalization of [CST21, Theorem 4.8], which only studied the deformed supergravity c-map space associated to the PSK manifold $\mathbb{C}\mathrm{H}^{n-1}$.

Corollary 5.2.7. *Let $(\bar{M},\bar{g})$ be a simply connected PSK manifold and $(\bar{N}, g_{\bar{N}}^c, \mathscr{Q})$ the associated deformed supergravity c-map space. Assume that* $\mathrm{Aut}(M)$ *acts transitively on $\bar{M}$. Then $(\bar{N}, g_{\bar{N}}^c)$ is complete and of cohomogeneity one for all $c > 0$.*

Proof. Before beginning the proof we give an overview of its steps. First we explain that under the above transitivity assumption on $\mathrm{Aut}(M)$ the corresponding undeformed supergravity c-map space $(\bar{N}, g_{\bar{N}})$ is complete. Using this property, we show then that $\bar{N}$ is homogeneous. As a third step we prove that $\bar{N}$ is an Alekseevsky space. Therefore $\bar{N}$ is either a supergravity q-map space associated with a homogeneous PSR manifold, a symmetric space of non-compact type dual to a complex Grassmannian of 2-planes or a quaternionic hyperbolic space. In the fourth step of the proof we show that the the deformed supergravity c-map space is complete in the first two cases. The third case is excluded in the fifth step of the proof, in which we show that the quaternionic hyperbolic space is not a supergravity c-map space. Finally, we conclude the proof using Theorem 5.2.6 together with the fact that $(\bar{N}, g^c_{\bar{N}})$ admits a group of isometries acting with cohomogeneity one.

(1) Since the Riemannian manifold $(\bar{M}, \bar{g})$ is complete, the corresponding undeformed supergravity c-map space $(\bar{N}, g_{\bar{N}})$ is complete in virtue of Theorem 4.3.3. Next, we will show that $(\bar{N}, g_{\bar{N}})$ is not only complete but is in fact homogeneous.

(2) The group of isometries $\mathrm{Aut}(\bar{M}) \subset \mathrm{Isom}(\bar{M}, \bar{g})$ induced by $\mathrm{Aut}(M)$ extends canonically to a group of isometries of $(\bar{N}, g_{\bar{N}})$. This is stated in [CDJL21, Proposition 26] for CASK domains but holds in general as a consequence of [CHM12, Lemma 4]. It can be also seen as a special case ($c = 0$) of the results of [CST21, CRT21, MS22] mentioned above. The group $\mathrm{Aut}(\bar{M})$ acts transitively on the base of the fiber bundle $\bar{N} \longrightarrow \bar{M}$ mapping fibers to fibers. In addition, there is a fiber-preserving isometric action of the solvable Iwasawa subgroup G_{2n+2} of $\mathrm{SU}(1, n+1)$ on $\bar{N}|_{\bar{U}}$ [CHM12, Theorem 5] for every domain $U \subset M$, which is isomorphic to a CASK domain, where $\bar{U}$ denotes the image of U under the projection $M \longrightarrow \bar{M}$ (recall that every CASK manifold is locally isomorphic to a CASK domain). Note that $\dim G_{2n+2} = 2n+2$, where $\dim_{\mathbb{R}} \bar{M} = 2n-2$. This solvable group action on $\bar{N}|_{\bar{U}}$ is simply transitive on each fiber. In particular, for every such $\bar{U}$ there is a Lie algebra $\mathfrak{g}_{\bar{U}} \cong \mathfrak{g}_{2n+2} = \mathrm{Lie}(G_{2n+2})$ of Killing fields of $\bar{N}|_{\bar{U}}$ transitive on each fiber. Moreover, $\mathfrak{g}_{\bar{U}}$ can be identified with the space of parallel sections over $\bar{U}$ of a flat symplectic vector bundle over $\bar{M}$ (with Lie algebras as fibers), compare with [CHM12, Theorem 9]. Since $\bar{M}$ is simply connected the above vector bundle has a global parallel frame. Thus we obtain a globally defined Lie algebra of Killing fields $\mathfrak{g} \cong \mathfrak{g}_{2n+2}$ of $\bar{N}$, which restricts to $\mathfrak{g}_{\bar{U}}$ on the domain $\bar{N}|_{\bar{U}} \subset \bar{N}$. Since $\bar{N}$ is complete, there is a corresponding Lie group G acting on $\bar{N}$, which together with $\mathrm{Aut}(\bar{M})$ generates a transitive group of isometries of $\bar{N}$.

(3) Now that we know that $(\bar{N}, g_{\bar{N}})$ is a homogeneous quaternionic Kähler manifold of negative scalar curvature, we apply Theorem 2.4.6 to conclude that it is an Alekseevsky space. These spaces are described in Theorem 2.4.5.

(4) We claim that the one-loop deformation of any supergravity c-map space which is an Alekseevsky space is complete if the deformation parameter c is positive (for $c = 0$ it holds by homogeneity). First we note that all of the Alekseevsky spaces with exception of the quaternionic hyperbolic spaces and the Hermitian symmetric

spaces of non-compact type dual to complex Grassmannians of 2-planes can be represented as supergravity q-map spaces [dWVP92]. By Theorem 4.3.17 the one-loop deformation of a complete supergravity q-map space is complete if $c > 0$. In particular, the one-loop deformed Alekseevsky supergravity q-map spaces with $c > 0$ are complete. Furthermore, the Hermitian symmetric Alekseevsky spaces (20) come from complex hyperbolic spaces, which were shown to have regular boundary behavior, implying the completeness of their one-loop deformation for $c > 0$ by Theorem 4.3.7 and Example 4.3.8.

(5) Finally, we are left with the quaternionic hyperbolic spaces $\mathbb{H}\mathrm{H}^n$. These manifolds cannot be represented as supergravity c-map spaces by Proposition 4.3.19 and hence cannot occur in our setting. This finishes the proof of the completeness of $(\bar{N}, g^c_{\bar{N}})$ for $c > 0$.

(6) Now the corollary follows from the fact that $(\bar{N}, g^c_{\bar{N}})$ has a group of isometries acting with cohomogeneity one but no such group acting with cohomogeneity zero by Theorem 5.2.6.

□

It was conjectured in [Thu20, Conjecture 5.38] that the deformed supergravity c-map space associated to a homogeneous PSK manifold is of cohomogeneity one. We see that Corollary 5.2.7 confirms this conjecture.

5.3 Symmetries of rigid c-map spaces

As we have mentioned in Section 4.2, there are two ways to obtain Hamiltonian automorphisms of a rigid c-map space. In this section, which contains the results of [CGT24, Section 2], we explain how to construct a canonical subgroup of infinitesimal automorphisms of the rigid c-map structure which are moreover ω_{H}-Hamiltonian. We first describe the canonical lifts of infinitesimal CASK automorphisms, then the translations in the fibers, and finally how these two interact. Here we use the Einstein summation convention when working on local coordinates.

5.3.1 Canonical lifts

Let $\mathrm{Aut}(M)$ be the group of CASK automorphisms of the CASK manifold M. Canonically lifting to $N = T^*M$, we obtain a group of ω_{H}-Hamiltonian automorphisms of the rigid c-map structure, as proven in [CST21]. It was remarked in [CRT21, Proposition 2.10] that this group even admits an equivariant moment map $\mu : N \longrightarrow \mathfrak{g}^*$ with respect to ω_{H}. We recall the result in Proposition 5.3.1 below adding details and fixing notation.

Let X denote a vector field on M generating a one-parameter family of automorphisms of the CASK structure, and Y its canonical lift to N. Then, with respect to the splitting

$T(T^*M) = T^{\mathrm{H}}N \oplus T^{\mathrm{V}}N \cong \pi^*(TM) \oplus \pi^*(T^*M)$, we may write

$$Y = \pi^*X - \lambda \circ (\pi^*\nabla)(\pi^*X).$$

Here $\lambda : \xi \longmapsto \lambda_\xi$ is the tautological section of the vector bundle $\pi^*(T^*M)$, defined as

$$\lambda_\xi(v) = \xi(v), \quad \xi \in N_x = T_x^*M,\, v \in (\pi^*(TM))_\xi = T_xM,\, \pi(\xi) = x,$$

where $\pi^*X = X \circ \pi \in \Gamma(\pi^*(TM))$ and $\lambda \circ (\pi^*\nabla)(\pi^*X) \in \Gamma(\pi^*(T^*M)) \subset \Gamma(T^*N)$ is the 1-form sending a vector field $A \in \Gamma(TN)$ to the smooth function $\lambda((\pi^*\nabla)_A(\pi^*X))$, and where we have identified $\pi^*(T^*M) = (\pi^*(TM))^* = (T^{\mathrm{H}}N)^* = (T^{\mathrm{V}}N)^0 \subset T^*N$.

Note that

$$(\pi^*\nabla)_Y\lambda = -\lambda \circ (\pi^*\nabla)(\pi^*X). \tag{25}$$

Indeed, for a vector field $Y \in \Gamma(TN)$ and $\lambda \in \Gamma(\pi^*(T^*M))$ the tautological section, $(\pi^*\nabla)_Y\lambda$ gives us the vertical component of Y, which is precisely $-\lambda \circ (\pi^*\nabla)(\pi^*X)$.

In local coordinates $\{q^i, p_i\}$ on N induced by local ∇-affine coordinates $\{q^i\}$ on M, the tautological section λ on N is given by $\lambda(\frac{\partial}{\partial q^i}) = p_i$ and the vector field $Y \in \Gamma(TN)$ by

$$Y = X^j \frac{\partial}{\partial q^j} - p_i \frac{\partial X^i}{\partial q^j}\frac{\partial}{\partial p_j}$$

in terms of the components $\{X^i\}$ of the vector field X in the local coordinates $\{q^i\}$.

Proposition 5.3.1. *Let X be an infinitesimal CASK automorphism and Y its canonical lift as explained above. Then*

$$\mu_Y = \frac{1}{2}\left(g_N(Z,Y) + \pi^*\omega^{-1}(\lambda \circ (\pi^*\nabla)(\pi^*X), \lambda)\right)$$

is a ω_{H}-Hamiltonian function for Y, where $Z = -\widetilde{J\xi}$. This assignment determines an equivariant (co)moment map $\mu : \mathfrak{aut}(M) \longrightarrow \mathscr{C}^\infty(N)$, $X \longmapsto \mu_X := \mu_Y$, for the action of $\mathfrak{aut}(M)$ on $N = T^*M$.

Proof. We have to show that $\iota_Y\omega_{\mathrm{H}} = -\mathrm{d}\mu_Y$. Recall that ω_{H} is given by (13) with respect to the splitting $T(T^*M) \cong \pi^*(TM) \oplus \pi^*(T^*M)$. Thus we have

$$\iota_Y\omega_{\mathrm{H}} = -\pi^*(\iota_X\omega) + \iota_{(\pi^*\nabla)_Y\lambda}\pi^*\omega^{-1}.$$

Let us compute these two terms. First notice that, by Proposition 4.1.5 (a) we have that $\mathscr{L}_\xi\omega = 2\omega$ and, hence

$$2\iota_X\omega = \iota_X\mathscr{L}_\xi\omega = \iota_X\mathrm{d}(\iota_\xi\omega) = \mathscr{L}_X(\iota_\xi\omega) - \mathrm{d}(\iota_X\iota_\xi\omega) = \mathrm{d}(g(-J\xi, X)),$$

since $\mathscr{L}_X(\iota_\xi\omega) = 0$ (recall that X is an infinitesimal CASK automorphism). So we see that

$$\pi^*(\iota_X\omega) = \frac{1}{2}\mathrm{d}(g_N(Z,Y)). \tag{26}$$

To compute the second term, let A be an arbitrary section of $\pi^*(T^*M) \cong T^{\mathrm{V}}(T^*M)$. Let $\{q^j\}$ be special real coordinates on M and $\{q^j, p_j\}$ the corresponding canonical coordinates on N. With respect to these, we can write $A = A_i \mathrm{d}q^i$ and, using $(\pi^*\nabla)_Y \lambda = -\lambda \circ (\pi^*\nabla)(\pi^*X) = -p_i \frac{\partial X^i}{\partial q^j} \mathrm{d}q^j$, we get

$$\pi^*\omega^{-1}((\pi^*\nabla)_Y \lambda, A) = -\omega^{jk} p_i \frac{\partial X^i}{\partial q^j} A_k.$$

Thus, $\iota_{(\pi^*\nabla)_Y \lambda} \pi^*\omega^{-1}$ is the 1-form on T^*M which vanishes when applied to a horizontal vector field and evaluates on a vertical vector field $A = A_i \frac{\partial}{\partial p_i}$ (corresponding to the A consider earlier by the canonical isomorphism $\pi^*(T^*M) \cong T^{\mathrm{V}}(T^*M)$) as above. This means that we can write it, in local coordinates, as

$$\iota_{(\pi^*\nabla)_Y \lambda} \pi^*\omega^{-1} = -\omega^{jk} p_i \frac{\partial X^i}{\partial q^j} \mathrm{d}p_k. \tag{27}$$

Now, we compute the differential of our proposed moment map:

$$\mathrm{d}\mu_Y = \frac{1}{2} \left(\mathrm{d}(g_N(Z,Y)) + \mathrm{d}(\pi^*\omega^{-1}(\lambda \circ (\pi^*\nabla)(\pi^*X), \lambda)) \right),$$

the second term of which can be computed (using (25) and (27)) in local coordinates as follows:

$$\begin{aligned} \frac{1}{2}\mathrm{d}(\pi^*\omega^{-1}(\lambda \circ (\pi^*\nabla)(\pi^*X), \lambda)) &= \frac{1}{2}\mathrm{d}\left(\omega^{jk} p_i p_k \frac{\partial X^i}{\partial q^j} \right) \\ &= \frac{1}{2} \omega^{jk} \frac{\partial X^i}{\partial q^j} (p_i \mathrm{d}p_k + p_k \mathrm{d}p_i) \\ &= \omega^{jk} p_i \frac{\partial X^i}{\partial q^j} \mathrm{d}p_k = -\iota_{(\pi^*\nabla)_Y \lambda} \pi^*\omega^{-1}, \end{aligned}$$

where, in passing to the third line, we used that both ω^{jk} and $\frac{\partial X^i}{\partial q^j}$ are anti-symmetric in their indices; the latter fact is just the equation $\mathscr{L}_X \omega = 0$ in (∇-affine) coordinates.

This computation, together with (26), concludes the proof that μ_Y is indeed a Hamiltonian function for Y.

To show that the map μ is equivariant, let $X_1, X_2 \in \mathfrak{aut}(M)$ and $Y_1, Y_2 \in \Gamma(TN)$ their corresponding canonical lifts. Note that the canonical lift of $[X_1, X_2]$ is precisely $[Y_1, Y_2]$. Using this and the fact that the moment map is constructed in a canonical way, we obtain

$$\begin{aligned} \mathscr{L}_{Y_1}(\mu_{Y_2}) &= \frac{1}{2} \left(g_N(Z, [Y_1, Y_2]) + \pi^*\omega^{-1}(\lambda \circ (\pi^*\nabla)(\pi^*([X_1, X_2])), \lambda) \right) \\ &= \mu_{[Y_1, Y_2]}. \end{aligned} \tag{28}$$

$\square$

Remark 5.3.2.

- Note that the ω_{H}-Hamiltonian function μ_Y given by Proposition 5.3.1 is the only moment map that is homogeneous, i.e. $\mathscr{L}_{\Xi}\mu_Y = 2\mu_Y$, where $\Xi \in \Gamma(TN)$ is the sum of the ∇-horizontal lifted Euler vector field ξ with the fiberwise Euler (or position) vector field of T^*M. In canonical coordinates associated to (conical) special real coordinates, Ξ takes the form $\Xi = q^j \frac{\partial}{\partial q^j} + p_j \frac{\partial}{\partial p_j}$.
- Note also that the equivariance of the moment map $\mu : \mathfrak{aut}(M) \longrightarrow \mathscr{C}^\infty(N)$ (without assuming homogeneity) fixes it uniquely up to adding a linear form $c : \mathfrak{aut}(M) \longrightarrow \mathbb{R}$, $X \longmapsto c_X$, invariant under the coadjoint representation. The space of such forms is trivial if and only if the Lie algebra is perfect, that is, it coincides with its derived ideal. This is the case, in particular, for semisimple Lie algebras.

Thus, we have a canonical action of $\mathrm{Aut}(M)$ on N which casts $\mathrm{Aut}(M)$ as a subgroup of $\mathrm{Ham}_{\mathbb{S}^1}(N)$, with a canonical choice of equivariant moment map.

5.3.2 Translations in the fibers

One of the crucial features of the rigid c-map metric on N is that it is semi-flat. This means that it is foliated by half-dimensional flat submanifolds. These are just the fibers of $\pi : N = T^*M \longrightarrow M$, and the reason they are flat is that the metric in each fiber is constant with respect to the affine structure induced by the vector space structure of the fiber, as can be seen directly from (12).

An important consequence is the following result.

Proposition 5.3.3. *The cotangent bundle $N = T^*M$ carries locally an ω_{H}-Hamiltonian action of the group $\mathbb{R}^{2n}$ by automorphisms of the rigid c-map structure, which preserves the fibers and acts on them by translations. If the holonomy group of the special connection ∇ on M is trivial, then the action is global.*

Proof. We first give a description of the local action. Choosing special real coordinates $\{q^j\}$ on M and using the associated canonical coordinates $\{q^j, p_j\}$ on N, the action of $\mathbb{R}^{2n}$ is generated by the locally defined vector fields $\frac{\partial}{\partial p_j}$. A quick look at (12) reveals that the full hyperkähler structure is preserved by these vector fields. Moreover, they commute with the rotating circle action as well, since $Z = -\widetilde{J\xi}$ only depends on the coordinates $\{q^j\}$ on the base M. This implies that they also preserve the ω_1-Hamiltonian function $f_Z^c = -\frac{1}{2}g_N(Z,Z) - \frac{1}{2}c$. Finally, let us give a local Hamiltonian with respect to ω_{H}. Writing (13) in the above coordinates, we have

$$\omega_{\mathrm{H}} = \frac{1}{2}(-\omega_{ij}\mathrm{d}q^i \wedge \mathrm{d}q^j + \omega^{ij}\mathrm{d}p_i \wedge \mathrm{d}p_j),$$

where each ω_{ij} is constant (and hence so are the components ω^{ij} of the inverse matrix), and we are omitting pullbacks when no confusion can arise.

Hence, we find

$$\iota_{\frac{\partial}{\partial p_k}} \omega_{\mathrm{H}} = \omega^{kj} \mathrm{d}p_j = -\mathrm{d}(-\omega^{kj} p_j).$$

We can thus assign the local Hamiltonian function

$$\mu_{\frac{\partial}{\partial p_k}} = -\omega^{kj} p_j \tag{29}$$

to $\frac{\partial}{\partial p_k}$. More generally, μ assigns to every vertical vector field $v = v_k \frac{\partial}{\partial p_k}$ with constant coefficients the function $\mu_v = -\omega^{kj} v_k p_j$. The vector fields $\frac{\partial}{\partial p_k}$ generate an action of the group $\mathbb{R}^{2n}$ with the claimed properties. In the local coordinates $\{q^j, p_j\}$ a vector $v = (v_j) \in \mathbb{R}^{2n}$ acts by $(q, p) \longmapsto (q, p + v)$.

To define a global moment map $\mu : \mathbb{R}^{2n} \longrightarrow \mathscr{C}^\infty(N)$ and a global group action of $\mathbb{R}^{2n}$ on N it suffices to have a global frame of the vertical bundle $T^{\mathrm{V}}N \cong \pi^*(T^*M)$ parallel with respect to the connection $\pi^*\nabla$. Such a frame exists if and only if the holonomy of the (flat) special connection is trivial. □

As a comment on the last step of the proof of Proposition 5.3.3 we note that locally (over the preimage $\pi^{-1}(U)$ of a suitable open set $U \subset M$) a parallel frame of $T^{\mathrm{V}}N \cong \pi^*(T^*M)$ can be chosen of the form $\pi^*(\mathrm{d}q^i)$, where $\{q^i\}$ are local ∇-affine coordinates.

5.3.3 The semidirect product

Under the assumption that the holonomy group of ∇ is trivial, we have constructed two subgroups of $\mathrm{Ham}_{\mathbb{S}^1}(N)$ and the next thing to do is to study how they interact. In this section, we check that the generators of the two groups combine into a semidirect product. Let us start by emphasizing that the group $\Gamma_\nabla(T^*M) \cong \mathbb{R}^{2n}$ of ∇-parallel sections acts by addition on $N = T^*M$, that is

$$\alpha \cdot \beta = \alpha(x) + \beta,$$

for all $\alpha \in \Gamma_\nabla(T^*M)$, $\beta \in T_x^*M$, $x \in M$. The group $\mathrm{Aut}(M)$ acts naturally on $N = T^*M$:

$$h \cdot \beta = h_* \beta = (h^{-1})^* \beta,$$

for all $h \in \mathrm{Aut}(M)$, $\beta \in T_x^*M$, $x \in M$.

Proposition 5.3.4. *The subgroups $\mathbb{R}^{2n}$ and $\mathrm{Aut}(M)$ of $\mathrm{Ham}_{\mathbb{S}^1}(N)$ generate a group $G \subset \mathrm{Ham}_{\mathbb{S}^1}(N)$ which is a semidirect product $\mathrm{Aut}(M) \ltimes \mathbb{R}^{2n}$.*

Proof. It is clear that the two subgroups have trivial intersection, as $\mathrm{Aut}(M)$ lifts a subgroup of diffeomorphisms of M whereas $\mathbb{R}^{2n}$ preserves each fiber of $N = T^*M$. The action of $\mathrm{Aut}(M)$ is linear on the fibers and preserves the space of ∇-parallel

1-forms. We check explicitly that the elements $h \in \mathrm{Aut}(M)$ normalize the group of translations:

$$(h \cdot \alpha \cdot h^{-1}) \cdot \beta = h_*(\alpha(h^{-1}(x)) + h^*\beta) = (h_*\alpha)(x) + \beta = (h_*\alpha) \cdot \beta,$$

for all $\alpha \in \Gamma_\nabla(T^*M)$, $\beta \in T^*_x M$, $x \in M$. This proves that $h \cdot \alpha \cdot h^{-1} = h_*\alpha \in \Gamma_\nabla(T^*M)$ for all $\alpha \in \Gamma_\nabla(T^*M)$. $\square$

Infinitesimally, we can describe the semidirect product structure in terms of structure constants if we choose a basis. Thus, let $\{Y_\alpha\}$ be infinitesimal generators of the action of $\mathrm{Aut}(M)$ on N, obtained by canonically lifting generators $\{X_\alpha\} \subset \mathfrak{aut}(M)$. With respect to canonical coordinates induced by (conical) special real coordinates on M, we can express Y_α as

$$Y_\alpha = X^j_\alpha \frac{\partial}{\partial q^j} - p_i \frac{\partial X^i_\alpha}{\partial q^j} \frac{\partial}{\partial p_j}, \tag{30}$$

where the component functions X^j_α of X_α are linear functions. The generators of the $\mathbb{R}^{2n}$-action are the vectors $\frac{\partial}{\partial p_k}$. The structure constants are now easily computed:

$$\left[Y_\alpha, \frac{\partial}{\partial p_k} \right] = \frac{\partial X^k_\alpha}{\partial q^j} \frac{\partial}{\partial p_j}.$$

Note that the coefficients multiplying $\frac{\partial}{\partial p_j}$ on the right-hand side are indeed constants, since any $X \in \mathfrak{aut}(M)$ is ∇-affine (i.e. $\mathscr{L}_X \nabla = 0$). Together with the structure constants of $\mathrm{Aut}(M)$ this determines the structure of $\mathfrak{aut}(M) \ltimes \mathbb{R}^{2n}$.

5.4 Symmetries under the HK/QK correspondence

Now that we have a completely explicit description of the group $\mathrm{Aut}(M) \ltimes \mathbb{R}^{2n} \subset \mathrm{Ham}_{\mathbb{S}^1}(N)$, the next step is to transfer the group action to the quaternionic Kähler manifold $\bar{N}$. Up to a so-called elementary modification explained below, this is essentially done by first lifting to the trivial circle bundle $P = N \times \mathbb{S}^1$ over N and subsequently studying the induced action on $\bar{N}$, which we realize as a submanifold of P. We will continue to assume that the holonomy group of the flat special connection ∇ is trivial to ensure the global $\mathbb{R}^{2n}$-action (see Proposition 5.3.3). This section contains the results of [CGT24, Section 3]. Here we use again the Einstein summation convention when working on local coordinates.

5.4.1 The twist construction for circle actions

For convenience of the exposition, we briefly recall here the twist construction explained in Section 3.3 particularized to the case $G = \mathbb{S}^1$.

Let N be a smooth manifold equipped with an $\mathbb{S}^1$-action generated by a vector field $Z \in \Gamma(TN)$, and let $\pi_N : P \longrightarrow N$ be a principal circle bundle over N with connection

1-form $\eta \in \Omega^1(P)$ and curvature $\omega \in \Omega^2(N)$ (i.e. $\mathrm{d}\eta = \pi_N^*\omega$). We want to lift the vector field Z to a vector field $Z_P \in \Gamma(TP)$ so that it preserves the connection η, i.e. $\mathscr{L}_{Z_P}\eta = 0$, and it commutes with the principal circle action on P generated by the vector field $X_P \in \Gamma(TP)$, i.e. $[Z_P, X_P] = 0$. It turns out that such a lift exists if and only if $\iota_Z\omega = -\mathrm{d}f_Z^c$ for some function $f_Z^c \in \mathscr{C}^\infty(N)$. The lift is given by

$$Z_P = \tilde{Z} + \pi_N^* f_Z^c X_P,$$

where $\tilde{Z}$ denotes the horizontal lift with respect to η. The triple (ω, Z, f_Z^c) with the above properties is called twist data. A manifold N equipped with twist data produces a new smooth manifold $\bar{N} := P/\langle Z_P\rangle$ called the twist of N with respect to the twist data (ω, Z, f_Z^c). We then have a double fibration structure on P:

$$\begin{tikzcd}
& P \arrow[loop left, "X_P"] \arrow[loop right, "Z_P"] \arrow[dl, "\pi_N"'] \arrow[dr, "\pi_{\bar{N}}"] & \\
N \arrow[loop left, "Z"] \arrow[rr, dashed, leftrightarrow, "\text{twist}"'] & & \bar{N} \arrow[loop right, "\bar{Z}"]
\end{tikzcd}$$

Recall that X_P generates the principal circle action on P with respect to the projection $\pi_N : P \longrightarrow N$ and note that Z_P plays the same role for the projection $\pi_{\bar{N}} : P \longrightarrow \bar{N}$. Note also that the twist construction produces a circle action on $\bar{N}$ generated by the vector field $\bar{Z} := \mathrm{d}\pi_{\bar{N}}(X_P) \in \Gamma(T\bar{N})$.

5.4.2 Infinitesimal description

The infinitesimal description of the transfer of symmetries under the HK/QK correspondence appears explained in [CST21]: one performs an (elementary) modification and then twists. More precisely, let $\mathfrak{g} \subset \mathfrak{ham}_{\mathbb{S}^1}(N)$ be a subalgebra with moment map $\mu : N \longrightarrow \mathfrak{g}^*$ with respect to ω_{H}. Denote the Hamiltonian function corresponding to $V \in \mathfrak{g}$ by μ_V. Then, the first step consists in modifying V to

$$V_{\mathrm{H}} := V - \frac{\mu_V}{f_{\mathrm{H}}^c} Z \in \Gamma(TN).$$

We will sometimes refer to V_{H} as the elementary deformation of V. The second step is twisting V_{H} to a vector field $\mathrm{tw}(V_H) \in \Gamma(T\bar{N})$, which we will denote by V_{Q}. Twisting is done by lifting V_{H} horizontally (with respect to a given connection η whose curvature is ω_{H}) to the trivial circle bundle $P = N \times \mathbb{S}^1$ and then projecting down to $\bar{N}$. In other words

$$\mathrm{tw}(V_{\mathrm{H}}) = V_{\mathrm{Q}} = \mathrm{d}\pi_{\bar{N}}(\tilde{V}_{\mathrm{H}}) = \mathrm{d}\pi_{\bar{N}}\left(\tilde{V} - \frac{\mu_V}{f_{\mathrm{H}}^c}\tilde{Z}\right),$$

where we have denoted the η-horizontal lift to P by a tilde.

This procedure gives rise to an injective, linear map $\varphi_\mu : \mathfrak{g} \longrightarrow \mathfrak{aut}_{\mathbb{S}^1}(\bar{N})$, dependent on the choice of moment map μ. Here, $\mathfrak{aut}_{\mathbb{S}^1}(\bar{N})$ denotes the space of Killing fields of the quaternionic Kähler manifold $(\bar{N}, g^c_{\bar{N}})$, $c \geq 0$, which commute with the canonical, isometric circle action on $\bar{N}$ generated by $\bar{Z} = \mathrm{tw}(-\frac{1}{f^c_{\mathrm{H}}} Z) \in \Gamma(T\bar{N})$.

It is shown in [CST21] that the linear map φ_μ is a homomorphism of Lie algebras if and only if the moment map μ is equivariant. In particular, they proved the following.

Theorem 5.4.1 ([CST21, Theorem 3.8]). *Let $\{V_i\}$ be a basis of a Lie subalgebra $\mathfrak{g} \subset \mathfrak{ham}_{\mathbb{S}^1}(N)$ with corresponding structure constants $\{c^k_{ij}\}$. Then*

$$[V^{\mathrm{Q}}_i, V^{\mathrm{Q}}_j] = c^k_{ij} V^{\mathrm{Q}}_k + A_{ij} \bar{Z}$$

for constants $A_{ij} = \mathrm{tw}(\omega_{\mathrm{H}}(V_i, V_j) - c^k_{ij} \mu_{V_k})$.

Therefore, the equivariance condition (28) can be expressed as

$$\omega_{\mathrm{H}}(V_i, V_j) - c^k_{ij} \mu_{V_k} = 0.$$

The left-hand side is precisely what measures the failure of φ_μ to be a homomorphism, according to Theorem 5.4.1.

In the case at hand, we are considering $\mathfrak{g} \cong \mathfrak{aut}(M) \ltimes \mathbb{R}^{2n}$, and we have a canonical moment map. Since this canonical moment map is equivariant for the action of the subgroup $\mathrm{Aut}(M)$ by Proposition 5.3.1, we obtain a subalgebra of $\mathfrak{aut}_{\mathbb{S}^1}(\bar{N})$ isomorphic to $\mathfrak{aut}(M)$. However, the canonical choice of moment map we gave for $\mathbb{R}^{2n}$ in (29) is not equivariant. Indeed

$$\omega_{\mathrm{H}} \left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j} \right) = \omega^{ij} \neq 0,$$

while $\mathbb{R}^{2n}$ is of course abelian and therefore has vanishing structure constants. Following Theorem 5.4.1, this implies that $\mathbb{R}^{2n}$ gives rise to a subalgebra of $\mathfrak{aut}_{\mathbb{S}^1}(\bar{N})$ which is a 1-dimensional central extension of $\mathbb{R}^{2n}$ by $\bar{Z}$, and whose non-trivial brackets are given by the coefficients of ω^{-1}. In other words,

$$\left[\left(\frac{\partial}{\partial p_i} \right)^{\mathrm{Q}}, \left(\frac{\partial}{\partial p_j} \right)^{\mathrm{Q}} \right] = \omega^{ij} \bar{Z}.$$

Since ω^{-1} is the natural symplectic form on the fibers of $N = T^*M$, the central extension in question is nothing but the Heisenberg algebra $\mathfrak{heis}_{2n+1}$.

We thus obtain two algebras of Killing fields, isomorphic to $\mathfrak{aut}(M)$ and $\mathfrak{heis}_{2n+1}$, respectively. Together, they generate an algebra which is once again a semidirect product $\mathfrak{aut}(M) \ltimes \mathfrak{heis}_{2n+1}$. To see that they indeed form a semidirect product, it suffices to show that $[\mathfrak{aut}(M), \mathfrak{heis}_{2n+1}] \subset \mathfrak{heis}_{2n+1}$. To check this, consider $Y^{\mathrm{Q}}_\alpha \in \mathfrak{aut}(M) \subset \mathfrak{aut}_{\mathbb{S}^1}(\bar{N})$ and $(\frac{\partial}{\partial p_k})^{\mathrm{Q}} \in \mathbb{R}^{2n} \subset \mathfrak{heis}_{2n+1}$ (we already know that $\bar{Z}$ is central in the full algebra), where the vector fields Y_α are as in (30). Then, again by Theorem 5.4.1, we have

$$\left[Y_\alpha^{\mathrm{Q}}, \left(\frac{\partial}{\partial p_k}\right)^{\mathrm{Q}}\right] = \frac{\partial X_\alpha^k}{\partial q^j}\left(\frac{\partial}{\partial p_j}\right)^{\mathrm{Q}} + \mathrm{tw}\left(\omega_{\mathrm{H}}\left(Y_\alpha, \frac{\partial}{\partial p_k}\right) - \frac{\partial X_\alpha^k}{\partial q^j}\mu_{\frac{\partial}{\partial p_j}}\right)\bar{Z},$$

where $X_\alpha \in \Gamma(TM)$ lifts to Y_α. Now we use local coordinates to compute

$$\omega_{\mathrm{H}}\left(Y_\alpha, \frac{\partial}{\partial p_k}\right) = -\frac{\partial X_\alpha^k}{\partial q^j}\omega^{j\ell}p_\ell = \frac{\partial X_\alpha^k}{\partial q^j}\mu_{\frac{\partial}{\partial p_j}}$$

and conclude that

$$\left[Y_\alpha^{\mathrm{Q}}, \left(\frac{\partial}{\partial p_k}\right)^{\mathrm{Q}}\right] = \frac{\partial X_\alpha^k}{\partial q^j}\left(\frac{\partial}{\partial p_j}\right)^{\mathrm{Q}},$$

so $\mathfrak{heis}_{2n+1}$ is an ideal inside the Lie algebra $\mathfrak{g}$ generated by $\mathfrak{aut}(M)$ and $\mathfrak{heis}_{2n+1}$ and we have $\mathfrak{g} \cong \mathfrak{aut}(M) \ltimes \mathfrak{heis}_{2n+1}$. Summarizing, we have obtained the following result.

Proposition 5.4.2. *There exists a subalgebra of* $\mathfrak{aut}_{\mathbb{S}^1}(\bar{N})$ *which is isomorphic to the semidirect product* $\mathfrak{aut}(M) \ltimes \mathfrak{heis}_{2n+1}$.

5.4.3 Global description

We would like to construct a global counterpart of the above construction, by lifting the action of the group $\mathrm{Aut}(M) \ltimes \mathbb{R}^{2n}$ on N to an action of the semidirect product $\mathrm{Aut}(M) \ltimes \mathrm{Heis}_{2n+1}$ on $P = N \times \mathbb{S}^1$ and then projecting to $\bar{N}$ directly, without passing to the generating vector fields and having to integrate them as intermediate steps. Here the product in the Heisenberg group Heis_{2n+1} is realized on the product manifold $\mathbb{R}^{2n} \times \mathbb{R}$, where $\mathbb{R}^{2n} \cong \Gamma_\nabla(T^*M)$,

$$(\alpha_1, \tau_1) \cdot (\alpha_2, \tau_2) = (\alpha_1 + \alpha_2, \tau_1 + \tau_2 + \tfrac{1}{2}\omega^{-1}(\alpha_1, \alpha_2)),$$

where the constant function $\omega^{-1}(\alpha_1, \alpha_2)$ is identified with a number.

Let us consider $h \in \mathrm{Aut}(M)$ and $(\alpha, \tau) \in \mathrm{Heis}_{2n+1}$. Then we have the following group action of $\mathrm{Aut}(M) \ltimes \mathrm{Heis}_{2n+1}$ on the trivial circle bundle P over $N = T^*M$:

$$\begin{aligned} h \cdot (\beta, s) &= (h_*\beta, s), \\ (\alpha, \tau) \cdot (\beta, s) &= (\alpha(x) + \beta, s + [\tau + \tfrac{1}{2}\omega^{-1}(\alpha(x), \beta)]), \end{aligned} \tag{31}$$

where $s \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, $\beta \in T_x^*M$, $x \in M$ and $[r] = r \pmod{2\pi}$.

This action covers the action of $\mathrm{Aut}(M) \ltimes \mathbb{R}^{2n}$ on N by means of the quotient homomorphism $\mathrm{Heis}_{2n+1} \longrightarrow \mathbb{R}^{2n} \cong \mathrm{Heis}_{2n+1}/\mathbb{R}$. Moreover, it induces the infinitesimal action on $\bar{N}$ from the previous section.

Proposition 5.4.3. *The infinitesimal group action on P corresponding to (31) descends to the action of $\mathfrak{aut}(M) \ltimes \mathfrak{heis}_{2n+1}$ on $\bar{N}$ as described above.*

Proof. Let $V \in \mathfrak{ham}_{\mathbb{S}^1}(N) \subset \Gamma(TN)$ and $\tilde{V}$ its η-horizontal lift to P. We define a lift of V to P by

$$\hat{V} := \tilde{V} + \varphi X_P \in \Gamma(TP),$$

where $\varphi \in \mathscr{C}^\infty(P)$ and $X_P = \frac{\partial}{\partial s}$ generates the principal circle action. We require that the lift $\hat{V}$ preserves the connection η. The condition $\mathscr{L}_{\hat{V}}\eta = 0$ then implies that $\varphi = \pi_N^* f$ for some ω_{H}-Hamiltonian function $f \in \mathscr{C}^\infty(N)$ for V. Thus $\hat{V} = \tilde{V} + \pi_N^* f X_P$. Such lift automatically commutes with X_P. We will omit the pullbacks in the notation from now on.

If we choose $f = \mu_V$, the canonical Hamiltonian, we recover the procedure described in the previous section. Indeed, note that

$$\tilde{V}_{\mathrm{H}} - \hat{V} = -\frac{\mu_V}{f_{\mathrm{H}}^c} Z_P,$$

where $Z_P = \tilde{Z} + f_{\mathrm{H}}^c X_P$. Since $\mathrm{d}\pi_{\bar{N}}(Z_P) = 0$, by definition of $\bar{N}$, we get $V_{\mathrm{Q}} = \mathrm{d}\pi_{\bar{N}}(\hat{V})$. This means that we obtain the same Killing vector field on $\bar{N}$ projecting the lift $\hat{V}$ or twisting the elementary deformation V_{H}.

Now let us work out explicitly what the infinitesimal lift $\hat{V}$ looks like. Since P is trivial, we may regard any vector field on N as a vector field on P which is horizontal with respect to the product structure (or equivalently with respect to the trivial connection $\mathrm{d}s$, with s a local coordinate on $\mathbb{S}^1$). Thus, we may write the η-horizontal lift $\tilde{V}$ as

$$\tilde{V} = V - \eta(V) X_P,$$

and similarly

$$\hat{V} = \tilde{V} + \mu_V X_P = V - (\eta(V) - \mu_V) X_P. \tag{32}$$

A canonical choice of connection η with curvature ω_{H} is given by

$$\eta = \mathrm{d}s + \frac{1}{2}\iota_\Xi \omega_{\mathrm{H}},$$

where $\Xi \in \Gamma(TN)$ is the vector field expressed in coordinates by $\Xi = q^j \frac{\partial}{\partial q^j} + p_j \frac{\partial}{\partial p_j}$, thus

$$\eta = \mathrm{d}s + \frac{1}{2}(-\omega_{ij} q^i \mathrm{d}q^j + \omega^{ij} p_i \mathrm{d}p_j).$$

Recall that a canonically lifted automorphism of M takes the form

$$Y = X^j \frac{\partial}{\partial p_j} - \frac{\partial X^i}{\partial q^j} p_i \frac{\partial}{\partial p_j} \in \mathfrak{ham}_{\mathbb{S}^1}(N)$$

with canonical Hamiltonian function (see Proposition 5.3.1)

$$\mu_Y = \frac{1}{2}\left(-\omega_{ij}q^iX^j + \omega^{jk}p_ip_k\frac{\partial X^i}{\partial q^j}\right).$$

Using these expressions, we find $\eta(Y) = \frac{1}{2}\iota_Y\iota_\Xi\omega_{\mathrm{H}} = \mu_Y$. The upshot is that $\hat{Y} = Y$ (see (32)), i.e. the lifted action of $\mathrm{Aut}(M)$ to $P = N \times \mathbb{S}^1$ is trivial on the $\mathbb{S}^1$-factor and it corresponds to the action of $\mathrm{Aut}(M)$ described in the first equation of (31).

Finally, we consider the lift of the group $\mathbb{R}^{2n} \cong \Gamma_\nabla(T^*M)$. For an element of its Lie algebra $v \in \mathbb{R}^{2n}$ we have the local expression $v = v_k\frac{\partial}{\partial p_k}$ and corresponding moment map $\mu_v = -\omega^{kj}v_kp_j$. This time, we find $\eta(v) = \frac{1}{2}\mu_v$, and consequently $\hat{v} = v + \frac{1}{2}\mu_vX_P$. These vector fields do not induce an action of $\mathbb{R}^{2n}$ since they no longer commute. Indeed, we have

$$\begin{aligned}[\hat{v},\hat{w}] &= \frac{1}{2}(v(\mu_w) - w(\mu_v))X_P = \frac{1}{2}(\mathrm{d}\mu_w(v) - \mathrm{d}\mu_v(w))X_P \\ &= \frac{1}{2}(-(\iota_w\omega_{\mathrm{H}})(v) + (\iota_v\omega_{\mathrm{H}})(w)) = \omega_{\mathrm{H}}(v,w)X_P,\end{aligned}$$

or, in local coordinates, $[\hat{v},\hat{w}] = \omega^{ij}v_iw_j\frac{\partial}{\partial s}$. Since ω^{ij} is constant and $\frac{\partial}{\partial s}$ is central, the conclusion is that we are now dealing with an infinitesimal action of a 1-dimensional central extension of $\mathbb{R}^{2n}$ whose non-trivial commutators are given by a symplectic form, i.e. a Heisenberg algebra $\mathfrak{heis}_{2n+1}$. Integrating, we obtain the action of Heis_{2n+1} described in the second line of (31). □

It is possible to describe the quaternionic Kähler manifold $\bar{N}$ as a submanifold of the circle bundle P. For that we define the following tensor fields on P:

$$g_P := -\frac{1}{f_{\mathrm{H}}^c}\eta^2 + \pi_N^*g_N,$$

$$\theta_0^P := \mathrm{d}f_Z^c, \quad \theta_1^P := \eta - \iota_Z g_N, \quad \theta_2^P := -\iota_Z\omega_3, \quad \theta_3^P := \iota_Z\omega_2.$$

With them, define the Z_P-invariant tensor field

$$\tilde{g}_P := g_P + \frac{1}{f_Z^c}\sum_{j=0}^{3}(\theta_j^P)^2,$$

where recall that $Z_P = \tilde{Z} + f_{\mathrm{H}}^cX_P$, and consider

$$g_{\bar{N}}^c := \frac{1}{4|f_Z^c|}\tilde{g}_P|_{\bar{N}},$$

where

$$\bar{N} := \{\arg(X^0) = 0\} \subset P = N \times \mathbb{S}^1 \tag{33}$$

is a codimension one submanifold of P which is transversal to the vector field Z_P and $(X^0,\dots,X^{n-1})$ are special holomorphic coordinates of the CASK manifold M. Then, by [ACDM15, Theorem 2 and Theorem 5], $(\bar{N}, g^c_{\bar{N}})$ is precisely the one-loop deformed c-map space.

To state Theorem 5.4.4, we have to focus on a particular class of CASK manifolds, namely those coming from projective special real manifolds (see Section 4.3 and (21)). In this particular situation, where the CASK manifold M is determined by a PSR manifold $\mathcal{H} \subset \mathbb{R}^{n-1}$, a group of isometries preserving the CASK structure was described in [CDJL21, Appendix A]. More precisely, the group we are considering is

$$\mathrm{Aff}_{\mathcal{H}}(\mathbb{R}^{n-1}) := (\mathbb{R}_{>0} \times \mathrm{Aut}(\mathcal{H})) \ltimes \mathbb{R}^{n-1} \hookrightarrow \mathrm{Aut}(M) \subset \mathrm{Sp}(\mathbb{R}^{2n}),$$

where

$$\mathrm{Aut}(\mathcal{H}) := \{A \in \mathrm{GL}(n-1,\mathbb{R}) \mid A\mathcal{H} = \mathcal{H}\}$$

and the arrow is a certain embedding [CDJL21, Proposition 23]. We then have the following result.

Theorem 5.4.4. *Let M be a CASK manifold determined by the PSR manifold $\mathcal{H} \subset \mathbb{R}^{n-1}$. Then the group*

$$\mathrm{Aff}_{\mathcal{H}}(\mathbb{R}^{n-1}) \ltimes (\mathrm{Heis}_{2n+1}/\mathcal{F}),$$

where $\mathcal{F}$ is an infinite cyclic subgroup of the Heisenberg center, acts effectively and isometrically on $(\bar{N}, g^c_{\bar{N}})$ for $c \geq 0$.

Proof. From the explicit description of the action of $\mathrm{Aff}_{\mathcal{H}}(\mathbb{R}^{n-1})$ on M given in [CDJL21, Appendix A] we see that the function $X^0 \in \mathscr{C}^\infty(P)$ changes under this action only by a real positive factor. It follows that the submanifold $\bar{N} \subset P$ given by (33) is invariant under the action of $\mathrm{Aff}_{\mathcal{H}}(\mathbb{R}^{n-1}) \ltimes \mathrm{Heis}_{2n+1}$, since the action of Heis_{2n+1} preserves the fibers of $\pi \circ \pi_N : P \longrightarrow M$, where $\pi : N = T^*M \longrightarrow M$ and $\pi_N : P \longrightarrow N$. Hence the group action on P described in (31) restricts to $\bar{N}$. From the explicit description, it is also easy to check that the action of $\mathrm{Aff}_{\mathcal{H}}(\mathbb{R}^{n-1})$ is effective on $\bar{N} \subset P$.

Finally, the action of Heis_{2n+1} is not quite effective, since its center acts by translations along the $\mathbb{S}^1$-factor of P. To obtain an effective action we need to divide out the infinite cyclic subgroup of the center, whose elements correspond to shifting $s \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ by $2\pi k$, $k \in \mathbb{Z}$.

The group $\mathrm{Aff}_{\mathcal{H}}(\mathbb{R}^{n-1}) \ltimes (\mathrm{Heis}_{2n+1}/\mathcal{F})$ acts by isometries on $\bar{N}$ by construction. □

Remark 5.4.5. The cyclic group $\mathcal{F}$ included in Theorem 5.4.4 to ensure effectiveness can be removed by considering the universal covering of the quaternionic Kähler manifold $\bar{N}$, which amounts to replacing the circle bundle P by an $\mathbb{R}$-bundle.

Remark 5.4.6. In [CRT21, Theorem 3.16] a similar result to that of Theorem 5.4.4 is obtained in the case where the CASK manifold M is determined by the PSK manifold $\bar{M} = \mathbb{C}\mathrm{H}^{n-1}$ with the transitive action of $\mathrm{Aut}(M) = \mathrm{SU}(1,n-1)$. This example does not belong to the above series of spaces determined by a PSR manifold $\mathcal{H}$ since $\mathbb{C}\mathrm{H}^{n-1}$ is not in the image of the supergravity r-map. They obtain that $\mathrm{SU}(1,n-1) \ltimes (\mathrm{Heis}_{2n+1}/\mathcal{F}')$ acts effectively and isometrically on $(\bar{N}'/\mathcal{F}', g^c_{\bar{N}})$, where $\bar{N}' \cong \mathbb{R}^{4n}$ is the universal covering of our $\bar{N}$, and $\mathcal{F}'$ is trivial if $c = 0$ and infinite cyclic for $c > 0$. In the latter case $\bar{N}'/\mathcal{F}' = \bar{N}$.

Chapter 6

Hypersurfaces of supergravity c-map spaces

In this second and last original chapter, we present the results of the study of the induced geometry on the hypersurface orbits of a cohomogeneity one group action on a deformed supergravity c-map space. More precisely, in Section 6.1 we first compute a general formula for the Ricci curvature tensor of a hypersurface of an Einstein manifold equipped with the induced metric. Then we apply this formula to the one-loop deformation of the symmetric space $\mathrm{SU}(n,2)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2))$, which is a supergravity c-map space, to obtain an explicit formula for the eigenvalues of the Ricci endomorphism of the induced metric on the hypersurface in terms of the deformation parameter $c \geq 0$. We conclude that the hypersurfaces are not homothetic for $c = 0$ and $c > 0$. In Section 6.2 we show that the hypersurface can be seen as a solvmanifold, that is, a solvable Lie group equipped with a left-invariant metric. We first determine the algebraic structure of the Lie group obtaining its structure constants and deducing some properties of it. Then we explain how to realize the induced metric on the hypersurface as a left-invariant metric on the Lie group. Finally we show, by explicit computations, that for $c = 0$ we have an algebraic Ricci soliton on the Lie group, whereas this is not longer true for $c > 0$.

6.1 Ricci curvature of a hypersurface of an Einstein manifold

Let K be a smooth manifold and consider the smooth manifold $\bar{N} := (0,\infty)\times K$. Write $\rho : \bar{N} \longrightarrow (0,\infty)$ for the canonical projection. On $\bar{N}$ we suppose that we have an Einstein metric g of the form

$$g = f(\rho)\mathrm{d}\rho^2 + g_\rho,$$

where g_ρ is a Riemannian metric on $\bar{N}_\rho := \{\rho\}\times K$ and $f : \bar{N} \longrightarrow (0,\infty)$ a smooth positive function depending only on ρ. We may think of g_ρ as a one-parameter family of Riemannian metrics on K depending on ρ. In this section we give a general formula for the Ricci tensor of the hypersurface $\bar{N}_\rho$. Similar computations appear in a different context in [Koi81]. Of course, in the case when ρ is a distance function, i.e. $f \equiv 1$, the computations that follow are classical, but we nevertheless work out the formulas for general f, since these then readily apply to the family of quaternionic Kähler metrics we are interested in.

6.1.1 Formula for the Ricci tensor of a hypersurface

We will denote
$$\partial_\rho := \frac{\partial}{\partial \rho}.$$

Lemma 6.1.1. *The bilinear form $h \in \Gamma(T^*\bar{N} \otimes T^*\bar{N})$ given by $h(X,Y) := g(\nabla_X \partial_\rho, Y)$ is symmetric, i.e. $g(\nabla_X \partial_\rho, Y) = g(\nabla_Y \partial_\rho, X)$.*

Proof. Consider the function $F = \int f \mathrm{d}\rho : (0,\infty) \longrightarrow \mathbb{R}$. Then $F' = f$, so that, viewed as a function on $\bar{N}$, we have $\mathrm{d}F = f\mathrm{d}\rho$. It follows that $\operatorname{grad}(F) = \partial_\rho$, and h is by definition the Hessian of F, hence symmetric. $\square$

Lemma 6.1.2. *We have $\nabla_{\partial_\rho} \partial_\rho = \frac{1}{2}\frac{f'}{f}\partial_\rho$.*

Proof. First of all we have

$$g(\nabla_{\partial_\rho}\partial_\rho, \partial_\rho) = \frac{1}{2}\frac{\partial}{\partial\rho} g(\partial_\rho, \partial_\rho) = \frac{1}{2}\frac{\partial f}{\partial \rho} = \frac{1}{2} f'(\rho) = g(\tfrac{1}{2}\tfrac{f'}{f}\partial_\rho, \partial_\rho). \tag{34}$$

Now, if X is a vector field tangent to the fibers of ρ, i.e. $\mathrm{d}\rho(X) = 0$, then $g(X, \partial_\rho) = 0$ and

$$\mathrm{d}\rho([X,\partial_\rho]) = [X,\partial_\rho](\rho) = X(\partial_\rho(\rho)) - \partial_\rho(X(\rho)) = X(1) - \partial_\rho(0) = 0.$$

This implies $g([X,\partial_\rho],\partial_\rho) = 0$. Now we compute using the Koszul formula

$$\begin{aligned} 2g(\nabla_{\partial_\rho}\partial_\rho, X) &= \frac{\partial}{\partial\rho} g(\partial_\rho, X) + \frac{\partial}{\partial\rho} g(\partial_\rho, X) - X g(\partial_\rho,\partial_\rho) \\ &\quad - g([\partial_\rho,\partial_\rho],X) - g([\partial_\rho, X],\partial_\rho) + g([X,\partial_\rho],\partial_\rho) \\ &= -X(f) = 0. \end{aligned}$$

It follows that $\nabla_{\partial_\rho}\partial_\rho \perp \ker(\mathrm{d}\rho)$, i.e. $\nabla_{\partial_\rho}\partial_\rho = \frac{1}{2}\frac{f'}{f}\partial_\rho$ by (34). $\square$

To state the next lemma we use the following notation. If $\alpha \in \Gamma(T^*\bar{N} \otimes T^*\bar{N})$ is a bilinear form with associated endomorphism A, i.e. $\alpha(\cdot,\cdot) = g(A\cdot,\cdot)$, we write

$$\alpha^2(\cdot,\cdot) = g(A^2\cdot,\cdot) = g(A\cdot, A\cdot) = \alpha(\cdot, A\cdot) \in \Gamma(T^*\bar{N} \otimes T^*\bar{N}).$$

In particular, for the bilinear form h above, we have $h(\cdot,\cdot) = g(\nabla_\cdot \partial_\rho, \cdot)$ so that $h^2(\cdot,\cdot) = g(\nabla_\cdot \partial_\rho, \nabla_\cdot \partial_\rho) = h(\cdot, \nabla_\cdot \partial_\rho)$.

Note that if $\{E_i\}$ is an orthonormal basis of $T\bar{N}$, then

$$AX = \sum_i g(AX, E_i)E_i = \sum_i \alpha(X, E_i)E_i,$$

so that

$$\alpha^2(X,X) = g(AX,AX) = \sum_{i,j} g(\alpha(X,E_i)E_i, \alpha(X,E_j)E_j) = \sum_i \alpha(X,E_i)^2.$$

Lemma 6.1.3. *Let $X, Y \in \Gamma(T\bar{N})$ be two vector fields tangent to K, i.e. such that $\mathrm{d}\rho(X) = 0 = \mathrm{d}\rho(Y)$, and suppose $[X, \partial_\rho] = 0 = [Y, \partial_\rho]$. Then*

(a) $g(\nabla_X \partial_\rho, \partial_\rho) = 0$.

(b) $h(X,Y) = g(\nabla_X \partial_\rho, Y) = -g(\nabla_X Y, \partial_\rho) = \frac{1}{2}\frac{\partial}{\partial\rho} g_\rho(X,Y)$.

(c) *The second fundamental form* $\mathrm{II} : T\bar{N}_\rho \times T\bar{N}_\rho \longrightarrow T\bar{N}_\rho^\perp$ *is given by*

$$\mathrm{II}(X,Y) = -\frac{1}{f} h(X,Y)\partial_\rho = -\frac{1}{2f}\frac{\partial}{\partial\rho} g_\rho(X,Y)\partial_\rho.$$

(d) $g(R_{\bar{N}}(X,\partial_\rho)\partial_\rho, X) = \frac{1}{4}\frac{f'}{f}\frac{\partial}{\partial\rho} g_\rho(X,X) - \frac{1}{2}\frac{\partial^2}{\partial\rho^2} g_\rho(X,X) + h^2(X,X)$.

Proof. (a) We have $g(\nabla_X \partial_\rho, \partial_\rho) = \frac{1}{2} X g(\partial_\rho, \partial_\rho) = \frac{1}{2} X(f) = 0$.

(b) Since $g(\partial_\rho, Y) = 0$ and ∇ is metric, it follows that $g(\nabla_X \partial_\rho, Y) = -g(\nabla_X Y, \partial_\rho)$. To compute $g(\nabla_X \partial_\rho, Y)$ we apply the Koszul formula using that $g(\partial_\rho, X) = 0 = g(\partial_\rho, Y)$ and $[X, \partial_\rho] = 0 = [Y, \partial_\rho]$:

$$\begin{aligned} 2h(X,Y) &= 2g(\nabla_X \partial_\rho, Y) \\ &= Xg(\partial_\rho, Y) + \frac{\partial}{\partial\rho} g(X,Y) - Yg(X,\partial_\rho) \\ &\quad + g([X,\partial_\rho],Y) - g([\partial_\rho,Y],X) + g([Y,X],\partial_\rho) \\ &= \frac{\partial}{\partial\rho} g_\rho(X,Y). \end{aligned}$$

(c) By definition we have $\mathrm{II}(X,Y) = (\nabla_X Y)^\perp = g(\nabla_X Y, \nu)\nu$, where $\nu = \frac{1}{\sqrt{f}}\partial_\rho$ is the unit normal vector field. Thus, using part (b):

$$\mathrm{II}(X,Y) = \frac{1}{f} g(\nabla_X Y, \partial_\rho)\partial_\rho = -\frac{1}{f} h(X,Y)\partial_\rho = -\frac{1}{2f}\frac{\partial}{\partial\rho} g_\rho(X,Y)\partial_\rho.$$

(d) We compute using part (a) and (b):

$$\begin{aligned}
g(R_{\bar N}(X,\partial_\rho)\partial_\rho,X) &= g(\nabla_X\nabla_{\partial_\rho}\partial_\rho-\nabla_{\partial_\rho}\nabla_X\partial_\rho-\nabla_{[X,\partial_\rho]}\partial_\rho,X)\\
&= g(\nabla_X(\tfrac{1}{2}\tfrac{f'}{f}\partial_\rho),X)-g(\nabla_{\partial_\rho}\nabla_X\partial_\rho,X)\\
&= \frac{1}{2}\frac{f'}{f}g(\nabla_X\partial_\rho,X)-\frac{\partial}{\partial\rho}g(\nabla_X\partial_\rho,X)+g(\nabla_X\partial_\rho,\nabla_{\partial_\rho}X)\\
&= \frac{1}{4}\frac{f'}{f}\frac{\partial}{\partial\rho}g_\rho(X,X)-\frac{1}{2}\frac{\partial^2}{\partial\rho^2}g_\rho(X,X)+h(X,\nabla_{\partial_\rho}X)\\
&= \frac{1}{4}\frac{f'}{f}\frac{\partial}{\partial\rho}g_\rho(X,X)-\frac{1}{2}\frac{\partial^2}{\partial\rho^2}g_\rho(X,X)+h(X,\nabla_X\partial_\rho)\\
&= \frac{1}{4}\frac{f'}{f}\frac{\partial}{\partial\rho}g_\rho(X,X)-\frac{1}{2}\frac{\partial^2}{\partial\rho^2}g_\rho(X,X)+h^2(X,X).
\end{aligned}$$

□

Lemma 6.1.4. *Suppose that $(\bar N,g)$ is an Einstein manifold with Einstein constant $\lambda\in\mathbb{R}$. Consider the hypersurface $\bar N_\rho=\{\rho\}\times K$ with induced metric g_ρ, unit normal vector field $\nu\in\Gamma(T\bar N_\rho^\perp)$ and second fundamental form $\mathrm{II}\in\Gamma(T^*\bar N_\rho\otimes T^*\bar N_\rho\otimes T\bar N_\rho^\perp)$. Then for any $x\in\bar N_\rho$ and $X\in T_x\bar N_\rho$ we have, with an orthonormal basis $\{E_i\}$ of $T_x\bar N_\rho$:*

$$\begin{aligned}
\mathrm{Ric}_{\bar N_\rho}(X,X) &= \lambda g_\rho(X,X)+g(\mathrm{II}(X,X),\mathrm{tr}(\mathrm{II}))\\
&\quad -\sum_i g(\mathrm{II}(X,E_i),\mathrm{II}(X,E_i))-\frac{1}{f}\mathrm{Rm}_{\bar N}(X,\partial_\rho,\partial_\rho,X).
\end{aligned}$$

Proof. Let $x\in\bar N_\rho$ and $X,Y,Z,W\in T_x\bar N_\rho$. Then writing the curvature tensors as $\mathrm{Rm}_{\bar N}(X,Y,Z,W)=g(R_{\bar N}(X,Y)Z,W)$ and $\mathrm{Rm}_{\bar N_\rho}(X,Y,Z,W)=g_\rho(R_{\bar N_\rho}(X,Y)Z,W)$ we have by the Gauss equation:

$$\mathrm{Rm}_{\bar N}(X,Y,Z,W)=\mathrm{Rm}_{\bar N_\rho}(X,Y,Z,W)-g(\mathrm{II}(X,W),\mathrm{II}(Y,Z))+g(\mathrm{II}(X,Z),\mathrm{II}(Y,W)).$$

Now let $\{E_i\}$ be an orthonormal basis of $T_x\bar N_\rho$, so that $\{E_i\}\cup\{\nu_x\}$ is an orthonormal basis of $T_x\bar N$, where $\nu=\frac{1}{\sqrt f}\partial_\rho$. Then we have

$$\begin{aligned}
\mathrm{Ric}_{\bar N}(X,X) &= \mathrm{tr}(V\longmapsto R_{\bar N}(V,X)X)\\
&= \sum_i \mathrm{Rm}_{\bar N}(X,E_i,E_i,X)+\mathrm{Rm}_{\bar N}(X,\nu_x,\nu_x,X)\\
&= \sum_i\left(\mathrm{Rm}_{\bar N_\rho}(X,E_i,E_i,X)-g(\mathrm{II}(X,X),\mathrm{II}(E_i,E_i))\right)\\
&\quad +\sum_i g(\mathrm{II}(X,E_i),\mathrm{II}(E_i,X))+\mathrm{Rm}_{\bar N}(X,\nu_x,\nu_x,X)\\
&= \mathrm{Ric}_{\bar N_\rho}(X,X)-g(\mathrm{II}(X,X),\mathrm{tr}(\mathrm{II}))\\
&\quad +\sum_i g(\mathrm{II}(X,E_i),\mathrm{II}(X,E_i))+\frac{1}{f}\mathrm{Rm}_{\bar N}(X,\partial_\rho,\partial_\rho,X).
\end{aligned}$$

Now $\mathrm{Ric}_{\bar N}=\lambda g$ and therefore we get the claimed formula. □

We write $h_\rho \in \Gamma(T^*\bar{N}_\rho \otimes T^*\bar{N}_\rho)$ for the restriction of h to $\bar{N}_\rho$. We refine the formula obtained in Lemma 6.1.4 as follows.

Lemma 6.1.5. *Suppose that $(\bar{N}, g)$ is an Einstein manifold with Einstein constant $\lambda \in \mathbb{R}$. Consider the hypersurface $\bar{N}_\rho = \{\rho\} \times K$ with induced metric g_ρ, unit normal vector field $\nu \in \Gamma(T\bar{N}_\rho^\perp)$ and second fundamental form $\mathrm{II} \in \Gamma(T^*\bar{N}_\rho \otimes T^*\bar{N}_\rho \otimes T\bar{N}_\rho^\perp)$. Then we have*

$$\mathrm{Ric}_{\bar{N}_\rho} = \lambda g_\rho + \left(\frac{1}{4f}\mathrm{tr}\left(\frac{\partial}{\partial\rho}g_\rho\right) - \frac{f'}{4f^2}\right)\frac{\partial}{\partial\rho}g_\rho - \frac{2}{f}h_\rho^2 + \frac{1}{2f}\frac{\partial^2}{\partial\rho^2}g_\rho.$$

Proof. Let $x \in \bar{N}_\rho$ and $X \in T_x \in \bar{N}_\rho$. Using Lemma 6.1.3 we obtain:

$$\begin{aligned}
\mathrm{Ric}_{\bar{N}_\rho}(X,X) &= \lambda g_\rho(X,X) + g(\mathrm{II}(X,X),\mathrm{tr}(\mathrm{II})) - \sum_i g(\mathrm{II}(X,E_i),\mathrm{II}(X,E_i)) \\
&\quad - \frac{1}{f}\mathrm{Rm}_{\bar{N}}(X,\partial_\rho,\partial_\rho,X) \\
&= \lambda g_\rho(X,X) + \frac{1}{f}h_\rho(X,X)\,\mathrm{tr}(h_\rho) - \frac{1}{f}\sum_i h_\rho(X,E_i)h_\rho(X,E_i) \\
&\quad - \frac{1}{f}\mathrm{Rm}_{\bar{N}}(X,\partial_\rho,\partial_\rho,X) \\
&= \lambda g_\rho(X,X) + \frac{1}{f}h_\rho(X,X)\,\mathrm{tr}(h_\rho) - \frac{1}{f}h_\rho^2(X,X) \\
&\quad - \frac{1}{f}\left(\frac{1}{4}\frac{f'}{f}\frac{\partial}{\partial\rho}g_\rho(X,X) - \frac{1}{2}\frac{\partial^2}{\partial\rho^2}g_\rho(X,X) + h_\rho^2(X,X)\right) \\
&= \lambda g_\rho(X,X) + \frac{1}{f}h_\rho(X,X)\,\mathrm{tr}(h_\rho) - \frac{2}{f}h_\rho^2(X,X) \\
&\quad - \frac{f'}{4f^2}\frac{\partial}{\partial\rho}g_\rho(X,X) + \frac{1}{2f}\frac{\partial^2}{\partial\rho^2}g_\rho(X,X) \\
&= \lambda g_\rho(X,X) + \frac{1}{4f}\frac{\partial}{\partial\rho}g_\rho(X,X)\,\mathrm{tr}\left(\frac{\partial}{\partial\rho}g_\rho\right) - \frac{2}{f}h_\rho^2(X,X) \\
&\quad - \frac{f'}{4f^2}\frac{\partial}{\partial\rho}g_\rho(X,X) + \frac{1}{2f}\frac{\partial^2}{\partial\rho^2}g_\rho(X,X) \\
&= \lambda g_\rho(X,X) + \left(\frac{1}{4f}\mathrm{tr}\left(\frac{\partial}{\partial\rho}g_\rho\right) - \frac{f'}{4f^2}\right)\frac{\partial}{\partial\rho}g_\rho(X,X) \\
&\quad - \frac{2}{f}h_\rho^2(X,X) + \frac{1}{2f}\frac{\partial^2}{\partial\rho^2}g_\rho(X,X).
\end{aligned}$$

□

6.1.2 Application to a particular series of examples

Let us consider the one-loop deformation (with $c > 0$) of the non-compact symmetric space

$$\mathrm{SU}(n,2)/\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(2)).$$

We identify the underlying manifold $\bar{N}$ as

$$\bar{N} = (0,\infty)\times(B_1(0)\times\mathbb{R}\times\mathbb{C}^n),$$

where $B_1(0)\subset\mathbb{C}^{n-1}$ is the unit open ball. On $\bar{N}$ we have the global coordinate system

$$(\rho, X^a, \tilde{\phi}, w^0, w^a)\in(0,\infty)\times(\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{C}\times\mathbb{C}^{n-1}),$$

where $a = 1,\dots,n-1$ and $\|X\|^2 = \sum_a |X^a|^2 < 1$. If $n = 1$ we adopt the convention that the family $(X^a, w^a)_{a=1}^0$ is empty, so that for $\bar{N} = (0,\infty)\times\mathbb{R}\times\mathbb{C}$ the global coordinate system is just $(\rho,\tilde{\phi},w^0)$. With the convention $\sum_{a=1}^0 = 0$, the one-loop deformed metric is then explicitly given by

$$g^c_{\bar{N}} = \frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}\mathrm{d}\rho^2 + g^c_\rho,$$

where

$$\begin{aligned} g^c_\rho = {} & \frac{\rho+c}{\rho}\frac{1}{1-\|X\|^2}\left(\sum_{a=1}^{n-1}|\mathrm{d}X^a|^2 + \frac{1}{1-\|X\|^2}\left|\sum_{a=1}^{n-1}\bar{X}^a\mathrm{d}X^a\right|^2\right) \\ & + \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}\left(\mathrm{d}\tilde{\phi} - 4\,\mathrm{Im}\left(\bar{w}^0\mathrm{d}w^0 - \sum_{a=1}^{n-1}\bar{w}^a\mathrm{d}w^a\right)\right. \\ & \left. + \frac{2c}{1-\|X\|^2}\mathrm{Im}\left(\sum_{a=1}^{n-1}\bar{X}^a\mathrm{d}X^a\right)\right)^2 \\ & - \frac{2}{\rho}\left(\mathrm{d}w^0\mathrm{d}\bar{w}^0 - \sum_{a=1}^{n-1}\mathrm{d}w^a\mathrm{d}\bar{w}^a\right) + \frac{\rho+c}{\rho^2}\frac{4}{1-\|X\|^2}\left|\mathrm{d}w^0 + \sum_{a=1}^{n-1}X^a\mathrm{d}w^a\right|^2. \end{aligned} \tag{35}$$

Remark 6.1.6. The metric (35) is obtained by applying the (deformed) supergravity c-map to the PSK manifold $\bar{M} = \mathbb{C}\mathrm{H}^{n-1}$ with quadratic holomorphic prepotential (see Example 4.3.8). The expressions (35) and (19) (applied to this particular case) are equivalent ways to express the quaternionic Kähler metric, but using different conventions (see [CT22b, Remark 2.7]).

The metric $g^c_{\bar{N}}$ fits into the framework of the previous subsection with

$$K = B_1(0)\times\mathbb{R}\times\mathbb{C}^n \quad \text{and} \quad f(\rho) = \frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}.$$

The metric $g^c_{\bar{N}}$ is quaternionic Kähler with reduced scalar curvature $\nu = \frac{\mathrm{scal}}{4n(n+2)} = -2$ (see [CDS17, Page 89]). In particular, it is Einstein with Einstein constant equal to $\lambda = \frac{\mathrm{scal}}{4n} = -2(n+2)$.

We are interested in computing the Ricci curvature tensor of the hypersurface $\bar{N}_\rho$ with metric g^c_ρ. For that we want to apply Lemma 6.1.5 and therefore we need to compute $\frac{\partial}{\partial\rho} g^c_\rho$, $\frac{\partial^2}{\partial\rho^2} g^c_\rho$ and the bilinear form h^2_ρ.

The level sets $\bar{N}_\rho$ of ρ are homogeneous, thus it suffices to perform all computations at a point $p_\rho = (\rho, 0) \in \bar{N}_\rho = \{\rho\} \times K$, obtained by fixing ρ and setting all the other coordinates to zero. Moreover, we are only differentiating in the direction ρ, so we may first evaluate g^c_ρ at $0 \in K$ and then differentiate with respect to ρ. We may thus work with the metric g^c_ρ, expressed in real coordinates, in the following simplified form:

$$
\begin{aligned}
g^c_\rho = {} & \frac{\rho+c}{4\rho} \sum_{a=1}^{n-1} \left((\mathrm{d}b^a)^2 + (\mathrm{d}t^a)^2 \right) + \frac{1}{4\rho^2} \frac{\rho+c}{\rho+2c} \mathrm{d}\tilde{\phi}^2 \\
& + \frac{1}{2\rho} \frac{\rho+2c}{\rho} \left((\mathrm{d}\tilde{\zeta}_0)^2 + (\mathrm{d}\zeta^0)^2 \right) + \frac{1}{2\rho} \sum_{a=1}^{n-1} \left((\mathrm{d}\tilde{\zeta}_a)^2 + (\mathrm{d}\zeta^a)^2 \right),
\end{aligned}
$$

where

$$
X^a = \frac{1}{2}(b^a + it^a), \quad w^0 = \frac{1}{2}(\tilde{\zeta}_0 + i\zeta^0), \quad w^a = \frac{1}{2}(\tilde{\zeta}_a - i\zeta^a).
$$

Now we compute the derivative of the metric g^c_ρ:

$$
\begin{aligned}
\frac{\partial}{\partial\rho} g^c_\rho = {} & -\frac{c}{4\rho^2} \sum_{a=1}^{n-1} \left((\mathrm{d}b^a)^2 + (\mathrm{d}t^a)^2 \right) - \frac{1}{4\rho^3} \frac{2\rho^2 + 5c\rho + 4c^2}{(\rho+2c)^2} \mathrm{d}\tilde{\phi}^2 \\
& - \frac{1}{2\rho^2} \frac{\rho+4c}{\rho} \left((\mathrm{d}\tilde{\zeta}_0)^2 + (\mathrm{d}\zeta^0)^2 \right) - \frac{1}{2\rho^2} \sum_{a=1}^{n-1} \left((\mathrm{d}\tilde{\zeta}_a)^2 + (\mathrm{d}\zeta^a)^2 \right) \\
= {} & -\frac{1}{\rho} \left(h_1(\rho) \frac{\rho+c}{4\rho} \sum_{a=1}^{n-1} \left((\mathrm{d}b^a)^2 + (\mathrm{d}t^a)^2 \right) + h_2(\rho) \frac{1}{4\rho^2} \frac{\rho+c}{\rho+2c} \mathrm{d}\tilde{\phi}^2 \right) \\
& - \frac{1}{\rho} \left(h_3(\rho) \frac{1}{2\rho} \frac{\rho+2c}{\rho} \left((\mathrm{d}\tilde{\zeta}_0)^2 + (\mathrm{d}\zeta^0)^2 \right) + \frac{1}{2\rho} \sum_{a=1}^{n-1} \left((\mathrm{d}\tilde{\zeta}_a)^2 + (\mathrm{d}\zeta^a)^2 \right) \right),
\end{aligned}
$$

where

$$
h_1(\rho) := \frac{c}{\rho+c} > 0, \quad h_2(\rho) := \frac{2\rho^2 + 5c\rho + 4c^2}{(\rho+c)(\rho+2c)} > 0, \quad h_3(\rho) := \frac{\rho+4c}{\rho+2c} > 0.
$$

Note that the Gram matrix of g^c_ρ is diagonal in these coordinates:

$$
g^c_\rho = \mathrm{diag}\left(\frac{\rho+c}{4\rho} \mathbb{1}_{2n-2}, \frac{\rho+c}{4\rho^2(\rho+2c)}, \frac{\rho+2c}{2\rho^2} \mathbb{1}_2, \frac{1}{2\rho} \mathbb{1}_{2n-2} \right),
$$

where we denote by $\mathbb{1}_k$ the $k\times k$ identity matrix. Sometimes we will also write $\mathbb{O}_k$ for the $k\times k$ zero matrix. If $n=1$, then $2n-2=0$ and we adopt the convention to interpret the Gram matrix of g^c_ρ as

$$g^c_\rho = \operatorname{diag}\left(\frac{\rho+c}{4\rho^2(\rho+2c)}, \frac{\rho+2c}{2\rho^2}\mathbb{1}_2\right),$$

which is consistent with our conventions of choosing coordinates on $\bar{N}$ explained above. We apply analogous conventions to the various other Gram matrices that appear below and henceforth we will not explicitly distinguish between the cases $n=1$ and $n>1$. It follows that the Gram matrix of the bilinear form $h_\rho(\cdot,\cdot)=g^c_\rho(\nabla.\partial_\rho,\cdot)=\frac{1}{2}\frac{\partial}{\partial\rho}g^c_\rho$ is

$$\begin{aligned}
H_\rho &= \frac{1}{2}\frac{\partial}{\partial\rho}g^c_\rho \\
&= -\frac{1}{2\rho}\operatorname{diag}\left(h_1(\rho)\frac{\rho+c}{4\rho}\mathbb{1}_{2n-2}, h_2(\rho)\frac{\rho+c}{4\rho^2(\rho+2c)}, h_3(\rho)\frac{\rho+2c}{2\rho^2}\mathbb{1}_2, \frac{1}{2\rho}\mathbb{1}_{2n-2}\right) \\
&= g^c_\rho A_\rho,
\end{aligned}$$

where A_ρ is the diagonal matrix

$$A_\rho = -\frac{1}{2\rho}\operatorname{diag}\left(h_1(\rho)\mathbb{1}_{2n-2}, h_2(\rho), h_3(\rho)\mathbb{1}_2, \mathbb{1}_{2n-2}\right) \tag{36}$$

corresponding to the endomorphism $\nabla\partial_\rho$. From this computation and Lemma 6.1.3 we deduce the following.

Proposition 6.1.7. *The eigenvalues of the shape operator S^c_ρ of the hypersurface $(\bar{N}_\rho, g^c_\rho)\subset(\bar{N}, g^c_{\bar{N}})$ with respect to the unit normal vector field $\frac{1}{\sqrt{f}}\partial_\rho$ are given by*

$$\begin{aligned}
\sigma_1 &= \frac{c}{\rho+c}\sqrt{\frac{\rho+c}{\rho+2c}}, \\
\sigma_2 &= \frac{2\rho^2+5c\rho+4c^2}{(\rho+c)(\rho+2c)}\sqrt{\frac{\rho+c}{\rho+2c}}, \\
\sigma_3 &= \frac{\rho+4c}{\rho+2c}\sqrt{\frac{\rho+c}{\rho+2c}}, \\
\sigma_4 &= \sqrt{\frac{\rho+c}{\rho+2c}},
\end{aligned}$$

where the multiplicities of σ_1 and σ_4 are $2n-2$, the multiplicity of σ_2 is 1 and the multiplicity of σ_3 is 2. In particular, if $c>0$, then $(\bar{N}_\rho, g^c_\rho)$ is strictly convex. Furthermore, the mean curvature of $(\bar{N}_\rho, g^c_\rho)$ is

$$\operatorname{tr}(S^c_\rho) = \frac{(2n+2)\rho^2+(8n+7)c\rho+(8n+4)c^2}{(\rho+c)(\rho+2c)}\sqrt{\frac{\rho+c}{\rho+2c}}.$$

Proof. By Lemma 6.1.3 (c) we know that the second fundamental form II_ρ of the hypersurface $(\bar{N}_\rho, g^c_\rho) \subset (\bar{N}, g^c_{\bar{N}})$ evaluates on tangent vectors X, Y to

$$\mathrm{II}_\rho(X,Y) = -\frac{1}{f} h_\rho(X,Y)\partial_\rho = g(S^c_\rho(X),Y)\frac{1}{\sqrt{f}}\partial_\rho,$$

where S^c_ρ is the shape operator. At the point p_ρ we see from the above discussion that

$$g^c_\rho(S^c_\rho(X),Y) = -\frac{1}{\sqrt{f}} h_\rho(X,Y) = g^c_\rho((-\tfrac{1}{\sqrt{f}}A_\rho)X,Y).$$

That is, from (36) we get

$$S^c_\rho = -\frac{1}{\sqrt{f}}A_\rho = \frac{1}{2\rho\sqrt{f}}\operatorname{diag}\left(h_1(\rho)\mathbb{1}_{2n-2}, h_2(\rho), h_3(\rho)\mathbb{1}_2, \mathbb{1}_{2n-2}\right).$$

The eigenvalues are then explicitly given by

$$\begin{aligned}
\sigma_1 &= \frac{h_1(\rho)}{2\rho\sqrt{f}} = \frac{c}{\rho+c}\sqrt{\frac{\rho+c}{\rho+2c}},\\
\sigma_2 &= \frac{h_2(\rho)}{2\rho\sqrt{f}} = \frac{2\rho^2+5c\rho+4c^2}{(\rho+c)(\rho+2c)}\sqrt{\frac{\rho+c}{\rho+2c}},\\
\sigma_3 &= \frac{h_3(\rho)}{2\rho\sqrt{f}} = \frac{\rho+4c}{\rho+2c}\sqrt{\frac{\rho+c}{\rho+2c}},\\
\sigma_4 &= \frac{1}{2\rho\sqrt{f}} = \sqrt{\frac{\rho+c}{\rho+2c}}.
\end{aligned}$$

The mean curvature

$$\begin{aligned}
\operatorname{tr}(S^c_\rho) &= (2n-2)\sigma_1 + \sigma_2 + 2\sigma_3 + (2n-2)\sigma_4\\
&= \frac{(2n+2)\rho^2 + (8n+7)c\rho + (8n+4)c^2}{(\rho+c)(\rho+2c)}\sqrt{\frac{\rho+c}{\rho+2c}}
\end{aligned}$$

is obtained from a straightforward computation. □

From the previous computations we also deduce that the trace of the bilinear form $\frac{\partial}{\partial\rho}g^c_\rho = 2h_\rho$ is given by

$$\operatorname{tr}\left(\frac{\partial}{\partial\rho}g^c_\rho\right) = 2\operatorname{tr}(A_\rho) = -\frac{1}{\rho}\left((2n-2)h_1(\rho) + h_2(\rho) + 2h_3(\rho) + 2n-2\right). \tag{37}$$

Moreover, it follows that $(\nabla\partial_\rho)^2$ is represented by

$$A^2_\rho = \frac{1}{4\rho^2}\operatorname{diag}\left(h^2_1(\rho)\mathbb{1}_{2n-2}, h^2_2(\rho), h^2_3(\rho)\mathbb{1}_2, \mathbb{1}_{2n-2}\right).$$

The Gram matrix of h_ρ^2, the bilinear form corresponding to the endomorphism $(\nabla\partial_\rho)^2$, is therefore

$$\begin{aligned} H_\rho^2 &= A_\rho^2 g_\rho^c \\ &= \frac{1}{4\rho^2}\operatorname{diag}\left(h_1^2(\rho)\frac{\rho+c}{4\rho}\mathbb{1}_{2n-2}, h_2^2(\rho)\frac{\rho+c}{4\rho^2(\rho+2c)}, h_3^2(\rho)\frac{\rho+2c}{2\rho^2}\mathbb{1}_2, \frac{1}{2\rho}\mathbb{1}_{2n-2}\right). \end{aligned}$$

It remains to compute the second derivative $\frac{\partial^2}{\partial\rho^2}g_\rho^c$. Its Gram matrix is given by

$$\begin{aligned} \frac{\partial^2}{\partial\rho^2}g_\rho^c &= \frac{\partial}{\partial\rho}\left(\frac{\partial}{\partial\rho}g_\rho^c\right) = 2\frac{\partial}{\partial\rho}H_\rho = 2\frac{\partial}{\partial\rho}(A_\rho g_\rho^c) \\ &= 2\left(\frac{\partial A_\rho}{\partial\rho}g_\rho^c + A_\rho\frac{\partial g_\rho^c}{\partial\rho}\right) = 2\left(\frac{\partial A_\rho}{\partial\rho}g_\rho^c + 2A_\rho H_\rho\right) \\ &= 2\left(\frac{\partial A_\rho}{\partial\rho} + 2A_\rho^2\right)g_\rho^c. \end{aligned}$$

We have already computed all terms except $\frac{\partial A_\rho}{\partial\rho}$, which is

$$\frac{\partial A_\rho}{\partial\rho} = \frac{1}{2\rho^2}\operatorname{diag}\left((h_1-\rho h_1')\mathbb{1}_{2n-2}, h_2-\rho h_2', (h_3-\rho h_3')\mathbb{1}_2, \mathbb{1}_{2n-2}\right).$$

We then find that

$$\begin{aligned} \frac{\partial^2}{\partial\rho^2}g_\rho^c &= 2\left(\frac{\partial A_\rho}{\partial\rho} + 2A_\rho^2\right)g_\rho^c \\ &= \frac{1}{\rho^2}\operatorname{diag}\left((h_1^2+h_1-\rho h_1')\mathbb{1}_{2n-2}, h_2^2+h_2-\rho h_2', (h_3^2+h_3-\rho h_3')\mathbb{1}_2, 2\mathbb{1}_{2n-2}\right). \end{aligned}$$

Since we have that $\operatorname{Ric}(g_{\bar N}^c) = -2(n+2)g_{\bar N}^c$, we must take $\lambda = -2(n+2)$ in the formula of Lemma 6.1.5. All other quantities in that formula are now computed, so putting everything together we find the following.

Proposition 6.1.8. *Let $n\in\mathbb{N}$ and $c\geq 0$. The Ricci curvature tensor of $(\bar N_\rho, g_\rho^c)$ at the point $p_\rho = (\rho,0)$ is given by*

$$\operatorname{Ric}_{\bar N_\rho} = -2(n+2)g_\rho^c - 2n\rho\frac{\partial}{\partial\rho}g_\rho^c + \frac{1}{f}g_\rho^c\left(\tfrac{\partial A_\rho}{\partial\rho}\cdot,\cdot\right).$$

If $n=1$, then in the global real coordinates $(\tilde\phi, \tilde\zeta_0, \zeta^0)$ the Ricci endomorphism is represented by the diagonal matrix

$$\operatorname{ric}_{\bar N_\rho} = \operatorname{diag}\left(r_2, r_3\mathbb{1}_2\right).$$

If $n>1$, then in the global real coordinates $(b^a, t^a, \tilde\phi, \tilde\zeta_0, \zeta^0, \tilde\zeta_a, \zeta^a)$, $a=1,\dots,n-1$, the Ricci endomorphism is represented by the diagonal matrix

$$\operatorname{ric}_{\bar N_\rho} = \operatorname{diag}\left(r_1\mathbb{1}_{2n-2}, r_2, r_3\mathbb{1}_2, r_4\mathbb{1}_{2n-2}\right).$$

The principal Ricci curvatures are, for any $n \in \mathbb{N}$ and $c \geq 0$, given by

$$
\begin{aligned}
r_1 &= -\frac{2(n+2)\rho^2 + 4(n+2)c\rho + 6c^2}{(\rho+c)(\rho+2c)},\\
r_2 &= \frac{2n\rho^4 + 4(3n-2)c\rho^3 + 2(14n-13)c^2\rho^2 + 32(n-1)c^3\rho + 16(n-1)c^4}{(\rho+c)(\rho+2c)^3},\\
r_3 &= \frac{-2\rho^3 + 2(2n-3)c\rho^2 + 16(n-1)c^2\rho + 16(n-1)c^3}{(\rho+2c)^3},\\
r_4 &= -\frac{2(\rho+3c)}{\rho+2c}.
\end{aligned}
$$

Proof. The general formula of Lemma 6.1.5, with $\lambda = -2(n+2)$, becomes

$$\mathrm{Ric}_{\bar{N}_\rho} = -2(n+2)g_\rho + \frac{1}{4f}\left(\mathrm{tr}\left(\frac{\partial}{\partial\rho}g_\rho\right) - \frac{f'}{f}\right)\frac{\partial}{\partial\rho}g_\rho - \frac{2}{f}h_\rho^2 + \frac{1}{2f}\frac{\partial^2}{\partial\rho^2}g_\rho,$$

where recall that $f = \frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}$. Using (37) we can therefore compute that

$$\mathrm{tr}\left(\frac{\partial}{\partial\rho}g_\rho\right) - \frac{f'}{f} = -8n\rho f.$$

Thus we get

$$\frac{1}{4f}\left(\mathrm{tr}\left(\frac{\partial}{\partial\rho}g_\rho\right) - \frac{f'}{f}\right) = \frac{1}{4f}(-8n\rho f) = -2n\rho$$

and the formula for $\mathrm{Ric}_{\bar{N}_\rho}$ simplifies to

$$\mathrm{Ric}_{\bar{N}_\rho} = -2(n+2)g_\rho - 2n\rho\frac{\partial}{\partial\rho}g_\rho - \frac{2}{f}h_\rho^2 + \frac{1}{2f}\frac{\partial^2}{\partial\rho^2}g_\rho.$$

We next simplify the last two terms. In terms of Gram matrices we have

$$-2h_\rho^2 + \frac{1}{2}\frac{\partial^2}{\partial\rho^2}g_\rho^c = -2A_\rho^2 g_\rho^c + \frac{1}{2}\cdot 2\left(\frac{\partial A_\rho}{\partial\rho} + 2A_\rho^2\right)g_\rho^c = \frac{\partial A_\rho}{\partial\rho}g_\rho^c.$$

Hence we find

$$\mathrm{Ric}_{\bar{N}_\rho} = -2(n+2)g_\rho^c - 2n\rho\frac{\partial}{\partial\rho}g_\rho^c + \frac{1}{f}g_\rho^c(\tfrac{\partial A_\rho}{\partial\rho}\cdot,\cdot).$$

The Gram matrix of the bilinear form $\mathrm{Ric}_{\bar{N}_\rho}$ is then given by

$$
\begin{aligned}
\mathrm{Ric}_{\bar{N}_\rho} &= -2(n+2)g_\rho^c - 2n\rho\cdot 2A_\rho g_\rho^c + \frac{1}{f}\frac{\partial A_\rho}{\partial\rho}g_\rho^c\\
&= \left(-2(n+2)\mathbb{1}_{4n-1} - 4n\rho A_\rho + \frac{1}{f}\frac{\partial A_\rho}{\partial\rho}\right)g_\rho^c\\
&= \mathrm{diag}\left(r_1\mathbb{1}_{2n-2}, r_2, r_3\mathbb{1}_2, r_4\mathbb{1}_{2n-2}\right)g_\rho^c,
\end{aligned}
$$

where

$$
\begin{aligned}
r_1 &= -2(n+2) + 2nh_1 + \frac{2(\rho+c)(h_1 - \rho h_1')}{\rho+2c} \\
&= -\frac{2(n+2)\rho^2 + 4(n+2)c\rho + 6c^2}{(\rho+c)(\rho+2c)}, \\
r_2 &= -2(n+2) + 2nh_2 + \frac{2(\rho+c)(h_2 - \rho h_2')}{\rho+2c} \\
&= \frac{2n\rho^4 + 4(3n-2)c\rho^3 + 2(14n-13)c^2\rho^2 + 32(n-1)c^3\rho + 16(n-1)c^4}{(\rho+c)(\rho+2c)^3}, \\
r_3 &= -2(n+2) + 2nh_3 + \frac{2(\rho+c)(h_3 - \rho h_3')}{\rho+2c} \\
&= \frac{-2\rho^3 + 2(2n-3)c\rho^2 + 16(n-1)c^2\rho + 16(n-1)c^3}{(\rho+2c)^3}, \\
r_4 &= -2(n+2) + 2n + \frac{2(\rho+c)}{\rho+2c} = -\frac{2(\rho+3c)}{\rho+2c}.
\end{aligned}
$$

□

Remark 6.1.9. We gather some comments about the nature of the principal Ricci curvatures computed in Proposition 6.1.8:

(1) If $n = 1$, then g_ρ^c is a left-invariant metric on the 3-dimensional Heisenberg group (see Section 6.2). For any $c \geq 0$, the Ricci endomorphism has just two eigenvalues, namely

$$r_2 = \frac{2\rho^4 + 4c\rho^3 + 2c^2\rho^2}{(\rho+c)(\rho+2c)^3} = \frac{2\rho^2(\rho+c)}{(\rho+2c)^3} = -r_3.$$

Thus, in this case $\mathrm{ric}_{\bar{N}_\rho} = r_2 \operatorname{diag}(1,-1,-1)$. In fact, the Ricci endomorphism of any left-invariant metric on the 3-dimensional Heisenberg group may be put in this form (see [Mil76]).

(2) If $c = 0$ and $n > 1$, the principal Ricci curvatures simplify to

$$r_1 = -2(n+2), \quad r_2 = 2n, \quad r_3 = r_4 = -2.$$

Note in particular that $r_3 = r_4$ in this case, so the spectrum of $\mathrm{ric}_{\bar{N}_\rho}$ consists only of three distinct eigenvalues. It follows that $\mathrm{ric}_{\bar{N}_\rho}$ restricts to a multiple of the identity on the subspace of $T_{p_\rho}\bar{N}_\rho$ spanned by $\frac{\partial}{\partial \zeta^0}, \frac{\partial}{\partial \tilde{\zeta}_0}, \ldots, \frac{\partial}{\partial \zeta^{n-1}}, \frac{\partial}{\partial \tilde{\zeta}_{n-1}}$.

(3) In the case $n > 1$ and $c > 0$ the eigenvalues r_1, r_2, r_3, r_4 are distinct and, in contrast to the case $c = 0$, the vectors $\frac{\partial}{\partial \zeta^0}, \frac{\partial}{\partial \tilde{\zeta}_0}$ and $\frac{\partial}{\partial \zeta^1}, \frac{\partial}{\partial \tilde{\zeta}_1}, \ldots, \frac{\partial}{\partial \zeta^{n-1}}, \frac{\partial}{\partial \tilde{\zeta}_{n-1}}$ now span distinct eigenspaces of $\mathrm{ric}_{\bar{N}_\rho}$.

From these remarks we deduce the following important consequence.

Corollary 6.1.10. *Let $n > 1$. The metrics $g_\rho = g_\rho^0$ and g_ρ^c, $c > 0$, are not homothetic.*

6.2 The level sets $(\bar{N}_\rho, g^c_\rho)$ as Riemannian solvmanifolds

In [CRT21] it was shown that the level sets $\bar{N}_\rho$ are the orbits for an isometric and c-dependent action of the simply connected Lie group with Lie algebra $\mathfrak{u}(1,n-1) \ltimes \mathfrak{heis}_{2n+1}$. The Lie algebra of the stabilizer of the point $p_\rho = (\rho, 0)$ with respect to this action is a subalgebra of $\mathfrak{u}(1,n-1) \ltimes \mathfrak{heis}_{2n+1}$ isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{u}(n-1)$. In this section we shall see that the subgroup L with Lie algebra $\mathfrak{l} = \mathfrak{b} \ltimes \mathfrak{heis}_{2n+1}$, where $\mathfrak{b} \subset \mathfrak{su}(1,n-1)$ is the (solvable) Iwasawa subalgebra, acts simply transitive and isometrically on $\bar{N}_\rho$. We may thus regard g^c_ρ as a left-invariant metric on the Lie group L, and in this subsection we shall determine the Lie algebra $\mathfrak{l}$ and the inner product on $\mathfrak{l}$ corresponding to g^c_ρ explicitly.

6.2.1 The Iwasawa decomposition of $\mathfrak{su}(1,n-1)$

Decomposing $\mathbb{C}^n = \mathbb{C}e_0 \oplus \mathbb{C}^{n-1}$ with an orthonormal basis $e_0, e_1, \ldots, e_{n-1}$ of $\mathbb{C}^n$ with respect to the pseudo-Hermitian inner product h of signature $(1,n-1)$ such that $h(e_0,e_0) = -1$, we may write

$$\mathfrak{u}(1,n-1) = \mathbb{R}C \oplus \mathfrak{su}(1,n-1),$$

where

$$C = \begin{pmatrix} i & 0 \\ 0 & i\mathbb{1}_{n-1} \end{pmatrix}, \quad \mathfrak{su}(1,n-1) = \left\{ \begin{pmatrix} -\operatorname{tr}(A) & \bar{v}^\top \\ v & A \end{pmatrix} \mid v \in \mathbb{C}^{n-1}, A \in \mathfrak{u}(n-1) \right\}.$$

We view $\mathfrak{u}(1,n-1) \subset \mathfrak{gl}(n,\mathbb{C})$ as the fixed-point set of the anti-linear involutive Lie algebra automorphism

$$\sigma : \mathfrak{gl}(n,\mathbb{C}) \longrightarrow \mathfrak{gl}(n,\mathbb{C}), \quad \sigma(A) := A^\sigma := -I\bar{A}^\top I, \quad \text{where } I = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1}_{n-1} \end{pmatrix}.$$

Note that $(AB)^\sigma = -B^\sigma A^\sigma$. Given $A \in \mathfrak{gl}(n,\mathbb{C})$ we then write

$$\operatorname{Re}(A) = \frac{1}{2}(A + A^\sigma) \quad \text{and} \quad \operatorname{Im}(A) = \frac{1}{2i}(A - A^\sigma).$$

For $a = 1, \ldots, n-1$ we write further

$$U_a = \begin{pmatrix} 0 & e_a^\top \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad U_a^\sigma := \sigma(U_a) = \begin{pmatrix} 0 & 0 \\ e_a & 0 \end{pmatrix}.$$

We observe that

$$[U_a, U_b] = 0, \quad [U_a^\sigma, U_b^\sigma] = 0, \quad [U_a, U_b^\sigma] = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & -e_b e_a^\top \end{pmatrix}.$$

Then

$$\{C, \operatorname{Re}(U_a), \operatorname{Im}(U_a), \operatorname{Re}([U_a, U_b^\sigma]), \operatorname{Im}([U_a, U_b^\sigma]) \mid a, b = 1, \ldots, n-1\}$$

is a basis for $\mathfrak{u}(1,n-1)$.

The real vector space underlying the Lie algebra $\mathfrak{heis}_{2n+1}$ is given by $\mathbb{C}^n \oplus \mathbb{R}$, where we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and write Z for the generator of the center $\mathbb{R}$.

We now fix $\rho \in (0,\infty)$, take as basepoint $p_\rho := (\rho,0) \in \bar{N}_\rho$ and consider the infinitesimal action of $\mathfrak{su}(1,n-1) \ltimes \mathfrak{heis}_{2n+1}$ on $\bar{N}_\rho$. The Lie algebra $\mathfrak{g}'$ of the stabilizer of p_ρ, i.e. the kernel of the map $\mathfrak{u}(1,n-1) \ltimes \mathfrak{heis}_{2n+1} \longrightarrow T_{p_\rho}\bar{N}_\rho$ given by evaluating the Killing fields at p_ρ, was computed in [CRT21, Lemma 3.5]:

$$\mathfrak{g}' = \operatorname{span}_{\mathbb{R}}\{C + 2cZ, \operatorname{Re}([U_a, U_b^\sigma]), \operatorname{Im}([U_a, U_b^\sigma]) + 2c\delta_{ab}Z \mid a,b = 1,\dots,n-1\},$$

which is isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{u}(n-1)$ and has trivial intersection with $\mathfrak{heis}_{2n+1}$.

We briefly review the Iwasawa decomposition of $\mathfrak{su}(1,n-1)$. We may choose the following Cartan decomposition

$$\mathfrak{u}(1,n-1) = \mathfrak{k} \oplus \mathfrak{p},$$

where

$$\begin{aligned}
\mathfrak{k} &= \mathbb{R}C \oplus \left\{ \begin{pmatrix} -\operatorname{tr}(A) & 0 \\ 0 & A \end{pmatrix} \mid A \in \mathfrak{u}(n-1) \right\} \cong \mathfrak{u}(1) \oplus \mathfrak{u}(n-1), \\
\mathfrak{p} &= \left\{ \begin{pmatrix} 0 & \bar{v}^\top \\ v & 0 \end{pmatrix} \mid v \in \mathbb{C}^{n-1} \right\}.
\end{aligned}$$

Define for $a \in \{1,\dots,n-1\}$:

$$B_a := (1+\delta_{1a})U_a - [U_a, U_1^\sigma] = \begin{pmatrix} -\delta_{1a} & (1+\delta_{1a})e_a^\top \\ 0 & e_1 e_a^\top \end{pmatrix}$$

and

$$\begin{aligned}
B_a^R &:= \operatorname{Re}(B_a) = \frac{1}{2}\begin{pmatrix} 0 & (1+\delta_{1a})e_a^\top \\ (1+\delta_{1a})e_a & e_1 e_a^\top - e_a e_1^\top \end{pmatrix}, \\
B_a^I &:= \operatorname{Im}(B_a) = \frac{1}{2i}\begin{pmatrix} -2\delta_{1a} & (1+\delta_{1a})e_a^\top \\ -(1+\delta_{1a})e_a & e_1 e_a^\top + e_a e_1^\top \end{pmatrix}.
\end{aligned}$$

Then

$$B_1^R = \begin{pmatrix} 0 & e_1^\top \\ e_1 & 0 \end{pmatrix}, \quad B_1^I = \begin{pmatrix} i & -i & 0 \\ i & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and for $a > 1$ we have

$$B_a^R = \frac{1}{2}\begin{pmatrix} 0 & 0 & \tilde{e}_{a-1}^\top \\ 0 & 0 & \tilde{e}_{a-1}^\top \\ \tilde{e}_{a-1} & -\tilde{e}_{a-1} & 0 \end{pmatrix}, \quad B_a^I = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i\tilde{e}_{a-1}^\top \\ 0 & 0 & -i\tilde{e}_{a-1}^\top \\ i\tilde{e}_{a-1} & -i\tilde{e}_{a-1} & 0 \end{pmatrix},$$

where $\tilde{e}_{a-1} \in \mathbb{C}^{n-2}$ is such that $e_a^\top = \begin{pmatrix} 0 & \tilde{e}_{a-1}^\top \end{pmatrix} \in \mathbb{C}^{n-1}$.

A maximal abelian subalgebra of $\mathfrak{p}$ is given by

$$\mathfrak{a} := \left\{ \begin{pmatrix} 0 & ae_1^\top \\ ae_1 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\} = \mathrm{span}_{\mathbb{R}}\{B_1^R\}.$$

The positive eigenvalues of $\mathrm{ad}(B_1^R) = [B_1^R, \cdot] \in \mathrm{End}(\mathfrak{u}(1,n-1))$ are 2 and 1 with eigenspaces given by

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} ia & -iae_1^\top \\ iae_1 & -iae_1e_1^\top \end{pmatrix} \mid a \in \mathbb{R} \right\} = \mathrm{span}_{\mathbb{R}}\{B_1^I\},$$

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & 0 & \bar{z}^\top \\ 0 & 0 & \bar{z}^\top \\ z & -z & 0 \end{pmatrix} \mid z \in \mathbb{C}^{n-2} \right\} = \mathrm{span}_{\mathbb{R}}\{B_a^R, B_a^I \mid a = 2, \dots, n-1\}.$$

With

$$\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \left\{ \begin{pmatrix} ia & -ia & \bar{z}^\top \\ ia & -ia & \bar{z}^\top \\ z & -z & 0 \end{pmatrix} \mid z \in \mathbb{C}^{n-2}, a \in \mathbb{R} \right\}$$

we get $\mathfrak{u}(1,n-1) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and the Iwasawa decomposition

$$\mathfrak{su}(1,n-1) = \mathfrak{u}(n-1) \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

We observe the following bracket relations for $i = 1,2$:

$$[\mathfrak{a},\mathfrak{a}] = 0, \quad [\mathfrak{a},\mathfrak{g}_i] \subset \mathfrak{g}_i, \quad [\mathfrak{g}_1,\mathfrak{g}_1] \subset \mathfrak{g}_2, \quad [\mathfrak{g}_1,\mathfrak{g}_2] = 0 = [\mathfrak{g}_2,\mathfrak{g}_2],$$

which imply

$$[\mathfrak{b},\mathfrak{b}] \subset \mathfrak{n}, \quad [\mathfrak{n},\mathfrak{n}] \subset \mathfrak{g}_2, \quad [\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] = 0,$$

where $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{n}$ is the solvable part of the Iwasawa decomposition.

Lemma 6.2.1. *The Lie algebra $\mathfrak{n}$ is isomorphic to $\mathfrak{heis}_{2n-3}$ and the basis vectors B_1^I, B_a^R, B_a^I, $a = 2, \dots, n-1$, satisfy the following non-trivial bracket relations (all other brackets are zero):*

$$[B_a^R, B_a^I] = \frac{1}{2} B_1^I.$$

Proof. Since $[\mathfrak{g}_1,\mathfrak{g}_2] = 0 = [\mathfrak{g}_2,\mathfrak{g}_2]$ we see that $\mathfrak{g}_2$ is contained in the center of $\mathfrak{n}$. Let $z_1, z_2 \in \mathbb{C}^{n-2}$. We can compute directly

$$\left[\begin{pmatrix} 0 & 0 & \bar{z}_1^\top \\ 0 & 0 & \bar{z}_1^\top \\ z_1 & -z_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \bar{z}_2^\top \\ 0 & 0 & \bar{z}_2^\top \\ z_2 & -z_2 & 0 \end{pmatrix} \right] = \begin{pmatrix} 2i\,\mathrm{Im}(\bar{z}_1^\top z_2) & -2i\,\mathrm{Im}(\bar{z}_1^\top z_2) & 0 \\ 2i\,\mathrm{Im}(\bar{z}_1^\top z_2) & -2i\,\mathrm{Im}(\bar{z}_1^\top z_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 2\,\mathrm{Im}(\bar{z}_1^\top z_2) B_1^I.$$

It follows then that the eigenspace $\mathfrak{g}_2 = \mathrm{span}_{\mathbb{R}}\{B_1^I\}$ is the center of $\mathfrak{n}$ and we note that $(z_1, z_2) \longmapsto 2\,\mathrm{Im}(\bar{z}_1^\top z_2)$ is a non-zero multiple of the standard symplectic (Kähler) form on $\mathbb{C}^{n-2}$. Thus $\mathfrak{n} \cong \mathfrak{heis}_{2n-3}$.

Since B_a^R corresponds to choosing $z = \frac{1}{2}\tilde{e}_{a-1}$ and B_a^I corresponds to choosing $z = \frac{i}{2}\tilde{e}_{a-1}$, if $a > 1$ then we find

$$[B_a^R, B_b^R] = [B_a^I, B_b^I] = \frac{1}{2}\mathrm{Im}(\tilde{e}_{a-1}^\top \tilde{e}_{b-1})B_1^I = 0,$$
$$[B_a^R, B_b^I] = \frac{1}{2}\mathrm{Im}(i\tilde{e}_{a-1}^\top \tilde{e}_{b-1})B_1^I = \frac{1}{2}\delta_{ab}B_1^I.$$

□

Consider the $(2n-2)$-dimensional real solvable Lie algebra

$$\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{n} = \left\{ \begin{pmatrix} ia & b - ia & \bar{z}^\top \\ b + ia & -ia & \bar{z}^\top \\ z & -z & 0 \end{pmatrix} \mid z \in \mathbb{C}^{n-2}, a, b \in \mathbb{R} \right\} \subset \mathfrak{su}(1, n-1).$$

A basis for $\mathfrak{b}$ is given by

$$\{B_a^R, B_a^I \mid a = 1, \ldots, n-1\}.$$

We have by construction

$$\mathfrak{a} = \mathrm{span}_{\mathbb{R}}\{B_1^R\}, \quad \mathfrak{g}_2 = \mathrm{span}_{\mathbb{R}}\{B_1^I\}, \quad \mathfrak{g}_1 = \mathrm{span}_{\mathbb{R}}\{B_a^R, B_a^I \mid a = 2, \ldots, n-1\}.$$

The subalgebra $\mathfrak{n} \cong \mathfrak{heis}_{2n-3}$ is an ideal in $\mathfrak{b}$ and we have computed the brackets of the basis vectors of $\mathfrak{n}$ in Lemma 6.2.1. The next lemma is then clear for the definition of $\mathfrak{g}_1, \mathfrak{g}_2$.

Lemma 6.2.2. *We have*

$$[B_1^R, B_1^I] = 2B_1^I, \quad [B_1^R, B_a^R] = B_a^R, \quad [B_1^R, B_a^I] = B_a^I$$

for any $a \in \{2, \ldots, n-1\}$.

Note that $\mathfrak{b} \cap \mathfrak{g}' = \{0\}$. It follows that the action of the $(4n-1)$-dimensional real solvable Lie algebra

$$\mathfrak{l} := \mathfrak{b} \ltimes \mathfrak{heis}_{2n+1}$$

on $\bar{N}_\rho$ is free. Here $\mathfrak{b}$ acts on $\mathfrak{heis}_{2n+1} \cong \mathbb{C}^n \oplus \mathbb{R}$ by the standard representation of $\mathfrak{u}(1, n-1)$ on $\mathbb{C}^n$.

6.2.2 The Lie algebra $\mathfrak{l}$

Now let e_0, f_0, e_a, f_a, $a = 1, \dots, n-1$, be the standard basis of $\mathbb{R}^{2n}$. We define the 1-dimensional central extension $\mathfrak{heis}_{2n+1}$ of $\mathbb{R}^{2n}$ by setting

$$[e_k, e_\ell] = 0, \quad [f_k, f_\ell] = 0, \quad [e_k, f_\ell] = \left(\delta_{k0}\delta_{\ell 0} - \sum_{a=1}^{n-1} \delta_{ka}\delta_{\ell a}\right) Z$$

for every $k, \ell = 0, 1, \dots, n-1$, where Z denotes the generator of the center. Complexifying and extending the Lie bracket complex-bilinearly, we obtain $\mathfrak{heis}^{\mathbb{C}}_{2n+1}$.

Set $E_k := e_k - i f_k$.

Lemma 6.2.3. *The non-trivial bracket relations between the elements of the basis* $\{B^R_a, B^I_a \mid a = 1, \dots, n-1\} \subset \mathfrak{b}$ *and of the complex basis* $\{E_k, \bar{E}_k, Z \mid k = 0, \dots, n-1\}$ *are as follows:*

$$\begin{aligned}
[B^R_1, E_k] &= \overline{[B^R_1, \bar{E}_k]} = -\delta_{k0}E_1 - \delta_{k1}E_0, \\
[B^R_a, E_k] &= \overline{[B^R_a, \bar{E}_k]} = -\frac{1}{2}(\delta_{k0} + \delta_{k1})E_a - \frac{1}{2}\delta_{ka}(E_0 - E_1), \\
[B^I_1, E_k] &= \overline{[B^I_1, \bar{E}_k]} = -i(\delta_{k0} + \delta_{k1})(E_0 - E_1), \\
[B^I_a, E_k] &= \overline{[B^I_a, \bar{E}_k]} = \frac{i}{2}(\delta_{k0} + \delta_{k1})E_a - \frac{i}{2}\delta_{ka}(E_0 - E_1).
\end{aligned}$$

Proof. By definition of the semidirect product structure, the bracket of $\mathfrak{gl}(n,\mathbb{C}) \ltimes \mathfrak{heis}^{\mathbb{C}}_{2n+1}$ evaluated on $A \in \mathfrak{u}(1, n-1)$ and $v \in \mathbb{R}^{2n}$ is just $[A, v] = -A^\top v$ (where $A^\top$ is identified with a real $2n \times 2n$-matrix), while we have $[A, Z] = 0$. Using this prescription, the following brackets were computed in [CRT21, Proposition 3.4]:

$$\begin{aligned}
[U_a, E_k] &= -\delta_{k0}E_a, & [U^\sigma_a, E_k] &= -\delta_{ka}E_0, \\
[U_a, \bar{E}_k] &= -\delta_{ka}\bar{E}_0, & [U^\sigma_a, \bar{E}_k] &= -\delta_{k0}\bar{E}_a.
\end{aligned}$$

From these identities, we may deduce the brackets between elements of $\mathfrak{b}$ and $\mathfrak{heis}_{2n+1}$ using the Jacobi identity:

$$\begin{aligned}
[B_1, E_k] &= -(2\delta_{k0} + \delta_{k1})E_1 + \delta_{k0}E_0, & [B^\sigma_1, E_k] &= -(2\delta_{k1} + \delta_{k0})E_0 + \delta_{k1}E_1, \\
[B_1, \bar{E}_k] &= -(2\delta_{k1} + \delta_{k0})\bar{E}_0 + \delta_{k1}\bar{E}_1, & [B^\sigma_1, \bar{E}_k] &= -(2\delta_{k0} + \delta_{k1})\bar{E}_1 + \delta_{k0}\bar{E}_0, \\
[B_a, E_k] &= -(\delta_{k0} + \delta_{k1})E_a, & [B^\sigma_a, E_k] &= -\delta_{ka}(E_0 - E_1), \\
[B_a, \bar{E}_k] &= -\delta_{ka}(\bar{E}_0 - \bar{E}_1), & [B^\sigma_a, \bar{E}_k] &= -(\delta_{k0} + \delta_{k1})\bar{E}_a.
\end{aligned}$$

Using these relations and that $A^R = \mathrm{Re}(A) = \frac{1}{2}(A + A^\sigma)$ and $A^I = \mathrm{Im}(A) = \frac{1}{2i}(A - A^\sigma)$ for any $A \in \mathfrak{gl}(n,\mathbb{C})$ we get the claimed result. $\square$

For $n \in \mathbb{N}$, we work in the ordered basis

$$\mathcal{B}_n := \begin{cases} (e_0, f_0, Z) & n = 1 \\ (B^R_1, B^I_1, \dots, B^R_{n-1}, B^I_{n-1}, e_0, f_0, e_1, f_1, \dots, e_{n-1}, f_{n-1}, Z) & n > 1 \end{cases} \tag{38}$$

Proposition 6.2.4. *For $n > 1$, the Lie algebra $\mathfrak{l}$ is completely solvable and non-unimodular.*

Proof. The adjoint operator of B_1^R with respect to the basis $\mathcal{B}_n$ defined in (38) is given by

$$\operatorname{ad}(B_1^R) = \operatorname{diag}\left(0, 2, \mathbb{1}_{2n-4}, \mathbf{V}_4, \mathbb{O}_{2n-4}, 0\right),$$

where

$$\mathbf{V}_4 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Then $\operatorname{tr}(\operatorname{ad}(B_1^R)) = 2n-2 \neq 0$, which implies that $\mathfrak{l}$ is non-unimodular. Moreover, the eigenvalues of $\operatorname{ad}(B_1^R)$ are $2, 1, 0, -1$ and for any $X \in \mathcal{B}_n \setminus \{B_1^R\}$ the operator $\operatorname{ad}(X)$ is nilpotent, so its only eigenvalue is zero. Hence $\mathfrak{l}$ is completely solvable. □

Remark 6.2.5. Note that the case $n = 1$ corresponds to $\mathfrak{l} = \mathfrak{heis}_3$, so $\mathfrak{l}$ is nilpotent, hence unimodular and completely solvable.

6.2.3 The metric g_ρ^c as a left-invariant metric on L

Our goal here is to write the induced metric g_ρ^c as a left-invariant metric on the Lie group L, and for this it is enough to compute the induced inner product on its Lie algebra $\mathfrak{l}$, under the identification $T_eL \cong T_{p_\rho}\bar{N}_\rho$ given by the infinitesimal action of $\mathfrak{u}(1, n-1) \ltimes \mathfrak{heis}_{2n+1}$ on $\bar{N}_\rho$. The complexification of the infinitesimal action is given by the anti-homomorphism $\alpha^{\mathbb{C}} : \mathfrak{gl}(n, \mathbb{C}) \ltimes \mathfrak{heis}_{2n+1}^{\mathbb{C}} \longrightarrow \Gamma(T\bar{N})^{\mathbb{C}}$, where (see [CRT21, Proposition 3.1]):

$$\begin{aligned}
Y_a &= \alpha^{\mathbb{C}}(U_a) = \frac{\partial}{\partial \bar{X}^a} - X^a \sum_{b=1}^{n-1} X^b \frac{\partial}{\partial X^b} - w^0 \frac{\partial}{\partial w^a} - \bar{w}^a \frac{\partial}{\partial \bar{w}^0} + icX^a \frac{\partial}{\partial \tilde{\phi}}, \\
\bar{Y}_a &= \alpha^{\mathbb{C}}(U_a^\sigma), \\
[Y_a, \bar{Y}_b] &= -\alpha^{\mathbb{C}}([U_a, U_b^\sigma]) \\
&= \delta_{ab}\left(\sum_j \left(X^j \frac{\partial}{\partial X^j} - \bar{X}^j \frac{\partial}{\partial \bar{X}^j}\right) + w^0 \frac{\partial}{\partial w^0} - \bar{w}^0 \frac{\partial}{\partial \bar{w}^0} - 2ic \frac{\partial}{\partial \tilde{\phi}}\right) \\
&\quad + X^a \frac{\partial}{\partial X^b} - \bar{X}^b \frac{\partial}{\partial \bar{X}^a} + \bar{w}^a \frac{\partial}{\partial \bar{w}^b} - w^b \frac{\partial}{\partial w^a}.
\end{aligned}$$

The action of the complexified Heisenberg Lie algebra $\mathfrak{heis}_{2n+1}^{\mathbb{C}}$ is then generated by the vector fields (see [CRT21, Proposition 3.3])

$$\begin{aligned} V_0 = \alpha^{\mathbb{C}}(E_0) &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial w^0} + 2i\bar{w}^0\frac{\partial}{\partial\tilde{\phi}}\right), \\ V_a = \alpha^{\mathbb{C}}(E_a) &= \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial w^a} - 2i\bar{w}^a\frac{\partial}{\partial\tilde{\phi}}\right), \\ \frac{\partial}{\partial\tilde{\phi}} &= \alpha^{\mathbb{C}}(Z) \end{aligned}$$

for $a = 1, \dots, n-1$.

Define the map

$$\alpha^{\mathbb{C}}_\rho : \mathfrak{l}^{\mathbb{C}} \longrightarrow T^{\mathbb{C}}_{p_\rho}\bar{N}_\rho, \quad \ell \longmapsto \alpha^{\mathbb{C}}(\ell)|_{p_\rho},$$

i.e. evaluation of the corresponding complex Killing field at p_ρ. Then the formulas above allow us to explicitly evaluate $\alpha^{\mathbb{C}}_\rho$ on the basis vectors of $\mathfrak{l}$:

$$\begin{aligned} \alpha^{\mathbb{C}}_\rho(B_a) &= \alpha^{\mathbb{C}}_\rho((1+\delta_{1a})U_a - [U_a, U_1^\sigma]) = (1+\delta_{1a})Y_a|_{p_\rho} + [Y_a, \bar{Y}_1]|_{p_\rho} \\ &= (1+\delta_{1a})\frac{\partial}{\partial\bar{X}^a} - 2ic\delta_{1a}\frac{\partial}{\partial\tilde{\phi}}, \\ \alpha^{\mathbb{C}}_\rho(E_k) &= \alpha^{\mathbb{C}}_\rho(e_k - if_k) = V_k|_{p_\rho} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial w^k}, \\ \alpha^{\mathbb{C}}_\rho(Z) &= \frac{\partial}{\partial\tilde{\phi}}. \end{aligned}$$

Let us consider again the real coordinates $X^a := \frac{1}{2}(b^a + it^a)$, $w^0 := \frac{1}{2}(\tilde{\zeta}_0 + i\zeta^0)$ and $w^a := \frac{1}{2}(\tilde{\zeta}_a - i\zeta^a)$. Then we find for $a = 2, \dots, n-1$ and $j = 1, \dots, n-1$:

- $\alpha_\rho(B_1^R) = \mathrm{Re}\left((2Y_1 + [Y_1, \bar{Y}_1])|_{p_\rho}\right) = 2\frac{\partial}{\partial b^1}$,
- $\alpha_\rho(B_1^I) = \mathrm{Im}\left((2Y_1 + [Y_1, \bar{Y}_1])|_{p_\rho}\right) = 2\frac{\partial}{\partial t^1} - 2c\frac{\partial}{\partial\tilde{\phi}}$,
- $\alpha_\rho(B_a^R) = \mathrm{Re}\left((Y_a + [Y_a, \bar{Y}_1])|_{p_\rho}\right) = \frac{\partial}{\partial b^a}$,
- $\alpha_\rho(B_a^I) = \mathrm{Im}\left((Y_a + [Y_a, \bar{Y}_1])|_{p_\rho}\right) = \frac{\partial}{\partial t^a}$,
- $\alpha_\rho(e_0) = \mathrm{Re}\left(V_0|_{p_\rho}\right) = \frac{1}{\sqrt{2}}\frac{\partial}{\partial\tilde{\zeta}_0}$,
- $\alpha_\rho(f_0) = -\mathrm{Im}\left(V_0|_{p_\rho}\right) = \frac{1}{\sqrt{2}}\frac{\partial}{\partial\zeta^0}$,
- $\alpha_\rho(e_j) = \mathrm{Re}\left(V_j|_{p_\rho}\right) = \frac{1}{\sqrt{2}}\frac{\partial}{\partial\tilde{\zeta}_j}$,
- $\alpha_\rho(f_j) = -\mathrm{Im}\left(V_j|_{p_\rho}\right) = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial\zeta^j}$,
- $\alpha_\rho(Z) = \frac{\partial}{\partial\tilde{\phi}}$,

where the vector fields on the right-hand side are evaluated at the point p_ρ.

At the point p_ρ, the metric expressed in these real coordinates is the following:

$$g^c_\rho = \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}\mathrm{d}\tilde{\phi}^2 + \frac{\rho+c}{4\rho}\sum_{a=1}^{n-1}\left((\mathrm{d}b^a)^2+(\mathrm{d}t^a)^2\right)$$
$$+\frac{\rho+2c}{2\rho^2}\left((\mathrm{d}\tilde{\zeta}_0)^2+(\mathrm{d}\zeta^0)^2\right)+\frac{1}{2\rho}\sum_{a=1}^{n-1}\left((\mathrm{d}\tilde{\zeta}_a)^2+(\mathrm{d}\zeta^a)^2\right).$$

Let us denote by $E_{i,j}$ the matrix with 1 in the (i,j)-position and zero elsewhere.

Proposition 6.2.6. *Let $n\in\mathbb{N}$ and consider the basis $\mathcal{B}_n$ of $\mathfrak{l}$ defined in (38). Then:*

(a) *If $n>1$, the Gram matrix of the inner product corresponding to g^c_ρ is given by*

$$g^c_\rho = \mathrm{diag}\left(\frac{\rho+c}{\rho}, \frac{(\rho+c)^3}{\rho^2(\rho+2c)}, \frac{\rho+c}{4\rho}\mathbb{1}_{2n-4}, \mathbf{G}_4, \frac{1}{4\rho}\mathbb{1}_{2n-4}, \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}\right)$$
$$-\frac{c}{2\rho^2}\frac{\rho+c}{\rho+2c}\left(E_{2,4n-1}+E_{4n-1,2}\right),$$

where

$$\mathbf{G}_4 := \begin{pmatrix} \frac{\rho+2c}{4\rho^2}\mathbb{1}_2 & 0 \\ 0 & \frac{1}{4\rho}\mathbb{1}_2 \end{pmatrix}.$$

(b) *If $n>1$, the Ricci endomorphism ric^c_ρ is in the above basis represented by the matrix:*

$$\mathrm{ric}^c_\rho = \mathrm{diag}\left(r_1\mathbb{1}_{2n-2}, r_3\mathbb{1}_2, r_4\mathbb{1}_{2n-2}, r_2\right) + 2c(r_1-r_2)E_{4n-1,2}.$$

(c) *If $n=1$, the Gram matrix of g^c_ρ and the matrix of the Ricci endomorphism are*

$$g^c_\rho = \begin{pmatrix} \frac{\rho+2c}{4\rho^2} & 0 & 0 \\ 0 & \frac{\rho+2c}{4\rho^2} & 0 \\ 0 & 0 & \frac{\rho+c}{4\rho^2(\rho+2c)} \end{pmatrix} \quad \textit{and} \quad \mathrm{ric}^c_\rho = \frac{2\rho^2(\rho+c)}{(\rho+2c)^3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. The expressions for g^c_ρ follow directly by plugging the explicit tangent vectors into g^c_ρ.

We have computed the Ricci endomorphism with respect to the basis of $T_{p_\rho}\bar{N}_\rho$ given by coordinate vector fields in Proposition 6.1.8. It is then straightforward to evaluate ric^c_ρ on the basis $\mathcal{B}_n$. □

In general, the Ricci endomorphism of $(\mathfrak{l}, g_\rho^c)$ is given by (see e.g. [Lau11, Equation 21])

$$\mathrm{ric}_\rho^c = R - \tfrac{1}{2}B - \mathrm{ad}(H)^s.$$

Here

$$\mathrm{ad}(H)^s = \tfrac{1}{2}(\mathrm{ad}(H) + \mathrm{ad}(H)^*)$$

is the symmetric part of $\mathrm{ad}(H)$ and $H \in \mathfrak{a}$ is the **mean curvature vector**, characterized by

$$g_\rho^c(H,A) = \mathrm{tr}(\mathrm{ad}(A))$$

for all $A \in \mathfrak{a}$. The term $B \in \mathrm{End}(\mathfrak{l})$ denotes the symmetric endomorphism defined by the Killing form of $\mathfrak{l}$ relative to g_ρ^c, that is

$$g_\rho^c(BX,X) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(X))$$

for all $X \in \mathfrak{l}$. The symmetric endomorphism R is defined by

$$g_\rho^c(RX,X) = -\tfrac{1}{2}\sum g_\rho^c([X,L_i],L_j)^2 + \tfrac{1}{4}\sum g_\rho^c([L_i,L_j],X)^2,$$

where $\{L_i\}$ is an orthonormal basis of $(\mathfrak{l}, g_\rho^c)$. In our situation we already know ric_ρ^c, so we can compute $R = \mathrm{ric}_\rho^c + \frac{1}{2}B + \mathrm{ad}(H)^s$ if we are able to determine B and H.

Lemma 6.2.7. *With respect to our explicit choice of basis $\mathcal{B}_n$ for $\mathfrak{l}$ we have:*

(a) *For $n > 1$, the mean curvature vector is*

$$H = \frac{\mathrm{tr}(\mathrm{ad}(B_1^R))}{g_\rho^c(B_1^R, B_1^R)} B_1^R = (2n-2)\frac{\rho}{\rho+c} B_1^R$$

and we find

$$\begin{aligned}\mathrm{ad}(H)^s = (2n-2)\,\mathrm{diag}\Big(0, \tfrac{2\rho^2+4c\rho+c^2}{(\rho+c)(\rho+2c)}, \tfrac{\rho}{\rho+c}\mathbb{1}_{2n-4}, \mathbf{S}_4, \mathbb{O}_{2n-4}, -\tfrac{c^2}{(\rho+c)(\rho+2c)}\Big)\\ + (2n-2)\Big(-\tfrac{c}{2(\rho+c)(\rho+2c)}E_{2,4n-1} + \tfrac{2c(\rho+c)}{\rho+2c}E_{4n-1,2}\Big),\end{aligned}$$

where

$$\mathbf{S}_4 := \begin{pmatrix} 0 & 0 & -\frac{\rho}{\rho+2c} & 0 \\ 0 & 0 & 0 & -\frac{\rho}{\rho+2c} \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

(b) *If $n > 1$, the symmetric endomorphism associated with the Killing form is*

$$B = (2n+4)\frac{\rho}{\rho+c}E_{1,1}.$$

(c) *If $n = 1$, then $B = 0 = H$.*

Proof. These assertions are proved by direct computations. The details are as follows.

(a) Since in our case $\mathfrak{a} = \mathrm{span}_{\mathbb{R}}\{B_1^R\}$, we have that H is a multiple of B_1^R. In particular

$$H = \frac{\operatorname{tr}(\operatorname{ad}(B_1^R))}{g_\rho^c(B_1^R, B_1^R)} B_1^R = (2n-2)\frac{\rho}{\rho+c} B_1^R.$$

We have computed $\operatorname{ad}(B_1^R)$ and g_ρ^c, in the basis $\mathcal{B}_n$, in Proposition 6.2.4 and in Proposition 6.2.6, respectively. Hence we can compute the adjoint operator of $\operatorname{ad}(B_1^R)$:

$$\begin{aligned}\operatorname{ad}(B_1^R)^* = \operatorname{diag}&\left(0, \frac{2(\rho+c)^2}{\rho(\rho+2c)}, \mathbb{1}_{2n-4}, \mathbf{V}_4^*, \mathbb{O}_{2n-4}, -\frac{2c^2}{\rho(\rho+2c)}\right)\\ &-\frac{c}{\rho(\rho+2c)}E_{2,4n-1} + \frac{4c(\rho+c)^2}{\rho(\rho+2c)}E_{4n-1,2},\end{aligned}$$

where

$$\mathbf{V}_4^* := \begin{pmatrix} 0 & 0 & -\frac{\rho}{\rho+2c} & 0 \\ 0 & 0 & 0 & -\frac{\rho}{\rho+2c} \\ -\frac{\rho+2c}{\rho} & 0 & 0 & 0 \\ 0 & -\frac{\rho+2c}{\rho} & 0 & 0 \end{pmatrix}.$$

We then have

$$\begin{aligned}\operatorname{ad}(B_1^R)^s = \operatorname{diag}&\left(0, \frac{2\rho^2+4c\rho+c^2}{\rho(\rho+2c)}, \mathbb{1}_{2n-4}, \tilde{\mathbf{S}}_4, \mathbb{O}_{2n-4}, -\frac{c^2}{\rho(\rho+2c)}\right)\\ &-\frac{c}{2\rho(\rho+2c)}E_{2,4n-1} + \frac{2c(\rho+c)^2}{\rho(\rho+2c)}E_{4n-1,2},\end{aligned}$$

where

$$\tilde{\mathbf{S}}_4 := \begin{pmatrix} 0 & 0 & -\frac{\rho+c}{\rho+2c} & 0 \\ 0 & 0 & 0 & -\frac{\rho+c}{\rho+2c} \\ -\frac{\rho+c}{\rho} & 0 & 0 & 0 \\ 0 & -\frac{\rho+c}{\rho} & 0 & 0 \end{pmatrix}.$$

Then

$$\operatorname{ad}(H)^s = (2n-2)\frac{\rho}{\rho+c}\operatorname{ad}(B_1^R)^s.$$

(b) Let us consider the Killing form $\beta(X,Y) = \operatorname{tr}(\operatorname{ad}(X)\circ\operatorname{ad}(Y))$. Then $\beta(B_1^R, B_1^R) = 2n+4$ and $\beta(X,Y) = 0$ for all $Y \in \mathcal{B}_n$, $X \in \mathcal{B}_n \setminus \{B_1^R\}$. This implies that

$$B = (g_\rho^c)^{-1}\beta = (2n+4)\frac{\rho}{\rho+c}E_{1,1}.$$

(c) This is clear since $\mathfrak{a} = 0$ and $\mathfrak{l} = \mathfrak{heis}_3$ is nilpotent. □

6.2.4 Existence of solvsolitons on $\bar{N}_\rho$

Ricci solitons were introduced by Hamilton in [Ham88] and they are a generalization of Einstein manifolds. More precisely, a Riemannian manifold (M,g) is a **Ricci soliton** if there exists a vector field X on M such that

$$\mathrm{Ric} = \lambda g + \mathscr{L}_X g,$$

where Ric denotes the Ricci curvature of g and $\lambda \in \mathbb{R}$. Note that if X is Killing then (M,g) is Einstein.

We are interested in study the existence of Ricci solitons in the homogeneous setting, in particular in Lie groups equipped with left-invariant metrics.

Definition 6.2.8. Let (G,g) be a simply connected Lie group equipped with a left-invariant Riemannian metric g, and let $\mathfrak{g}$ denote the Lie algebra of G. Then the pair (G,g) is called an **algebraic Ricci soliton** if it satisfies

$$\mathrm{ric} = \lambda\,\mathrm{Id} + D, \tag{39}$$

where ric denotes the Ricci endomorphism of g, $\lambda \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$. In particular, an algebraic Ricci soliton on a solvable (resp. nilpotent) Lie group is called **solvsoliton** (resp. **nilsoliton**).

The relationship between left-invariant Ricci solitons on simply connected Lie groups and algebraic Ricci solitons was studied by Lauret in [Lau01, Lau11]. He shows that any algebraic Ricci soliton gives rise to a Ricci soliton.

Proposition 6.2.9. *Let (G,g) be a simply connected Lie group equipped with a left-invariant Riemannian metric g. If (G,g) is an algebraic Ricci soliton, then it is a Ricci soliton.*

Proof. Suppose that $\mathrm{ric} = \lambda\,\mathrm{Id} + D$ for $\lambda \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$. Let X_D be the vector field on G defined by $X_D(p) := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\varphi_t(p)$, where $p \in G$ and $\varphi_t \in \mathrm{Aut}(G)$ is the unique automorphism such that $\mathrm{d}\varphi_t|_e = e^{\frac{t}{2}D} \in \mathrm{Aut}(\mathfrak{g})$. Then

$$\mathscr{L}_{X_D} g = \tfrac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\varphi_t^* g = \tfrac{\mathrm{d}}{\mathrm{d}t}|_{t=0} g(e^{\frac{t}{2}D}\cdot, e^{\frac{t}{2}D}\cdot) = g(D\cdot,\cdot),$$

which implies that $\mathrm{Ric} = \lambda g + \mathscr{L}_{X_D} g$. □

He proves that the converse also holds in the case of completely solvable Lie groups. Recall that G is called **completely solvable** if G is solvable and the eigenvalues of $\mathrm{ad}(X)$ are real for all $X \in \mathfrak{g}$. Note that nilpotent Lie groups are completely solvable.

Proposition 6.2.10. *Let (G,g) be a simply connected completely solvable Lie group equipped with a left-invariant Riemannian metric g. If (G,g) is a Ricci soliton, then it is a solvsoliton.*

Remark 6.2.11. It is shown in [Lau11, Proposition 4.6] that if (G,g) is a solvsoliton with $\lambda \geq 0$, then $\mathrm{ric} = 0$. Thus the corresponding left-invariant Ricci soliton is Ricci-flat and hence flat by Theorem 3.1.6.

Since we have determined the structure of the Lie algebra $\mathfrak{l}$ and we have obtained explicit formulas for the metric g^c_ρ and its Ricci endomorphism ric^c_ρ in the previous subsections, we can determine whether $(\mathfrak{l}, g^c_\rho)$ is a solvsoliton or not.

Lemma 6.2.12. *Consider the Lie algebra $\mathfrak{l} = \mathfrak{b} \ltimes \mathfrak{heis}_{2n+1}$ and write $\mathfrak{heis}_{2n+1} = \mathbb{C}^n \oplus \mathbb{R}Z$. Consider $\delta \in \mathrm{End}(\mathfrak{l})$ given by $\delta|_{\mathfrak{b}} = 0$, $\delta Z = 2Z$ and $\delta V = V$ for all $V \in \mathbb{C}^n$. Then δ is a derivation of $\mathfrak{l}$.*

Proof. First note that the endomorphism δ acts as zero on $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_1$. For the Heisenberg Lie algebra $\mathfrak{heis}_{2n+1}$ we have $[\mathfrak{heis}_{2n+1}, \mathfrak{heis}_{2n+1}] \subset \mathbb{R}Z$ and $\delta Z = 2Z$. This implies that for all $V + tZ, W + uZ \in \mathfrak{heis}_{2n+1} = \mathbb{C}^n \oplus \mathbb{R}Z$ we have

$$\delta[V + tZ, W + uZ] = 2[V + tZ, W + uZ] = 2[V, W].$$

On the other hand

$$\begin{aligned}[\delta(V + tZ), W + uZ] + [V + tZ, \delta(W + uZ)] &= [V + 2tZ, W + uZ] + [V + tZ, W + 2uZ] \\ &= [V, W] + [V, W] = 2[V, W].\end{aligned}$$

Using that $\delta B = 0$ and $[B, V + tZ] \in \mathbb{C}^n \oplus \{0\} \subset \mathfrak{heis}_{2n+1}$ for all $B \in \mathfrak{b}$ and all $V + tZ \in \mathfrak{heis}_{2n+1}$, and a straightforward computation, we conclude that $\delta \in \mathrm{Der}(\mathfrak{l})$. $\square$

As we have pointed out in Remark 6.2.5, the case $n = 1$ is special, so we consider it separately.

Theorem 6.2.13. *Let $n = 1$. Then the pair $(\mathfrak{l}, g^c_\rho)$ is a nilsoliton for any $c \geq 0$ and $\rho > 0$.*

Proof. In the case $n = 1$ we have that $\mathfrak{b}$ is 0-dimensional and then $\mathfrak{l} = \mathfrak{heis}_3$. By Proposition 6.2.6 (c) the Ricci endomorphism with respect to the basis $\mathcal{B}_1$ is

$$\mathrm{ric}^c_\rho = K \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $K := \frac{2\rho^2(\rho+c)}{(\rho+2c)^3}$. If we take $\lambda = -3K$ and

$$D = 2K\delta = 2K \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \mathrm{Der}(\mathfrak{heis}_3),$$

then (39) holds. $\square$

Remark 6.2.14. The result of Theorem 6.2.13 is not new since the existence of a nil-soliton metric on $\mathfrak{heis}_3$ is well-known in the literature (see e.g. [Mil76, Corollary 4.6]). In fact, it follows from this result of Milnor that any left-invariant metric on $\mathfrak{heis}_3$ is a Ricci soliton.

Theorem 6.2.15. *Let $n > 1$. Then the pair $(\mathfrak{l}, g^c_\rho)$ is a solvsoliton for $c = 0$ and $\rho > 0$.*

Proof. In this case the metric $g_\rho = g^0_\rho$ is diagonal with respect to the basis $\mathcal{B}_n$ and its Ricci endomorphism is computed in Proposition 6.2.6:

$$\begin{aligned}\mathrm{ric}_\rho &= \mathrm{diag}\left(r_1 \mathbb{1}_{2n-2}, r_3 \mathbb{1}_2, r_4 \mathbb{1}_{2n-2}, r_2\right)\\ &= \mathrm{diag}\left(-2(n+2)\mathbb{1}_{2n-2}, -2\mathbb{1}_2, -2\mathbb{1}_{2n-2}, 2n\right).\end{aligned}$$

If we consider the derivation δ from Lemma 6.2.12, choose $\lambda = -2(n+2)$ and put

$$D = (2n+2)\delta = \mathrm{diag}\left(\mathbb{O}_{2n-2}, (2n+2)\mathbb{1}_2, (2n+2)\mathbb{1}_{2n-2}, 2(2n+2)\right),$$

then (39) holds. □

Remark 6.2.16. Let S be a codimension one connected Lie subgroup of the solvable Iwasawa group AN of an irreducible symmetric space of non-compact type. It was shown in [DST21, Theorem A] that if S contains the nilpotent part N, then S is a Ricci soliton with respect to the metric induced by the left-invariant Einstein metric on AN. For the case $c = 0$ we have $\mathrm{SU}(n,2)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(2))$ equipped with its symmetric metric. Thus Theorem 6.2.15 shows that our explicit computations agree with this general result.

Theorem 6.2.17. *Let $n > 1$. Then the pair $(\mathfrak{l}, g^c_\rho)$ is not a solvsoliton for $c > 0$ and $\rho > 0$.*

Proof. We have the orthogonal decomposition $\mathfrak{l} = \mathfrak{a} \oplus [\mathfrak{l}, \mathfrak{l}]$, where $[\mathfrak{l}, \mathfrak{l}]$ coincides with the nilradical of $\mathfrak{l}$. In this situation [Lau11, Theorem 4.8] provides a characterization of the existence of a solvsoliton on $(\mathfrak{l}, g^c_\rho)$. In particular, $\mathrm{ad}(A)$ must be a normal operator for all $A \in \mathfrak{a}$, that is $[\mathrm{ad}(A), \mathrm{ad}(A)^*] = 0$, where $\mathrm{ad}(A)^* = (g^c_\rho)^{-1}\,\mathrm{ad}(A)^\top g^c_\rho$ denotes the adjoint operator of $\mathrm{ad}(A)$ with respect to the metric g^c_ρ. In Lemma 6.2.18 below we will show that

$$[\mathrm{ad}(B^R_1), \mathrm{ad}(B^R_1)^*] \neq 0,$$

therefore the pair $(\mathfrak{l}, g^c_\rho)$ is not a solvsoliton. □

Lemma 6.2.18. $[\mathrm{ad}(B^R_1), \mathrm{ad}(B^R_1)^*] \neq 0$.

Proof. We work in the basis $\mathcal{B}_n$ given by (38). We have computed $\mathrm{ad}(B^R_1)$ and $\mathrm{ad}(B^R_1)^*$ in the proofs of Proposition 6.2.4 and Lemma 6.2.7, respectively. Then we compute that

$$\begin{aligned}[\mathrm{ad}(B^R_1), \mathrm{ad}(B^R_1)^*] = \mathrm{diag}&\left(0, 0, \mathbb{O}_{2n-4}, \frac{4c(\rho+c)}{\rho(\rho+2c)}\mathbb{1}_2, -\frac{4c(\rho+c)}{\rho(\rho+2c)}\mathbb{1}_2, \mathbb{O}_{2n-4}, 0\right)\\ &- \frac{2c}{\rho(\rho+2c)}E_{2,4n-1} - \frac{8c(\rho+c)^2}{\rho(\rho+2c)}E_{4n-1,2},\end{aligned}$$

which is zero if and only if $c = 0$. □

Remark 6.2.19. It was shown in [Lau11, Theorem 5.1] that if two solvsolitons are isomorphic as Lie groups, then they are isometric up to scaling. This result, together with Corollary 6.1.10, provides an alternative proof to Theorem 6.2.17.

It was shown in [LL14] that if (S,g) is an Einstein solvmanifold, then S admits a unimodular codimension one closed subgroup S_0 which is a solvsoliton with the induced metric. Conversely, if (S_0,g_0) is a solvsoliton, where S_0 is an unimodular solvable Lie group, then $S=\mathbb{R}\times S_0$ admits a homogeneous Einstein metric extending g_0. Recently, it was proved by Thompson [Tho24, Theorem A] that the results of [LL14] can be extended to the inhomogeneous setting. More precisely, given a unimodular solvsoliton (S_0,g_0), he shows that there exists a one-parameter family of complete Ricci soliton metrics on $M=\mathbb{R}\times S_0$, with $\lambda<0$, that are of cohomogeneity one and exactly one of the metrics on the family is Einstein.

Our results differ from the ones of Thompson. In Theorem 6.2.15 we have a non-unimodular solvsoliton $(\mathfrak{l},g_\rho)$ which admits a rank-one extension to a quaternionic Kähler homogeneous metric (in fact symmetric, see Remark 6.2.16). Whereas for $c>0$ we have a metric g_ρ^c on $\mathfrak{l}$, which is not a solvsoliton by Theorem 6.2.17, but such that $(0,\infty)\times L$ admits a complete quaternionic Kähler metric of cohomogeneity one. Determine which are the precise properties of the metric g_ρ^c will be the object of a future study.

References

[AC97a] D. V. Alekseevsky and V. Cortés, *Classification of N-(super)-extended Poincaré algebras and bilinear invariants of the spinor representation of* $\mathrm{Spin}(p,q)$, Commun. Math. Phys. **183** (1997), no. 3, 477–510. (cited on p. 11)

[AC97b] ______, *Homogeneous quaternionic Kähler manifolds of unimodular group*, Boll. Un. Mat. Ital. B (7) **11** (1997), no. 2, 217–229. (cited on p. 32)

[AC99] ______, *Isometry groups of homogeneous quaternionic Kähler manifolds*, J. Geom. Anal. **9** (1999), no. 4, 513–545. (cited on p. 36)

[AC00] ______, *Classification of stationary compact homogeneous special pseudo-Kähler manifolds of semisimple groups*, Proc. London Math. Soc. **81** (2000), no. 1, 211–230. (cited on p. 66)

[ACD02] D. V. Alekseevsky, V. Cortés, and C. Devchand, *Special complex manifolds*, J. Geom. Phys. **42** (2002), 85–105. (cited on p. 2, 62, 66, 67, 68, 69, 70, 74)

[ACD03] ______, *Yang-Mills connections over manifolds with Grassmann structure*, J. Math. Phys. **44** (2003), no. 12, 6047–6076. (cited on p. 26)

[ACDM15] D. V. Alekseevsky, V. Cortés, M. Dyckmanns, and T. Mohaupt, *Quaternionic Kähler metrics associated with special Kähler manifolds*, J. Geom. Phys. **92** (2015), 271–287. (cited on p. 2, 3, 4, 44, 52, 77, 79, 80, 111)

[ACDP03] D. V. Alekseevsky, V. Cortés, C. Devchand, and A. Van Proeyen, *Flows on quaternionic-Kähler and very special real manifolds*, Commun. Math. Phys. **238** (2003), no. 3, 525–543. (cited on p. 48)

[ACM13] D. V. Alekseevsky, V. Cortés, and T. Mohaupt, *Conification of Kähler and hyper-Kähler manifolds*, Commun. Math. Phys. **324** (2013), no. 2, 637–655. (cited on p. 2, 51, 52, 57, 74)

[AF81] L. Alvarez-Gaumé and D. Z. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric* σ*-model*, Commun. Math. Phys. **80** (1981), no. 3, 443–451. (cited on p. 11)

[AHS78] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. R. Soc. Lond. A **362** (1978), no. 1711, 425–461. (cited on p. 28)

[AK75] D. V. Alekseevsky and B. N. Kimel'fel'd, *Structure of homogeneous Riemann spaces with zero Ricci curvature*, Funct. Anal. Appl. **9** (1975), no. 2, 97–102. (cited on p. 42)

[Ale68a] D. V. Alekseevsky, *Compact quaternion spaces*, Funct. Anal. Appl. **2** (1968), no. 2, 106–114. (cited on p. 32)

[Ale68b] ———, *Riemannian spaces with exceptional holonomy groups*, Funct. Anal. Appl. **2** (1968), no. 2, 97–105. (cited on p. 20, 27)

[Ale75] ———, *Classification of quaternionic spaces with a transitive solvable group of motions*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 2, 315–362. (cited on p. 2, 35)

[Ale07] S. Alexandrov, *Quantum covariant c-map*, J. High Energy Phys. **2007** (2007), no. 5, 094. (cited on p. 4)

[Ale13] ———, *Twistor approach to string compactifications: a review*, Phys. Rep. **522** (2013), no. 1, 1–57. (cited on p. 4, 12)

[AM01] D. V. Alekseevsky and S. Marchiafava, *Hermitian and Kähler submanifolds of a quaternionic Kähler manifold*, Osaka J. Math. **38** (2001), no. 4, 869–904. (cited on p. 84)

[Ama09] M. Amann, *Positive quaternion Kähler manifolds*, Ph.D. thesis, WWU Münster, 2009. (cited on p. 29, 33)

[AMP98] D. V. Alekseevsky, S. Marchiafava, and M. Pontecorvo, *Compatible almost complex structures on quaternion Kähler manifolds*, Ann. Glob. Anal. Geom. **16** (1998), no. 5, 419–444. (cited on p. 31)

[APP11] S. Alexandrov, D. Persson, and B. Pioline, *Wall-crossing, Rogers dilogarithm, and the QK/HK correspondence*, J. High Energy Phys. **2011** (2011), no. 12, 027. (cited on p. 50)

[AS53] W. Ambrose and I. M. Singer, *A theorem on holonomy*, Trans. Amer. Math. Soc. **75** (1953), no. 3, 428–443. (cited on p. 17)

[Bat99] F. Battaglia, *Circle actions and Morse theory on quaternion-Kähler manifolds*, J. London Math. Soc. **59** (1999), no. 1, 345–358. (cited on p. 51)

[Bea83] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782. (cited on p. 3, 41, 42, 43)

[Bea11] ———, *Holomorphic symplectic geometry: a problem list*, Complex and differential geometry, Springer proceedings in mathematics, vol. 8, Springer, Berlin, Heidelberg, 2011, pp. 49–63. (cited on p. 43)

[Ber55] M. Berger, *Sur les groupes d'holonomie homogènes de variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955), 279–330. (cited on p. 1, 20)

[Ber66] ______, *Remarques sur les groupes d'holonomie des variétés riemanniennes*, C. R. Acad. Sci. Paris **262** (1966), 1316–1318. (cited on p. 1, 26, 27)

[Bes87] A. L. Besse, *Einstein manifolds*, Springer Berlin, Heidelberg, 1987. (cited on p. 15, 21, 23, 28, 34, 39, 42, 48)

[BG72] R. Brown and A. Gray, *Riemannian manifolds with holonomy group* $\mathrm{Spin}(9)$, Differential geometry (in honor of Kentaro Yano), Kinokuniya, Tokyo, 1972, pp. 41–59. (cited on p. 20)

[BG97] O. Biquard and P. Gauduchon, *Hyperkähler metrics on cotangent bundles of Hermitian symmetric spaces*, Geometry and physics (Aarhus, 1995), Dekker, New York, 1997, pp. 287–298. (cited on p. 43)

[BG08] C. P. Boyer and K. Galicki, *Sasakian geometry*, Oxford University Press, 2008. (cited on p. 21, 24, 29, 48, 49)

[BL52] A. Borel and A. Lichnerowicz, *Groupes d'holonomie des variétés riemanniennes*, C. R. Acad. Sci. Paris **234** (1952), 1835–1837. (cited on p. 17)

[BL23] C. Böhm and R. A. Lafuente, *Non-compact Einstein manifolds with symmetry*, J. Amer. Math. Soc. **36** (2023), no. 3, 591–651. (cited on p. 2, 36)

[Boo62] W. M. Boothby, *A note on homogeneous complex contact manifolds*, Proc. Amer. Math. Soc. **13** (1962), no. 2, 276–280. (cited on p. 31)

[Bor63] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963), 111–122. (cited on p. 2, 35)

[BW83] J. Bagger and E. Witten, *Matter couplings in* $\mathcal{N} = 2$ *supergravity*, Nucl. Phys. B **222** (1983), no. 1, 1–10. (cited on p. 11)

[BWW22] J. Buczyński, J. A. Wiśniewski, and A. Weber, *Algebraic torus actions on contact manifolds*, J. Differential Geom. **121** (2022), no. 2, 227–289. (cited on p. 2, 33)

[Cal79] E. Calabi, *Métriques kählériennes et fibrés holomorphes*, Ann. Sci. Éc. Norm. Supér. **12** (1979), no. 2, 269–294. (cited on p. 43)

[Car26] E. Cartan, *Sur une classe remarquable d'espaces de Riemann I*, Bull. Soc. Math. France **54** (1926), 214–264. (cited on p. 19)

[Car27] ______, *Sur une classe remarquable d'espaces de Riemann II*, Bull. Soc. Math. France **55** (1927), 114–134. (cited on p. 19)

[CDJL21] V. Cortés, M. Dyckmanns, M. Jüngling, and D. Lindemann, *A class of cubic hypersurfaces and quaternionic Kähler manifolds of co-homogeneity one*, Asian J. Math. **25** (2021), no. 1, 1–30. (cited on p. 6, 36, 81, 84, 99, 111)

[CDL14] V. Cortés, M. Dyckmanns, and D. Lindemann, *Classification of complete projective special real surfaces*, Proc. London Math. Soc. **109** (2014), no. 2, 423–445. (cited on p. 81)

[CDM18] V. Cortés, P.-S. Dieterich, and T. Mohaupt, *ASK/PSK-correspondence and the r-map*, Lett. Math. Phys. **108** (2018), no. 5, 1279–1306. (cited on p. 82)

[CDS17] V. Cortés, M. Dyckmanns, and S. Suhr, *Completeness of projective special Kähler and quaternionic Kähler manifolds*, Special metrics and group actions in geometry, Springer INdAM series, vol. 23, Springer, Cham, 2017, pp. 81–106. (cited on p. 3, 80, 81, 83, 84, 119)

[Cec15] S. Cecotti, *Supersymmetric field theories*, Cambridge University Press, 2015. (cited on p. 6)

[CFG89] S. Cecotti, S. Ferrara, and L. Girardello, *Geometry of type II superstrings and the moduli of superconformal field theories*, Int. J. Mod. Phys. A **4** (1989), no. 10, 2475–2529. (cited on p. 2, 12, 74)

[CGS23] V. Cortés, A. Gil-García, and A. Saha, *A class of locally inhomogeneous complete quaternionic Kähler manifolds*, Commun. Math. Phys. **403** (2023), no. 3, 1611–1626. (cited on p. 5, 76, 85, 92)

[CGT24] V. Cortés, A. Gil-García, and D. Thung, *Symmetries of one-loop deformed q-map spaces*, https://arxiv.org/abs/2402.16178, 2024. (cited on p. 5, 100, 105)

[CH17] V. Cortés and A. S. Haupt, *Mathematical methods of classical physics*, Springer Cham, 2017. (cited on p. 7)

[CHM12] V. Cortés, X. Han, and T. Mohaupt, *Completeness in supergravity constructions*, Commun. Math. Phys. **311** (2012), no. 1, 191–213. (cited on p. 36, 62, 78, 79, 82, 83, 99)

[CHSW85] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, *Vacuum configurations for superstrings*, Nucl. Phys. B **258** (1985), 46–74. (cited on p. 12)

[CM09] V. Cortés and T. Mohaupt, *Special geometry of Euclidean supersymmetry III: the local r-map, instantons and black holes*, J. High Energy Phys. **2009** (2009), no. 7, 066. (cited on p. 62)

[CM15] D. Conti and T. B. Madsen, *Harmonic structures and intrinsic torsion*, Transform. Groups **20** (2015), no. 3, 699–723. (cited on p. 26)

[CNS16] V. Cortés, M. Nardmann, and S. Suhr, *Completeness of hyperbolic centroaffine hypersurfaces*, Commun. Anal. Geom. **24** (2016), no. 1, 59–92. (cited on p. 83)

[Cor96a] V. Cortés, *Alekseevskian spaces*, Differential Geom. Appl. **6** (1996), no. 2, 129–168. (cited on p. 2, 35)

[Cor96b] ———, *Homogeneous special geometry*, Transform. Groups **1** (1996), no. 4, 337–373. (cited on p. 66)

[Cor98] ———, *On hyper-Kähler manifolds associated to Lagrangian Kähler submanifolds of $T^*\mathbb{C}^n$*, Trans. Amer. Math. Soc. **350** (1998), no. 8, 3193–3205. (cited on p. 65)

[Cor02] ———, *Special Kähler manifolds: a survey*, Rend. Circ. Mat. Palermo (2) **69** (2002), 11–18. (cited on p. 65)

[Cor23] ———, *A note on quaternionic Kähler manifolds with ends of finite volume*, Q. J. Math. **74** (2023), no. 4, 1489–1493. (cited on p. 3, 37)

[CRT21] V. Cortés, M. Röser, and D. Thung, *Complete quaternionic Kähler manifolds with finite volume ends*, https://arxiv.org/abs/2105.00727, 2021. (cited on p. 3, 37, 98, 99, 100, 111, 125, 126, 129, 130, 131)

[CST21] V. Cortés, A. Saha, and D. Thung, *Symmetries of quaternionic Kähler manifolds with S^1-symmetry*, Trans. London Math. Soc. **8** (2021), no. 1, 95–119. (cited on p. 5, 64, 98, 99, 100, 106, 107)

[CST22] ———, *Curvature of quaternionic Kähler manifolds with S^1-symmetry*, Manuscripta Math. **168** (2022), 35–64. (cited on p. 36, 58, 75, 98)

[CT22a] V. Cortés and I. Tulli, *Quaternionic Kähler metrics associated to special Kähler manifolds with mutually local variations of BPS structures*, Ann. Henri Poincaré **23** (2022), no. 6, 2025–2067. (cited on p. 4)

[CT22b] ———, *S-duality and the universal isometries of q-map spaces*, Commun. Math. Phys. **394** (2022), no. 2, 833–885. (cited on p. 4, 77, 79, 81, 118)

[CT24] ———, *S-duality and the universal isometries of instanton corrected q-map spaces*, Commun. Math. Phys. **405** (2024), no. 3, 68 (72 pp). (cited on p. 4)

[Dan99] A. S. Dancer, *Hyper-Kähler manifolds*, Essays on Einstein manifolds, Surveys in differential geometry, vol. 6, International Press of Boston, 1999, pp. 15–38. (cited on p. 39)

[dR52] G. de Rham, *Sur la réductibilité d'un espace de Riemann*, Comment. Math. Helv. **26** (1952), 328–344. (cited on p. 18)

[DST21] M. Domínguez-Vázquez, V. Sanmartín-López, and H. Tamaru, *Codimension one Ricci soliton subgroups of solvable Iwasawa groups*, J. Math. Pures Appl. **152** (2021), 69–93. (cited on p. 6, 137)

[DW96] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, Nucl. Phys. B **460** (1996), no. 2, 299–334. (cited on p. 66)

[dWVP84] B. de Wit and A. Van Proeyen, *Potentials and symmetries of general gauged $\mathcal{N} = 2$ supergravity-Yang-Mills models*, Nucl. Phys. B **245** (1984), 89–117. (cited on p. 11, 61)

[dWVP92] ———, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, Commun. Math. Phys. **149** (1992), no. 2, 307–333. (cited on p. 2, 3, 13, 35, 66, 81, 83, 100)

[Dyc15] M. Dyckmanns, *The hyper-Kähler/quaternionic Kähler correspondence and the geometry of the c-map*, Ph.D. thesis, Universität Hamburg, 2015. (cited on p. 3)

[Fei01] B. Feix, *Hyperkähler metrics on cotangent bundles*, J. Reine Angew. Math. **532** (2001), 33–46. (cited on p. 44, 74)

[FK82] T. Friedrich and H. Kurke, *Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature*, Math. Nachr. **106** (1982), no. 1, 271–299. (cited on p. 2, 33)

[Fow23] U. Fowdar, *Einstein metrics on bundles over hyperkähler manifolds*, Commun. Math. Phys. **403** (2023), no. 1, 1–35. (cited on p. 37)

[Fre99a] D. S. Freed, *Five lectures on supersymmetry*, American Mathematical Society, 1999. (cited on p. 6)

[Fre99b] ———, *Special Kähler manifolds*, Commun. Math. Phys. **203** (1999), no. 1, 31–52. (cited on p. 2, 61, 64, 66, 70, 74)

[FS90] S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces*, Nucl. Phys. B **332** (1990), no. 2, 317–332. (cited on p. 2, 13, 36, 77, 78)

[Fuj83] A. Fujiki, *On primitively symplectic compact Kähler V-manifolds of dimension four*, Classification of algebraic and analytic manifolds, Progress in mathematics, vol. 39, Birkhäuser Boston, MA, 1983, pp. 71–250. (cited on p. 43)

[FVP12] D. Z. Freedman and A. Van Proeyen, *Supergravity*, Cambridge University Press, 2012. (cited on p. 6)

[GJH03] M. Gross, D. Joyce, and D. Huybrechts, *Calabi-Yau manifolds and related geometries*, Springer Berlin, Heidelberg, 2003. (cited on p. 41)

[GL88] K. Galicki and H. B. Lawson, *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. **282** (1988), no. 1, 1–21. (cited on p. 48)

[GMS11] P. Gauduchon, A. Moroianu, and U. Semmelmann, *Almost complex structures on quaternion-Kähler manifolds and inner symmetric spaces*, Invent. Math. **184** (2011), no. 2, 389–403. (cited on p. 31)

[Ham88] R. S. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity, Contemporary mathematics, vol. 71, American Mathematical Society, 1988, pp. 237–262. (cited on p. 135)

[Ham17] M. J. D. Hamilton, *Mathematical gauge theory*, Springer Cham, 2017. (cited on p. 6, 8)

[Hay08] A. Haydys, *HyperKähler and quaternionic Kähler manifolds with S^1-symmetries*, J. Geom. Phys. **58** (2008), no. 3, 293–306. (cited on p. 2, 51, 52, 57)

[Heb98] J. Heber, *Noncompact homogeneous Einstein spaces*, Invent. Math. **133** (1998), no. 2, 279–352. (cited on p. 36)

[Hel78] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, 1978. (cited on p. 19)

[HH02] H. Herrera and R. Herrera, *$\hat{A}$-genus on non-spin manifolds with S^1 actions and the classification of positive quaternion-Kähler 12-manifolds*, J. Differential Geom. **61** (2002), no. 3, 341–364. (cited on p. 33)

[Hit81] N. J. Hitchin, *Kählerian twistor spaces*, Proc. London Math. Soc. (3) **43** (1981), no. 1, 133–150. (cited on p. 2, 33)

[Hit87] ———, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. **55** (1987), no. 3, 59–126. (cited on p. 40)

[Hit92] N. Hitchin, *Hyperkähler manifolds*, Astérisque **206** (1992), 137–166. (cited on p. 43)

[Hit99] N. J. Hitchin, *The moduli space of complex Lagrangian submanifolds*, Asian J. Math. **3** (1999), no. 1, 77–92. (cited on p. 65)

[Hit09] N. Hitchin, *Quaternionic Kähler moduli spaces*, Riemannian topology and geometric structures on manifolds, Progress in mathematics, vol. 271, Birkhäuser Boston, 2009, pp. 49–61. (cited on p. 2, 78)

[Hit13] ———, *On the hyperkähler/quaternion Kähler correspondence*, Commun. Math. Phys. **324** (2013), no. 1, 77–106. (cited on p. 47, 51, 57)

[HKLR87] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, *Hyperkähler metrics and supersymmetry*, Commun. Math. Phys. **108** (1987), no. 4, 535–589. (cited on p. 40, 43, 46, 48)

[Ion19] R. A. Ionas, *An analogue of the Gibbons-Hawking ansatz for quaternionic Kähler spaces*, https://arxiv.org/abs/1901.11166, 2019. (cited on p. 44)

[Ish74] S. Ishihara, *Quaternion Kählerian manifolds*, J. Differential Geom. **9** (1974), no. 4, 483–500. (cited on p. 25)

[Joy92] D. Joyce, *Compact hypercomplex and quaternionic manifolds*, J. Differential Geom. **35** (1992), no. 3, 743–761. (cited on p. 52)

[Joy96a] D. D. Joyce, *Compact 8-manifolds with holonomy* $\mathrm{Spin}(7)$, Invent. Math. **123** (1996), no. 3, 507–552. (cited on p. 3)

[Joy96b] ______, *Compact Riemannian 7-manifolds with holonomy* G_2 *I*, J. Differential Geom. **43** (1996), no. 2, 291–328. (cited on p. 3)

[Joy96c] ______, *Compact Riemannian 7-manifolds with holonomy* G_2 *II*, J. Differential Geom. **43** (1996), no. 2, 329–375. (cited on p. 3)

[Joy00] ______, *Compact manifolds with special holonomy*, Oxford University Press, 2000. (cited on p. 15, 39)

[Kal01] D. Kaledin, *A canonical hyperkähler metric on the total space of a cotangent bundle*, Quaternionic structures in mathematics and physics (Rome, 1999), World Scientific Publishing, Singapore, 2001, pp. 195–230. (cited on p. 44, 74)

[Kob59] S. Kobayashi, *Remarks on complex contact manifolds*, Proc. Amer. Math. Soc. **10** (1959), no. 1, 164–167. (cited on p. 30)

[Kod64] K. Kodaira, *On the structure of compact complex analytic surfaces I*, Amer. J. Math. **86** (1964), no. 4, 751–798. (cited on p. 43)

[Koi81] N. Koiso, *Hypersurfaces of Einstein manifolds*, Ann. Sci. Éc. Norm. Supér. **14** (1981), no. 4, 433–443. (cited on p. 114)

[Kon75] M. Konishi, *On manifolds with Sasakian 3-structure over quaternion Kaehler manifolds*, Kodai Math. Sem. Rep. **26** (1975), 194–200. (cited on p. 48)

[Lau01] J. Lauret, *Ricci soliton homogeneous nilmanifolds*, Math. Ann. **319** (2001), no. 4, 715–733. (cited on p. 135)

[Lau10] ______, *Einstein solvmanifolds are standard*, Ann. Math. **172** (2010), no. 3, 1859–1877. (cited on p. 36)

[Lau11] ______, *Ricci soliton solvmanifolds*, J. Reine Angew. Math. **650** (2011), 1–21. (cited on p. 133, 135, 136, 137, 138)

[LeB89] C. LeBrun, *Quaternionic-Kähler manifolds and conformal geometry*, Math. Ann. **284** (1989), no. 3, 353–376. (cited on p. 29)

[LeB91] ______, *On complete quaternionic-Kähler manifolds*, Duke Math. J. **63** (1991), no. 3, 723–743. (cited on p. 2, 36)

[LeB95] ______, *Fano manifolds, contact structures, and quaternionic geometry*, Internat. J. Math. **6** (1995), no. 3, 419–437. (cited on p. 33)

[LL14] R. Lafuente and J. Lauret, *Structure of homogeneous Ricci solitons and the Alekseevskii conjecture*, J. Differential Geom. **98** (2014), no. 2, 315–347. (cited on p. 138)

[LM20] G. Lopes Cardoso and T. Mohaupt, *Special geometry, Hessian structures and applications*, Phys. Rep. **855** (2020), 1–141. (cited on p. 13)

[LS94] C. LeBrun and S. Salamon, *Strong rigidity of positive quaternion-Kähler manifolds*, Invent. Math. **118** (1994), no. 1, 109–132. (cited on p. 1, 33)

[Lu99] Z. Lu, *A note on special Kähler manifolds*, Math. Ann. **313** (1999), no. 4, 711–713. (cited on p. 65)

[Man21] M. Mantegazza, *Construction of projective special Kähler manifolds*, Ann. Mat. Pura Appl. **200** (2021), no. 6, 2645–2687. (cited on p. 64)

[Mar90] S. Marchiafava, *Su alcune sottovarietà che ha interesse considerare in una varietà Kähleriana quaternionale*, Rend. Mat. Appl. (7) **10** (1990), no. 3, 493–529. (cited on p. 24)

[Mas62] W. S. Massey, *Non-existence of almost-complex structures on quaternionic projective spaces*, Pacific J. Math. **12** (1962), no. 4, 1379–1384. (cited on p. 31)

[Mil76] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. **21** (1976), no. 3, 293–329. (cited on p. 124, 137)

[MS14] O. Macia and A. Swann, *Elementary deformations and the hyperKähler-quaternionic Kähler correspondence*, Real and complex submanifolds, Springer proceedings in mathematics & statistics, vol. 106, Springer, Tokyo, 2014, pp. 339–347. (cited on p. 55, 56)

[MS15] ———, *Twist geometry of the c-map*, Commun. Math. Phys. **336** (2015), no. 3, 1329–1357. (cited on p. 2, 56, 57, 74)

[MS22] M. Mantegazza and A. Saha, *The c-map as a functor on certain variations of Hodge structure*, Geom. Dedicata **216** (2022), no. 3, 32 (42 pp). (cited on p. 73, 98, 99)

[O'G99] K. G. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, J. Reine Angew. Math. **512** (1999), 49–117. (cited on p. 43)

[O'G03] ———, *A new six-dimensional irreducible symplectic variety*, J. Algebraic Geom. **12** (2003), no. 3, 435–505. (cited on p. 43)

[ORSW21] G. Occhetta, E. A. Romano, L. E. Solá Conde, and J. A. Wiśniewski, *High rank torus actions on contact manifolds*, Sel. Math. **27** (2021), no. 1, 10 (33 pp). (cited on p. 33)

[Pen76] R. Penrose, *Nonlinear gravitons and curved twistor theory*, Gen. Relativity Gravitation **7** (1976), 31–52. (cited on p. 27, 28)

[PS91] Y. S. Poon and S. M. Salamon, *Quaternionic Kähler 8-manifolds with positive scalar curvature*, J. Differential Geom. **33** (1991), no. 2, 363–378. (cited on p. 2, 33)

[PW22] P. Petersen and M. Wink, *Tachibana-type theorems and special holonomy*, Ann. Glob. Anal. Geom. **61** (2022), no. 4, 847–868. (cited on p. 33)

[RSV06] D. Robles Llana, F. Saueressig, and S. Vandoren, *String loop corrected hypermultiplet moduli spaces*, J. High Energy Phys. **2006** (2006), no. 3, 081. (cited on p. 2, 4, 36, 79)

[Sah20] A. Saha, *Twists of quaternionic Kähler manifolds*, Ph.D. thesis, Universität Hamburg, 2020. (cited on p. 3, 24)

[Sal82] S. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), no. 1, 143–171. (cited on p. 1, 21, 29, 30)

[Sal99] S. M. Salamon, *Quaternion-Kähler geometry*, Essays on Einstein manifolds, Surveys in differential geometry, vol. 6, International Press of Boston, 1999, pp. 83–121. (cited on p. 21, 29, 51)

[Sal01] S. Salamon, *Almost parallel structures*, Global differential geometry: the mathematical legacy of Alfred Gray, Contemporary mathematics, vol. 288, American Mathematical Society, 2001, pp. 162–181. (cited on p. 26)

[Str90] A. Strominger, *Special geometry*, Commun. Math. Phys. **133** (1990), no. 1, 163–180. (cited on p. 65)

[SW02] U. Semmelmann and G. Weingart, *Vanishing theorems for quaternionic Kähler manifolds*, J. Reine Angew. Math. **544** (2002), 111–132. (cited on p. 34)

[Swa91] A. Swann, *HyperKähler and quaternionic Kähler geometry*, Math. Ann. **289** (1991), 421–450. (cited on p. 26, 44, 46, 47, 48)

[Swa10] ______, *Twisting Hermitian and hypercomplex geometries*, Duke Math. J. **155** (2010), no. 2, 403–431. (cited on p. 2, 52, 53, 54, 55)

[Swa16] A. F. Swann, *Twists versus modifications*, Adv. Math. **303** (2016), 611–637. (cited on p. 57)

[Tan14] Y. Tanii, *Introduction to supergravity*, Springer Tokyo, 2014. (cited on p. 6)

[Tho24] A. Thompson, *Inhomogeneous deformations of Einstein solvmanifolds*, J. London Math. Soc. **109** (2024), no. 5, e12904 (22 pp). (cited on p. 138)

[Thu20] D. K. Thung, *Symmetries of deformed c-map metrics and the HK/QK correspondence*, Ph.D. thesis, Universität Hamburg, 2020. (cited on p. 3, 5, 25, 100)

[vdA08] T. A. F. van der Aalst, *A geometric interpretation of the c-map*, Master's thesis, Universiteit Utrecht, 2008. (cited on p. 12)

[Wol65] J. A. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech. **14** (1965), no. 6, 1033–1047. (cited on p. 1, 31, 34)

[Yau78] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I*, Commun. Pure Appl. Math. **31** (1978), no. 3, 339–411. (cited on p. 3)

Publication list

This dissertation is based in part on the following article and preprint co-authored by me:

1. *A class of locally inhomogeneous complete quaternionic Kähler manifolds*
   (with V. Cortés and A. Saha)
   Communications in Mathematical Physics
   https://doi.org/10.1007/s00220-023-04830-6
   https://arxiv.org/abs/2210.10097

2. *Symmetries of one-loop deformed q-map spaces*
   (with V. Cortés and D. Thung)
   https://arxiv.org/abs/2402.16178

During my time as a PhD student I co-authored one further article and two preprints which are unrelated to this dissertation:

1. *Symmetric and skew-symmetric complex structures*
   (with G. Bazzoni and A. Latorre)
   Journal of Geometry and Physics
   https://doi.org/10.1016/j.geomphys.2021.104348
   https://arxiv.org/abs/2101.11953

2. *Pseudo-Kähler and hypersymplectic structures on semidirect products*
   (with D. Conti)
   https://arxiv.org/abs/2310.20660

3. *Moduli spaces of (co)closed* G_2*-structures on nilmanifolds*
   (with G. Bazzoni)
   https://arxiv.org/abs/2307.04732

Declaration of personal contribution

Chapters 1 to 4 of this dissertation are introductory material and do not contain original results.

My contribution to the paper [CGS23], on which half of Chapter 5 is based, is comparable to that of my co-authors.

My contribution to the paper [CGT24], on which the other half of Chapter 5 is based, is comparable to that of the first-named author and significantly larger than that of the last-named author. The contribution of the last-named author ended in the middle of the project, when he left academia in November 2022.

My contribution to the collaborative work in progress on which Chapter 6 is based is comparable to that of my collaborators Vicente Cortés and Markus Röser.

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Alejandro Gil García
Hamburg, 2024