Letters in Mathematical Physics (2026) 116:18
https://doi.org/10.1007/s11005-026-02049-9

# Relationship between a $\Phi^4$ matrix model and harmonic oscillator systems

**Harald Grosse[1,2] · Naoyuki Kanomata[3] · Akifumi Sako[3] · Raimar Wulkenhaar[4]**

Received: 20 July 2025 / Revised: 15 January 2026 / Accepted: 19 January 2026


**Abstract**

A Hermitian $\Phi^4$ matrix model with a Kontsevich-type kinetic term is studied. It was recently discovered that the partition function of this matrix model satisfies the Schrödinger equation of the $N$-body harmonic oscillator, and that eigenstates of the Virasoro operators can be derived from this partition function. We extend these results and obtain an explicit formula for such eigenstates in terms of the free energy. Furthermore, the Schrödinger equation for the $N$-body harmonic oscillator can also be reformulated in terms of connected correlation functions. The $U(1)^N$-symmetry allows us to derive loop equations.

## 1 Introduction

In the 1990s, numerous connections between matrix models and two-dimensional quantum gravity were discovered, and many important developments have been made. We refer to [1] for an early review that covers most of these achievements. Of particular importance is the Kontsevich model [12]. Its action is given by $S_K = N\, Tr\{E\Phi^2 + \frac{\lambda}{3}\Phi^3\}$, where $\Phi$ is an $N \times N$ Hermitian matrix, $E$ is a positive diagonal $N \times N$ matrix $E := diag(E_1, E_2, \cdots, E_N)$ without degenerate eigenvalues, and $\lambda$ is a complex number as a coupling constant. This model was proposed to prove the Witten conjecture [14]. The model we will study in this paper is given by replacing the interaction term $\frac{1}{3}\Phi^3$ by $\frac{1}{4}\Phi^4$. The motivation to consider such kind of matrix models comes from quantum field theories on noncommutative spaces.

✉ Akifumi Sako
sako@rs.tus.ac.jp

1 Erwin Schrödinger International Institute for Mathematics and Physics, Boltzmanngasse 9, 1090 Vienna, Austria

2 Faculty of Physics, University of Vienna, Boltzmanngasse 5, 1090 Vienna, Austria

3 Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, 162-8601 Tokyo, Japan

4 Mathematisches Institut, Universität Münster, Einsteinstraße 62, 48149 Münster, Germany

Published online: 17 February 2026 

A noncommutative space is studied in many ways. For example, we consider a noncommutative function algebra as a noncommutative space obtained by deforming a commutative function algebra into a noncommutative one. In the process, we replace the commutative product with a noncommutative product. Then, by using an appropriate matrix representation to the noncommutative algebra, the field theory can be described as a matrix model.

Most of quantum field theories on noncommutative spaces are not renormalizable because of the UV/IR problem. However, the scalar $\Phi^4$ field theory on the Moyal space proposed by Grosse and Wulkenhaar [9] and the scalar $\Phi^3$ theory developed by Grosse and Steinacker [6, 8] appeared as exceptions that could be renormalized. In other words, they showed that those field theories to which certain counter-Lagrangian terms are added are renormalizable. This scalar $\Phi^3$ theory is basically equivalent to the Kontsevich model. This scalar $\Phi^4$ quantum field theory on the Moyal space (Grosse-Wulkenhaar model) is the model we discuss in this paper. (It is also worth noting here that alternative formulations of renormalizable quantum field theory on noncommutative spaces, different from the Grosse-Wulkenhaar or Grosse-Steinacker types, have also been discussed in recent years [13].)

The $\Phi^3$ theory (the Kontsevich model) has been known from the outset to correspond to the KdV hierarchy, and it has been expected that a model in which the interactions are simply replaced by $\Phi^4$ would also be related to integrable systems. This is because, if the Feynman diagrams of the $\Phi^3$ matrix model are mapped onto triangulations of a surface, it is natural to think that the $\Phi^4$ matrix model would simply correspond to quadrangulation, with no essential difference. Actually, it was found that the partition functions of the Hermitian $\Phi^4$-matrix model correspond to zero-energy solutions of a Schrödinger-type equation with an $N$-body harmonic oscillator Hamiltonian. Furthermore, the partition functions of the real symmetric $\Phi^4$-matrix model corresponds to zero-energy solutions of a Schrödinger type equation with the Calogero-Moser Hamiltonian [3–5].

In this paper, we extend the result that the partition function satisfies the Schrödinger equation for the $N$-body harmonic oscillator system, and obtain an explicit formula for such eigenstates in terms of the free energy. Furthermore, since the free energy serves as the generating function for connected multi-point correlation functions, the differential equation for the harmonic oscillator can also be reformulated in terms of these connected correlation functions. The corresponding equations for the connected two- and four-point functions are derived. These results are further confirmed perturbatively up to first order in the coupling constant of the interaction. The process of obtaining the Schrödinger equation for the $N$-body harmonic oscillator is constructed from a set of Schwinger-Dyson equations. The contribution of additional Schwinger-Dyson equations is often discussed using loop equations in Hermitian matrix models with $U(N)$ symmetry. Although this model lacks $U(N)$ symmetry due to the presence of a kinetic term, it retains $U(1)^N$ symmetry, enabling us to derive equations similar to loop equations, as described in [2].

## 2 Setup of $\Phi^4$ Matrix Model and preparations

In this section, we review the $\Phi^4$ matrix model based on previous studies [3–5], and we provide the notation in this paper.

Let $\Phi = (\Phi_{ij})$ be a Hermitian matrix for $i, j = 1, 2, \ldots, N$ and $E$ be a real diagonal $N \times N$ matrix $E := diag(E_1, E_2, \cdots, E_N)$ without degenerate eigenvalues, i.e. $E_i \neq E_j$ if $i \neq j$. Let us consider the following action:

$$S[\Phi] = N\mathrm{tr}\left(E\Phi^2 + \frac{\eta}{4}\Phi^4\right), \tag{2.1}$$

where $\eta$ is a coupling constant that is a positive real number. Since the diagonal matrix $E$ is not proportional to the unit matrix in general, there is no symmetry for the unitary transformation in $\Phi \to U\Phi U^\dagger$. Here $U$ is a unitary matrix, and $U^\dagger$ is its Hermitian conjugate.

Let $\mathcal{D}\Phi$ be the ordinary Lebesgue measure,

$$\mathcal{D}\Phi := \prod_{i=1}^{N} d\Phi_{ii} \prod_{1\leq i<j\leq N} d\mathrm{Re}\Phi_{ij} d\mathrm{Im}\Phi_{ij}, \tag{2.2}$$

where each variable is divided into real and imaginary parts $\Phi_{ij} = \mathrm{Re}\Phi_{ij} + i\mathrm{Im}\Phi_{ij}$ with $\mathrm{Re}\Phi_{ij} = \mathrm{Re}\Phi_{ji}$ and $\mathrm{Im}\Phi_{ij} = -\mathrm{Im}\Phi_{ji}$. Let us consider the following partition function:

$$Z(E, \eta) := \int_{H_N} \mathcal{D}\Phi \exp\left(-S[\Phi]\right), \tag{2.3}$$

where $H_N$ is the space of $N \times N$ Hermitian matrices.

Let $\Delta(E)$ be the Vandermonde determinant $\Delta(E) := \prod_{k<l}(E_l - E_k)$. Then the function

$$\Psi(E, \eta) := e^{-\frac{N}{2\eta}\sum_{i=1}^{N} E_i^2} \Delta(E) Z(E, \eta)$$

is a zero-energy solution of the Schrödinger type equation for the $N$-body harmonic oscillator system.

**Theorem 2.1** *[5] Let $\Psi(E, \eta)$ be the function defined above. Then $\Psi(E, \eta)$ is a zero-energy solution of the Schrödinger type equation*

$$\mathcal{H}_{HO}\Psi(E, \eta) = 0.$$

*Here $\mathcal{H}_{HO}$ is the Hamiltonian for the $N$-body harmonic oscillator system*

$$\mathcal{H}_{HO} := -\frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{N}{\eta}\sum_{i=1}^{N}(E_i)^2. \tag{2.4}$$

It is also known that by replacing the Hermitian matrix $\Phi$ with a real symmetric matrix, the Hamiltonian of the above harmonic oscillator is replaced with the Calogero model Hamiltonian [3].

With the aim of naturally introducing the formulae we will use later, we give in this section a rough outline of the proof of this theorem. Details are given in [5]. To derive the above differential equation, we introduce $H$ as a positive Hermitian $N \times N$ matrix with nondegenerate eigenvalues $\{E_1, E_2, \cdots, E_N \mid E_i \neq E_j \text{ for } i \neq j\}$. Using this $H$, we consider the new action

$$\begin{aligned} S &= N\, Tr\{H\Phi^2 + \frac{\eta}{4}\Phi^4\} \\ &= N\left(\sum_{i,j,k}^{N} H_{ij}\Phi_{jk}\Phi_{ki} + \frac{\eta}{4}\sum_{i,j,k,l}^{N}\Phi_{ij}\Phi_{jk}\Phi_{kl}\Phi_{li}\right). \end{aligned} \tag{2.5}$$

The partition function defined by this $S$

$$Z(E,\eta) := \int_{H_N} \mathcal{D}\Phi\; e^{-S}, \tag{2.6}$$

is the same one defined by (2.3), because the integral measure is $U(N)$ invariant. We use the symbol $\langle O\rangle$ as a non-normalized vacuum expectation value defined by $\langle O\rangle := \int_{H_N} \mathcal{D}\Phi\; Oe^{-S}$.

The Schwinger–Dyson equation is derived from

$$\int_{H_N} \mathcal{D}\Phi\, \frac{\partial}{\partial\Phi_{ij}}\left(\Phi_{ij}e^{-S}\right) = 0, \tag{2.7}$$

which is expressed as

$$Z(E,\eta) - N\sum_k(\langle H_{ki}\Phi_{ij}\Phi_{jk}\rangle + \langle H_{jk}\Phi_{ki}\Phi_{ij}\rangle) - N\eta\sum_{k,l}\langle\Phi_{jk}\Phi_{kl}\Phi_{li}\Phi_{ij}\rangle = 0. \tag{2.8}$$

To obtain the desired partial differential equation, we use the fact that the following expectation values can be expressed in terms of partial derivatives:

$$\frac{\partial Z(E,\eta)}{\partial H_{ij}} = -N\sum_k\langle\Phi_{jk}\Phi_{ki}\rangle, \quad \frac{\partial^2 Z(E,\eta)}{\partial H_{ij}\partial H_{mn}} = N^2\sum_{k,l}\langle\Phi_{jk}\Phi_{ki}\Phi_{nl}\Phi_{lm}\rangle. \tag{2.9}$$

After summing (2.8) over indices $i$ and $j$ and substituting (2.9) for it,

$$\mathcal{L}^H_{SD} Z(E,\eta) = 0. \tag{2.10}$$

is obtained, where $\mathcal{L}^H_{SD}$ is a second order differential operator defined by

$$\mathcal{L}^H_{SD} := N^2 + 2\sum_{i,k} H_{ki}\frac{\partial}{\partial H_{ki}} - \frac{\eta}{N}\sum_{i,k}\left(\frac{\partial}{\partial H_{ki}}\frac{\partial}{\partial H_{ik}}\right). \tag{2.11}$$

To rewrite this Schwinger–Dyson equation in terms of $E_n (n = 1, 2, \cdots, N)$, the following formulae for the second term,

$$\sum_{i,j} H_{ij}\frac{\partial Z(E,\eta)}{\partial H_{ij}} = \sum_k E_k \frac{\partial Z(E,\eta)}{\partial E_k}, \tag{2.12}$$

and the third term,

$$\sum_{i,k}\left(\frac{\partial}{\partial H_{ki}}\frac{\partial}{\partial H_{ik}}\right) Z(E,\eta) = \left\{\sum_{i=1}^N\left(\frac{\partial}{\partial E_i}\right)^2 + \sum_{i\neq j}\frac{1}{E_i - E_j}\left(\frac{\partial}{\partial E_i} - \frac{\partial}{\partial E_j}\right)\right\} Z(E,\eta) \tag{2.13}$$

are used. Here $\sum_{i\neq j}$ means $\sum_{i,j=1,i\neq j}^{N}$. From (2.10) , (2.12) , and (2.13), we find that the partition function defined by (2.3) satisfies

$$\mathcal{L}_{SD} Z(E,\eta) = 0, \tag{2.14}$$

where

$$\mathcal{L}_{SD} := \left\{\frac{\eta}{N}\sum_{i=1}^N\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{\eta}{N}\sum_{i\neq j}\frac{1}{E_i - E_j}\left(\frac{\partial}{\partial E_i} - \frac{\partial}{\partial E_j}\right) - 2\sum_k E_k\frac{\partial}{\partial E_k} - N^2\right\}. \tag{2.15}$$

We can diagonalize $\mathcal{L}_{SD}$ by some kind of gauge transformation as

$$-e^{-\frac{N}{2\eta}\sum_i E_i^2}\Delta(E)\mathcal{L}_{SD}\Delta^{-1}(E)e^{\frac{N}{2\eta}\sum_i E_i^2} = -\frac{\eta}{N}\sum_{i=1}^N\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{N}{\eta}\sum_{i=1}^N (E_i)^2. \tag{2.16}$$

The right-hand side is the Hamiltonian of the $N$-body harmonic oscillator system and we denote it by $\mathcal{H}_{HO}$ as (2.4). To cancel the gauge transformation $\Delta^{-1}(E)e^{\frac{N}{2\eta}\sum_i E_i^2} =: g$, we introduce a transformed partition function $\Psi(E,\eta)$ by

$$\Psi(E,\eta) := e^{-\frac{N}{2\eta}\sum_i E_i^2}\Delta(E)Z(E,\eta) = g^{-1}Z(E,\eta). \tag{2.17}$$

From (2.16), we find that $\Psi(E,\eta)$ is a zero-energy solution of the Schrödinger-type differential equation:

$$\mathcal{H}_{HO}\Psi(E,\eta)=0. \tag{2.18}$$

Theorem 2.1 is thus proved. This $N$-body harmonic oscillator system has no interaction terms between the oscillators, so it is a trivial quantum integrable system.

We have one remark here. It is known that there are no zero eigenvalue solutions of the $N$-body harmonic oscillator system in $L^2(\mathbb{R}^N)$, so the solution obtained here is not such a function [4].

We will also use the partition function and the free energy with external fields, so let us introduce them here. Let $J=(J_{mn})$ be a Hermitian matrix for $m,n=1,\ldots,N$ as an external field. Let us consider the following partition function and the free energy with this $J$:

$$\begin{aligned}\mathcal{Z}[E,J]&:=\int\mathcal{D}\Phi\exp\left(-S[\Phi]+N\mathrm{tr}(J\Phi)\right)\\&=\int\mathcal{D}\Phi\exp\left(-N\mathrm{tr}\left(E\Phi^2+\frac{\eta}{4}\Phi^4\right)\right)\exp\left(N\mathrm{tr}\left(J\Phi\right)\right). \end{aligned}\tag{2.19}$$

$$F[E,J]:=\log\mathcal{Z}[E,J] \tag{2.20}$$

Note that $\mathcal{Z}[E,0]=Z(E,\eta)$.

We saw that the partition function corresponds to the solution of the Schrödinger equation for the $N$-body harmonic oscillator. It is known that the harmonic oscillator system leads to a representation of the Virasoro (Witt) algebra. We see that this yields an infinite sequence of partial differential equations to be satisfied by the partition function.

For simplicity, we use variables $y_i:=\sqrt{\frac{N}{\eta}}E_i$. The Hamiltonian (2.4) in this coordinate is written as

$$\mathcal{H}_{HO}=\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial y_i^2}+y_i^2\right)=\sum_{i=1}^{N}\{a_i\ ,\ a_i^\dagger\},$$

where

$$a_i=\frac{1}{\sqrt{2}}\left(y_i+\frac{\partial}{\partial y_i}\right),\quad a_i^\dagger=\frac{1}{\sqrt{2}}\left(y_i-\frac{\partial}{\partial y_i}\right).$$

We introduce the Virasoro generators with a free parameter $\alpha$:

$$L_{-n} = \sum_{i=1}^{N} \left( \alpha \left( a_i^{\dagger} \right)^{n+1} a_i + (1-\alpha) a_i \left( a_i^{\dagger} \right)^{n+1} \right), \quad (n \geq -1) \tag{2.21}$$

which satisfy the following commutation relations:

$$[L_n, L_m] = (n-m) L_{n+m}.$$

In particular, $L_0$ is written by using $\mathcal{H}_{HO}$:

$$L_0 = \frac{1}{2}\mathcal{H}_{HO} + \left( \frac{1}{2} - \alpha \right) N.$$

Then, we find that $\mathcal{H}_{HO}$ satisfies

$$\left[ \frac{1}{2}\mathcal{H}_{HO}, L_{-m} \right] = m L_{-m}.$$

Using $g = \Delta^{-1}(E) e^{\frac{N}{2\eta}\sum_i E_i^2} = \left( \frac{N}{\eta} \right)^{\frac{N(N-1)}{4}} \Delta^{-1}(y) e^{\frac{1}{2}\sum_i y_i^2}$, we define $\tilde{L}_n$ by $\tilde{L}_n := g L_n g^{-1}$ satisfying $\left[ \tilde{L}_n, \tilde{L}_m \right] = (n-m)\tilde{L}_{n+m}$. Using this $\tilde{L}_n$, we immediately obtain the following result,

$$\left[ \mathcal{L}_{SD}, \tilde{L}_{-m} \right] = -2 \left[ \tilde{L}_0, \tilde{L}_{-m} \right] = -2m \tilde{L}_{-m},$$

which implies the following theorem.

**Theorem 2.2** *The partition function defined by (2.3) satisfies*

$$\mathcal{L}_{SD}(\widetilde{L}_{-m} Z(E,\eta)) = -\,2m(\widetilde{L}_{-m} Z(E,\eta)) \quad (m \geq -1). \tag{2.22}$$

## 3 Schwinger–Dyson equation for free energy

In this section, we will rewrite the results of Sect. 2 using the free energy, which is the generating function of the connected Green's function.

As we saw in (2.14) the partition function $\mathcal{Z}[E,0] = Z(E,\eta)$ satisfies the partial differential equation $\mathcal{L}_{SD}\mathcal{Z}[E,0] = 0$.

Let $F[E,J] := \log \mathcal{Z}[E,J]$ be the free energy. Then the following is obtained immediately.

**Proposition 3.1** *The Schwinger–Dyson equation $\mathcal{L}_{SD}\mathcal{Z}[E,0] = 0$ is equivalent to*

$$\frac{\eta}{N}\sum_{i=1}^{N} \left( \frac{\partial^2}{\partial E_i^2} F[E,0] \right) + \frac{\eta}{N}\sum_{i=1}^{N} \left( \frac{\partial}{\partial E_i} F[E,0] \right) \left( \frac{\partial}{\partial E_i} F[E,0] \right)$$

$$
+\frac{\eta}{N}\sum_{i,j=1,i\neq j}^{N}\frac{1}{E_i-E_j}\left(\frac{\partial}{\partial E_i}F[E,0]-\frac{\partial}{\partial E_j}F[E,0]\right)
$$
$$
-2\sum_{k=1}^{N}E_k\left(\frac{\partial}{\partial E_k}F[E,0]\right)-N^2=0. \tag{3.1}
$$

In the following, the eigenfunctions of $\mathcal{H}_{HO}$ with eigenvalue $-2m$ given by Theorem 2.2, will also be rewritten using the free energy.

**Proposition 3.2** *Let us introduce* $\Psi_m := L_{-m}(Z[E,0]g^{-1}) = L_{-m}\Psi$ $(m = 0,1,2,\cdots)$.

$$
\mathcal{H}_{HO}\Psi_m = 2m\Psi_m \tag{3.2}
$$

***Proof*** From the Schwinger–Dyson equation $\hat{\mathcal{L}}_{SD}Z[E,0]=0$, or equivalent equation $\mathcal{H}_{HO}\Psi=0$, and $[\mathcal{H}_{HO},L_{-m}]=2mL_{-m}$, this proposition follows immediately. □

**Proposition 3.3** *For a non-negative integer m,* $\Psi_m = L_{-m}e^{F[E,0]}g^{-1}$ *is given by*

$$
\Psi_m = -\left(-\frac{1}{\sqrt{2}}\right)^m\left(\frac{\eta}{N}\right)^{\frac{N(N-1)}{4}}e^{\frac{1}{2}\sum_{k=1}^N y_k^2}\times
$$
$$
\sum_{l=1}^{N}\left\{y_l\left(\frac{\partial}{\partial y_l}\right)^{m+1}+\frac{1}{2}\left(\frac{\partial}{\partial y_l}\right)^{m+2}+\alpha(m+1)\left(\frac{\partial}{\partial y_l}\right)^{m}\right\}e^{F[E,0]-V[E]}. \tag{3.3}
$$

*Here* $V[y] := V[E] := \sum_{k=1}^N y_k^2 + \sum_{1\le i<j\le N}\log\left(y_j-y_i\right)$ *with* $y_i = \sqrt{\frac{N}{\eta}}E_i$.

***Proof*** Using (2.21),

$$
\begin{aligned}
\Psi_m =& L_{-m}(e^{F[E,0]}g^{-1}) = \sum_{i=1}^{N}\left(\alpha(a_i^\dagger)^{m+1}a_i+(1-\alpha)a_i(a_i^\dagger)^{m+1}\right)e^{F[E,0]}g^{-1}\\
=&\sum_{i=1}^{N}\left(a_i(a_i^\dagger)^{m+1}-\alpha(m+1)(a_i^\dagger)^m\right)e^{F[E,0]}g^{-1}\\
=&\left(\frac{\eta}{N}\right)^{\frac{N(N-1)}{4}}e^{\frac{1}{2}\sum_{k=1}^N y_k^2}\left(-\frac{1}{\sqrt{2}}\right)^m\times\\
&\left\{-\sum_{l=1}^{N}y_l\left(\frac{\partial}{\partial y_l}\right)^{m+1}-\frac{1}{2}\sum_{l=1}^{N}\left(\frac{\partial}{\partial y_l}\right)^{m+2}-\alpha(m+1)\sum_{l=1}^{N}\left(\frac{\partial}{\partial y_l}\right)^{m}\right\}\\
&e^{F[E,0]-\sum_{k=1}^N y_k^2+\sum_{1\le i<j\le N}\log(y_j-y_i)}.
\end{aligned}
$$

To derive the third equality, the following formula was used:

$$a_i^\dagger = \frac{1}{\sqrt{2}}\left(y_i - \frac{\partial}{\partial y_i}\right) = -\frac{1}{\sqrt{2}} e^{\frac{1}{2}\sum_{k=1}^N y_k^2} \frac{\partial}{\partial y_i} e^{-\frac{1}{2}\sum_{k=1}^N y_k^2}. \tag{3.4}$$

□

The following proposition requires a special case of the Faà di Bruno's formula

$$\left\{e^{f(z)}\right\}^{(n)} = e^{f(z)} \sum_{r=0}^{n} B_{n,r}(f^{(1)}, f^{(2)}, \cdots, f^{(n-r+1)}) \tag{3.5}$$

in terms of the Bell polynomials

$$B_{n,r}(f_1, f_2, \cdots, f_{n-r+1}) = \sum_{\substack{j_1+j_2+\cdots+j_{n-r+1}=r,\\ j_1+2j_2+\cdots+(n-r+1)j_{n-r+1}=n}} \frac{n!}{j_1!j_2!\cdots j_{n-r+1}!}\left(\frac{f_1}{1!}\right)^{j_1}\left(\frac{f_2}{2!}\right)^{j_2}\cdots\left(\frac{f_{n-r+1}}{(n-r+1)!}\right)^{j_{n-r+1}}.$$

We adopt the slightly extended Bell polynomials with $B_{0,0} = 1$ and $B_{n,0} = 0$ $(n > 0)$ so that the equality holds even when $n = 0$.

From Propositions 3.3 and (3.5), we obtain the following expression of the eigenfunction $\Psi_m$ of $\mathcal{H}_{HO}$.

**Proposition 3.4** *The eigenfunction $\Psi_m := L_{-m}\Psi$ $(m \geq 0)$ is expressed by using Bell polynomials as follows:*

$$\Psi_m = -\left(-\frac{1}{\sqrt{2}}\right)^m \Psi \sum_{l=1}^{N}\left\{y_l \sum_{r=0}^{m+1} B^l_{m+1,r} + \frac{1}{2}\sum_{r=0}^{m+2} B^l_{m+2,r} + \alpha(m+1)\sum_{r=0}^{m} B^l_{m,r}\right\} \tag{3.6}$$

*where*

$$B^l_{m,r} := B_{m,r}\left(\left(\frac{\partial}{\partial y_l}\right)^1 (F[E,0] - V[E]), \cdots, \left(\frac{\partial}{\partial y_l}\right)^{m-r+1} (F[E,0] - V[E])\right),$$

*and*

$$V[E] = \sum_{k=1}^{N} y_k^2 - \sum_{1\leq i<j\leq N} \log(y_j - y_i) = \frac{N}{\eta}\sum_{k=1}^{N} E_k^2 - \sum_{1\leq i<j\leq N} \log(E_j - E_i) - \frac{N(N-1)}{4}\log\frac{N}{\eta}.$$

## 4 Loop equation

Let us consider more general Schwinger–Dyson equations called loop equations. For details on loop equations and related calculations, refer to [2].

$$I_{k_1,k_2,\cdots,k_n} := \sum_{i,j=1}^{N} \frac{1}{\mathcal{Z}[E,0]} \int_{H(N)} d\Phi \frac{\partial}{\partial \Phi_{ij}} \left\{ \left(\Phi^{k_1}\right)_{ij} \mathrm{Tr}\left[\Phi^{k_2}\right] \cdots \mathrm{Tr}\left[\Phi^{k_n}\right] e^{-S[\Phi]} \right\} \\ = 0,$$

where $\frac{\partial}{\partial \Phi_{ij}} = \frac{1}{2}\left(\frac{\partial}{\partial \Phi^{\mathrm{Re}}_{ij}} - i \frac{\partial}{\partial \Phi^{\mathrm{Im}}_{ij}}\right)$ for $i \neq j$. Recall the following useful formulae

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial \Phi_{ij}} \left(\Phi^{k}\right)_{ij} = \begin{cases} 0 \quad (k=0) \\ N^2 \quad (k=1) \\ \sum_{l=0}^{k-1} \mathrm{Tr}\Phi^{l} \mathrm{Tr}\Phi^{k-1-l} \quad (k>1) \end{cases} \tag{4.1}$$

$$\sum_{i,j=1}^{N} \left(\Phi^{k_1}\right)_{ij} \frac{\partial}{\partial \Phi_{ij}} \mathrm{Tr}\Phi^{k} = \sum_{i,j=1}^{N} \left(\Phi^{k_1}\right)_{ij} k \left(\Phi^{k-1}\right)_{ji} = k\mathrm{Tr}\Phi^{k+k_1-1}. \tag{4.2}$$

In addition, using the computation

$$\begin{aligned} \sum_{i,j=1}^{N} \left(\Phi^{k_1}\right)_{ij} \frac{\partial}{\partial \Phi_{ij}} e^{-S[\Phi]} &= -N \sum_{i,j=1}^{N} \left(\Phi^{k_1}\right)_{ij} \frac{\partial}{\partial \Phi_{ij}} \left(\mathrm{Tr}E\Phi^2 + \frac{\eta}{4}\mathrm{Tr}\Phi^4\right) e^{-S[\Phi]} \\ &= -N\mathrm{Tr}\left(2E\Phi + \eta\Phi^3\right) \Phi^{k_1} e^{-S[\Phi]}, \end{aligned} \tag{4.3}$$

and definition $L'(\Phi, E) := N\left(2E\Phi + \eta\Phi^3\right)$, we obtain

$$\sum_{i,j=1}^{N} \left(\Phi^{k_1}\right)_{ij} \frac{\partial}{\partial \Phi_{ij}} e^{-S[\Phi]} = -N\mathrm{Tr}\left(L'(\Phi, E)\, \Phi^{k_1}\right) e^{-S[\Phi]}. \tag{4.4}$$

From (4.4),(4.1),(4.2), the loop equations

$$\begin{aligned} &N\langle \mathrm{Tr}(\Phi^{k_2}) \cdots \mathrm{Tr}(\Phi^{k_n}) \mathrm{Tr}\left(L'(\Phi, E)\Phi^{k_1}\right)\rangle \\ =&(1-\delta_{k_1 0}) \sum_{l=0}^{k_1-1} \langle \mathrm{Tr}(\Phi^{l}) \mathrm{Tr}(\Phi^{k_1-1-l}) \mathrm{Tr}(\Phi^{k_2}) \cdots \mathrm{Tr}(\Phi^{k_n})\rangle \end{aligned}$$

$$+\sum_{l=2}^{n} k_l \langle \mathrm{Tr}(\Phi^{k_2}) \cdots \mathrm{Tr}(\Phi^{k_{l-1}})(\mathrm{Tr}\Phi^{k_l+k_1-1})\mathrm{Tr}(\Phi^{k_{l+1}}) \cdots \mathrm{Tr}(\Phi^{k_n})\rangle \tag{4.5}$$

are obtained for any non-negative integers $k_1, k_2, \cdots, k_n$.

We introduce resolvents by

$$R(u) = \mathrm{Tr}\left(\frac{1}{u-\Phi}\right) = \frac{1}{u}\sum_{k=0}^{\infty} \frac{1}{u^k}\mathrm{Tr}\Phi^k.$$

After multiplying equation (4.5) by $1/u^{k_1+1}, 1/u_2^{k_2+1}, \cdots, 1/u_n^{k_n+1}$, taking sum for each $k_1, \cdots, k_n$, then we get the loop equation.

$$\begin{aligned}
&N\left\langle \mathrm{Tr}\left(L'(\Phi, E)\frac{1}{u-\Phi}\right) R(u_2)\cdots R(u_n)\right\rangle \\
=&\left\langle R^2(u)R(u_2)R(u_3)\cdots R(u_n)\right\rangle \\
&+\sum_{l=2}^{n}\frac{\partial}{\partial u_l}\left(\frac{\langle R(u)R(u_2)\cdots R(u_{l-1})R(u_{l+1})\cdots R(u_n)\rangle - \langle R(u_2)\cdots R(u_n)\rangle}{u-u_l}\right).
\end{aligned} \tag{4.6}$$

We introduce $\mathcal{U} = \{u_2, u_3, \cdots, u_n\}$,

$$\hat{P}_{n-1}(u, \mathcal{U}) := \left\langle \mathrm{Tr}\left(\left(L'(u, E) - L'(\Phi, E)\right)\frac{1}{u-\Phi}\right) R(u_2)\cdots R(u_n)\right\rangle, \tag{4.7}$$

and

$$\hat{R}_n(u_1, \cdots, u_n) = \left\langle \mathrm{Tr}\frac{1}{u_1-\Phi}\cdots \mathrm{Tr}\frac{1}{u_n-\Phi}\right\rangle = \langle R(u_1)\cdots R(u_n)\rangle. \tag{4.8}$$

Using these symbols, (4.6) is rewritten as

$$N\left(\mathrm{Tr}(L'(u, E))\hat{R}_n(u, \mathcal{U}) - \hat{P}_{n-1}(u, \mathcal{U})\right) = \hat{R}_{n+1}(u, u, \mathcal{U}) + \sum_{l=2}^{n}\frac{\partial}{\partial u_l}\frac{\hat{R}_{n-1}(u, \mathcal{U}\backslash\{u_l\}) - \hat{R}_{n-1}(\mathcal{U})}{u-u_l}. \tag{4.9}$$

We define the multipoint cumulant resolvent,

$$\begin{aligned}
R_n(u_1, \cdots, u_n) =& \left\langle \mathrm{Tr}\frac{1}{u_1-\Phi}\cdots \mathrm{Tr}\frac{1}{u_n-\Phi}\right\rangle_c \\
=& \langle R(u_1)\cdots R(u_n)\rangle_c,
\end{aligned} \tag{4.10}$$

where

$$R_n(u_1.\cdots,u_n) := \partial_{s_1}\cdots\partial_{s_n}\log\left\langle\exp\left(\sum_{i=1}^{N}s_iR(u_i)\right)\right\rangle\Bigg|_{s_1=\cdots=s_n=0}. \tag{4.11}$$

Similarly $P_{n-1}(u,\mathcal{U})$ is defined by

$$\begin{aligned}P_{n-1}(u,\mathcal{U}) &= \left\langle\mathrm{Tr}\left(\left(L'(u,E)-L'(\Phi,E)\right)\frac{1}{u-\Phi}\right)R(u_2)\cdots R(u_n)\right\rangle_c \\ &:= \partial_{s_1}\cdots\partial_{s_n}\log\left\langle\exp\left(s_1\mathrm{Tr}\left(\left(L'(u,E)-L'(\Phi,E)\right)\frac{1}{u-\Phi}\right)+\sum_{i=2}^{N}s_iR(u_i)\right)\right\rangle\Bigg|_{s_1=\cdots=s_n=0}.\end{aligned} \tag{4.12}$$

The above discussions give in the simplest case $n=1$ (we put $k_2=k_3=\cdots=k_n=0$ from the begining of Sect. 4)

$$N\left\langle\mathrm{Tr}\left(L'(\Phi,E)\frac{1}{u-\Phi}\right)\right\rangle = \left\langle R^2(u)\right\rangle. \tag{4.13}$$

This equation is rewritten in terms of cumulants as

$$N\left\langle\mathrm{Tr}\left(L'(\Phi,E)\frac{1}{u-\Phi}\right)\right\rangle_c = \left\langle R^2(u)\right\rangle_c + \langle R(u)\rangle_c\,\langle R(u)\rangle_c\,. \tag{4.14}$$

Similarly, (4.6) or (4.9) is rewritten as follows:

**Proposition 4.1**

$$\begin{aligned}&N\left(\mathrm{Tr}\left(L'(u,E)\right)R_n(u,\mathcal{U})-P_{n-1}(u,\mathcal{U})\right) = R_{n+1}(u,u,\mathcal{U}) \\ &+\sum_{J(\vec{k})\cup J(\vec{l})=\mathcal{U}}R_{|\vec{k}|+1}(u,J(\vec{k}))R_{|\vec{l}|+1}(u,J(\vec{l}))+\sum_{u_l\in\mathcal{U}}\frac{\partial}{\partial u_l}\frac{R_{n-1}(u,\mathcal{U}\backslash\{u_l\})-R_{n-1}(\mathcal{U})}{u-u_l}.\end{aligned} \tag{4.15}$$

*Here* $\mathcal{U}=\{u_2,\cdots,u_n\}$ *and* $J(\vec{k}):=J_{k_1,\cdots,k_j}:=\{u_{k_1},\cdots,u_{k_j}\}\subset\mathcal{U}$.

The result is almost the same as the loop equation of the normal Hermitian one-matrix model [2], despite the fact that $E$ is attached to the kinetic term of the matrix model we are considering. However, as we have been unable to find a paper that rigorously details the process of rewriting it in terms of cumulants, we provide a proof of this for the convenience of the reader.

***Proof*** $n=2$ case of (4.9) is given by

$$N\left\langle\mathrm{Tr}\left(L'(\Phi,E)\frac{1}{u-\Phi}\right)R(u_2)\right\rangle = \left\langle R^2(u)R(u_2)\right\rangle + \frac{\partial}{\partial u_2}\left(\frac{\langle R(u)\rangle-\langle R(u_2)\rangle}{u-u_2}\right) \tag{4.16}$$

By definition of cumulants, this is rewritten as

$$N\left(\left\langle \mathrm{Tr}\left(L'(\Phi, E)\frac{1}{u-\Phi}\right)R(u_2)\right\rangle_c + \left\langle \mathrm{Tr}\left(L'(\Phi, E)\frac{1}{u-\Phi}\right)\right\rangle_c \langle R(u_2)\rangle_c\right) =$$
$$\left\langle R^2(u)R(u_2)\right\rangle_c + \left\langle R^2(u)\right\rangle_c \langle R(u_2)\rangle_c + \langle R(u)\rangle_c^2 \langle R(u_2)\rangle_c + 2\langle R(u)\rangle_c \langle R(u)R(u_2)\rangle_c$$
$$+\frac{\partial}{\partial u_2}\left(\frac{\langle R(u)\rangle_c - \langle R(u_2)\rangle_c}{u-u_2}\right). \tag{4.17}$$

Using (4.14), this is reduced as

$$N\left\langle \mathrm{Tr}\left(L'(\Phi, E)\frac{1}{u-\Phi}\right)R(u_2)\right\rangle_c =$$
$$\left\langle R^2(u)R(u_2)\right\rangle_c + 2\langle R(u)\rangle_c \langle R(u)R(u_2)\rangle_c + \frac{\partial}{\partial u_2}\left(\frac{\langle R(u)\rangle_c - \langle R(u_2)\rangle_c}{u-u_2}\right). \tag{4.18}$$

This equation is the case of $n = 2$ in (4.15).

We assume the above equations (4.15) for $n = 2, 3, \cdots, n$ to be satisfied. As we proved it above, the $n+1$ case of (4.9) is satisfied.

$$N\left(\mathrm{Tr}(L'(u, E))\hat{R}_{n+1}(u, \mathcal{U}') - \hat{P}_n(u, \mathcal{U}')\right) = \hat{R}_{n+2}(u, u, \mathcal{U}') + \sum_{l=2}^{n+1}\frac{\partial}{\partial u_l}\frac{\hat{R}_n(u, \mathcal{U}'\backslash\{u_l\}) - \hat{R}_n(\mathcal{U}')}{u-u_l}, \tag{4.19}$$

where $\mathcal{U}' = \{u_2, \cdots, u_{n+1}\}$.

Note that $\mathcal{U}' = \{u_2, \cdots, u_{n+1}\} = \left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right) \amalg \left(\{u_{n+1}\} \cup J_{k_1\cdots k_j}\right)$ and $J_{k_1\cdots k_j} = \{u_{k_1}, \cdots, u_{k_j} \mid 2 \le k_j \le n\} \subset \mathcal{U}$. Then the following is obtained:

$$\hat{R}_n(\mathcal{U}') = \sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}} \hat{R}_{n-j-1}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right) R_{j+1}\left(J_{k_1\cdots k_j} \cup \{u_{n+1}\}\right). \tag{4.20}$$

Similarly, we use the following identities. [1]

$$\hat{R}_{n+1}(u, \mathcal{U}') = \sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}} \hat{R}_{n-j}(u, \mathcal{U}\backslash J_{k_1\cdots k_j}) R_{j+1}\left(J_{k_1\cdots k_j} \cup \{u_{n+1}\}\right) \tag{4.21}$$

[1] (In the following, it appears that formula numbers have been assigned excessively. The purpose is to assign a number to each term on the right side of each formula and to clearly indicate which term is being referred to by the number.)

$$+\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n-j-1}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right)R_{j+2}\left(\{u,u_{n+1}\}\cup J_{k_1\cdots k_j}\right),\tag{4.22}$$

$$\hat{P}_n(u,\mathcal{U}') := \left\langle \mathrm{Tr}\left(\left(L'(u,E)-L'(\Phi,E)\right)\frac{1}{u-\Phi}\right)R(u_2)\cdots R(u_{n+1})\right\rangle$$

$$=\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{P}_{n-j-1}(u,\mathcal{U}\backslash J_{k_1\cdots k_j})R_{j+1}\left(J_{k_1\cdots k_j}\cup\{u_{n+1}\}\right)\tag{4.23}$$

$$+\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n-1-j}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right)P_{j+1}(u,J_{k_1\cdots k_j},u_{n+1}),\tag{4.24}$$

$$\hat{R}_{n+2}(u,u,\mathcal{U}') = \sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n+1-j}\left(u,u,\mathcal{U}\backslash J_{k_1\cdots k_j}\right)R_{j+1}\left(J_{k_1\cdots k_j}\cup\{u_{n+1}\}\right)\tag{4.25}$$

$$+2\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n-j}\left(u,\mathcal{U}\backslash J_{k_1\cdots k_j}\right)R_{j+2}\left(J_{k_1\cdots k_j},u,u_{n+1}\right)\tag{4.26}$$

$$+\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n-1-j}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right)R_{j+3}\left(J_{k_1\cdots k_j},u,u,u_{n+1}\right),\tag{4.27}$$

$$\sum_{l=2}^{n+1}\frac{\partial}{\partial u_l}\frac{\hat{R}_n\left(u,\mathcal{U}'\backslash\{u_l\}\right)}{u-u_l} = \sum_{l=2}^{n+1}\hat{R}_n(\mathcal{U}')\bigg|_{u_l=u}\frac{\partial}{\partial u_l}\frac{1}{u-u_l}$$

$$=\sum_{l=2}^{n+1}\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n-j-1}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right)R_{j+1}(u_{n+1},J_{k_1\cdots k_j})\bigg|_{u_l=u}\frac{\partial}{\partial u_l}\frac{1}{u-u_l}$$

$$=\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\left\{\sum_{u_l\in\mathcal{U}\backslash J_{k_1\cdots k_j}}\hat{R}_{n-j-1}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right)\bigg|_{u_l=u}R_{j+1}\left(u_{n+1},J_{k_1\cdots k_j}\right)\frac{\partial}{\partial u_l}\frac{1}{u-u_l}\right.\tag{4.28}$$

$$+\sum_{u_l\in\{u_{n+1}\}\cup J_{k_1\cdots k_j}} \hat{R}_{n-j-1}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right) R_{j+1}(u_{n+1}, J_{k_1\cdots k_j})\Big|_{u_l=u} \frac{\partial}{\partial u_l}\frac{1}{u-u_l}\Big\}, \tag{4.29}$$

and

$$\sum_{l=2}^{n+1}\frac{\partial}{\partial u_l}\frac{\hat{R}_n(\mathcal{U}')}{u-u_l} =$$
$$\sum_{l=2}^{n+1}\frac{\partial}{\partial u_l}\left(\frac{1}{u-u_l}\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n-j-1}(\mathcal{U}\backslash J_{k_1\cdots k_j})R_{j+1}\left(J_{k_1\cdots k_j}\cup\{u_{n+1}\}\right)\right)$$
$$=\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\Big\{\sum_{u_l\in\mathcal{U}\backslash J_{k_1\cdots k_j}}\frac{\partial}{\partial u_l}\left(\frac{1}{u-u_l}\hat{R}_{n-j-1}(\mathcal{U}\backslash J_{k_1\cdots k_j})\right)R_{j+1}\left(J_{k_1\cdots k_j}\cup\{u_{n+1}\}\right)$$
$$+\hat{R}_{n-j-1}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right)\sum_{u_l\in\{u_{n+1}\}\cup J_{k_1\cdots k_j}}\frac{\partial}{\partial u_l}\frac{1}{u-u_l}R_{j+1}\left(J_{k_1\cdots k_j}\cup\{u_{n+1}\}\right)\Big\} \tag{4.30}$$

are obtained.

By the assumption,

$$N\mathrm{Tr}\left(L'(u,E)\right)\times(4.21)-N\times(4.23)-(4.25)-(4.28)+(4.30)=0, \tag{4.31}$$

then (4.19) are rewritten as

$$N\Big(\mathrm{Tr}\left(L'(u,E)\right)\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n-j-1}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right)R_{j+2}\left(\{u,u_{n+1}\}\cup J_{k_1\cdots k_j}\right)$$
$$-\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n-1-j}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right)P_{j+1}(u,J_{k_1\cdots k_j},u_{n+1})\Big)$$
$$-2\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n-j}\left(u,\mathcal{U}\backslash J_{k_1\cdots k_j}\right)R_{j+2}\left(J_{k_1\cdots k_j},u,u_{n+1}\right)$$
$$-\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n-1-j}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right)R_{j+3}\left(J_{k_1\cdots k_j},u,u,u_{n+1}\right)$$
$$-\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\Big\{\sum_{u_l\in\{u_{n+1}\}\cup J_{k_1\cdots k_j}}\hat{R}_{n-j-1}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right)R_{j+1}(u_{n+1},J_{k_1\cdots k_j})\Big|_{u_l=u}\frac{\partial}{\partial u_l}\frac{1}{u-u_l}$$

$$-\hat{R}_{n-j-1}\left(\mathcal{U}\backslash J_{k_1\cdots k_j}\right)\sum_{u_l\in\{u_{n+1}\}\cup J_{k_1\cdots k_j}}\frac{\partial}{\partial u_l}\frac{1}{u-u_l}R_{j+1}\left(J_{k_1\cdots k_j}\cup\{u_{n+1}\}\right)\Bigg\}=0. \tag{4.32}$$

Expanding the third line further into cumulants, we obtain

$$\begin{aligned}0=&\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n-j-1}(\mathcal{U}\backslash J_{k_1\cdots k_j})\Big[N\left(\mathrm{Tr}\left(\mathrm{L}'(\mathrm{u},\mathrm{E})\right)R_{j+2}(u,u_{n+1},J_{k_1\cdots k_j})-P_{j+1}(u,u_{n+1},J_{k_1\cdots k_j})\right)\\&-\Bigg\{\sum_{u_l\in\{u_{n+1}\}\cup J_{k_1\cdots k_j}}R_{j+1}(u_{n+1},J_{k_1\cdots k_j})\Bigg|_{u_l=u}\frac{\partial}{\partial u_l}\frac{1}{u-u_l}\\&-\sum_{u_l\in\{u_{n+1}\}\cup J_{k_1\cdots k_j}}\frac{\partial}{\partial u_l}\frac{1}{u-u_l}R_{j+1}\left(J_{k_1\cdots k_j}\cup\{u_{n+1}\}\right)\Bigg\}-R_{j+3}\left(J_{k_1\cdots k_j},u,u,u_{n+1}\right)\Big]\\&-\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\Bigg\{2\sum_{i=0}^{n-1-j}\sum_{J(\vec{l}_i)\subset\left(\mathcal{U}-J(\vec{k}_j)\right)}\hat{R}_{n-j-i-1}(\mathcal{U}-J(\vec{k}_j)-J(\vec{l}_i))R_{i+1}(\mathcal{U}-J(\vec{l}_i))R_{j+2}(u,u_{n+1},J_{k_1\cdots k_j})\Bigg\},\end{aligned} \tag{4.33}$$

where $J(\vec{k}_j)=J_{k_1\cdots k_j}$. We transform the last line in (4.33).

$$\begin{aligned}&-2\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\sum_{i=0}^{n-1-j}\sum_{J(\vec{l}_i)\subset\left(\mathcal{U}-J(\vec{k}_j)\right)}\hat{R}_{n-j-i-1}(\mathcal{U}-J(\vec{k}_j)-J(\vec{l}_i))R_{i+1}(\mathcal{U}-J(\vec{l}_i))R_{j+2}(u,u_{n+1},J_{k_1\cdots k_j})\\&=-2\sum_{j=0}^{n-1}\sum_{i=0}^{n-1-j}\sum_{J(\vec{k}_j)\subset\mathcal{U}}\sum_{J(\vec{k}_j+\vec{l}_i)\subset\mathcal{U}}\hat{R}_{n-j-i-1}(\mathcal{U}-J(\vec{k}_j)-J(\vec{l}_i))R_{i+1}(\mathcal{U}-J(\vec{l}_i))R_{j+2}(u,u_{n+1},J_{k_1\cdots k_j})\\&=-2\sum_{m=0}^{n-1}\sum_{J(\vec{k}_m)\subset\mathcal{U}}\sum_{i+j=m}\sum_{J(k_j)\amalg J(\vec{l}_i)=J(\vec{k}_m)}\hat{R}_{n-j-i-1}(\mathcal{U}-J(\vec{k}_j)-J(\vec{l}_i))R_{i+1}(\mathcal{U}-J(\vec{l}_i))R_{j+2}(u,u_{n+1},J_{k_1\cdots k_j})\\&=-2\sum_{m=0}^{n-1}\sum_{J(\vec{k}_m)\subset\mathcal{U}}\hat{R}_{n-1-m}\left(\mathcal{U}\backslash J(\vec{k}_m)\right)\sum_{i+j=m}\sum_{J(\vec{k}_i)\amalg J(\vec{k}_i)=J(\vec{k}_m)}R_{i+1}(u,J(\vec{k}_i))R_{j+2}\left(u,u_{n+1},J(\vec{k}_j)\right)\\&=-\sum_{m=0}^{n-1}\sum_{J(\vec{k}_m)\subset\mathcal{U}}\hat{R}_{n-1-m}\left(\mathcal{U}\backslash J(\vec{k}_m)\right)\sum_{i+j=m+1}\sum_{J(\vec{k}_i)\amalg J(\vec{k}_j)=J(\vec{k}_m)\cup\{u_{n+1}\}}R_{i+1}(u,J(\vec{k}_i))R_{j+1}(u,J(\vec{k}_j)).\end{aligned} \tag{4.34}$$

Substituting (4.34) in the last line in (4.33), the following is obtained.

$$\begin{aligned}0=&\sum_{j=0}^{n-1}\sum_{J_{k_1\cdots k_j}\subset\mathcal{U}}\hat{R}_{n-j-1}(\mathcal{U}\backslash J_{k_1\cdots k_j})\Big[N\left(\mathrm{Tr}\left(\mathrm{L}'(\mathrm{u},\mathrm{E})\right)R_{j+2}(u,u_{n+1},J_{k_1\cdots k_j})-P_{j+1}(u,u_{n+1},J_{k_1\cdots k_j})\right)\\&-\Bigg\{\sum_{u_l\in\{u_{n+1}\}\cup J_{k_1\cdots k_j}}R_{j+1}(u_{n+1},J_{k_1\cdots k_j})\Bigg|_{u_l=u}\frac{\partial}{\partial u_l}\frac{1}{u-u_l}\end{aligned}$$

$$
- \sum_{u_l \in \{u_{n+1}\} \cup J_{k_1 \cdots k_j}} \frac{\partial}{\partial u_l} \frac{1}{u - u_l} R_{j+1} \left( J_{k_1 \cdots k_j} \cup \{u_{n+1}\} \right) \Big\} - R_{j+3} \left( J_{k_1 \cdots k_j}, u, u, u_{n+1} \right) \Big]
$$
$$
- \sum_{m=0}^{n-1} \sum_{J(\vec{k}_m) \subset \mathcal{U}} \hat{R}_{n-1-m} \left( \mathcal{U} \backslash J(\vec{k}_m) \right) \sum_{i+j=m+1} \sum_{J(\vec{k}_i) \cup J(\vec{k}_j) = J(\vec{k}_m) + \{u_{n+1}\}} R_{i+1}(u, J(\vec{k}_i)) R_{j+1}(u, J(\vec{k}_j)) \tag{4.35}
$$

When $J_{k_1 \cdots k_j} \neq \mathcal{U}$, the coefficient of $\hat{R}_{n-j-1}(\mathcal{U} \backslash J_{k_1 \cdots k_j})$ is

$$
\begin{aligned}
& N \left( \mathrm{Tr} \left( \mathrm{L}'(\mathrm{u}, \mathrm{E}) \right) R_{j+2}(u, u_{n+1}, J_{k_1 \cdots k_j}) - P_{j+1}(u, u_{n+1}, J_{k_1 \cdots k_j}) \right) \\
& - \Bigg\{ \sum_{u_l \in \{u_{n+1}\} \cup J_{k_1 \cdots k_j}} R_{j+1}(u_{n+1}, J_{k_1 \cdots k_j}) \Bigg|_{u_l = u} \frac{\partial}{\partial u_l} \frac{1}{u - u_l} \\
& - \sum_{u_l \in \{u_{n+1}\} \cup J_{k_1 \cdots k_j}} \frac{\partial}{\partial u_l} \frac{1}{u - u_l} R_{j+1} \left( J_{k_1 \cdots k_j} \cup \{u_{n+1}\} \right) \Bigg\} - R_{j+3} \left( J_{k_1 \cdots k_j}, u, u, u_{n+1} \right) \\
& - \sum_{i+l=j} \sum_{J(\vec{k}_i) \amalg J(\vec{k}_l) = J(\vec{k}_j) \cup \{u_{n+1}\}} R_{l+1}(u, J(\vec{k}_l)) = 0,
\end{aligned} \tag{4.36}
$$

because the assumption for $j + 1 \leq n - 1$. From (4.36), the case $J_{k_1 \cdots k_j} = \mathcal{U}$ gives still survived term. The expression is

$$
\begin{aligned}
0 = & N \left( \mathrm{Tr} \left( L'(u, E) \right) R_{n+1}(u, u_{n+1}, \mathcal{U}) - P_n(u, u_{n+1}, \mathcal{U}) \right) \\
& - R_{n+2}(u, u, u_{n+1}, \mathcal{U}) - \sum_{J(\vec{k}) \cup J(\vec{l}) = \mathcal{U} \cup \{u_{n+1}\}} R_{|\vec{k}|+1}(u, J(\vec{k})) R_{|\vec{l}|+1}(u, J(\vec{l})) \\
& - \Bigg\{ \sum_{u_l \in \{u_{n+1}\} \cup \mathcal{U}} R_n(u_{n+1}, \mathcal{U})|_{u_l = u} \frac{\partial}{\partial u_l} \frac{1}{u - u_l} - \sum_{u_l \in \{u_{n+1}\} \cup \mathcal{U}} \frac{\partial}{\partial u_l} \frac{1}{u - u_l} R_n(\mathcal{U} \cup \{u_{n+1}\}) \Bigg\},
\end{aligned} \tag{4.37}
$$

where $|\vec{k}| = i$ for $\vec{k} = \{k_1, \cdots, k_i\}$. This equation can be written as

$$
\begin{aligned}
0 = & N \left( \mathrm{Tr} \left( L'(u, E) \right) R_{n+1}(u, \mathcal{U}') - P_n(u, \mathcal{U}') \right) \\
& - R_{n+2}(u, u, \mathcal{U}') \\
& - \sum_{J(\vec{k}) \cup J(\vec{l}) = \mathcal{U}'} R_{|\vec{k}|+1}(u, J(\vec{k})) R_{|\vec{l}|+1}(u, J(\vec{l})) - \sum_{u_l \in \mathcal{U}'} \frac{\partial}{\partial u_l} \frac{R_n(u, \mathcal{U}' \backslash \{u_l\}) - R_n(\mathcal{U}')}{u - u_l}.
\end{aligned} \tag{4.38}
$$

It was thus proved by mathematical induction. □

The genus expansion of (4.15) is given by

$$
\begin{aligned}
0 =& \left(\mathrm{Tr}\left(L'(u,E)\right) R_{g,n}(u,\mathcal{U}) - P_{g,n-1}(u,\mathcal{U})\right)\\
&- R_{g-1,n+1}(u,u,\mathcal{U}) - \sum_{J(\vec{k})\cup J(\vec{l})=\mathcal{U}, h_1+h_2=g} R_{h_1,|\vec{k}|+1}(u, J(\vec{k})) R_{h_2,|\vec{l}|+1}(u, J(\vec{l}))\\
&- \sum_{l=2}^{n} \frac{\partial}{\partial u_l} \frac{R_{g,n-1}(u,\mathcal{U}\backslash\{u_l\}) - R_{g,n-1}(\mathcal{U})}{u-u_l}.
\end{aligned}
\tag{4.39}
$$

## 5 Schwinger–Dyson equation for connected Green's function

### 5.1 Connected Green's function

Using $\log \dfrac{\mathcal{Z}[E,J]}{\mathcal{Z}[E,0]}$, the connected $\sum_{i=1}^{B} N_i$-point function $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$ is defined as

$$
\log \frac{\mathcal{Z}[E,J]}{\mathcal{Z}[E,0]} := \sum_{B=1}^{\infty} \sum_{1\le N_1\le\cdots\le N_B}^{\infty} \sum_{p_1^1,\ldots,p_{N_B}^B=0}^{\mathcal{N}} N^{2-B} \frac{G_{|p_1^1\ldots p_{N_1}^1|\ldots|p_1^B\ldots p_{N_B}^B|}}{S_{(N_1,\ldots,N_B)}} \prod_{\beta=1}^{B} \frac{\mathbb{J}_{p_1^\beta\ldots p_{N_\beta}^\beta}}{N_\beta},
\tag{5.1}
$$

where $N_i$ is the identical valence number for $i=1,\ldots,B$, $\mathbb{J}_{p_1\ldots p_{N_i}} := \prod_{j=1}^{N_i} J_{p_j p_{j+1}}$ with $N_i+1\equiv 1$, $(N_1,\ldots,N_B) = (\underbrace{N_1',\ldots,N_1'}_{\nu_1},\ldots,\underbrace{N_s',\ldots,N_s'}_{\nu_s})$, and $S_{(N_1,\ldots,N_B)} = \prod_{\beta=1}^{s}\nu_\beta!$. The $\sum_{i=1}^{B} N_i$-point function denoted by $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$ is given by the sum over all Feynman diagrams (ribbon graphs) on Riemann surfaces with $B$-boundaries, and each $|a_1^i\cdots a_{N_i}^i|$ corresponds to the Feynman diagrams having $N_i$-external ribbons from the $i$-th boundary[10]. (See Fig. 1.)

Some connected $\sum_{i=1}^{B} N_i$ points function $\langle \Phi_{a_1^1a_2^1}\cdots\Phi_{a_{N_1}^1a_1^1}\Phi_{a_1^2a_2^2}\cdots\Phi_{a_{N_2}^2a_1^2}\cdots \Phi_{a_1^Ba_2^B}\cdots\Phi_{a_{N_B}^Ba_1^B}\rangle_c$ might include contributions from several types of surfaces classified by their boundaries. For example, let us consider $\langle\Phi_{aa}\Phi_{aa}\rangle_c$. From (5.1), $\langle\Phi_{aa}\Phi_{aa}\rangle_c = \dfrac{1}{N}G_{|aa|} + \dfrac{1}{N^2}G_{|a|a|}$. This means that $\langle\Phi_{aa}\Phi_{aa}\rangle_c$ includes contributions from two types of surfaces which are surfaces with one boundary and ones with two boundaries.

We prepare a connected oriented surface with $B$ boundaries for drawing each Feynman diagram to calculate $G_{|a_1^1\ldots a_{N_1}^1|\ldots|a_1^B\ldots a_{N_B}^B|}$. We draw a Feynman diagram with

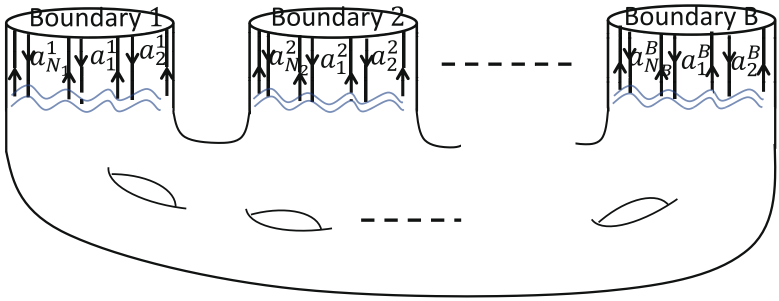



**Fig. 1** The relationship between external ribbons of Feynman diagrams and boundaries as expressed in $G_{|a_1^1...a_{N_1}^1|...|a_1^B...a_{N_B}^B|}$

external ribbons with $(a_1^i a_2^i), \cdots, (a_{N_i}^i a_1^i)$ subscripted to each boundary $i$. For any connected segments in a Feynman diagram, both ends are on the same boundary. $G_{|a_1^1...a_{N_1}^1|...|a_1^B...a_{N_B}^B|}$ is given by the sum over all such Feynman diagrams.

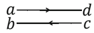
Let $F[E, J] := \log \mathcal{Z}[E, J]$ be the free energy. Then the terms up to the fourth order for $J$ are written out explicitly as follows:

$$\begin{aligned}
F[E,J] =& F[E,0] \\
&+ \sum_{B=1}^{\infty} \sum_{N_1,\cdots,N_B=1}^{\infty} \sum_{p_1^1,\cdots,p_{N_B}^B=1}^{N} N^{2-B} \frac{G_{|p_1^1\cdots p_{N_1}^1|\cdots|p_1^B\cdots p_{N_B}^B|}}{B!} \prod_{\beta=1}^{B} \frac{\mathbb{J}_{p_1^\beta\cdots p_{N_\beta}^\beta}}{N_\beta} \\
=& F[E,0] + \frac{N}{2}\sum_{m,n=1}^{N} G_{|mn|} J_{mn} J_{nm} + \frac{1}{2}\sum_{m,n=1}^{N} G_{|m|n|} J_{mm} J_{nn} \\
&+ \frac{1}{24N^2}\sum_{m,n,k,r=1}^{N} G_{|m|n|k|r|} J_{mm} J_{nn} J_{kk} J_{rr} \\
&+ \frac{1}{4N}\sum_{m,n,k,r=1}^{N} G_{|m|n|kr|} J_{mm} J_{nn} J_{kr} J_{rk} + \frac{1}{8}\sum_{m,n,k,r=1}^{N} G_{|mn|kr|} J_{mn} J_{nm} J_{kr} J_{rk} \\
&+ \frac{1}{3}\sum_{m,n,k,r=1}^{N} G_{|m|nkr|} J_{mm} J_{nk} J_{kr} J_{rn} \\
&+ \frac{N}{4}\sum_{m,n,k,r=1}^{N} G_{|mnkr|} J_{mn} J_{nk} J_{kr} J_{rm} + \mathcal{O}(J^6). \qquad (5.2)
\end{aligned}$$

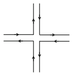
We have taken into account that a correlation function $G_{|a_1^1...a_{N_1}^1|...|a_1^B...a_{N_B}^B|}$ in the $\Phi^4$ matrix model is identically zero if $\sum_{i=1}^{B} N_i$ is odd.

The Feynman rules are given as follows. The propagator for the free Lagrangian is given by

$$\langle \Phi_{ab}\Phi_{cd}\rangle_f = \frac{1}{N}\frac{\delta_{ad}\delta_{bc}}{E_a+E_b} =: \tag{5.3}$$



the interaction is given as



$$:= -\frac{\eta}{4}, \tag{5.4}$$

and each loop corresponds to a sum $\sum_n$.

### 5.2 Formula for connected Green's function

We will rewrite the Schwinger–Dyson equation (3.1) using a connected Green's function. First, the first-order derivative terms are calculated. Using

$$\frac{\partial}{\partial E_i}F[E,J] = -\frac{1}{N\mathcal{Z}[E,J]}\sum_{k=1}^{N}\frac{\partial^2}{\partial J_{ik}\partial J_{ki}}\mathcal{Z}[E,J], \tag{5.5}$$

we find

$$\begin{aligned}\frac{\partial}{\partial E_i}F[E,0] &= -\frac{1}{N\mathcal{Z}[E,0]}\left\{\sum_{k=1}^{N}\left(\frac{\partial^2}{\partial J_{ik}\partial J_{ki}}F[E,J]\right)e^{F[E,J]}\right\}\Bigg|_{J=0}\\ &= -\sum_{k=1}^{N}G_{|ik|} - \frac{1}{N}G_{|i|i|}.\end{aligned} \tag{5.6}$$

Using (5.6), we obtain

$$\begin{aligned}\frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}F[E,0]\right)\left(\frac{\partial}{\partial E_i}F[E,0]\right) =& \frac{\eta}{N}\sum_{i,k,l=1}^{N}G_{|ik|}G_{|il|} + \frac{2\eta}{N^2}\sum_{i,k=1}^{N}G_{|ik|}G_{|i|i|}\\ &+\frac{\eta}{N^3}\sum_{i=1}^{N}G_{|i|i|}^2,\end{aligned} \tag{5.7}$$

and

$$\frac{\eta}{N}\sum_{i,j=1,i\neq j}^{N}\frac{1}{E_i-E_j}\left(\frac{\partial}{\partial E_i}F[E,0] - \frac{\partial}{\partial E_i}F[E,0]\right)$$

$$= -\frac{\eta}{N}\sum_{i,j,k=1,i\neq j}^{N}\frac{1}{E_i - E_j}(G_{|ik|} - G_{|jk|}) - \frac{\eta}{N^2}\sum_{i,j=1,i\neq j}^{N}\frac{1}{E_i - E_j}(G_{|i|i|} - G_{|j|j|}). \tag{5.8}$$

Next let us transform the Laplacian term $\frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial^2}{\partial E_i^2}F[E,J]\right)$ to express it in terms of a connected Green's function. Note that

$$\frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial^2}{\partial E_i^2}F[E,J]\right) \tag{5.9}$$
$$= \frac{\eta}{N}\sum_{i=1}^{N}\left\{\frac{\frac{1}{N^2}\sum_{k,l=1}^{N}\frac{\partial^2}{\partial J_{ik}\partial J_{ki}}\frac{\partial^2}{\partial J_{il}\partial J_{li}}\mathcal{Z}[E,J]}{\mathcal{Z}[E,J]} - \left(\frac{-\frac{1}{N}\sum_{k=1}^{N}\frac{\partial^2}{\partial J_{ik}\partial J_{ki}}\mathcal{Z}[E,J]}{\mathcal{Z}[E,J]}\right)^2\right\},$$

and one point functions vanish.

The second term with $J = 0$ is given as

$$-\frac{\eta}{N}\sum_{i=1}^{N}\left\{\left(\frac{-\frac{1}{N}\sum_{k=1}^{N}\frac{\partial^2}{\partial J_{ik}\partial J_{ki}}\mathcal{Z}[E,J]}{\mathcal{Z}[E,J]}\right)^2\right\}\Bigg|_{J=0}$$
$$= -\frac{\eta}{N^3(\mathcal{Z}[E,0])^2}\sum_{i=1}^{N}\left\{\sum_{k=1}^{N}\left(\frac{\partial^2 F[E,J]}{\partial J_{ik}\partial J_{ki}}\right)e^{F[E,J]}\right\}^2\Bigg|_{J=0}$$
$$= -\frac{\eta}{N}\sum_{i,l,t=1}^{N}G_{|il|}G_{|it|} - \frac{2\eta}{N^2}\sum_{i,l=1}^{N}G_{|il|}G_{|i|i|} - \frac{\eta}{N^3}\sum_{i=1}^{N}G^2_{|i|i|}. \tag{5.10}$$

The first term with $J = 0$ is complex:

$$\sum_{k,l=1}^{N}\frac{\partial^2}{\partial J_{ik}\partial J_{ki}}\frac{\partial^2}{\partial J_{il}\partial J_{li}}\mathcal{Z}[E,J]\Bigg|_{J=0}$$
$$= \sum_{k,l=1}^{N}\Bigg\{\left(\frac{\partial^4}{\partial J_{il}\partial J_{li}\partial J_{ik}\partial J_{ki}}F[E,J]\right)e^{F[E,J]}$$
$$+\left(\frac{\partial^2}{\partial J_{ik}\partial J_{ki}}F[E,J]\right)\left(\frac{\partial^2}{\partial J_{il}\partial J_{li}}F[E,J]\right)e^{F[E,J]}$$
$$+\left(\frac{\partial^2}{\partial J_{li}\partial J_{ki}}F[E,J]\right)\left(\frac{\partial^2}{\partial J_{il}\partial J_{ik}}F[E,J]\right)e^{F[E,J]}$$

$$+\left(\frac{\partial^2}{\partial J_{il}\partial J_{ki}}F[E,J]\right)\left(\frac{\partial^2}{\partial J_{li}\partial J_{ik}}F[E,J]\right)e^{F[E,J]}\Bigg\}\Bigg|_{J=0}. \tag{5.11}$$

For example, the first term of (5.11) is obtained as

$$\begin{aligned}
&\frac{\eta}{N^3}\sum_{i,l,m=1}^{N}\frac{\partial^4 F[E,J]}{\partial J_{im}\partial J_{mi}\partial J_{il}\partial J_{li}}\Bigg|_{J=0}\\
&=\frac{\eta}{N^5}\sum_{i=1}^{N}G_{|i|i|i|i|}+\frac{2\eta}{N^4}\sum_{i,l=1}^{N}G_{|i|i|il|}+\frac{4\eta}{N^4}\sum_{i=1}^{N}G_{|i|i|ii|}+\frac{\eta}{N^3}\sum_{i,m,l=1}^{N}G_{|im|il|}+\frac{\eta}{N^3}\sum_{i=1}^{N}G_{|ii|ii|}\\
&+\frac{\eta}{N^3}\sum_{l,i=1}^{N}G_{|il|il|}+\frac{4\eta}{N^3}\sum_{i=1}^{N}G_{|i|iii|}+\frac{4\eta}{N^3}\sum_{i,l=1}^{N}G_{|i|iil|}+\frac{2\eta}{N^2}\sum_{i=1}^{N}G_{|iiii|}+\frac{2\eta}{N^2}\sum_{i,l=1}^{N}G_{|ilii|}\\
&+\frac{\eta}{N^2}\sum_{l,i=1}^{N}G_{|ilil|}+\frac{\eta}{N^2}\sum_{i,m,l=1}^{N}G_{|limi|}.
\end{aligned} \tag{5.12}$$

After similar calculations, $\frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial^2}{\partial E_i^2}F[E,0]\right)$ is obtained as follows:

$$\begin{aligned}
&\frac{\eta}{N^5}\sum_{i=1}^{N}G_{|i|i|i|i|}+\frac{2\eta}{N^4}\sum_{i,l=1}^{N}G_{|i|i|il|}+\frac{4\eta}{N^4}\sum_{i=1}^{N}G_{|i|i|ii|}+\frac{\eta}{N^3}\sum_{i,m,l=1}^{N}G_{|im|il|}\\
&+\frac{\eta}{N^3}\sum_{i=1}^{N}G_{|ii|ii|}+\frac{\eta}{N^3}\sum_{i,l=1}^{N}G_{|il|il|}+\frac{4\eta}{N^3}\sum_{i=1}^{N}G_{|i|iii|}+\frac{4\eta}{N^3}\sum_{i,l=1}^{N}G_{|i|iil|}\\
&+\frac{2\eta}{N^2}\sum_{i=1}^{N}G_{|iiii|}+\frac{2\eta}{N^2}\sum_{i,l=1}^{N}G_{|ilii|}+\frac{\eta}{N^2}\sum_{i,l=1}^{N}G_{|ilil|}+\frac{\eta}{N^2}\sum_{i,m,l=1}^{N}G_{|limi|}\\
&+\frac{\eta}{N}\sum_{i=1}^{N}G_{|ii|}^2+\frac{4\eta}{N^2}\sum_{i=1}^{N}G_{|ii|}G_{|i|i|}+\frac{2\eta}{N^3}\sum_{i=1}^{N}G_{|i|i|}^2+\frac{\eta}{N}\sum_{i,m=1}^{N}G_{|im|}^2.
\end{aligned} \tag{5.13}$$

Summarizing the results from (5.6) to (5.13), the equivalent equation with $\mathcal{L}_{SD}\mathcal{Z}[E,0]=\mathcal{L}_{SD}e^{F[E,0]}=0$ in the form of connected Green function is obtained.

**Proposition 5.1** *The connected* 4*-point functions and the connected* 2*-point functions defined in (5.1) satisfy the following relation.*

$$\begin{aligned}
&\frac{\eta}{N^5}\sum_{i=1}^{N}G_{|i|i|i|i|}+\frac{2\eta}{N^4}\sum_{i,l=1}^{N}G_{|i|i|il|}+\frac{4\eta}{N^4}\sum_{i=1}^{N}G_{|i|i|ii|}+\frac{\eta}{N^3}\sum_{i,m,l=1}^{N}G_{|im|il|}+\frac{\eta}{N^3}\sum_{i=1}^{N}G_{|ii|ii|}\\
&+\frac{\eta}{N^3}\sum_{i,l=1}^{N}G_{|il|il|}+\frac{4\eta}{N^3}\sum_{i=1}^{N}G_{|i|iii|}+\frac{4\eta}{N^3}\sum_{i,l=1}^{N}G_{|i|iil|}+\frac{2\eta}{N^2}\sum_{i=1}^{N}G_{|iiii|}+\frac{2\eta}{N^2}\sum_{i,l=1}^{N}G_{|ilii|}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\eta}{N^2}\sum_{i,l=1}^{N} G_{|ilil|}+\frac{\eta}{N^2}\sum_{i,m,l=1}^{N} G_{|limi|}\\
&+\frac{\eta}{N}\sum_{i,l,m=1}^{N} G_{|il|}G_{|im|}+\frac{2\eta}{N^2}\sum_{i,l=1}^{N} G_{|il|}G_{|i|i|}+\frac{3\eta}{N^3}\sum_{i=1}^{N} G_{|i|i|}^2+\frac{\eta}{N}\sum_{i=1}^{N} G_{|ii|}^2+\frac{4\eta}{N^2}\sum_{i=1}^{N} G_{|ii|}G_{|i|i|}\\
&+\frac{\eta}{N}\sum_{i,k=1}^{N} G_{|ik|}^2+\frac{\eta}{N}\sum_{i,j,k=1,i\neq j}^{N}\frac{1}{E_i-E_j}\left(G_{|jk|}-G_{|ik|}\right)+\frac{\eta}{N^2}\sum_{i,j=1,i\neq j}^{N}\frac{1}{E_i-E_j}(G_{|j|j|}-G_{|i|i|})\\
&+2\sum_{i,k=1}^{N} E_i G_{|ik|}+\frac{2}{N}\sum_{i=1}^{N} E_i G_{|i|i|}-N^2=0.
\end{aligned}
\tag{5.14}
$$

### 5.3 Perturbative check for Schwinger–Dyson equation

Let us perturbatively check Proposition 5.1 up to the first order of $\eta$ in this subsection.

First we calculate $G_{|ik|}$.

$$
\begin{aligned}
G_{|ik|}=&\frac{1}{N}\left.\frac{\partial^2}{\partial J_{ik}\partial J_{ki}}\log\frac{\mathcal{Z}[E,J]}{\mathcal{Z}[E,0]}\right|_{J=0}=\left.\frac{1}{N\mathcal{Z}[E,0]}\frac{\partial^2\mathcal{Z}[E,J]}{\partial J_{ik}\partial J_{ki}}\right|_{J=0}\\
=&\frac{1}{N\mathcal{Z}[E,0]}\frac{\partial^2}{\partial J_{ik}\partial J_{ki}}\left\{\sum_{t=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{t!}\frac{1}{l!}\int\mathcal{D}\Phi\left(-\frac{N\eta}{4}\right)^t\left(\sum_{n_1,n_2,n_3,n_4=1}^{N}\Phi_{n_1n_2}\Phi_{n_2n_3}\Phi_{n_3n_4}\Phi_{n_4n_1}\right)^t\right.\\
&\left.\left.\times N^l\left(\sum_{m_1,m_2=1}^{N}J_{m_1m_2}\Phi_{m_2m_1}\right)^l\exp\left(-N\mathrm{tr}\left(E\Phi^2\right)\right)\right\}\right|_{J=0}\\
=&\frac{1}{E_k+E_i}-\frac{\eta}{N}\sum_{n_3=1}^{N}\frac{1}{(E_k+E_i)^2(E_{n_3}+E_i)}\\
&-\frac{\eta}{N}\sum_{n_3=1}^{N}\frac{1}{(E_k+E_i)^2(E_{n_3}+E_k)}+\mathcal{O}(\eta^2)
\end{aligned}
\tag{5.15}
$$

Second we calculate $G_{|i|k|}$.

$$
\begin{aligned}
G_{|i|k|}=&\left.\frac{\partial^2}{\partial J_{ii}\partial J_{kk}}\log\frac{\mathcal{Z}[E,J]}{\mathcal{Z}[E,0]}\right|_{J=0}=\left.\frac{1}{\mathcal{Z}[E,0]}\frac{\partial^2\mathcal{Z}[E,J]}{\partial J_{ii}\partial J_{kk}}\right|_{J=0}\\
=&\frac{1}{\mathcal{Z}[E,0]}\frac{\partial^2}{\partial J_{ii}\partial J_{kk}}\left\{\sum_{t=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{t!}\frac{1}{l!}\int\mathcal{D}\Phi\left(-\frac{N\eta}{4}\right)^t\left(\sum_{n_1,n_2,n_3,n_4=1}^{N}\Phi_{n_1n_2}\Phi_{n_2n_3}\Phi_{n_3n_4}\Phi_{n_4n_1}\right)^t\right.\\
&\left.\left.\times N^l\left(\sum_{m_1,m_2=1}^{N}J_{m_1m_2}\Phi_{m_2m_1}\right)^l\exp\left(-N\mathrm{tr}\left(E\Phi^2\right)\right)\right\}\right|_{J=0}\\
=&-\frac{\eta}{4E_iE_k(E_k+E_i)}+\mathcal{O}(\eta^2)
\end{aligned}
\tag{5.16}
$$

Note that from the Feynmann rule given in the end of Sect. 5.1, the following terms in Proposition 5.1 is $\mathcal{O}(\eta^2)$:

$$
\begin{aligned}
\mathcal{O}(\eta^2) =& \frac{\eta}{N^5}\sum_{i=1}^{N} G_{|i|i|i|i|} + \frac{2\eta}{N^4}\sum_{i,l=1}^{N} G_{|i|i|il|} + \frac{4\eta}{N^4}\sum_{i=1}^{N} G_{|i|i|ii|} + \frac{\eta}{N^3}\sum_{i,m,l=1}^{N} G_{|im|il|} + \frac{\eta}{N^3}\sum_{i=1}^{N} G_{|ii|ii|} \\
&+ \frac{\eta}{N^3}\sum_{i,l=1}^{N} G_{|il|il|} + \frac{4\eta}{N^3}\sum_{i=1}^{N} G_{|i|iii|} + \frac{4\eta}{N^3}\sum_{i,l=1}^{N} G_{|i|iil|} + \frac{2\eta}{N^2}\sum_{i=1}^{N} G_{|iiii|} + \frac{2\eta}{N^2}\sum_{i,l=1}^{N} G_{|ilii|} \\
&+ \frac{\eta}{N^2}\sum_{i,l=1}^{N} G_{|ilil|} + \frac{\eta}{N^2}\sum_{i,m,l=1}^{N} G_{|limi|} + \frac{\eta}{N^2}\sum_{i,j=1,i\neq j}^{N} \frac{1}{E_i - E_j}(G_{|j|j|} - G_{|i|i|}) \\
&+ \frac{4\eta}{N^2}\sum_{i=1}^{N} G_{|ii|}G_{|i|i|} + \frac{2\eta}{N^2}\sum_{i,l=1}^{N} G_{|il|}G_{|i|i|} + \frac{3\eta}{N^3}\sum_{i=1}^{N} G_{|i|i|}^2.
\end{aligned}
$$

From (5.16),(5.15), we can calculate perturbatively (5.14) as follows

$$
\begin{aligned}
& L.H.S.of\,(5.14) \\
=& \Bigg\{ -N^2 + \frac{2}{N}\sum_{i=1}^{N} E_i G_{|i|i|} + 2\sum_{i,k=1}^{N} E_i G_{|ik|} + \frac{\eta}{N}\sum_{i,j,k=1,i\neq j}^{N} \frac{1}{E_i - E_j}\left(G_{|jk|} - G_{|ik|}\right) \\
&+ \frac{\eta}{N}\sum_{i=1}^{N} G_{|ii|}^2 + \frac{\eta}{N}\sum_{i,l,m=1}^{N} G_{|il|}G_{|im|} + \frac{\eta}{N}\sum_{i,k=1}^{N} G_{|ik|}^2 \Bigg\} + \mathcal{O}(\eta^2) \\
=& -N^2 - \frac{\eta}{4N}\sum_{i=1}^{N}\frac{1}{E_i^2} + 2\sum_{i,k=1}^{N} E_i \frac{1}{E_k + E_i} - \frac{2\eta}{N}\sum_{i,k,n_3=1}^{N} \frac{E_i}{(E_k + E_i)^2(E_{n_3} + E_i)} \\
&- \frac{2\eta}{N}\sum_{i,k,n_3=1}^{N} \frac{E_i}{(E_k + E_i)^2(E_{n_3} + E_k)} + \frac{\eta}{N}\sum_{i,k,j=1,i\neq j}^{N} \frac{1}{(E_j + E_k)(E_i + E_k)} \\
&+ \frac{\eta}{4N}\sum_{i=1}^{N}\frac{1}{E_i^2} + \frac{\eta}{N}\sum_{i,l,m=1}^{N} \frac{1}{(E_i + E_l)(E_i + E_m)} + \frac{\eta}{N}\sum_{i,k=1}^{N} \frac{1}{(E_i + E_k)^2} + \mathcal{O}(\eta^2) \\
=& 0 + \mathcal{O}(\eta^2).
\end{aligned}
$$

Thus, it was also confirmed that there is no inconsistency in the perturbation calculation for $\eta$ up to order $\eta^2$.

## 6 Summary

It was recently discovered that the partition function of the $\Phi^4$ matrix model with a Kontsevich-type kinetic term satisfies the Schrödinger equation of the $N$-body harmonic oscillator, and that eigenstates of the Virasoro operators can be derived from this partition function. In this paper, we build upon these findings and obtain an explicit formula for such eigenstates in terms of the free energy, as demonstrated in Sect. 3. Furthermore, since the free energy serves as the generating function for connected multi-point correlation functions, the differential equation for the harmonic oscillator can also be reformulated in terms of these connected correlators. The corresponding equations for the connected two- and four-point functions are derived in Section 5. These results are further confirmed perturbatively up to first order in the coupling constant of the interaction.

The process of obtaining the Schrödinger equation for the $N$-body harmonic oscillator is constructed from a set of Schwinger–Dyson equations. The contribution of additional Schwinger–Dyson equations is often discussed using loop equations in matrix models with $U(N)$ symmetry. Although this model lacks $U(N)$ symmetry due to the presence of a kinetic term, it retains $U(1)^N$ symmetry, enabling us to derive equations similar to loop equations, as described in [2]. This is done in Sect. 4.

## 7 Notations

For the reader's convenience, we provide a list of notations in this appendix. The two coordinate systems, $E_i$ and $y_i$, are related by $y_i = \sqrt{\frac{N}{\eta}} E_i$.

- $\mathcal{L}_{SD} := \frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}\right)^2 + \frac{\eta}{N}\sum_{i,j=1,i\neq j}^{N}\frac{1}{E_i - E_j}\left(\frac{\partial}{\partial E_i} - \frac{\partial}{\partial E_j}\right) - 2\sum_{k=1}^{N} E_k \frac{\partial}{\partial E_k} - N^2$
- $\mathcal{L}_{SD} = \sum_{i=1}^{N}\left(\frac{\partial}{\partial y_i}\right)^2 + \sum_{i,j=1,i\neq j}^{N}\frac{1}{y_i - y_j}\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_j}\right) - 2\sum_{i=1}^{N} y_i \frac{\partial}{\partial y_i} - N^2$
- $F[E,0] = \log \mathcal{Z}[E,0]$ : Free energy
- $\hat{\mathcal{L}}_{SD} = e^{-F[E,0]}\mathcal{L}_{SD}e^{F[E,0]}$
- $a_i = \frac{1}{\sqrt{2}}\left(y_i + \frac{\partial}{\partial y_i}\right) = \frac{1}{\sqrt{2}} e^{-\frac{1}{2}\sum_{k=1}^{N} y_k^2}\frac{\partial}{\partial y_i}e^{\frac{1}{2}\sum_{k=1}^{N} y_k^2}$
- $a_i^\dagger = \frac{1}{\sqrt{2}}\left(y_i - \frac{\partial}{\partial y_i}\right) = -\frac{1}{\sqrt{2}} e^{\frac{1}{2}\sum_{k=1}^{N} y_k^2}\frac{\partial}{\partial y_i}e^{-\frac{1}{2}\sum_{k=1}^{N} y_k^2}$
- $L_{-m} := \sum_{i=1}^{N}\left(\alpha(a_i^\dagger)^{m+1}a_i + (1-\alpha)a_i(a_i^\dagger)^{m+1}\right)$
  $= \sum_{i=1}^{N}\left\{a_i(a_i^\dagger)^{m+1} - \alpha(m+1)(a_i^\dagger)^m\right\}$
- $\Delta(E) = \prod_{1\le i<j\le N}(E_j - E_i)$
- $\Delta(y) = \prod_{1\le i<j\le N}(y_j - y_i)$
  $= \exp\left(\log\prod_{1\le i<j\le N}(y_j - y_i)\right) = \exp\left(\sum_{1\le i<j\le N}\log(y_j - y_i)\right)$
- $g := e^{\frac{N}{2\eta}\sum_{i=1}^{N} E_i^2}\Delta^{-1}(E) = \left(\frac{N}{\eta}\right)^{\frac{N(N-1)}{4}} e^{\frac{1}{2}\sum_{i=1}^{N} y_i^2}\Delta^{-1}(y)$

- $\widetilde{L}_{-m} := gL_{-m}g^{-1}$
- $\hat{L}_{-m} = e^{-F}\widetilde{L}_{-m}e^{F}$
- $e^{-\frac{N}{2\eta}\sum_{i=1}^{N}E_i^2}\Delta(E)\mathcal{L}_{SD}\Delta^{-1}(E)e^{\frac{N}{2\eta}\sum_{i=1}^{N}E_i^2} = \frac{\eta}{N}\sum_{i=1}^{N}\left(\frac{\partial}{\partial E_i}\right)^2 - \frac{N}{\eta}\sum_{i=1}^{N}(E_i)^2 = -\mathcal{H}_{HO}$

**Acknowledgements** A.S. was supported by JSPS KAKENHI Grant Number 21K03258. R.W. was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 427320536 – SFB 1442, as well as under Germany's Excellence Strategy EXC 2044 – 390685587, Mathematics Münster: Dynamics – Geometry – Structure. We would like to thank the Erwin Schrödinger International Institute for Mathematics and Physics (ESI) for supporting the realization of this collaborative research.

**Funding** Open Access funding provided by Tokyo University of Science

**Data Availability** Data sharing is not applicable to this article as no new data were created or analyzed in this study.

## Declarations

**Conflicts of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.



## References

1. Di Francesco, P., Ginsparg, P., Zinn-Justin, J.: "2-D Gravity and random matrices," Phys. Rept. **254** pp 1-133 https://doi.org/10.1016/0370-1573(94)00084-G [arXiv:hep-th/9306153 [hep-th]]
2. Eynard, B., Kimura, T., Ribault, S.: "Random matrices," [arXiv:1510.04430 [math-ph]]
3. Grosse, H., Kanomata, N., Sako, A., Wulkenhaar, R.: "Real symmetric $\Phi^4$-matrix model as Calogero–Moser model," Lett. Math. Phys. **114** no.1, 25 (2024) https://doi.org/10.1007/s11005-024-01772-5 [arXiv:2311.10974 [hep-th]]
4. Grosse, H., Kanomata, N., Sako, A., Wulkenhaar, R.: Relationship between $\Phi^4$-matrix model and N-body harmonic oscillator or Calogero–Moser model. J. Phys: Conf. Ser. **2912**(1), 012014 (2024). https://doi.org/10.1088/1742-6596/2912/1/012014
5. Grosse, H., Sako, A.: "Integrability of $\Phi^4$ matrix model as N-body harmonic oscillator system," Lett. Math. Phys. **114** no.2, 48 (2024) https://doi.org/10.1007/s11005-024-01783-2 [arXiv:2308.11523 [math-ph]]
6. Grosse, H., Steinacker, H.: Renormalization of the noncommutative $\phi^3$ model through the Kontsevich model, Nucl. Phys. B **746**, 202–226 (2006) https://doi.org/10.1016/j.nuclphysb.2006.04.007 [arXiv:hep-th/0512203 [hep-th]]
7. Grosse, H., Steinacker, H.: A Nontrivial solvable noncommutative $\phi^3$ model in 4 dimensions, JHEP **08**, 008 (2006) https://doi.org/10.1088/1126-6708/2006/08/008 [arXiv:hep-th/0603052 [hep-th]]

8. Grosse, H., Steinacker, H.: Exact renormalization of a noncommutative $\phi^3$ model in 6 dimensions. Adv. Theor. Math. Phys. **12**(3), 605–639 (2008). https://doi.org/10.4310/ATMP.2008.v12.n3.a4 [arXiv:hep-th/0607235 [hep-th]]
9. Grosse, H., Wulkenhaar, R.: "Renormalisation of $\phi^4$-theory on noncommutative $\mathbb{R}^4$ in the matrix base," Commun. Math. Phys. **256** (2005), 305-374 https://doi.org/10.1007/s00220-004-1285-2 [arXiv:hep-th/0401128]
10. Grosse, H., Wulkenhaar, R.: Self-Dual Noncommutative $\phi^4$ -Theory in Four Dimensions is a Non-Perturbatively Solvable and Non-Trivial Quantum Field Theory. Commun. Math. Phys. **329**, 1069–1130 (2014). https://doi.org/10.1007/s00220-014-1906-3 [arXiv:1205.0465 [math-ph]]
11. Hock, A., Grosse, H., Wulkenhaar, R.: A Laplacian to Compute Intersection Numbers on $\overline{\mathcal{M}}_{g,n}$ and Correlation Functions in NCQFT. Commun. Math. Phys. **399**(1), 481–517 (2023). https://doi.org/10.1007/s00220-022-04557-w [arXiv:1903.12526 [math-ph]]
12. Kontsevich, M.: Intersection theory on the moduli space of curves and the matrix Airy function. Commun. Math. Phys. **147**, 1–23 (1992). https://doi.org/10.1007/BF02099526
13. Nguyen, H., Schenkel, A., Szabo, R.J.: "Batalin–Vilkovisky quantization of fuzzy field theories," Lett. Math. Phys. **111** no.6, 149 (2021) https://doi.org/10.1007/s11005-021-01490-2 [arXiv:2107.02532 [hep-th]]
14. Witten, E.: Two-dimensional gravity and intersection theory on moduli space. Surveys Diff. Geom. **1**, 243–310 (1991). https://doi.org/10.4310/SDG.1990.v1.n1.a5