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# Thèse de doctorat de l'Université Paris-Saclay préparée à l'Université Paris-Sud au sein de l'Institut de Physique Théorique, CEA, Saclay

Ecole doctorale n°564
Ecole Doctorale Physique en Ile-de-France
Spécialité de doctorat: Physique

par

**M. Soumya SASMAL**

## Composite Anomaly in Supergravity and String Amplitude Comparison

Thèse présentée et soutenue à l'Institut de Physique Théorique, CEA, Saclay, le 06 Septembre 2017.

Composition du Jury :

| | | | |
|---|---|---|---|
| M. | Ulrich Ellwanger | Professeur<br>LPT Orsay | (Président du jury) |
| M. | Carlo Angelantonj | Chargé de recherche<br>Turin University | (Rapporteur) |
| M. | Guillaume Bossard | Chargé de recherche<br>CPHT Ecole Polytechnique | (Examinateur) |
| M. | Stefan Hohenegger | Maître de conférence<br>IPNL, Université de Lyon | (Rapporteur) |
| M. | Ruben Minasian | Directeur de recherche<br>IPHT, CEA, Saclay | (Directeur de thèse) |
| M. | Daniel Waldram | Professeur<br>Imperial College London | (Examinateur) |

ÉCOLE DOCTORALE
Physique
en île-de-France
(EDPIF)

**Titre : Autour de Supergravité par l'Anomalie Composée et l'Amplitude en Théorie de Cordes**

**Keywords :** QFT Anomalie, Supergravité, Amplitude en théorie de cordes

**Résumé :** Dans ce projet de thèse, nous étudions le rôle joué par l'anomalie dû à la connexion composée dans les théories de supergravités dans l'espace-temps à huit dimensions. Ce genre d'anomalie est en effet engendrée par la structure quotient d'espace des champs moduli de la supergravité là où le nombre des super-charges posent des contraintes rigoureux. Notre accomplissement principal est de proposer des termes supplémentaires pour annuler cette anomalie dans la théorie de supergravité en huit dimensions avec seize supercharges. Ces termes, en outre, peuvent être considérés comme une manifestation des corrections provenant de la théorie de super-cordes et nous montrons par des calculs explicites qu'une amplitude sur une boucle dans la théorie de cordes correspondante reproduit ces termes. Motivés par cette démonstration de la cohérence entre la supergravité et la théorie de cordes, nous proposons un seuil mathématique pour la compactification de ces théories dans huit dimensions vers six dimensions sur une sphère en présence des branes de co-dimension 2. Ceci est une simulation de compactification sur une surface K3 à l'aide des branes. Nous montrons que la présence d'anomalie composée ne peut être justifiée que par des branes de co-dimensions deux. Nous discutons la dualité entre la théorie Heterotic et la théorie-F sous la lumière de 7-branes et puis la compactification des supergravités de dix dimensions sur K3 en présence des 5-branes. Tous cela nous ouvrent nouvelles voies pour étudier des aspects non-perturbatives de la théorie de cordes. Nous concluons avec un calcul sur deux boucles dans la théorie de cordes Heterotic de dix dimensions qui n'était pas beaucoup exploré dans la littérature.

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Abstract : We examine the structure of composite anomaly in maximal and half-maximal supergravity theories especially in eight space-time dimensions. The number of super-charges dictates the structure of the coset space of the moduli fields of the theory which in turn engenders the composite anomaly in such theories. Our main achievement lies in proposing counter-terms for such anomalies. These terms are of stringy nature and we show by explicit 1-loop amplitude calculations in corresponding string theories that those counter-terms are consistently provided by string amplitude. In the light of non-perturbative higher dimensional theories like F-theory, the anomaly cancelling counter-terms are seen to be related to co-dimension two branes e.g. 7-branes. We then use these results of 8-dimensional theories to provide for supergravity theories in six-dimensions by compactifying on a sphere in the presence of 5-branes. This is in fact a simulation of K3 compactification and our knowledge of composite connection provide us with threshold conditions to achieve such compactifications. All these analysis provide for greater insight into the non-perturbative regime of string theory. We then conclude with a calculation of 2-loop Heterotic string amplitude which has been very less explored in the literature.

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# Contents

*To my parents Jugal Kisor and Renuka for your unconditional love and infinite sacrifice.*

# Résumé en Français

La théorie quantique des champs (QFT) a reconnu un énorme succès en étant le moyen pour décrire la structure microscopique des interactions fondamentales de la nature, notamment l'interaction électrofaible et l'interaction forte qui constituent le modèle standard des particules élémentaires. Cet accomplissement de la QFT tente à prédire une description quantique de l'interaction gravitationnelle. Ce dernier, au niveau macroscopique, est décrit par l'équation d'Einstein dont la conception est fondée sur l'idée de la courbure de l'espace-temps. Par conséquence, une quantification de l'espace-temps semble conceptuellement non-faisable. Néanmoins, la théorie quantique de la gravité propose l'existence d'un quanta, nommé graviton, comme porteur de l'interaction gravitationnelle dont la superposition cohérente reconstruit la courbure d'espace-temps : c'est dans l'esprit de la théorie quantique de l'électromagnétisme, dont le porteur d'interaction est le quanta, nommé photon dont la superposition cohérente donne lieu aux champs magnétique de puissance gigantesque. La théorie quantique de gravitation subit un défaut sérieux pour être classifiée comme une « bonne théorie quantique des champs » ; cette théorie n'est pas renormalizable. Ce défaut provient du fait que le quanta d'interaction, le graviton, en soi-même engendre la courbure de l'espace-temps. Or, dans des interactions avec d'autres particule, la prescription traditionnelle de quantification à la Feynman impose l'interaction dans un point particulier dans l'espace-temps. Cette prescription n'est donc point conforme avec la nature de graviton. La théorie de cordes propose un point de vu radical pour éviter cette difficulté de la gravité quantique : elle rejet l'idée du point mathématique des particules naturelles et la remplace par une seule corde. L'axiome est que les particules observées dans les microstructures des interactions fondamentales ne sont que des différentes modes de vibrations de cette corde. Les interactions entre différentes particules sont vues comme interactions entre les cordes en différentes modes de vibration dont la description à la Feynman soit décrite par une membrane dans l'espace-temps. C'est alors la singularité provenant de la localisation du graviton soit remédiée. De plus, la théorie des cordes, en outre qu'une théorie quantique de gravitation complètement renormalisée, est aussi une théorie qui a l'essence d'unifications des quatre forces fondamentales de la nature. Le besoin pour trouver une telle unification est bidirectionnelle. D'une part, le modèle standard n'est pas une théorie complète en soi-même, c'est qui soit signalé par le problème de saveurs lourdes, le problème de matière noire et l'énergie noire. D'autre part, la théorie de gravitation basée sur l'équation d'Einstein ne peut pas expliquer l'entropie des trous-noirs. C'est donc une théorie plus générale semble exister dont une approximation soit le modèle standard et une réalisation macroscopique soit la description Einsteinienne de la gravité. La versatilité de la théorie des cordes semble apporter des réponses aux toutes ses questions. Cependant la « limitation » dite de cette théorie soit sa richesse. En effet, cette théorie vit dans un espace-temps de dimension dix. Pour qu'on puisse tirer des conclusions sur la phénoménologie, il faut compactifier les six dimensions spatiales sur une variété compactes. La propriété géométrique contrôle fortement le modèle final, aperçu après la compactification. Notamment, la brisure de la supersymétrie, le nombre des champs scalaires sans

masse etc. sont fortement dictés par la structure géométrique de cette variété compacte. L'une des voies principales dans la recherche dans la théorie des cordes est d'étudier la propriété géométrique de la compactification.

Dans l'espace-temps de dix dimensions, il existe cinq différentes théories des cordes. Bien que cela semble gênant en face de la philosophie de l'unification, en tenant compte aux effets non-perturbatifs, il ne reste qu'une seule théorie, définie dans l'espace-temps à onze dimensions, la théorie M qui engendre, dans les régimes perturbatifs les cinq théories des cordes. Cette relation entre différentes théories des cordes et leur origine non-perturbative commune depuis théorie-M est appelé la dualité dans la littérature de la théorie des cordes.

Etant donné que la théorie des cordes est une généralisation des interactions de point de vu des « particules » vers des objets ayant des dimensions finis, dans une échelle d'énergie très faible là où on ne peut pas percevoir la longueur physique des cordes élémentaires, la physique semblera celle des interactions particulaires. C'est donc dans la basse énergie, en dimension dix, les actions effectives des théories des cordes reproduisent celles des théories quantiques de gravité avec supersymétrie, dite la supergravité. Cette correspondance entre la théorie des cordes et la supergravité n'est pas aussi évident lorsqu'on compactifie certaines dimensions spatiales. L'ensemble des théories de la supergravité admissible dans un espace-temps de dimensions strictement inférieurs à dix soit plus large que des théories de supergravités qui peuvent être récupérées depuis la compactification des théories des cordes. Cette identification d'origine permet de suivre la compatibilité et les contraintes sous-jacents des théories des cordes ainsi retrouver les options des supergravités plus rigoureuses dans différents dimensions d'espace-temps. L'un des buts principaux de ce projet de thèse est de comparer l'action effective de la supergravité avec celle obtenue depuis la limite de basse énergie de la théorie de corde en huit dimensions.

Les calculs des interactions du côté de la théorie des cordes sont plus faciles à effectuer par rapport aux calculs des amplitudes des interactions dans la supergravité et donc il est plus facile à tirer les corrections quantiques aux équations classiques de la gravité depuis la théorie des cordes. Néanmoins, ces corrections quantiques dans l'action effective de la supergravité soit plus difficile à expliquer de manière physique. Dans ce projet de thèse nous comparons l'amplitude entre les gravitons par les méthodes de calcul perturbatif dans la théorie de cordes et les corrections quantique dans la théorie des supergravités qui sont de l'origine d'anomalie dans ces dernières théories quantiques des champs. Effectivement, les anomalies dans une théories quantique des champs est le manque de la symétrie classique du système. Les puissants théorèmes d'indices donnent les corrections quantiques à ajouter à l'action effective pour remédier ce manque de symétrie. Nous montrons par des calculs explicits qu'en des théories des supergravités en huit dimensions avec seize super-charges, il existe une anomalie engendrée par la structure quotient d'espace de moduli et puis la correction quantique qu'on doit ajouter à l'action effective est reproduit par le calcul d'amplitude dans la théorie des cordes correspondantes.

L'anomalie dont on vient de parler précédemment dans la supergravité est un peu particulier. Nous considérons plutôt les théories des supergravités ayant des champs moduli paramétrisant l'espace quotient $SL(2,\mathbb{R})/U(1)$. La théorie en considération a la symétrie globale $SL(2,\mathbb{R})$ pour les participant bosonique tandis que les fermions acquièrent une charge sous la jauge Abélienne $U(1)$. Ces fermions chargés donne lieu aux interactions ou amplitudes a une boucle qui ne conservent pas la symétrie $U(1)$ des fermions. On ajoute à l'action effective un terme dont la variation annule le résidu anomal provenant des fermions chargés. Or, la jauge Abélienne ne dépend que sur un facteur de phase si bien que le terme correctif, contenant le même facteur comme un champ de la théorie ne peut pas être considéré comme une correction physique. On regard alors la variation des paramètres de la partie $SL(2,\mathbb{R})$ en tant que des fonctions du facteur de la phase Abélienne. En autre, la quantification de charge dans la théorie des cordes correspondante brise la symétrie $SL(2,\mathbb{R})$ à un sous-groupe discret $SL(2,\mathbb{Z})$ et donc on remplace le facteur de phase dans le terme correctif par une fonction modulaire $SL(2,\mathbb{Z})$. En vue de cette construction du terme correctif, on baptise ce dernier comme le terme correctif d'anomalie composée.

Nous avons rencontré ce genre de démarche par Green et Gaberdiel qui ont tester la théorie de supergravité type IIB dans l'espace-temps à dix dimensions et d'où ils ont tenté à tirer une relation avec la théorie-F, une généralisation non-perturbative de la théorie de corde de type IIB. Cette dernière théorie est une théorie qui décrit de manière non-perturbative la réaction des éléments nommé 7-branes sur la géométrie de l'espace sous-jacent. Le résultat de cette analyse nous a tenter à proposer un couplage de type Wess-Zumino pour les 7-branes dans le régime non-perturbatif. Pour cela il nous faudrait examiner la théorie de supergravité en huit dimensions avec seize supercharges car c'est la théorie qui décrit l'interaction (gravité) quantique de la matière reposant sur les 7-branes.

Nous donc commençons notre investigation sur la théorie de supergravité D=8, N=1 pour pouvoir extraire les informations sur la propriété non-perturbative de 7-brane ce qui nous permettrai à gagner plus d'informations sur la théorie-F. La difficulté formidable devant nous est qu'il y a peu d'outils mathématiques à notre disponibilité pour explorer le régime non-perturbatif de la théorie quantique de gravité. Heureusement, la théorie dont nous explorons, permets d'avoir une version duale purement perturbative, la théorie Heterotique compactifiée sur un tore. La limite supergravité de cette théorie a pour son espace de moduli l'espace $SO(2,n)/SO(2)\times SO(n)$. Cette espace n'a pas une partie $SL(2,\mathbb{R})/U(1)$ pour pouvoir posséder une anomalie composée. Néanmoins, le groupe $SO(2,n)$ possède un sous-groupe $SL(2,\mathbb{R})$ et la partie $SO(2)$ qui est équivalent avec $U(1)$ rend les fermions de la théorie chargés. Ce dernier est encore responsable pour une anomalie dans la théorie. Cette fois ci, on ne peut pas écrire toute de suite un terme pour annuler l'anomalie composée dû au $SL(2,\mathbb{Z})$ car ceci n'est pas le groupe de quotient dans ce cas mais on peut restaurer la partie $SL(2,\mathbb{Z})$ de la symétrie $SO(2,n)$ de la théorie. Nous examinons l'anomalie composée dans le cas de n=18 c'est-à-dire la théorie Heterotique compactifiée sur un tore sans ou avec les lignes de Wilson ce qui donne lieu aux groupes de jauges $SO(32)$, $E_8 \times E_8$, $SO(16)^2$ et $SO(8)^4$.

Après avoir calculer les termes dû à l'anomalie composée, nous essayons à conclure leur connexion avec la théorie-F. Pour la première observation, nous démontrons que ces termes sont aussi bien reproduits par un calcul sur un boucle dans la théorie de cordes. Ce genre de calcul dans la théorie de cordes n'est pas neuf dans la littérature mais une interprétation physique par la notion d'anomalie dans la supergravité est un apport original que nous avons porté. Pour la deuxième observation, nous démontrons que ces termes d'anomalie sont fortement liés aux fermions résident sur 24 les 7-branes qui sont les éléments dans la théorie-F dans ce cas. En effet, nous démontrons une relation avec les modes de cordes tirer entre les deux de ce genre de branes. Cependant, nous ne pouvons point conclure comment ce genre de termes soient liés aux couplages de type Wess-Zumino de 7-branes.

Dans la course de l'investigation de la théorie Heterotique compactifiée sur un tore, nous avons procéder à un genre de compactification particulière de la théorie-F. En effet, la théorie Heterotique compactifiée sur un tore est duale avec la théorie-F compactifiée sur une surface K3. Si bien que la théorie-F soit génériquement non-perturbative, nous ne pouvions point compactifiée cette théorie sur une surface K3 mais on procède différemment. Nous utilisons du fait que la théorie-F est la limite non-perturbative de la théorie de cordes type IIB en dix dimensions d'espace-temps, dans presque le même sens que la théorie-M soit la limite non-perturbative de la théorie type IIA. Alors, nous compactifions la théorie type IIB sur une sphère avec le champ axio-dilaton variable. Pour que cette compactification soit compatible avec la supersymétrie, il nous faut ajouter 24 7-branes dans le fond. Ceci nous donne une simulation de la compactification sur une K3 elliptique dont la base est une sphère. Nous allons donc proceder à un même genre de compactification des théories N=1 et 2 dans la dimension D=8 vers les théories D=6, N=1 et N=2. La technique de compactification à l'aide des branes que nous avions utilisé est considérée comme la compactification non-géométrique à cause de présence des branes. Nous procédons dans la voie similaire pour compactifier la théorie D=8, N=1 sur une sphère en présence des 24 5-branes cette fois-ci. Ceci nous emmène à une conclusion spectaculaire : cette technique de compactification non-géométrique est compatible avec les contraintes de supergravité de dimensions d'espace-temps six si les modes massives de la théorie dans la dimension supérieure perdent la masse et contribuent dans la théorie. Bien que le mécanisme exacte par laquelle les modes massives de multiplets de supergravité deviennent sans masse à cause des 24 singularités du fond, désignant les positions de 24 5-branes nous n'est pas encore claire, nous concluons que cette tactique de compactification peut donner lieu aux théories de supergravités dans D=6 avec 8 supercharges couplées avec des jauges de Yang-Mills $SO(32)$, $E_8 \times E_8$, $SO(16)^2$ et $SO(8)^4$ ce qui n'ont pas été étudier dans la littérature auparavant.

Après avoir étudier la théorie de supergravité dans les dimensions d'espace-temps dix, huit et six de manière respective, nous nous intéressons au calcul d'amplitude aux 2-boucles dans la théorie Heterotique dans dimensions dix. Ce genre de calcul ont été effectué pour les théories type IIA et IIB ainsi que pour la théorie Heterotique avec la méthode de calcul proposée par D'Hocker et Phong. Nous faisons une fusion de cette nouvelle méthode avec le calcul d'amplitude par la méthode hyperelliptique. Notre but était de découvrir la propriété de groupe $Sp(4)$ et l'apparence des fonctions modulaires comme ceci était pour le calcul à une boucle. Notre résultat, cependant, est si compliqué dans la forme qu'on ne peut pas conclure sur ses propriétés modulaires. Ce travail, nous voulons persuader dans le futur.

En conclusion, nous avons investigué la relation entre la théorie de cordes et la théorie de supergravité en huit dimensions d'espace-temps. Nous avons expliqué le terme dû au calcul à une boucle dans la théorie de cordes par le terme nécessaire pour annuler l'anomalie composée dans la théorie de supergravité correspondante. Cette relation, nous avons permit à conclure sur le positionnement de 7-branes lorsqu'on considère la théorie duale c'est-à-dire la théorie-F compactifiée sur une surface K3 elliptique. Ensuite, nous étudions la compactification non-géométrique des théories D=8, N=1 vers la dimension six en présence des 5-branes en concluons la possibilité d'avoir la théorie couplée avec nouvelle jauge de type Yang-Mills. Finalement, nous faisons une fusion de la méthode de la projection fermionique et la définition hyperelliptique pour calculer l'amplitude aux deux boucles dans la théorie Heterotique en dix dimensions dix. Ces travaux ouvrent les voies pour une étude plus profonde sur les aspets non-perturbatifs de la théorie de cordes.

# Chapter 1

# Introduction

## 1.1 Quantization of gravity and quest for unification

Quantum field theory has proven to be of spectacular success in the description of three of the fundamental interactions of nature, namely the electromagnetic interaction plus the strong and weak nuclear forces. The field theory description exploits the elements of symmetry of the theory concerned and with powerful conservation principles, for example the conservation of Noether's current, dictates the physics. As a matter of fact, the three interactions mentioned above are described by a renormalizable gauge field theory where the gauge symmetry is $SU(3) \times SU(2) \times U(1)$. This is the celebrated Standard Model of particle physics which has been verified upto extreme accuracy in the high energy physics experiments. The modern approach to quantum field theory is the description of the quantized theory in terms of an effective action which is valid only upto a certain energy scale. What makes the standard model successful is that, being a renormalized theory it retains its power of prediction even in the higher energy regimes. The field interactions are successfully described by the so called perturbative expansion which is at the heart of particle collision experiments. Another important aspect of Standard Model is that it unifies the weak nuclear interaction and the electromagnetic interaction. One would not exaggerate in saying that almost all the success of classical physics and modern physics lies on the idea of unification. Maxwell's idea of unifying electricity and magnetism, Einstein's effort of unifying space and time plus the geometry with electro-magneto-mechanical dynamics, quantum mechanics unifying electromagnetic interaction with matter-wave duality giving rise to quantum electrodynamics are only a few examples of this long endeavour of science. It is thus intriguing to look for a quantum field theory description of the last of the four fundamental interactions, Gravity.

Einstein's classical theory of general relativity has been proven, in its own credit, to be extremely successful in describing all known celestial dynamics plus the so called concordance model (that is the $\Lambda$CDM model) of cosmology which describes the history of the universe from the Big-Bang nucleo-synthesis up to present day. Very recently, the signal of gravitational wave has been claimed to have been detected (with an experimental set up that draws a formidable accuracy of $10^{-23}/\sqrt{Hz}$ from the quantum electromagnetic coherence of laser!) which was one of the astonishing prediction of Einstein's theory of general relativity.

The naive approach of the quantization of the gravitational field fails to the disappointment of the scientific community. The description of general relativity contains in its Einstein-Hilbert action, the Planck energy parameter, which derives itself in turn from Newton's gravitational constant and the smallness of the latter renders the quantum description of gravity non-renormalizable: that is to say, the description of gravitational interaction in terms of exchange of its quanta, called the graviton, does not retain its predictive power if the energy scale of interaction is boosted. One can nevertheless gather valuable information from such quantum gravity interactions by properly dealing with the singularities.

It is in fact not just an aesthetic question to find a quantum description of gravity or to find a unified framework describing all four fundamental interactions but it happens to be, for modern physics, an immediate necessity because both Standard Model and the concordance model of cosmology have their own deficiencies. To mention a few for the Standard Model, this theory requires almost 20 parameters whose values are needed to be fixed by experimental input. Some of these parameters are needed to be fine-tuned upto very non-practical degrees of accuracy, which is the so-called hierarchy problem of Standard Model. In addition, the Standard Model does not give satisfactory answers to neutrino mass spectrum and the strong CP problem. An ingenious way to answer the hierarchy problem is to appeal to the supersymmetry, which puts both bosonic and fermionic degrees of freedoms in terms of an underlying symmetry of the quantum theory. This set-up provides bosons and fermions of same quantum characters thereby adding elements to solve for the hierarchy problem. Moreover, the localization of supersymmetry gives rise to a supersymmetric theory of quantum gravity, called supergravity which is however again crippled with non-renormalizability issues, but its degree is milder to that of naive quantum gravity. In the observable world, supersymmetry is not manifest and thus this symmetry should be broken at the phenomenological energy scale. Thus instead of discouraging, it cues to look forward for the physics at the energy scales higher than that of common interactions which might reveal these interesting symmetry structures.

The principal defect of cosmology based on Einstein's relativity lies on the fact that it does not account for the dark matter and dark energy of the universe. It is predicted that the unobserved supersymmetric partners of known particles might provide an answer for the dark matter. Even though, Einstein's equations cannot explain the discrepancy between the theoretically predicted value of dark energy content with the observed value. Singular objects in gravity also point to the limitations of the classical set-up of general relativity. Black-holes provide such examples: at classical level, the event horizon of black-holes causally disconnects the rational regime of space-time geometry. In fact, the collapse of massive gravitating objects towards singularity breaks the very notion of classical geometry itself. If one should consider the quantum effects like Hawking radiation, the rupture due to event horizon poses a puzzle for the loss of information.

It is thus clear that a consistent description of gravity should submit itself to a quantum description and that the Standard Model physics should be thought of as a limit of a more fundamental theory. These two aspects hint towards a theory of everything and string theory comes out to be the most promising candidate for such a unification.

String theory revolutionizes the understanding of fundamental interactions in that, instead of particles, it is based on the dynamics of the one-dimensional objects called string. The only free parameter (apart from those arising from the compactification, to be explained shortly) this theory requires is its length scale and thus providing for the essence of the unification at its very foundation. Consistency of string dynamics however seems to require ten space-time dimensions instead of four but this is not discouraging as the very old idea of Kaluza-Klein compactification accounted for unification of gravity and electromagnetism quite successfully. Incorporating this idea with string theory one finds many a interesting scenarios for the phenomenological four-dimensional physics. String theory draws upon powerful tools of conformal field theory, classical geometry and even way to geometrize physical degrees of freedom to answer for supersymmetry breaking, black-hole thermodynamics and above all an unified quantum theory of gravity which is completely renormalized. In addition, the effective action of string theory seems to give in low energy limit the effective action of supergravity actions in ten dimensions. The understanding of the non-perturbative aspects of string theory has also been vastly improved by the incorporation of soliton-like branes in the theory.

## 1.2 What this thesis stands for?

Any quantum theory of fields draws heavily upon symmetries of the theory. It is in fact the symmetry of the theory that dictates the interaction part of the quantum theory. Thus the loss of symmetry in any quantum theory should be investigated closely. It happens to be that after quantization of a classical field theory, the classical symmetry is lost due to the incorporation of quantum corrections arising from the dynamical interactions of the fields. This is what we call anomaly. We have stated previously that consistent string theories require ten space-time dimensions to live in. One can however compactify this theory to lower dimensions for example eight, six, four etc. The supersymmetric structure of the string theory dictates specific group coset structure to the space parametrized by the massless scalar fields of the corresponding supergravity theory. It has been observed that if the group coset contains Abelian factors the fermions of the theory might get charged under this Abelian factor and their interaction might give rise to anomalous quantum correction to the effective supergravity theories. The standard process of curbing such anomalies is to find a counter-term whose anomalous variation would cancel the anomaly due to quantum correction. This might smell like a fine-tuning of the quantum theory. However, string theory in fact automatically provides for such anomaly cancelling counter-terms. Our motivation and achievement in this thesis is to look into the quantum anomalies in supergravity theories in eight space-time dimensions with maximal and half-maximal supersymmetry. The meaning of "maximal" and "half-maximal" supersymmetry shall be made clear in due course of time: for instance, we can say that for a supersymmetric theory of quantum gravity to have states not higher than spin-2, which is graviton, only a certain maximal number of fermionic superpartners for a given bosonic degree of freedom are allowed. In case such a maximum number of super-partners are present, we say that the theory has maximal supersymmetry. If however only half of those super-partners stay in the theory we call it half-maximal. The anomaly counter-term for the coset structure mentioned above reveal an important structure in terms of discretization of the coset group. Such mechanism has been observed long ago but has never been tested consistently in eight-dimensional supergravity theories. Our achievement in this thesis is to examine the class of such discrete anomalies and find the origin of the anomaly cancelling counter-term from string theory amplitudes. This examination provides important insight into the effect of higher dimensional solitonic branes into the supergravity effective actions. All such results culminate into providing further directions of research into the so-called F-theory, which is generically a non-perturbative theory. We observe further how the coset structure of eight dimensional supergravity theories put stringent constraint on the compactification of such theories down to six dimensions in the presence of branes of particular kind. Thus in this thesis, instead of delving deep into the phenomenological prospect of string theory, we concentrate upon the consistency of string theory in higher dimensions. We finally launch ourselves into the academic interest of two-loop string amplitude.

## 1.3 Outline of the thesis

We conclude this introductory chapter with the structure of the thesis. We start by describing the necessary concepts of supersymmetry and supergravity in chapter 2 where we also introduce the concept and mathematical tools for the analysis of anomaly in quantum theory. In chapter 3 we give a brief yet self-consistent introduction to the string theory along with the non-pertutbative theories like M-theory and F-theory. A large part of this thesis is devoted to the computation of string amplitude and hence we also provide some details of the process of amplitude calculation in string theory. Next in chapter 4 we start reviewing the origin of discrete anomaly in 10-dimensional string theory and its interpretation in terms of non-perturbative aspects related to branes. From this motivation we compute anomalies in eight dimensional supergravity theories in chapter 5 and provide the computations of string one-loop calculation to corroborate our conclusion that string theory accounts for the quantum consistency of the low energy supergravity effective action. In this course, we discover an interesting paradigm of compactification with aid of the non-perturbative degrees of string theory which we apply for the compactification towards six-dimensions in chapter 6. Finally in chapter 7 we shall discuss the particularity of two-loop Heterotic string amplitude in the light of newly developed method for evaluating the path-integral measure. This was a subject studied long before however never accomplished with the correct form of path integral except in a few references. We retake this computations in order to pave way for further study in this direction.

The work presented in this thesis is based on the following publications

1. Discrete anomalies in supergravity and consistency of string backgrounds: R Minasian, S SASMAL, R Savelli, arxiv: 1611.09575 [hep-th], Published in JHEP 1702 (2017) 025, DOI: 10.1007/JHEP02(2017)025.

2. One loop amplitude for Heterotic string on $T^2$: S SASMAL, arxiv: 1611.09808 [hep-th].

# Chapter 2

# Supersymmetry and Supergravity

We shall begin our discussion with a very compact review of supersymmetry and supergravity theories in dimensions greater than 4. The reason behind this is that we shall show in certain supergravity theories, there are quantum anomalies and that the counter-term to be included in the supergravity effective action to curb those anomalies are in fact provided by string theory loop amplitudes. Therefore to prepare the ground we give a very precise introduction to supersymmetry and supergravity theories in 11 and 10 space-time dimensions to be able to make comparison with string theory and M-theory later. For more details, we refer to the standard texts of Wess & Bagger [1] and West [2]. Here we shall outline the importance of supersymmetry algebra as being an extension to that of the Poincaré algebra and the promotion of supersymmetry to a local symmetry to give rise to the supersymmetric quantum gravity that is supergravity. We then provide the low energy effective actions to 11 and 10 dimensional supergravity theories. We then discuss the coset space structure of the scalar fields living in supergravity theories and the possibility of gauging some coset degrees of freedom. This in turn shall be of immense importance for the discussion of chapter 4, 5 and 6 where the gauging of Abelian factors in the denominator of the coset shall be responsible for rendering the theory anomalous. We shall thus also give a brief introduction to anomalies in quantum field theory and the way to calculate them from the perspective of index theorems and characteristic classes.

## 2.1 Supersymmetry

Our discussion in course of this thesis shall be centered on supersymmetric theories where the bosonic degrees of freedom are related to the fermionic degrees of freedom with aid of operators whose algebra extends that of the Poincaré algebra. The importance of supersymmetry in standard model phenomenology is of course colossal, however we are studying the theories with supersymmetry because it poses very stringent constraints on QFT and in particular in string theory which makes the analysis simpler. Let us introduce the idea of supersymmetry as an extention of Poincaré invariance. Consider any quantum field theory (on a flat Minkowski metric background) with the space-time Lorentz invariance described by the Poincaré algebra

$$\begin{aligned}
&[P_\mu, P_\nu] = 0, \\
&[P_\mu, M_{\rho\lambda}] = \eta_{\mu\lambda} P_\rho - \eta_{\mu\rho} P_\lambda, \\
&[M_{\mu\nu}, M_{\rho\lambda}] = \eta_{\mu\lambda} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\lambda} - \eta_{\mu\rho} M_{\nu\lambda} - \eta_{\nu\lambda} M_{\mu\rho}.
\end{aligned} \tag{2.1}$$

There can be internal symmetries in the theory for example the symmetry under a group in gauge field theories whose generators follow the Lie algebra

$$[T_a, T_b] = f^c_{ab} T_c \tag{2.2}$$

where $f_{ab}^c$ are the so-called structure constants of the Lie algebra of the symmetry group. The generators $T_a$ commute with the generators $P_\mu$ and $M_{\mu\nu}$ of the Poincaré algebra. This is due to a no-go theorem proposed by Coleman & Mandula [3] stating that for the QFT in question having a non-trivial S-matrix where all continuous symmetries are described by Lie algebras, the space-time and internal symmetries of S-matrix do not mix

$$[P_\mu, T_a] = 0, \quad [M_{\mu\nu}, T_a] = 0. \tag{2.3}$$

To evade the Coleman-Mandula theorem and the restriction it poses on the symmetries of the theory, we may relax the assumption that these symmetries are described by ordinary Lie algebras and allow for graded Lie algebras or super-algebras. This is most readily done by introducing a set of fermionic generators $Q_\alpha$, $\bar{Q}_\alpha$ satisfying the anti-commutation relation

$$\{Q_\alpha, \bar{Q}_\beta\} = 2P_\mu \Gamma^\mu_{\alpha\beta} \tag{2.4}$$

with $\alpha, \beta$ the spinor indices. These generators, called the supercharges mix with the Poincaré generators and internal symmetry generators according to

$$\begin{aligned} [M_{\mu\nu}, Q_\alpha] &= (\Gamma_{\mu\nu})_\alpha^\beta Q_\beta, \\ [P_\mu, Q_\alpha] &= 0. \end{aligned} \tag{2.5}$$

Thus the bosonic and fermionic fields transform into each other by supersymmetry transformation, in infinitesimal schematic form

$$\delta\phi \sim \bar{\epsilon}\psi, \quad \delta\psi \sim \Gamma^\mu \epsilon \partial_\mu \phi \tag{2.6}$$

where $\epsilon$ is the fermionic supersymmetry transformation parameter. Some theories have multiple supersymmetries with generators $Q^I_\alpha$ parametrized by an index $I = 1, \cdots, N$ where N is the number of supersymmetries of the theory. In this case the supersymmetry algebra generalizes and the algebra with the internal symmetry is also non-trivial

$$\{Q^I_\alpha, \bar{Q}^J_\beta\} = 2\delta^{IJ} P_\mu \Gamma^\mu_{\alpha\beta} + Z^{IJ}\delta_{\alpha\beta}, \tag{2.7}$$

$$[T_a, Q^I_\alpha] = (t^I)^A_B Q^B_\alpha. \tag{2.8}$$

The first line in above is the extension of the supersymmetry algebra with the term $Z^{IJ}$ called the central charge commuting with all other generators. For the moment we shall neglect this factor and shall discuss the emergences of central charges in 10D supersymmetry multiplets and RR-forms in string excitations. The second equation (2.8) denotes the representation of the internal symmetry in terms of supercharges. This symmetry of the supercharges is called the R-symmetry. Supersymmetry regroups bosonic and fermionic degrees of freedom according to helicity states as they are created from the vacuum. In the next section, we shall demonstrate the massless representation of supersymmetry algebras in diverse dimensions and the related supermultiplets.

## 2.2 Supersymetry algebra in various dimensions

The construction of massless representation of the supersymmetry algebra in D-space-time can be accomplished with the following algorithm. First note that in D space-time dimensions, the Dirac-spinors contain $\triangle = 2^{\lfloor D/2 \rfloor}$ complex components.

The Weyl, Majorana or both conditions can be imposed on the Dirac-fermions reducing the effective number of spinor components (by half, half and one-quarter respectively). For the massless representation of interest, one passes to the light-like frame where for the momentum states look like $P_\mu = (-E, E, 0, \cdots, 0)$ and thus the little group is SO(D-2). Hence the massless representation of the supersymmetry should be built up from the representation of SO(D-2). To this end, the fermionic oscillators can be grouped into $\triangle$ pairs of annihilation and creation operators $(b^I, b^{\dagger I})$, $I = 1, \cdots, \triangle$ and the vacuum $|0>$ is such that it is annihilated by all the annihilation operators $b^I|0>= 0$. Thus applying the creation operators to the vacuum we can create the representation of the Clifford algebra forming a spinor of $SO(2\triangle)$ with $2^{\triangle}$ components . One then embeds the irreps of the little group SO(D-2) into $SO(2\triangle)$ to determine the massless representation. It may be noted that the representation so constructed should not have helicity states with helicity greater than 2 as otherwise the theory cannot be coupled consistently with gravity. This restricts the maximal dimensions of supersymmetric and Lorentz invariant theory to be $D \leq 11$ so that the maximal number of supercharges in any dimensions is 32. We are in fact assuming in advance the presence of graviton states in the massless representation of supersymmetry. In fact, in the next section, we shall argue that supersymmetry, which is a global symmetry of the QFT concerned can be promoted to a local symmetry and then it couples automatically to the gravity states present in the massless representation of the supersymmetry algebra. The resulting theory is known as supergravity. Before discussing the supergravity, we construct the massless representations of supersymmetry in 11 and 10 dimensions.

**Supersymmetry in D=11**

The spinor representation in 11D has 32 complex non-zero components. However, in dimensions $D = 1, 3$ mod 8, the Majorana condition can be imposed thereby giving only 16 complex non-zero components i.e. $2\triangle = 16$ or $\triangle = 8$. The Clifford algebra has thus $2^8 = 256$ states arranged in a Dirac spinor **256** of SO(16) which can be decomposed into two irreps $\mathbf{128}_1$ and $\mathbf{128}_2$. Embedding the little group SO(9) into SO(16), we thus get from the state $\mathbf{128}_{\ 1} \rightarrow$ **44 + 84** so that we get the graviton state $g_{\mu\nu}$ as the state **44** and an anti-symmetric 3-form $C_{\mu\nu\rho}$ as the state **84**. The rest $\mathbf{128}_2$ state is the spin-(3/2) gravitino state $\psi_\mu$. Thus we get the gravity multiplet in 11D $g_{\mu\nu}$, $C_{\mu\nu\rho}$ and $\psi_\mu$.

**Supersymmetry in D=10**

In dimensions $D = 2$ mod 8, the fermions are Majorana-Weyl and thus a spinor, for example in 10D has 16 real components. The condition of 32 maximal supercharges thus allow the maximal supersymmetry N=2 in 10D with 32 supercharges and the minimal supersymmetry N=1 with 16 supercharges.

Let us first construct the supermultiplets in the minimal N=1 case. The little group is SO(8) and the fermion from the $2^{\triangle}$ with $\triangle = 4$ transforms as a spinor **16** of SO(8)=$SO(2\triangle)$. Thus embedding the little group irreps $\mathbf{8}_V$ and $\mathbf{8}_+$ according to $\mathbf{16} \rightarrow \mathbf{8}_V + \mathbf{8}_+$ we see that we can construct a vector multiplet containing gauge boson $\mathbf{8}_V$ and positive chirality gaugini $\mathbf{8}_+$. Next consider the multiplet obtained from the tensor product $\mathbf{8}_V \times (\mathbf{8}_V + \mathbf{8}_+) = \mathbf{35} + \mathbf{28} + \mathbf{1} + \mathbf{56}_+ + \mathbf{8}_-$ containing graviton (**35**), anti-symmetric 2-form (**28**), 1 scalar (**1**), positive chirality gravitino $\mathbf{56}_+$ and a fermion $\mathbf{8}_-$ of negative chirality. This multiplet, because it contains spin-2 graviton, is called the gravity multiplet.

For N=2, we now have two spinors in **16** of the spinor automorphic group $SO(2\triangle) = SO(8)$. We can however embed the Majorana-Weyl **8** spinors of the little group SO(8) in two ways : we can have $\mathbf{16} \rightarrow \mathbf{8}_V + \mathbf{8}_+$ for both the **16** such that the irreps of SO(8) spinor states $\mathbf{8}_+$ in both of N=2 are of same chirality or have $\mathbf{16} \to \mathbf{8}_V + \mathbf{8}_+$ for one and $\mathbf{16} \to \mathbf{8}_V + \mathbf{8}_-$ for the other supercharge. The former case is said to be type IIB whereas the latter is called the type IIA.

The spectrum of type IIB is $(\mathbf{8}_V + \mathbf{8}_+) \times (\mathbf{8}_V + \mathbf{8}_+) = 1$ graviton $g_{\mu\nu}$ in **35**+ 2-form $B_{\mu\nu}$ in **28** + scalar $\phi$ in **1**+ two gravitini $\psi_i^\mu$ in $2 \times \mathbf{56}_+$ + in two fermions $\lambda_i$ $2 \times \mathbf{8}_-$ + another scalar $C_0$ in **1** + another two-form $C_2$ in **28** + 4-form $C_4$ in **35**.

The spectrum of type IIA is $(\mathbf{8}_V + \mathbf{8}_+) \times (\mathbf{8}_V + \mathbf{8}_-) = 1$ graviton $g_{\mu\nu}$ in **35**+ 2-form $B_{\mu\nu}$ in **28** + scalar $\phi$ in **1**+ two gravitini $\psi^\mu$ and $\bar{\psi}^\mu$ in $\mathbf{56}_+$ +$\mathbf{56}_-$+ two fermions $\lambda$ and $\bar{\lambda}$ in $\mathbf{8}_+$ + $\mathbf{8}_-$ + 1-form $C_1$ in $\mathbf{8}_V$ + 3-form $C_3$ in **56**.
We shall discuss more about these spectrums in the next section in the context of supergravity.

## 2.3 Supergravity

The end of the discussion in the previous section might seem like adding cart before horse as we have described the supergravity multiplets before describing supergravity. The reason is that in localizing the supersymmetry thereby promoting the full super-Poincaré symmetry to a local one necessarily leads to a theory of quantum gravity and the connection one-form to construct the covariant derivative for the gauged theory will now be played by gravitino. Let us make this statement more clear.

A general super-Poincaré transformation is of the form

$$\delta_\lambda \phi = i\lambda\phi \tag{2.9}$$

where $\phi$ is any bosonic field in the theory and the infinitesimal transformation parameter $\lambda$ is of the form

$$\lambda = \frac{1}{2}\lambda^{ab}M_{ab} + \xi^a P_a + \bar{\epsilon}_\alpha Q^\alpha \tag{2.10}$$

in terms of the super-Poincaré generators (2.1) and (2.4) ($\epsilon$ being the infinitesimal fermionic constant parameter). Note that we are using latin indices for the Poincaré generators to emphasize that they are evaluated in a flat-background whereas the greek indices shall denote the curved background which incorporates the graviton in the metric as quantum fluctuations. Thus in short we are implicitly using the vielbein formalism where the curved metric in terms of flat metric is

$$g_{\mu\nu}(x) = e_\mu^a(x) e_\nu^b(x) \eta_{ab} \tag{2.11}$$

with $e_\mu^a$ being the vielbein. In this formalism the curved space covariant derivative acting on spinors $\psi$ (in case of ordinary general relativity)

$$\mathcal{D}_\mu \psi = \partial_\mu \psi - i\frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\psi \tag{2.12}$$

with $\omega_\mu^{ab}$ being the so called spin connection and $\Gamma_{ab}$ being the spinor representation of the Lorentz generators $M_{ab}$. The vanishing of the torsion in general relativity dictates the relation

$$de_\mu^a + \omega_\mu^{ab} \wedge e_\mu^a = 0 \tag{2.13}$$

between spin connection and vielbein thereby fixing spin connection in terms of the latter. Going back to (2.9), in case one promotes a global symmetry to a local one, one introduces a gauge field or a connection 1-form $A_\mu$ such that it transforms according to

$$\delta_\lambda A_\mu = \partial_\mu \lambda - i[A_\mu, \lambda] \tag{2.14}$$

and hence the covariant derivative of $\phi$ defined as

$$\mathcal{D}_\mu \phi = (\partial_\mu - iA_\mu)\phi \tag{2.15}$$

transforms in the same way as $\phi$. For the super-Poincaré symmetry at hand the localization needs the connection one-form

$$A_\mu = \frac{1}{2}\omega_\mu^{ab} M_{ab} + e_\mu^a P_a + \bar{\psi}_{\mu\alpha} Q^\alpha \tag{2.16}$$

where along with the familiar spin connection for Lorentz transformations, vielbeins for translations we have to use the gravitino field $\psi_{\mu\alpha}$ for the supersymmetry transformations. For example the supersymmetry transformation of the gravitino in the local gauge shall now be (using (2.14))

$$\delta_\lambda \psi_\mu = \partial_\mu \epsilon - i\frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\epsilon + i\frac{1}{4}\lambda^{ab}\Gamma_{ab}\psi_\mu \tag{2.17}$$

where now all the parameters $\epsilon$, $\lambda^{ab}$ and $\xi^a$ are functions of space-time coordinates x. In case there is a gauge symmetry in the theory, its connection also appears in the supersymmetry variation of the fermions like in (2.17) giving the charges of the fermions under such couplings. One can then define the super-Poincaré covariant derivative

$$\mathcal{D}_\mu \phi = \partial_\mu \phi - i\frac{1}{2}\omega_\mu^{ab} M_{ab}\phi - ie_\mu^a P_a \phi + \bar{\psi}_\mu^\alpha Q_\alpha \phi \tag{2.18}$$

from which one can compute the curvature

$$R = dA - iA \wedge A \tag{2.19}$$

to derive the Riemann tensor $R^{\mu\nu}_{\alpha\beta}(e, \omega, \epsilon)$ which depends upon all the connection components. Conditions, similar to those of the vanishing torsion, however more complicated in case with fermionic parameters, are to be imposed which make spin connection to be determined completely by vielbein and gravitino i.e. $\omega \equiv \omega(e, \psi)$. Imposing this into the Riemann curvature brings out, from the second order formalism of gravity (where one starts typically with an action $S = \int trR^2$ and then uses the torsion constraints to reduce it to the familiar Einstein-Hilbert action $S = \int R$), one gets the Einstein-Rarita-Schwinger action

$$S = \frac{1}{2\kappa^2}\int d^d x (R - \frac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho}\mathcal{D}_\nu \psi_\rho). \tag{2.20}$$

With these preliminaries, we are now ready to address the the supergravity effective actions in 11 and 10 dimensions.

Let us first write the low energy effective action for the 11D supergravity containing graviton $g_{\mu\nu}$, gravitino $\psi_\mu$ and a 3-form $C_{\mu\nu\rho}$. The effective action is [4, 5]

$$\begin{aligned} S_{11} = \quad & \frac{1}{2\kappa_{11}^2}\int \left[\mathcal{R} *_{11} 1 - \frac{1}{2} G \wedge *_{11} G - \frac{1}{6} C_3 \wedge G_4 \wedge G_4\right] \\ & - \frac{2\pi}{(4\pi\kappa_{11}^2)^{1/3}}\int C_3 \wedge X_8. \end{aligned} \tag{2.21}$$

with $G_4 = dC_3$, $\kappa_{11}^2$ is the 11-D gravitational coupling constant and the eight form (related to M5 brane anomaly [6, 7], (see M-theory in 3.7) $X_8^- = \frac{1}{192(2\pi)^4}\left[TrR^4 - \frac{1}{4}(TrR^2)^2\right]$.

In 10D, the type IIA supergravity multiplet contains the graviton $g_{\mu\nu}$, anti-symmetric 2-form $B_{\mu\nu}$, a scalar called dilaton $\phi$, a 1-form $C_\mu$, a 3-form $C_{\mu\nu\rho}$, two Majorana-Weyl gravitini $\psi_\mu^i$ (i=1, 2) of opposite chirality and two Majorana-Weyl fermions (dilatini) $\lambda^i$ of opposite chirality. The effective action of this theory is [9]

$$\begin{aligned} S_{\text{IIA}} = &\frac{1}{2\kappa_{10}^2}\int \{\mathcal{R} - \frac{1}{2}(\nabla\phi)^2\} *_{10} 1 \\ &-\frac{1}{2\kappa_{10}^2}\int \frac{1}{12}e^{-\phi}H_3\wedge *_{10}H_3 + \frac{1}{2}e^{\phi/2}F_4\wedge *_{10}F_4 + \frac{1}{4}e^{3\phi/2}F_2\wedge *_{10}F_2 \\ &+\frac{1}{4\kappa_{10}^2}\int B_2\wedge dC_3\wedge dC_3 \end{aligned} \tag{2.22}$$

where

$$F_2 = dC_1, \;\; H_3 = dB_2, \;\; F_4 = dC_3 - C_1\wedge H_3. \tag{2.23}$$

The action (2.22) can be seen as a circle compactification of the eleven-dimensional supergravity action (2.21) and this relation will be useful to define the relation between type IIA string theory and M-theory subsequently.

The type IIB supergravity multiplet contains graviton $g_{\mu\nu}$, anti-symmetric 2-form $B_{\mu\nu}$, a scalar called dilaton $\phi$, another scalar $C_0$ called axion, another two-form $C_2$ and a 4-form $C_4$ plus two Majorana-Weyl gravitini $\psi_\mu^i$, (i=1, 2) of same chirality and two Majorana-Weyl dilatini $\lambda^i$ of same chirality but opposite to that of the gravitini. The effective action is

$$\begin{aligned} S_{\text{IIB}} = &\frac{1}{2\kappa_{10}^2}\int\{\mathcal{R} - \frac{1}{2}\frac{\partial_\mu\tau\partial^\mu\bar{\tau}}{\tau_2^2}\} *_{10} 1 - \frac{1}{12}G_3\wedge *_{10}\bar{G}_3 - \frac{1}{2}F_5\wedge *_{10}F_5 \\ &+\frac{1}{2i}\int C_4\wedge G_3\wedge\bar{G}_3, \end{aligned} \tag{2.24}$$

where the dilaton field $\phi$ and the axion $C_0$ makes up the complex axio-dilaton scalar $\tau = C_0 + ie^{-\phi}$ and

$$H_3 = dB_2, \;\; F_3 = dC_2, \;\; G_3 = i\frac{F_3 + \tau H_3}{\sqrt{\tau_2}}, \;\; F_5 = dC_4 + C_2\wedge H_3. \tag{2.25}$$

The action (2.24) is incomplete in the sense that the equations of motions derived from it does not provide the self-duality condition of the $F_5$ i.e.

$$F_5 = *F_5. \tag{2.26}$$

The condition (2.26) should be provided separately with the effective action or one can use somewhat different forms of the type IIB action, as discussed by Schwarz [8] and Howe et al. [9] which incorporates the above self-duality condition within it so as to provide it as an equation of motion. The usefulness of using the form (2.24) for the effective action is that it makes the global $SL(2,\mathbb{R})$ symmetry of the action manifest as proposed by Schwarz [10]. In fact, considering a matrix representative of $SL(2,\mathbb{R})$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \;\; ad - bc = 1, \;\; a, b, c, d \in \mathbb{R} \tag{2.27}$$

such that it acts on the axio-dilaton field as

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{2.28}$$

and that the 2-forms $B_2$ and $C_2$ transform as a doublet under $SL(2,\mathbb{R})$

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \tag{2.29}$$

whereas the metric, the 4-form field $C_4$ remain invariant, the action (2.24) is seen to be invariant under $SL(2,\mathbb{R})$ transformation. This symmetry, in string quantization level shall be discretized leading to the so-called S-duality of type IIB string theory and its eventual geometrical meaning in terms of F-theory in sections 3.6.3 and 3.8. One more important remark is that the presence of chiral fermions and self-dual forms apparently makes this theory anomalous (see section 2.5) however a direct computation of the anomaly polynomial shows that the fermionic anomaly is countered by that of the self-dual form and thus the theory is anomaly free. The moduli space of the theory, that is the manifold parametrized by the scalars $\phi$ and $C_0$ is of the form $\frac{SL(2,\mathbb{R})}{U(1)}$ and the composite connection due to the U(1) factor in the coset denominator seems to make the theory anomalous in a very particular manner. This point shall be the essence of this thesis and shall be discussed in details in chapters 4, 5.

Finally we come to the discussion of 10D N=1 supergravity theory. The supergravity multiplet contains a gravity multiplet with graviton $g_{\mu\nu}$, 2-form $B_{\mu\nu}$, dilaton $\phi$ and a Majorana-Weyl Gravitino $\psi_\mu$ and a Majorana-Weyl dilatino $\lambda$ which is by the way, of opposite chirality to that of the gravitino. There is however a vector multiplet too in the theory with gauge bosons $A_\mu$ and Majorana-Weyl gaugini $\chi$ of same chirality as of gravitino. The consistent coupling of vector and gravity multiplet leads to anomaly which can only be resolved by the Green-Schwarz mechanism (see section 5.1) as discussed by Green & Schwarz [11], if the gauge group of the theory is of dimensions 496 and should not have any independent 6th order Casimir invariant i.e. $TrF^6$ should be decomposable in terms of $TrF^4$ and $TrF^2$ (F is of course the gauge field-strength). The only gauge groups allowing for these two conditions are SO(32), $E_8 \times E_8$, $U(1)^{496}$ and $E_8 \times U(1)^{248}$. Of these, the first one describe the type I theory, the second and third choice denote respectively SO(32) and $E_8 \times E_8$ Heterotic theories and the last two options are possible only from the pure supergravity point of view and cannot be seen to have relations with consistent superstring theories. These two latter theories however are trivial in the sense that all Abelian gauge traces in the anomaly polynomial (see 2.61 of section 2.5) are zero and hence the effective anomaly is just for the gravity part only. Thus the groups $U(1)^{496}$ and $E_8 \times U(1)^{248}$ are not considered for any useful supergravity vacua. In fact in 10D there are 5 consistent supergravity vacua (type IIA, type IIB, Heterotic SO(32) and $E_8 \times E_8$ plus type I) which are low energy effective theories of 5 consistent string theories in 10D. Below we give the effective action for the D=10, N=1 supergravity theory to which we shall add loop contributions in course of this work

$$S_{\text{D=10, N=1}} = \frac{1}{2\kappa^2}\int \{\mathcal{R} - \frac{1}{2}(\nabla\phi)^2\} *_{10} 1 - \frac{1}{12}e^{-\phi}H_3 \wedge *_{10}H_3 - \frac{1}{2}e^{-\phi/2}tr(F_2 \wedge *_{10}F_2). \tag{2.30}$$

## 2.4 Composite connection in supergravity

Supersymmetry and thus also supergravity multiplets seem to accommodate certain types of scalar fields (called the moduli by some abuse of the word), which are required to parametrize certain types of coset manifolds of the form $G/H$ called the moduli space of the theory. The allowed forms of G and H depend on the particular supersymmetry algebra. It will be clear after our exposition to string theory and duality therein that the numerator group $G$ is the so-called U-duality group whose discrete version gives rise to an exact symmetry after (string)quantization whereas the denominator is regarded as the gauge symmetry of the theory. In particular, the supersymmetry variations of all fermionic fields, inert under G, involve the gauge composite connection corresponding to H. To make this point clear, consider a given group G and a subgroup $H \subset G$. The right coset $G/H$ is defined as the set of equivalence classes of elements of G under the right action of H that is the set of points $g \in G$ modulo the identification $g \cong gh$ for all $h \in H$. Now let the group manifold of the coset $G/H$ be of real dimensions n(=dimG-dimH) and let $\varphi^\alpha$, $\alpha = 1, \cdots, n$ be n coordinates parametrizing the coset manifold: these are in fact the real scalars in the supersymmetry theory concerned. From these scalars, one can form a G-valued matrix L with rigid symmetry under left multiplication with elements $g \in G$

$$L \to g^{-1}L. \tag{2.31}$$

L is called the coset representative which has the local symmetry under right multiplications with elements $h \in H$

$$L \to Lh(\varphi) \tag{2.32}$$

where the elements $h$ are dependent on the coordinates $\varphi$ (we are using the conventions of Salam & Strathdee [12]). In this construction, the choice of L is not unique and to isolate the physical degrees of freedom we can choose L as a particular function of $\varphi$ thereby fixing the gauge. The rigid symmetry of L (2.31) does not preserve this gauge choice and thus the complete transformation of L under G becomes

$$L(\varphi) \to g^{-1}L(\varphi)h(\varphi, g) \tag{2.33}$$

with $h(\varphi, g)$ being an element of H selected so that the transformation retains the functional form of L.
To bring forward the notion of composite connection, consider the G-valued left invariant Maurer-Cartan form constructed from the coset representative L

$$L^{-1}dL = A^i H_i + V^a K_a. \tag{2.34}$$

The G-valued one form $A = A^i_\alpha H_i dx^\alpha$ is the connection one-form parametrizing the tangent space rotations of the representative with respect to H while $V = V^i_\alpha H_i dx^\alpha$ defines an orthonormal frame on the coset space and is identified with the coset vielbein. Note also that $H_i$ are the generators of the Lie algebra of the subgroup H while $K_a$ are those for the group $G/H$. The following transformation of the Maurer-Cartan form corroborates to these identifications

$$L^{-1}\partial_\alpha L \to h^{-1}(A_\alpha + \partial_\alpha)h + h^{-1}V_\alpha h. \tag{2.35}$$

The Maurer-Cartan form is very helpful in constructing the kinetic term of the scalars $\varphi^\alpha$ in the supergravity effective action. The scalar fields $\varphi^\alpha(x)$ define a map from the space-time manifold with coordinates $x^\mu$ to the coset manifold. Thus the $\varphi$

dependent transformations $h(\varphi)$ are associated with x dependent transformations parametrized by $h(x)$. We can thus pull-back the one-form A and V to the space-time manifold via

$$Q_\mu = A_\alpha \partial_\mu \varphi^\alpha, \quad P_\mu = V_\alpha \partial_\mu \varphi^\alpha \tag{2.36}$$

and write the pull-back of the Maurer-Cartan form (2.34)

$$L^{-1}\partial_\mu L = Q_\mu + P_\mu . \tag{2.37}$$

In (2.37) $Q_\mu$ is the antisymmetric part of $L^{-1}\partial_\mu L$ and $P_\mu$ is its symmetric part. From these connections, one can construct the following Lagrangian density for the scalars $\varphi$ which is invariant under rigid G transformation as well as under local H transformation

$$\mathcal{L} = \frac{1}{2} tr(P_\mu P^\mu) \tag{2.38}$$

which describes the non-linear sigma model Lagrangian in supergravity effective action.

We illustrate the above with the example of $SL(2,\mathbb{R})/SO(2)$ coset space which occur in 10 and 8 dimensional supergravity theories. The coset space is parametrized by 2 real scalars $U_1$ and $U_2$ which can be combined into a complex scalar $U = U_1 + iU_2$. The coset representative L is in matrix form (following the results from Gilmore [13])

$$L = \frac{1}{\sqrt{U_2}} \begin{pmatrix} U_2 & -U_1 \\ 0 & 1 \end{pmatrix} \tag{2.39}$$

with the inverse

$$L^{-1} = \frac{1}{\sqrt{U_2}} \begin{pmatrix} 1 & U_1 \\ 0 & U_2 \end{pmatrix} . \tag{2.40}$$

From this we construct the Maurer-Cartan form

$$L^{-1}dL = -\frac{1}{2U_2} \begin{pmatrix} 0 & dU_1 \\ -dU_1 & 0 \end{pmatrix} - \frac{1}{2U_2} \begin{pmatrix} -dU_2 & dU_1 \\ dU_1 & dU_2 \end{pmatrix} \tag{2.41}$$

from which we read the composite U(1) connection

$$Q_\mu = -\frac{\partial_\mu U_1}{2U_2} \tag{2.42}$$

and the vielbein

$$P_\mu = -\frac{1}{2U_2} \begin{pmatrix} -\partial_\mu U_2 & \partial_\mu U_1 \\ \partial_\mu U_1 & \partial_\mu U_2 \end{pmatrix} \tag{2.43}$$

which, in turn, gives the following scalar kinetic term in the Lagrangian

$$\mathcal{L} = \frac{\partial_\mu U \partial^\mu \bar{U}}{4U_2^2} . \tag{2.44}$$

We shall see in course of this work that in case when H contains U(1) factors, the fermions of the theory are coupled with the U(1) composite connection (2.42) giving rise to chiral anomalies as discussed by Marcus [14]. We shall describe the anomaly in QFT in the next section.

## 2.5 Anomaly in supergravity

Anomaly in quantum field theory is the loss of classical symmetry in the quantized theory. Anomalies in QFT are seen to arise from the 1-loop regularization. Consider for example a 1-loop diagram with chiral states flowing inside the loop as loop momentum states. The UV regularization (e.g. Pauli-Villars) shall interfere with the gauge symmetry of the chiral states and shall give rise to current non-conservation when the effect of such 1-loop amplitude is taken as correction to the effective action. Another easier way to realize the anomaly is to see the path-integral formulation of a QFT which possess at the classical level certain local symmetry. In the path-integral quantization, the classical action still retains the symmetry but the path integral measure is not invariant under the classical symmetry: this gives rise to the phase variation of the path-integral which is in turn identified as the anomalous phase variation or anomaly for short.
Let us demonstrate this in mathematical terms (our analysis closely follows the steps of the standard references on anomaly in QFT as in Peskin & Schroeder [15], Alvarez-Gaumé & Witten [16], Alvarez-Gaumé & Ginsparg [17] and Bilal [18]). Consider a theory with a set of fields $\phi^r$ which has at classical level the local symmetry under the transformation

$$\phi^r \to \phi'^r = \phi^r(x) + \delta\phi^r(x) = \phi^r(x) + \epsilon^\alpha(x) F^r_\alpha(x, \phi(r)) \tag{2.45}$$

such that the classical action be invariant

$$S[\phi^r + \epsilon^\alpha(x) F^r_\alpha] = S[\phi^r]. \tag{2.46}$$

The integration measure in path-integral formulation is not invariant and acquires a phase as

$$\mathcal{D}[\phi^r] \to \mathcal{D}[\phi^r + \epsilon^\alpha(x) F^r_\alpha] = \mathcal{D}[\phi^r] e^{i\epsilon^\alpha \int \mathcal{A}_\alpha}. \tag{2.47}$$

It will be helpful to see the relation of the anomalous phase variation $\mathcal{A}_\alpha$ in terms of the quantum effective action (the generator of the 1-PI diagrams in field theory) $\Gamma[\phi]$ to reproduce the so-called anomalous Taylor-Slavnov identities. First consider the generating functional of connected diagrams $W[J]$ which can be defined by

$$e^{iW[J]} = \int \mathcal{D}[\phi^r] e^{iS[\phi^r] + i\int J_r(x)\phi^r(x)}. \tag{2.48}$$

Under the transformation (2.45) the variation of (2.48) is given by

$$\begin{aligned} e^{iW[J]} &= \int \mathcal{D}[\phi^r] e^{iS[\phi^r] + i\int d^dx J_r(x)\phi^r(x)} \to \int \mathcal{D}[\phi'^r] e^{iS[\phi'^r] + i\int d^dx J_r(x)\phi'^r(x)} \\ &= \int \mathcal{D}[\phi^r] e^{iS[\phi^r] + i\int d^dx J_r(x)\phi^r(x)} \left[1 + i\epsilon^\alpha \int d^dx \left(\mathcal{A}_\alpha(x) + J_r F^r_\alpha(x, \phi)\right)\right]. \end{aligned} \tag{2.49}$$

The modified current conservation law then reads as

$$\epsilon^\alpha \int d^dx \left(\mathcal{A}_\alpha(x) + < F^r_\alpha(x, \phi) >_J\right) = 0. \tag{2.50}$$

Next, note that with current $J_r$ the 1PI effective action $\Gamma[\phi]$ is the Legendre transform of the connected Green's function $W[J]$ according to

$$\Gamma[\phi] = W[J] - \int d^d x \phi^r J_r(x) \tag{2.51}$$

so that

$$J_r(x) = -\frac{\delta\Gamma[\phi]}{\delta\phi^r} \tag{2.52}$$

thus the Taylor-Slavnov identity (2.50) becomes

$$\int d^d x \epsilon^\alpha \mathcal{A}_\alpha(x) \equiv \delta_A \Gamma[\phi] = \int d^d x \epsilon^\alpha F^r_\alpha(x) \frac{\delta\Gamma[\phi]}{\delta\phi^r} = \int d^d x \delta\phi^r \frac{\delta\Gamma[\phi]}{\delta\phi^r} \tag{2.53}$$

where in the last equation we have used (2.45). This states that in the presence of anomalies, the anomalous phase variation of the path-integral measure is same as the variation of the effective action.
For a theory with chiral matter coupled to some gauge field $A^\alpha_\mu$ with the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^\alpha_{\mu\nu} F^{\alpha\mu\nu} + \mathcal{L}_{\text{matter}} \tag{2.54}$$

where $F = dA + A \wedge A$ is the (non-Abelian or Abelian whatever it may) gauge field strength and the matter Lagrangian

$$\mathcal{L}_{\text{matter}} = \bar{\psi} \not{D} \psi = \bar{\psi}(\not{d} + A)\psi \tag{2.55}$$

is such that the classical matter current reads

$$J^{\alpha\mu} = \frac{\partial \mathcal{L}_{\text{matter}}}{\partial A^\alpha_\mu}. \tag{2.56}$$

(Note that $D$ is the covariant derivative in the anti-Hermitian formalism which we shall explain shortly.) Instead of the 1PI effective action $\Gamma[\phi]$, one can use the current effective action $\tilde{\Gamma}[A]$ where the external legs in the interaction Feynman diagrams are composed of matter currents instead of particle states. The definition of $\tilde{\Gamma}[A]$ is given by

$$e^{\tilde{\Gamma}[A]} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i \int d^d x \mathcal{L}_{\text{matter}}}. \tag{2.57}$$

Using the steps of derivation of Taylor-Slavnov identity (2.53) for $\Gamma[\psi]$ now for the current effective action $\tilde{\Gamma}[A]$ we get

$$\delta_A \tilde{\Gamma}[A] = \int d^d x \epsilon^\alpha D_\mu \left( \frac{\delta\tilde{\Gamma}[A]}{\delta A^\alpha_\mu} \right) = \int d^d x \epsilon^\alpha D_\mu \left( J^\mu_\alpha \right) = \int d^d x \epsilon^\alpha \mathcal{A}_\alpha(x). \tag{2.58}$$

Thus (2.58) shows that the non-conservation of matter current due to correction terms from interactions is proportional to the anomaly as defined in (2.47).
In case of chiral fermions, the Atiyah-Patodi-Singer index theorem (or in a different guise, the Witten index theorem) relates the covariant derivative of matter current $\int D_\mu \left( J^\mu_\alpha \right)$ in (2.58) to the Dirac index $ind(\not{D})$ and hence the anomalous phase variation $\mathcal{A}$ too is related to the latter [19, 20, 21, 22, 23, 24, 25]. To be precise, the so-called anomaly polynomial for a theory in (real) dimensions d=2r is a 2r+2 form polynomial

$I_{2r+2}$ whose descents $I_{2r+1}$ and $Q^1_{2r}$ are defined by

$$I_{2r+2} = dI_{2r+1}, \tag{2.59}$$
$$\delta I_{2r+1} = dQ^1_{2r}. \tag{2.60}$$

The APS index theorem then states [16, 17]

$$\mathcal{A} = -\int Q^1_{2r}. \tag{2.61}$$

Throughout this work, we shall be working in the Hermitian format for the gauge fields which is convenient in case of calculating and using chiral gauge charges. This formalism is different from that of the standard references of [16, 17] so that we list below our convention and that of Alvarez-Gaumé & Ginsparg [16] and the conversions between these two conventions

**Table 1: Gauge theory dictionary**

| **Group quantities** | **Anti-hermitian convention of [16]** | **Hermitian** | **Relation** |
|---|---|---|---|
| Generators | $T_a$ | $t_a$ | $iT_a = t_a$ |
| Transformation of field $\phi$ in a rep. of G | $\delta_v\phi = -v\phi$ | $\delta_\epsilon\phi = i\epsilon\phi$ | $iv = \epsilon$; $v_a = \epsilon_a$ |
| Gauge connection | $A' = A_a T^a$ | $A = A_a t^a$ | $iA' = A$; $A_a = A_a$ |
| Gauge connection variation | $\delta A' = dv + [A', v]$ | $\delta A = d\epsilon - i[A, \epsilon]$ | |
| Gauge field-strength | $F' = F_a T^a$ | $F = F_a t^a$ | $iF' = F$; |
| Gauge field-strength | $F' = dA' + A' \wedge A'$ | $F = dA - iA \wedge A$ | |
| F variation | $\delta F' = [F', v]$ | $\delta F = -i[F, \epsilon]$ | |
| Covariant derivative | $D = d + A'$ | $D = d - iA$ | |

In any QFT it is the chiral fermions who participate in the anomalous couplings due to the index theorems. The (anti)self-dual forms being tensor products of chiral fermion states are also responsible for generating the anomaly. We give below the appropriate index formuli for the anomaly polynomials. We shall give a brief review of characteristic classes used below in appendix A.

1. **Spin-1/2 fermion anomaly polynomial:**

$$I_{1/2} = (2\pi) \times [\widehat{A}(\mathcal{M}_d)] \times [ch(-iF)]\,, \tag{2.62}$$

where

$$\widehat{A}(\mathcal{M}_d) = 1 + \frac{1}{12\,(4\pi)^2} trR^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{360} trR^4 + \frac{1}{288}(trR^2)^2\right] + \frac{1}{(4\pi)^6}\left[\frac{1}{5670} trR^6 + \frac{1}{4320} trR^4\, trR^2 + \frac{1}{10368}(trR^2)^3\right] + \dots \quad (2.63)$$

and

$$ch(-iF) = \sum_{k=0} \frac{1}{k!(2\pi)^k} TrF^k. \quad (2.64)$$

2. **Gravitino (Spin-3/2) anomaly polynomial:**

$$\begin{aligned} I^d_{3/2} &= (2\pi) \times [\widehat{A}(\mathcal{M})][Tr(e^{\frac{iR}{2\pi}}) - 1] \times [ch(-iF)] \\ &= (2\pi) \times [ch(-iF)] \\ &\times \Big[(d-1) + \frac{d-25}{12\,(4\pi)^2} trR^2 + \frac{1}{(4\pi)^4}\left(\frac{d+239}{360} trR^4 + \frac{d-49}{288}(trR^2)^2\right) \\ &+ \frac{1}{(4\pi)^6}\left(\frac{d-505}{5670} trR^6 + \frac{d+215}{4320} trR^4\, trR^2 + \frac{d-73}{10368}(trR^2)^3\right) \\ &+ \dots\Big]. \end{aligned} \quad (2.65)$$

3. **Self-dual form:**

$$I_{\text{form}} = (2\pi) \times [\widehat{L}(\mathcal{M}_d)] \times [-\frac{x}{4}], \quad (2.66)$$

where

$$x = \begin{cases} 1 \text{ if the base fermions are Weyl or Majorana,} \\ 1/2 \text{ if the base fermions are Majorana-Weyl} \end{cases} \quad (2.67)$$

and

$$\widehat{L}(\mathcal{M}_d) = 1 - \frac{1}{6\,(2\pi)^2} trR^2 + \frac{1}{(2\pi)^4}\left(-\frac{7}{180} trR^4 + \frac{1}{72}(trR^2)^2\right) + \frac{1}{(2\pi)^6}\left(-\frac{31}{2835} trR^6 + \frac{7}{1080} trR^4\, trR^2 - \frac{1}{1296}(trR^2)^3\right) + \dots \quad (2.68)$$

We end our discussion with a few examples of Chern-Simons forms and descents forms:

1. $TrF = dQ_1$, $Q_1 = TrA$, $\delta Q_1 = Trd\Sigma(x)$, $Q^1_2 = Tr\Sigma(x)$.

2. $TrF^2 = dQ_3$, $Q_3 = Tr(A \wedge F - i\frac{1}{3}A^3)$, $\delta Q_3 = Trd\Sigma(x)(dA)$, $Q^1_4 = Tr\Sigma(x)(dA)$.

3. $TrF^3 = dQ_5$, $Q_5 = Tr(A \wedge F^2 - \frac{1}{2}A^3F + \frac{1}{10}A^5)$, $\delta Q_5 = Trd\Sigma(x)(dAdA - i\frac{1}{2}dA^3)$, $Q^1_6 = Tr\Sigma(x)(dAdA - i\frac{1}{2}dA^3)$.

In the next chapter we shall introduce the general features of string theory and the relation of supergravity vacua with string vacua.

# Chapter 3

# Perturbative and non-perturbative aspects of string theory

Despite the success of supergravity as a quantum field theory of gravity, the loop amplitudes are still crippled with UV divergences. Heuristically, the origin of such divergence is that in a local theory of point particles, interactions take place at definite spacetime events which effectively means that spacetime is probed to arbitrarily high resolution when virtual states are considered in the path integral that defines the quantized theory. A natural way of remedy is to consider the motion of an extended object rather than a particle and the simplest one being a string. The string dynamics were known to include a spin-2 mode corresponding to graviton and the corresponding theory of quantum gravity thus obtained is not only free from UV divergences but has only one parameter namely its length $l_s = \sqrt{\alpha'} = 1.22 \times 10^{19}$ GeV which makes it by far the most promising candidate for the unified theory of everything. The propagation of string in space-time has two distinct topologies: we can have closed strings or open strings, the world-sheet swept by the former being a cylinder while for the latter it is sheet with two boundaries. It is a generic feature of string theory that one of the oscillation modes of the closed string is associated to massless spin-2 particle that is graviton and that for an open string one finds a massless spin-1 particle which is interpreted as gauge boson. Gravitational and Yang-Mills interactions are thus unified thanks to the two different topologies of the string. Open strings can however always join to make closed strings so that closed strings and hence gravity must always be included in a sensible string theory. The quantum theory of relativistic strings is much more constrained than that of relativistic particles because symmetries determine the dynamics of the string completely. The absence of quantum anomaly and stability of vacuum single out five fundamental (super)string theories in 10 space-time dimensions. When non-perturbative arguments are taken into account, these five theories are seen to be limits of a single unifying theory, called the M-theory. The requirement of 10 spacetime dimensions for consistent string theories may seem unrealistic, but the argument to bring about the effective 4 spacetime dimensional physics is to compactify the 6 extra space-dimensions in a compact manifold, the volume of which is still beyond the scope of the penetrative resolution of high energy experimental set-ups. When compactified on compact manifolds, the seemingly different 5 string theories are seen to be related to each other. Such relations is called as duality in string theory. As we shall discuss further, there are dualities which relate non-perturbative aspects of a string theory to the perturbative aspect of another, thereby providing the tool to probe into the non-perturbative aspects of string theory. Much of this thesis shall rely upon the duality of Heterotic string theory compactified on a torus with that of F-theory compactified on a K3 surface, a Calabi-Yau 2-fold. In the rest of this chapter, we shall present and explain the fundamental notions of string theory and the non-perturbative theories like M-and F-theory to pave the way for the subsequent discussions. Our introduction will thus be minimalist and for the more mathematical details and subtle issues of string quantization, we shall refer to the standard textbooks like Green, Schwarz & Witten [26], Polchinski [27], Kiritsis [28].

## 3.1 Worldsheet perspective of (super)string

A covariant description of the string motion is furnished by the embedding of its world-sheet $\Sigma$ in spacetime. The latter is usually referred to as target space which is of d-spacetime dimensions with Minkowsky signature $(-\underbrace{++\cdots+}_{d-1})$. In a flat coordinate system $x^\mu$, $\mu = 0, 1, \cdots, d-1$, the world-sheet is described by a set of functions $X^\mu(\tau, \sigma)$ where $\tau, \sigma$ are respectively the time and space coordinates on $\Sigma$.

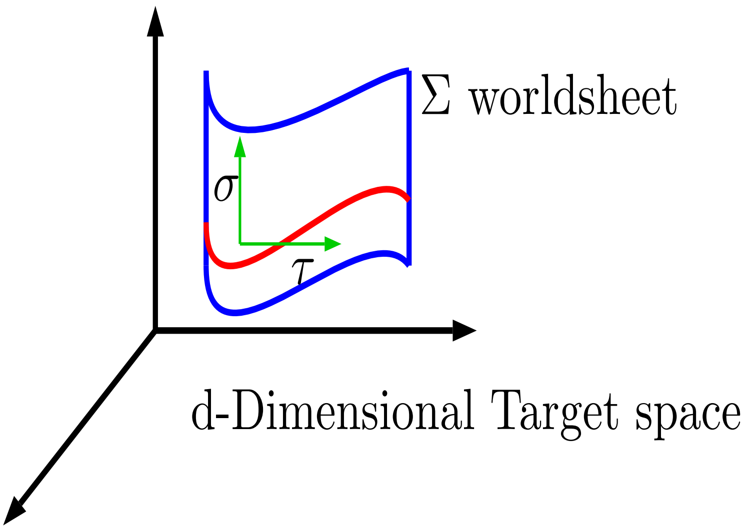


FIGURE 3.1: An impression of string propagation in target space

Generalizing the action of the relativistic point particle, one might write down the action of the relativistic string as proportional to the the volume of the world-sheet: the action thus obtained is called the Nambu-Goto action

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_\Sigma d\tau d\sigma \sqrt{\left(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \sigma}\right)^2 - \left(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \tau}\right)\left(\frac{\partial X^\mu}{\partial \sigma}\frac{\partial X_\mu}{\partial \sigma}\right)} \tag{3.1}$$

In the above, $T = \frac{1}{2\pi\alpha'}$ is the string tension. Although a classical description of string dynamics is easily furnished by the action (3.1), the difficulty lies in the quantization because (1) of the square root, (2) the Hamiltonian is zero so that kinematical constraints exclusively govern the dynamics. One thus seeks the classically equivalent Polyakov action

$$S_P = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-h} h^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \tag{3.2}$$

with $\eta_{\mu\nu}$ the Minkowski target-space-metric, $h_{\alpha\beta}$ is a Lorentzian metric on the world-sheet $\Sigma$ and $h = det(h_{\alpha\beta})$. At the classical level, the world-sheet metric $h_{\alpha\beta}$ enters algebraically and can be removed with aid of the equations of motion, thereby getting back the Nambu-Goto action (3.1). The Polyakov action (3.2) can be visualized as 2D quantum gravity coupled with d scalar fields $X^\mu(\tau,\sigma)$.

The Polyakov action is invariant under the following transformation and hence have the following symmetries (note that in the following the small case Greek indices from the beginning of the alphabet e.g. $\alpha$, $\beta$, $\gamma$, $\delta$ etc are used to denote the two world-sheet indices while the letters like $\mu$, $\nu$ etc are used for target space indices)
(1) Diffeomorphism symmetries: Under the reparametrization $\sigma^\alpha \to \sigma'^\alpha(\sigma)$ of the world-sheet coordinates $\sigma^\alpha = \tau, \sigma$ the metric $h_{\alpha\beta}$ transforms according to

$$h_{\alpha\beta}(\sigma) \to h'_{\alpha\beta}(\sigma') = \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\delta}{\partial\sigma'^\beta}h_{\gamma\delta}(\sigma) \tag{3.3}$$

or in the infinitesimal form where $\xi^\alpha = \sigma'^\alpha - \sigma^\alpha$

$$\delta h_{\alpha\beta} = \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha. \tag{3.4}$$

The change in the target space scalars $X^\mu$ are of the form

$$\delta X^\mu = \xi^\alpha\partial_\alpha X^\mu. \tag{3.5}$$

The group of such reparametrizations is called the diffeomorphism group $Diff(\Sigma)$.
(2) Weyl resaclaing invariance: Under the scale transformation of the metric $h_{\alpha\beta}$

$$h_{\alpha\beta}(\sigma) \to h'_{\alpha\beta}(\sigma) = e^{2\Lambda(\sigma)}h_{\alpha\beta}(\sigma), \tag{3.6}$$

or in the differential form

$$\delta h_{\alpha\beta} = 2\Lambda(\sigma)h_{\alpha\beta}. \tag{3.7}$$

The group of all such rescaling is called $Weyl(\Sigma)$.
(3) Poincaré invariance: Under the usual Poincaré transformation of the target-space-time scalars $X^\mu$

$$X^\mu \to X'^\mu = \Lambda^\mu_\nu X^\nu + X^\mu_0, \;\; \eta_{\mu\nu}\Lambda^\mu_\kappa\Lambda^\nu_\lambda = \eta_{\kappa\lambda}. \tag{3.8}$$

Among all of the above symmetries, Weyl symmetry is particular only to string dynamics and this is one reason why strings are chosen at first place to smear out the point-like interactions of quantum gravity theory. Weyl symmetry in string theory is precious for the following reason: the world-sheet metric $h_{\alpha\beta}$ has three independent degrees of freedom which is the same number of parameters for the diffeomorphism (2) plus Weyl invariance (1) so that one can gauge fix $h_{\alpha\beta}$ locally to the flat metric $\eta_{\alpha\beta}$; this is indeed needed because the metric $h_{\alpha\beta}$ is not physical as is demonstrated by the classical equivalence of the Polyakov and Nambu-Goto action. The absence of Weyl symmetry would then imply the presence of non-physical degrees of freedom in the theory and thus in the quantization of the Polyakov action, the lack of the Weyl symmetry induces an anomaly which can only be cancelled if the space-time dimension d be equal to 26 for pure bosonic strings.

Even after gauge fixing the world-sheet metric, one is left with unfixed symmetries due to diffeomorphisms and admits an infinite number of charges. This property allows one to use the powerful methods of conformal field theory (CFT) to compute in particular perturbative interactions of string theory.

So far we have discussed only the world-sheet perspective of bosonic strings living in 26 space-time dimensions. To include fermions in the target space (and also to get rid of the negative square-mass tachyonic excitation living in the spectrum of the quantized bosonic string), one considers the Polyakov action (3.2) this time with scalars $X^\mu$ along with two 2D Majorana-Weyl world sheet fermions $\psi^\mu, \bar{\psi}^\mu$ of opposite chirality coupled (in Weyl invariant way of course) to 2D superconformal gravity. The new action will be (with 2d-gravitino $\chi_a$ and 2d gamma matrices $\gamma^a$)

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \left[ h^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \frac{i}{2}h^{\alpha\beta}\eta_{\mu\nu}\bar{\psi}^\mu \not{\partial}\psi^\nu + \frac{i}{2}(\bar{\chi_a}\gamma^b\gamma^a\psi^\mu)(\partial_b X^{m-\frac{i}{4}\bar{\chi_b}\psi^\mu}) \right] \tag{3.9}$$

which possess the diffeomorphism and Lorentz invariance as in (3.4) and (3.8) as well as the supersymmetric generalization of the Weyl invariance (the super-Weyl transformation) plus the 2D N=(1,1) supersymmetry transformations which intertwines the world-sheet bosonic and fermionic degrees of freedom. Once more the theory allows for residual diffeomorphism invariance after necessary gauge fixing and it is described by $N = (1,1)$ super-conformal field theory (sCFT). The demand to have the Weyl symmetry in quantized theory necessitates the spacetime dimensions d=10.

The generic solution to the equations of motion arising from the gauge-fixed classical Polyakov action (i.e. $h_{\alpha\beta}$ fixed to flat $\eta_{\alpha\beta}$) for the closed string is of the form

$$X^\mu(\tau,\sigma) = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma),\ \psi^\mu(\tau,\sigma) = \psi^\mu(\tau+\sigma),\ \bar{\psi}^\mu(\tau,\sigma) = \bar{\psi}^\mu(\tau-\sigma) \tag{3.10}$$

with

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau,\sigma) \text{ as } \sigma + 2\pi \sim \sigma. \tag{3.11}$$

Thus the dynamics of the string can be analysed in terms of right-moving that is holomorphic on the variable $\sigma^- = (\tau - \sigma)$ and left-moving that is holomorphic on the variable $\sigma^+ = (\tau + \sigma)$ where the two sectors do not have any local interactions. The left-moving and right-moving sectors are related only via considerations about the global topology of the world-sheet. This shall in fact enables us to use N=(0,1) supersymmetry in 2D world-sheet theory where only the right moving sector is supersymmetric and the left-moving sector is bosonic. This is at the heart at the construction of Heterotic string theory which we shall consider in the next section. Consistent quantization of closed string determines the target-space metric $g_{\mu\nu}$ along with anti-symmetric 2-form $B_{\mu\nu}$ plus a scalar $\phi$ called the dilaton.

Let us end this section with a very brief discussion of open and unoriented strings. Open string dynamics can again be described by the Polyakov action (3.2), however, instead of identification (3.11) of $\sigma$ coordinate of closed strings, we need to impose two boundary conditions

$$\text{Neumann:} \quad \partial_\sigma X^\mu|_{\sigma=0,\pi} = 0, \tag{3.12}$$

$$\text{Dirichlet:} \quad \partial_\tau X^\mu|_{\sigma=0,\pi} = 0 \tag{3.13}$$

where we have $\sigma$ coordinate to vary from $0$ to $\pi$. The Neumann condition above is consistent with the string equations of motions derived from the Polyakov action as well as with the Poincaré symmetry of target-space however Dirichlet condition, though compatible with equations of motion, is not compatible with the Poincaré invariance. Thus in case of the latter, one needs to add extra objects at the ends of open strings to absorb the excess momentum: these are called the D-branes which we shall discuss in detail in section 3.4. One can even add non-dynamical degrees of freedom at the end of open strings, called the Chan-Paton factors, in complete consistency with world-sheet and target-space symmetries. Chan-Paton factors $i, j$, $i, j = 1, \cdots, N$ can be added to each end of open strings and the world-sheet thus seen to carry a U(N) gauge symmetry. Consistent quantization of open string dynamically determines a U(N) gauge field $A_\mu$ and the scalar dilaton $\phi$. In the fermionic sector, one gets the gaugini.

The other important ingredient in string theory is the unoriented string. Consider for example the closed string theory with the following world-sheet coordinate $(\tau, \sigma)$ transformation

$$\tau' \to \tau, \;\; \sigma' \to 2\pi - \sigma. \tag{3.14}$$

This transformation changes the orientation of the world-sheet and thus can be attributed to the action of an operator $\Omega$. Keeping then the massless states which are invariant under the action of $\Omega$ gives rise to string theory known as unoriented string theory. For example, an oriented closed string theory contains $g_{\mu\nu}$, $B_{\mu\nu}$ and $\phi$ in the massless excitation modes (to be discussed in the next section) whereas the unoriented closed string does not contain the antisymmetric $B_{\mu\nu}$ field due to its odd parity under $\Omega$ operation. Similarly, the unoriented open string sector contains no Abelian field $A_\mu$ whereas the oriented open strings do.

## 3.2 String theories in 10D

Let us now explore the vacua of consistent string theories in 10 spacetime dimensions. For the closed string the world-sheet is a cylinder and therefore we require the scalars $X^\mu$ to be periodic along the spatial direction $\sigma$ that is

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma). \tag{3.15}$$

Since any physical observable is quadratic in fermions, for $\psi^\mu$ we can choose either periodic or anti-periodic boundary conditions which are referred to as Ramond and Neveu-Schwarz sector respectively

$$\text{Ramond:} \qquad \psi^\mu(\tau, \sigma + 2\pi) = \psi^\mu(\tau, \sigma), \tag{3.16}$$

$$\text{Neveu-Schwarz:} \qquad \psi^\mu(\tau, \sigma + 2\pi) = -\psi^\mu(\tau, \sigma). \tag{3.17}$$

The fermions above are $\psi(\sigma^+)$ or $\bar{\psi}(\sigma^-)$ as we are discussing the 2D N=(1,1) supersymmetry. To preserve the Lorentz invariance, one is forced to choose the same periodicity for all values of $\mu$ but one is free to chose different periodicities for left- and right- movers. Thus for the case of N=(1,1) supersymmetry, we have four sectors (R,R), (R,NS), (NS,NS), (NS,R) while for the N=(0,1) case we have two sectors (NS,NS) and (R, NS).

There is one more constraint to be applied to the above sectors of string states: that is due to Gliozzi-Scherk-Olive or GSO projection in short. In principle, this is a grading according to fermion number operators $\mathcal{F}$ (the definition of $\mathcal{F}$ depends on the R and NS sectors accordingly) which grades the R and NS sector according to eigenstates of the operator $(-1)^{\mathcal{F}}$ which gives rise to four classes in each right and left moving sectors: $R_\pm$ times $NS_\pm$. The consistent grouping of left and right moving sectors should take place so that we get supersymmetric spectrum for the target space fields. Indeed spacetime supersymmetry is the rational behind the consistency of the GSO truncation[1]. The truncation requires the NS sectors in both left and right moving sector be $NS_+$ while one can choose left moving and right moving R sectors to be of opposite parity $R_+$ and $R_-$ giving rise to type IIA theory or both of the same parity e.g. $R_+$ giving the type IIB theory in 10D.

[1]From a more mathematical point of view, the GSO projection is dictated by the modular invariance of string partition function. We shall elaborate more on modular invariance of string partition function later.

We have already discussed the field content of type IIA and type IIB theories in the 10D supergravity discussion 2.2. We give it here once more for convenience

**Type IIA:** $(NS_+, NS_+) \rightarrow g_{\mu\nu},\ \ B_{\mu\nu},\ \ \phi$,
$(NS_+, R_-) \rightarrow \psi^\mu,\ \ \lambda$,
$(R_+, NS_+) \rightarrow \bar{\psi}^\mu,\ \ \bar{\lambda}$,
$(R_+, R_-) \rightarrow C_1,\ \ C_3$.

**Type IIB:** $(NS_+, NS_+) \rightarrow g_{\mu\nu},\ \ B_{\mu\nu},\ \ \phi$,
$(NS_+, R_+) \rightarrow \psi^\mu,\ \ \lambda$,
$(R_+, NS_+) \rightarrow \psi^\mu,\ \ \lambda$,
$(R_+, R_+) \rightarrow C_0,\ \ C_2,\ \ C_4$.

As we see, the (NS, NS) sectors are same in both the theories while from the (NS,R) and (R,NS) sector we get two opposite chirality gravitini $\psi^\mu$, $bar\psi^\mu$ and fermions $\lambda$, $\bar{\lambda}$ in type IIA while in type IIB the two gravitini are of same chirality as well as the two fermions (note that in 10D all fermions are considered in Majorana-Weyl basis).

Let us now discuss the Heterotic string theory. This is also a closed string theory with N=(0,1) world-sheet supersymmetry so that only the right moving ($\sigma^-$ holomorphic) sector is supersymmetric and lives in 10D target space while the left moving sector is purely bosonic and lives in 26 spacetime target space. In order to make a consistent theory out of left and right moving sectors, one needs to compactify the extra 16 space dimensions in the right moving sectors in a compact space of 16 dimensions or in a lattice of 16 dimensions. The modular invariance of the string partition function (we shall discuss the modularity of the string partition function in due course) dictates that this lattice should be unimodular and self-dual. In 10D there are only two such 16 dimensional lattices $\Gamma_{16}$ at our disposal: that is of group $Spin(32)/\mathbb{Z}_2$ and $E_8 \times E_8$. Thus in the massless spectrum we obtain

$(NS_+, NS_+) \rightarrow g_{\mu\nu},\ \ B_{\mu\nu},\ \ \phi$,
$(R_+, NS_+) \rightarrow \psi^\mu,\ \ \lambda$,
$(NS_+, \Gamma_{16}) \rightarrow A_\mu$,
$(R_+, \Gamma_{16}) \rightarrow \lambda_\mu$.

The effect of combing the 16 world-sheet scalars on the gauge lattice $\Gamma_{16}$ gives rise to the vector multiplet $(A_\mu, \lambda_\mu)$ with $A^\mu$ transforming in the adjoint representation of either $SO(32)$ or $E_8 \times E_8$. Thus we get the 10D heterotic string theory with gauge group either SO(32) (called HO in short) or $E_8 \times E_8$ (called HE in short). It is worthwhile to note that the GSO projection automatically removes the negative mass-squared tachyonic state from the string spectrum.

It remains to discuss the superstring theory of the open strings. We have however noted that as open strings can join to form a closed string, any consistent string theory should contain closed strings. The way to include both open and closed string in a consistent theory is to have a theory of unoriented closed and open strings. One takes the theory of unoriented closed string: the problem with this theory is that the one-point diagram of creation of a closed string state out of vacuum. The Poincaré invariance of the theory requires the amplitude of such a process to be coupled to the RR form $C_{10}$. Such diagrams are known as tadpoles and the non-vanishing amplitude of such processes indicates the instability of the vacuum. The way out of this difficulty is to include unoriented open strings in the theory which also have tadpole due to cross-cap coupling with RR forms but the amplitude is just opposite to that of the closed unoriented diagram. Moreover, the orientation projection kills half of the supersymmetry so that to construct a superstring theory, one takes the orientation projection of type IIB theory in 10D and couples it with the unoriented open string theory the massless spectrum of which includes gauge bosons in the adjoint representation of $SO(32)$. This is called the type I theory having SO(32) vector multiplet as in HO (along with gaugini) theory plus one graviton, one dilton, one 2-form, one gravitino and one dilatino in gravity multiplet.

It is interesting to note that in 10 spacetime dimensions, the vacua of consistent supergravity theories (2.2, 2.3) is exactly the same as that of the consistent superstring theories. In lower dimensions however, the consistent supergravity vacua is larger than the ones obtained from the compactifications of string theory and the supergravity vacua not constructible from the stringy ones are called the swampland.

## 3.3 String interactions and effective actions

Once the particle states of QFT are replaced by string excitation states, one can also replace the world-line construction of Feynman diagrams of the interaction by the world-sheet diagrams. The problem with the infinite precision of the probing of space-time interaction point is now resolved by the smearing of the interaction position which, in string interaction, lies on a patch of world-sheet (see figure 3.2). Thus

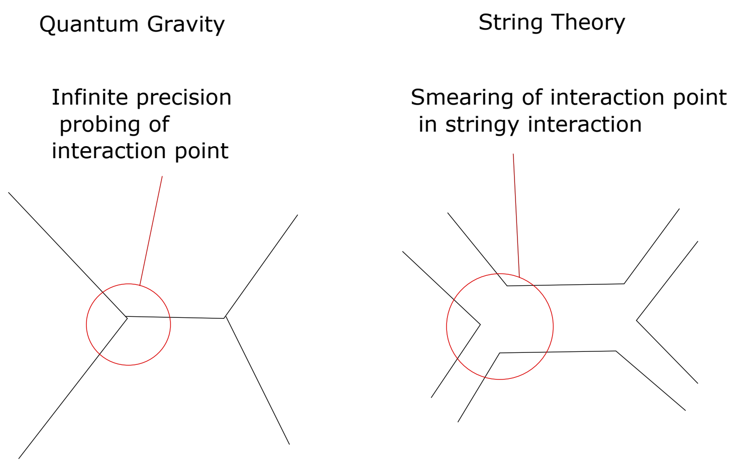


FIGURE 3.2: Quantum gravity interaction vs String interaction

the non-renormalization problem of the quantum gravity is resolved in string theory. As a consistent description of the second quantization of the string theory is still lacking, one however has to look for the conformal field theory description of the Polyakov action (3.2) which, as mentioned earlier, can also be thought as the theory of 2D gravity coupled to target space scalars and fermions. Even after gauge fixing the symmetries (3.3), (3.6) of this action, there remain infinite number of conserved charges which allow for the CFT description of dynamical string interaction. We shall be considering mostly the interaction of closed oriented strings and for the world-sheet diagrams, we shall take the initial and final string states to be on shell and at infinity. Under conformal transformations, the world-sheet Feynman diagrams can be transformed to a compact surface of genus g with the initial and final string states at infinity represented as punctures on these surfaces (see figure 3.3). The equivalent of Lehmann-Symanzik-Zimmermann (LSZ) reduction formalism in string theory states that the punctures i.e. the initial and final string states are to be represented by a quantum wave function, which are called the vertex operators and that the string S-matrix components are to be calculated for each world-sheet diagram of a given

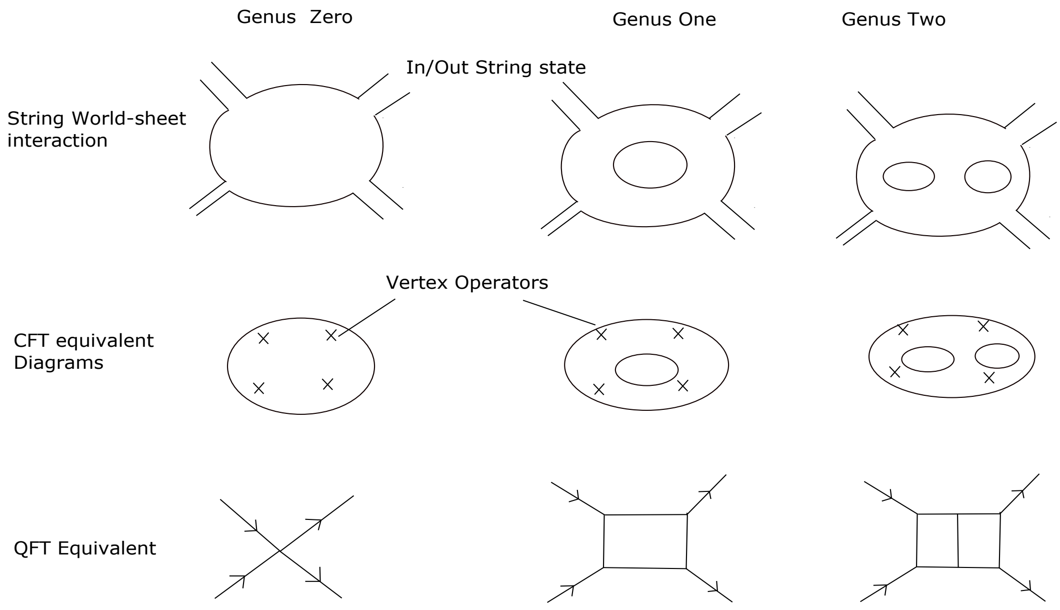


FIGURE 3.3: String Feynman diagrams

conformally inequivalent topology by the time ordered product of these vertex operators on the world-sheet surface. Just as in case of ordinary QFT we need to sum over all possible S-matrix elements for a set of given initial and final states to determine the so-called Green's function of those states, in string theory we need to sum over all possible topologically inequivalent world-sheet surfaces with vertex operators inserted to compute the stringy Green's function. Each S-matrix elements are then the string scattering amplitudes. It is however very difficult to compute string amplitudes beyond spherical (i.e. zero loop genus zero) and toroidal (i.e. one loop genus one) topologies (we shall give details of the 2-loop computation in chapter 7). Before we address the amplitude results, which in turn provide for the stringy correction terms in the low energy supergravity effective actions, we shall briefly discuss the complexity of gauge fixing the partition function obtained from the Polyakov action.

### 3.3.1 Scattering amplitudes in string theory

Let us start by giving the expression of the string partition function. Recall that in an ordinary QFT of some set of fields $\phi^i$ described by a free plus interaction Lagrangian $\mathcal{L} + \mathcal{L}'$, one gets the n-point Green's function $G(\phi^1, \cdots \phi^n)$, i.e. the complete interaction of n fields as

$$G(\phi^1, \cdots \phi^n) = \frac{< 0|T\mathcal{S}\phi^1 \cdots \phi^n|0 >}{< 0|\mathcal{S}|0 >} \tag{3.18}$$

with $\mathcal{S}$ being the S-matrix obtained from the interaction Lagrangian $\mathcal{L}'$. These Green's functions are also conveniently found from the variation of the so called partition function $Z$ defined by

$$Z[\phi^i] = N \int \mathcal{D}[\phi^i] e^{i\int(\mathcal{L}+\mathcal{L}')} = N \int \mathcal{D}[\phi^i] e^{iS} \tag{3.19}$$

where $S = \int(\mathcal{L} + \mathcal{L}')$ is the action of the theory. From the partition function $Z$, one gets the Green's function using

$$\begin{aligned} G(\phi^1, \cdots \phi^n) &= \frac{\delta^n Z[\phi^i, \sigma^i]}{\delta\sigma^1 \cdots \delta\sigma^n}, &(3.20)\\ &= N \int \mathcal{D}[\phi^i] e^{iS} \phi^1 \cdots \phi^n &(3.21)\end{aligned}$$

with $\sigma^i$ some auxiliary fields in the functional formalism of QFT [15]. Thus all the details of the interacting field theory is contained in the partition function $Z$. In string theory, one can similarly write the partition function

$$\begin{aligned} Z_s &= \int \frac{\mathcal{D}X\mathcal{D}h}{\text{Diff} \times \text{Weyl}} e^{iS_P(X,h)}, &(3.22)\\ &= \sum_{\text{inequivalent topologies}} \int \frac{\mathcal{D}X\mathcal{D}h}{\text{Diff} \times \text{Weyl}} e^{iS_P(X,h)}, &(3.23)\\ &= \sum_{\chi} Z_{\chi}. &(3.24)\end{aligned}$$

In the first line of (3.22), we have used the Polyakov action (3.2) and have divided the integral measure by the volume of diffeomorphism and Weyl symmetry in order to have finite path integral measure. In fact it is well-known from the QFT of gauge theory that unfixed gauge symmetry makes the path-integral infinite and one needs to imply for example Faddeev-Popov method of gauge fixing. In the second line of (3.22) we have broken the S-matrix elements as sum over CFT inequivalent world-sheet topologies and the partition function for each such topologies, characterized by the Euler characteristic $\chi$, we write $Z_{\chi}$. The Euler characteristic for a closed orientable surface is defined by $\chi = 2 - 2g$ where g is the number of handles or genus in the topology. In case of a non-compact (non)orientable world-sheet topology, we find $\chi = 2 - 2g - b - c$ where b is the number of boundaries and c the number of cross-caps. Thus when we shall speak of tree level, one loop level closed string string amplitudes, we shall have in back of our mind the topology of sphere and torus respectively. To complete the analogy with the QFT Green's function, we write the on-shell initial and final string states represented by the vertex operators $\mathcal{V}_i(k_i, \sigma_i)$ which carry quantum state $k_i$ (say momentum) and are to be inserted at the positions $\sigma_i = (\tau_i, \sigma_i)$ on the world-sheet diagram. The expression for the Green's function then looks like

$$G(1, \cdots, n) = \sum_{\text{inequivalent topologies}} \int \frac{\mathcal{D}X\mathcal{D}h}{\text{Diff} \times \text{Weyl}} e^{iS_P(X,h)} \prod_i^n \int d^2\sigma_i \mathcal{V}_i(k_i, \sigma_i). \tag{3.25}$$

The implementation of the Faddeev-Popov gauge fixing method to choose for a gauge slice of diffeomorphism and Weyl symmetry necessitates the introduction of Faddeev-Popov determinant and the ghost states coming there-from. We shall only sketch the facts instead of giving detailed mathematics. Recall from the discussion of symmetries of the Polyakov action for a closed oriented bosonic string have the diffeomorphism and Weyl symmetries, the infinitesimal form of which are

$$\delta h_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha. \tag{3.26}$$

and

$$\delta h_{\alpha\beta} = 2\Lambda(\sigma) h_{\alpha\beta}. \tag{3.27}$$

Such symmetries of the world-sheet metrics $h_{\alpha\beta}$ define the so-called moduli space for a given topology

$$\mathcal{M} = \frac{\mathcal{G}}{\text{Diff} \times \text{Weyl}} \tag{3.28}$$

where $\mathcal{G}$ is the space of all world-sheet metrics $h_{\alpha\beta}$ for a given topology. For example, the genus one world-sheet is a torus which is parametrized by a complex structure $\tau$. Any torus is however invariant under the $SL(2,\mathbb{Z})$ transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) = \frac{SL(2,\mathbb{Z})}{\mathbb{Z}_2}. \tag{3.29}$$

Thus though the complex structure is à priori allowed to take values in the upper half-plane $\mathcal{H}_2$, the moduli space is the fundamental domain

$$\mathcal{F} = \frac{\mathcal{H}_2}{PSL(2,\mathbb{Z})}. \tag{3.30}$$

The space $\mathcal{F}$ is thus parametrized by the complex scalar $\tau$. In case of complicated topologies, there may be a subgroup of Diff $\times$ Weyl which leaves the metric invariant and is called the conformal Killing group (CKG in short). The infinitesimal elements of CKG are called the conformal Killing vectors (CKV) which generate infinitesimal Diff $\times$ Weyl changes in the metric $h_{\alpha\beta}$. The way to quantify the CKVs is to combine the infinitesimal Diff $\times$ Weyl changes of the metric in the following form

$$\delta h_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - h_{\alpha\beta} \nabla^\mu \xi_\mu + (2\Lambda(\sigma) - \nabla^\mu \xi_\mu) h_{\alpha\beta} \tag{3.31}$$

$$= (\widehat{P}\xi)_{\alpha\beta} + (2\Lambda(\sigma) - \nabla^\mu \xi_\mu) h_{\alpha\beta} \tag{3.32}$$

where we define the operator $\widehat{P}$ and its adjoint $\widehat{P}^\dagger$ as

$$(\widehat{P}\xi)_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - h_{\alpha\beta} \nabla^\mu \xi_\mu, \tag{3.33}$$

$$(\widehat{P}^\dagger \delta h)_\mu = -2\nabla^\nu \delta h_{\mu\nu}. \tag{3.34}$$

The CKVs are transformations such that $\delta h_{\alpha\beta} = 0$ and the trace of this equation gives the conformal Killing equation

$$(\widehat{P}\xi)_{\alpha\beta} = 0. \tag{3.35}$$

Also the zero modes of $\widehat{P}^\dagger$ i.e. $\delta h$ such that

$$(\widehat{P}^\dagger \delta h)_\mu = 0 \tag{3.36}$$

are in fact deformations which cannot be compensated by Diff $\times$ Weyl symmetry. The elements of $Ker\widehat{P}$ are the CKVs while the elements of $Ker\widehat{P}^\dagger$ are called the moduli deformations or metric moduli. The Riemann-Roch theorem on the world-sheet relates them to the Euler characteristic $\chi = 2 - 2g$ according to

$$dim Ker\widehat{P} - dim Ker\widehat{P}^\dagger = 3\chi \tag{3.37}$$

which in turn states that for an orientable compact topology with genus g

$$dimKer\widehat{P}^{\dagger} = \begin{cases} 0, & \text{if } g = 0, \\ 2, & \text{if } g = 1, \\ 6g - 6, & \text{if } g \geq 2. \end{cases} \tag{3.38}$$

The process of gauge fixing is in fact completely specified by manipulation of the CKVs and metric moduli. If the moduli space contains only CKVs, then the gauge fixing of the amplitude can be done by adjusting the vertex positions. This is the case for spherical topology or tree level closed string amplitude. If there are metric deformations in the moduli space then one needs a further step. It can be shown that the Faddeev-Popov Jacobian to be included in the path integral (3.22) for the gauge fixing is given by

$$\Delta_{FP} = \sqrt{\widehat{P}\widehat{P}^{\dagger}} \tag{3.39}$$

in the operator form. This inclusion in the path integral is in fact equivalent to the inclusion of the bc-ghost system [27] in the Polyakov action

$$S_{bc} = \frac{1}{2\pi} \int d^2\sigma b\partial\bar{c} \tag{3.40}$$

where b and c are anti-commuting bosonic fields (that is why they are ghosts).

In case of super-string theory, in addition to the bc-ghost system, another ghost system, the $\beta\gamma$-system (commuting fermionic fields) should be included in order to account for the full supersymmetric Polyakov action (3.9). The moduli space $\mathcal{M}$ (3.28) is now to be understood as a super-moduli space.

The sum of Polyakov action (3.2) and bc-ghost action $S = S_P + S_{bc}$ (or bc- plus $\beta\gamma$-system in super-string case) possess the Becchi-Rouet-Stora-Tyutin (BRST) symmetry. This is in fact an implicit effect of the Faddeev-Popov gauge fixing method and is the elegant alternative to the Gupta-Blauler gauge fixing conditions. In short, in the Gupta-Blauler method (for Abelian gauge theories), one truncates the physical states in order to decouple the non-physical states (e.g. ghost states) from the spectrum of the physical theory. In case of non-Abelian gauge symmetries, one asks for the physical states to be BRST invariant thereby decoupling the non-physical states out of the physical spectrum. In string amplitude computations, one thus needs to implement the BRST invariance, that is the string amplitudes should be BRST invariant. This is rather easily achieved in super-string amplitude computation by introducing the so called picture changing operators (PCO) $\mathcal{O}(k_i, \sigma_i)$ in the Green's function (3.41)

$$G(1, \cdots, n) = \sum_{\text{inequivalent topologies}} \int \frac{\mathcal{D}X\mathcal{D}h}{\text{Diff} \times \text{Weyl}} e^{iS_P(X,h)} \prod_i^n \int d^2\sigma_i \mathcal{V}_i(k_i, \sigma_i) \prod_j^m \mathcal{O}(k_j, \sigma_j). \tag{3.41}$$

This is once more equivalent to evaluating the string amplitude of states represented by n vertex operators $\mathcal{V}_i(k_i, \sigma_i)$ along with b and c ghost operators. The algorithm of picture formalism in string amplitude [29] is the following: for a string loop amplitude on a genus g world-sheet with $n_B$ bosonic vertices in -1 picture i.e. vertex with BRST charge -1, the BRST consistency condition for the amplitude forces one to insert $N = 2g - 2 + n_B$ picture changing operators. One then coincides these PCOs with the bosonic vertices provided that the complex structure of the string world-sheet allows for such coincidence (for example an infinitesimal patch of the world-sheet around the bosonic vertex position accommodates the insertion of the PCO such that they come infinitely close together). If $N = n_B$ i.e. for genus one case, all the vertices now become zero picture BRST vertices. If $N < n_B$ i.e. for genus zero case, $n_B - 2$ vertices turn to zero picture and 2 remaining states in -1 picture. In case of genus $g \geq 2$ there are more picture changing operators than the physical vertices. One thus needs further considerations to solve for the ambiguities arising from the PCO insertions. We shall address this issue in chapter 7.

The 10D string theories with N=2 supersymmetry contains states from NS-NS sectors. In an amplitude computation, one needs to consider the PCO insertions for each NS sectors separately. We shall discuss the particularities of string 1-loop amplitudes in course of this work and shall provide the details for the vertex operators, kinetic structures, partition functions, fermionic characters in terms of world-sheet $SL(2,\mathbb{Z})$ genus-one modular functions in due places. The goal for the above general discussion was to make the foundation for the more complicated discussion of two-loop amplitudes in chapter 7.

### 3.3.2 String coupling to background fields and low energy effective action

Starting from the Polyakov action (3.2)

$$S_p = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \tag{3.42}$$

with the target space metric assumed the flat Minkowski one i.e. $\eta_{\mu\nu}$, one finds, after the quantization of string excitations, the massless modes contain the graviton $g_{\mu\nu}$ which, from the general relativity perspective, is the fluctuation around the flat metric i.e.

$$G_{\mu\nu} = \eta_{\mu\nu} + g_{\mu\nu}. \tag{3.43}$$

The coherent superposition of graviton creates the generic curved background similar to the photon reconstruction of macroscopic electro-magnetic field. Thus the consistent background for Polyakov action should couple with $g_{\mu\nu}$ as well as with $B_{\mu\nu}$ and the dilaton field $\phi$ so that it becomes

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \left[\sqrt{-h} h^{\alpha\beta} g_{\mu\nu}(X) + \epsilon^{\alpha\beta} B_{\mu\nu}(X)\right] \partial_\alpha X^\mu \partial_\beta X^\nu + \frac{1}{4\pi} \int d^2\sigma \sqrt{-h} R^{(2)} \phi(X), \tag{3.44}$$

where $R^{(2)}$ is the Ricci-scalar of the intrinsic world-sheet metric $h_{\alpha\beta}$. Note that the coupling of dilaton field in the above is dictated by the diffeomorphism and Weyl invariance of the original Polyakov action. This coupling is however not consistent with the Weyl invariance if the higher order corrections, that is, the string loop corrections to the Polyakov action are taken into account. The Weyl anomaly in the non-linear sigma model (3.44) is controlled by three $\beta$- functions $\beta^G_{\mu\nu}$, $\beta^B_{\mu\nu}$, $\beta^\phi$ and the Weyl consistency is restored if those functions in the variables of Ricci-tensor $R_{\mu\nu}$ of the target space metric $g_{\mu\nu}$, the field strength $dB_2 = H_3$ and $\phi$ derivatives vanish. The resulting equations are in fact

$$\frac{\beta^G_{\mu\nu}}{\alpha'} = R_{\mu\nu} - \frac{1}{4}H_{\mu\nu\rho}H^{\mu\nu\rho} + 2\nabla_\mu\nabla_\nu\phi = 0, \tag{3.45}$$

$$\frac{\beta^B_{\mu\nu}}{\alpha'} = -\frac{1}{2}\nabla_\rho\left[e^{-2\phi}H_{\mu\nu\rho}\right] = 0, \tag{3.46}$$

$$\frac{\beta^\phi}{\alpha'} = \frac{3}{2}\left[4(\nabla\phi)^2 - 4\Box\phi - R + \frac{1}{12}H^2\right] = 0 \tag{3.47}$$

and they can be thought of as equations of motions coming from the action

$$S_{\text{tree}} = \frac{1}{\kappa^2}\int_{10} d^{10}x\sqrt{-det(G)}e^{-2\phi}\left[R + 4(\nabla\phi)^2 - \frac{1}{12}H^2\right] \tag{3.48}$$

which can be compared with the 10D supergravity actions (2.22), (2.24) and (2.30) with $\kappa^2 = e^{2<\phi>}\alpha'^4$ being the 10D gravitational constant. The factor $< \phi >$ is the vacuum expectation value of $\phi$ such that $\phi =< \phi > +\Phi$. To make the comparison more clear, we absorb the exponential factor inside the metric with $g^E_{\mu\nu} = e^{-\frac{\Phi}{2}}g_{\mu\nu}$ to bring the action (3.48), said to be in string frame, to Einstein frame

$$S^E_{\text{tree}} = \frac{1}{2\kappa^2}\int_{10} d^{10}x\sqrt{g^E}\left[R - \frac{1}{2}(\nabla\Phi)^2 - e^{-\Phi}\frac{1}{12}H^2\right]. \tag{3.49}$$

The reason we have inserted the label "tree" in (3.48) and (3.49) is that the same action can be obtained from tree level string scattering amplitude. This is evident also from the dilaton factor $e^{-2\phi}$ in (3.48) which is indicative of the fact that the action is a result of sphere amplitude for closed strings. This point of string coupling constant shall be discussed in more detail in the next section. We conclude this section by pointing out that the low energy effective action of the string massless modes can be derived, at the lowest order in perturbation theory by CFT arguments as presented above or from direct tree level amplitude calculation and the action so obtained is same as that for the supergravity effective action in the first order formulation. This corroborates our previous assertion that supergravity theories in 10D are in fact low energy limits of 10D string theories.

### 3.3.3 String loop expansion in $g_s$ and $\alpha'$

The coupling of the dilaton field $\phi$ in the background coupled Polyakov action (3.44) dictates the string coupling constant. Consider the coupling $\frac{1}{4\pi}\int d^2\sigma\sqrt{h}R^{(2)}\phi(X)$ with $\phi(x)$ replaced by its constant vacuum expectation value $< \phi >$ so that

$$\frac{1}{4\pi}\int d^2\sigma\sqrt{h}R^{(2)} < \phi(X) >=< \phi(X) > \chi \tag{3.50}$$

where $\chi$ is the Euler characteristic of the world-sheet: $\chi = 2 - 2g - c - b$ where g stands for genus, b for boundaries and c for cross-caps. For oriented closed strings b and c are zero and a tree level amplitude, that is a spherical world-sheet with vertex functions of incoming and outgoing modes shall have $\chi = 2$. At 1-loop level, i.e. for a torus diagram $\chi = 0$ and negative for higher loop terms. In the Euclidean formulation of Feynman path integral, the scattering amplitude shall have the structure

$$< out|in >= \sum_{\text{world-sheet topologies}} \int \frac{[\mathcal{D}X][\mathcal{D}h]}{\text{diff } \times \text{ Weyl invariance}} e^{-S_P} V_{in} V_{out}. \qquad (3.51)$$

Thus it is easy to see that each topology shall be weighted by the factor $e^{-<\phi(X)>\chi}$ in the path integral and comparison with ordinary QFT (e.g. QED) (or most readily, writing the Green's function (3.51) in terms of sum over string S-matrix elements) shows that $g_s = e^{<\phi(X)>}$ shall act as the coupling constant of the string perturbation theory in the world-sheet Feynman diagrams. The power of $g_s$ in the string frame effective action thus determines the order of the perturbation: for example, the effective action (3.48) of the previous section can be recovered from tree level amplitude of closed strings as $\chi = 2$. We shall see for one loop, the string coupling constant appears to be $g_s = e^{<\phi(X)>0} = 1$. Thus the string S-matrix can be seen as an expansion in $g_s$ when $g_s << 1$ and is called the perturbative expansion. A string theory is non-perturbative or strongly coupled if $g_s > 1$.
Next looking at the gravitational constant $\kappa^2 = g_s^2 \alpha'^4$ with $\alpha' = l_s^2$, we see that the tree level effective action in string frame (3.48) or in the Einstein frame (3.49) takes into account the string length scale $l_s$ within the definition of Einstein-Hilbert action of 10D supergravity. In fact, in natural units, the action should be dimensionless and thus looking at (3.49) we see that the denominator comes with a length scale dependence $(l_s)^8$ which is counter-balanced with $(l_s)^{10}$ in $d^{10}x$ and $(l_s)^{-2}$ from each of R, $H^2$, $(\nabla\Phi)^2$. Hence each of the latter terms are called two-derivative terms and the action (3.49) is called "two-derivative" action. When one incorporates the corrections due to higher loops into the effective action, one sees the emergence of higher-derivative terms. For example, in one loop, as we shall see in the course of this work, the effective action receives correction of the form

$$S = \frac{1}{\alpha'} \int d^{10}x R^4 \qquad (3.52)$$

An example of such terms can be readily seen from the CP-even partners of the anomaly polynomials as given in section 2.5 for the 10D case. To make contact with the Einstein Hilbert action (3.49) one thus needs to incorporate in the denominator of (3.52) the gravitational constant $\kappa^2$ and the modified Einstein-Hilbert action would look like

$$S = \frac{1}{2\kappa^2} \int d^{10}x \left[R + \alpha'^3 R^4 + ...\right] \qquad (3.53)$$

Thus the correction term, which in this case is of 8-derivative, can be seen as an expansion in $\alpha'$ parameter. The higher derivative corrections from still higher loops are then further terms in the expansion in $\alpha'$. Note that $\alpha'$ denotes the stringy nature of correction, that is one delves the supergravity effective action from string perspective with an energy resolution to visualize the stringy emergence of gravity. In case the energy resolution too low, the stringy nature cannot be detected and corrections of order $\alpha'$ cannot be "felt" at the low energy effective interaction. We shall devote the main contribution of this work to the $\alpha'^3$ corrections to 10D and 8D supergravity theories.

## 3.4 Branes in string theory

The massless spectrum of 10D string theories contain p-form fields e.g. the NS-NS 2 form $B_2$ and RR forms in type II theories. These fields couple to extended objects in a similar manner to the 4D electromagnetic field couple to point charges. The natural objects which are charged under (p+1)-form $C_{p+1}$ are p-branes, extended objects with p spatial dimensions. A point particle thus corresponds to a zero-brane and a string corresponds to a one-brane. The natural or minimal coupling of $C_{p+1}$ to a p-brane is given by

$$S_p = Q_p \int_{p-\text{brane world volume}} C_{p+1} \tag{3.54}$$

which generalizes the ordinary gauge theory coupling of point particle source that is

$$S = \int d^D x A_\mu j^\mu \tag{3.55}$$

where the current $j^\mu = \int d\tau \delta^D(x_\mu - X_\mu(\tau))\partial_\tau X^\mu(\tau)$ with $X_\mu(\tau)$ labelling the location of the point particle. For p-brane the equivalent of point particle charge density is $Q_p$ which is the string tension for one-brane and brane tension (mass per unit volume) for p-branes with $p \geq 2$. One can also define the magnetic dual $\tilde{C}_{D-p-3}$ for a given field $C_{p+1}$ by the Hodge dual of the field strength $F_{p+2}$ by

$$* \, dC_{p+1} = *F_{p+2} = \tilde{F}_{D-p-2} = d\tilde{C}_{D-p-3} \tag{3.56}$$

where D is the dimension of the space-time. Thus the dual form couples to (D-p-4)-branes or $C_{p+1}$ couples "magnetically" with (D-p-4)-branes with magnetic charge $Q'_{D-p-4}$. A Dirac quantization similar to that of 4D case relates the electric and magnetic charges according to

$$Q_p Q'_{D-p-4} \in 2\pi\mathbb{Z}. \tag{3.57}$$

Of the important branes in string theory, the Dp-branes are of particular interest. For theories with open strings, suitable boundary conditions for the equations of motion obtained from Polyakov action are to be imposed on $X^\mu$ and there are two such conditions: (1) the Neumann condition: $\partial_\sigma X^\mu|_{\sigma=0,\pi} = 0$ and (2) Dirichlet condition: $X^\mu|_{\sigma=0,\pi}$ =const. One can impose these boundary conditions independently at each end-point of the open string and along different directions in space-time. The imposition of Dirichlet boundary condition on an endpoint breaks the Poincaré invariance on the directions of constant $X^\mu$s. Thus for example imposing Neumann condition in p+1-directions and Dirichlet conditions in D-p-1 orthogonal directions require this end of open string to lie on a (p+1)-dimensional subspace of target space. Such a subspace is referred to as a Dp-brane. In the world-volume of each Dp-brane lives a $U(1)$ gauge field and when N such branes coincide the open strings stretched between them give rise to the gauge enhancement $U(1)^n \to U(n)$ (we defer from a general discussion of Chan-Paton factors as these notions will be of minimal use in the rest of the thesis: for more information we refer to Polchinski [30] and Bachas [31] lecture notes).

Dp-branes are in fact the objects of type II theories which are charged under RR-forms. They can also be obtained from the supergravity equations of type II theories. Here however, we discuss their BPS (Bogomolny-Prasad-Sommerfeld) nature. In fact the 10D supersymmetry algebra of different string theories can be extended with tensorial central charges so that the supersymmetry algebra has the structure

$$\{Q^A_\alpha, Q^{B\dagger}_\beta\} = -2\delta^{AB} P_\mu \Gamma^\mu_{\alpha\beta} - 2iZ^{AB}_{\mu_1\cdots\mu_{p+1}} (\Gamma^\mu_1 \cdots \Gamma^\mu_{p+1})_{\alpha\beta} \tag{3.58}$$

where the operators $Z^{AB}_{\mu_1\cdots\mu_{p+1}}$ are the central charges which commute with Q's and behave as tensors with respect to the generators of the Lorentz group. Their commutation with the Hamiltonian makes them proportional to p-brane charges. These branes are BPS states because they saturate the BPS condition "brane tension $\geq$ maximal eigenvalue of Z matrix". The BPS states are absolutely stable at a generic point of moduli space because its decomposition to daughter states would violate the simultaneous conservation mass and charges. These BPS branes however preserve only a fraction of the original supersymmetry because of the central charges Z.

The effective action for Dp-branes is given by the Dirac-Born-Infeld action which describes the coupling of Dp-brane world-volume degrees of freedom to the bulk NS-NS fields by

$$S_{DBI} = -T_p \int_{p+1} d^{p+1}\xi e^{-\phi} \sqrt{-det(g_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} \tag{3.59}$$

where $\phi$ is the dilaton restricted to brane, $\xi^a$ are the brane world-volume coordinates, $g_{ab}$ and $B_{ab}$ are the pull-backs of bulk metric and $B_2$ field

$$g_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} g_{\mu\nu}, \quad B_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}. \tag{3.60}$$

Finally $F_{ab}$ is the gauge field strength of the Abelian field living in the brane world-volume. The brane tension $T_p$ is equal to

$$T_p = (2\pi)^{-p}(\alpha')^{-\frac{p+1}{2}} g_s^{-1} \tag{3.61}$$

which is also equal to the RR charge $\mu_p$ of Dp-brane.

In addition to the DBI action, there is a Chern-Simons action for Dp-branes which are related to the cancellation of anomaly due to the chiral matter living in the intersection of two branes. This is given by [32, 33, 34, 35]

$$S^D_{WZ} = \mu_p \int \sum_n C_n e^{\mathcal{F}} \sqrt{\widehat{A(\mathcal{R}_T)}/\widehat{A(\mathcal{R}_N)}}\ |_{p+1-form} \tag{3.62}$$

where $\sum_n C_n$ is the formal sum of the RR potentals pulled-back to Dp-brane world-volume, $\mathcal{F} = 2\pi\alpha' F - B$ with F being the gauge-field strength living on the Dp-brane world-volume and B is the NS-NS 2-form field pulled-back to the brane world-volume. $\mathcal{R} = 4\pi^2\alpha' R$ is the normalized curvature (T and N respective for the tangent and normal bundle) and the square-root of Dirac genus is given by

$$\sqrt{\widehat{A(\mathcal{R})}} = 1 - \frac{(4\pi^2\alpha')^2}{48} p_1(R) + \frac{(4\pi^2\alpha')^4}{2560} p_1^2(R) - \frac{(4\pi^2\alpha')^4}{2880} p_2(R) + \dots \tag{3.63}$$

with $p_1(R) = -\frac{1}{8\pi^2} TrR^2$, and $p_2 = \frac{1}{4(2\pi)^4}[\frac{1}{2}(TrR^2)^2 - TrR^4]$.

Apart from Dp-branes, another important BPS state in string theory is the NS5-brane which is magnetic dual of fundamental (or perturbative) string according to (3.56) and couples magnetically to NS-NS two form $B_2$ with which the perturbative string couples electrically. The tension of NS5 brane is $T_5 = (2\pi)^{-5}\alpha'^3 g_s^{-2}$ which is heavier than the corresponding D5-brane in (3.61). The world-volume theory of NS5 brane contains D=6 N=1 supergravity hypermultiplet (to be described fully in chapter 6).

We shall end this section with a discussion of orientifold planes. Although a consistent discussion of orientifold planes require knowledge of T-duality which we shall discuss in section 3.6.1, we present it here for the sake of better classification. The idea is related to the orientofold projection which is a world-sheet operator $\Omega$ implementing a parity transformation on world-sheet coordinates

$$\Omega X^\mu(\tau,\sigma)\Omega^{-1} = X^\mu(\tau, 2\pi - \sigma) \tag{3.64}$$

for the closed strings with $\sigma \sim \sigma + 2\pi$. For closed strings, the original coordinates are $X^\mu = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$ and their T-dual coordinates (see section 3.6.1) are $X'\mu = X_L'^\mu(\sigma^+) - X_R'^\mu(\sigma^-)$. Thus the action of world-sheet parity (3.64) exchanges right-moving coordinates $X_L^\mu(\sigma^+)$ with the left-moving $X_R^\mu(\sigma^-)$ sectors and for the dual coordinates $X'^m$

$$X'^\mu(\tau,\sigma) \leftrightarrow -X'^\mu(\tau, 2\pi-\sigma) \tag{3.65}$$

which can be thought of as a product of a world-sheet and a space-time parity operations. Taking now n-space-dimensions ($x^\mu$, $\mu = 0, \cdots, n$) out of D=10 to have the even parity as in (3.64) and the rest 10-n ($x^m$, $m = 10-n, \cdots, 9$) coordinates to have parity odd property (3.65), one can see that the string excitation modes (massless) decomposes under the eigenstates of the $\Omega$ operator e.g., for the metric and anti-symmetric states ([30])

$$\begin{aligned} g_{\mu\nu}(x^\mu, -x^m) &= g_{\mu\nu}(x^\mu, x^m), \;\; g_{\mu m}(x^\mu, -x^m) = -g_{\mu m}(x^\mu, x^m), & (3.66)\\ g_{mn}(x^\mu, -x^m) &= g_{mn}(x^\mu, x^m), & \\ B_{\mu\nu}(x^\mu, -x^m) &= -B_{\mu\nu}(x^\mu, x^m), \;\; B_{\mu m}(x^\mu, -x^m) = B_{\mu m}(x^\mu, x^m), & (3.67)\\ B_{mn}(x^\mu, -x^m) &= -B_{mn}(x^\mu, x^m). & \end{aligned}$$

The T-dual space-time is then seen to have the geometry of the $\mathbb{R}^{10-n}/\mathbb{Z}_2$ quotient space and the resulting target-space theory is given by $\mathbb{R}^n \times \mathbb{R}^{10-n}/\mathbb{Z}_2$, the origin of $\mathbb{R}^{10-n}/\mathbb{Z}_2$ corresponds to an n-dimensional subspace of $\mathbb{R}^{1,9}$ fixed under the $\mathbb{Z}_2$ parity operation. The projection or orientation-parity operation also necessitates the inclusion of unoriented strings in the theory. The effect of these are accounted by the inclusion of non-dynamical orientifols planes or O-planes for short: for the case of 10-n dual coordinates, the O-plane is of dimension 10-n. Orientifold planes are not dynamical in the sense that no string modes are tied to it to represent fluctuation of its shape. In case of orientifold compactification with D-branes, one observes that the cancellation of RR-tadpoles needs the O-planes to couple with RR-forms in CP-odd manner: intuitively, the Chern-Simons coupling of RR-forms with D-branes as in (3.62) shows that to counter-balance the net RR-flux one needs to add O-planes within the stack of D-branes having Chern-Simons action of the form [32, 36, 34, 35]

$$S^{Op\pm} = \pm\mu_p' \int \sum_n C_n \sqrt{L(\widehat{\mathcal{R}_T/4})/L(\widehat{\mathcal{R}_N/4})}\;|_{p+1-form}, \tag{3.68}$$

with $\mu_p' = 2^{p-5}\mu_p$ and

$$\sqrt{L(\widehat{\mathcal{R}/4})} = 1 + \frac{(4\pi^2\alpha')^2}{96}p_1(R) - \frac{(4\pi^2\alpha')^4}{10240}p_1^2(R) + \frac{7(4\pi^2\alpha')^4}{23040}p_2(R) + \ldots. \tag{3.69}$$

In the next section we shall elaborate in more details the compactification paradigm in string theory.

## 3.5 Compactification in string theory

The consistency of the string theory requires it to live in a target spacetime of dimensions 10 where the string dynamics determines the gravitational degrees of freedom dynamically. In order to make contact with the 4D standard model physics with possibly minimal supersymmetry, it is thus necessary to require that the dynamically determined space-time geometry $\mathcal{M}_{10}$ should allow for the product form $\mathcal{M}_{10} = \mathcal{M}_d \times \mathcal{M}_c$ where $\mathcal{M}_d$ is Minkowski space of dimension d< 10 (for the phenomenological purpose d=4 in fact) and $\mathcal{M}_c$ is a compact manifold of dimension c such that $c + d = 10$. To escape the detection by the actual resolution of high energy physics experiment, the size of $\mathcal{M}_c$ must be smaller than the length scale probed by such experiments. We are also assuming implicitely that we are admitting the methods of the classical geometry in a limit where the length scale is much bigger than the string length scale $l_s$. In fact various string dualities, like T-duality suggest that the notion of stringy geometry which is different from the geometry probed by point particle. In the latter case, one needs to consider geometrization of the internal string degrees of freedom. The space-time solution of the form $\mathcal{M}_{10} = \mathcal{M}_d \times \mathcal{M}_c$ are not in general supersymmetry preserving but can leave a set of original supersymmetry parameters preserved. There are phenomenological and technical reasons to consider compactifications which leave some supersymmetry unbroken at the TeV scale. For example, it can explain the existence of rather light scalar fields (like the Higgs boson) by protecting their mass from large quantum corrections. Such a solution is invariant under supersymmetry if the infinitesimal transformations of the background fields vanish. In particular the supersymmetric variation of fermionic fields determine the shape of compact manifold $\mathcal{M}_c$. Take for example the gravitino variation which is schematically (see for example (2.17) in section 2.3)

$$\delta_\epsilon \psi_M = \nabla_M \epsilon + \cdots \tag{3.70}$$

where $\nabla_M$ is the covariant derivative on spinors and $\epsilon$ is a supersymmetry parameter. In a vacua in which all matter fields are set to zero, there shall remain a fraction of supersymmetry preserved if and only if there exists non-trivial solution to the Killing spinor equation $\nabla_M \epsilon = 0$. In order for such a solution to exist for the Killing spinor equation, one decomposes the 10D spinor $\epsilon$ into d and c dimensional spinor in the factorized form $\epsilon = \epsilon_d \bigotimes \eta_c$. Putting this into the Killing spinor equation $\nabla_M \epsilon = 0$ one finds $\nabla_d \eta = 0$ where the spinor covariant derivative $\nabla_d$ is now constructed from the internal metric $g_{mn}$ of the manifold $\mathcal{M}_c$. This equation in turn implies the Ricci tensor $R_{mn}$ of the internal manifold to vanish. Thus the requirement for the unbroken supersymmetry requires the Ricci flatness of the internal manifold $\mathcal{M}_c$.

We shall not explore the vast variety of compactification to Ricci-flat spaces for compactification but shall only sketch the circle and toroidal compactification which we shall use in the thesis. Amongst the interesting cases of Calabi-Yau compactification we shall emphasize on the K3 surface.

### 3.5.1 Circle compactification

The simplest example of compactification is obtained by choosing a one-dimensional internal space with the topology of a circle. Thus the background manifold of D=d+1 space-time dimensions can be written in the product form $\mathcal{M}_D = \mathcal{M}_d \times S^1$ where $\mathcal{M}_d$ is the d-dimensional Minkowski space-time. Consider the compactification ansatz

$$G_{\hat{\mu}\hat{\nu}} d\hat{x}^{\hat{\mu}} d\hat{x}^{\hat{\nu}} = g_{\mu\nu} dx^\mu dx^\nu + R^2 dy^2 \tag{3.71}$$

where careted indices $\hat{x}^{\hat{\mu}}$, $\hat{\mu} = 0, \cdots, d$ are D=d+1 dimensional coordinates of the manifold $\mathcal{M}_D$ and the non-careted indices $x^\mu$, $\mu = 0, \cdots, d-1$ are that of d-dimensional Minkowski space $\mathcal{M}_d$. The coordinate "y" parametrizes the circle with the identification $y \sim y + 2\pi$ and R is the radius of the compactification circle of circumference $2\pi R$. In general, the metric functions $g_{\mu\nu}, R$ are allowed to depend on $x^\mu$ and y. However, in order to retain only the zero-modes of excitation, one restricts to the case where $g_{\mu\nu}, R$ depend only on the external coordinates $x^\mu$. Under this assumption the D=d+1 dimensional fields $\hat{\phi}(x, y)$ which depend on both external coordinates $x^\mu$ and the internal coordinate y, should be decomposed in the Fourier expansion with x-dependent coefficients because of the identification $y \sim y + 2\pi$. Thus for a real scalar $\hat{\phi}$ we get

$$\hat{\phi}(x, y) = \sum_{n \in \mathbb{Z}} \phi^{(n)}(x) e^{iny}. \tag{3.72}$$

the Fourier coefficients $\phi^{(n)}(x)$ are interpreted as d-dimensional scalar fields and are referred to as Kaluza-Klein (KK for short) modes of $\hat{\phi}$. If we suppose that the dynamics of $\hat{\phi}$ in D=d+1 dimensions is governed by the massless Klein-Gordon equation $G_{\hat{\mu}\hat{\nu}} \partial^{\hat{\mu}} \partial^{\hat{\nu}} \hat{\phi} = 0$ , the reduction to the d-dimension dynamics of KK modes according to (3.72) is governed by the equation

$$g_{\mu\nu} \partial^\mu \partial^\nu \phi^{(n)}(x) - \frac{n^2}{R^2} \phi^{(n)}(x) = 0. \tag{3.73}$$

Thus the zero-mode $\phi^{(0)}$ is a free massless scalar in d dimensions, while excited modes are massive modes of mass $m_n = n/R$. We also note the mode expansion basis function $[e^{iny}]_{n \in \mathbb{Z}}$ are the set of complete and orthogonal eigenfunctions of the internal Laplacian $g_{yy} \partial^y \partial^y$ such that

$$g_{yy} \partial^y \partial^y e^{iny} = -\frac{n^2}{R^2} e^{iny} \tag{3.74}$$

This feature will be recurrent in compactification to more complicated internal manifold where the higher dimensional fields are expanded into eigenfunctions of suitable differential operator in the internal space and the modes associated to non-vanishing eigenvalues are massive with mass inversely proportional to the typical length scale of the internal geometry.

### 3.5.2 Toroidal compactification

For a general toroidal compactification that is where the internal manifold $\mathcal{M}_c = T^c = \prod_1^c S^1$, the metric ansatz for the reduction is

$$G_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} + A^i_\mu G_{ij} A^j_\nu & G_{ij}A^j_\mu \\ G_{ij}A^j_\nu & G_{ij} \end{pmatrix} \tag{3.75}$$

with the careted indices for the D-dimensional manifold, non-careted greek indices are used for the non-comapact D-c dimensional Minkowsky space-time and latin indices for the c dimensional internal torus manifold $T^c$. Using this ansatz the reduction of the D-dimensional Einstein Hilbert action yields the following action in D-c dimensions (see for example Bailin & Love [37])

$$\int d^D x \sqrt{det(G_{\hat{\mu}\hat{\nu}})}\hat{R} \rightarrow \int d^{D-c}x\sqrt{det(g)}\left[R - \frac{1}{4}\partial_\mu\phi\partial^\mu\phi + \frac{1}{4}\partial_\mu G_{ij}\partial^\mu G^{ij} - \frac{1}{4}G_{ij}F^i_{\mu\nu}F^{j\mu\nu}\right] \tag{3.76}$$

in the above, $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu$ is the field strength of the $A^i_\mu$ gauge fields originating from the metric and $\phi = log(detG_{ij})$. In particular the case of two torus $T^2$ is important as we shall use $T^2$ compactification quite a lot in this work. Its metric is specified by its complex structure $U = U_1 + iU_2$ and volume $V_2$ such that

$$G_{ij} = \begin{pmatrix} g_{88} & g_{89} \\ g_{89} & g_{99} \end{pmatrix} = \begin{pmatrix} R_1^2 & R_1R_2cos\omega \\ R_1R_2cos\omega & R_2^2 \end{pmatrix} = \frac{V}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}. \tag{3.77}$$

where $R_1$ and $R_2$ are the radii of the two one-cycles of the torus and $\omega$ is the angle of inclination between these two directions: see figure 3.4. Thus the compactification of

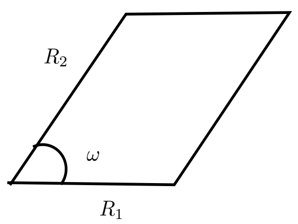


FIGURE 3.4: Torus complex structure and radii

the higher dimensional Einstein-Hilbert action on $T^2$ yields

$$\int d^D x\sqrt{det(G_{\hat{\mu}\hat{\nu}})}\hat{R} \rightarrow \int d^{D-2}x\sqrt{det(g)}\left[R - \frac{1}{4}\partial_\mu V\partial^\mu V + \frac{\partial_\mu U\partial^\mu \bar{U}}{4U_2^2} + \cdots\right] \tag{3.78}$$

### 3.5.3 Calabi-Yau compactification

A more complicated example of the compact internal spaces are the Calabi-Yau manifolds (CY for short) which are compact n-complex dimensional Kähler Ricci-flat manifold with holonomy SU(n). The restriction of holonomy group to SU(n) instead of general SO(2n) endows the CY manifolds the property that the compactification upon such manifolds preserves only half of the original supersymmetry. In case of circle and torus compactification, there were no such constraint on the holonomy group and thus compactification on such internal manifolds breaks no supersymmetry. From the phenomenological point of view, CY 3-folds are of immense importance because compactification of 10D string theories upon CY 3-folds yields the 4D theories. We shall however concentrate mostly on the CY 2-fold K3 surface.

A K3 surface is a Calabi-Yau 2-fold with vanishing first Chern class $c_1(K3) = 0$ and holonomy SU(2). The geometric informations of any CY manifold are contained in the Hodge numbers $h^{p,q}$, which are the dimensions of the Dolbeault cohomology groups ($h^{p,q} = dim(H^{p,q})$)

$$H^{p,q} = \frac{\text{space of } \bar{\partial} - \text{closed (p,q) forms}}{\text{space of } \bar{\partial} - \text{exact (p,q) forms}} \tag{3.79}$$

and satisfy

$$h^{p,q} = h^{q,p}, \quad h^{p,q} = h^{n-q,n-p}, \quad h^{p,p} > 0. \tag{3.80}$$

For any connected CY manifold $h^{0,0} = 1$ and $h^{0,n} = h^{n,0} = 1$ whereas $h^{p,0} = 0$ for $p \neq 0, n$ (we refrain from giving detailed proofs of these relations which can be found in the standard references of Eguchi, Gilkey & Hanson [38], Nakahara [39], Nash & Sen [40]). The Hodge numbers are represented conveniently in Hodge diamond as in figure 3.5 (example for a CY-3-fold).

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}$$

FIGURE 3.5: Hodge diamond for a CY 3-fold

Using the natural inner product $< \eta_1, \eta_2 >= \int_{CY} \eta_1 \wedge *\eta_2$ for (p,q)-forms $\eta_1, \eta_2 \in H^{p,q}$ one can define the adjoint operators $\partial^\dagger$ and $\bar{\partial}^\dagger$ for $\partial$ and $\bar{\partial}$ respectively and construct the two Laplacians

$$\triangle_\partial = \partial\partial^\dagger + \partial^\dagger\partial, \quad \triangle_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}. \tag{3.81}$$

A (p,q)-form $X_{p,q}$ is called $\triangle_{\bar{\partial}}$ harmonic if

$$\triangle_{\bar{\partial}} X_{p,q} = 0 \tag{3.82}$$

and are in one-to-one correspondence with the generators of $H^{p,q}(K)$.

A somewhat coarser information of geometry of the complex manifold is contained in the de Rham cohomology defined with respect to the exterior derivative $d$. The r-th cohomology group $H^r(K)$ of 2n-dimensional manifold K is the space

$$H^r = \frac{\text{space of d-closed r-forms}}{\text{space of d-exact r-forms}}. \tag{3.83}$$

The dimension of $H^r$ is known as r-th Betti number $b_r$ which are in fact

$$b_r = \sum_{r=p+q} h^{p,q} \tag{3.84}$$

in terms of Hodge numbers (3.79). The Laplacian on p-forms can be written in terms of exterior derivative as

$$\triangle = (d + *d*)^2. \tag{3.85}$$

A p-form $X_p$ is harmonic if $\triangle X_p = 0$ and are in one-to-one correspondence with the generators of $H^p(K)$. Finally, the Euler characteristic of the manifold K is expressed as

$$\chi(K) = \sum_r^{2n} (-1)^r b_r. \tag{3.86}$$

In case of K3 the Hodge numbers are $h^{1,1} = 20$, $h^{2,0} = h^{0,2} = 1$, $h^{1,0} = h^{0,1} = 0$, $h^{2,2} = h^{0,0} = 1$ and hence for K3 we have $\chi = h^{1,1} + 2h^{0,0} + 2h^{0,2} = 24$ in terms of Hodge numbers.

For a generic field $\Phi$ in the higher dimensional theory, whose dynamics, say, is governed by an equation

$$\mathcal{D}_D \Phi = 0 \tag{3.87}$$

in D-dimensions with $\mathcal{D}_D$ some D-dimensional differential operator. Under decomposition $\mathcal{M}_D = \mathcal{M}_d \times CY_n$ such that $\mathcal{M}_d$ a Minkowski space and $CY_n$ be a compact Calabi-Yau n-fold such that $D = d + 2n$ the differential operator decomposes as $\mathcal{D}_D = \mathcal{D}_d + \mathcal{D}_n$. the solution for the d-dimensional fields can now be obtained by taking the product ansatz $\Phi = \phi_d \otimes f_m$ where $f_m$ are the eigenfunctions of the CY differential operator $\mathcal{D}_n$ so that

$$\mathcal{D}_n f_m = m^2 f_m. \tag{3.88}$$

Then the dynamics of the KK modes $\phi_d$ are governed by

$$(\mathcal{D}_d + m^2)\phi_d^m = 0. \tag{3.89}$$

For the reduction of symmetric tensor fields like $g_{\mu\nu}$, the corresponding Laplace operator is the Lichnerowicz operator.

In case of a complex Kähler manifold (and therefore for a CY manifold $CY_n$) the differential operator $\mathcal{D}_n$ for the internal manifold can be shown to be de Rham Laplacian (3.85) (upto sign). Therefore the the number of massless modes of the KK reduction over CY manifold is provided by the Betti numbers (or with Hodge numbers) which were defined in (3.84).

One more point of interest is the moduli space of the internal manifold $\mathcal{M}_c$. It is defined as the space of metric deformations which preserve the internal manifold $\mathcal{M}_c$. The scalars parametrizing the moduli space appear as massless scalars in the action of the reduced $\mathcal{M}_d$ theory. For example, in circle compactification, the radius of the circle R is a free parameter and appear in the lower dimensional theory as a free massless scalar. In case of $T^2$ compactification, the reduced action (3.78) contains the kinetic term $\frac{\partial_\mu U \partial^\mu \bar{U}}{4U_2^2}$ for the complex structure U which is a moduli of $T^2$. In case of CY n-fold, there are two different class of moduli called the complex structure moduli and Kähler moduli which are deformations of the CY metric $\delta g_{ab}$ preserving the Calabi-Yau condition. It can be shown that the total moduli space of CY metrics is a direct product of complex moduli space and Kähler moduli space with dimensions $h^{2,1}$ and $h^{1,1}$ respectively.

In case of K3 surface ($h^{2,1} = 0$), the harmonic forms can be split into self-dual and anti-self-dual forms with respect to the 4D metric defined on it. There are 19 self-dual forms out of 20 (1,1) forms and 3 anti-self-dual forms from the combination of 1 (1,1) form with (0,2) and (2,0) forms. The moduli space of K3 is 58 dimensional with 20 (=$h^{1,1}$) real parameters specifying the Kähler class and 38 real or 19 complex parameters specifying the complex structure.

## 3.6 Duality in string theory

Although there are 5 different string theories in 10D, compactifying these theories on some suitably chosen compact manifolds, one can get same lower dimensional theories from two different 10D theories. This particularity of string theory is called the duality. Duality relations map the string coupling constant, the data of the shape and size of the compactification manifold and various other background fields of different string theories. In cases it happens to be that perturbative regime of one string theory compactified on a certain manifold is dual to the perturbative regime of another string theory compactified on a certain manifold. There are also cases where the strong-coupling regimes of some theory is mapped to the weak-coupling regime of some other theory thereby providing tools for probing the non-perturbative physics of the former theory. In the paradigm of effective theory, one also needs to consider the non-perturbative aspects of the theory too, and for the case of string theory, it is thus necessary to consider the non-perturbative M-theory and F-theory to gain complete knowledge string interactions (in terms of low energy effective action). We shall discuss the M-theory and F-theory in better details in subsequent sections. For instance we shall discuss the well known T-duality, S-duality and U-duality relations amongst string theory. The duality network between different string theories is in fact complemented by the inclusion of the compactifications of M and F-theory. We give a schematic view of the duality relations in figure 3.6.

### 3.6.1 T-duality

T-duality is the acronym for "target space duality" which relates string compactifications on spaces that admit continuous isometries. The classic example of this duality is the duality between compactification of type IIA string theory on a circle of radius R and the compactification of type IIB in the dual circle of radius $\alpha'/R$. As a matter of fact, taking the $X^9$ space dimension to be the compactification circle for type IIA theory, one finds the following identification

$$X^9(\tau, \sigma + 2\pi) = X^9(\tau, \sigma) + 2\pi w R \tag{3.90}$$

where the integer $w$ denotes the winding number, i.e., the number of times the closed string wraps the compactified dimension. The compactification along $X^9$ space-dimension leads to the quantization of the momentum in the 9th dimension

$$p^9 = \frac{n}{R}. \tag{3.91}$$

The closed string sectors are now labelled by the couple $(n, w)$. Compactifying the type IIB theory on the dual circle of radius $R' = \frac{\alpha'}{R}$ in fact exchanges the winding number $w$ of type IIA with the momentum number n of type IIA and vice versa. The Kaluza-Klein mass level is given by

$$m^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{wR}{\alpha'}\right)^2 + \text{oscillator modes}. \tag{3.92}$$

Thus under the exchange of $(n, w)$ couple and the radius $R \leftrightarrow \alpha'/R$ one obtains the same tower of states in 9D.

One can also repeat the same argument for the Heterotic SO(32) string theory compactified on a circle which happens to be dual to the compactification of $E_8 \times E_8$ Heterotic theory on the dual circle. In this case, the particular point of interest is that

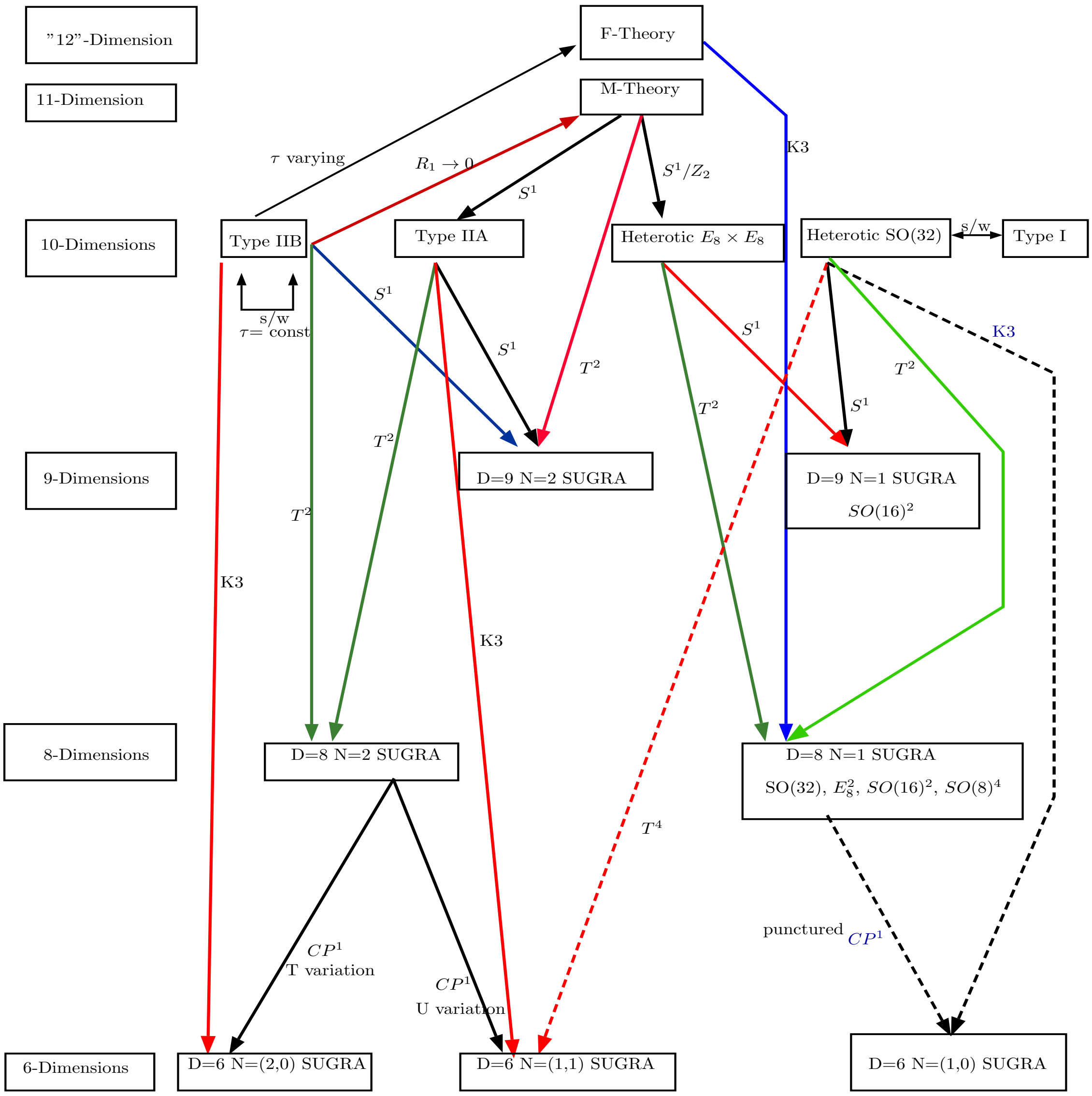


FIGURE 3.6: String and SUGRA duality network

one has to consider the Wilson lines along the compactification circle which breaks both SO(32) and $E_8 \times E_8$ theory to the subgroup $SO(16) \times SO(16)$. The T-duality relation then relates the compactification along dual radius as in case of type II theories along with the exchange of momenta and winding numbers plus the reordering of the non-zero Wilson lines along the compactification circle.

In case of open strings, the exchange of winding and momenta numbers interchanges Neumann and Dirichlet boundary conditions 3.12. As a result, a Dp-brane in i-th direction is turned into a D(p-1) brane localized at a point along the dual $\bar{i}-$th direction and vice versa. In case of a stack of N coincident Dp-branes, the information about the relative positions of the dual D(p-1) branes is included in Wilson lines which is a non-trivial constant vacuum expectation value of the i-th component of the non-Abelian gauge field living on the branes.

### 3.6.2 S-duality

S-duality is the acronym for "self-duality" or "strong/weak duality" which relates the perturbative regime with the non-perturbative regime of the same or another theory respectively. In the weak coupling limit, the non-perturbative states e.g. D-branes or other BPS states become massive and decouple from the theory whereas in the strong coupling limit they become light and participate in the effective action. The strong-weak duality thus relates the fundamental strings of the perturbative theory to the non-perturbative states e.g. D1 branes of the dual strongly coupled theory. The example of the strong/weak duality case is the duality between the strongly coupled type I theory with that of the weakly coupled SO(32) Heterotic theory in 10D. The classic example of self-duality is in fact provided by the 10D type IIB theory which we shall discuss below in detail and this shall pave the way for the discussion of F-theory in section 3.8.

### 3.6.3 S duality of type IIB string theory

Low energy effective actions of superstring theories in 10D are that of the effective actions of the supergravity theories in 10D. In case of type IIB, the supergravity effective action is invariant under $SL(2,\mathbb{R})$ action while the superstring effective action is invariant only under the discrete subgroup $SL(2,\mathbb{Z})$ due to the quantization [10].

The effective action of type IIB theory in string frame is

$$
\begin{aligned}
S_{IIB} \;&=\; \frac{1}{2\kappa^2}\int d^{10}x\sqrt{g_{10}}\left[e^{-2\phi}\left(R+4\partial_\mu\phi\partial_\mu\phi-\frac{1}{2}|H_3|^2\right)\right] \\
&\quad -\int \frac{1}{12}G_3\wedge *_{10}\bar{G}_3+\frac{1}{2}F_5\wedge *_{10}F_5 \\
&\quad +\frac{1}{2i}\int C_4\wedge G_3\wedge\bar{G}_3,
\end{aligned}
\tag{3.93}
$$

Changing towards the Einstein frame with the metric redefinitions

$$
g^E_{\mu\nu}=e^{-\Phi/2}g_{\mu\nu} \tag{3.94}
$$

and with the complex combination of axion and dilaton field: called the axi-dilaton field collectively

$$
\tau=C_0+ie^{-\Phi} \tag{3.95}
$$

plus the combination of field strengths of $B_2$, $C_2$ and $C_4$

$$
H_3=dB_2,\quad F_3=dC_2,\quad G_3=i\frac{F_3+\tau H_3}{\sqrt{\tau_2}},\quad F_5=dC_4+C_2\wedge H_3 \tag{3.96}
$$

as usual (see (2.24)), the type IIB effective action can be recast in the form

$$
S_{IIB}=\frac{1}{2\kappa_{10}^2}\int_{\mathcal{M}_{10}} R*1-\frac{1}{2}\frac{d\tau\wedge *d\bar{\tau}}{\tau_2^2}-\frac{1}{2}G_3\wedge *\bar{G}_3-\frac{1}{4}F_5\wedge *F_5-\frac{i}{2}C_4\wedge G_3\wedge\bar{G}_3. \tag{3.97}
$$

The above action is now can be seen to be invariant under the action of $SL(2,\mathbb{R})$: using the $SL(2,\mathbb{R})$ matrix

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{R}),\quad ad-bc=1, \tag{3.98}
$$

under which the axi-dilaton field $\tau$ transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{3.99}$$

and the $B_2$ and $C_2$ fields rotate into one another

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \tag{3.100}$$

the action is invariant provided that the Einstein frame metric and the RR four form $C_4$ and hence the field strength $F_5$ are invariant. Semi-classical arguments suggest that only the discrete subgroup $SL(2,\mathbb{Z})$ of the classical invariance group $SL(2,\mathbb{R})$ can be realized in the quantum theory. In fact the $B_2$ field couples electrically to the fundamental type IIB string F1 with field strength $dB_2$ and the path integral quantization of the theory imposes that the flux $\int_{X3} dB_2$ should be quantized i.e.

$$\int_{X3} dB_2 \in \mathbb{Z} \tag{3.101}$$

where $X3$ is any 3-cycle in space-time. Similar argument shows that $C_2$ field couples electrically with the D1 brane, called the D-string in type IIB and its flux should also be quantized in accordance with the consistency of the path integral quantization $\int_{X3} dC_2 \in \mathbb{Z}$. Thus both $B_2$ and $C_2$ are integrally quantized and in view of their inter-mixing under $SL(2,\mathbb{R})$ as in (3.100), only the $SL(2,\mathbb{Z})$ subgroup preserves the quantization conditions.
The group $SL(2,\mathbb{Z})$ is generated by the transformations

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.102}$$

which act on $\tau$ as

$$\tau \underset{T}{\to} \tau + 1, \quad \tau \underset{S}{\to} -\frac{1}{\tau}. \tag{3.103}$$

In a simple background with $C_0 = 0$ the S transformation above implies

$$g_s \to \frac{1}{g_s} \tag{3.104}$$

which is in fact the weak/strong duality that is weakly coupled perturbative theory is mapped to a strongly coupled non-perturbative theory. In the case at present both the theories happen to be type IIB: the weakly coupled one being the theory of F1 strings while the strongly coupled one being that with the D1 string. In the weak coupling limit D-branes do not participate in the dynamics as they acquire infinite tension. For finite or large coupling, one can consider the dynamical BPS objects that can be thought of as bound states of F1 and D1 strings called the (p,q)-strings and couple electrically to $B_2$ with charge p and to $C_2$ with charge q. Thus a (1,0) string is the fundamental string while (0,1) string is a D1 brane. The expression for the tension of the (p,q) string is

$$\tau_{(p,q)} = \frac{|p + \tau q|}{\sqrt{\tau_2}} \tag{3.105}$$

which is $SL(2,\mathbb{Z})$ invariant if under $SL(2,\mathbb{Z})$ the charges (p,q) transform as

$$\begin{pmatrix} p & q \end{pmatrix} \to \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \tag{3.106}$$

from which $p\int B_2 + q\int C_2$ is also seen to be invariant. Fundamental strings can end on D1-brane so that (p,q) strings can have several prongs, a configuration referred to as string junctions [41, 42, 43].

Under $SL(2,\mathbb{Z})$ transformation, NS5 branes are mixed with D5 branes much similar to the mixing of fundamental strings and D1 branes. D3 branes are unaffected by $SL(2,\mathbb{Z})$ as they are self-dual under the action of the latter group. A D7 brane couple magnetically to $C_0 = Re\tau$ which transforms non-trivially under $SL(2,\mathbb{Z})$ as in (3.103). Thus one expects the presence of (p,q)-7 branes on which a (p,q) string can end.

### 3.6.4 U-duality

We end this section with the discussion of U-duality. So far we have seen the perturbative connection between type IIA and IIB theories on a circle and the non-perturbative connection of type IIB with itself. As we shall discuss in the next section, these two aspects can be unified via the compactification of the non-perturbative M-theory which is an eleven-dimensional theory. In short, the low-energy limit of M-theory compactified on a circle yields type IIA theory in 10D. We shall also elaborate that the torus compactification of M-theory explains the $SL(2,\mathbb{Z})$ symmetry of type IIB, i.e. compactify on a torus to get type IIA in 9D and then decompactify along the dual one-circle of the torus. It turns out that under further compactification on tori of type II theories, the perturbative T-duality symmetries mix with the non-perturbative symmetries inherited from M-theory and generate a larger duality symmetry called the U-duality. We illustrate this point once more with type II theories. Compactification of type II theories on torus $T^2$ of complex structure $U = U_1 + iU_2$ and of volume $V_2$ happen to have the perturbative symmetry under the exchange of complex structure U with the Kähler structure $T = B_{89} + iV_2$ which is the complexification of the torus volume $V_2$ with the scalar $B_{89}$ obtained from the compactification of the type II $B_2$ field on with both of its legs on the two one-cycles of the torus (which are along the 8th and 9th space-dimensions). The resulting perturbative symmetry being $SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U$. However, it happens to be that the $SL(2,\mathbb{Z})_T$ symmetry mixes with the $SL(2,\mathbb{Z})_\tau$ symmetry of the type IIB theory yielding for the larger symmetry $SL(3,\mathbb{Z})$. Therefore, the U-duality group becomes $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})_U$.

In the next sections we shall explore M-theory and F-theory and their connection with type II and Heterotic theories in better details thereby providing proofs of the duality network 3.6.

## 3.7 M-theory: relation with type IIA and Heterotic theory

Looking at the effective action of type IIA supergravity (2.22) and that of the 11D one (2.21), we see that the former is the circle compactification of the latter. To see this relation in string theoretic level, we note that the NS-NS two form $B_2$ couples electrically to the perturbative type-IIA string and magnetically with the NS5-Brane, the RR 1-form $C_1$ can couple electrically to $D0$ branes (or particles) and magnetically to $D6$ branes. Finally the RR three form $C_3$ can couple electrically to a membrane and magnetically to four-branes. The emergence of an eleventh dimension can be seen from the spectrum of $D0$ branes of type IIA. The D0 brane tension or mass is given by $\tau_{D0} = \frac{2\pi}{\sqrt{\alpha' g_s}}$ and these states can form bound states whose energy is just the sum of the energy of the constituents i.e. a bound state of n D0 branes has mass $n\tau_D$. This seems like a tower of equally spaced massive states that become light as the string coupling constant increases and thus can be interpreted as a Kaluza-Klein spectrum of a circle compactification. Thus the effective type IIA field theory is a dimensional reduction of the 11D supergravity with the $g_{11,11}$ component of the 11D metric identified with the type IIA string coupling (this can be easily verified using the circle compactification ansatz as in section 3.5.1). Thus taking the string coupling constant $g_S \to \infty$ type IIA theory becomes 11D theory whose low energy limit is the 11D supergravity: this 11D non-perturbative theory is known as M-theory [44, 45, 46].

M-theory contains the three form that can couple to a membrane $M2$ and its magnetic dual 5-brane $M5$. An M2 brane not winding around the circle is equivalent to the type IIA D2 brane coupling to the three form $C_3$. Also the M2 brane wrapped around the circle becomes the perturbative type IIA string coupling electrically with $B_{\mu\nu}$. Wrapping the M5 brane around the compactification circle provides the D4 brane coupling magnetically to $C_3$ (the D6 brane of the type IIA theory can be explained as KK monopole) [31].

There is an interesting relation between M-theory and type IIB theory [47]. We have seen that type IIB string compactified on a circle of radius R is dual to type IIA theory compactified on a circle of dual radius $\alpha'/R$. The latter is dual to M-theory compactified on a circle. Thus M-theory compactified on a torus $T^2$ is expected to be dual to type IIB theory compactified on a circle. The construction of 10D type IIB theory from M-theory can be seen from the following: compactify M-theory on a torus of complex structure $U = U_1 + iU_2$ which can be characterized by the radii of the two one cycles $R_1$ and $R_2$ of the torus by $U_1 = \frac{R_2}{R_1} cos\omega$, $U_2 = \frac{R_2}{R_1} sin\omega$ (see 3.4). The type IIA compactification circle may be identified with say $R_1$ then shrink this one-cycle to zero size while keeping the complex structure $U$ fixed. The complex structure gives rise to the axi-dilaton field $U = U_1 + iU_2 = C_0 + ie^{-\phi}$ while compactification of $C_3$ RR form of M-theory compactified to $C_1$ decompactifies to $C_2$ in 10D and the uncompactified $C_3$ gives rise to the RR form $C_4$. The 10D metric is automatically recovered from the non-shrinked $R_2$ cycle of the torus. Thus one recovers the type IIB theory in 10D.

### 3.7.1 Heterotic theory from M-theory: Horava-Witten mechanism

A classic example of orbifold compactification which provides an impressive demonstration of the role played by anomaly cancellation is provided by the Horava-Witten mechanism in M-theory compactified on $S^1/\mathbb{Z}_2$ giving rise to $E_8 \times E_8$ Heterotic theory in 10D [48, 49]. The compactification on this orbifold results in a chiral spectrum which is crippled with anomaly, the way out to cancel these anomalies is through the introduction of 10D vector multiplets on the orbifold fixed planes and the anomaly cancellation condition fixes the gauge group on each fixed plane to be $E_8$. The dynamics of the resulting theory corresponds to a strongly coupled version of $E_8 \times E_8$ Heterotic string theory. Let us compactify the 11D supergravity on a circle which can be taken as the compactification of the 11th space-time dimension that is $x_{10}$. Next imply the $\mathbb{Z}_2$ parity transformation acting on $x_{10}$ as $\mathbb{Z}_2 : x_{10} \to -x_{10}$. From the reduction of 11D metric $g_{MN}$ one gets $g_{MN} \to (g_{\mu\nu}, g_{\mu,10}, g_{10,10}) = (g_{\mu\nu}, A_\mu, \phi)$. From the reduction of 11D 3-form $C_3$ one gets $C_{MNP} \to c_{\mu\nu\rho}, B_{\mu\nu}$. However, under the

$\mathbb{Z}_2$ parity operation $c_{\mu\nu\rho} \to -c_{\mu\nu\rho}$. Thus we have to mod this state out of the spectrum. From the 11d gravitino, we get 10D gravitino and a dilatino: $\psi_M \to \psi_\mu^+, \chi^-$. The parity even spectrum in 10D theory is then $g_{\mu\nu}, B_{\mu\nu}, \phi, \psi_\mu^+, \chi^-$ which is the spectrum of the gravity multiplet of the 10D N=1 supergravity theory. The appearance of the chiral fermions in the theory gives rise to the chiral anomaly. The proposition of Horava-Witten mechanism to curb this anomaly is to introduce 10D N=1 vector multiplet $(A_\mu, \lambda^+)$ in each of the orbifold fixed planes or on the two boundaries with $A_\mu$ transforming under the adjoint representation of a gauge group $G_{1,2}$. The total anomaly of the gravity multiplet i.e. the bulk anomaly (using the anomaly index polynomials given in (2.65) and (2.62))

$$\begin{aligned} I_{12}^{\text{bulk}} &= \frac{1}{2} I_{3/2}^{d=10} - \frac{1}{2} I_{1/2}^{dilatino} & (3.107)\\ &= \frac{2\pi}{2(4\pi)^6}\left[\frac{-495}{5670} trR^6 + \frac{225}{4320} trR^4 trR^2 - \frac{63}{10368}(trR^2)^3\right] \\ &\quad - \frac{2\pi}{2(4\pi)^6}\left[\frac{1}{5670} trR^6 + \frac{1}{4320} trR^4 trR^2 + \frac{1}{10368}(trR^2)^3\right] \\ &= -\frac{2\pi}{2(4\pi)^6}\left[\frac{496}{5670} trR^6 - \frac{224}{4320} trR^4 trR^2 + \frac{64}{10368}(trR^2)^3\right] & (3.108) \end{aligned}$$

The anomaly from the boundary vector multiplet

$$\begin{aligned} I_{12}^{\text{boundary}} &= (dim(G_1) + dim(G_2))\frac{2\pi}{2(4\pi)^6}\left[\frac{1}{5670} trR^6 + \frac{1}{4320} trR^4 trR^2 + \frac{1}{10368}(trR^2)^3\right] \\ &\quad + \frac{2\pi}{2(4\pi)^4}\frac{TrF_1^2 + TrF_2^2}{2(2\pi)^2}\left[\frac{1}{360} trR^4 + \frac{1}{288}(trR^2)^2\right] \\ &\quad + (2\pi)\left[\frac{TrF_1^4 + TrF_2^4}{2\times 4!(2\pi)^4}\frac{trR^2}{12(4\pi)^2}\right] \\ &\quad + (2\pi)\left[\frac{TrF_1^6 + TrF_2^6}{2\times 6!(2\pi)^6}\right] & (3.109) \end{aligned}$$

The total anomaly is the sum of the bulk and two boundary terms (3.107) and (3.109)

$$\begin{aligned} I_{12}^{d=10,N=1} &= I_{12}^{\text{bulk}} + I_{12}^{\text{boundary}} \\ &= \frac{2\pi}{2(4\pi)^6}\{\frac{(dim(G_1) + dim(G_2)) - 496}{5670} trR^6 \\ &\quad + \frac{224 + (dim(G_1) + dim(G_2))}{4320} trR^4 trR^2 \\ &\quad + \frac{(dim(G_1) + dim(G_2)) - 64}{10368}(trR^2)^3 \\ &\quad + \left[\frac{1}{180} trR^4 + \frac{1}{144}(trR^2)^2\right](TrF_1^2 + TrF_2^2) \\ &\quad + \frac{1}{18} trR^2 (TrF_1^4 + TrF_2^4) + \frac{4}{45}(TrF_1^6 + TrF_2^6)\} \end{aligned}$$
(3.110)

The anomaly term above can be cancelled according to the Green-Schwarz anomaly cancellation in 10D N=1 supergravity theory [50, 51, 52] (see section 5.1 for more details). The requirement is that the anomaly polynomial $I_{12}$ can be factorised as

$I_4 \times I_8$ so that one can write the counter-term

$$S_c = \frac{g_s^2}{\kappa^2} \int B_2 \wedge I_8 \tag{3.111}$$

with $B_2$ transforming non-trivially under the gauge and Lorentz transformation according to

$$\delta B_2 = \frac{\kappa^2}{g_s^2} Q_4^1 \tag{3.112}$$

where $I_4 = dI_3$ and $\delta I_3 = dQ_4^1$ according to (2.59) with $g_s$ is the Heterotic string coupling constant so that the anomalous phase variation due to the factorized anomaly polynomial, (using the descent equations (2.59))

$$\mathcal{A} = \int \delta I_4 \wedge I_8 = \int Q_4^1 \wedge I_8 = -\delta S_c \tag{3.113}$$

is cancelled by the gauge-gravity variation of the counter-term (3.111). Thus to use this mechanism, the requirements are that $trR^6$ term must vanish i.e. $(dim(G_1) + dim(G_2)) = 2 \times 248$ and that the 6th order Casimir invariant $TrF^6$ (Tr in adjoint) must be decomposed in terms of 4th order and 2nd order Casimirs $TrF^4$ and $TrF^2$. These conditions are in fact met by choosing the gauge group $G_{1,2}$ at each boundary to be $E_8$ and thus one recovers the $E_8 \times E_8$ Heterotic theory in 10D. Using the gauge group $E_8$ living at each boundary we get the factorized anomaly polynomial

$$\begin{aligned} I_{12} \quad &= \quad \frac{2\pi}{2(4\pi)^6} \frac{1}{6} (trR^2 + trF_1^2 + trF_2^2) \wedge \\ & (trR^4 + \frac{1}{4}(trR^2)^2 + trR^2(trF_1^2 + trF_2^2) - 2trF_1^2 trF_2^2 + 2(trF_1^2)^2 + 2(trF_2^2)^2) \end{aligned} \tag{3.114}$$

so that the form of the anomaly cancelling term is

$$S_c = \frac{g_s^2}{\kappa^2} \int B_2 \wedge (trR^4 + \frac{1}{4}(trR^2)^2 + trR^2(trF_1^2 + trF_2^2) - 2trF_1^2 trF_2^2 + 2(trF_1^2)^2 + 2(trF_2^2)^2) \tag{3.115}$$

It is not difficult to see that the anomaly cancelling counter-term (3.115) is essentially furnished by the compactification of M-theory. Recall that the effective action of the 11D supergravity contains the following Chern-Simons terms in its effective action [6, 7]

$$S_{11} \supset -\frac{1}{2\kappa_{11}^2} \int \frac{1}{6} C_3 \wedge G_4 \wedge G_4 - \frac{2\pi}{(4\pi\kappa_{11}^2)^{1/3}} \int C_3 \wedge X_8. \tag{3.116}$$

The $X_8$ term in the above is $X_8 = \frac{1}{192(2\pi)^4}(trR^4 - \frac{1}{4}(trR^2)^2)$. The restriction or the boundary condition on $G_4$ term for the consistency of the vector multiplet leaving in the boundary surface reads as

$$G_4|_{\text{i-th boundary}} = \frac{\kappa^2}{g_s^2}(trF_i^2 + \frac{1}{2} trR^2) \tag{3.117}$$

which is in fact at the heart of the relation (3.112). However, $G_4 = dC_3$ so that solving for $C_3$ from (3.117) gives

$$C_3 = \Omega_3 + dO_2 \tag{3.118}$$

where $\Omega_3$ is the descent from $\frac{1}{2}trR^2 + TrF^2$ such that ($A$ being the gauge connection and $\omega$ the spin-connection (2.12))

$$\Omega_3 = Q_3 = Tr(A_i \wedge F_i - i\frac{A^3}{3}) + \frac{1}{2}tr(\omega \wedge R - i\frac{\omega^3}{3}) \tag{3.119}$$

hence the gauge gravity variation of $C_3$ is

$$(\delta^{gauge} + \delta^{grav})C_3 = \frac{\kappa^2}{g_s^2}d(Tr\Sigma_i dA_i + \frac{1}{2}trvd\omega). \tag{3.120}$$

Now compactifying the Chern-Simons term (3.116) of M-theory on the interval $S^1/\mathbb{Z}_2$ enforcing the boundary condition (3.117) for both the boundaries with $C_3 \rightarrow B_2$ give rise to the term

$$S_c = \frac{\kappa^2}{g_s^2}\int B_2 \wedge (G_4|_1 \wedge G_4|_2 + X_8) = \frac{g_s^2}{\kappa^2}\int B_2 \wedge I_8 \tag{3.121}$$

which is exactly the desired anomaly cancelling term (3.115). Thus we see the compactification of M-theory on $S^1/\mathbb{Z}_2$ gives rise to $E_8 \times E_8$ Heterotic theory in a self-consistent manner.

We shall refer to the amplitude computation in Horava-Witten background in section 5.5.

## 3.8 F-theory: relation with type IIB and Heterotic theory

The relation between M-theory and type IIA theory shows that the non-perturbative compactification of type IIA theory can be described by the suitable compactification of M-theory. Likewise, the theory which describes the non-perturbative compactification of type IIB string theory is called the F-theory [53, 54, 47]. F-theory is the geometrization of the $SL(2,\mathbb{Z})$ symmetry of type IIB string theory which we have noted in section 3.6.3. This symmetry in conventional type IIB is global and to define F-theory one has to gauge or localize this particular symmetry. The $SL(2,\mathbb{Z})$ transformation properties hint to the symmetry of a torus and thus localization of $SL(2,\mathbb{Z})$ would mean to have elliptic fibration or torus fibration on a certain manifold. Let us view this idea more clearly. Consider an elliptically fibered Calabi-Yau manifold $\mathcal{M}$ with a $2 \times d$-real dimensional base $\mathcal{B}$ (that is a base of complex dimensions d) obtained by defining torus fiber at each point of the base manifold so that the manifold $\mathcal{M}$ is of real dimensions 2d+2. Now in conventional type IIB string theory, the axio-dilaton field is taken to be constant and in particular at weak coupling ($g_s << 1$). Instead if we now compactify type IIB theory on the base manifold $\mathcal{B}$ such that the torus fiber is identified with the varying axio-dilaton field, that is, if we denote $U(z_1, \cdots, z_{2d}) = U_1(z_1, \cdots, z_{2d}) + iU_2(z_1, \cdots, z_{2d})$ to be the complex structure of the torus fiber at the point on the base manifold defined by the coordinates $(z_1, \cdots, z_{2d})$ and we identify this varying complex structure with the varying axio-dilaton field $\tau(z_1, \cdots, z_{2d}) = \tau_1(z_1, \cdots, z_{2d}) + i\tau_2(z_1, \cdots, z_{2d})$ i.e.

$$U(z_1, \cdots, z_{2d}) = \tau(z_1, \cdots, z_{2d}) \tag{3.122}$$

then the above compactification of type IIB on $\mathcal{B}$ is said to define the compactification of F-theory on the manifold $\mathcal{M}$.

The generic non-perturbative property of F-theory seems to be forbidding for any perturbative information to be drawn out of it. However certain compactification of F-theory are dual to compactifications of other perturbative theories and we can exploit these duality relations to shed light on F-theory and its properties. In particular we shall be exploring the duality between F-theory compactified on elliptically fibered K3 and Heterotic string theory compactified on a torus $T^2$ both leading to D=8 N=1 supergravity theories in low-energy limit.

It is worthwhile to note that geometrization of the $SL(2,\mathbb{Z})$ symmetry of type IIB gives a 12 dimensional appearance to F-theory. One should however bear in mind that the 2D "torus" in question is a book-keeping device only used to visualize the elliptic nature of the fibration [55].

In what follows, we shall concentrate mostly on the K3 compactification of F-theory and the generalization to compactification on other elliptically fibered Calabi-Yau can be accomplished easily. An important feature of type IIB $SL(2,\mathbb{Z})$ symmetry is the presence of (p,q)-strings which couple electrically with $(B_2, C_2)$ fields and the presence of (p,q) 7-Branes which couple magnetically to the axio-dilaton field $\tau$ accounting for the non-trivial monodromy transformations of the $\tau$ while encircling the 7-branes. In fact, encircling the 7-brane position on the transverse space, the axion field changes by $C_0 \to C_0 + 1$. Because of this coupling the 7-branes backreact on the geometry and F-theory can be thought of as a geometric framework to incorporate seven-branes in a type IIB compactification in a fully consistent, backreacted and non-perturbative way.

Consider an elliptically fibered K3 with base $\mathbb{CP}^1$. This fibration can be defined by the Weierstrass model

$$y^2 = x^3 + f(z)x + g(z) \tag{3.123}$$

with the discriminant

$$\Delta = 4f^3 + 27g^2 \tag{3.124}$$

where $z, \bar{z}$ are the complex coordinates on $\mathbb{CP}^1$, f and g are respectively 8 and 12 degree polynomials in z. The complex structure of the torus (the varying axio-dilaton field of type IIB compactified on $\mathbb{CP}^1$) varies over the base space $\mathbb{CP}^1$ according to the relation

$$j(\tau(z)) = \frac{4(24f(z))^3}{27g^2(z) + 4f^3(z)} = \prod_{i=1}^{24}(z - z_i). \tag{3.125}$$

where $j$ is the Leech-j function defined in appendix B. There are 24 point on the $\mathbb{CP}^1$ where the torus fiber degenerates (see figure 3.7) and are in fact point where $\Delta = 0$. These are infact the positions of the 7-branes on $\mathbb{CP}^1$. The 7-brane solution to type IIB supergravity equations of motion can be inferred from the compactification of type IIB on $\mathbb{CP}^1$ with the metric ansatz [56]

$$ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_7^2 + \underbrace{e^{\phi(z,\bar{z})} dz d\bar{z}}_{\mathbb{CP}^1}. \tag{3.126}$$

The presence of 7-branes can be seen also intuitively: the first Chern class of the base $\mathbb{CP}^1$ is non-zero: $c_1(\mathbb{CP}^1) = 2$ and this cannot be a supersymmetry preserving background. The remedy is to add 7-branes to the theory which sit at arbitrary points $z_i$ on $\mathbb{CP}^1$ and otherwise fill the 7+1 non-compact space-time dimensions (figure 3.7). From the Wierestrass description above, the $z_i$ positions of the 7-branes are in fact roots of the polynomial $\Delta = 0$ which is of degree 24 and thus there are 24 singular points on $\mathbb{CP}^1$ i.e. 24 7-branes in the theory each contributing to $-1/12$ to the first Chern class due to their back-reaction on the geometry. This combination $24 \times \frac{-1}{12} = -2$ combines

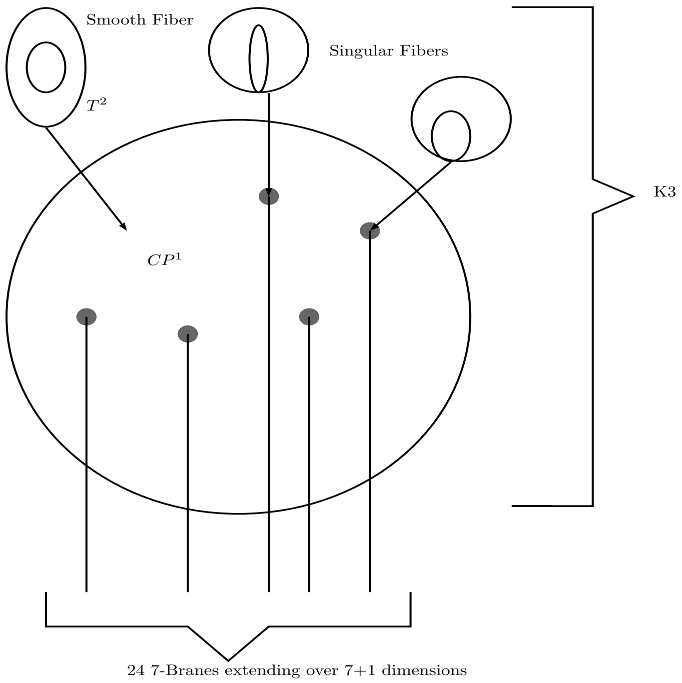


FIGURE 3.7: An impression (inspired from [57]) of elliptically fibered K3

with that of $c_1(\mathbb{CP}^1)$ providing a consistent background for compactification.

The D7-branes are co-dimension-2 branes in the above theory i.e. branes with 2 transverse direction and they couple electrically to $C_8$ (dual to $C_0$ scalar) and magnetically to $C_0$ according to

$$d * F_9 = dF_1 = \delta_7(z, \bar{z}) \tag{3.127}$$

where $F_1$ and $F_9$ are the field strengths of $C_0$ and $C_8$ respectively and $\delta_7 = \delta(z)\delta(\bar{z})dz \wedge d\bar{z}$ is a source two-form on $\mathbb{CP}^1$ which is the support of those branes. The magnetic coupling of $C_0$ with 7-branes result in the fact that encircling any of the support point $z_i$ i.e. position of 7-brane on $\mathbb{CP}^1$ induce a non-trivial monodromy that is a transformation of the field $\tau \to \frac{a\tau+b}{c\tau+d}$ (this is similar to the Dirac string in 4D electromagnetic theory). In fact from the asymptotic solution of the equation (3.125) it can be shown that in the neighborhood of a D7-brane, one must have a non-constant string coupling of the form $\tau(z) = \frac{1}{2\pi i} ln[z - z_i]$. For a generic (p,q)-7 brane (of which D7 branes are the category (1,0) as discussed in 3.6.3), encircling its position once induces a monodromy $\mathcal{M}(p, q)$ written in $SL(2, \mathbb{Z})$ matrix form [57]

$$\mathcal{M}(p, q) = \begin{pmatrix} 1 + pq & q^2 \\ p^2 & 1 - pq \end{pmatrix} \in SL(2, \mathbb{Z}) \tag{3.128}$$

The global consistency condition requires that the total monodromy on $\mathbb{CP}^1$ base must be trivial

$$\prod_{i=1}^{24} \mathcal{M}(p_i, q_i) = 1 \tag{3.129}$$

which forbids all 24 branes to be of (1,0)-type.

A given pair of 7-branes is said to be mutually local if their monodromies commute i.e. when k in

$$p_1 q_2 - p_2 q_1 = k \in \mathbb{Z} \tag{3.130}$$

is equal to zero. In that case, the pair of branes can be treated simultaneously at weak coupling. In case $k \neq 0$ then the pair cannot be treated simultaneously at weak coupling. Thus we see that if one considers each brane carrying a U(1) gauge group in the world volume, the gross effect will result in a $U(1)^{24}$ theory, but because of the constraint (3.129) a maximum of 18 mutually commuting monodromies are permitted and hence there cannot be more than 18 $U(1)$ factors in the full theory. This is where the duality with heterotic string theory on $T^2$ comes into play. Compactifying Heterotic string theory on a $T^2$ with all the Wilson lines switched with non-zero vacuum expectation values gives rise to $U(1)^{18}$ gauge group 16 of which come from the Wilson line and the rest two from the compactification of 10D metric $g_{\mu\nu}$ and $B_{\mu\nu}$ fields. In terms of scalar fields of the theory, the moduli space of Heterotic on $T^2$ is

$$M = \frac{SO(2,18)}{SO(2) \times SO(18)} \tag{3.131}$$

which is parametrized by 18 complex scalars, two from the torus moduli T and U and the rest 16 from the Wilson lines. In the context of F-theory on K3, the polynomials f(z) and g(z) (which are respectively of order 8 and 12 with respect to z) have 18 independent free complex parameters thereby confirming the duality with the Heterotic theory.

In the context of Heterotic string compactified on $T^2$ it is known that the 10D gauge group can be broken (or left unbroken) by appropriate switching of the Wilson lines. In the context of F-theory on K3, this is explained with coincidence of the singularities that is the enhancement of gauge symmetry when more than one singular points out of 24 $z_i$s coincide. This is defined according to the Kodaira classification of singularities which we represent (in a brief format) in the table below

**Table 2:Kodaira classification of elliptic singularities**

| Type | Gauge symmetry | 7-Brane content |
|---|---|---|
| $\mathcal{A}_n$ | $U(n+1)$ | $A^{n+1}$ |
| $\mathcal{D}_{n+4}$ | $SO(2n+8)$ | $A^{n+4}BC$ |
| $\mathcal{E}_6$ | $E_6$ | $A^5BC^2$ |
| $\mathcal{E}_7$ | $E_7$ | $A^6BC^2$ |
| $\mathcal{E}_8$ | $E_8$ | $A^7BC^2$ |

Instead of the classic discussion about these singularities (see for example [47, 53, 57, 58]) we discuss the gauge theory enhancement mechanism due to stretched open strings [59, 60].

When n D7-branes come to the same point on $\mathbb{CP}^1$, the open strings those are stretched between them in all n(n-1) possible ways, become massless and give rise to the enhancement $U(1)^n \to U(n)$. In case of generic (p,q)-7 branes it can be shown that these branes allow only the same (p,q)-charged strings to end on them [61]. However, if a given stack of 7-branes contains for example orientifold 7 planes, which inverts the (p,q) charges of the open strings, then two 7-branes of different charges may be connected by an open string through complicated path. We exemplify the above statement with $SO(2n+8)$ and $E_8$ groups below.

We denote the fundamental 7-branes participating in the scenario to be A brane (monodromy (1,0)) which is in fact a D7-brane, B brane (monodromy (3,1)) and C brane (monodromy (1,1)). Looping a fundamental string around a path encircling a B and a C brane changes the charge from (1,0) to (-1,0). The sign of the charges are related to the gauge charges of the ends of the open string on the 7-branes and hence they are related to the orientations of the string. Thus, at weak-coupling limit an ensemble of BC brane acts as an orientifold plane. This ensemble of BC brane commute with the monodromy of that of 4 A branes according to (3.130). To construct a stack of branes giving rise to $SO(2n+8)$ gauge theory, one needs $n+4$ A branes and a B and a C brane i.e.

$$A^{n+4}BC \rightarrow SO(2n+8) \tag{3.132}$$

The perturbative group for the above configuration is $SU(n+4) \times U(1)$ where the U(1) factor lives on the open string stretched between B and C branes and $SU(n+4)$ is due to the enhancement of n+4 coincident A branes. However, the strings which encircle the BC ensemble once get their charges inverted and couple to different A branes giving rise to the enhancement $SU(n+4) \rightarrow SO(2n+8)$. We shall discuss the particular cases of F-theory on K3 with gauge theory $SO(16)^2$ and $SO(8)^4$ in chapter 5 and draw relation to this brane scenarion in terms of amplitude results of Heterotic string compactified on $T^2$. Note that the above analysis forbids the gauge group $SO(32)$ which can be obtained from 10D SO(32) heterotic theory compactified on $T^2$ with all Wilson lines switched to zero. In F-theory on K3 context, one needs n=16 and the total of 18 branes for such a configuration which is equal to the allowed number of commuting monodromies. Thus the singularity with 18 coincident brane destroys the triviality condition of the normal bundle of K3 and we consider only the K3 surfaces with trivial normal bundle.

The case of $SO(8)^4$ is of particular interest as it remains on a constant coupling slice of the moduli space that is $\tau$ = arbitrary constant [62]. In particular one can take the $T^4/\mathbb{Z}_2$ orientifold limit of K3 and the compactification of type IIB theory on $T^2/\mathbb{Z}_2$ orientifold. The constant coupling can be taken to be in the weak coupling limit thereby providing a perturbative probe into the F-theory.

To construct the $E_8$ singularity one needs 7 A branes, one B brane and 2 C branes i.e.

$$A^7BC^2 \rightarrow E_8. \tag{3.133}$$

The perturbative subgroup is $SU(7) \times SU(2) \times U(1)$ which is enhanced to $E_8$ by the open strings stretched between different A branes encircling the BC (one of the two C branes) ensemble plus the open strings stretched between the two C branes encircling once the charge reversing ensemble $A^4B$. This is in particular $T^4/\mathbb{Z}_6$ orientifold limit of K3, the compactification of F-theory on which provides $E_8 \times E_8$ singularity. The corresponding Heterotic theory is $E_8 \times E_8$ Heterotic theory compactified on a $T^2$ with all Wilson lines switched to zero.

# Chapter 4

# Discrete anomaly in maximal supergravity

In this chapter we shall setup the most important keystone of this thesis: the discrete anomaly in supergravity theories. Our motivation to look for such anomalies and study the anomaly couterterm in those theories came from the original article by Green & Gaberdiel [63] where the $SL(2, \mathbb{Z})$ anomaly in type IIB supergravity in D=10 is analyzed and its possible interpretation in terms of F-theory was given. As we shall elaborate the origin of such anomaly due to the chiral coupling of the fermions to the composite abelian factor in the factor H of the scalar coset $G/H$ and the process of generating a counterterm, which we have coined as the Green-Gaberdiel counterterm, we shall see that the couterterm in D=10 type IIB supergravity is a strong reminiscent of the higher derivative coupling to 7-branes which are in fact the essential building blocks of F-theory. With our knowledge of the article Bachas, Bain & Green [64] where the authors discussed similar anomaly generating chiral couplings of the fermions in D=4, N=4 supergravity theory and its combination with a bosonic anomaly due to the presence of a self-dual two-form field, a result of the Montonen-Olive duality [65, 66, 67] giving rise to the D-instanton induced strongly coupled world-volume higher derivative coupling of the D3 branes, we endeavoured to draw a similar conclusion for the strong coupling limit of the D7-brane world-volume coupling from the anomaly analysis of D=10 type IIB supergravity. The task is ambitious one in view of the monodromy of the 7-branes which we have discussed in section 3.8. We shall be eventually led to consider supergravity theories in D=8, N=1 Yang-Mills theories to draw satisfactory conclusions which in turn yet pose open questions about open string modes stretched between 7-branes while considering the duality between Heterotic string compactified on $T^2$ and F theory on K3 in section 5.4.

Another important aspect of such composite anomalies in D=8 dimensional supergravities is that the anomaly cancelling counterterm is essentially furnished by a one loop string amplitude calculation where one allows only the massless modes to circulate in the string loop, that is the string threshold corrections. This is one of our main achievements in the course of this work, especially in D=8, N = 1 supergravities coupled with a gauge group of rank 16. This correspondence between discrete anomaly in supergravity and string one loop amplitude will be studied in greater detail for the case of Heterotic strings compactified on $T^2$ in the next chapter. We shall however illustrate the analysis in case of type II strings compactified on $T^2$ and its relation with M-theory at the end of this chapter.

We shall thus begin by considering discrete anomaly in D=10 type IIB theory and its possible relation to F theory and 7-branes and conclude this chapter by drawing comparison between discrete anomaly in D=8, N=2 supergravity, the corresponding string amplitude and M theory compactified on $T^3$.

## 4.1 The Green-Gaberdiel counterterm

The 10D type IIB supergravity theory has a global $SL(2,\mathbb{R})$ symmetry. The $SL(2,\mathbb{R})$ group manifold can be parametrized by a complex scalar $\tau = \tau_1 + i\tau_2$ (identified with the axio-dilaton field) taking values in the upper half plane, and a real (angular) scalar $0 \leq \phi \leq 2\pi$, which is a pure-gauge degree of freedom charged under the local symmetry group $U(1) \subset SL(2,\mathbb{R})$. The scalar manifold of the theory, i.e. the coset space $SL(2,\mathbb{R})/U(1)$, is then usually described by the following vielbein

$$V_i^a = \frac{1}{\sqrt{-2i\tau_2}} \begin{pmatrix} \bar{\tau} e^{-i\phi} & \tau e^{i\phi} \\ e^{-i\phi} & e^{i\phi} \end{pmatrix} . \tag{4.1}$$

Under a general $SL(2,\mathbb{R}) \times U(1)$ transformation, the vielbein $V_i^a$ transforms as

$$V_i^{\prime a} = A_b^a V_j^b u_i^j , \tag{4.2}$$

where $A$ is the $SL(2,\mathbb{R})$ matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad a,b,c,d \in \mathbb{R}, \quad ad - bc = 1, \tag{4.3}$$

and $u$ is

$$u = \begin{pmatrix} e^{-i\Sigma} & 0 \\ 0 & e^{i\Sigma} \end{pmatrix} , \qquad 0 \leq \Sigma \leq 2\pi \, , \tag{4.4}$$

which thus shifts $\phi$ as $\phi \to \phi + \Sigma$. The composite $U(1)$ connection is locally the $SL(2,\mathbb{R})$-invariant combination (see section 2.4)

$$Q_\mu = \partial_\mu \phi - \frac{\partial_\mu \tau_1}{2\tau_2}, \tag{4.5}$$

so that its $U(1)$ field strength takes the local form

$$F = dQ = \frac{d\tau \wedge d\bar{\tau}}{4i\tau_2^2}. \tag{4.6}$$

The two gravitini form a complex conjugate pair which carry charges $\pm\frac{1}{2}$ under $U(1)$, and the two dilatini form a complex conjugate pair of opposite chirality and carry charges $\pm\frac{3}{2}$. Due to these chiral couplings, the theory may suffer from an anomaly for the $U(1)$ gauge symmetry. At the perturbative level, this anomaly can be detected from one-loop hexagon diagrams containing at least one composite gauge field (4.5). Alternatively, it can be seen to descent from a 12-form anomaly polynomial, which, according to the rules summarized in section 2.5, takes the form:

$$I_{12} = \frac{F^2}{2\pi} \left[ 2X_8^-(R) + \frac{p_1(R)}{48} \left( \frac{F}{2\pi} \right)^2 - \frac{1}{32} \left( \frac{F}{2\pi} \right)^4 \right] , \tag{4.7}$$

where we defined

$$X_8^{\pm}(R) = \frac{1}{192(2\pi)^4} \left( trR^4 \pm \frac{1}{4} (trR^2)^2 \right) , \tag{4.8}$$

in terms of the 10D Einstein-frame curvature two-form $R$, and $p_1(R) = -\frac{1}{2} trR^2/(2\pi)^2$ is the first Pontryagin class. The absence of an $F^0$-term in the expression (4.7) is clearly due to the well-known freedom of the type IIB theory from pure local gravitational anomalies. Moreover, the absence of a linear term in $F$ implies that the new $U(1)$ anomaly vanishes for a pure supergravity theory (i.e. without brane sources). Indeed, if no 7-brane is present, $F$ is an exact form and, because of its composite structure (4.6), it squares to zero. On the contrary, when 7-branes are there, the expression (4.6) is only valid away from them, because the background value of $\tau$ undergoes monodromies around such sources.

From the anomaly polynomial (4.7) one deduces the anomalous phase variation of the partition function

$$\mathcal{A} = -\int \Sigma \left[2X_8^-(R) + \frac{p_1(R)}{48}\left(\frac{F}{2\pi}\right)^2 - \frac{1}{32}\left(\frac{F}{2\pi}\right)^4\right]\frac{F}{2\pi}, \tag{4.9}$$

which can clearly be cancelled by the addition of the following counterterm in the 10D action:

$$S_\phi = \int \phi \left[2X_8^-(R) + \frac{p_1(R)}{48}\left(\frac{F}{2\pi}\right)^2 - \frac{1}{32}\left(\frac{F}{2\pi}\right)^4\right]\frac{F}{2\pi}. \tag{4.10}$$

Upon fixing the gauge (say by setting $\phi \equiv 0$), symmetries will be realized non-linearly. Nevertheless, one can still describe the transformation properties of all fermion fields as local phase shifts, by specifying their charge under the $U(1)$ gauge symmetry. This is achieved simply by exploiting the property of the vielbein (4.1) to convert $SL(2,\mathbb{R})$ indices into $U(1)$ indices. The result is that, in the gauge fixed theory, any field $\Psi$ with charge $q$ under the local $U(1)$ will have the following transformation under $SL(2,\mathbb{R})$:

$$SL(2,\mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad \Psi \longrightarrow e^{iq\Sigma(\tau)}\Psi \qquad \text{with} \qquad \Sigma(\tau) = -\arg(c\tau + d). \tag{4.11}$$

Therefore, in a gauge fixed formulation, one needs to add to the 10D action an appropriate counterterm compensating for the non-trivial transformation of the fermion path integral measure under (4.11). The quantum theory is expected to be symmetric only under the discrete subgroup $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$, and hence anomaly cancellation requires a Chern-Simons-like counterterm with suitable modular properties, such as

$$S_{GG}^{10} = i\int ln\left(\frac{\eta(\tau)\bar{j}^{1/12}(\bar{\tau})}{\bar{\eta}(\bar{\tau})j^{1/12}(\tau)}\right)\left[2X_8^-(R) + \frac{p_1(R)}{48}\left(\frac{F}{2\pi}\right)^2 - \frac{1}{32}\left(\frac{F}{2\pi}\right)^4\right]\frac{F}{2\pi}. \tag{4.12}$$

The $\eta$ and $j$ functions are defined in appendix B whose modular properties we note below for convenience

$$\begin{aligned}
log\,\eta(\tau+1) &= log\,\eta(\tau) + i\frac{\pi}{12}, & (4.13)\\
log\,\eta(-\frac{1}{\tau}) &= log\,\eta(\tau) - i\frac{\pi}{4} + \frac{log\,\tau}{2} & (4.14)\\
\frac{j(\tau+1)}{\bar{j}(\bar{\tau}+1)} &= e^{-4i\pi}\frac{j(\tau)}{\bar{j}(\bar{\tau})}, & (4.15)\\
\left(\frac{j(-1/\tau)}{\bar{j}(-1/\bar{\tau})}\right)^{1/12} &= -\left(\frac{j(\tau)}{\bar{j}(\bar{\tau})}\right)^{1/12}. & (4.16)
\end{aligned}$$

Note that, as explained in [63], anomaly cancellation is not enough to completely fix the modular function of the counterterm. Here we adopt the choice proven in [63] to be consistent with compactications of the 8D theory.

### 4.1.1 F theory on K3 and 7-brane coupling

We shall now argue that the Green-Gaberdiel counter-term (4.12) codifies the structure of higher-derivative $R^4$ couplings on D7-brane world-volumes in the regime of strong string coupling. To this end, we shall confine our attention to the piece

$$S^{10}_{GG} = i \int ln\left(\frac{\eta(\tau)\bar{j}^{1/12}(\bar{\tau})}{\bar{\eta}(\bar{\tau})j^{1/12}(\tau)}\right)\left[2X_8^-(R)\right]\frac{F}{2\pi} \tag{4.17}$$

of the Green-Gaberdiel term which encodes the compactification of F-theory on K3 where the vacuum expectation value of the axi-dilaton field $<\tau>$ is only allowed to vary on the 2-sphere $\mathbb{CP}^1$ in the space-time decomposition $\mathcal{M}_{10} = \mathcal{M}_8 \times \mathbb{CP}^1$ of the D=10 type IIB string theory, the relation of which with F-theory we have detailed in section 3.8. The terms proportional to $F^3$ and $F^5$ in (4.12) are relevant in the cases of compactification of F-theory on Calabi-Yau fourfold and sixfold respectively which we shall not discuss in this thesis.

In view of the simple form of the $U(1)$ field strength (4.6) $F = dQ = \frac{d\tau\wedge d\bar{\tau}}{4i\tau_2^2}$ we have $F \wedge F = 0$ so that the anomaly 12-form polynomial (4.7) vanishes automatically. Nevertheless, if we still write down the anomaly cancelling counter-term (4.17), the trivial form of the $U(1)$ field strength renders this term harmless, that is its $S \in SL(2,\mathbb{Z})$ variation is a total derivative

$$\delta_S S = \frac{i}{2\pi}\int ln\left(\frac{\tau}{\bar{\tau}}\right)X_8^- \wedge F \tag{4.18}$$

$$= \frac{-1}{\pi}\int arg(\tau)d(X_7)dQ \tag{4.19}$$

$$= \frac{-1}{\pi}\int d\left(\theta d(X_7)dQ\right) - d\left(X_7 d\left[ln|\tau|\right]d\left[ln|sin(\theta)|\right]\right) \tag{4.20}$$

where we have used $\theta = arg(\tau)$, $F = dQ$ and $X_8^- = dX_7$ in the above. Hence its presence is harmless in the effective action as there is no anomaly to cancel: we remember that the $SL(2,\mathbb{Z})$ variation of the counter-term is meant to cancel the $U(1)$ induced $SL(2,\mathbb{Z})$ anomaly. The field strength $F$ will represent a non-trivial cohomology class only in the presence of 7-branes. To show this we take back the lines of our previous discussion of the relation between type IIB in D=10 and the formulation of F-theory there-from (following the discussion of Greene, Shapere, Vafa & Yau [56]).

The low energy effective action of type IIB theory (neglecting for the moment the matter contributions from the NS-NS and RR 2-form potentials and the self-dual RR 4-form potential)

$$S^{10D}_{IIB} = \frac{1}{2\kappa_{10}^2}\int_{\mathcal{M}_{10}} R*1 - \frac{1}{2}\frac{d\tau\wedge *d\bar{\tau}}{\tau_2^2} \tag{4.21}$$

$$= \frac{1}{2\kappa_{10}^2}\int_{\mathcal{M}_{10}} R*1 - L_{\text{matter}} \tag{4.22}$$

where in the second line of above we have written the kinetic term of the axi-dilaton field as the matter Lagrangian. We remember once more that this action is invariant under the $SL(2,\mathbb{Z})$ transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{4.23}$$

Next we take the metric ansatz

$$ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_7^2 + \underbrace{e^{\phi(z,\bar{z})} dz d\bar{z}}_{\mathbb{CP}^1} \tag{4.24}$$

for the decomposed space-time $\mathcal{M}_{10} = \mathcal{M}_8 \times \mathbb{CP}^1$ so as to obtain solution of the action (4.21) corresponding to compactifications of F-theory on elliptic K3 over the base space $\mathbb{CP}^1 \equiv S^2$. We set $\tau = \tau(z)$ a holomorphic function over the affine coordinates $z = x_8 + ix_9$ on $\mathbb{CP}^1$. The only non-trivial Riemann tensor is

$$R_{89} = -\left(\frac{\partial_8^2 \phi}{2} + \frac{\partial_9^2 \phi}{2}\right) dx_8 \wedge dx_9 \tag{4.25}$$

so that the Ricci scalar is

$$R = -\left(\frac{\partial_8^2 \phi}{2} + \frac{\partial_9^2 \phi}{2}\right) e^{-\phi}. \tag{4.26}$$

Hence the only non-zero component of the Einstein tensor is (here $\partial = \partial_{x_8} + i\partial_{x_9}$)

$$G_{00} = -2e^{-\phi}\partial\bar{\partial}\phi = -e^{-\phi}\left(\partial_8^2\phi + \partial_9^2\phi\right). \tag{4.27}$$

Next, the stress-tensor from the metric ansatz (4.24) and the action (4.21) is

$$T = T_{\mu\nu} = -\frac{2}{\sqrt{g_8}} \frac{\delta\sqrt{g_8} L_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{4.28}$$

However, using the Cauchy-Riemann conditions of holomorphicity of $\tau$ we get the only non-zero component of the stress-energy tensor

$$T_{00} = \frac{e^{-\phi}}{2\tau_2^2} \partial\tau\bar{\partial}\bar{\tau}. \tag{4.29}$$

Einstein equation of motion relates the Einstein tensor (4.27) with the stress-energy tensor (4.29) so that, for the first Chern class of $\mathbb{CP}^1$ we get

$$c_1 = \left[\frac{i}{2\pi} trR\right] = -\frac{i}{2\pi} 2\partial\bar{\partial}\phi dx_8 \wedge dx_9 = -\frac{1}{8\pi\tau_2^2} d\tau \wedge d\bar{\tau} = \frac{i}{2\pi} F \tag{4.30}$$

so that the non-trivial vacuum profile for the axi-dilaton induces a non-trivial Ricci curvature. Thus in the presence of 7-branes, the Green-Gaberdiel anomaly cancelling term is relevant. Indeed, we have argued in section 3.8 that F-theory compactification on K3 are a class of vaccua preserving half of the original 32 supercharges in D=8 and involves exactly 24 7-branes localized on the $\mathbb{CP}^1$ [56] of which only 18 are mobile due to certain global constraints. We shall show that the Green-Gaberdiel term (4.17) is sensitive to these gravitational constraints in section 5.4.

We would now like to point out another intriguing implication of the 10D $SL(2,\mathbb{Z})$ anomaly cancellation. We will indeed argue that the Green-Gaberdiel counterterm (4.12) codifies the structure of higher-derivative $R^4$ couplings on D7-brane worldvolumes in the regime of strong string coupling. The story is analogous to the one of $R^2$ couplings on D3-branes [64], whose expression for any value of the string coupling is dictated by the cancellation of an $SL(2,\mathbb{Z})$ anomaly of the $N=4$ Maxwell theory living on the D3-brane. Here, however, things are more involved, as D7-branes are not singlets under $SL(2,\mathbb{Z})$. But F-theory teaches us how to handle this problem: as long as gravitational effects are concerned, at strong coupling the physics of D7-branes is completely codified by a non-Ricci-flat 10D bulk geometry together with a non-trivial axio-dilaton profile. Therefore from this perspective it is very reasonable to look at 10D $SL(2,\mathbb{Z})$ anomaly cancelation to seek for the strong coupling completion of $R^4$ terms on D7-branes. In the following we provide strong evidence that the counterterm (4.12) plays the role of such a completion.

To this end, we first take a weak string coupling limit $g_s = \langle\tau_2\rangle^{-1} \to 0$ of (4.12) and bring the coupling down to 8D by using, as before, $\int_{S^2} F/2\pi = -2$ (all higher $F$-powers being zero). We thus obtain:

$$2\pi \int \tau_1\, X_8^-(R)\,, \tag{4.31}$$

where we have used that in this limit $ln\left(\frac{\eta(\tau)\bar{j}^{1/12}(\bar{\tau})}{\bar{\eta}(\bar{\tau})j^{1/12}(\tau)}\right) \to \frac{i\pi\tau_1}{2}$.

Let us now compare (4.31) with the weak coupling expectation of the higher-derivative couplings to the Ramond-Ramond axion $\tau_1$. To do that we have to compute the total D(-1)-brane charge induced by the brane content of the theory. In a regime of weak coupling the 24 7-branes arrange themselves in 4 O7$^-$-planes and 16 D7-branes plus 16 D7-images [62]. Since an integral (mobile) D(-1)-brane charge is made up of a pair D(-1)/image-D(-1) brane, we must compute it on the orientifold double cover of the 2-sphere. We use the well known formulae for the induced brane charges [32, 33], which for a single Dp-brane (with trivial normal and gauge bundle) read $\Gamma_{-1}^{\mathrm{Dp}} = 2\pi\sqrt{\widehat{A}(R)}$ and for a single O7$^-$-plane (with trivial normal bundle) read [32, 36, 34, 35] $\Gamma_{-1}^{\mathrm{O7}} = -16\pi\sqrt{\widehat{L}(R/4)}$ (see sections 2.5, 3.4 for the relevant definitions). In addition to that, there is a density of D3-brane charge which (if part of $\mathcal{M}_8$ is compactified) needs to be added to cancel the one induced by the 24 7-branes. This amounts to $p_1(R)/2$. Of course these D3-branes also induce D(-1)-brane charge and, if we take into account that too, we obtain the following axion coupling:

$$\begin{aligned} &\int \tau_1 \left(32 \times \Gamma_{-1}^{\mathrm{D7}} + 4 \times \Gamma_{-1}^{\mathrm{O7}} + \frac{p_1(R)}{2} \times \Gamma_{-1}^{\mathrm{D3}}\right) \\ =\;& \frac{2\pi}{192(2\pi)^4}\int \tau_1 \{32 \times \frac{1}{32\times 15}\left[8trR^4 + 5(trR^2)^2\right] \\ &-4\times\frac{1}{16\times 15}\left[5(trR^2)^2 - 28trR^4\right] - \frac{(trR^2)^2}{2}\} \\ =\;& \frac{2\pi}{192(2\pi)^4}\int \tau_1\left(trR^4 + \frac{1}{4}(trR^2)^2 - \frac{(trR^2)^2}{2}\right) \\ =\;& 2\pi\int \tau_1\, X_8^-(R)\,, \end{aligned} \tag{4.32}$$

i.e. exactly what is predicted by the 10D $SL(2,\mathbb{Z})$ anomaly cancelation.

This remarkable match comes with an annoying puzzle which remains to be explained: Why should the F-theory coupling (4.12) “know" about D3-branes, which do not backreact on the axio-dilaton. Notice that the D3-brane contribution, i.e. the last piece in the l.h.s. of (4.32), just flips the sign of $(trR^2)^2$ in (4.8) from $+$ to $-$. The sign flip could presumably be explained in an alternative way, by a suitable redefinition of the Ramond-Ramond four-form potential $C_4$. Such a redefinition, at any value of the string coupling, should look like

$$C_4 \quad \longrightarrow \quad C_4 + \frac{i\,\alpha'^2}{192(2\pi)}\, ln \left( \frac{j(\tau)}{\bar{j}(\bar{\tau})} \right) \frac{trR^2}{(2\pi)^2}\,, \tag{4.33}$$

which respect the $SL(2,\mathbb{Z})$ invariance of $C_4$. Operating this redefinition adds an additional contribution to the induced D(-1)-brane charge on D7-branes and explains the sign flip from $-$ to $+$ when going to weak coupling, without relying on explicitly added D3-branes. We hope to clarify this issue in the future.

## 4.2 Discrete anomaly in D=8, N=2 supergravity

We now turn to the case of type IIB theory compactified on a $T^2$ with complex structure $U = U_1 + iU_2$ which is T-dual to the type IIA theory compactified on a $T^2$ under the exchange $T \leftrightarrow U$ where $T$ is the Kähler structure $T = B_{89} + iV_{T^2}$.
The moduli space of the theory is

$$\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SL(3,\mathbb{R})}{SO(3)}\,, \tag{4.34}$$

where the first factor is parametrized by $U$.

The field content of this 8D theory is given by [68, 69] a gravity multiplet comprising 1 graviton $g_{\mu\nu}$, 2 gravitino $\psi^a_\mu$(a=1,2), 6 vectors $A_\mu$, 2 dilatini $\lambda^i$, 4 gaugini $\chi^i$, 7 real scalars $\phi$, 3 2-forms $A_{\mu\nu}$ and 1 3-form $A_{\mu\nu\rho}$.
The supergarvity effective action contains covariant derivatives of the form (recall the sugra covariant derivative (2.18)) [68, 70, 69]

$$\mathcal{D}_\mu \psi_\nu \;=\; D_\mu \psi_\nu - \frac{i}{2} Q^{ab}_\mu T_{ab} \psi_\nu - \frac{i}{2} Q_\mu \gamma^9 \psi_\nu \tag{4.35}$$

$$\mathcal{D}_\mu \lambda \;=\; D_\mu \lambda - \frac{i}{2} Q^{ab}_\mu T_{ab} \lambda - \frac{3i}{2} Q_\mu \gamma^9 \lambda \tag{4.36}$$

$$\mathcal{D}_\mu \chi^i \;=\; D_\mu \chi^i + \frac{i}{2} \epsilon_{jab} Q^{ab}_\mu \epsilon^{icd} T_{cd} \chi^j - \frac{3i}{2} Q^{ab}_\mu T_{ab} \chi^i + \frac{i}{2} Q_\mu \gamma^9 \chi^i \tag{4.37}$$

where $Q_\mu$ is the composite $U(1)$ connection of the coset factor $\frac{SL(2,\mathbb{R})}{U(1)}$ (not to be confused with the supercharges) and $Q^{ab}_\mu$ is the composite $SO(3)$ connection of the coset factor $\frac{SL(3,\mathbb{R})}{SO(3)}$ ($T_{ab}$ being the SO(3) generators). $D_\mu$ is the ordinary covariant derivative $D_\mu = \partial_\mu - i\frac{1}{2}\omega^{ab}_\mu M_{ab} - ie^a_\mu P_a + \bar{\psi}^\alpha_\mu Q_\alpha$ containing the super-Poincaré connections. Thus the fermions have chiral couplings (with respect to the $\gamma^9$ projection) with respect to the composite $U(1)$ connection and hence, as in the case of D=10 type IIB supergravity with scalar coset $\frac{SL(2,\mathbb{R})}{U(1)}$ shall induce an $SL(2,\mathbb{Z})$ anomaly.

The U(1) charges of the gravitini, dilatini and gaugini are respectively (they are all positive chiral): $\frac{1}{2}$, $\frac{3}{2}$ and $-\frac{1}{2}$. The self dual 4-form field strength is initially uncharged under $U(1)$ but charged under $SL(2,\mathbb{R})$. As discussed by Basu [69] Bossard & Verschinin [71], and Marcus [14], this self-dual 4-form can be manipulated to become charged under the $U(1)$ for the quantum consistency and thus carry charge 1.

Hence the total anomalous phase variation is [71]

$$\begin{aligned}\mathcal{A} &= -\int \left[2\times\frac{1}{2}I^{d=8}_{3/2} - 4\times\frac{1}{2}I_{1/2} + 2\times\frac{3}{2}I_{1/2} - \frac{1}{2(2\pi)^4}\widehat{L(R)}\right]\Sigma(x) \\ &= -\frac{1}{16(2\pi)^4}\int\left(trR^4 - \frac{1}{4}(trR^2)^2\right)\Sigma(x).\end{aligned} \tag{4.38}$$

As in section 4.1 we first write a $U(1)$ counter-term

$$S_{U(1)} = \frac{1}{16(2\pi)^4}\int\phi\left(trR^4 - \frac{1}{4}(trR^2)^2\right), \tag{4.39}$$

where $\phi$ is the $U(1)$ parameter of the $SL(2,\mathbb{R})$ matrix transforming under $U(1)$ as $\phi \to \phi + \Sigma$. One then gauge fixes the $U(1)$ in terms of the parameter $U$ ($T$ in case of type IIA) $\phi \equiv \phi(U)$. The counter-term (4.39) is now not invariant under the $SL(2,\mathbb{R})$ as

$$\delta\phi = -\frac{i}{2}ln\left(\frac{cU+d}{c\bar{U}+d}\right). \tag{4.40}$$

Once more the quantum theory is expected to be symmetric only under the discrete subgroup $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ and thus we can write a counter-term of the form (using the modular properties listed in (4.13))

$$\begin{aligned}S_{IIB} &= \frac{i}{24\times 16(2\pi)^4}\int\left(ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right) + ln\left(\frac{j(U)}{\bar{j}(\bar{U})}\right)\right)\left(trR^4 - \frac{1}{4}(trR^2)^2\right) \\ &= \frac{i}{2}\int\left(ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right) + ln\left(\frac{j(U)}{\bar{j}(\bar{U})}\right)\right)X_8^-,\end{aligned} \tag{4.41}$$

with

$$X_8^- = \frac{1}{192}\left(trR^4 - \frac{1}{4}(trR^2)^2\right). \tag{4.42}$$

Though the anomalous variation (4.40) is cancelled by the $ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right)$ piece only, the reason of adding a modular invariant $ln\left(\frac{j(U)}{\bar{j}(\bar{U})}\right)$ term is to hinder the decompactification limit of (4.41) as this $U(1)$ anomaly and its counter-term are local and relevant to D=8 only and should not give rise to any contributions in the higher dimensions. This is because the massless particles states which flow in the anomaly generating one-loop amplitude in D=8 are not the same as the massless states which flow in the anomaly generating (if there exists any) one-loop amplitude in D=10. In the lower dimensions it shall only provide a non-anomalous correction term in the effective action. For the case of complex structure $U$ we note from (B.16) that in the decompactification limit $ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right) = \frac{i\pi U_1}{6} \to 0$ so that even leaving the counter-term as $\frac{i}{2}\int ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right)X_8^-$ shall serve the two-fold purpose: first to provide the $SL(2,\mathbb{Z})/U(1)$ counter-term and second to have no decompactification limit. However in view of the T-duality, i.e. $U \leftrightarrow T$ exchange, we add the modular invariant $ln\left(\frac{j(U)}{\bar{j}(\bar{U})}\right)$ term so that in the T dual side there shall be no risque of decompactification.

In the case of type IIA on $T^2$ we get by T-duality ($T \leftrightarrow U$)

$$\begin{aligned}S_{IIA} &= \frac{i}{24\times 16(2\pi)^4}\int\left(ln\left(\frac{\eta^{24}(T)}{\bar{\eta}^{24}(\bar{T})}\right) + ln\left(\frac{j(T)}{\bar{j}(\bar{T})}\right)\right)\left(trR^4 - \frac{1}{4}(trR^2)^2\right) \\ &= \frac{i}{2}\int\left(ln\left(\frac{\eta^{24}(T)}{\bar{\eta}^{24}(\bar{T})}\right) + ln\left(\frac{j(T)}{\bar{j}(\bar{T})}\right)\right)X_8^-,\end{aligned} \tag{4.43}$$

The above terms (4.41) and (4.43) provide $\alpha'^3$ term in the D=8, N=2 Supergravity effective action are consistent with M-theory and with the string amplitude results as we shall now show.

### 4.2.1 Comparison with 5-point String amplitude and M-theory

Before comparing our result with the string theory amplitude which have been calculated by Kiritsis & Pioline [72] and the steps of which we have rephrased in the appendix 4.2.2 at the end of this chapter, we first note that the reduction of the 5-Brane anomaly cancelling term in M-theory action (2.21) $-\frac{1}{(2\pi)^2(\alpha')^{3/2}}\int C_3\wedge X_8^-$ on a circle $S^1$ gives

$$S_{M5} = -\frac{2\pi l_s}{(2\pi)^2(\alpha')^{3/2}}\int B_2\wedge X_8^-. \tag{4.44}$$

in type IIA theory in D=10. Next compactifying this type IIA theory on a 2-torus $T^2$ of complex structure $U$ and Kähler moduli $T$ to get

$$\begin{aligned} S_{D=8,N=2A} &= -(2\pi)\int B_{89}\wedge X_8^- \qquad (4.45)\\ &= -(2\pi)\int T_1\wedge X_8^- \qquad (4.46)\end{aligned}$$

Now we T-dualize on one cycle of the torus, which amounts to the exchange $U\longleftrightarrow T$ to get the following in the type IIB theory compactified on a torus of complex structure $U$

$$S_{D=8,N=2B} = -(2\pi)\int U_1 X_8^- \tag{4.47}$$

The above terms (4.45) and (4.47) provide the “trivial orbit" (of $SL(2,\mathbb{Z})$) term in the D=8, N=2 string amplitude in type IIA and type IIB case respectively as we shall see now. It is thus instructive to write the string threshold result as the sum of “trivial orbit" plus “regularised 1-loop level" where the latter comes from the contributions of the degenerate and non-degenerate orbits of $SL(2,\mathbb{Z})$ from the decomposition of the $\Gamma_{2,2}$ lattice in the partition function under trivial, degenerate and non-degenerate orbits of $SL(2,\mathbb{Z})$. It is this “regularized 1-loop" term which shall correspond to the $SL(2,\mathbb{Z})$ terms (4.45) and (4.47) because of the fact that the infra-red divergence in string theory corresponds to the quantum anomaly in the corresponding supergravity theory.

Now we quote the result of the CP-odd part of the string threshold calculation as given in appendix 4.2.2 and [72] along with the “trivial orbit" plus “regularised 1-loop level" decomposition of the result.

$$\begin{aligned} S_{IIA}^{CP-odd} &= \underbrace{-N\int(4\pi T_1)\left[trR^4-\frac{1}{4}(trR^2)^2\right]}_{\text{trivial orbit}} \qquad (4.48)\\ &\quad \underbrace{+N\int i\left(ln\left(\frac{\eta^{24}(T)}{\bar\eta^{24}(\bar T)}\right)-4i\pi T_1\right)\left[trR^4-\frac{1}{4}(trR^2)^2\right]}_{\text{“Reg. 1-loop"}}\\ &= \underbrace{N\int i\left(ln\left(\frac{\eta^{24}(T)}{\bar\eta^{24}(\bar T)}\right)\right)\left[trR^4-\frac{1}{4}(trR^2)^2\right]}_{\text{total amplitude}}.\end{aligned}$$

$$
\begin{aligned}
S_{IIB}^{CP-odd} &= \underbrace{-N\int (4\pi U_1)\left[trR^4-\frac{1}{4}(trR^2)^2\right]}_{\text{trivial orbit}} \\
&\quad \underbrace{+N\int i\left(ln\left(\frac{\eta^{24}(U)}{\bar\eta^{24}(\bar U)}\right)-4i\pi U_1\right)\left[trR^4-\frac{1}{4}(trR^2)^2\right]}_{\text{"Reg. 1-loop"}} \\
&= \underbrace{N\int i\left(ln\left(\frac{\eta^{24}(U)}{\bar\eta^{24}(\bar U)}\right)\right)\left[trR^4-\frac{1}{4}(trR^2)^2\right]}_{\text{total amplitude}}.
\end{aligned}
\tag{4.49}
$$

By noting that the U-linear term in q-expansion of $ln\left(\frac{j(U)}{\bar j(\bar U)}\right)$ is $-4i\pi U_1$ we can conclude that the supergravity result (4.45) and (4.47) are consistent with the string amplitude result (4.48) and (4.49) apart from the non-harmonic terms which arise from the complete q-expansion of $ln\left(\frac{j(U)}{\bar j(\bar U)}\right)$ and $ln\left(\frac{j(T)}{\bar j(\bar T)}\right)$. We also note that the functions of T and U in front of the "regularised 1-loop term"s in (4.48) and (4.49) respectively do not have the correct modular property to cancel the $SL(2,\mathbb{Z})$ anomaly but this lack of proper modular transformation stems from the fact that we had split the amplitude integral into three orbits of $SL(2,\mathbb{Z})$ thereby distributing the modular invariance of the property into 3 different orbits. The sum of the contributions from all the orbits should bear the correct modular property which is the case for the total amplitude as we can see from complete result given in the last lines of (4.48) and (4.49).

### 4.2.2 Five point string amplitude for type II strings on $T^2$ : a complemntary review

In this subsection we summarise the steps of one-loop string amplitude calculation for type II strings compactified on a $T^2$. Though the amplitude calculation has been carried out in great detail in [72] we summarize the steps here to make the decomposition of the amplitude in terms of the trivial, degenerate and non-degenerate orbits of the world-sheet $SL(2,\mathbb{Z})$ and thereby justifying the interpretation of (4.48) and (4.49). Different notions of the world-sheet fermionic characters and the construction of the string partition function have been detailed in chapter 5 in course of construction of elliptic genus in case of Heterotic strings (and also in chapter 3). Thus here we give only a very compact mathematical details of the computation.

From world-sheet point of view, we have $\mathcal{N}=(1,1)_2$ super-conformal field theory (SCFT) whose action is the supersymmetric Polyakov action 3.9

$$
S_P=\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\eta_{\mu\nu}\left[\partial_\alpha X^\mu\partial_\beta X^\nu+\frac{i}{2}\bar\psi^\mu\partial\!\!\!/\psi^\nu\right]. \tag{4.50}
$$

In this string theory, we have two space-dimensions compactified on the torus $T^2$ with complex structure $U$ and volume $V_2=T_2$. From the world-sheet point of view, these two space-scalars are wrapped inside a two dimensional lattice $\Gamma_{2,2}$ whose character, in the absence of Wilson lines, can be written as

$$\Gamma_{2,2} = \frac{T_2}{\tau_2} \sum_{A \in ML(2,\mathbb{Z})} exp\left[ -2\pi i T det(A) - \frac{\pi T_2}{\tau_2 U_2} |(1 \;\; U) A \begin{pmatrix} \tau \\ 1 \end{pmatrix} |^2 \right] = \sum_{\vec{m},\vec{n} \in \mathbb{Z}^N} q^{P_L^2/2} \bar{q}^{P_R^2/2}. \tag{4.51}$$

with

$$p_L^2 = \frac{|Um_1 - m_2 + Tn^1 + TUn^2|^2}{2T_2U_2}, \tag{4.52}$$

$$p_L^2 - p_R^2 = 2m_I n^I, \tag{4.53}$$

$$m^1, m^2, n^1, n^2 \in \mathbb{Z}. \tag{4.54}$$

The string partition function is

$$Z_{d=8 \text{ type } \mathbf{K}} = \frac{iV_8}{4(2\pi l_s)^{10}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{(\sqrt{\tau}\eta\bar{\eta})^8} \frac{1}{2} \sum_{a,b=0}^{1} (-1)^{a+b} \frac{1}{2} \sum_{\bar{a},\bar{b}=0}^{1} (-1)^{\bar{a}+\bar{b}+\mathbf{K}\bar{a}\bar{b}} \frac{\theta^4 \left[{a \atop b}\right] \bar{\theta}^4 \left[{\bar{a} \atop \bar{b}}\right]}{\eta^4 \bar{\eta}^4} \Gamma_{2,2}. \tag{4.55}$$

where $\mathbf{K} = 1, 0$ stands for type IIA and type IIB respectively.

Next we have the vertex operators for the NS-NS massless states (i.e. dilaton, Graviton and B-field):

$$V_{-1,-1}^{\mu\nu}(z,\bar{z}) = \frac{g_{closed}}{l_s^2} \psi^\mu(z) \bar{\psi}^\nu(\bar{z}) e^{ip.X(z,\bar{z})} (\partial X^\alpha \psi_\alpha(0)) (\bar{\partial} X^\beta \bar{\psi}_\beta(0)). \tag{4.56}$$

$$V_{0,0}^{\mu\nu}(z,\bar{z}) = \frac{2g_{closed}}{l_s^2} \left[\partial X^\mu(z,\bar{z}) + ip.\psi(z)\psi^\mu(z)\right] \left[\bar{\partial} X^\nu(z,\bar{z}) + ip.\psi(\bar{z})\bar{\psi}^\nu(z)\right] e^{ip.X(z,\bar{z})} \tag{4.57}$$

We also note that while computing the amplitude with the vertex functions inserted with the partition function, one should contract the above vertices with appropriate polarization tensors. For example, for a graviton vertex the vertices $V^{\mu\nu}$ in (5.30) above should be contracted with a transverse symmetric traceless polarization tensor $e_{\mu\nu}$ and the Riemann tensor (calculated for a gravitational fluctuation around a flat background) can be retrieved from the momentum representation

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2} \left[ e_{\alpha\gamma} p_\beta p_\delta - (\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) + \{(\alpha,\gamma) \leftrightarrow (\beta,\delta)\} \right]. \tag{4.58}$$

We shall also need the vertex operators for the moduli which can be chosen in (0,0), (-1,-1), (0,-1) or (-1,0) ghost picture (see chapter 12 of Polchinski [27] for details) given by

$$V_{0,0}(\phi_i, p, z, \bar{z}) = v_{IJ}(\phi_i) \left[\partial X^I(z,\bar{z}) + ip.\psi(z)\psi^I(z)\right] \left[\bar{\partial} X^J(z,\bar{z}) + ip.\psi(\bar{z})\bar{\psi}^J(z)\right] e^{ip.X(z,\bar{z})}. \tag{4.59}$$

$$V_{-1,-1}(\phi_i, p, z, \bar{z}) = v_{IJ}(\phi_i) \psi^I(z) \bar{\psi}^J(\bar{z}) e^{ip.X(z,\bar{z})} \left[\partial X^\alpha \psi_\alpha(0) + G_{KL} \partial X^K(0) \psi^L(0)\right] \left[\bar{\partial} X^\beta \bar{\psi}_\beta(0) + G_{KL} \bar{\partial} X^K(0) \bar{\psi}^L(0)\right]. \tag{4.60}$$

$$V_{-1,0}(\phi_i, p, z, \bar{z}) = v_{IJ}(\phi_i) \psi^I(z) \left[\bar{\partial} X^J(z,\bar{z}) + ip.\psi(\bar{z})\bar{\psi}^J(z)\right] \left[\partial X^\alpha \psi_\alpha(0) + G_{KL} \partial X^K(0) \psi^L(0)\right]. \tag{4.61}$$

$$V_{0,-1}(\phi_i, p, z, \bar{z}) = \text{exchange I and J in the above and then take complex conjugate of the whole expression.} \tag{4.62}$$

In the above

$$v_{IJ}(\phi_i) = \frac{\partial(G_{IJ}+B_{IJ})}{\partial\phi_i}. \quad (4.63)$$

$$\Longrightarrow \quad (4.64)$$

$$v(T) = -\frac{i}{2U_2}\begin{pmatrix} 1 & U \\ \bar{U} & |U|^2 \end{pmatrix},$$

$$v(U) = \frac{iT_2}{U_2^2}\begin{pmatrix} 1 & \bar{U} \\ \bar{U} & \bar{U}^2 \end{pmatrix}. \quad (4.65)$$

Note that the derivative with respect to moduli in $v_{IJ}$ plays the role of killing the massless modes in the loop calculation which is necessary to make comparison with supergravity anomaly because in the latter arises from the supergravity one loop calculation with massless particles circulating inside the loop. The string amplitude to be calculated is

$$\mathfrak{A} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\prod_{i=1}^{4}\langle \int \frac{d^2 z_i}{\pi} \underbrace{\epsilon^i_{\bar{\alpha}_i\alpha_i} V^{\bar{\alpha}_i\alpha_i}(p_i, z_i, \bar{z}_i)}_{\text{4 graviton vertices}} \int \frac{d^2 z_5}{\pi} \underbrace{V_\phi(p_5, z_5, \bar{z}_5)}_{\text{one moduli vertex } \phi=T \text{ or } U} \rangle, \quad (4.66)$$

We shall only evaluate the CP-odd part of the threshold correction above as the anomaly polynomial (4.41) involves the wedge product and hence an $\epsilon_8$ tensor. This CP-odd part comprises of odd-even and even-odd parity with respect to the left and right moving world-sheet fermions so that the moduli vertex will be chosen respectively $(-1,0)$-picture or $(0,-1)$-picture. The following identities will determine the moduli vertex insertion[1]

$$\langle V_\phi \rangle = \quad (4.67)$$

$$v_{IJ}(\phi)\langle \psi^I \bar{\partial} X^J G_{KL} \partial X^K \psi^L \rangle = \frac{\chi_\phi}{\pi\tau_2}\partial_\phi \Gamma_{2,2} \quad \chi_\phi = \begin{cases} 1, & \phi = T, U \\ -1, & \phi = \bar{T}, \bar{U}. \end{cases}$$

$$\text{Or} \quad v_{IJ}(\phi)\langle \bar{\psi}^I \partial X^J G_{KL} \bar{\partial} X^K \bar{\psi}^L \rangle = \frac{\sigma_\phi \chi_\phi}{\pi\tau_2}\partial_\phi \Gamma_{2,2}. \quad (4.68)$$

Hence the moduli vertex insertion can be integrated out using $\int \frac{d^2 z_5}{\tau_2} = 1$. We need to find momenta in eighth order, i.e. $O(p^8)$ in order to reconstruct 4th order topological quantities in Riemann tensor. For the fermionic contraction, in the odd side one will get the $\epsilon_8$ tensor and $\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \to 2\pi\eta^3$ and shall gather $p^4$ order by this process. In any even side we shall use the contraction of the 4-fermionic pairs $p.\psi\psi$ of the NS-NS vertex operators to give us rest $p^4$ order of momenta. This constructions provides us with the $t_8$ tensor structure, which when combined with $\epsilon_8$ tensor gives

$$\star 1 \epsilon_8 t_8 R^4 = 24 tr R^4 - 6(tr R^2)^2 \quad (4.69)$$

where $\star 1$ denotes the D=8 differential volume element and the terms $R^4$ or $R^2$ are to be understood in terms of wedge product. The contraction of the fermionic pairs

[1] Note that our presentation for the construction of the string partition function is minimal i.e. it affords a smooth flow of reading with enough technical materials for its self-sufficiency. The detailed theoretical introduction can be found in the standard text of Green, Schwarz & Witten [26], Polchinsky [27], Francesco, Mathieu & Senechal [73].

yields in the summation of the spin-structure summation yielding position independent $-\frac{1}{2}(2\pi\eta^3)^4$ factor. Thus the final amplitude shall be[2]

$$\mathfrak{A} = N(trR^4 - \frac{1}{4}(trR^2)^2)\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\partial_\phi\Gamma_{2,2} \tag{4.70}$$

$$= N(trR^4 - \frac{1}{4}(trR^2)^2)\partial_\phi\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\Gamma_{2,2} \tag{4.71}$$

where in the last line we have taken the derivative $\partial_\phi$ with respect to moduli $\phi$ out so as to account for the loop amplitude with only massless states circulating inside the loop. Now we can evaluate the integral on the fundamental domain $\mathcal{F}$ of world-sheet $SL(2,\mathbb{Z})$ using

$$\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\Gamma_{2,2} = -log(T_2|\eta(T)|^4U_2|\eta(U)|^4), \tag{4.72}$$

upto an irrelevant IR ambiguity . We shall however decompose the $\Gamma_{2,2}$ lattice in terms of the $SL(2,\mathbb{Z})$ orbits as follows: the lattice $\Gamma_{2,2}$ is of the form

$$\Gamma_{2,2} = \frac{T_2}{\tau_2}\sum_{A\in ML(2,\mathbb{Z})} exp\left[-2\pi i T det(A) - \frac{\pi T_2}{\tau_2 U_2}|(1\ \ U)A\begin{pmatrix}\tau\\1\end{pmatrix}|^2\right] = \sum_{\vec{m},\vec{n}\in\mathbb{Z}^N} q^{P_L^2/2}\bar{q}^{P_R^2/2}.$$

One then decomposes the $2\times 2$ matrices $A$ in the lattice sum into the orbits of $PSL(2,\mathbb{Z})$ (see [74, 75, 76]) :

| Orbits | Defining properties | Canonical representative |
|---|---|---|
| Invariant | $A = 0$ | $\begin{pmatrix}0&0\\0&0\end{pmatrix}$ |
| Degenerate | $A \neq 0;\ detA = 0$ | $\begin{pmatrix}0&j\\0&p\end{pmatrix};\ j,p\neq 0.$ |
| Non-degenerate | $A \neq 0;\ detA \neq 0$ | $\begin{pmatrix}k&j\\0&p\end{pmatrix};\ 0\leq j<k;\ p\neq 0.$ |

The modular integration will now look like

$$A = V_8T_2\times\{\underbrace{\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}}_{\text{Trivial orbit}} \tag{4.73}$$

$$+\underbrace{\int_{\text{strip-boundary of }PSL(2,\mathbb{Z})}\frac{d^2\tau}{\tau_2^2}\sum_{(j,p)\neq(0,0)}e^{-\frac{\pi T_2}{\tau_2U_2}|j+pU|^2}}_{\text{Degenerate orbit}}$$

$$+\underbrace{2\int_{\mathbb{C}+}\frac{d^2\tau}{\tau_2^2}\sum_{0\leq j<k,p\neq 0}e^{-2\pi iTpk}e^{-\frac{\pi T_2}{\tau_2U_2}|k\tau+j+pU|^2}\}.}_{\text{Non-degenerate orbit}}$$

The integral on the trivial orbit is just the volume of the domain $\mathcal{F}$ which is given by

$$\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2} = \frac{2\pi}{3}. \tag{4.74}$$

[2] The world-sheet torus complex structure is $\tau$ (not to be confused with 10D type IIB axio-dilaton field).

Next we shall use the following result of [74] to find the harmonic part from the integral on the strip-boundary of $PSL(2,\mathbb{Z})$

$$\int_{\text{strip-boundary of } PSL(2,\mathbb{Z})} \frac{d^2\tau}{\tau_2^2} \sum_{(j,p)\neq(0,0)} e^{-\frac{\pi T_2}{\tau_2 U_2}|j+pU|^2} \tag{4.75}$$
$$= \left[ logU_2|\eta(U)|^2 + \frac{\pi U_2}{6} \right] + \text{terms with } V_{T^2} \text{ in denominator.}$$

To determine the non-volume suppressed part of the amplitude coming from the non-degenerate orbit, we use the integral [74], [77]

$$T_2 \sum_{0\leq j<k, p\neq 0} e^{-2\pi i T pk} \int_{\mathbb{C}+} \frac{d^2\tau}{\tau_2^2} e^{-\frac{\pi T_2}{\tau_2 U_2}|k\tau+j+pU|^2} c_0 \tag{4.76}$$
$$= \sum_j \sum_{k>0,p>0} \frac{e^{2\pi i k p T}}{k|p|} c_0 + \text{cc.} + \text{volume suppressed terms.}$$

Thus we find the form of the integral given in (4.48) and (4.49)

$$\begin{aligned} S_{IIA}^{CP-odd} &= \underbrace{-N\int (4\pi T_1)\left[trR^4 - \frac{1}{4}(trR^2)^2\right]}_{\text{trivial orbit}} \\ &\quad \underbrace{+N\int i\left(ln\left(\frac{\eta^{24}(T)}{\bar{\eta}^{24}(\bar{T})}\right) - 4i\pi T_1\right)\left[trR^4 - \frac{1}{4}(trR^2)^2\right]}_{\text{“Reg. 1-loop"}} \\ &= \underbrace{N\int i\left(ln\left(\frac{\eta^{24}(T)}{\bar{\eta}^{24}(\bar{T})}\right)\right)\left[trR^4 - \frac{1}{4}(trR^2)^2\right]}_{\text{total amplitude}}. \end{aligned} \tag{4.77}$$

$$\begin{aligned} S_{IIB}^{CP-odd} &= \underbrace{-N\int (4\pi U_1)\left[trR^4 - \frac{1}{4}(trR^2)^2\right]}_{\text{trivial orbit}} \\ &\quad \underbrace{+N\int i\left(ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right) - 4i\pi U_1\right)\left[trR^4 - \frac{1}{4}(trR^2)^2\right]}_{\text{“Reg. 1-loop"}} \\ &= \underbrace{N\int i\left(ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right)\right)\left[trR^4 - \frac{1}{4}(trR^2)^2\right]}_{\text{total amplitude}}. \end{aligned} \tag{4.78}$$

# Chapter 5

# Discrete anomalies in half-maximal Supergravity and string amplitude

We now come to the core chapter of this thesis where we shall be discussing the $U(1)$ composite connection induced $SL(2,\mathbb{Z})$ anomaly in case of $D=8$, $N=1$ supergravity theories which are obtained from $D=10$ Heterotic theories with gauge groups $SO(32)$ or $E_8 \times E_8$ compactified on a torus $T^2$ with or without Wilson lines. We shall come to the conclusion that in these theories, the anomaly cancelling term in the supergravity effective action is provided from a string theory one loop computations like the anomaly cancelling Green-Schwarz term in $D=10$, $N=1$ supergravity theories which is also obtained from a string 1-loop computation. Such anomaly cancelling terms shall provide us with $\alpha'^3$ terms in the supergravity effective action. We shall start the discussion of this chapter with a brief introduction to the Green-Schwarz mechanism [50] in $D=10$, $N=1$ theory and the construction of the elliptic genus [78, 79] which we shall be using extensively in computing the string amplitudes in the case of Heterotic string theory compactified on a torus $T^2$. We shall then give an account of the tentative 7-Brane interpretation of these terms in the light of the duality between F-theory on $K3$ and Heterotic on $T^2$. Next we shall look for a possible M-theory origin of these $\alpha'^3$ correction terms in the perspective of recently developed Horava-Witten background amplitude computation [80], a work which is ongoing at the time of writing this thesis. The last two sections however are only openings towards further work and hence shall pose more questions than providing new answers.

## 5.1 The Green-Schwarz term in D=10, N=1 supergravity effective action

The massless spectrum of $D=10$, $N=1$ supergravity contains the gravity multiplet with 1 graviton $g_{\mu\nu}$, 1 anti-symmetric 2-form $B_{\mu\nu}$, 1 real scalar: dilaton $\phi$, 1-gravitino $\psi^+_\mu$, 1 dilatino $\xi^-$ and a vector multiplet with gauge bosons $A_\mu$ in the adjoint representation of a gauge group $G$ of dimensions n plus n gaugini $\lambda^+_i$ $(i=1,\cdots n)$. The $\pm$ signs for the fermions denote their chiralities. The rank of the gauge group G in case of the Heterotic theories in $D=10$ is needed to be 16 because in the construction of the Heterotic string one has a supersymmetric right moving sector living in 10-spacetime dimensions however the non-supersymmetric left moving sector lives in 26-spacetime dimensions. To make the theory consistent one thus needs to comb the extra 16 $(=26-10)$ space-dimensions on a compact lattice $\Gamma_G$ of dimensions 16: hence the rank of the ensuing gauge group be 16. The modular invariance of the partition function of such theory puts the further constraint that $\Gamma_G$ should be unimodular (i.e. the dual of the lattice be equal to the lattice itself) and thus further restricting the gauge group $G$ to be either $SO(32)$ or $E_8 \times E_8$. The constraint of unimodularity of the lattice however breaks down when one compactifies the heterotic theories to lower dimensions because the departure from the modular invariance of the lattice can be compensated by the toroidal lattice $\Gamma_{n,n}$ so that the string partition function in lower dimensions remain modular invariant. The constraint of $rank(G) = 16$ however have to be maintained. This is why we are allowed to break the $D = 10$ Heterotic gauge group $G_{10} = SO(32)$ or $E_8 \times E_8$ to $G_8 = SO(32)$, $E_8 \times E_8$, $SO(16) \times SO(16)$, $SO(8)^4$, $E_8 \times E_7 \times U(1)$ etc in $D < 10$.

Returning back to the discussion of Heterotic string theory in $D = 10$, we see that the chiral fermions $\psi^+_\mu$, $\xi^-$ and $\lambda^+$ engender gauge-gravity type anomaly in the supergravity theory. The amount of the anomaly can be easily computed using the anomaly polynomials listed in section 2.5. The $10+2$ dimensional anomaly polynomial is

$$\begin{aligned} I_{12}^{d=10,N=1} &= \frac{1}{2} I_{3/2}^{d=10} - \frac{1}{2} I_{1/2}^{dilatino} + \frac{1}{2} I_{1/2}^{gaugino}|_{adj(G)} \qquad (5.1)\\ &= \frac{2\pi}{2(4\pi)^6}\left[\frac{n-496}{5670} trR^6 + \frac{224+n}{4320} trR^2 trR^4 + \frac{n-64}{10368}(trR^2)^3\right] \\ &+ \frac{1}{180} trR^2 TrF^2 + \frac{1}{144}(trR^2)^2 TrF^2 + \frac{1}{18} trR^2 TrF^4 + \frac{4}{45} TrF^6 \end{aligned}$$

In the above "Tr" denotes the group traces in adjoint representations while "tr" denotes group traces in fundamental representation. The way to compensate for this anomaly is to factorise the anomaly polynomial as

$$I_{12}^{d=10,N=1} = I_4 \times I_8 \qquad (5.2)$$

and use the gauge-gravity variation of the 2-form field $B_2$ to compensate for the gauge-gravity descent factor for $I_4$ (see (2.59) for the descent formalism in computing the anomaly). For the factorisation (5.2) to happen, one needs to chose the gauge group G such that (i) the $trR^6$ term vanishes and (ii) the 6th order Casimir $TrF^6$ should be reducible in terms of the 2nd and 4th Casimir that is

$$\begin{aligned} n &= 496 \qquad (5.3)\\ TrF^6 &= aTrF^2 TrF^4 + b(TrF^2)^3. \qquad (5.4) \end{aligned}$$

The above two conditions are satisfied by the gauge groups $G = SO(32)$, $E_8 \times E_8$, $E_8 \times U(1)^{248}$ and $U(1)^{496}$. The condition of $rank(G) = 16$ however restricts one's attention only to the case $G = SO(32)$ and $E_8 \times E_8$ which were found in the Heterotic string argument for the consistency of spacetime dimensions and the modular invariance of the partition function. This hints to the intricate relation between anomaly in supergravity and modularity in string theory. We shall come back to this issue in the next section.

From (5.1) using for $G = SO(32)$ we get

$$\begin{aligned} I_{12}^{d=10,N=1} &= \frac{1}{768(2\pi)^5}\{(trR^2 + trF^2) \\ &\times(trR^4 + \frac{1}{4}(trR^2)^2 + trR^2 trF^2 + 8trF^4)\} \qquad (5.5)\\ &= \frac{1}{768(2\pi)^5} I_4 I_8 \qquad (5.6) \end{aligned}$$

where we have transformed the adjoint traces "Tr" in terms of the fundamental trace "tr" for the group SO(32)

$$TrF^2 = 30trF^2,\ \ TrF^4 = 24trF^4 + 3(trF^2)^2,\ \ TrF^6 = 15trF^4trF^2. \tag{5.7}$$

Doing the gauge-gravity descent only for the $I_4$ part, we get the anomalous phase variation:

$$\mathcal{A}_{GS} = -\frac{1}{768(2\pi)^5}\int (trvd\omega + tr\Sigma dA)I_8 \tag{5.8}$$

using

$$\begin{aligned} I_4 &= (trR^2 + trF^2) = dQ_3, && (5.9)\\ Q_3 &= tr(A\wedge F - \frac{i}{3}A^3) + tr(\omega R - \frac{i}{3}\omega^3), && (5.10)\\ \delta Q_3 &= tr\Sigma dA + trvd\omega. && (5.11)\end{aligned}$$

This anomaly will be cancelled effectively by the G-S action

$$\begin{aligned} S_{GS} &= \frac{1}{192(2\pi)^5\alpha'}\int B_2\wedge I_8 && (5.12)\\ &= \frac{1}{192(2\pi)^5\alpha'}\int B_2\wedge (trR^4 + \frac{1}{4}(trR^2)^2 + trR^2trF^2 + 8trF^4) && (5.13)\end{aligned}$$

if the gauge-gravity variation of the 2-form B is

$$(\delta^{YM} + \delta^{grav})B_2 = \frac{\alpha'}{4}(tr\Sigma dA + trvd\omega). \tag{5.14}$$

The $I_8$ part is invariant under the gauge-gravity variation as it is composed of the invariant forms from F and R.

Indeed, the consistent coupling of the B-field happens through the modified field strength $H = dB - \frac{\alpha'}{4}(Q_3^{YM} + Q_3^{grav})$. This gives us the following Bianchi identity

$$dH = -\frac{\alpha'}{4}(trR^2 + trF^2). \tag{5.15}$$

For the gauge group $E_8\times E_8$ we shall have two copies of the gauge field strength $F_1$ and $F_2$ for two different $E_8$s. However, $E_8$ does not have a vector representation so it is standard to define its trace $trF^2_{E_8}$ in "fundamental" by using that of the group $SO(32)$ so that they have a uniform effect in the Green-Scharz term of the Heterotic theory. Thus we have

$$TrF^2_{E_8} = 30trF^2_{E_8},\ \ TrF^4_{E_8} = \frac{1}{100}(TrF^2_{E_8})^2 = 9(trF^2_{E_8})^2. \tag{5.16}$$

Working out the anomaly polynomial as before, we get

$$\begin{aligned} I_{12}^{d=10,N=1} &= I_4\wedge I_8, && (5.17)\\ I_4 &= trR^2 + trF_1^2 + trF_2^2, && (5.18)\\ I_8 &= trR^4 + \frac{1}{4}(trR^2)^2 + trR^2(trF_1^2 + trF_2^2) \\ &\quad -2trF_1^2trF_2^2 + 2(trF_1^2)^2 + 2(trF_2^2)^2. && (5.19)\end{aligned}$$

## 5.2 Green-Schwarz term from string 1-loop amplitude: Elliptic genus

In this section we shall repeat the computations of Green & Schwarz[50] and Lerche, Nilsson, Schellekens & Warner [78] showing that the Green-Schwarz anomaly cancelling term (5.12) is provided by the 1-loop 5 point amplitude computation in Heterotic string theory when the string amplitude is traded for the higher derivative correction term for the low energy effective action (i.e. $D = 10$, $N = 1$ supergarvity with gauge group $G = SO(32)$ or $E_8 \times E_8$). The calculation of string amplitude and the process of the construction of the elliptic genus is a very classical technique [78, 79]. We, however, outline the process because it shall help to understand the amplitude result of the case Heterotic string theory compactified on $T^2$ with minimal effort. We also wish to emphasize that both string 1-loop and 2-loop computations have been carried out in the language of hyperelliptic functions [81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91]. The superiority of using the elliptic genus lies in the fact that by construction it gives the relative numerical factors between different components e.g. for 1-loop case between $trR^4$, $(trR^2)^2$, $trR^2trF^2$, $(trF^2)^2$ and $trF^4$ etc whereas piecewise calculation of these amplitudes does not fix this relative numerical factors so easily. For the comparison of the string loop amplitude with anomaly polynomials these numerics however is indispensable. In fact the genus-1 elliptic genus can be shown to be the same as the anomaly generating functional [79]. For the genus two case, 1-loop anomaly is not a constraint at all as the local anomaly is a result of the UV divergence of the 1-loop Feynman diagrams in quantum field theory. It is à priori not clear what form of "anomaly" might occur in case of 2-loop Feynman diagrams in the quantum field theory perspective as the renormalization is already notoriously difficult. Hence in case of genus-2 amplitude, we shall content ourself by only providing a possible construction of such amplitude results which were not explored much in the literature.

For the Heterotic string theory, the world-sheet quantum description is the $d = 2$, $\mathcal{N} = (1,0)$ superconformal field theory or $\mathcal{N} = (1,0)_2$ SCFT for short so that the right moving (holomorphic) sector is supersymmetric but the left moving (anti-holomorphic) sector is only bosonic. Calculation for the 1-loop string amplitude necessitates this CFT on the world-sheet torus of complex structure $\tau = \tau_1 + i\tau_2$ with $\tau_2 \geq 0$ and volume=1. The metric of the torus is:

$$g_{ij} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}. \tag{5.20}$$

The torus coordinates are $\sigma_i, i = 1, 2$.
The moduli space of the inequivalent tori is the fundamental domain $\mathcal{F} = \mathcal{H}_2/PSL(2, \mathbb{Z})$ and the generic form of the partition function for a torus of complex structure $\tau \in \mathcal{F}$ is

$$Z = \int e^{-S} = Tr\left[q^{L_0 - c/24}\bar{q}^{\bar{L_0} - c/24}\right], \tag{5.21}$$

the trace is over all of the spectrum (here $q = e^{2\pi i\tau}$ and $L_0(\bar{L_0})$ is the (anti)holomorphic Hamiltonian and $c/24$ factor is the shift in central charge due to toric CFT). The

Polyakov action for $\mathcal{N} = (1,0)_2$ SCFT is of the form[1]

$$S = \frac{1}{4\pi\alpha'} \int d^\sigma \sqrt{g} \left[ g^{ab}\partial_a X^\mu \partial_b X^\nu + \frac{i}{2}\psi\bar{\partial}\psi \right], \tag{5.22}$$

where $X^\mu$ are bosonic spacetime coordinates and $\psi$ is a Majorana-Weyl fermion in $d = 2$ world-sheet conformal field theory. This theory has a manifest $\mathbb{Z}_2$ symmetry $\psi^i \to -\psi^i$.
In the NS-sector we have vacuum and vector representation with the following affine characters with respect to the affine parameters $v_i$ (note that while constructing the partition function, we will set $v_i = 0$, the only reason to use the affine characters instead of true characters is to make the difference between various representation manifest)[2]:

$$\chi_{vac-rep.}(v_i) = \frac{1}{2}\left[\prod_{i=1}^{N/2}\frac{\theta_3(v_i)}{\eta} + \prod_{i=1}^{N/2}\frac{\theta_4(v_i)}{\eta}\right]. \tag{5.23}$$

$$\chi_{vect-rep.}(v_i) = \frac{1}{2}\left[\prod_{i=1}^{N/2}\frac{\theta_3(v_i)}{\eta} - \prod_{i=1}^{N/2}\frac{\theta_4(v_i)}{\eta}\right]. \tag{5.24}$$

Our convention for the $\tau-$functions have been detailed in the appendix B.

The vacuum of the Ramond sector is fermionic and hence we get two spinor representation in this sector: the positive-chiral S representation and negative-chiral C representation. Their affine characters are:

$$\chi_S(v_i) = \frac{1}{2}\left[\prod_{i=1}^{N/2}\frac{\theta_2(v_i)}{\eta} + e^{-i\pi N/4}\prod_{i=1}^{N/2}\frac{\theta_1(v_i)}{\eta}\right]. \tag{5.25}$$

$$\chi_C(v_i) = \frac{1}{2}\left[\prod_{i=1}^{N/2}\frac{\theta_2(v_i)}{\eta} - e^{-i\pi N/4}\prod_{i=1}^{N/2}\frac{\theta_1(v_i)}{\eta}\right]. \tag{5.26}$$

Fermions on a torus can have a combination of periodic (P) and anti-periodic (A) boundary conditions along the two cycles of the torus (see figure 5.1).

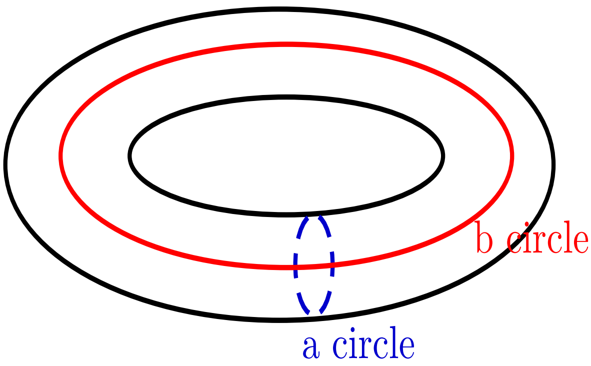


FIGURE 5.1: One cycles of the torus

[1]The $\mathcal{N} = (1,0)_2$ SCFT also contains a 2d gravitino but for instance we do not need its explicit appearance in the Polyakov action.
[2]In all of the following $q = e^{2\pi i\tau}$.

We use a=0,1 to denote A, P boundary condition along the first cycle and b=0,1 to denote A, P boundary condition along the second cycle. Then the above equations defining the characters can be expressed as combination of $\frac{\theta\left[{a \atop b}\right]}{\eta}$, using the convention that $\theta\left[{0 \atop 0}\right] = \theta_3, \theta\left[{0 \atop 1}\right] = \theta_4, \theta\left[{1 \atop 0}\right] = \theta_2, \theta\left[{1 \atop 1}\right] = \theta_1$.

The (P,P) spin structure is called the odd spin structure and one immediately sees that $\theta\left[{1 \atop 1}\right] = \theta_1 = 0$ which is due to the presence of zero modes of the fermionic determinant. Hence, in CP-odd sector, one soaks up the fermionic zero modes with an N-point correlator giving rise to $\epsilon^{a_1 \cdots a_N}$ and $\theta_1$ replaced by $\theta_1' = 2\pi\eta^3(\tau)$. Using these characters values of the world-sheet fermions, we can right down the partition function of the 10-dimensional Heterotic string for toric CFT

$$Z_{d=10\text{ heterotic}} = \frac{iV_{10}}{4(2\pi l_s)^{10}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8} \frac{1}{2} \sum_{a,b=0}^{1} (-1)^{a+b+ab} \frac{\theta^4\left[{a \atop b}\right]}{\eta^4} \frac{\bar{\Gamma}_{16}}{\bar{\eta}^{16}}, \tag{5.27}$$

where $\bar{\Gamma}_{16}$ is the lattice of $Spin(32)/\mathbf{Z}_2$ or $E_8 \times E_8$. In terms of the genus-1 Eisenstein series, these lattices are

$$\bar{\Gamma}_{O(32)/\mathbb{Z}_2} = \frac{1}{2} \sum_{a,b=0,1} \bar{\theta}\left[{a \atop b}\right]^{16} = \bar{E}_8 = \bar{E}_4^2, \tag{5.28}$$

$$\bar{\Gamma}_{E_8\times E_8} = \left[\frac{1}{2} \sum_{a,b=0,1} \bar{\theta}\left[{a \atop b}\right]^{8}\right]^2 = \bar{E}_4 \times \bar{E}_4. \tag{5.29}$$

We now come to the vertex functions for different massless excitations of the Heterotic string. The NS-NS massless state vertices (i.e. dilaton, Graviton and B-field) in respectively -1 ghost and zero-ghost pictures are[3]

$$V_{-1}^{\mu\nu}(z,\bar{z}) = e^{-\phi} \frac{\sqrt{2}g_{closed}}{l_s^2} \psi^\mu(z)\bar{\partial}X^\nu(z,\bar{z})(\partial X^\alpha \psi_\alpha(0))e^{ip.X(z,\bar{z})}. \tag{5.30}$$

$$V_{0}^{\mu\nu}(z,\bar{z}) = \frac{2g_{closed}}{l_s^2} \left[\partial X^\mu(z,\bar{z}) + ip.\psi(z)\psi^\mu(z)\right] \bar{\partial}X^\nu(z,\bar{z})e^{ip.X(z,\bar{z})}. \tag{5.31}$$

For the gauge-Bosons:

$$V_{-1}^{a,\nu}(z,\bar{z}) = e^{-\phi} \frac{\sqrt{2}g_{closed}}{l_s^2} \psi^\nu(z)\bar{J}^a(z,\bar{z})(\bar{\partial}X^\alpha \bar{\psi}_\alpha(0))e^{ip.X(z,\bar{z})}. \tag{5.32}$$

$$V_{0}^{a,\nu}(z,\bar{z}) = \frac{2g_{closed}}{l_s^2} \left[\partial X^\nu(z,\bar{z}) + ip.\psi(z)\psi^\nu(z)\right] \bar{J}^a(z,\bar{z})e^{ip.X(z,\bar{z})} \tag{5.33}$$

$\bar{J}^a$ is an anti-holomorphic O(32) or $E_8 \times E_8$ current which are normalized as $\langle J^a(z_1)J^b(0)\rangle = \delta^{ab}$. We also note that while computing the amplitude with the vertex functions inserted with the partition function, one should contract the above vertices with appropriate polarization tensors. For example, for a graviton vertex the vertices $V^{\mu\nu}$ in (5.30) above should be contracted with a transverse symmetric traceless polarization tensor $e_{\mu\nu}$ and the Riemann tensor (calculated for a gravitational fluctuation around a flat background) can be retrieved from the momentum representation

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left[e_{\alpha\gamma}p_\beta p_\delta - (\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) + \{(\alpha,\gamma) \leftrightarrow (\beta,\delta)\}\right]. \tag{5.34}$$

[3] The $e^{-\phi}$ factor is due to the bosonisation with a chiral scalar field $\phi$.

Similarly, for the B-field, the related polarization tensor $\epsilon_{\mu\nu}$ will be an anti-symmetric tensor and for the gauge bosons one introduces a polarization vector $\epsilon_\mu$ such that $p^\mu\epsilon_\mu = 0$ and $\epsilon^\mu\epsilon_\mu = 1$.
Finally we give the expressions for the 2-point Green's function in case of the toric SCFT necessary to apply the Wick's theorem in case of 5-point amplitude:

$$\begin{aligned}\langle X^\mu(z_1,\bar z_1)X^\nu(z_2,\bar z_2)\rangle &= g^{\mu\nu}\triangle(z_1-z_2,\bar z_1-\bar z_2) \quad (5.35)\\ &\equiv -g^{\mu\nu}log\{e^{-2\pi\frac{[Im(z_1-z_2)]^2}{\tau_2}}|\frac{\theta_1(z_1-z_2)}{\theta'(0)}|^2\}.\\ \langle\bar\partial X^\mu(z_1,\bar z_1)\partial X^\nu(z_2,\bar z_2)\rangle &= -g^{\mu\nu}\bar\partial\partial\triangle(z_{12},\bar{z_{12}}) = -\frac{\pi}{\tau_2}g^{\mu\nu}. \quad (5.36)\\ \langle\bar\partial X^I(z_1,\bar z_1)\partial X^J(z_2,\bar z_2)\rangle &= p_R^I p_L^J. \quad (5.37)\\ \langle\psi^\mu(z_1)\psi^\nu(z_2)\rangle\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right] &= g^{\mu\nu}\frac{\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](z_{12})\theta'(0)}{\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](0)\theta_1(z_{12})}. \quad (5.38)\\ \langle p_1.\psi(z_1),\psi^\mu(z_2)\rangle\langle\psi^\nu(z_1),p_2.\psi(z_2)\rangle &= p_1^\mu p_2^\nu\langle\psi(z_1)\psi(z_2)\rangle^2. \quad (5.39)\end{aligned}$$

With these preparations we are now ready to compute the 5-point string amplitude of 1 B-field with 4 gravitons, 4-gauge bosons and 2 gravitons + 2 gauge-bosons

$$\mathcal{A} = V_{10}\prod_{i=1}^{4}\langle\int\frac{d^2z_i}{\pi}\underbrace{e^i_{\alpha_i\beta_i}V^{\alpha_i\beta_i}(p_i,z_i,\bar z_i)}_{\text{4 graviton/gauge vertices}}\int\frac{d^2z_5}{\pi}\underbrace{\epsilon_{\mu\nu}V^{\mu\nu}(p_5,z_5,\bar z_5)\rangle}_{\text{one B vertex}} \quad (5.40)$$

In (5.40) $z_i$, $i = 1,\cdots 4$ are the (world-sheet) positions of the gauge/gravity vertex insertions while $z_5$ is the position of the B-field vertex. To compare with the anomaly structure (5.12) we need to calculate the CP-odd amplitude, that is the amplitude containing a spacetime $\epsilon-$tensor. For this, we are bound to use 1-vertex in (-1)-ghost picture for which we choose the B-vertex and the rest in zero-ghost picture as given in (5.30)[4].

In CP-odd sector, one soaks up the fermionic zero modes with an N-point correlator giving rise to $\epsilon^{a_1\cdots a_N}$ and $\theta_1$ replaced by $\theta_1' = 2\pi\eta^3(\tau)$ (that is the character sum $\sum_{a,b=0}^{1}(-1)^{a+b+ab}\theta^4\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ to contribute only $\theta_1^4$ which is replaced by $(2\pi)^4\eta^{12}(\tau)$). Hence, a 5-point function (1 vertex in -1-picture) gives the following structure in CP-odd sector

$$\langle\underbrace{:(\partial X\psi)(\bar\partial X\psi):}_{\text{B-vertex in -1 picture}}\underbrace{:p.\psi\psi::p.\psi\psi::p.\psi\psi::p.\psi\psi:\rangle}_{\text{4 graviton vertices in zero picture}} \longrightarrow 2\epsilon_{10}\langle\partial X\bar\partial X\rangle p^4. \quad (5.41)$$

Using the contractions given in (5.35) we can integrate out the $\langle\partial X\bar\partial X\rangle$ part at the $z_5$ vertex which gives

$$\int d^2z_5\,\langle\partial X^\mu\bar\partial X^\nu\rangle(p_5,z_5,\bar z_5) = \int\frac{d^2z_5}{\tau_2}p^\mu p^\nu = B^{\mu\nu} \quad (5.42)$$

[4]For a string loop amplitude on a genus g world-sheet with $n_B$ bosonic vertices in -1 picture and $n_F$ fermionic vertices in -1/2 picture, the BRST consistency condition for the amplitude forces one to insert $N = 2g-2+n_B+n_F/2$ picture changing operators (PCO) (see e.g. [27]) which transforms each -1 ghost vertex to a zero vertex. In case of CP-odd sector, one needs to add another extra PCO.

where we have used our convention

$$\frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - i\partial_2), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2). \tag{5.43}$$
$$d^2z = idz \wedge d\bar{z} = 2dz_1dz_2. \tag{5.44}$$
$$\int \frac{d^2z}{\tau_2} = 1. \tag{5.45}$$
$$\int_{\mathcal{F}} d^2\tau/\tau_2^2 = 2\pi/3. \tag{5.46}$$

From the structure of the Riemann tensor in momentum mode (5.34) (and similarly for gauge group) it is clear that if we want terms containing eight derivatives, that is terms like $trR^4$, $trF^4$ etc, we need 8 powers of momentum from the 5-point Green's function (5.40). $p^4$ term is obtained from the 5 pairs of fermions contracted in the CP-odd case as stated in (5.41). We thus need to extract further $p^4$ term from the 4-point piece $\prod_{i=1}^4 \langle \int \frac{d^2z_i}{\pi} \underbrace{e^i_{\alpha_i\beta_i} V^{\alpha_i\beta_i}(p_i, z_i, \bar{z}_i)}_{\text{4 graviton/gauge vertices}}$ of the amplitude (5.40). Each such vertex operators contain term of the form $\partial X \bar{\partial} X_i e^{ip.X(z_i)}$ so that we have to expand the exponential part and extract relevant powers of p. We shall consider first the pure gravity sector and then the gauge sector.
Consider the general n-point function (with n-even) of the form

$$\begin{aligned}\prod_{i=1}^{n} \langle \int \frac{d^2z_i}{\pi} e_{\alpha_i\beta_i} V^{\alpha_i\beta_i}(p_i, z_i, \bar{z}_i)\rangle &= \int \prod_{i=1}^{n} \frac{d^2z_i}{\pi} \langle e_{\alpha_i\beta_i} \partial X^{\alpha_i} \bar{\partial} X^{\beta_i} e^{ip.X(z_i)}\rangle \\ &= \int \frac{d^2z_i}{\pi} \prod_{1\le i<j}^{n} e_{\alpha_i\beta_i} g^{\alpha_i\beta_i} e^{-p_i.p_j\triangle^{ij}/2}.\end{aligned} \tag{5.47}$$

In the first line of (5.47) we have used the bosonic 2-point correlator

$$\langle X^\alpha(z_1,\bar{z}_1)X^\beta(z_2,\bar{z}_2)\rangle = g^{\alpha\beta}\triangle(z_1-z_2,\bar{z}_1-\bar{z}_2) \equiv -g^{\alpha\beta} log\{e^{-2\pi\frac{[Im(z_1-z_2)]^2}{\tau_2}} |\frac{\theta_1(z_1-z_2)}{\theta'(0)}|^2\}. \tag{5.48}$$

In the second line we have used the Wick contraction (or the Koba-Nielsen formula)

$$\langle \prod_{i=1}^{4} e^{ip_i.X(z_i)}\rangle = \prod_{1\le i<j}^{4} e^{-p_i.p_j\triangle^{ij}/2}, \tag{5.49}$$

and $\langle \bar{\partial} X^\mu(z_1,\bar{z}_1)\partial X^\nu(z_2,\bar{z}_2)\rangle \sim g^{\mu\nu}$.
Now expanding the exponential part in (5.47) we come across the following integral

$$\int d^2z_1 \cdots d^2z_n \triangle(z_1,z_n)\cdots\triangle(z_{n-1},z_n). \tag{5.50}$$

However given the form of the correlator $\triangle(z_i,z_j)$ the above integral is divergent: so that using suitable regularization for the correlator (see for example [78]) one can evaluate the integral

$$\int_{\text{n even}} d^2z_1 \cdots d^2z_n \triangle(z_1,z_n)\cdots\triangle(z_{n-1},z_n) = (\tau_2)^n \hat{\bar{G}}_n(\bar{\tau}) \tag{5.51}$$

with

$$\hat{G}_{2m}(\tau) = -\frac{(2\pi i)^{2m}}{(2m)!} B_{2m} E_{2m} \tag{5.52}$$

for $m > 1$. For $m = 1$ we have

$$\hat{G}_2 = 2\pi^2 B_2 E_2 - \frac{\pi}{\tau_2}. \tag{5.53}$$

This departure from the holomorphicity for the integral of the two-point function is due to the regularization process of the integral. Here $E_{2m}$ is the Eisenstein series of $2m$ and $B_{2m}$ is the $2m-$th Bernoulli number (see appendix B for the details on Eisenstein series). Using this result the n-point function can be recast in the form

$$\begin{aligned} \prod_{i=1}^{n} \langle \int \frac{d^2 z_i}{\pi} V^{\alpha_i \beta_i}(p_i, z_i, \bar{z}_i) \rangle &= \prod_{i=1}^{n} \frac{d^2 z_i}{\pi} \langle \int \partial X^{\alpha_i} \bar{\partial} X^{\beta_i} e^{ip.X(z_i)} \rangle \\ &= \int \frac{d^2 z_i}{\pi} \prod_{1 \le i < j}^{n} e^{-p_i.p_j \triangle^{ij}/2} \\ &= exp\left[ -\sum_{m=1}^{\infty} \frac{1}{2m} tr\,(R)^{2m} \frac{(2\pi i)^{2m}}{(2m)!} B_{2m} E_{2m} \right]. \end{aligned} \tag{5.54}$$

The term $exp\left[-\sum_{m=1}^{\infty} \frac{1}{2m} tr\,(R)^{2m} \frac{(2\pi i)^{2m}}{(2m)!} B_{2m} E_{2m}(\bar{\tau})\right]$ is the **gravitational elliptic genus**. Note that the term $\frac{1}{2m}$ inside the exponential is added by hand as in the expansion (5.54) any particular contraction is counted $2m$ times. The piece $m = 1$ being non-holomorphic, we factor this term out as

$$\begin{aligned} & exp\left[ -\sum_{m=1}^{\infty} \frac{1}{2m} tr\,(R)^{2m} \frac{(2\pi i)^{2m}}{(2m)!} B_{2m} E_{2m}(\tau) \right] \\ = & \underbrace{exp\left[ \frac{trR^2}{(2\pi)^2} \frac{\hat{E}_2(\bar{\tau}, \tau_2)}{48} \right]}_{I_4} \times \underbrace{exp\left[ \frac{trR^4}{(2\pi)^4} \frac{E_4}{2^7\, 3^2\, 5}(\bar{\tau}) \right]}_{I_8} \\ & \times exp\left[ -\sum_{m=3}^{\infty} \frac{1}{2m} tr\,(R)^{2m} \frac{(2\pi i)^{2m}}{(2m)!} B_{2m} E_{2m}(\bar{\tau}) \right]. \end{aligned} \tag{5.55}$$

In this elliptic genus we have marked the pieces $I_4$ and $I_8$ because these terms provide those corresponding factors in the factorized Green-Schwarz anomaly cancelling (5.1) term as we shall show in an instance. But before that we shall compute the gauge part elliptic genus.

Once more for the CP-odd amplitude we use a (-1)-ghost B vertex and 4 zero-ghost gauge vertices whose expressions are listed in (5.30) and (5.32). One needs to saturate ten-fermionic modes from the total correlator similar to the expression (5.41) that is

$$\langle \underbrace{: (\partial X \psi)(\bar{\partial} X \psi) :}_{\text{B-vertex in -1 picture}} \underbrace{: p.\psi\psi :: p.\psi\psi :: p.\psi\psi :: p.\psi\psi :}_{\text{4 gauge vertices in zero picture}} \rangle \longrightarrow 2\epsilon_{10} \langle \partial X \bar{\partial} X \rangle p^4 \tag{5.56}$$

from which we gather $p^4$ order of momentum and the B-state can be integrated out as in

$$\int d^2z_5 \, \langle \partial X^\mu \bar{\partial} X^\nu \rangle (p_5, z_5, \bar{z}_5) = \int \frac{d^2 z_5}{\tau_2} p^\mu p^\nu = B^{\mu\nu}. \tag{5.57}$$

To reconstruct 4th order terms in gauge field strength of the form $trF^4$ etc. we need to gather another $p^4$ order in momentum. One might tempted to act like we have done for the case of graviton vertex, trying to collect order $p^4$ from the exponential piece of the gauge boson vertex

$$V_0^{a,\nu}(z,\bar{z}) = \frac{2g_{closed}}{l_s^2} \left[\partial X^\nu(z,\bar{z}) + ip.\psi(z)\psi^\nu(z)\right] \bar{J}^a(z,\bar{z}) e^{ip.X(z,\bar{z})}. \tag{5.58}$$

There is however another term which can supply momentum orders, namely the gauge lattice character function $\bar{\Gamma}_{16}$. We remember that the 16 extra left moving bosonic degrees of freedom have been confined in this 16-dimensional lattice so as to make the theory consistent as a whole. These extra 16 left moving momentum states can be seen if we rewrite the covariant expression for the lattice [92, 78, 76]

$$\bar{\Gamma}_{16} = \sum \bar{q}^{p_L^2}, \tag{5.59}$$

where the sum is over all vectors $\vec{p}_L$ whose basis is composed by the root vectors of the (i) $O(16) \times O(16)$ in case of $E_8 \times E_8$ and (ii) $O(32)$ in case of $SO(32)$. Thus in our manipulation for the exponential as in the case of gravitons, we must include these momentum modes. The scalar product on the lattice however is defined in terms of the generators $H^I$ of the Cartan subalgebra of the gauge group G whereas, in the exponential part of the vertices the scalar product is with respect to the space-time metric. We thus have to make an orthogonal change of basis and write the momentum as

$$p_j = \sum_I k_j^I H^I. \tag{5.60}$$

The vertex contraction for the 4-gauge bosons, (taking care only of the exponential part as the current pieces contract according to $\langle J^a(z_1) J^b(0) \rangle = \delta^{ab}$) gives rise to

$$\begin{aligned} \prod_{i=1}^4 \langle \int \frac{d^2 z_i}{\pi} \bar{\Gamma}_{16} \epsilon_{\alpha_i} V^{\alpha_i}(p_i, z_i, \bar{z}_i) \rangle &= \prod_{i=1}^4 \frac{d^2 z_i}{\pi} \langle \int \sum \bar{q}^{p_L^2} e^{ip.X(z_i)} \rangle \\ &= \sum \bar{q}^{p_L^2} e^{k.p_L}. \end{aligned} \tag{5.61}$$

In the last line, we have integrated the position vectors out by replacing them with their Fourier components with respect to the lattice momenta. Now the expression $\sum \bar{q}^{p_L^2} e^{k.p_L}$ can be seen as the "gauged" character of the lattice, that is

$$\sum \bar{q}^{p_L^2} e^{k.p_L} = \left[ \frac{1}{2} \sum_{l=1}^4 \prod_{i=1}^{16} \theta_l(u_i, \bar{\tau}) \right], \tag{5.62}$$

where $u_i$, $i = 1, \cdots, 16$ are the skew-eigenvalues of the gauge field strength tensor $F$ of SO(32) such that

$$\sum_{i=1}^{16} u_i^n = \frac{(i)^n}{2} trF^n, \tag{5.63}$$

and

$$\sum \bar{q}^{p_L^2} e^{k.p_L} = \left[ \frac{1}{2} \sum_{l=1}^{4} \prod_{i=1}^{8} \theta_l(v_i, \bar{\tau}) \right], \tag{5.64}$$

with $v_i$, $i = 1, \cdots, 8$ are the skew-eigenvalues of the gauge field strength tensor $F$ of $E_8$ (for each copy of $E_8 \times E_8$) such that

$$\sum_{i=1}^{8} v_i^n = \frac{(i)^n}{2} trF^n. \tag{5.65}$$

The quantities defined in (5.62) and (5.64) are called the **gauge elliptic genus** for $SO(32)$ and $E_8 \times E_8$ respectively.
Combining the pure gravity elliptic genus (5.55) and the pure gauge elliptic genus (5.62) we find the total elliptic genus $A(\bar{q} = e^{2\pi i \bar{\tau}}, R, F)$ for 1-loop Heterotic (G=SO(32)) string amplitude

$$A(\bar{q}, R, F) \tag{5.66}$$

$$= \underbrace{exp\left[ \frac{trR^2}{(2\pi)^2} \frac{\hat{E}_2(\bar{\tau}, \tau_2)}{48} \right]}_{I_4} \times \underbrace{exp\left[ \frac{trR^4}{(2\pi)^4} \frac{E_4(\bar{\tau})}{2^7\ 3^2\ 5} \right]}_{I_8} \tag{5.67}$$

$$\times exp\left[ -\sum_{m=3}^{\infty} \frac{1}{2m} tr\,(R)^{2m} \frac{(2\pi i)^{2m}}{(2m)!} B_{2m} E_{2m}(\bar{\tau}) \right] \times \left[ \frac{1}{2} \sum_{l=1}^{4} \prod_{i=1}^{16} \theta_l(u_i, \bar{\tau}) \right],$$

with

$$\sum_{i=1}^{16} u_i^2 = -\frac{1}{2} trF^2, \tag{5.68}$$

$$\sum_{i=1}^{16} u_i^4 = \frac{1}{2} trF^4 \tag{5.69}$$

etc. For $G = E_8 \times E_8$ we find similarly

$$A(\bar{q}, R, F_1, F_2) \tag{5.70}$$

$$= \underbrace{exp\left[ \frac{trR^2}{(2\pi)^2} \frac{\hat{E}_2(\bar{\tau}, \tau_2)}{48} \right]}_{I_4} \times \underbrace{exp\left[ \frac{trR^4}{(2\pi)^4} \frac{E_4(\bar{\tau})}{2^7\ 3^2\ 5} \right]}_{I_8} \tag{5.71}$$

$$\times exp\left[ -\sum_{m=3}^{\infty} \frac{1}{2m} tr\,(R)^{2m} \frac{(2\pi i)^{2m}}{(2m)!} B_{2m} E_{2m}(\bar{\tau}) \right]$$

$$\times \left[ \frac{1}{2} \sum_{l=1}^{4} \prod_{i=1}^{8} \theta_l(u_i, \bar{\tau}) \right] \times \left[ \frac{1}{2} \sum_{n=1}^{4} \prod_{j=1}^{8} \theta_n(v_j, \bar{\tau}) \right],$$

this time with the skew eigenvalues $u_i$, $i = 1, \cdots 8$ for the first $E_8$ and $v_j$, $j = 1, \cdots 8$ for the second $E_8$. Note that in the very first lines of (5.66) and (5.70) we have inserted the values of the Bernoulli numbers $B_2 = 1/6$ and $B_4 = -1/30$.

From the above elliptic genera (5.66) and (5.70) we need to extract the 8-form polynomial on $R$ and $F$ to draw the conclusion about matching anomaly. To "gauge" the gauge group lattice part $\left[\frac{1}{2}\sum_{l=1}^{4}\prod_{i=1}^{16}\theta_l(u_i)\right]$ etc. we shall use the following identities realting $\theta$ functions and Eisenstein series given in appendix B (B.7) (a standard proof of which can be found in Abramowitz & Stegun [93]

$$\frac{\theta_2(\nu|\tau)}{\theta_2(0|\tau)} = exp\left\{\sum_{k=1}^{\infty}\frac{(2\pi i)^{2k}B_{2k}\nu^{2k}}{(2k+1)!-(2k)!}\left[E_{2k}(q)-2^{2k}E_{2k}(q^2)\right]\right\} \tag{5.72a}$$

$$\frac{\theta_3(\nu|\tau)}{\theta_3(0|\tau)} = exp\left\{\sum_{k=1}^{\infty}\frac{(2\pi i)^{2k}B_{2k}\nu^{2k}}{(2k+1)!-(2k)!}\left[E_{2k}(q)-E_{2k}(-\sqrt{q})\right]\right\} \tag{5.72b}$$

$$\frac{\theta_4(\nu|\tau)}{\theta_4(0|\tau)} = exp\left\{\sum_{k=1}^{\infty}\frac{(2\pi i)^{2k}B_{2k}\nu^{2k}}{(2k+1)!-(2k)!}\left[E_{2k}(q)-E_{2k}(\sqrt{q})\right]\right\} \tag{5.72c}$$

where $B_k$ are the Bernoulli numbers: $B_2 = 1/6,\ B_4 = -1/30,\ B_6 = 1/42$. The 8-form elliptic genera are then

$$\begin{aligned}
A(\bar{q},R,F)^{SO(32)} &= \frac{E_4^3}{2^7 3^2 5}\frac{trR^4}{(2\pi)^4} + \frac{\hat{E}_2^2E_4^2}{2^9 3^2}\frac{(trR^2)^2}{(2\pi)^4} && (5.73a)\\
&+ \frac{trR^2trF^2}{2^8 3^2(2\pi)^4}\left(\hat{E}_2E_4E_6-\hat{E}_2^2E_4^2\right)\\
&+ \frac{trF^4}{(2\pi)^4} + \frac{(trF^2)^2}{2^9 3^2(2\pi)^4}\left(E_4^3-2\hat{E}_2E_4E_6+\hat{E}_2^2E_4^2-2^7 3^2\eta^{24}\right),\\
A(\bar{q},R,F)^{E_8\times E_8} &= \frac{E_4^3}{2^7 3^2 5}\frac{trR^4}{(2\pi)^4} + \frac{\hat{E}_2^2E_4^2}{2^9 3^2}\frac{(trR^2)^2}{(2\pi)^4}\\
&+ \frac{trR^2(trF_1^2+trF_2^2)}{2^8 3^2(2\pi)^4}\left(\hat{E}_2E_4E_6-\hat{E}_2^2E_4^2\right)\\
&+ \frac{trF_1^2trF_2^2}{2^8 3^2(2\pi)^4}\left(\hat{E}_2^2E_4^2-2\hat{E}_2E_4E_6+E_6^2\right) && (5.73b)\\
&+ \frac{(trF_2^2)^2+(trF_2^2)^2}{2^8 3^2(2\pi)^4}\left(E_4^3-2\hat{E}_2E_4E_6+\hat{E}_2^2E_4^2\right).
\end{aligned}$$

In the above, all group traces "tr" are in fundamental or vector representation.
We now finalize the computation of the amplitude (5.40)

$$\begin{aligned}
\mathcal{A} &= V_{10}\prod_{i=1}^{4}\langle\int\frac{d^2z_i}{\pi}\underbrace{e^i_{\alpha_i\beta_i}V^{\alpha_i\beta_i}(p_i,z_i,\bar{z}_i)}_{\text{4 graviton/gauge vertices}}\int\frac{d^2z_5}{\pi}\underbrace{\epsilon_{\mu\nu}V^{\mu\nu}(p_5,z_5,\bar{z}_5)}_{\text{one B vertex}}\rangle && (5.74)\\
&= V_{10}ZA(\bar{q},R,F)\wedge B_2\\
&= V_{10}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{\eta^{12}(\tau)}{\eta^4(\tau)}(\tau_2)^4\frac{A(\bar{q},R,F)}{\bar{\eta}^{16}}\wedge B_2\\
&= V_{10}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\frac{A(\bar{q},R,F)}{\bar{\eta}^{24}}\wedge B_2.
\end{aligned}$$

A few explanations are in order: in the second line above we have made the partition function (5.27) to appear explicitly and the fact that in CP-odd amplitude we have saturated 5-pairs of fermionic modes giving rise to the $\epsilon^{10}$ tensor which gives rise to the wedge product with $B_2$ field as well as wedge product in the polynomial structures inside $A(\bar{q},R,F)$. Next we write the complete expression of Z as $Z_{d=10\,\text{heterotic}} = \frac{iV_{10}}{4(2\pi l_s)^{10}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\frac{\theta^4\left[{a \atop b}\right]}{\eta^4}\frac{\bar{\Gamma}_{16}}{\bar{\eta}^{16}}$ with $\theta^4\left[{a \atop b}\right]$ contributing $\eta^{12}$ in the

CP-odd case. We also find the factor $(\tau_2)^4$ in the numerator from the 4-point vertex integral (5.51). Cancellation of the holomorphic $\eta$ functions in numerator and denominator happens and gives rise to the final line in (5.74). To evaluate the final line of (5.74) we use the integration formuli for the integration of modular functions on the fundamental domain $\mathcal{F}$ of $SL(2,\mathbb{Z})$ below [94, 78]

$$I(a,b,c) \equiv \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{\hat{E}_2^a E_4^b E_6^c}{\eta^{24p}}(\tau), \tag{5.75}$$

$$\text{where} \quad 6p = a + 2b + 3c, \text{ and thus} \tag{5.76}$$

$$\begin{aligned} I(0,0,0) &= \frac{2\pi}{3}, \\ I(0,3,0) &= 2^5 35\pi, \\ I(0,0,2) &= -2^5 37\pi, \\ I(1,1,1) &= -2^5 3\pi, \\ I(2,2,0) &= 2^5 3\pi, \\ I(3,0,1) &= -2^5 3\pi, \\ I(4,1,0) &= 2^5 35^{-1}\pi, \\ I(6,0,0) &= -2^5 37^{-1}\pi. \end{aligned}$$

The final 5-point amplitude for the Heterotic $SO(32)$ and $E_8 \times E_8$ theories are respectively

$$\mathcal{A}_{SO(32)} = V_{10} B_2 \wedge \left( trR^4 + \frac{1}{4}(trR^2)^2 + trR^2 trF^2 + 8trF^4 \right), \tag{5.77}$$

$$\begin{aligned} \mathcal{A}_{E_8\times E_8} = {} & V_{10} B_2 \wedge \{(trR^4 + \frac{1}{4}(trR^2)^2 + trR^2 \\ & (trF_1^2 + trF_2^2) - 2trF_1^2 trF_2^2 + 2(trF_1^2)^2 + 2(trF_2^2)^2\}. \end{aligned} \tag{5.78}$$

These are, when traded to the supergravity effective action for the D=10, N=1 Heterotic $SO(32)$ and $E_8 \times E_8$ theories, respectively the same anomaly cancelling terms which we have encountered before in (5.9) and (5.17). In fact one can show by using the index theorems relating anomalies and characteristic classes that the elliptic genus for string 1-loop is same as the anomaly generating functions [92]. The quantum theoretical (that is the supergravity) anomaly is the result of the UV divergence of the related QFT and the loss of classical symmetry in course of renormalization. The string theory amplitude, on the other hand is free from UV divergence but might suffer the lack of modular non-invariance due to the IR modes circulating in the string loop. Going towards the low energy limit, this departure from modularity translates itself to the UV divergence of the limiting supergravity theory.

A note about the CP-even sector of the above amplitude may be of interest. In case of 16-supercharges one finds certain super-invariants whose bosonic parts are [95, 74, 72]

$$I_1 = t_8 trF^4 - \frac{1}{4}\epsilon_{10}B_2 trF^4, \tag{5.79a}$$

$$I_2 = t_8 trF^2 TrF^2 - \frac{1}{4}\epsilon_{10}B_2 trF^2 trF^2, \tag{5.79b}$$

$$I_3 = t_8 trR^4 - \frac{1}{4}\epsilon_{10}B_2 trR^4, \tag{5.79c}$$

$$I_4 = t_8 trR^2 trR^2 - \frac{1}{4}\epsilon_{10}B_2 trR^2 trR^2, \tag{5.79d}$$

$$I_5 = t_8 trR^2 trF^2 - \frac{1}{4}\epsilon_{10}B_2 trR^2 trF^2. \tag{5.79e}$$

We see our familiar CP-odd terms as 5-particle states like $B_2 \wedge trR^4$ etc. The CP-even partners are in fact 4-point Green's functions pieces whose tensor structure is the familiar $t_8$ tensor defined to be

$$\begin{aligned} t^{ijklmnpq} \quad = \quad & -\frac{1}{2}\epsilon^{ijklmnpq} - \frac{1}{2}[(g^{ik}g^{jl} - g^{il}g^{jk})(g^{mp}g^{nq} - g^{mq}g^{np}) \\ & +(g^{km}g^{ln} - g^{kn}g^{lm})(g^{pi}g^{qj} - g^{pj}g^{qi}) \\ & +(g^{im}g^{jn} - g^{in}g^{jm})(g^{kp}g^{lq} - g^{kq}g^{lp})] \\ & +\frac{1}{2}[g^{jk}g^{lm}g^{np}g^{qi} + g^{jm}g^{kn}g^{lp}g^{qi} + g^{jm}g^{np}g^{kq}g^{il} \\ & +45 \text{ more terms by anti-symmetrizing on} (ij), (kl), (mn), (pq)] \end{aligned} \tag{5.80}$$

We end this section with the effective action of SO(32) Heterotic theory incorporating corrections upto 1-loop. This effective action does not receive any further loop corrections as it is associated with the anomaly cancelling mechanism. Thus in D=10, the tree level plus 1-loop level effective action is given in string frame by [96, 97]

$$\begin{aligned} S_{\text{tree + 1-loop}} \quad = \quad & \int d^{10}x\sqrt{G}e^{-2\phi}\{R + 4(\partial\phi)^2 - \frac{1}{12}H^2_{\mu\nu\lambda} + \frac{1}{8}(TrF^2 - trR^2) \\ & -\frac{1}{2^8}t_8(TrF^2 - trR^2)^2 + \frac{\zeta(3)}{3.2^9}(t_8t_8R^4 - \frac{1}{8}\epsilon_{10}\epsilon_{10}R^4) \\ & -\int d^{10}x\sqrt{G}\frac{1}{4}\beta(t_8X_8)\}. \end{aligned} \tag{5.81}$$

The term $J_0 = t_8t_8R^4 - \frac{1}{8}\epsilon_{10}\epsilon_{10}R^4$ is the bosonic part of a d=10, N=1 super-invariant and

$$X_8 \quad = \quad 32TrF^4 - 4TrF^2trR^2 + 4trR^4 + (trR^2)^2 \tag{5.82}$$

$$\beta \quad = \quad \frac{1}{3 \times 2^{14} \times \pi^5}. \tag{5.83}$$

In the next section we shall discover this interesting relation between string one-loop amplitude and supergravity anomalies for the case of D=8, N=1 theories coupled with rank-16 Yang-Mills gauge groups.

## 5.3 Discrete anomaly in D=8, N=1 supergravity and string amplitude comparison

After making ourselves familiar with the construction of the Heterotic elliptic genus we now turn to the computation of discrete anomalies in supergravity theories and compare the string theory amplitude, a course which we have explored for the maximal supergravity theories in D=10 and D=8 in chapter 2. This time we consider D=8 theories with half-maximal, that is, with 16 supercharges. We obtain this type of theories by compactifying Heterotic theories with gauge group either $SO(32)$ or $E_8 \times E_8$ with or without Wilson lines on a $T^2$ down to D=8. The T-duality relation between the HO and HE string theories, discussed in section 3.6.1, tells us that if we break one of the gauge groups between $SO(32)$ and $E_8 \times E_8$ with symmetrical Wilson line cofigurations, then one might rearrange the Wilson lines to reach the same new group from the other one. This point we shall explore when we shall be dealing with the gauge groups $SO(16)\times SO(16)$ and $SO(8)^4$ in D=8. We shall however restrict our attention for the time being to the cases where the new gauge group does not contain any $U(1)$ factors(we shall try to sketch some complexities regarding the groups $U(1)^{16}$ and $E_8 \times E_7 \times U(1)$ while discussing the duality relationship between Heterotic on $T^2$ and F theory on K3). Generically, $T^2$ compactifications of the Heterotic string theories have the following contents in their massless spectrum: [98]

**The gravity multiplet:** 1 graviton $g_{\mu\nu}$, 1 antisymmetric 2-form $B_{\mu\nu}$, 1 gravitino $\psi^i_\mu$, 2 graviphotons $A^I_\mu$, 1 dilatino $\chi^i$, 1 real scalar $\phi$.

**n vector multiplet:** n photons $L^J_\nu$, n gauginos $\lambda^{iJ}$, $n \times 2$ real scalars $\phi^J$.

The 2n real scalars $\phi^J$ parametrize the coset space $\frac{SO(2,n)}{SO(2)\times SO(n)}$. For the cases when the D=8 N=1 Yang-Mills gauge group $G_8$ does not contain any factor of $U(1)$ we get only n=2 abelian vector multiplets (from the reduction of metric $g_{\mu\nu}$ and the 2-form $B_{\mu\nu}$). We also get the scalars Kähler structure $T = B_{89} + iV_2$ and the complex structure $U = U_1 + iU_2$ of the torus $T^2$ from the reduction of NS-NS sector bosons and they parametrize the coset space $\frac{SO(2,2)}{U(1)\times U(1)}$.

To clarify the notation further, we start with Heterotic string theories in 10 spacetime dimensions and compactify the 8-th and 9-th space dimensions to form the torus $T^2$. The torus metric is specified by

$$G_{ij} = \frac{T_2}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}, B_{ij} = \begin{pmatrix} 0 & T_1 \\ -T_1 & 0 \end{pmatrix} \tag{5.84}$$

with $T_2, U_2 \geq 0$ . $T_1 = B_{89}$ is the $B_2$ field with both its legs on the toric cycles while $T_2 = V_2$ is the volume of this space-torus.

For the generic n vector multiplet case, the fermions have chiral coupling to the $U(1)$ of the coset $\frac{SO(2,n)}{SO(2)\times SO(n)}$ [98]. The representative vielbein metric $L$ as discussed in section 2.4 for the case of $\frac{SO(2,n)}{SO(2)\times SO(n)}$ whose form, in terms of the 2n scalars $\phi^J$ is [13]

$$L = exp\begin{pmatrix} 0 & \phi \\ \phi^T & 0 \end{pmatrix} = \begin{pmatrix} cosh\sqrt{\phi\phi^T} & \phi\frac{sinh\sqrt{\phi^T\phi}}{\sqrt{\phi^T\phi}} \\ \phi^T\frac{sinh\sqrt{\phi\phi^T}}{\sqrt{\phi\phi^T}} & cosh\sqrt{\phi^T\phi} \end{pmatrix} \tag{5.85}$$

with $\phi$ being a $n \times 2$ matrix composed of the 2n real scalars $\phi^J$

$$\phi = \begin{pmatrix} \phi^1 & \psi^2 \\ \psi^3 & \psi^4 \\ \vdots & \vdots \\ \psi^{2n-1} & \psi^{2n} \end{pmatrix}. \tag{5.86}$$

The Maurer-Cartan form looks like

$$L^{-1}\partial_\mu L = \begin{pmatrix} Q^b_{\mu a} & P^i_{\mu a} \\ P^a_{\mu i} & Q^j_{i\mu} \end{pmatrix} \tag{5.87}$$

where $Q_{\mu ab}$ $(a, b = 1, \cdots n)$ is the composite $SO(n)$ connection and $Q_{\mu ij}$ $(i, j = 1, 2)$ is the composite $SO(2) \equiv U(1)$ connection. The local supersymmetry actions contain the covariant derivatives of the fermionic fields

$$\mathcal{D}_\mu\psi_\nu = D_\mu\psi_\nu + \frac{1}{4}\omega_{\mu rs}\gamma^{rs}\psi_\nu + \frac{1}{2}i\gamma_9 Q_\mu\psi_\nu, \tag{5.88}$$

$$\mathcal{D}_\mu\chi = D_\mu\chi + \frac{1}{4}\omega_{\mu rs}\gamma^{rs}\chi - \frac{1}{2}i\gamma_9 Q_\mu\chi, \tag{5.89}$$

$$\mathcal{D}_\mu\lambda_J = D_\mu\lambda_J + \frac{1}{4}\omega_{\mu rs}\gamma^{rs}\lambda_J + \frac{1}{2}i\gamma_9 Q_\mu\lambda_J + Q^J_{\mu a}\lambda_J \tag{5.90}$$

with $D_\mu$ the super-Poincaré covariant derivative (2.18).

The supergravity action is invariant under the composite chiral $U(1)$ transformations

$$\delta\psi_\nu = \frac{1}{2}\Sigma(\phi)\gamma_9\psi_\nu, \tag{5.91}$$

$$\delta\chi = -\frac{1}{2}\Sigma(\phi)\gamma_9\chi, \tag{5.92}$$

$$\delta\lambda_J = \frac{1}{2}\Sigma(\phi)\gamma_9\lambda_J. \tag{5.93}$$

Thus working in the Weyl basis for the fermions in D=8, we read of the $U(1)$ charges[5] of the positive chiral gravitino, negative chiral dilatino and positive chiral gaugini to be a11 $\frac{1}{2}$. Note that, in case the coset space is $\frac{SO(2,2)}{U(1)\times U(1)}$ so that the D=8 gauge group G does not contain any $U(1)$ factor, the fermions of the theory chiral couplings to one of the two $U(1)$s of the coset: this is evident from the form of the covariant derivatives given in (5.88) above. Since in this type of compactification there is an exchange symmetry between T and U we can discuss the ensuing $SL(2, \mathbb{Z})$ anomaly (discussed in chapter 2) in terms of either of these two moduli and there will be corresponding counterterm involving the other moduli as well. The exchange symmetry between T and U is in fact the result of the perturbative symmetry $O(2, 2, \mathbb{Z}) = SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \rtimes \mathbb{Z}_2$ of the T-duality group.

Let us first discuss the $SL(2, \mathbb{Z})$ anomaly in case of $\frac{SO(2,2)}{U(1)\times U(1)}$ coset following the Green-Gaberdiel method discussed in chapter 4. Using the $U(1)$ charges of the fermions and the index formuli in section 2.5 we find the following anomalous phase variation of the path integral under the composite $U(1)_m$ (m is either $T$ or $U$)

$$\mathcal{A} = -\int \frac{\Sigma}{32(2\pi)^4}\left[(248 + \text{dim}G)\left[\frac{trR^4}{360} + \frac{(trR^2)^2}{288}\right] - (trR^2)^2 + \frac{1}{6}trR^2TrF^2 + \frac{2}{3}TrF^4\right] \tag{5.94}$$

[5]Note that our convention for the $\gamma_9$ matrix, which we have adhered to from the convention of calculating anomaly polynomials in accordance with Alvarez-Gaumé & Ginsparg [16] is negative of the convention used in Salam & Sezgin [98].

where the "Tr" means group traces in the adjoint of the gauge group G.

Following the Green-Gaberdiel method we can write the following $U(1)$ counter-term

$$S = \int \frac{\xi_m}{32(2\pi)^4}\left[(248+\text{dim}G)\left[\frac{trR^4}{360}+\frac{(trR^2)^2}{288}\right]-(trR^2)^2+\frac{1}{6}trR^2TrF^2+\frac{2}{3}TrF^4\right] \tag{5.95}$$

with $\xi_m$ transforms as $\xi_m \to \xi_m + \Sigma_m$ under the $U(1)_m$ transformation. $\xi_m$ however is not a physical field of the supergravity theory as we have mentioned in the case of Green-Gaberdiel anomaly for D=10 type IIB supergravity. We thus have to gauge fix it in terms of the discrete U-duality group $SO(2,2,\mathbb{Z})$. This group however does not afford any one dimensional non-trivial representation like $SL(2,\mathbb{Z})$. Nevertheless, one can exploit the local isomorphism of the continuous group $SO(2,2,\mathbb{R}) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and write the following non-trivial $SL(2,\mathbb{Z})$ transformation of the "gauged" $U(1)$ field $\xi_m$

$$\delta\xi_m = -\frac{i}{2}ln\left(\frac{cm+d}{c\bar{m}+d}\right). \tag{5.96}$$

Hence we can write a sum of two $SL(2,\mathbb{Z})$ counter-terms for $T$ and $U$ and sum them up so that we can retrieve the perturbative symmetry group $SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U \rtimes \mathbb{Z}_2$. Hence we shall propose the following $\alpha'^3$ CP-odd correction term in D=8, N=1 supergravity coupled with a Yang-Mills gauge group G

$$S_{\text{Het on }T^2} = \frac{i}{24}\int\left(ln\left(\frac{\eta^{24}(T)}{\bar{\eta}^{24}(\bar{T})}\right)+ln\left(\frac{j(T)}{\bar{j}(\bar{T})}\right)+ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right)+ln\left(\frac{j(U)}{\bar{j}(\bar{U})}\right)\right)Y_8, \tag{5.97}$$

where

$$Y_8 = \frac{1}{32(2\pi)^4}\left[(248+dimG)I_{1/2}-(trR^2)^2+\frac{1}{6}trR^2TrF^2+\frac{2}{3}TrF^4)\right]. \tag{5.98}$$

By explicit calculation we shall show that this term is also reproduced in the CP-odd sector of the 5-point string amplitude with 1 $U(1)_m$ current (m= T or U) and 4 gravitons, 4 gauge bosons, 2 gravitons plus 2 gauge bosons. Because this term in 1-loop amplitude is related to the anomaly counter-term in supergarvity, this term does not receive any further renormalization from higher string loops.

In the general case of n vector multiplet with the coset $\frac{SO(2,n)}{SO(2)\times SO(n)}$, the $U(1)$ current can be chosen to be generated by any of the n-complex scalars. In that case we can follow the procedure of "gauging" the n scalars [98] to obtain the subgroup $SO(1,2)\times H \subset SO(2,n)$ with $H$ being an (n-1)-dimensional compact subgroup of $SO(2,n)$ and then noting the isomorphism $SO(1,2) \simeq SL(2,\mathbb{R})$ we can write a $SL(2,\mathbb{Z})$ counter-term for the gauge fixed scalar. As the gauge fixed scalar can be any of the n-complex scalars, we can write a sum of n such counter-terms.

In the case of 5-point amplitude calculation in string theory side we have just two changes to incorporate in our development of the string amplitude and elliptic genus whose construction have been detailed in section 5.2. Instead of $B_2$ field we have to insert the vertex operator for the $U(1)$ current generated by T or U moduli whose form is given by

$$\begin{aligned} V_0(\phi_i, z, \bar{z}) &= v_{IJ}(\phi_i)\left[\partial X^I(z,\bar{z}) + ip.\psi(z)\psi^I(z)\right] \\ &\quad \bar{\partial} X^J(z,\bar{z}) e^{ip.X(z,\bar{z})}. \end{aligned} \tag{5.99}$$

$$\begin{aligned} V_{-1}(\phi_i, z, \bar{z}) &= v_{IJ}(\phi_i)\psi^I(z)\left[\partial X^\alpha \psi_\alpha(0) + G_{KL}\partial X^K(0)\psi^L(0)\right] \\ &\quad \bar{\partial} X^J(z,\bar{z}) e^{ip.X(z,\bar{z}}). \end{aligned} \tag{5.100}$$

respectively for the zero-ghost and the (-1)-ghost picture and the moduli vertices are

$$v_{IJ}(\phi_i) = \frac{\partial(G_{IJ} + B_{IJ})}{\partial \phi_i}. \tag{5.101}$$

$$\Longrightarrow \tag{5.102}$$

$$\begin{aligned} v(T) &= -\frac{i}{2U_2}\begin{pmatrix} 1 & U \\ \bar{U} & |U|^2 \end{pmatrix}, \\ v(U) &= \frac{iT_2}{U_2^2}\begin{pmatrix} 1 & \bar{U} \\ \bar{U} & \bar{U}^2 \end{pmatrix}. \end{aligned} \tag{5.103}$$

These are simply NS sector vertices for the Heterotic string as given in (5.30) but with proper moduli polarization tensor given by $v_{IJ}$. Next in this string theory, we have two space-dimensions compactified on the torus $T^2$ with complex structure $U$ and volume $V_2 = T_2$. From the world-sheet point of view, these two space-scalars are wrapped inside a two dimensional lattice $\Gamma_{2,2}$ whose character, in the absence of Wilson lines, can be written as

$$\Gamma_{2,2} = \frac{T_2}{\tau_2}\sum_{A\in ML(2,\mathbb{Z})} exp\left[-2\pi i T det(A) - \frac{\pi T_2}{\tau_2 U_2}|(1\ \ U)A\begin{pmatrix}\tau \\ 1\end{pmatrix}|^2\right] = \sum_{\vec{m},\vec{n}\in\mathbb{Z}^N} q^{P_L^2/2}\bar{q}^{P_R^2/2}. \tag{5.104}$$

with

$$p_L^2 = \frac{|Um_1 - m_2 + Tn^1 + TUn^2|^2}{2T_2U_2}, \tag{5.105}$$

$$p_L^2 - p_R^2 = 2m_I n^I, \tag{5.106}$$

$$m^1, m^2, n^1, n^2 \in \mathbb{Z}. \tag{5.107}$$

This lattice has to be included in the string partition function so that in D=8, the new string partition function reads

$$Z_{d=8 \text{ heterotic on } T^2} = \frac{iV_8}{4(2\pi l_s)^8}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\frac{1}{(\sqrt{\tau}\eta\bar{\eta})^6}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\frac{\theta^4\left[{a \atop b}\right]}{\eta^4}\frac{\Gamma_{2,2}\bar{\Gamma}_{16}}{\eta^2\bar{\eta}^2\bar{\eta}^{16}}. \tag{5.108}$$

In case there are non-zero Wilson lines along the $Y_I^\alpha$ , I=1,2 denoting the 1-cycles of the torus $T^2$ and $\alpha = 1, \cdots, 16$ denoting the Cartan components of the Cartan subalgebra of original gauge group $G = SO(32)$ or $E_8 \times E_8$ of the D=10 theory, the effect of the Wilson lines $Y_I^\alpha$ ($\alpha = 1, 2$ denoting the two 1-cycles of the torus and $I = 1, \cdots, 16$ the 16 roots of Cartan lattice) are to be included in the gauge lattice $\Gamma_{16}$

according to [99, 28]

$$\Gamma_{16}(T,U,Y_I^\alpha) = \frac{T_2}{\tau_2}\sum_{\substack{A\in ML(2,\mathbb{Z})\\ m^1,m^2,n^1,n^2\in\mathbb{Z}}} exp\left[-2\pi i T det(A) - \frac{\pi T_2}{\tau_2 U_2}|(1\ \ U)A\begin{pmatrix}\tau\\1\end{pmatrix}|^2\right] \times \sum_{a,b=0}^{1}\prod_{I=1}^{16} e^{i\pi(m_\alpha Y_I^\alpha Y_I^\beta n_\beta - b n_\alpha Y_I^\alpha)}\bar\theta\begin{bmatrix}a-2n_\alpha Y_I^\alpha\\ b-2m_\beta Y_I^\beta\end{bmatrix}(0,\bar\tau). \quad (5.109)$$

Therefore, one has to gauge the lattice according to the new D=8 gauge group $G_8$. There is however one conceptual difference in the string one loop calculation in D=8 case than in D=10 case. Because of the compactification, the particle, especially the fermion spectrum contains a whole mass tower of Kaluza-Klein states from the reduction of the massless states in D=10 theory. In the supergravity one loop computation, all these massive and the massless states will be circulating in the supergravity loop. However, the supergravity anomaly arising from one loop calculation is due to the loop divergence of the massless states only: the massive tower divergence can be renormalized without disturbing the classical symmetry. Thus, if we wish to find the corresponding anomaly cancelling terms from the string loop amplitude, we have to integrate out the contribution from the massive modes. This type of amplitude and the correction there-from is called the threshold correction. The implementation of this principle of integrating out the massive states is however relatively easy for the CP-even amplitude. We have noted that the vertex operator of the moduli (5.101) contains a derivative with respect to the moduli $\partial_m$. By manipulating this derivative out of the final $SL(2,\mathbb{Z})$ integral, we can keep only the contribution from the massless modes. To find the CP-odd case, we have to find a suitable integrability condition from the form of the threshold correction in CP-even amplitude as mere use of the tensor super-invariants (5.79) does not suffice to relate the moduli dependent functions in CP-even and CP-odd amplitudes. We shall illustrate this process in detail in case of $SO(32)$ and $E_8\times E_8$ and subsequently use the same in case of the gauge groups $SO(16)\times SO(16)$ and $SO(8)^4$. But before doing so we give the generic form of the 1-loop amplitude below.

The string amplitude in the CP-even sector will be of the form

$$\mathfrak{A} = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\prod_{i=1}^{4}\langle\int\frac{d^2 z_i}{\pi}\underbrace{\epsilon^i_{\bar\alpha_i\alpha_i}V^{\bar\alpha_i\alpha_i}(p_i,z_i,\bar z_i)}_{\text{4 gauge/graviton vertices}}\int\frac{d^2 z_5}{m}\underbrace{V_m(p_5,z_5,\bar z_5)}_{\text{one moduli vertex } m=T \text{ or } U}\rangle, \quad (5.110)$$

where all vertex operators are in zero-ghost picture,

$$V_0(R) = \frac{2g_{closed}}{l_s^2}\left[\partial X^\mu(z,\bar z) + ip.\psi(z)\psi^\mu(z)\right]\bar\partial X^\nu(z,\bar z)e^{ip.X(z,\bar z)} \quad (5.111)$$

$$V_0(F) = \frac{2g_{closed}}{l_s^2}\left[\partial X^\nu(z,\bar z) + ip.\psi(z)\psi^\nu(z)\right]\bar J^a(z,\bar z)e^{ip.X(z,\bar z)}. \quad (5.112)$$

To construct 8-derivative terms we again need momenta of eighth order i.e. $O(p^8)$. Consider then the part $ip.\psi\psi$ part of the gauge/graviton vertex: we get $p^4$ order directly from the fermionic contractions of the 4 fermion-couples in 4 gauge/graviton vertices. This will contribute to the $t_8$ tensor structure in the kinematical factor of the amplitude. The spin-structure-sum of the fermionic 2-point functions with the $\theta$

piece of the partition function yields again a position independent $\eta^{12}$ factor:

$$\frac{1}{2}\sum_{(a,b)\ even:(0,0),(0,1),(1,0)}(-1)^{a+b}\theta^4\left[{a \atop b}\right]\prod_{i,j}^{4}\frac{\theta\left[{a \atop b}\right](z_{ij})\theta'(0)}{\theta\left[{a \atop b}\right](0)\theta_1(z_{ij})}=-\frac{1}{2}(2\pi\eta^3)^4. \tag{5.113}$$

The rest $p^4$ order can be obtained from the exponential parts of the gauge/ gravity vertices leading to the gauge elliptic genus similar to (5.62)

$$\mathcal{E}(F)=\sum_{a,b=0}^{1}\prod_{I=1}^{16}e^{i\pi(m^\alpha Y^I_\alpha Y^I_\beta n^\beta-bn^\alpha Y^I_\alpha)}\bar{\theta}\left[{a-2n^\alpha Y^I_\alpha \atop b-2m^\beta Y^I_\beta}\right](v_i,\bar{\tau}) \tag{5.114}$$

in which we have inserted the Wilson lines too and the gravity elliptic genus similar to (5.2)

$$\begin{aligned}\mathcal{E}(R) &= exp\left[\frac{trR^2}{(2\pi)^2}\frac{\hat{E}_2(\bar{\tau},\tau_2)}{48}\right]\times exp\left[\frac{trR^4}{(2\pi)^4}\frac{E_4}{2^7\,3^2\,5}(\bar{\tau})\right] \\ &\quad\times exp\left[-\sum_{m=3}^{\infty}\frac{1}{2m}tr\,(R)^{2m}\frac{(2\pi i)^{2m}}{(2m)!}B_{2m}E_{2m}(\bar{\tau})\right]\end{aligned} \tag{5.115}$$

We thus do not need any momentum mode from the moduli vertex and it can be combined with the $\Gamma_{2,2}$ lattice to give

$$\langle V(m)\rangle=\frac{1}{\pi\tau_2}\partial_m\Gamma_{2,2} \tag{5.116}$$

and we can perform the $z_5$ integral using

$$\int_\Sigma d^2z_5\frac{1}{\tau_2}=1. \tag{5.117}$$

Thus we are led to the CP-even amplitude

$$\mathfrak{A}=V_8t_8\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}A(q,R,F)|_{\text{8-form}} \tag{5.118}$$

with $A(q,R,F)$ being the combined gauge gravity elliptic genus

$$A(q,R,F)=\mathcal{E}(R)\mathcal{E}(F)|_{\text{8-form}}. \tag{5.119}$$

Now to integrate the massive modes out and denote the threshold correction to be $\triangle_{\text{CP-even}}$, then

$$\begin{aligned}\partial_m\triangle_{\text{CP-even}}|_{\text{Gravitational}} &= 4t_8V_8\partial_m\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\Gamma_{2,2}A(q,R)|_{8-forms} \\ &\Longrightarrow \\ \triangle_{\text{CP-even}}|_{\text{Gravitational}} &= 4t_8V_8\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\Gamma_{2,2}A(q,R)|_{8-forms}.\end{aligned} \tag{5.120}$$

The calculation of the CP-odd part is quite similar as above but with the necessary changes: 4-fermionic couples contracted to $\epsilon_8$ tensor, the $\theta_1$ replaced by $(2\pi\eta^3)$ and using the moduli vartex in -1-picture. The moduli vertex can again be integrated out combining it with the $\Gamma_{2,2}$ lattice using this time the following identities

$$\langle V_m \rangle \quad = \tag{5.121}$$

$$v_{IJ}(m)\langle\psi^I\bar{\partial}X^J G_{KL}\partial X^K\psi^L\rangle = \frac{\chi_m}{\pi\tau_2}\partial_m\Gamma_{2,2} \quad \chi_m = \begin{cases}1, & m = T, U\\ -1, & m = \bar{T}, \bar{U}.\end{cases}$$

$$\text{Or} \quad v_{IJ}(m)\langle\bar{\psi}^I\partial X^J G_{KL}\bar{\partial}X^K\bar{\psi}^L\rangle = \frac{\sigma_m\chi_m}{\pi\tau_2}\partial_m\Gamma_{2,2}. \tag{5.122}$$

The final CP-odd amplitude will be of the form

$$\mathfrak{A} = 2i\epsilon_8 V_8 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\partial_m\Gamma_{2,2}A(q,R,F)|_{8-forms}. \tag{5.123}$$

The extraction of the threshold correction is now a little delicate [100, 101, 102]: noting the threshold correction as $\Theta_{\text{CP-odd}}$, we have to demand that this correction should reverse its sign according to the insertion of the moduli $m$ or its complex conjugate $\bar{m}$ so that the identities (5.121) and the contraction with $\epsilon$-tensor be consistent with each other and form the super-invariant complex function with the CP-even threshold $\triangle_{\text{CP-even}}$. Thus we need to have

$$\partial_m\Theta \quad = \quad -i\partial_m \triangle_{\text{CP-even}} \tag{5.124}$$

$$\partial_{\bar{m}}\Theta \quad = \quad i\partial_{\bar{m}} \triangle_{\text{CP-even}}\,. \tag{5.125}$$

If the functional dependence of the CP-even threshold coefficient $\triangle_{\text{CP-even}}$ be holomorphic in T and U, the above integrability condition gives rise to

$$\Theta = \text{Im}\,\triangle_{\text{CP-even}}\,. \tag{5.126}$$

We shall however find, as in the case of loop amplitude in type II theories on $T^2$ case (section 4.2.2) $\triangle_{\text{CP-even}}$ is not completely holomorphic in T and U. We shall need the functional form of the amplitude $\triangle_{\text{CP-even}}$ in (5.118) to clarify the process of integrability further. We postpone it for the moment and shall come back to it once we have the amplitude results.

In the following subsections, we shall explore the anomaly counter-terms (5.97) for the case of D=8, N=1 supergravity coupled to the gauge groups $G = SO(32),\ E_8\times E_8,\ SO(16)^2$ and $SO(8)^4$.

### 5.3.1 Case 1: G= $SO(32)$ and $E_8 \times E_8$

#### The anomaly counterterm

For the case of SO(32) and $E_8\times E_8$ we use once again the group trace rules to convert the adjoint traces "Tr" to the fundamental traces "tr" in the formula (5.97) and (5.98). We restate the group traces once again

$$TrF^2_{SO(N)} = (N-2)\,trF^2_{SO(N)} \tag{5.127a}$$

$$TrF^2_{E_8} = 30\,trF^2_{E_8} \tag{5.127b}$$

$$TrF^4_{SO(N)} = (N-8)\,trF^4_{SO(N)} \;+\; 3\,(trF^2_{SO(N)})^2 \tag{5.127c}$$

$$TrF^4_{E_8} = \frac{1}{100}(TrF^2_{E_8})^2 = 9\,(trF^2_{E_8})^2. \tag{5.127d}$$

As $E_8$ does not have a vector representation, it is standard to define its trace $trF^2_{E_8}$ in "fundamental" by using that of the group $SO(32)$ so that they have a uniform effect in the Green-Schwarz term of the D=10 Heterotic theory with gauge group either $SO(32)$ or $E_8 \times E_8$ [11].
Hence the $Y_8$ polynomial in (5.98) takes the form

$$Y_8^{SO(32)} = \frac{1}{32(2\pi)^4}\left(\frac{31}{15}trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2trF^2 + 2(trF^2)^2 + 16trF^4\right), \tag{5.128a}$$

$$Y_8^{E_8\times E_8} = \frac{1}{32(2\pi)^4}\left(\frac{31}{15}trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2(trF_1^2 + trF_2^2) + 6((trF_1^2)^2 + (trF_2^2)^2)\right). \tag{5.128b}$$

Putting this in (5.97) we get the following $\alpha'^3$ correction terms in D=8, N=1, $G = SO(32)$ and $E_8 \times E_8$ effective action

$$\begin{aligned} S_{SO(32)} &= \frac{i}{768(2\pi)^4} \times \{ \\ & \int \left(ln\left(\frac{\eta^{24}(T)}{\bar\eta^{24}(\bar T)}\right) + ln\left(\frac{j(T)}{\bar j(\bar T)}\right)\right) \times \\ & \left(\frac{31}{15}trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2trF^2 + 2(trF^2)^2 + 16trF^4\right) \\ &+ \int \left(ln\left(\frac{\eta^{24}(U)}{\bar\eta^{24}(\bar U)}\right) + ln\left(\frac{j(U)}{\bar j(\bar U)}\right)\right) \times \\ & \left(\frac{31}{15}trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2trF^2 + 2(trF^2)^2 + 16trF^4\right)\}. \end{aligned} \tag{5.129}$$

$$\begin{aligned} S_{E_8\times E_8} &= \frac{i}{768(2\pi)^4} \times \{ \\ & \int \left(ln\left(\frac{\eta^{24}(T)}{\bar\eta^{24}(\bar T)}\right) + ln\left(\frac{j(T)}{\bar j(\bar T)}\right)\right) \times \\ & \left(\frac{31}{15}trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2(trF_1^2 + trF_2^2) + 6((trF_1^2)^2 + (trF_2^2)^2)\right) \\ &+ \int \left(ln\left(\frac{\eta^{24}(U)}{\bar\eta^{24}(\bar U)}\right) + ln\left(\frac{j(U)}{\bar j(\bar U)}\right)\right) \times \\ & \left(\frac{31}{15}trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2(trF_1^2 + trF_2^2) + 6((trF_1^2)^2 + (trF_2^2)^2)\right)\}. \end{aligned} \tag{5.130}$$

**Comparison with 5-point String amplitude**

We now compare the above results (5.129), (5.130) with the $SO(32)$ and $E_8 \times E_8$ 5-point string amplitude [74]. We first evaluate the CP-even amplitude, the process of which has been sketched before. The CP-even threshold amplitude has the form (5.118)

$$\mathfrak{A} = V_8 t_8 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\Gamma_{2,2}A(q,R,F)|_{8-forms}, \tag{5.131}$$

where the elliptic genus[6] are the same as in (5.66) and (5.70)

$$A(q,R,F)^{SO(32)} = \frac{E_4^3}{2^7 3^2 5 \eta^{24}} \frac{trR^4}{(2\pi)^4} + \frac{\hat{E}_2^2 E_4^2}{2^9 3^2 \eta^{24}} \frac{(trR^2)^2}{(2\pi)^4} \tag{5.132a}$$
$$+ \frac{trR^2 trF^2}{2^8 3^2 (2\pi)^4} \left( \frac{\hat{E}_2 E_4 E_6}{\eta^{24}} - \frac{\hat{E}_2^2 E_4^2}{\eta^{24}} \right)$$
$$+ \frac{trF^4}{(2\pi)^4} + \frac{(trF^2)^2}{2^9 3^2 (2\pi)^4} \left( \frac{E_4^3}{\eta^{24}} - \frac{2\hat{E}_2 E_4 E_6}{\eta^{24}} + \frac{\hat{E}_2^2 E_4^2}{\eta^{24}} - 2^7 3^2 \right),$$

$$A(q,R,F)^{E_8\times E_8} = \frac{E_4^3}{2^7 3^2 5 \eta^{24}} \frac{trR^4}{(2\pi)^4} + \frac{\hat{E}_2^2 E_4^2}{2^9 3^2 \eta^{24}} \frac{(trR^2)^2}{(2\pi)^4}$$
$$+ \frac{trR^2(trF_1^2 + trF_2^2)}{2^8 3^2 (2\pi)^4} \left( \frac{\hat{E}_2 E_4 E_6}{\eta^{24}} - \frac{\hat{E}_2^2 E_4^2}{\eta^{24}} \right)$$
$$+ \frac{trF_1^2 trF_2^2}{2^8 3^2 (2\pi)^4} \left( \frac{\hat{E}_2^2 E_4^2}{\eta^{24}} - \frac{2\hat{E}_2 E_4 E_6}{\eta^{24}} + \frac{E_6^2}{\eta^{24}} \right) \tag{5.132b}$$
$$+ \frac{(trF_2^2)^2 + (trF_2^2)^2}{2^8 3^2 (2\pi)^4} \left( \frac{E_4^3}{\eta^{24}} - \frac{2\hat{E}_2 E_4 E_6}{\eta^{24}} + \frac{\hat{E}_2^2 E_4^2}{\eta^{24}} \right).$$

In view of the above elliptic genus, the amplitude $\mathfrak{A}$ can be viewed as the sum of integrals of the type

$$I(T,U) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \Gamma_{2,2}(T,U)\Phi(q)$$

with $\Phi(q)$ being the modular form coefficient of each of the 8-form components $trR^4$, $(trR^2)^2$, $trR^2 trF^2$, $trF^4$ and $(trF^2)^2$ . We then use the q-expansion (with $q = e^{2\pi i\tau}$) of $\Phi(q)$

$$\Phi(q) = \sum_{n=-1}^{\infty} c_n q^n, \tag{5.133}$$

and decomposes the $2\times 2$ matrices $B$ in the lattice sum

$$\Gamma_{2,2} = \frac{T_2}{\tau_2} \sum_{B\in ML(2,\mathbb{Z})} exp\left[ 2\pi i T det(B) - \frac{\pi T_2}{\tau_2 U_2} |(1 \;\; U) B \begin{pmatrix} \tau \\ 1 \end{pmatrix}|^2 \right] = \sum_{\vec{m},\vec{n}\in\mathbb{Z}^N} q^{P_L^2/2} \bar{q}^{P_R^2/2}. \tag{5.134}$$

into the orbits of $PSL(2,\mathbb{Z})$ (see [74, 75, 76])[7] :

| Orbits | Defining properties | Canonical representative |
|---|---|---|
| Trivial | $B = 0$ | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ |
| Degenerate | $B \neq 0;\ detB = 0$ | $\begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix}$ ; $j, p \neq 0.$ |
| Non-degenerate | $B \neq 0;\ detB \neq 0$ | $\begin{pmatrix} k & j \\ 0 & p \end{pmatrix}$ ; $0 \leq j < k;\ p \neq 0.$ |

[6] All group traces "tr" are in fundamental or vector representation.

[7] More sophisticated methods for such $SL(2,\mathbb{Z})$ integration calculations are available for example Florakis & Pioline [103], Angelantonj, Florakis & Pioline [104, 105]. We are however demonstrating the orbit decomposition method because it is easy to use in case of Wilson lines switched on.

The modular integration will now look like

$$
\begin{aligned}
I \;=\; & V_8 T_2 t_8 \times \{ \underbrace{\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} A(q,R,F)}_{\text{trivial orbit}} \\
& + \underbrace{\int_{\text{strip-boundary of } PSL(2,\mathbb{Z})} \frac{d^2\tau}{\tau_2^2} \sum_{(j,p)\neq(0,0)} e^{-\frac{\pi T_2}{\tau_2 U_2}|j+pU|^2} A(q,R,F)}_{\text{degenerate orbit}} \\
& + \underbrace{2\int_{\mathbb{C}+} \frac{d^2\tau}{\tau_2^2} \sum_{0\le j<k, p\neq 0} e^{-2\pi i T p k} e^{-\frac{\pi T_2}{\tau_2 U_2}|k\tau+j+pU|^2} A(q,R,F)\}.}_{\text{non-degenerate orbit}}
\end{aligned} \tag{5.135}
$$

To determine the leading part (non-volume suppressed part) of the amplitude coming from the degenerate orbit, that is to evaluate the integral

$$
\int_{\text{strip-boundary of } PSL(2,\mathbb{Z})} \frac{d^2\tau}{\tau_2^2} \sum_{(j,p)\neq(0,0)} e^{-\frac{\pi T_2}{\tau_2 U_2}|j+pU|^2} c_0, \tag{5.136}
$$

where $c_0$ is the coefficient of $q^0$ of the $q$ expansion of the elliptic genus $A(q,R,F)$, we use result of [74] to obtain the following harmonic part

$$
\begin{aligned}
& \int_{\text{strip-boundary of } PSL(2,\mathbb{Z})} \frac{d^2\tau}{\tau_2^2} \sum_{(j,p)\neq(0,0)} e^{-\frac{\pi T_2}{\tau_2 U_2}|j+pU|^2} c_0 \\
= & \left[ log U_2 |\eta(U)|^2 + \frac{\pi U_2}{6} \right] c_0 + \text{terms with } V_{T^2} \text{ in denominator}
\end{aligned} \tag{5.137}
$$

Note that the seemingly non-harmonic $log U_2$ piece in (5.137) comes from taking the appropriate renormalization scheme against the infra-red divergence of the above amplitude calculation.

To determine the non-volume suppressed part of the amplitude coming from the non-degenerate orbit, we use the integral [74], [77]

$$
\begin{aligned}
& T_2 \sum_{0\le j<k, p\neq 0} e^{-2\pi i T p k} \int_{\mathbb{C}+} \frac{d^2\tau}{\tau_2^2} e^{-\frac{\pi T_2}{\tau_2 U_2}|k\tau+j+pU|^2} c_0 = \\
& \sum_j \sum_{k>0, p>0} \frac{e^{2\pi i k p T}}{k|p|} c_0 + \text{cc.} + \text{volume suppressed terms.}
\end{aligned} \tag{5.138}
$$

We then sum up the leading order non-volume suppressed terms from all the three orbits which gives us

$$
\begin{aligned}
I(T,U) \;=\; & \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \Gamma_{2,2}(T,U)\Phi(q) \\
=\; & \frac{\pi T_2}{3}[c_0 - 24 c_{-1}] + \left[ log U_2 |\eta(U)|^2 + \frac{\pi U_2}{6} \right] c_0 + \left[ log T_2 |\eta(T)|^2 + \frac{\pi T_2}{6} \right] c_0 \\
& + \text{non-harmonic terms with } T_2^s \text{ in denominator with s=1, 2.}
\end{aligned} \tag{5.139}
$$

Using the above plus the $q-$expansion of different modular functions which we have summarized in (B.13) in appendix B we find the CP-even amplitude for SO(32)

$$\mathfrak{A}^{SO(32)}_{\text{CP-even}} = V_8 T_2 N \frac{\pi}{24} t_8 \underbrace{\left( trR^4 + \frac{1}{4}(trR^2)^2 + trR^2 trF^2 + 8trF^4 \right)}_{\text{Trivial orbit}} \tag{5.140}$$
$$+ V_8 N \frac{1}{48} \left[ log U_2 |\eta(U)|^2 \right]$$
$$\times\, t_8 \underbrace{\left( \frac{31}{15} trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2 trF^2 + 2(trF^2)^2 + 16trF^4 \right)}_{\text{Harmonic term from the degenerate orbit}}$$
$$+ V_8 N \frac{1}{48} \left[ log T_2 |\eta(T)|^2 + \frac{\pi T_2}{6} \right]$$
$$\times\, t_8 \underbrace{\left( \frac{31}{15} trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2 trF^2 + 2(trF^2)^2 + 16trF^4 \right)}_{\text{Harmonic term from the non-degenerate orbit}}$$
$$+ \text{non-harmonic terms.}$$

Similarly, using the elliptic genus for $E_8 \times E_8$ we find

$$\mathfrak{A}^{E_8\times E_8}_{\text{CP-even}} = V_8 T_2 N \frac{\pi}{24} t_8 \underbrace{\left( trR^4 + \frac{1}{4}(trR^2)^2 + trR^2(trF_1^2 + trF_2^2) - 2trF_1^2 trF_2^2 + 2(trF_1^2)^2 + 2(trF_2^2)^2 \right)}_{\text{Trivial orbit}} \tag{5.141}$$
$$+ V_8 N \frac{1}{48} \left[ log U_2 |\eta(U)|^2 \right]$$
$$\times\, t_8 \underbrace{\left( \frac{31}{15} trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2(trF_1^2 + trF_2^2) + 6((trF_1^2)^2 + (trF_2^2)^2) \right)}_{\text{Harmonic term from the degenerate orbit}}$$
$$+ V_8 N \frac{1}{48} \left[ log T_2 |\eta(T)|^2 + \frac{\pi T_2}{6} \right]$$
$$\times\, t_8 \underbrace{\left( \frac{31}{15} trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2(trF_1^2 + trF_2^2) + 6((trF_1^2)^2 + (trF_2^2)^2) \right)}_{\text{Harmonic term from the non-degenerate orbit}}$$
$$+ \text{non-harmonic terms.}$$

Next we determine the CP-odd amplitude from the above CP-even amplitude. It is clear from our discussion of the CP-odd threshold following (5.121) that the tensor structure in terms of the 8-form components $trR^4$, $(trR^2)^2$, $trR^2trF^2$, $trF^4$ and $(trF^2)^2$ remain the same, only the $t_8$ tensor is now replaced by the $\epsilon-$tensor so that each "tr" includes "wedge" products as in the anomaly polynomial (5.9), (5.17), (5.97) and (5.98). However the coefficients depending on U and T moduli have to be treated carefully according to the discussion following (5.121) which we shall complete here. The CP-odd coupling has to be of odd parity in accordance with the insertion of holomorphic or anti-holomorphic moduli (5.124). As the CP-even coefficients are $T_2$ in the trivial orbit part and $log m_2|\eta(m)|^2$ (m=T, U) in the leading harmonic parts coming from the degenerate and non-degenerate parts in (5.140) and (5.141) and are not holomorphic in m, we can not integrate the equations (5.124). Instead we write the CP-odd threshold in the form

$$\Theta = \Omega_1 \wedge (i\partial_T T_2 dT - i\partial_{\bar{T}} T_2 d\bar{T}) + \Omega_2 \wedge (i\partial_m logm_2|\eta(m)|^2 dm - i\partial_{\bar{m}} logm_2|\eta(m)|^2 d\bar{m}) \quad (5.142)$$

where $\Omega_1$ and $\Omega_2$ are Chern-Simons 7-forms such that, in case of G=SO(32)

$$d\Omega_1 = \left(trR^4 + \frac{1}{4}(trR^2)^2 + trR^2 trF^2 + 8trF^4\right) \quad (5.143)$$

$$d\Omega_2 = \left(\frac{31}{15}trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2trF^2 + 2(trF^2)^2 + 16trF^4\right). \quad (5.144)$$

Doing a partial integration in (5.142) we find the following form of the CP-odd (leading order) amplitude in case of G=SO(32)

$$\begin{aligned}\mathfrak{A}^{SO(32)}_{\text{CP-odd}} = \; & V_8 T_2 N\frac{\pi}{24}\underbrace{\left(trR^{\wedge 4} + \frac{1}{4}(trR^2)^{\wedge 2} + trR^{\wedge 2}trF^{\wedge 2} + 8trF^{\wedge 4}\right)}_{\text{Trivial orbit}} - \frac{1}{T_2}\Omega_1 \wedge dT_1 \quad (5.145)\\ & + V_8 N\frac{1}{48}\left[ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right)\right]\\ & \left(\frac{31}{15}trR^{\wedge 4} + \frac{19}{12}(trR^2)^{\wedge 2} + 5trR^{\wedge 2}trF^{\wedge 2} + 2(trF^2)^{\wedge 2} + 16trF^{\wedge 4}\right) - \frac{1}{U_2}\Omega_2 \wedge dU_1\\ & + V_8 N\frac{1}{48}\left[ln\left(\frac{\eta^{24}(T)}{\bar{\eta}^{24}(\bar{T})}\right) - 4i\pi T_1\right]\\ & \times\left(\frac{31}{15}trR^{\wedge 4} + \frac{19}{12}(trR^2)^{\wedge 2} + 5trR^{\wedge 2}trF^{\wedge 2} + 2(trF^2)^{\wedge 2} + 16trF^{\wedge 4}\right) - \frac{1}{T_2}\Omega_2 \wedge dT_1\\ & + \text{volume suppressed terms.}\end{aligned}$$

The above amplitude (5.145) gives rise to the following $\alpha'^3$ term in the effective action (where we also note that $T = B_{89} + iV_{T^2} = T_1 + iT_2$)

$$\begin{aligned}S^{SO(32)}_{\text{amp}} = \; & \frac{1}{192(2\pi)^3}\int B_{89}\underbrace{\left(trR^{\wedge 4} + \frac{1}{4}(trR^2)^{\wedge 2} + trR^{\wedge 2}trF^{\wedge 2} + 8trF^{\wedge 4}\right)}_{\text{Trivial orbit}} \quad (5.146)\\ & + \frac{1}{4\times 192(2\pi)^4}\int\left[ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right)\right]\times\\ & \left(\frac{31}{15}trR^{\wedge 4} + \frac{19}{12}(trR^2)^{\wedge 2} + 5trR^{\wedge 2}trF^{\wedge 2} + 2(trF^2)^{\wedge 2} + 16trF^{\wedge 4}\right)\\ & + \frac{1}{4\times 192(2\pi)^4}\int\left[ln\left(\frac{\eta^{24}(T)}{\bar{\eta}^{24}(\bar{T})}\right) - 4i\pi T_1\right]\\ & \times\left(\frac{31}{15}trR^{\wedge 4} + \frac{19}{12}(trR^2)^{\wedge 2} + 5trR^{\wedge 2}trF^{\wedge 2} + 2(trF^2)^{\wedge 2} + 16trF^{\wedge 4}\right).\end{aligned}$$

In the above, the term in the first line, which comes from the trivial orbit is in fact the $T^2$ reduction of the Green-Schwarz term of the D=10, N=1, G=SO(32) Heterotic theory (5.9)[11]

$$S_{GS} = \frac{1}{192(2\pi)^5\alpha'}\int B_2 \wedge \left(trR^{\wedge 4} + \frac{1}{4}(trR^2)^{\wedge 2} + trR^{\wedge 2}trF^{\wedge 2} + 8trF^{\wedge 4}\right) \quad (5.147)$$

Next we compare the remaining terms in (5.146) with the $\alpha'^3$ term (5.129) for this theory that we got from anomaly consideration and we see that apart from the $j$-function part (whose decompactification limit for $T_2 \to \infty$ is $-4\pi i T_1$) we have the exact matching between the string amplitude result and the anomaly counter-term in supergravity action. The matching is in fact the result of the fact that the low

energy limit of the 5-point 1-loop string amplitude is the inverse of the supergravity 1-loop amplitude and that the IR divergence in the string loop amplitude renders itself to the quantum anomaly in the low energy effective action.
For $E_8 \times E_8$ case we find similarly

$$\begin{aligned} S^{E_8\times E_8}_{\text{amp}} &= \frac{1}{192(2\pi)^3}\int B_{89}\{trR^4 + \frac{1}{4}(trR^2)^2 + trR^2(trF_1^2 + trF_2^2) \\ & -2trF_1^2 trF_2^2 + 2(trF_1^2)^2 + 2(trF_2^2)^2\} \\ & + \frac{1}{4\times 192(2\pi)^4}\int \left[ ln\left(\frac{\eta^{24}(U)}{\bar\eta^{24}(\bar U)}\right)\right] \times \\ & \left(\frac{31}{15}trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2(trF_1^2+trF_2^2) + 6((trF_1^2)^2 + (trF_2^2)^2)\right) \\ & + \frac{1}{4\times 192(2\pi)^4}\int \left[ ln\left(\frac{\eta^{24}(T)}{\bar\eta^{24}(\bar T)}\right) - 4i\pi T_1\right] \\ & \times\left(\frac{31}{15}trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2(trF_1^2+trF_2^2) + 6((trF_1^2)^2 + (trF_2^2)^2)\right). \end{aligned} \tag{5.148}$$

Once more this can be compared with (5.130) which matches apart from the $j$-function and the terms in the first two lines are the $T^2$ reduction of the $E_8\times E_8$ Green-Schwarz term

$$S_{GS} = \frac{1}{192(2\pi)^5\alpha'}\int B_2\wedge\left(trR^4 + \frac{1}{4}(trR^2)^2 + trR^2(trF_1^2+trF_2^2) - 2trF_1^2trF_2^2 + 2(trF_1^2)^2 + 2(trF_2^2)^2\right). \tag{5.149}$$

### 5.3.2 Case 2: G= $SO(16)\times SO(16)$

**The anomaly counter-term and group traces**

Now we consider D=10 Heterotic string theory with gauge group $E_8\times E_8$ compactified on a $T^2$ with Kähler structure $T = B_{89} + iV_{T^2}$ and complex structure $U = U_1 + iU_2$ and with the following Wilson line on $T^2$

$$Y_i^1 = (0^4, \frac{1}{2}^4, 0^4, \frac{1}{2}^4), \qquad Y_i^2 = (0^8, 0^8), \qquad i = 1, \cdots, 16, \tag{5.150}$$

so that the gauge group is broken to $SO(16)\times SO(16)$ in D=8. One can of course rearrange the 8 non-zero values of the Wilson lines so that one can start from $SO(32)$ gauge group in D=10 and again obtain $SO(16)\times SO(16)$ in D=8.

First we discuss the group decomposition $E_8\times E_8 \supset SO(16)\times SO(16)$ which we shall find extremely useful to understand the string amplitude part.
For the decomposition $E_8\times E_8\supset SO(16)\times SO(16)$ we have

$$\mathbf{248}\oplus\mathbf{248} = \underbrace{\mathbf{(120,1)}\oplus\mathbf{(1,120)}}_{\text{adjoint rep. of } SO(16)\times SO(16)} \oplus \underbrace{\mathbf{(128,1)}\oplus\mathbf{(1,128)}}_{\text{spinor rep. of } SO(16)\times SO(16)}. \tag{5.151}$$

The rules for transcribing group trace "Tr" in the adjoint representation towards the group trace "tr" in the fundamental representation for $SO(N)$ groups [106, 107]

$$TrF^2_{SO(N)} = (N-2)\, trF^2_{SO(N)}, \tag{5.152}$$

$$TrF^4_{SO(N)} = (N-8)\, trF^4_{SO(N)} + 3\,(trF^2_{SO(N)})^2. \tag{5.153}$$

For the **(128,1)** $\oplus$ **(1,128)** representation, we write the traces formula

$$tr_{128}F_1^2 + tr_{128}F_2^2 = 16trF_1^2 + 16trF_2^2, \quad (5.154a)$$
$$tr_{128}F_1^4 + tr_{128}F_2^4 = 6(trF_1^2)^2 + 6(trF_2^2)^2 - 8trF_1^4 - 8trF_2^4. \quad (5.154b)$$

For the sake of completeness, we also provide the branching rule for the decomposition $SO(32) \supset SO(16) \times SO(16)$

$$\mathbf{496} = \underbrace{\mathbf{(120,1)} \oplus \mathbf{(1,120)}}_{\text{adjoint rep. of } SO(16)\times SO(16)} \oplus \underbrace{\mathbf{(16,16)}}_{\text{cospinor rep. of } SO(16)\times SO(16)} . \quad (5.155)$$

For the **(16,16)** representation, we write the traces formula

$$tr_{(16,16)}F^2 = 16trF_1^2 + 16trF_2^2, \quad (5.156a)$$
$$tr_{(16,16)}F^4 = 16trF_1^4 + 16trF_2^4 + 6(trF_1^2)(trF_2^2). \quad (5.156b)$$

Adapting formula (5.98) to the adjoint representation **(120,1)** $\oplus$ **(1,120)** of $SO(16)^2$ and using the trace-conversion formulae (5.152) we can write the 8-form polynomial for the case $G = SO(16)^2$ as follows

$$Y_8^{SO(16)^2} = \frac{1}{32(2\pi)^4}\left(\frac{488}{360}trR^4 + \frac{200}{288}(trR^2)^2 + \frac{7}{3}trR^2\sum_{i=1}^{2}trF_i^2 + \frac{16}{3}\sum_{i=1}^{2}trF_i^4 + 2\sum_{i=1}^{2}(trF_i^2)^2\right). \quad (5.157)$$

In the following we shall compare this supergravity result with the string amplitude calculation.

### Comparison with 5-point String amplitude

We now elaborate the process of the CP-even 5-point string amplitude for the $SO(16)\times SO(16)$ following the lines of Gutperle [108] where the pieces of the calculation have been provided e.g. the coefficient of $trR^4$, $trF^4$ and $(trF^2)^2$ for the non-degenerate orbit (5.135). We shall provide the CP-even part of the amplitude in the leading order non-volume suppressed harmonic forms in trivial, degenerate and non-degenerate orbits.
The amplitude will be derived from

$$\mathfrak{A} = V_8 t_8 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\Gamma_{2,2}A(q,R,F)|_{8-forms}, \quad (5.158)$$

where $\Gamma_{2,2}$ is the $T^2$ lattice sum as before

$$\Gamma_{2,2} = \frac{T_2}{\tau_2}\sum_{B\in ML(2,\mathbb{Z})} exp\left[2\pi i T det(B) - \frac{\pi T_2}{\tau_2 U_2}|(1 \;\; U)B\begin{pmatrix}\tau\\1\end{pmatrix}|^2\right] = \sum_{\vec{m},\vec{n}\in\mathbb{Z}^N} q^{P_L^2/2}\bar{q}^{P_R^2/2} \quad (5.159)$$

with $B$ being the $2\times 2$ matrix

$$B = \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix}. \quad (5.160)$$

The form of the elliptic genus $A(q,R,F)$ shall depend on the spin structure as we shall describe shortly and hence there are 3 different elliptic genus for trivial, degenerate and non-degenerate orbits which we shall note by $A_{\text{trivial}}(q,R,F)$, $A_{\text{degenerate}}(q,R,F)$

and $A_{\text{non-degenerate}}(q,R,F)$ respectively. The general elliptic genus is obtained [78] from the gauging of

$$A(q,R,F) = \frac{1}{\eta^{24}} Exp\left(\frac{trR^2}{(2\pi)^2}\frac{\hat{E}_2}{48}\right) \times Exp\left(\frac{trR^4}{(2\pi)^4}\frac{E_4}{2^7 3^2 5}\right) \times \sum_{a,b=1}^{2} \underbrace{\theta^8 \begin{bmatrix} a \\ b \end{bmatrix}}_{SO(16)_1} \times \underbrace{\theta^8 \begin{bmatrix} a+m_1 \\ b+n_1 \end{bmatrix}}_{SO(16)_2}. \tag{5.161}$$

We have summarised our convention of Jacobi theta functions in appendix B. The labels $SO(16)_1$ and $SO(16)_2$ in (5.161) denote the gauging of the theta functions according to two $SO(16)$s.
The trivial orbit is characterised by $B=0$ so that the elliptic genus will be

$$\begin{aligned} A_{\text{trivial}}(q,R,F) &= \frac{E_4^3}{2^7 3^2 5 \eta^{24}}\frac{trR^4}{(2\pi)^4} + \frac{\hat{E}_2^2 E_4^2}{2^9 3^2 \eta^{24}}\frac{(trR^2)^2}{(2\pi)^4} \\ &+ \frac{trR^2(trF_1^2+trF_2^2)}{2^8 3^2 (2\pi)^4}\left(\frac{\hat{E}_2 E_4 E_6}{\eta^{24}} - \frac{\hat{E}_2^2 E_4^2}{\eta^{24}}\right) \\ &+ \frac{trF_1^2 trF_2^2}{2^8 3^2 (2\pi)^4}\left(\frac{\hat{E}_2^2 E_4^2}{\eta^{24}} - \frac{2\hat{E}_2 E_4 E_6}{\eta^{24}} + \frac{E_6^2}{\eta^{24}}\right) \\ &+ \frac{(trF_1^2)^2+(trF_2^2)^2}{2^8 3^2 (2\pi)^4}\left(\frac{E_4^3}{\eta^{24}} - \frac{2\hat{E}_2 E_4 E_6}{\eta^{24}} + \frac{\hat{E}_2^2 E_4^2}{\eta^{24}}\right). \end{aligned} \tag{5.162}$$

The degenerate orbit is characterised by $B \neq 0, \quad det(B)=0$, for which choose the two following sectors

$$B^{(1)} = \begin{pmatrix} 0 & 2j \\ 0 & p \end{pmatrix}, \qquad B^{(2)} = \begin{pmatrix} 0 & 2j+1 \\ 0 & p \end{pmatrix}, \qquad j,p \in \mathbb{Z}. \tag{5.163}$$

For the gauging, we use the identities (B.7) and the definitions of Eisenstein series given in (B.6) plus the combinations $f_1$, $f_2$, $f_3$ of theta functions

$$f_1 = \theta_3^4 + \theta_4^4, \quad f_2 = \theta_2^4 - \theta_4^4, \quad f_3 = -\theta_2^4 - \theta_3^4. \tag{5.164}$$

The elliptic genus for degenerate orbit is then

$$\begin{aligned} A_{\text{degenerate}}(q,R,F) &= \frac{trR^4}{(2\pi)^4}\frac{E_4}{2^7 3^2 5 \eta^{24}}\left(B^{(1)}\sum_{a=2}^{4}\theta_a^{16} + B^{(2)} 2\theta_3^8\theta_4^8\right) \\ &+ \frac{(trR^2)^2}{(2\pi)^4}\frac{\hat{E}_2^2}{2^9 3^2 \eta^{24}}\left(B^{(1)}\sum_{a=2}^{4}\theta_a^{16} + B^{(2)} 2\theta_3^8\theta_4^8\right) \\ &- \frac{trR^2(trF_1^2+trF_2^2)}{2^8 3^2 (2\pi)^4 \eta^{24}}\{2B^{(1)}(\hat{E}_2 E_4 E_6 - \hat{E}_2^2 E_4^2) - B^{(2)}(\hat{E}_2 \theta_3^8\theta_4^8)(f_2+f_3+2\hat{E}_2)\} \\ &+ \frac{trF_1^4+trF_2^4}{2^7 3 (2\pi)^4 \eta^{24}}\{B^{(1)}(-\theta_2^{16}\theta_3^4\theta_4^4 + \theta_3^{16}\theta_2^4\theta_4^4 - \theta_4^{16}\theta_2^4\theta_3^4) + B^{(2)}(\theta_3^8\theta_4^8(\theta_2^4\theta_4^4 - \theta_2^4\theta_3^4))\} \\ &+ \frac{(trF_1^2)^2+(trF_2^2)^2}{2^9 3^2 (2\pi)^4 \eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16}(\hat{E}_2+f_{a-1})^2 + B^{(2)}\theta_3^8\theta_4^8\left[(f_2+\hat{E}_2)^2 + (f_3+\hat{E}_2)^2\right]\} \\ &+ \frac{(trF_1^2)(trF_2^2)}{2^8 3^2 (2\pi)^4 \eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16}(\hat{E}_2+f_{a-1})^2 + B^{(2)}\theta_3^8\theta_4^8\left[(f_2+\hat{E}_2)^2 + (f_3+\hat{E}_2)^2 - 9\theta_2^8\right]\}. \end{aligned} \tag{5.165}$$

In the above elliptic genus the $B^{(1)}$ and $B^{(2)}$ act as operators such that in the amplitude integration (5.158) one should take into account the values of the matrix $B$ as given in (5.163).

Finally we come to the non-degenerate orbit ($B \neq 0$, $det(B) \neq 0$) whose matrix representative is

$$B = \left(\begin{smallmatrix} k & j \\ 0 & p \end{smallmatrix}\right); \qquad 0 \leq j < k;\ p \neq 0.$$

We have to use the following 4 sectors of this representative matrix because of the spin structure (5.161)

$$\begin{aligned} B^{(1)} &= \begin{pmatrix} 2k & 2j \\ 0 & p \end{pmatrix}, \qquad B^{(2)} = \begin{pmatrix} 2k & 2j+1 \\ 0 & p \end{pmatrix}, \\ B^{(3)} &= \begin{pmatrix} 2k+1 & 2j \\ 0 & p \end{pmatrix}, \qquad B^{(4)} = \begin{pmatrix} 2k+1 & 2j+1 \\ 0 & p \end{pmatrix}, \qquad 0 \leq j < k, \quad j,k,p \in \mathbb{Z}. \end{aligned} \tag{5.166}$$

The elliptic genus for the non-degenerate orbit is then

$$\begin{aligned} &A_{\text{non-degenerate}}(q,R,F) = \\ &\frac{trR^4}{(2\pi)^4}\frac{E_4}{2^7 3^2 5\eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16} + B^{(2)}2\theta_3^8\theta_4^8 + B^{(3)}2\theta_2^8\theta_3^8 + B^{(4)}2\theta_2^8\theta_4^8\} \\ &+\frac{(trR^2)^2}{(2\pi)^4}\frac{\hat{E}_2^2}{2^9 3^2\eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16} + B^{(2)}2\theta_3^8\theta_4^8 + B^{(3)}2\theta_2^8\theta_3^8 + B^{(4)}2\theta_2^8\theta_4^8\} \\ &-\frac{trR^2(trF_1^2+trF_2^2)}{2^8 3^2(2\pi)^4\eta^{24}}\{(2B^{(1)}(\hat{E}_2E_4E_6 - \hat{E}_2^2E_4^2) - B^{(2)}(\hat{E}_2\theta_3^8\theta_4^8)(f_2+f_3+2\hat{E}_2) \\ &\qquad - B^{(3)}(\hat{E}_2\theta_2^8\theta_3^8)(f_1+f_2+2\hat{E}_2) - B^{(4)}(\hat{E}_2\theta_2^8\theta_4^8)(f_1+f_3+2\hat{E}_2)\} \\ &+\frac{trF_1^4+trF_2^4}{2^7 3(2\pi)^4\eta^{24}}\{B^{(1)}(-\theta_2^{16}\theta_3^4\theta_4^4 + \theta_3^{16}\theta_2^4\theta_4^4 - \theta_4^{16}\theta_2^4\theta_3^4) + B^{(2)}(\theta_3^8\theta_4^8(\theta_2^4\theta_4^4 - \theta_2^4\theta_3^4)) \\ &\qquad + B^{(3)}(\theta_2^8\theta_3^8(\theta_2^4\theta_4^4 - \theta_3^4\theta_4^4)) + B^{(4)}(\theta_2^8\theta_4^8(-\theta_3^4\theta_4^4 - \theta_2^4\theta_3^4))\} \\ &+\frac{(trF_1^2)^2+(trF_2^2)^2}{2^9 3^2(2\pi)^4\eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16}(\hat{E}_2+f_{a-1})^2 + B^{(2)}\theta_3^8\theta_4^8\left[(f_2+\hat{E}_2)^2+(f_3+\hat{E}_2)^2\right] \\ &\qquad + B^{(3)}\theta_2^8\theta_3^8\left[(f_1+\hat{E}_2)^2+(f_2+\hat{E}_2)^2\right] + B^{(4)}\theta_2^8\theta_4^8\left[(f_1+\hat{E}_2)^2+(f_3+\hat{E}_2)^2\right]\} \\ &+\frac{(trF_1^2)(trF_2^2)}{2^8 3^2(2\pi)^4\eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16}(\hat{E}_2+f_{a-1})^2 + B^{(2)}\theta_3^8\theta_4^8\left[(f_2+\hat{E}_2)^2+(f_3+\hat{E}_2)^2-9\theta_2^8\right] \\ &\qquad + B^{(3)}\theta_2^8\theta_3^8\left[(f_1+\hat{E}_2)^2+(f_2+\hat{E}_2)^2-9\theta_4^8\right] + B^{(4)}\theta_2^8\theta_4^8\left[(f_1+\hat{E}_2)^2+(f_3+\hat{E}_2)^2-9\theta_3^8\right]\}. \end{aligned} \tag{5.167}$$

Once again in the above, the terms $B^{(i)}$ with $i = 1,2,3,4$ denote the sector operators so that one takes into account correctly the values of the matrix elements $B$ according to the convention (5.166).

The complete amplitude is then

$$\begin{aligned}
\mathfrak{A} &= T_2V_8t_8 \times \{\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} A(q,R,F)_{\text{trivial}} && (5.168)\\
&+\int_{\text{strip-boundary of } PSL(2,\mathbb{Z})} \frac{d^2\tau}{\tau_2^2} \sum_{(n_1,n_2)\neq(0,0)} e^{-\frac{\pi T_2}{\tau_2 U_2}|n_1+n_2U|^2} A(q,R,F)_{\text{degenerate}}\\
&+2\int_{\mathbb{C}^+} \frac{d^2\tau}{\tau_2^2} \sum_{0\le n_1<m_1, n_2\neq 0} e^{-2\pi i Tpk} e^{-\frac{\pi T_2}{\tau_2 U_2}|m_1\tau+n_1+n_2U|^2} A(q,R,F)_{\text{non-degenerate}}\},\\
&= \mathfrak{A}_{\text{trivial}} + \mathfrak{A}_{\text{degenerate}} + \mathfrak{A}_{\text{non-degenerate}} && (5.169)
\end{aligned}$$

The trivial orbit amplitude gives

$$\begin{aligned}
\mathfrak{A}_{\text{trivial}} = T_2V_8t_8\frac{1}{(2\pi)^4}\{trR^4 + \frac{1}{4}(trR^2)^2 + trR^2(trF_1^2 + trF_2^2) \qquad (5.170)\\
- 2trF_1^2trF_2^2 + 2(trF_1^2)^2 + 2(trF_2^2)^2\}
\end{aligned}$$

To evaluate the degenerate amplitude, we q-expand the modular function in the elliptic genus (5.165) and take the constant coefficients which we have noted in (B.13) which shall provide the harmonic part of the amplitude. In this respect we also note that we can sum up the contributions of $B^{(1)}$ and $B^{(2)}$ sectors in (5.163) so that the sum run in the complete set of integers for $n_1$ and $n_2$ so that the CP-even modular coefficient will be $logU_2|\eta(U)|^2$ (the $logU_2$ follows from the renormalization scheme). Also we note that the sum of the coefficients of $1/q$ is zero so that there are no poles in $q$. Using (5.137) we find the harmonic part of the CP-even amplitude coming from the degenerate orbit

$$\begin{aligned}
\mathfrak{A}_{\text{degenerate}} = \frac{1}{(2\pi)^4}logU_2|\eta(U)|^2V_8t_8\{\frac{488}{360}trR^4 + \frac{200}{288}(trR^2)^2 + \frac{7}{3}trR^2(trF_1^2 + trF_2^2)\\
(5.171)\\
+ \frac{16}{3}(trF_1^4 + trF_2^4) + 2((trF_1^2)^2 + (trF_2^2)^2)\}.
\end{aligned}$$

Finally for the non-degenerate amplitude we again q-expand the modular functions in the elliptic genus (5.167) and check that there is no pole in $q$. Next we note that the leading term in the harmonic part for $B^{(1)}$ and $B^{(2)}$ sectors are the same and is equal to $\left[logT_2|\eta(2T)|^2 + \frac{\pi T_2}{3}\right]$. We then sum the constant coefficients which shall provide the leading term (which are not volume suppressed) in the harmonic part. The constant coefficients in $B^{(3)}$ and $B^{(4)}$ are the same and hence the sum over $m_1$, $n_1$ and $n_2$ can be extended to the complete $\mathbb{Z}$ with the contribution $\left[logT_2|\eta(2T)|^2 - logT_2|\eta(T)|^2 + \frac{\pi T_2}{6}\right]$. Once again, we evaluate the CP-even integral using (5.138) and the leading term (harmonic) in the non-degenerate amplitude will be (we write only the non-volume suppressed harmonic part of the amplitude)

$$\begin{aligned}
&\mathfrak{A}_{\text{non-degenerate}} = \\
&\frac{1}{(2\pi)^4}\left[logT_2|\eta(2T)|^2 + \frac{\pi T_2}{3}\right] V_8 t_8\{\frac{488}{360}trR^4 + \frac{200}{288}(trR^2)^2 + \frac{7}{3}trR^2(trF_1^2 + trF_2^2) \\
&\qquad + \frac{16}{3}(trF_1^4 + trF_2^4) + 2((trF_1^2)^2 + (trF_2^2)^2)\} \\
&+\frac{1}{2(2\pi)^4}\left[logT_2|\eta(2T)|^2 - logT_2|\eta(T)|^2 + \frac{\pi T_2}{6}\right] \times \\
&\qquad V_8 t_8\{256\left(\frac{trR^4}{360} + \frac{(trR^2)^2}{288}\right) + \frac{8}{3}trR^2(trF_1^2 + trF_2^2) \\
&\qquad - \frac{16}{3}(trF_1^4 + trF_2^4) + 4((trF_1^2)^2 + (trF_2^2)^2)\}
\end{aligned} \tag{5.172}$$

Finally, to extract the CP-odd amplitude from the above CP-even amplitude, we deploy the same technique of moduli integrability (see discussion following (5.140) of the previous sub-section 5.3.1) and we get

$$\mathcal{A}_{\text{cp-odd}} = \mathcal{A}_{\text{trivial}} + \mathcal{A}_{\text{degenerate}} + \mathcal{A}_{\text{non-degenerate}} \tag{5.173}$$

with

$$\mathcal{A}_{\text{trivial}} = T_1\left(trR^4 + \frac{1}{4}(trR^2)^2 + trR^2\sum_{i=1}^{2}trF_i^2 - 2trF_1^2trF_2^2 + 2\sum_{i=1}^{2}(trF_i^2)^2\right), \tag{5.174}$$

$$\begin{aligned}
\mathcal{A}_{\text{degenerate}} = \frac{1}{96(2\pi)^4}ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right)&\left(\frac{488}{360}trR^4 + \frac{200}{288}(trR^2)^2 + \frac{7}{3}trR^2\sum_{i=1}^{2}trF_i^2\right. \\
&\left. + \frac{16}{3}\sum_{i=1}^{2}trF_i^4 + 2\sum_{i=1}^{2}(trF_i^2)^2\right),
\end{aligned} \tag{5.175}$$

$$\begin{aligned}
\mathcal{A}_{\text{non-degenerate}} = &\frac{1}{96(2\pi)^4}\left[8i\pi T_1 - ln\left(\frac{\eta^{24}(2T)}{\bar{\eta}^{24}(2\bar{T})}\right)\right]\left(\frac{488}{360}trR^4 + \frac{200}{288}(trR^2)^2 + \frac{7}{3}trR^2\sum_{i=1}^{2}trF_i^2\right. \\
&\left. + \frac{16}{3}\sum_{i=1}^{2}trF_i^4 + 2\sum_{i=1}^{2}(trF_i^2)^2\right) +
\end{aligned} \tag{5.176a}$$

$$\begin{aligned}
&+ \frac{1}{192(2\pi)^4}\left[-4i\pi T_1 + ln\left(\frac{\eta^{24}(2T)}{\bar{\eta}^{24}(2\bar{T})}\right) - ln\left(\frac{\eta^{24}(T)}{\bar{\eta}^{24}(\bar{T})}\right)\right] \times \\
&\times\left[256\left(\frac{trR^4}{360} + \frac{(trR^2)^2}{288}\right) + \frac{8}{3}trR^2\sum_{i=1}^{2}trF_i^2 - \frac{16}{3}\sum_{i=1}^{2}trF_i^4 + 4\sum_{i=1}^{2}(trF_i^2)^2\right].
\end{aligned} \tag{5.176b}$$

The above expressions translate into the following $\alpha'^3$ corrections to the 8D effective action

$$S^{SO(16)}_{\text{amp}} = \int \left[N_1 \mathcal{A}_{\text{trivial}} + N_2 \mathcal{A}_{\text{deg.}} + N_3 \mathcal{A}_{\text{non-deg.}}\right] , \tag{5.177}$$

with appropriate normalization factors $N_1,\ N_2,\ N_3$. As usual, the trivial orbit term (5.174) is the same as the $T^2$ reduction of the Heterotic Green-Schwarz term, as can be most easily seen by starting from the $E_8 \times E_8$ one in (5.149). The degenerate orbit term (5.175) and the first piece of the non-degenerate orbit term (5.176a) involve exactly the same 8-form polynomial dictated by anomaly cancelation (5.157). Note that the modular coefficients in front do not appear to have the correct modular properties to cancel the corresponding anomalous $SL(2,\mathbb{Z})$ phase variations, but this is only an artefact of having split the amplitude in three different orbits and of having adopted appropriate renormalization schemes [74]. Finally, the second piece of the non-degenerate orbit term (5.176b) is the contribution from the massive vector multiplets in the **(128,1)** $\oplus$ **(1,128)** representation of $SO(16)^2$, as can be verified using the trace formulae (5.154a), (5.154b).

One can of course make a totally analogous analysis starting from $SO(32)$, and using the branching rules (5.155) and the trace formulae (5.156a).

### 5.3.3 Case 3: G= $SO(8)^4$

#### Anomaly cancelling term and group trace

Finally we come to the case of the D=8, N=1 theory with gauge group $SO(8)^4$ which can be obtained from D=10, N=1 theory with gauge group either $SO(32)$ or $E_8 \times E_8$ compactified on a $T^2$ with appropriate Wilson lines along the two 1-cycles of the torus. Following the reasoning outlined in 5.3 we can write the composite anomaly cancelling term for the gravitino and gaugini in the adjoint representation of $SO(8)^4$ as follows

$$\begin{aligned} S_{SO(8)^4} = &\frac{i}{768(2\pi)^4} \times \{ \\ &\int \left( ln\left(\frac{\eta^{24}(T)}{\bar\eta^{24}(\bar T)}\right) + ln\left(\frac{j(T)}{\bar j(\bar T)}\right)\right)\left[trR^4 + \frac{1}{4}(trR^2)^2 + trR^2\sum_i^4 trF_i^2 + \sum_i^4 2(trF_i^2)^2\right] \\ &+ \int \left( ln\left(\frac{\eta^{24}(U)}{\bar\eta^{24}(\bar U)}\right) + ln\left(\frac{j(U)}{\bar j(\bar U)}\right)\right)\left[trR^4 + \frac{1}{4}(trR^2)^2 + trR^2\sum_i^4 trF_i^2 + \sum_i^4 2(trF_i^2)^2\right]\} \end{aligned} \tag{5.178}$$

As in the case of $SO(16) \times SO(16)$ we shall now discuss the group traces originating from the group decompositions $SO(32) \to SO(8)^4$ and $E_8 \times E_8 \to SO(8)^4$ which shall prove indispensable to understand the string loop amplitude.
For the decomposition $E_8 \supset SO(8)^2$ we have

$$\begin{aligned} \mathbf{248} = &\underbrace{\mathbf{(28,1)} \oplus \mathbf{(1,28)}}_{\text{adjoint rep. of } SO(8)\times SO(8)} \oplus \underbrace{\mathbf{(8,8)}}_{\text{bifundamental rep. of } SO(8)\times SO(8)} \\ &\oplus \underbrace{\mathbf{(8,8)'}}_{\text{spinor rep. of } SO(8)\times SO(8)} \oplus \underbrace{\mathbf{(8,8)''}}_{\text{cospinor rep. of } SO(8)\times SO(8)} . \end{aligned} \tag{5.179}$$

Thus the complete decomposition $E_8^{(1)} \to SO(8)_{(1)} \times SO(8)_{(2)}$ plus $E_8^{(2)} \to SO(8)_{(3)} \times SO(8)_{(4)}$ gives

$$\begin{aligned} \mathbf{248} \oplus \mathbf{248} =& \mathbf{(28,1,1,1)} \oplus \mathbf{(1,28,1,1)} \oplus \mathbf{(1,1,28,1)} \oplus \mathbf{(1,1,1,28)} \\ & \oplus \mathbf{(8,8,1,1)} \oplus \mathbf{(1,1,8,8)} \\ & \oplus \mathbf{(8,8,1,1)'} \oplus \mathbf{(1,1,8,8)'} \\ & \oplus \mathbf{(8,8,1,1)''} \oplus \mathbf{(1,1,8,8)''}. \end{aligned} \tag{5.180}$$

For the decomposition $SO(32) \to SO(8)_{(1)} \times SO(8)_{(2)} \times SO(8)_{(3)} \times SO(8)_{(4)}$ we have

$$\begin{aligned} \mathbf{496} =& \mathbf{(28,1,1,1)} \oplus \mathbf{(1,28,1,1)} \oplus \mathbf{(1,1,28,1)} \oplus \mathbf{(1,1,1,28)} \\ & \oplus \mathbf{(8,8,1,1)} \oplus \mathbf{(1,1,8,8)} \\ & \oplus \mathbf{(8,1,8,1)} \oplus \mathbf{(1,8,1,8)} \\ & \oplus \mathbf{(1,8,8,1)} \oplus \mathbf{(8,1,1,8)}. \end{aligned} \tag{5.181}$$

From the decomposition (5.180) we see that $E_8^{(1)} \to SO(8)_{(1)} \times SO(8)_{(2)}$ plus $E_8^{(2)} \to SO(8)_{(3)} \times SO(8)_{(4)}$ has a preferred $trF_1^2 trF_2^2$ and $trF_3^2 trF_4^2$ interaction. The T-duality exchanges the spinor and co-spinor representation with the bi-fundamental representations and we shall see that this fact appears in the string 1-loop elliptic genus as the orbifold shift [109] which gives the mixed interaction of the type $trF_1^2 trF_3^2$ and $trF_1^2 trF_4^2$ etc. even if one starts with the decomposition $E_8^{(1)} \to SO(8)_{(1)} \times SO(8)_{(2)}$ and $E_8^{(2)} \to SO(8)_{(3)} \times SO(8)_{(4)}$.
We finally summarize the trace formuli for different states

$$Tr_{28}F^2 = 6trF^2, \quad Tr_{28}F^4 = 3(trF^2)^2, \tag{5.182a}$$
$$tr_{(8,8)}F^2 = 8trF_1^2 + 8trF_2^2, \quad tr_{(8,8)}F^4 = 8trF_1^4 + 8trF_2^4 + 6trF_1^2 trF_2^2, \tag{5.182b}$$
$$tr_{(8,8)'}F^2 = tr_{(8,8)''}F^2 = 8trF_1^2 + 8trF_2^2, \tag{5.182c}$$
$$tr_{(8,8)'}F^4 = tr_{(8,8)''}F^4 = 3(trF_1^2)^2 + 3(trF_2^2)^2 + 6trF_1^2 trF_2^2 - 4trF_1^4 - 4trF_2^4. \tag{5.182d}$$

Before moving towards comparing the supergravity result (5.178) with the string theory amplitude, we note that the $U(1)$ polynomial in (5.178) has an interesting rewriting as follows

$$\begin{aligned} & \left[ trR^4 + \frac{1}{4}(trR^2)^2 + trR^2 \sum_i^4 trF_i^2 + \sum_i^4 2(trF_i^2)^2 \right] \\ & = \left[ trR^4 - \frac{1}{4}(trR^2)^2 \right] + \sum_{i=1}^4 \left[ \frac{1}{2} trR^2 + 2trF_i^2 \right]^2. \end{aligned} \tag{5.183}$$

We see that $X_8^- = \left[ trR^4 - \frac{1}{4}(trR^2)^2 \right]$ polynomial appears in the context of type II theories and M5 branes while the decomposition (5.183) has a strong note of Horava-Witten like machanism [48, 49] (see section 3.7.1). We shall try to give more light on this issue in future.

#### String amplitude with G= $SO(8)^4$

Finally we come to the case of the D=8, N=1 theory with gauge group $SO(8)^4$ which can be obtained from D=10, N=1 theory with gauge group either $SO(32)$ or $E_8 \times E_8$ compactified on a $T^2$ with appropriate Wilson lines along the two 1-cycles of the

torus. We shall, for the moment use the same q-expansion method [77] as in the case of $SO(16)\times SO(16)$ (5.3.2).
As before, the amplitude has the generic form

$$\mathcal{A} = V_8 t_8 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\Gamma_{2,2}A(q,R,F)|_{8-forms}, \tag{5.184}$$

where $\Gamma_{2,2}$ is the $T^2$ lattice sum

$$\Gamma_{2,2} = \frac{T_2}{\tau_2}\sum_{B\in ML(2,\mathbb{Z})} exp\left[2\pi i T det(B) - \frac{\pi T_2}{\tau_2 U_2}|(1\ \ U)B\begin{pmatrix}\tau\\1\end{pmatrix}|^2\right] = \sum_{\vec{m},\vec{n}\in\mathbb{Z}^N} q^{P_L^2/2}\bar{q}^{P_R^2/2} \tag{5.185}$$

with $B$ being the $2\times 2$ matrix

$$B = \begin{pmatrix} m_1 & n_1\\ m_2 & n_2\end{pmatrix}. \tag{5.186}$$

To define the elliptic genus we shall start with the D=10 N=1 $E_8\times E_8$ Heterotic string compactified on a $T^2$ with the Wilson line

$$Y_i^1 = (0^4, \frac{1}{2}^4, 0^4, \frac{1}{2}^4), \quad Y_i^2 = (0^4, \frac{1}{2}^4, 0^4, \frac{1}{2}^4), \qquad i = 1,\cdots,16, \tag{5.187}$$

so that the gauge group decomposition $E_8^{(1)} \to SO(8)_{(1)}\times SO(8)_{(2)}$ and $E_8^{(2)} \to SO(8)_{(3)}\times SO(8)_{(4)}$ applies. Thus the elliptic genus is obtained by gauging

$$\begin{aligned} A(q,R,F) \;&=\; \frac{1}{\eta^{24}} Exp\left(\frac{trR^2}{(2\pi)^2}\frac{\hat{E}_2}{48}\right)\times Exp\left(\frac{trR^4}{(2\pi)^4}\frac{E_4}{2^7 3^2 5}\right) \\ &\times \sum_{a,b=1}^{2} \underbrace{\theta^4\begin{bmatrix}a\\b\end{bmatrix}\theta^4\begin{bmatrix}a+m_2\\b+n_2\end{bmatrix}}_{SO(8)_{(1)}\times SO(8)_{(2)}} \times \underbrace{\theta^4\begin{bmatrix}a+m_1\\b+n_1\end{bmatrix}\theta^4\begin{bmatrix}a+m_1+m_2\\b+n_1+n_2\end{bmatrix}}_{SO(8)_{(3)}\times SO(8)_{(4)}}. \end{aligned} \tag{5.188}$$

In the above (5.188) we have labelled the theta functions by $SO(8)_{(1)}\times SO(8)_{(2)}$ and $SO(8)_{(3)}\times SO(8)_{(4)}$ to denote that those functions are to be "gauged" accordingly by the 4 copies of $SO(8)$s. We now decompose the integration by now familiar method of the decomposition to trivial, degenerate and non-degenerate orbit. The elliptic genus for the trivial orbit ($B=0$) shall be

$$\begin{aligned} &A_{\text{trivial}}(q,R,F) = \\ &\frac{E_4^3}{2^7 3^2 5\eta^{24}}\frac{trR^4}{(2\pi)^4} + \frac{\hat{E}_2^2E_4^2}{2^9 3^2\eta^{24}}\frac{(trR^2)^2}{(2\pi)^4} + \frac{trR^2\sum_{i=1}^4 trF_i^2}{2^8 3^2(2\pi)^4}\left(\frac{\hat{E}_2E_4E_6}{\eta^{24}} - \frac{\hat{E}_2^2E_4^2}{\eta^{24}}\right) \\ &+\frac{(trF_1^2trF_3^2 + trF_2^2trF_4^2 + trF_1^2trF_4^2 + trF_2^2trF_3^2)}{2^8 3^2(2\pi)^4}\left(\frac{\hat{E}_2^2E_4^2}{\eta^{24}} - \frac{2\hat{E}_2E_4E_6}{\eta^{24}} + \frac{E_6^2}{\eta^{24}}\right) \\ &+\frac{\sum_{i=1}^4(trF_i^2)^2}{2^8 3^2(2\pi)^4}\left(\frac{E_4^3}{\eta^{24}} - \frac{2\hat{E}_2E_4E_6}{\eta^{24}} + \frac{\hat{E}_2^2E_4^2}{\eta^{24}}\right) \\ &+\frac{trF_1^2trF_2^2 + trF_3^2trF_4^2}{2^7 3^2(2\pi)^4}\left(\frac{E_4^3}{\eta^{24}} - \frac{2\hat{E}_2E_4E_6}{\eta^{24}} + \frac{\hat{E}_2^2E_4^2}{\eta^{24}}\right). \end{aligned} \tag{5.189}$$

For the degenerate orbit ($B \neq 0$ and $det(B) = 0$) we choose the following sectors

$$
\begin{aligned}
B^{(1)} &= \begin{pmatrix} 0 & 2j \\ 0 & 2p \end{pmatrix}, \qquad B^{(2)} = \begin{pmatrix} 0 & 2j \\ 0 & 2p+1 \end{pmatrix}, \\
B^{(3)} &= \begin{pmatrix} 0 & 2j+1 \\ 0 & 2p+1 \end{pmatrix}, \qquad B^{(4)} = \begin{pmatrix} 0 & 2j+1 \\ 0 & 2p \end{pmatrix}, \qquad 0 \le j < k, \quad j,k,p \in \mathbb{Z}.
\end{aligned} \tag{5.190}
$$

The sectors $B^{(2)}$, $B^{(3)}$ and $B^{(4)}$ in (5.190) generate the orbifold shifts which mix the $SO(8)_1$ and $SO(8)_2$ with $SO(8)_3$ and $SO(8)_4$ which arise from the decomposition of a different $E_8$. The elliptic genus for the degenerate orbit is

$$
\begin{aligned}
&A_{\text{degenerate}}(q, R, F) = \\
&\frac{trR^4}{(2\pi)^4}\frac{E_4}{2^7 3^2 5 \eta^{24}}\left(B^{(1)}\sum_{a=2}^{4}\theta_a^{16} + \sum_{j=2}^{4} B^{(j)} 2\theta_3^8\theta_4^8\right) + \frac{(trR^2)^2}{(2\pi)^4}\frac{\hat{E}_2^2}{2^9 3^2 \eta^{24}}\left(B^{(1)}\sum_{a=2}^{4}\theta_a^{16} + \sum_{j=2}^{4} B^{(j)} 2\theta_3^8\theta_4^8\right) \\
&-\frac{trR^2\sum_{i=1}^4 trF_i^2}{2^8 3^2 (2\pi)^4 \eta^{24}}\{2B^{(1)}(\hat{E}_2E_4E_6 - \hat{E}_2^2E_4^2) - \sum_{j=2}^{4} B^{(j)}(\hat{E}_2\theta_3^8\theta_4^8)(f_2 + f_3 + 2\hat{E}_2)\} \\
&+\frac{\sum_{i=1}^4 trF_i^4}{2^7 3 (2\pi)^4 \eta^{24}}\{B^{(1)}(-\theta_2^{16}\theta_3^4\theta_4^4 + \theta_3^{16}\theta_2^4\theta_4^4 - \theta_4^{16}\theta_2^4\theta_3^4) + \sum_{j=2}^{4} B^{(j)}(\theta_3^8\theta_4^8(\theta_2^4\theta_4^4 - \theta_2^4\theta_3^4))\} \\
&+\frac{\sum_{i=1}^4 (trF_i^2)^2}{2^9 3^2 (2\pi)^4 \eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16}(\hat{E}_2 + f_{a-1})^2 + \sum_{j=2}^{4} B^{(j)}\theta_3^8\theta_4^8\left[(f_2 + \hat{E}_2)^2 + (f_3 + \hat{E}_2)^2\right]\} \\
&+\frac{trF_1^2trF_2^2 + trF_3^2trF_4^2}{2^8 3^2 (2\pi)^4 \eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16}(\hat{E}_2 + f_{a-1})^2 + B^{(2)}\theta_3^8\theta_4^8\left[(f_2 + \hat{E}_2)^2 + (f_3 + \hat{E}_2)^2\right] \\
&\qquad + \sum_{j=3}^{4} B^{(j)}\left[(f_2 + \hat{E}_2)^2 + (f_3 + \hat{E}_2)^2 - 9\theta_2^8\right]\} \\
&+\frac{trF_1^2trF_3^2 + trF_2^2trF_4^2}{2^8 3^2 (2\pi)^4 \eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16}(\hat{E}_2 + f_{a-1})^2 + B^{(3)}\theta_3^8\theta_4^8\left[(f_2 + \hat{E}_2)^2 + (f_3 + \hat{E}_2)^2\right] \\
&\qquad + (B^{(2)} + B^{(4)})\left[(f_2 + \hat{E}_2)^2 + (f_3 + \hat{E}_2)^2 - 9\theta_2^8\right]\} \\
&+\frac{trF_1^2trF_4^2 + trF_2^2trF_3^2}{2^8 3^2 (2\pi)^4 \eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16}(\hat{E}_2 + f_{a-1})^2 + B^{(4)}\theta_3^8\theta_4^8\left[(f_2 + \hat{E}_2)^2 + (f_3 + \hat{E}_2)^2\right] \\
&\qquad + (B^{(2)} + B^{(3)})\left[(f_2 + \hat{E}_2)^2 + (f_3 + \hat{E}_2)^2 - 9\theta_2^8\right]\}.
\end{aligned} \tag{5.191}
$$

Once again in the above, the $B^{(i)}$s remind one to take into account the different sectors as in (5.190) while performing the final integration in (5.184).
Finally for the non-degenerate orbit ($B \neq 0$, $det(B) \neq 0$), we have to use the following sectors [77]

$$
\begin{aligned}
&B^{(1)} = \begin{pmatrix} 2k & 2j \\ 0 & 2p \end{pmatrix}, \\
&B^{(2,1)} = \begin{pmatrix} 2k & 2j \\ 0 & 2p+1 \end{pmatrix}, \quad B^{(2,2)} = \begin{pmatrix} 2k & 2j+1 \\ 0 & 2p+1 \end{pmatrix}, \quad B^{(2,3)} = \begin{pmatrix} 2k & 2j+1 \\ 0 & 2p \end{pmatrix} \\
&B^{(3)} = \begin{pmatrix} 2k+1 & 2j \\ 0 & 2p \end{pmatrix}, \qquad B^{(4)} = \begin{pmatrix} 2k+1 & 2j+1 \\ 0 & 2p \end{pmatrix}, \qquad 0 \le j < k, \quad j,k,p \in \mathbb{Z}.
\end{aligned} \tag{5.192}
$$

The sector $B^{(2)}$ has been divided in 3 subsectors $B^{(2,1)}$, $B^{(2,2)}$, $B^{(2,3)}$ because of the spin structure in the elliptic genus (5.188). To shorten the notation we shall use

$$
B^{(2)} = B^{(2,1)} + B^{(2,2)} + B^{(2,3)}, \tag{5.193}
$$

in the elliptic genus for the non-degenerate orbit (below) whenever the modular coefficients in front of $B^{(2,k)}, k = 1,2,3$ are same. We finally get the following elliptic genus for the non-degenerate orbit

$$A_{\text{non-degenerate}}(q,R,F) = \frac{trR^4}{(2\pi)^4}\frac{E_4}{2^7 3^2 5\eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16} + B^{(2)}2\theta_3^8\theta_4^8 + B^{(3)}2\theta_2^8\theta_3^8 + B^{(4)}2\theta_2^8\theta_4^8\} \tag{5.194}$$
$$+\frac{(trR^2)^2}{(2\pi)^4}\frac{\hat{E}_2^2}{2^9 3^2\eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16} + B^{(2)}2\theta_3^8\theta_4^8 + B^{(3)}2\theta_2^8\theta_3^8 + B^{(4)}2\theta_2^8\theta_4^8\}$$
$$-\frac{trR^2\sum_{i=1}^4 trF_i^2}{2^8 3^2(2\pi)^4\eta^{24}}\{(2B^{(1)}(\hat{E}_2E_4E_6 - \hat{E}_2^2E_4^2) - B^{(2)}(\hat{E}_2\theta_3^8\theta_4^8)(f_2+f_3+2\hat{E}_2)$$
$$-B^{(3)}(\hat{E}_2\theta_2^8\theta_3^8)(f_1+f_2+2\hat{E}_2) - B^{(4)}(\hat{E}_2\theta_2^8\theta_4^8)(f_1+f_3+2\hat{E}_2)\}$$
$$+\frac{\sum_{i=1}^4 trF_i^4}{2^7 3(2\pi)^4\eta^{24}}\{B^{(1)}(-\theta_2^{16}\theta_3^4\theta_4^4 + \theta_3^{16}\theta_2^4\theta_4^4 - \theta_4^{16}\theta_2^4\theta_3^4) + B^{(2)}(\theta_3^8\theta_4^8(\theta_2^4\theta_4^4 - \theta_2^4\theta_3^4))$$
$$+B^{(3)}(\theta_2^8\theta_3^8(\theta_2^4\theta_4^4 - \theta_3^4\theta_4^4)) + B^{(4)}(\theta_2^8\theta_4^8(-\theta_3^4\theta_4^4 - \theta_2^4\theta_3^4))\}$$
$$+\frac{\sum_{i=1}^4(trF_i^2)^2}{2^9 3^2(2\pi)^4\eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16}(\hat{E}_2+f_{a-1})^2 + B^{(2)}\theta_3^8\theta_4^8\left[(f_2+\hat{E}_2)^2 + (f_3+\hat{E}_2)^2\right]$$
$$+B^{(3)}\theta_2^8\theta_3^8\left[(f_1+\hat{E}_2)^2 + (f_2+\hat{E}_2)^2\right] + B^{(4)}\theta_2^8\theta_4^8\left[(f_1+\hat{E}_2)^2 + (f_3+\hat{E}_2)^2\right]\}$$
$$+\frac{trF_1^2trF_2^2 + trF_3^2trF_4^2}{2^8 3^2(2\pi)^4\eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16}(\hat{E}_2+f_{a-1})^2 + B^{(2)}\theta_3^8\theta_4^8\left[(f_2+\hat{E}_2)^2 + (f_3+\hat{E}_2)^2\right]$$
$$-(B^{(2,2)} + B^{(2,3)})2^8 3^2\eta^{24} + B^{(3)}\theta_2^8\theta_3^8\left[(f_1+\hat{E}_2)^2 + (f_3+\hat{E}_2)^2\right]$$
$$+B^{(4)}\theta_2^8\theta_4^8\left[(f_1+\hat{E}_2)^2 + (f_3+\hat{E}_2)^2\right]\}$$
$$+\frac{trF_1^2trF_3^2 + trF_2^2trF_4^2}{2^8 3^2(2\pi)^4\eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16}(\hat{E}_2+f_{a-1})^2 + B^{(2)}\theta_3^8\theta_4^8\left[(f_2+\hat{E}_2)^2 + (f_3+\hat{E}_2)^2\right]$$
$$-(B^{(2,1)} + B^{(2,3)})2^8 3^2\eta^{24} + B^{(3)}\theta_2^8\theta_3^8\left[2(f_1+\hat{E}_2)(f_3+\hat{E}_2)\right]$$
$$+B^{(4)}\theta_2^8\theta_4^8\left[2(f_1+\hat{E}_2)(f_3+\hat{E}_2)\right]\}$$
$$+\frac{trF_1^2trF_4^2 + trF_2^2trF_3^2}{2^8 3^2(2\pi)^4\eta^{24}}\{B^{(1)}\sum_{a=2}^{4}\theta_a^{16}(\hat{E}_2+f_{a-1})^2 + B^{(2)}\theta_3^8\theta_4^8\left[(f_2+\hat{E}_2)^2 + (f_3+\hat{E}_2)^2\right]$$
$$-(B^{(2,1)} + B^{(2,2)})2^8 3^2\eta^{24} + B^{(3)}\theta_2^8\theta_3^8\left[2(f_1+\hat{E}_2)(f_3+\hat{E}_2)\right]$$
$$+B^{(4)}\theta_2^8\theta_4^8\left[2(f_1+\hat{E}_2)(f_3+\hat{E}_2)\right]\}.$$

The complete CP-even amplitude will be (see (5.135) for the integration domains)

$$\mathfrak{A} \quad = \quad \mathfrak{A}_{\text{trivial}} + \mathfrak{A}_{\text{degenerate}} + \mathfrak{A}_{\text{non-degenerate}} \tag{5.195}$$

with

$$\mathfrak{A}_{\text{trivial}} = T_2V_8t_8\{trR^4 + \frac{1}{4}(trR^2)^2 + trR^2\sum_{i=1}^{4}trF_i^2 \tag{5.196}$$
$$-2trF_1^2trF_3^2 - 2trF_1^2trF_4^2 - 2trF_2^2trF_4^2 - 2trF_1^2trF_3^2 + 4trF_1^2trF_2^2 + 4trF_3^2trF_4^2 + 2\sum_{i=1}^{4}(trF_i^2)^2\}$$

being the trivial orbit amplitude. Note that by recombining the $SO(8)_1$ with $SO(8)_2$ and $SO(8)_3$ with $SO(8)_4$ we find back the $T^2$ reduction of the $E_8\times E_8$ Green-Schwarz term (5.149).

We now collect the constant parts of the q-expansion of the modular functions in the degenerate and non-degenerate elliptic genus from (B.13), verify that there are no poles and then use the integral (5.137) to evaluate the non-volume suppressed harmonic part of the degenerate amplitude and (5.138) to evaluate the non-volume suppressed harmonic part of the the non-degenerate amplitude in the CP-even sector.

$$\begin{aligned}
&\mathfrak{A}_{\text{degenerate}} = \qquad (5.197)\\
&\frac{1}{4(2\pi)^4} ln\left(U_2|\eta(U)|^2\right) V_8 t_8 \left[trR^4 + \frac{1}{4}(trR^2)^2 + trR^2\sum_i^4 trF_i^2 + \sum_i^4 2(trF_i^2)^2\right]\\
&+\frac{1}{4(2\pi)^4} ln\left(U_2|\eta(U)|^2\right) \times\\
&\quad V_8 t_8\left[2\times 64\left(\frac{trR^4}{360}+\frac{(trR^2)^2}{288}\right)+\frac{4}{3}trR^2\sum_{i=1}^4 trF_i^2 + 2\times\frac{2}{3}\left(4\sum_{i=1}^4 trF_i^4 + 3trF_1^2trF_2^2 + 3trF_3^2trF_4^2\right)\right]\\
&+\frac{1}{8(2\pi)^4} ln\left(U_2|\eta(U)|^2\right)\times V_8 t_8\{256\left(\frac{trR^4}{360}+\frac{(trR^2)^2}{288}\right)+\frac{8}{3}trR^2\sum_{i=1}^4 trF_i^2\\
&\qquad +\frac{8}{3}\left(-4\sum_{i=1}^4 trF_i^4 + 3\sum_{i=1}^4 (trF_i^2)^2 + 6trF_1^2trF_2^2 + 6trF_3^2trF_4^2\right)\}\\
&+\frac{1}{(2\pi)^4} ln\left(U_2|\eta(U)|^2\right) V_8 t_8\left[trF_1^2trF_2^2 + trF_3^2trF_4^2 + trF_1^2trF_3^2 + trF_2^2trF_4^2 + trF_1^2trF_4^2 + trF_2^2trF_3^2\right].
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{A}_{\text{non-degenerate}} = \qquad (5.198)\\
&\frac{1}{4(2\pi)^4}\left[\frac{\pi T_2}{3} + ln\left(T_2|\eta(2T)|^2\right)\right] V_8 t_8\left[trR^4 + \frac{1}{4}(trR^2)^2 + trR^2\sum_i^4 trF_i^2 + \sum_i^4 2(trF_i^2)^2\right]\\
&+\frac{1}{4(2\pi)^4}\left[\frac{\pi T_2}{3} + ln\left(T_2|\eta(2T)|^2\right)\right]\times\\
&\quad V_8 t_8\left[2\times 64\left(\frac{trR^4}{360}+\frac{(trR^2)^2}{288}\right)+\frac{4}{3}trR^2\sum_{i=1}^4 trF_i^2 + 2\times\frac{2}{3}\left(4\sum_{i=1}^4 trF_i^4 + 3trF_1^2trF_2^2 + 3trF_3^2trF_4^2\right)\right]\\
&+\frac{1}{8(2\pi)^4}\left[\frac{\pi T_2}{3} + ln\left(T_2|\eta(4T)|^2\right) - ln\left(T_2|\eta(2T)|^2\right)\right]\times V_8 t_8\{256\left(\frac{trR^4}{360}+\frac{(trR^2)^2}{288}\right)+\frac{8}{3}trR^2\sum_{i=1}^4 trF_i^2\\
&\qquad +\frac{8}{3}\left(-4\sum_{i=1}^4 trF_i^4 + 3\sum_{i=1}^4 (trF_i^2)^2 + 6trF_1^2trF_2^2 + 6trF_3^2trF_4^2\right)\}\\
&+\frac{1}{(2\pi)^4}\left[\frac{\pi T_2}{3} + ln\left(T_2|\eta(2T)|^2\right)\right] V_8 t_8\left[trF_1^2trF_2^2 + trF_3^2trF_4^2\right]\\
&+\frac{1}{(2\pi)^4}\left[\frac{\pi T_2}{3} + ln\left(T_2|\eta(4T)|^2\right) - ln\left(T_2|\eta(2T)|^2\right)\right]\times V_8 t_8\left[trF_1^2trF_3^2 + trF_2^2trF_4^2\right]\\
&+\frac{1}{(2\pi)^4}\left[ln\left(T_2|\eta(4T)|^2\right) - 2ln\left(T_2|\eta(2T)|^2\right)\right]\times V_8 t_8\left[trF_1^2trF_4^2 + trF_2^2trF_3^2\right].
\end{aligned}$$

We note that the 8-form polynomial

$$Y_8 = \left[trR^4 + \frac{1}{4}(trR^2)^2 + trR^2\sum_i^4 trF_i^2 + \sum_i^4 2(trF_i^2)^2\right] \qquad (5.199)$$

is due to the fermions transforming under the adjoint representation $(28,1,1,1)\oplus(1,28,1,1)\oplus(1,1,28,1)\oplus(1,1,1,28)$ of the $SO(8)^4$ and the CP-odd partner of the above provides with the discrete $SL(2,\mathbb{Z})$ anomaly cancelling counter-term in D=8, N=1 supergravity with gauge group $G = SO(8)^4$. The other two 8-form polynomials

$$Y_8' = \{2 \times 64 \left( \frac{trR^4}{360} + \frac{(trR^2)^2}{288} \right) + \frac{4}{3} trR^2 \sum_{i=1}^{4} trF_i^2 \tag{5.200}$$

$$+2 \times \frac{2}{3} \left( 4 \sum_{i=1}^{4} trF_i^4 + 3trF_1^2 trF_2^2 + 3trF_3^2 trF_4^2 \right) \} \tag{5.201}$$

$$Y_8'' = \{256 \left( \frac{trR^4}{360} + \frac{(trR^2)^2}{288} \right) + \frac{8}{3} trR^2 \sum_{i=1}^{4} trF_i^2 \tag{5.202}$$

$$+\frac{8}{3} \left( -4 \sum_{i=1}^{4} trF_i^4 + 3 \sum_{i=1}^{4} (trF_i^2)^2 + 6trF_1^2 trF_2^2 + 6trF_3^2 trF_4^2 \right) \}$$

are respectively the contributions from the massive vector multiplet transforming under the bi-fundamental representations $(8, 8, 1, 1) \oplus (1, 1, 8, 8)$ and (co)spinor representations $(8, 8, 1, 1)' \oplus (1, 1, 8, 8)'$ respectively. The last few pure gauge terms in (5.198) are due to the orbifold shifts [110, 109].

### Calculating string amplitude with Hecke operators

We now deploy the elegant method of Hecke operator to evaluate the degenerate plus non-degenerate CP-even amplitude $\mathfrak{A}_{\text{degenerate}} + \mathfrak{A}_{\text{non-degenerate}}$ which have been carried out by Kiritsis, Obers & Pioline in [109] and in the guise of modular identities by Lerche & Stieberger [110]. We complement the calculation of [109] where only the $\Gamma_2^-$ subgroup (of $SL(2, \mathbb{Z})$) invariant part has been computed using the Hecke image $H_{\Gamma_2^-}$ of the $\Gamma_2^-$ invariant part of $\mathfrak{A}_{\text{degenerate}} + \mathfrak{A}_{\text{non-degenerate}}$ in (5.197) and (5.198). We compute the $\Gamma_2^+$ and $\Gamma_2^0$ invariant parts of (5.197) and (5.198) using the method of Hecke operators $H_{\Gamma_2^+}$ and $H_{\Gamma_2^0}$. We shall see that in the pure gravitational and in mixed gauge gravity part we can separate the contribution from the adjoint representation (5.199) and the total contribution from 6 sets of bi-fundamental states like $(8, 8, 1, 1)$ etc. but in the pure gauge part we cannot separate these contributions: instead the sum from the 3 subgroups $\Gamma_2^-$, $\Gamma_2^+$ and $\Gamma_2^0$ of $SL(2, \mathbb{Z})$ we shall retrieve the total pure gauge contributions which have been investigated in detail in [110].

We now describe the method in brief. For exclusive details we refer to [109]. We note that subgroups $\Gamma_2^-$, $\Gamma_2^+$ and $\Gamma_2^0$ are the invariant subgroups of $\theta_2$, $\theta_4$ and $\theta_3$ modulo the phase and weight factors. Now using the (B.10) and (B.11a), (B.11b), (B.11b), (B.11d) summation identities we can decompose the $B^{(1)}$ part in both degenerate and non-degenerate elliptic genus (5.191) and (5.194) into sum of the form

$$B^{(1)}(\cdots) = B^{(1)}\theta_3^8\theta_4^8(\cdots) + B^{(1)}\theta_2^8\theta_3^8(\cdots) + B^{(1)}\theta_2^8\theta_4^8(\cdots). \tag{5.203}$$

One can now combine the part $B^{(1)}\theta_3^8\theta_4^8(\cdots)$ with $B^{(2)}$, $B^{(3)}$ and $B^{(4)}$ sectors (5.190) in the degenerate elliptic genus (5.191) and $B^{(2,1)}$, $B^{(2,2)}$ and $B^{(2,3)}$ sectors (5.192) in the non-degenerate elliptic genus (5.194). The sum over $\theta_3^8\theta_4^8(\cdots)$ is then of the form

$$\int_{\mathcal{F}^-} \frac{d^2\tau}{\tau_2^2} \Gamma_{2,2}(2T, U; 2\tau)\Phi_-(\tau) \tag{5.204}$$

where $\Phi_-(\tau)$ is $\Gamma_2^-$ invariant modular function and we restrict the integral domain to $\mathcal{F}^-$ which is the fundamental domain of $\Gamma_2^-$ subgroup and is a 6-fold cover of the $SL(2, \mathbb{Z})$ fundamental domain $\mathcal{F}$. One can now change the variable $2\tau = \rho$ and unfold the integral (5.204) to the fundamental domain $\mathcal{F}$ of $SL(2,\mathbb{Z})$ by the following unfolding

$$\int_{\mathcal{F}^-}\frac{d^2\tau}{\tau_2^2}\Gamma_{2,2}(2T,U;2\tau)\Phi_-(\tau)=\int_{\mathcal{F}}\frac{d^2\rho}{\rho_2^2}\Gamma_{2,2}(2T,U;\rho)\left(\Phi_-(\frac{\rho}{2})+\Phi_-(-\frac{1}{2\rho})+\Phi_-(\frac{\rho+1}{2})\right)$$
$$=\int_{\mathcal{F}}\frac{d^2\rho}{\rho_2^2}\Gamma_{2,2}(2T,U;\rho)H_{\Gamma_2^-}\Phi_-(\rho) \tag{5.205}$$

where in the last line we have used the definition of the Hecke operator for the $\Gamma_2^-$ subgroup.
We then combine the $\theta_2^8\theta_3^8(\cdots)$ piece in (5.203) with the $B^{(3)}$ sector of the non-degenerate elliptic genus to get the following combination of the partitions function

$$\int_{\mathcal{F}^+}\frac{d^2\tau}{\tau_2^2}\Gamma_{2,2}(2T,U;\tau/2)\Phi_+(\tau) \tag{5.206}$$

where $\Phi_+(\tau)$ is $\Gamma_2^+$ invariant modular function and we restrict the integral domain to $\mathcal{F}^+$ which is the fundamental domain of $\Gamma_2^+$ subgroup ($\mathcal{F}^+$ is also a 6-fold cover of $\mathcal{F}$). We make the change of variable $\tau/2=\rho$ and unfold the integral to the fundamental domain $\mathcal{F}$ to make appear the Hecke operator $H_{\Gamma_2^+}$ for the $\Gamma_2^+$ subgroup

$$\int_{\mathcal{F}^+}\frac{d^2\tau}{\tau_2^2}\Gamma_{2,2}(2T,U;\tau/2)\Phi_+(\tau)=\int_{\mathcal{F}}\frac{d^2\rho}{\rho_2^2}\Gamma_{2,2}(2T,U;\rho)\left(\Phi_+(2\rho)+\Phi_+(-\frac{1}{2\rho})+\Phi_+(\frac{1}{2\rho+1})\right) \tag{5.207}$$
$$=\int_{\mathcal{F}}\frac{d^2\rho}{\rho_2^2}\Gamma_{2,2}(2T,U;\rho)H_{\Gamma_2^+}\Phi_+(\rho).$$

It now rests to combine the $\theta_2^8\theta_4^8(\cdots)$ piece in (5.203) with the $B^{(4)}$ sector of the non-degenerate elliptic genus to get the following combination of the partitions function

$$\int_{\mathcal{F}^0}\frac{d^2\tau}{\tau_2^2}\Gamma_{2,2}(T,U;(\tau+1)/2)\Phi_0(\tau) \tag{5.208}$$

where $\Phi_0(\tau)$ is $\Gamma_2^0$ invariant modular function and we restrict the integral domain to $\mathcal{F}^0$ which is the fundamental domain of $\Gamma_2^0$ subgroup (and is a 3-fold cover of $\mathcal{F}$). Making the change of variable $(\tau+1)/2=\rho$ and unfold the integral to the fundamental domain $\mathcal{F}$ to make appear the Hecke operator $H_{\Gamma_2^0}$ for the $\Gamma_2^0$ subgroup

$$\int_{\mathcal{F}^0}\frac{d^2\tau}{\tau_2^2}\Gamma_{2,2}(T,U;(\tau+1)/2)\Phi_0(\tau)=\int_{\mathcal{F}}\frac{d^2\rho}{\rho_2^2}\Gamma_{2,2}(T,U;\rho)H_{\Gamma_2^0}\Phi_0(\rho) \tag{5.209}$$
$$=\int_{\mathcal{F}}\frac{d^2\rho}{\rho_2^2}\Gamma_{2,2}(T,U;\rho)\left(\Phi_0(2\rho-1)+\Phi_0(-\frac{1}{2\rho-1})+\Phi_0(-\frac{1}{2\rho})\right).$$

Now to get the harmonic part of the CP-even amplitude, we pick up the constant parts of the Hecke images of the related modular functions which we are enlisting below in (5.210), (5.211), (5.212) and (5.213)

$$\frac{1}{2^8 3\eta^{24}}\left(-\theta_2^{16}\theta_3^4\theta_4^4+\theta_3^{16}\theta_2^4\theta_4^4-\theta_4^{16}\theta_2^4\theta_3^4\right)=1,\qquad \frac{\theta_2^8\theta_3^8}{2^8 3\eta^{24}}\left(-\theta_3^4\theta_4^4+\theta_2^4\theta_4^4\right)=-\frac{1}{3}, \tag{5.210}$$
$$\frac{\theta_3^8\theta_4^8}{2^8 3\eta^{24}}\left(+\theta_2^4\theta_4^4-\theta_2^4\theta_3^4\right)=-\frac{1}{3},\qquad \frac{\theta_2^8\theta_4^8}{2^8 3\eta^{24}}\left(-\theta_3^4\theta_4^4-\theta_2^4\theta_3^4\right)=-\frac{1}{3}.$$

$$H_{\Gamma_2^-}\left[\frac{\theta_3^8\theta_4^8E_4}{\eta^{24}}\right]=360,\qquad H_{\Gamma_2^-}\left[\frac{E_2^2}{\eta^{24}}\theta_3^8\theta_4^8\right]=72 \tag{5.211}$$
$$H_{\Gamma_2^-}\left[\frac{E_2}{\eta^{24}}\theta_3^8\theta_4^8(2\hat{E}_2+f_2+f_3)\right]=288,\qquad H_{\Gamma_2^-}\left[\frac{\theta_3^8\theta_4^8}{\eta^{24}}\left((\hat{E}_2+f_2)(\hat{E}_2+f_3)\right)\right]=-576$$
$$H_{\Gamma_2^-}\left[\frac{\theta_3^8\theta_4^8}{\eta^{24}}\left((\hat{E}_2+f_2)^2+(\hat{E}_2+f_3)^2\right)\right]=2304$$

$$H_{\Gamma_2^+}\left[\frac{\theta_2^8\theta_3^8E_4}{\eta^{24}}\right]=384,\qquad H_{\Gamma_2^+}\left[\frac{E_2^2}{\eta^{24}}\theta_2^8\theta_3^8\right]=384, \tag{5.212}$$
$$H_{\Gamma_2^+}\left[\frac{E_2}{\eta^{24}}\theta_2^8\theta_3^8(2\hat{E}_2+f_1+f_2)\right]=1152,\qquad H_{\Gamma_2^+}\left[\frac{\theta_2^8\theta_3^8}{\eta^{24}}\left((\hat{E}_2+f_1)(\hat{E}_2+f_2)\right)\right]=3456$$

$$H_{\Gamma_2^0}\left[\frac{\theta_2^8\theta_4^8E_4}{\eta^{24}}\right]=384,\qquad H_{\Gamma_2^0}\left[\frac{E_2^2}{\eta^{24}}\theta_2^8\theta_4^8\right]=384, \tag{5.213}$$
$$H_{\Gamma_2^0}\left[\frac{E_2}{\eta^{24}}\theta_2^8\theta_4^8(2\hat{E}_2+f_1+f_3)\right]=1152,\qquad H_{\Gamma_2^0}\left[\frac{\theta_2^8\theta_4^8}{\eta^{24}}\left((\hat{E}_2+f_1)(\hat{E}_2+f_3)\right)\right]=3456$$

Combining these we find the result for degenerate and non-degenerate amplitude

$$\mathfrak{A}_{\text{degenerate}}+\mathfrak{A}_{\text{non-degenerate}}= \tag{5.214}$$
$$\frac{1}{(2\pi)^4}\left[\frac{\pi T_2}{3}+lnT_2|\eta(2T)|^2+ln\left(U_2|\eta(U)|^2\right)\right]t_8\underbrace{\left[\frac{2\times 360}{2^73^25}trR^4+\frac{2\times 72}{2^93^2}(trR^2)^2+\frac{288}{2^83^2}trR^2\sum_i^4 trF_i^2\right]}_{\text{adjoint of }SO(8)^4}$$
$$+\frac{1}{(2\pi)^4}\left[\frac{\pi T_2}{2}+lnT_2|\eta(2T)|^2+ln\left(T_2|\eta(T)|^2\right)+ln\left(U_2|\eta(U)|^2\right)\right]$$
$$\times t_8\underbrace{\left[\frac{2\times 384}{2^73^25}trR^4+\frac{2\times 384}{2^93^2}(trR^2)^2+\frac{1152}{2^83^2}trR^2\sum_i^4 trF_i^2\right]}_{\text{bi-fundamental and bi-spinor states}}$$
$$+\frac{1}{(2\pi)^4}\left[lnT_2|\eta(2T)|^2-2ln\left(T_2|\eta(T)|^2\right)+ln\left(U_2|\eta(U)|^2\right)\right]t_8\sum_{i=1}^4(trF_i^2)^2$$
$$+\frac{1}{(2\pi)^4}\left[\frac{\pi T_2}{3}+lnT_2|\eta(4T)|^2-lnT_2|\eta(2T)|^2\right]t_8\sum_{i=1}^4 trF_i^4$$
$$+\frac{1}{(2\pi)^4}\left[\frac{\pi T_2}{3}+lnT_2|\eta(4T)|^2-lnT_2|\eta(2T)|^2+ln\left(U_2|\eta(U)|^2\right)\right]\times t_8\{trF_1^2trF_3^2+trF_2^2trF_4^2\} \tag{5.215}$$
$$+\frac{1}{(2\pi)^4}\left[lnT_2|\eta(4T)|^2-2lnT_2|\eta(2T)|^2+ln\left(U_2|\eta(U)|^2\right)\right]\times t_8\{trF_1^2trF_4^2+trF_2^2trF_3^2\}. \tag{5.216}$$

From the above, we recognise the composite anomaly cancelling polynomial (5.199) in pure gravity and gauge-gravity sector in part "adjoint of $SO(8)^4$" (5.214) and the part "bi-fundamental and bi-spinor states in" corresponds to the pure gravity and gauge-gravity coupling of states in $(8,8,1,1)\oplus(1,1,8,8)$, $(8,8,1,1)'\oplus(1,1,8,8)'$ and $(8,8,1,1)''\oplus(1,1,8,8)''$ representations. However the pure gauge sector irons down the contributions from these representations to give the last terms in (5.214). One can also check that there is a "local conservation" of coefficients e.g. for $trR^4/(2^73^25)$

terms in both methods with constant coefficients and Hecke operators the total numerical coefficients are same if one sums them in the respective sectors

$$2\times 744 = \underbrace{2\times 360 + 2\times 384}_{\text{Hecke metheod}} = \underbrace{2\times 360 + 2\times 128 + 2\times 256}_{\text{adjpoint + bi-fundamental + spinor reps.}} \ . \qquad (5.217)$$

One can check the other numerical coefficients for the 8-forms $(trR^2)^2$, $(trF^2)^2$, $trR^2trF^2$ and $trF^4$. There is a nice interpretation for the modular forms in front of the pure gauge sector 8-forms as discussed in [110] and they correspond to the $C_4$ and $C_0 - C_8$ exchange between four $\mathcal{D}_4$ branes in the dual F-theory on K3 description in Sen limit [62].

## 5.4 Chasing duality with F-theory on K3 at $\alpha'^3$ level

Now that we have new data from Heterotic on $T^2$ side, we can take up the discussion of the relation between ten dimensional Green-Gaberdiel term (4.17) in Type IIB theory and F-theory by using the duality of the latter compactified on K3. As we have mentioned in section 4.1.1 F-theory compactifications on K3 are a class of type IIB vacua preserving minimal supersymmetry in D=8 and involving 24 7-branes localised on the base $\mathbb{CP}^1$ in case of elliptically fibered K3. We have also mentioned in section 3.8 that not all of these 24 7-branes are mobile due to global obstructions as described by Douglas & Argyres [111, 112, 113]. We shall show that the structure of the Green-gaberdiel term "knows" about these gravitational constraints.

Consider first the case of Heterotic string compactified on $T^2$ with Wilson lines switched on such that the gauge group is $U(1)^{18}$. In the dual F-theory on K3 side, this corresponds to 18 dynamical 7-branes (out of 24) giving rise to the $U(1)^{18}$ gauge group in a generic region of the moduli space. The moduli space of the theory is the coset $\frac{SO(2,18)}{SO(2)\times SO(18)}$ and the fermions of the theory are chirally charged under the $SO(2)$ as before. The anomaly counter-term resulting from the general formula (5.97), (5.98) is

$$S^8 = \frac{i}{24}\int \left( ln\left(\frac{\eta^{24}(T)}{\bar\eta^{24}(\bar T)}\right) + ln\left(\frac{j(T)}{\bar j(\bar T)}\right) + ln\left(\frac{\eta^{24}(U)}{\bar\eta^{24}(\bar U)}\right) + ln\left(\frac{j(U)}{\bar j(\bar U)}\right)\right) \times\frac{1}{32(2\pi)^4}\left[\frac{11}{15}trR^4 - \frac{1}{12}(trR^2)^2\right] . \qquad (5.218)$$

Remarkably, the quartic polynomial in 8D Riemann tensor appearing in (5.218) is exactly reproduced by adding the the contributions of 24 punctures to the polynomial $X_8^-$ in (4.17) fixed by the 10D anomaly cancelation, an observation made in the article by Green & Gaberdiel [63]. More precisely, using the equality of $F/2\pi$ with the first Chern class $c_1$ of the $\mathbb{CP}^1$ base, one integrates $F/2\pi$ to $-2$ on the base 2-sphere and then one adds a term due to 24 punctures, as if each one of them would contribute a dynamical gaugino of charge $\frac{1}{2}$ and one obtains

$$4X_8^-(R) + 24\times\frac{1}{2}\widehat{A}(\mathcal{M}_8)|_{8-\text{form}} \ = \ \frac{1}{32(2\pi)^4}\left[\frac{11}{15}trR^4 - \frac{1}{12}(trR^2)^2\right] , \qquad (5.219)$$

where $\hat{A}(\mathcal{M}_8) = \frac{1}{(2\pi)^4}\left[\frac{trR^4}{360} + \frac{(trR^2)^2}{288}\right]$ is the A-roof genus (see section 2.5).

Now we take up the case of HE theory compactified on a $T^2$ without any Wilson line so that the resulting 8D theory shall have the gauge group $E_8\times E_8$. We remember once more that the gauge group $SO(32)$ cannot be realized in the scenario of F-theory compactification on elliptically fibered K3 with trivial normal bundle which we are studying here. We have already detailed the supergravity anomaly generated by the chiral coupling with the composite $U(1)$ in (5.130). We now look at the dual F-theory side. For the enhancement of the gauge group, one needs to have 2 stacks of 10 7-branes with 7 D7-branes and 2 B brane and a C brane of respective monodromies (1,0), (3,1) and (1,1) (see section 3.8). From the property of the Douglas-Argyres singularities (D-A in short) in this set-up, we know that the rank of the gauge group (due to open strings stretched between 7-branes in each stack) is related to the number of branes present in the stack by [60, 112, 113]

$$\text{number of brane} = \text{rank(G)} + \text{number of D-A singularities.} \tag{5.220}$$

For each stack of $E_8$ we have rank($E_8$)=8 and the number of branes =10 so that in each stack we have 2 Douglas-Argyres type singularities which do not participate in gauge enhancement and would contribute like isolated punctures having a dynamical gaugino of $1/2$ charge. We thus have total 20 branes accounted, 10 each in 2-stacks of which 4 are of Douglas-Argyres type. The rest 4 (=24-20) are also isolated 7-branes accounting for the $U(1)^2$ graviphotons of the dual Heterotic theory. The 16 branes, 8 in each stack, which contribute to the gauge enhancement, would contribute a dynamical gaugino of charge $1/2$ but this time in the adjoint representation of the gauge group $E_8$ for each stack. Combining all these, we see again that the form of the anomaly polynomial of the supergravity side (5.148) is again reproduced by the combination of 10D Green-gaberdiel term $X_8^-$ and the enhanced plus isolated punctures according to

$$\left(\frac{31}{15}trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2(trF_1^2 + trF_2^2) + 6((trF_1^2)^2 + (trF_2^2)^2)\right) \\ = 4X_8^- + \frac{1}{2}\sum_{i=1,2}\widehat{A}(R,F_i)|_8 + \frac{1}{2}\times 8\times \widehat{A}(R)|_8 \tag{5.221}$$

with

$$\begin{aligned} \widehat{A}(R,F)|_8 &= ch(-iF)\widehat{A}(R)|_8 \qquad (5.222)\\ &= \text{dim(G)}\frac{1}{(2\pi)^4}\left[\frac{trR^4}{360} + \frac{(trR^2)^2}{288}\right] + \frac{1}{96(2\pi)^4}trR^2trF^2 + \frac{1}{24(2\pi)^4}trF^4 \end{aligned}$$

accounting for each $E_8$ gauge singularity with total rank of G =8+8=16 and dim(G)=248+248=496 (with $trF$ in fundamental representation).

Similar conclusion can be drawn for the 8D gauge group $SO(16)\times SO(16)$ where the 20 branes out of 24, align themselves in 2 stacks of 10 branes each consisting of 8 D7-branes and one B and C brane each. The number counting for the branes goes like-wise, i.e. 16 branes, 8 in each stack participating in gauge enhancements, contributing $\frac{1}{2}\hat{A}(R,F)$ in each stack in the adjoint representation of $SO(16)$ and the rest 8 isolated singularities contributing to $\frac{1}{2}\hat{A}(R)$ so that in total we recover again the anomaly polynomial for the $SO(16)\times SO(16)$ theory (with dim(G)= 128+128=256)

$$\frac{1}{32(2\pi)^4}\left(\frac{488}{360}trR^4 + \frac{200}{288}(trR^2)^2 + \frac{7}{3}trR^2\sum_{i=1}^{2}trF_i^2 + \frac{16}{3}\sum_{i=1}^{2}trF_i^4 + 2\sum_{i=1}^{2}(trF_i^2)^2\right) \\ = 4X_8^- + \frac{1}{2}\sum_{i=1,2}\widehat{A}(R,F_i)|_8 + \frac{1}{2}\times 8\times\widehat{A}(R)|_8. \tag{5.223}$$

For $SO(8)^4$ case we have 4 stacks with 4 D7 branes and a B and C branes each. However, due to the Sen-limit of constant coupling slice, the B and C branes come up infinitely close together to give rise an $O7^-$ plane in each stack so that, in each stack, effectively there are 5 7-branes of which only 4 (D7-) branes (along with their orientifold images) take part in gauge enhancement and there are two isolated singularities of $O7^-$ type and Douglas-Argyres type (here, different to the case of $E_8 \times E_8$ and $SO(16) \times SO(16)$ the 4(=24-20) isolated singularities live inside 4 stacks each). Hence one more we have the counting of 16 punctures contributing to $\frac{1}{2}\hat{A}(R,F)$ in each stack in the adjoint representation of $SO(8)$ and the rest 8 isolated singularities contributing to $\frac{1}{2}\hat{A}(R)$ so that once again (dim(G) = $4 \times 28 = 112$)

$$\left[trR^4 + \frac{1}{4}(trR^2)^2 + trR^2\sum_i^4 trF_i^2 + \sum_i^4 2(trF_i^2)^2\right]$$
$$= 4X_8^- + \frac{1}{2}\sum_{i=1}^4 \widehat{A}(R,F_i)|_8 + \frac{1}{2}\times 8\times \widehat{A}(R)|_8. \tag{5.224}$$

At this point one might ask if the $\alpha'^3$ correction terms in supergravity effective action that we have obtained for different gauge groups in equations (5.130), (5.157) and (5.178) can provide us with the strong coupling versions of the D7 brane worldvolume Wess-Zumino actions that we have outlined in section 3.4 in equations (3.62) and (3.68) which we are quoting here once more for convenience

$$D7 \;=\; \frac{1}{192\times 32\times 15(2\pi)^3}\left[8trR^4 + 5(trR^2)^2\right] \tag{5.225}$$
$$O7 \;=\; \frac{1}{192\times 16\times 15(2\pi)^3}\left[5(trR^2)^2 - 28trR^4\right]. \tag{5.226}$$

For different gauge groups we have different brane configurations and thus it is tempting to try to relate the strong coupling completion of above with (5.130), (5.157), (5.178). The resolution of this question remains however to be answered. In the weak coupling limit, the 7-brane couplings are seen to be related to the reduced Green-Schwarz term (5.12) that is the trivial orbit part in the string amplitude computation rather than to the degenerate and non-degenerate parts which are clearly related to the composite anomaly. To illustrate this point, we first take the case of $SO(8)^4$ where we can use the Sen-limit paradigm to consider the weak coupling regime and use the formuli (5.225) in full confidence. The world-volume coupling of 16 D7-branes along with their orientifold images and 4 $O7^-$ planes gives

$$S_{7-branes} \;=\; 16D7 - 4O7^- \tag{5.227}$$
$$= \frac{1}{192(2\pi)^3}\int C_0\Big[\Big(TrR^4 + \frac{1}{4}(TrR^2)^2\Big) + 4\sum_{i=1}^4 \left(8trF_i^4 + trF_i^2 TrR^2\right)\Big].$$

Combining the 4 volumina of 4 copies of $SO(8)$ one retrieves the volumina of $SO(32)$ or $E_8 \times E_8$ accordingly and giving rise to the 10D Green-Schwarz terms (5.12) and (5.17) reduced on $T^2$ (an observation originally due to Lerche [57]).

In case of $SO(16) \times SO(16)$ we have 2 stacks each having 8 D7 branes and a B and C brane each. This situation does not live in a constant coupling slice of the generic moduli space so that we cannot say that the B and C branes combine into an $O7$ plane. However, anticipating the lift to 9-dimensions and thus choosing monodromy cycle sufficiently large so that the B and C brane can be approximated to behave like an

$O7^-$ plane we have the Wess-Zumino coupling

$$16D7 \quad - \quad 2O7 = \tag{5.228}$$
$$\frac{1}{192(2\pi)^3}\int C_0[trR^4 + \frac{1}{4}(trR^2)^2 + trR^2(trF_1^2 + trF_2^2) - 2trF_1^2trF_2^2 + 2(trF_1^2)^2 + 2(trF_2^2)^2].$$

Once more combining the world-volumina of 2 $SO(16)$s we recover the Green-Schwarz term for either $SO(32)$ or for $E_8 \times E_8$.

It is interesting to ask whether the complete Green-Gaberdiel polynomial (4.12) might have a 12-dimensional origin. It can be shown that the complete Green-Gaberdiel polynomial in 10D (4.12)

$$S^{10}_{GG} = i\int ln\left(\frac{\eta(\tau)\bar{j}^{1/12}(\bar{\tau})}{\bar{\eta}(\bar{\tau})j^{1/12}(\tau)}\right)\left[2X_8^-(R) + \frac{p_1(R)}{48}\left(\frac{F}{2\pi}\right)^2 - \frac{1}{32}\left(\frac{F}{2\pi}\right)^4\right]\frac{F}{2\pi}. \tag{5.229}$$

cannot be derived from a 12D polynomial of Riemann curvatures tensors using the adjunction formula applicable to elliptically fibered K3. Existence of such a 12-form would have provided with a term in the effective action of "12D" F-theory. Such a term term does not exist as we shall illustrate below.

For a smooth elliptically fibered CY manifold M of complex dimensions d, one finds one finds the following relation between its total Chern class $C(M)$ and that of the d-1 (complex)dimensional base manifold B i.e. C(B) plus a top form $\omega$ on the fiber such that $\omega \wedge \omega = -c_1(B) \wedge \omega$:[47]

$$C(M) = C(B)\frac{(1+2c_1(B))(1+2c_1(B)+2\omega)(1+3c_1(B)+3\omega)}{(1+6c_1(B)+6\omega)}. \tag{5.230}$$

From the adjunction formula above, one can write down the Chern classes $c_i(M)$ in terms of the chern classes $c_i(B)$ and $\omega$ as follows:

$$c_1(M) = 0 \tag{5.231}$$
$$c_2(M) = c_2(B) + 11c_1(B)^2 + 12c_1(B)\wedge\omega, \tag{5.232}$$
$$c_3(M) = c_3(B) - c_1(B)c_2(B) - 60c_1(B)^3 - 60c_1(B)^2\omega, \tag{5.233}$$
$$c_4(M) = c_4(B) - c_1(B)c_3(B) + 12c_1(B)^2c_2(B) + 360c_1(B)^4 + 360c_1(B)^3\omega + 12c_1(B)c_2(B)\omega, \tag{5.234}$$
$$c_5(M) = c_5(B) - c_1(B)c_4(B) + 12c_1(B)^2c_3(B) - 72c_1(B)^3c_2(B) - 2160c_1(B)^5 - 2160c_1(B)^4\omega - 72c_1(B)^2c_2(B)\omega + 12c_1(B)c_3(B)\omega, \tag{5.235}$$
$$c_6(M) = c_6(B) - c_1(B)c_5(B) + 12c_1(B)^2c_4(B) - 72c_1(B)^3c_3(B) + 432c_1(B)^4c_2(B) + 12960c_1(B)^6 + 12960c_1(B)^5\omega + 432c_1(B)^3c_2(B)\omega - 72c_1(B)^2c_3(B)\omega + 12c_1(B)c_4(B)\omega. \tag{5.236}$$

Now we try to lift the Green-Gaberdiel anomaly counter-term (4.12) to a possible 12-d CP-odd coupling. We shall only deal with the topological polynomial part and leave aside the fate of the modular function $ln\left(\frac{\eta(\tau)\bar{j}^{1/12}(\bar{\tau})}{\bar{\eta}(\bar{\tau})j^{1/12}(\tau)}\right)$ aside for instance. We first start with a linear combination of 12-forms created out of Pontryagin classes $p_i$ defined on a 12-manifold

$$R^6 \sim ap_3(R) + bp_2(R)p_1(R) + cp_1(R)^3. \tag{5.237}$$

We then use the expressions of $p_i$s in terms of the Chern classes $c_i$ of the 12-d CY manifold assuming that a complex structure exists on the manifold (see appendix A). Then we use the adjunction formuli (5.231) to break from 12-d forms to 10-d forms so that the structure becomes $R^6 \to R^4 \wedge c_1(B)$. Noting that the base manifold is of real dimension 10, all terms not involving $c_1(B)$ are trivial by dimensionality. Then one converts Chern classes in terms of $trR$ and the fact $c_1(B) = \frac{i}{2\pi}F$ and compares with the Green-Gaberdiel term (4.12). This gives us a system of linear equations in a, b, c which does not allow for a consistent solution. Thus one needs further study to find the meaning of the Green-Gaberdiel term in 10D type IIB theory in terms of 7-branes.

## 5.5 A lift to D=9 and comparison of amplitude in Horava-Witten background

An interesting aspect of the CP-even 1-loop string amplitude in case of Heterotic string compactified on a $T^2$ is that in the decompactification limit towards nine space-time dimensions the CP-even partner of the composite anomaly cancelling counter-term like (5.130), (5.157), (5.178) exists but not towards the 10-dimensions. The locality of the anomaly cancelling term makes them meaningful only in the relevant dimensions that is in 8D whereas in 10D it does not exist at all. Its non-vanishing limit in 9D is however not related to anomaly as there is no chiral coupling of the fermions but as we shall discuss further below this might have a possible explanation in terms of recently developed string amplitude in Horava-Witten background [80].

First let us consider the case of string 1-loop amplitude for Heterotic $SO(32)$ string theory compactified on a torus $T^2$. We have evaluated the CP-even amplitude in (5.140) which we write once again for the convenience

$$\begin{aligned}\mathfrak{A}^{SO(32)}_{\text{CP-even}} &= V_8 T_2 N \frac{\pi}{24} t_8 \underbrace{\left( trR^4 + \frac{1}{4}(trR^2)^2 + trR^2 trF^2 + 8trF^4 \right)}_{\text{Trivial orbit}} \\ &+ V_8 N \frac{1}{48} \left[ log U_2 |\eta(U)|^2 \right] \\ &\times t_8 \underbrace{\left( \frac{31}{15} trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2 trF^2 + 2(trF^2)^2 + 16trF^4 \right)}_{\text{Harmonic term from the degenerate orbit}} \\ &+ V_8 N \frac{1}{48} \left[ log T_2 |\eta(T)|^2 + \frac{\pi T_2}{6} \right] \\ &\times t_8 \underbrace{\left( \frac{31}{15} trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2 trF^2 + 2(trF^2)^2 + 16trF^4 \right)}_{\text{Harmonic term from the non-degenerate orbit}} \\ &+ \text{non-harmonic terms.}\end{aligned} \tag{5.238}$$

Next suppose that the $T^2$ in the case above have radii $R_1$ and $R_2$ along the two cycles and the angle between them be $\omega$. We can then write the $T^2$ metric and its volume and complex structure in terms of $R_1$, $R_2$ and $\omega$ as follows

$$G_{ij} = \begin{pmatrix} g_{88} & g_{89} \\ g_{89} & g_{99} \end{pmatrix} = \begin{pmatrix} R_1^2 & R_1 R_2 cos\omega \\ R_1 R_2 cos\omega & R_2^2 \end{pmatrix} = \frac{V}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}, \tag{5.239}$$

$$V = R_1 R_2 sin\omega, \quad U_1 = \frac{R_2}{R_1} cos\omega, \quad U_2 = \frac{R_2}{R_1} sin\omega. \tag{5.240}$$

We use the above to decompactify the CP-even amplitude (5.140) by taking $\omega = \frac{\pi}{2}$ and $R_2 = V_1$ such that $V_9 = V_8 R_2$ and $V_{10} = V_8 R_1 R_2$ become the normalized world-volumes in D=9 and D=10 respectively. In this limit $U_1 = 0$ , $U_2 = R_2/R_1$, $logU_2|\eta(U)|^2 = -\frac{\pi U_2}{6}$ and the limit of the amplitude (5.140) gives

$$\begin{aligned}\mathfrak{A}^{SO(32)}_{\text{CP-even}} &= V_9 R_1 N t_8 \frac{\pi}{24}\left(trR^4 + \frac{1}{4}(trR^2)^2 + trR^2 trF^2 + 8trF^4\right) \\ &+V_9 N \frac{1}{48}\frac{1}{R_1} t_8 \left(\frac{31}{15} trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2trF^2 + 2(trF^2)^2 + 16trF^4\right) \\ &+\text{non-harmonic terms.}\end{aligned} \tag{5.241}$$

We can compare the above with the direct calculation of the string amplitude in D=9 as is calculated in [97]

$$\begin{aligned}\mathfrak{A}^{SO(32)}_{\text{CP-even}} &= V_{10}\{N t_8 \left(trR^4 + \frac{1}{4}(trR^2)^2 + trR^2trF^2 + 8trF^4\right) \\ &+\frac{N_1}{R_1^2} t_8 \left(\frac{31}{15}trR^4 + \frac{19}{12}(trR^2)^2 + 5trR^2trF^2 + 2(trF^2)^2 + 16trF^4\right) \\ &+\frac{N_2}{R_1^4} t_8 \left(3(trF^2)^2 + 5trR^2trF^2 + 2(trR^2)^2\right) + \frac{N_3}{R_1^6} t_8 \left(trR^2 + TrF^2\right)^2\}\end{aligned} \tag{5.242}$$

and we see that the first two lines match as they should. The polynomials in the 3rd line of (5.242) are also present in the volume suppressed part of (5.140). In both (5.241) and (5.242) the first line is the circle compactification of the CP-even Green-Schwarz term and the second line contains the CP-even partner of the $SL(2,\mathbb{Z})$ anomaly cancelling term in D=8 as we have seen.

The above observation has an interesting interpretation in terms of the Horava-Witten mechanism applied to M-theory compactified on $S^1/\mathbb{Z}_2 \times S^1$ [48, 49, 80]. We have described the Horava-Witten mechanism in section 3.7.1 where we have described the compactification of M-theory on the interval $S^1/\mathbb{Z}_2$ giving rise to the $E_8 \times E_8$ Heterotic theory in 10D. Compactification of M-theory on $S^1/\mathbb{Z}_2 \times S^1$ now can be thought of as compactification of M-theory with appropriate state truncation under the $\mathbb{Z}_2$ in the presence of two stacks of D8 branes. Recently, an analysis of string amplitude have been carried out partially in this background [80] where it was shown that the 9D bulk gives rise to the term $X_8^-$ in case of 1-loop 4 graviton amplitude. One then expect to have 1-loop 4 graviton amplitude at the two boundaries, that is, at the stacks of D8 branes giving rise to a contribution of the form $\Delta_i(R)$ (i=1,2 for two boundaries) which was not calculated in the reference [80] and is underway at the time of writing this thesis. These brane stacks also accommodate $E_8$ gauge fields within and hence one also get mixed gauge-gravity amplitude and 4 gauge bosons one-loop amplitudes residing in the boundary terms $\Delta_i \equiv \Delta_i(R,F)$. We expect that the sum of the bulk 1-loop amplitude and that of the 2 boudaries provide us the 9D string amplitude or equivalently the 9D lift of the 8D string one-loop amplitude

$$S(R,F) = \Delta_{\text{bulk}} + \sum_{i=1,2} \Delta_i(R,F). \tag{5.243}$$

In this way, we recover a similarity of the 8D case of singularity contribution to the gauge-gravity amplitude in case of F-theory on K3 (5.221) i.e. in 7-brane context and D8 brane contribution in case of 9D M-theory on $S^1/\mathbb{Z}_2 \times S^1$ case. In both cases, the bulk $X_8^-$ term is completed into the anomaly cancelling term of the corresponding supergravity theories: in the former the $X_8^-$ term is due to the Green-Gaberdiel anomaly cancelling term of 10D type IIB theory (4.17) and in the latter, the $X_8^-$ term is due to the bulk 4 graviton 1-loop amplitude or equivalently the 5-brane anomaly cancelling term of M-theory (3.116) [6]. It is only due to the $SL(2,\mathbb{Z})$ transformations properties of the 7-branes that in 8D the anomaly cancelling terms acquire the modular function of the form $\left(ln\left(\frac{\eta^{24}(T)}{\bar{\eta}^{24}(\bar{T})}\right)+ln\left(\frac{j(T)}{\bar{j}(\bar{T})}\right)+ln\left(\frac{\eta^{24}(U)}{\bar{\eta}^{24}(\bar{U})}\right)+ln\left(\frac{j(U)}{\bar{j}(\bar{U})}\right)\right)$.

We have now completed the description of composite anomaly in D=8, N=1 supergravity and we have shown that the anomaly cancelling term is effectively produced in string 1-loop threshold amplitude for the case of Heterotic on $T^2$. We shall now use these results and the similarity in terms of co-dimension-2 branes to describe non-geometric construction of half-maximal and quarter-maximal supergravity theories in six dimensional supergarvities in the next chapter.

# Chapter 6

# Constraints on (non)Geometric compactification to D=6, N=(1,0) theory

So far we have described the 10 and 8 dimensional maximal and half-maximal supergravity theories and the presence of composite connection anomaly therein. We have also shown that in case of 8 dimensional theories the corresponding string 1-loop amplitude provides with the necessary $SL(2,\mathbb{Z})$ anomaly cancelling term. Such terms can have very intriguing interpretations in terms of M- and F-theory for the maximal and half-maximal supersymmetries under suitable compactifications. We shall now go one step further in the compactification down to D=6 and shall consider theories with of 16 and 8 supercharges. The goal is to use the tadpole constraint condition of the type (4.30) $c_1 = \frac{i}{2\pi}F$ in these theories that is inherited from the 8D theories. This point is made more clear by considering the fact that while we have studied the 8D cases, we had to sought for the presence of 7-branes for a compactification of 10D string theories on a $\mathbb{CP}^1$ base for the F theory compactified on an elliptically fibered K3. This is in fact the strategy we have used in section 3.8 to describe the compactification of F-theory on an elliptically fibered K3 from the type IIB perpective where the latter was compactified on the base $\mathbb{CP}^1$ but to provide for a supersymmetry preserving background, one is forced have the co-dimension-2 branes that is 7-branes in these cases (and particularly 24 of them) to cancel the total Ricci tensor term. In the case of D=6, one can either consider the K3 compactification of 10D theories incorporating the necessary instantons (once again 24 SU(2) in case of K3) which we call to be the **geometric compactification** [114, 115] or we can again use the two step process of compactification of 10D theories to 8D theories on a $T^2$ and then compactify on $\mathbb{CP}^1$ base along with 24 co-dimension-2 branes, which are now the NS5 branes: we call this process **non-geometric compactification** [116, 117]. We shall see that for D=6, N=(1,0) supergravity theories the geometric compactification put strong constraints of the possible gauge groups whereas the tadpole constraints in case of non-geometric compactifications opens up for unexplored Yang-Mills gauge groups in D=6, N=(1,0) theories. We shall begin with a discussion of D=6, N=(1,0) supergarvity multiplets and the string theory realization of them. Then we shall discuss the well known geometric compactification of 10D Heterotic theories on elliptically fibered K3 providing for the constraints on the possible Yang-Mills theories in the D=6, N=(1,0) supergravity theories and finally we shall describe the non-geometric compactification of D=8, N=1 supergarvity on $\mathbb{CP}^1$ with NS5 branes. We shall then conclude the chapter with the discussion of D=6, N=2 supergravities from the geometric and non-geometric compactification schemes.

## 6.1 Generalities of D=6, N=(1,0) supergravity

The minimal supersymmerty algebra in six dimensions is the chiral N=1 algebra (note that we are using the Weyl basis for the supercharges, in case one uses the Majorana-Weyl basis, one then write the supersymmetry algebra as N=2 which is chiral and has USp(2) as R-symmetry group). The multiplets in the supergravity theory are

supergravity multiplet: graviton $g_{\mu\nu}$, self-dual two form $B^{+}_{\mu\nu}$, gravitino $\psi^{-}_{\mu}$,
tensor multiplet: anti-self-dual two form $B^{-}_{\mu\nu}$, 1 real scalar $\phi$, tensorino $\chi^{+}$,
vector multiplet: gauge boson $A_\mu$ and gaugino $\lambda^{-}$,
hyper-multiplet: four real scalar $\varphi$ with two fermions $\psi^{+}$.
(The $\pm$ signs of the fermions denote their chiralities.)

A general D=6, N=1 supergravity theory coupled to matter is constructed by combining one gravity multiplet with $n_T$ tensor multiplet, $n_V$ vector multiplet and $n_H$ hypermultiplet. The $n_T$ real scalars in the tensor multiplet parametrize the coset space $\frac{SO(1,n_T)}{SO(n_T)}$ while the $4n_H$ real scalars in the hypermultiplet parametrize a quaternionic manifold of the form $\frac{\mathcal{G}}{\mathcal{H}\times USp(2)}$. The allowed chices for $\frac{\mathcal{G}}{\mathcal{H}\times USp(2)}$ are [118, 119, 120, 121, 122] $\frac{USp(2n_H,2)}{USp(2n_H)\times USp(2)}$, $\frac{SU(n_H,2)}{SU(n_H)\times U(1)\times USp(2)}$, $\frac{SO(n_H,4)}{SO(n_H)\times SO(3)\times USp(2)}$, $\frac{E_8}{E_7\times USp(2)}$, $\frac{E_7}{SO(12)\times USp(2)}$, $\frac{E_6}{SU(6)\times USp(2)}$, $\frac{F_4}{USp(6)\times USp(2)}$ and $\frac{G_2}{USp(2)\times USp(2)}$.

The vector multiplets may belong to a gauge group $G$ under which the hypermultiplets may be charged. The case when the gauge group does not include the $USp(2)$ group of the coset $\frac{\mathcal{G}}{\mathcal{H}\times USp(2)}$ (which can also be identified with the R-symmetry group in case one considers the supercharges and fermions in the Majorana-Weyl basis), is called the ungauged theory while in case of gauged theories the total gauge group $G$ contains the factors $USp(2)$ or its $U(1)$ subgroup as a multiplicative factor. We can thus consider the D=6 gauge group taking the general form [58]

$$G = G_1 \times G_2 \times \cdots G_k \times U(1)^n. \tag{6.1}$$

A further quotient by a discrete group $\Gamma$ may be necessary i.e. to have $G = G_1 \times G_2 \times \cdots G_k \times U(1)^n/\Gamma$ if one soughts for the consistent quantum field theories in 4 and 2 spacetime dimension [58]. We shall neglect this issue in our analysis.

The spectrum of the general D=6, N=(1,0) supergravity model contains chiral fermions and (anti)self-dual two forms, hence at the 1-loop level, it is plagued with the gravity, gauge and mixed gauge-gravity anomaly. To curb this anomaly one uses the generalized Green-Schwarz mechanism due to Sagnotti [123] which we shall review briefly now.

Consider thus the D=6, N=(1,0) supergarvity with $n_T$ tensor multiplet and $n_H$ hypermultiplet coupled to the gauge group $G = \prod_i G_i$ so that the number of vector multiplet $n_V$ is given by the sum of the dimensions of $G_i$. The eight-form anomaly polynomial that one obtains using the index polynomials (2.62), (2.65), (2.66) is

$$\begin{aligned} 16(2\pi)^3 I_8 &= trR^4\left(\frac{n_H-n_V+29n_T-273}{360}\right)+(trR^2)^2\left(\frac{n_H-n_V-7n_T+51}{288}\right) \\ &+\frac{2}{3}trR^2\sum_i TrF_i^2+\frac{2}{3}\sum_i TrF_i^4+\frac{2}{3}\sum_{i<j}TrF_i^2TrF_j^2 \quad (6.2)\\ &= trR^4\left(\frac{n_H-n_V+29n_T-273}{360}\right)+(trR^2)^2\left(\frac{n_H-n_V-7n_T+51}{288}\right) \\ &+\frac{2}{3}trR^2\sum_i c_i trF_i^2+\frac{2}{3}\sum_i a_i trF_i^4+\frac{2}{3}\sum_i b_i(trF_i^2)^2 \\ &+\frac{2}{3}\sum_{i<j}c_{ij}trF_i^2trF_j^2. \quad (6.3)\end{aligned}$$

Note that in (6.2), we have first written the anomaly polynomial in terms of adjoint representation ("Tr") for the gauge groups and then wrote it in fundamental representation ("tr") using

$$\begin{aligned} TrF_i^2 &= c_i trF^2, \quad (6.4)\\ TrF_i^4 &= a_i trF_i^4+b_i(trF_i^2)^2, \quad (6.5)\\ c_{ij} &= c_i c_j. \quad (6.6)\end{aligned}$$

To cancel this anomaly one wish to use the principle of the Green-Schwarz mechanism, i.e. the tree-level exchange of the $B_2$ field as in 10D Heterotic theory. The problem however in this case is the presence of $n_T+1$ $B_2^\alpha$ fields. The Saggnotti generalization of the Green-Schwarz mechanism lies in demanding the 8-form anomaly polynomial $I_8$ in (6.2) to be factorised in the form

$$I_8=\sum_{\alpha,\beta}\Omega_{\alpha\beta}X_4^\alpha X_4^\beta \quad (6.7)$$

where $X_4^\alpha$ are the 4-forms

$$X_4^\alpha=a^\alpha trR^2+\sum_i b_i^\alpha trF_i^2. \quad (6.8)$$

The $a^\alpha$, $b_i^\alpha$ are known as the anomaly coefficients and transform as vectors in the space $\mathbb{R}^{1,n_T}$ with symmetric product $\Omega_{\alpha\beta}$.The group traces "tr" are in fundamental representattions of the respective factors $G_i$ in $G$.

For the generalized factorization condition (6.9) to be met, one finds a set of constraints on the structures of the $trR^4$, $tr(R^2)^2$, $trR^2trF_i^2$, $trF_i^4$ and $tr(F_i^2)^2$ of the eight-form $I_8$ in (6.2). Notably the term $trR^4$ and $trF^4$ should vanish if the 4th order casimirs cannot be written in terms of the 2nd order Casimirs. In total we have the following constraints to be satisfied

$$R^4 : n_H - n_V + 29n_T = 273, \tag{6.9a}$$
$$F^4 : a_i = \sum_R x^i_R, \tag{6.9b}$$
$$(R^2)^2 : \Omega_{\alpha\beta} a^\alpha a^\beta = 9 - n_T, \tag{6.9c}$$
$$F^2R^2 : \Omega_{\alpha\beta} a^\alpha b^\alpha_i = \frac{2}{3}(c_i - \sum_R x^i_R), \tag{6.9d}$$
$$(F^2)^2 : \Omega_{\alpha\beta} b^\alpha_i b^\beta_i = \frac{2}{3}(b_i - \sum_R x^i_R), \tag{6.9e}$$
$$F^2_i F^2_j : \Omega_{\alpha\beta} b^\alpha_i b^\beta_j = \frac{2}{3}(c_i c_j - \sum_{R,S} x^{ij}_{RS}). \tag{6.9f}$$

The $x^i_R$ and $x^{ij}_{RS}$ denote the number of matter fields that transform in the representation $R$ of the gauge group factor $G_i$ and $(R, S)$ of $G_i \times G_j$ respectively. For groups SU(2) and SU(3) there are no invariant 4th order invariant because

$$trF^4_{SU(2)} = \frac{1}{2}(trF^2_{SU(2)})^2, \tag{6.10}$$
$$trF^4_{SU(3)} = \frac{1}{2}(trF^2_{SU(3)})^2, \tag{6.11}$$

so that the condition $a_i = \sum_R x^i_R$ is absent.

When the above condition of the factorizations are met, one can introduce a generalized Green-Schwarz term

$$S = \frac{4}{\alpha'} \int \Omega_{\alpha\beta} B^\alpha_2 X^\beta_4, \tag{6.12}$$

to cancel the anomaly by noting that the $B^\alpha_2$ fields transform inhomogeneously under the local Lorentz and gauge transformations by (compare with 10D Heterotic case as in (5.14))

$$\delta B^\alpha = \frac{\alpha'}{4}(a^\alpha tr\Sigma dA + \sum_i b^\alpha_i tr v_i d\omega_i) = \frac{\alpha'}{4}(a^\alpha \delta Q^{grav}_3 + \sum_i b^\alpha_i \delta Q^{G_i}_3). \tag{6.13}$$

Hence in this case the consistent coupling of the B-field happens through the modified 3-form (compare with 10D Heterotic case (5.15))

$$H^\alpha_3 = dB^\alpha - \frac{\alpha'}{4}(a^\alpha Q^{grav}_3 + \sum_i b^\alpha_i Q^{G_i}_3) \tag{6.14}$$

and these 3-forms satisfies the Bianchi identity

$$\frac{4}{\alpha'} dH^\alpha = -X^\alpha_4 = a^\alpha trR^2 + \sum_i b^\alpha_i trF^2_i. \tag{6.15}$$

Along with the above local anomaly, the 6D theory can suffer a global anomaly. Suppose we have a 2n-dimensional theory with Weyl fermions in a representation of a gauge group G with a non-trivial 2n-th homotopy group $\pi_{2n}(G) \neq 0$. In such situations the partition function in the path integral formulation is changed by a phase factor under non-trivial gauge transformation. Lest this phase factor is a multiple of

$2\pi$, the partition function becomes ill-defined. The canonical example of this is the Witten's SU(2) anomaly [124] in D=4 theory for which we have $\pi_4(SU(2)) = \mathbb{Z}_2$ and the phase factor in question is $\pi$ so that under a non-trivial gauge transformation the partition function changes as $Z[A] \to e^{i\pi} Z[A] = -Z[A]$.
In the 6D case the possible gauge groups that may lead to the global anomalies are SU(2), SU(3) and $G_2$ as

$$\pi_6(SU(2)) = \mathbb{Z}_{12}, \tag{6.16a}$$
$$\pi_6(SU(3)) = \mathbb{Z}_6, \tag{6.16b}$$
$$\pi_6(G_2) = \mathbb{Z}_3. \tag{6.16c}$$

The conditions for the absence of these anomalies are in fact complementary to the absence of $a_i = \sum_R x_R^i$ condition in the general factorization criterion in case of SU(2) and SU(3). The gauge groups which we shall be discussing will not contain these particular groups as factors so that we can safely neglect this matter in the subsequent discussion.

The vacua of consistent D=6, N=(1,0) theories coupled to a Yang-Mills gauge group is constrained by the anomaly cancellation conditions and hence determined by the number of tensor, hyper and vector multiplets. Therefore, the supergravity vacua is larger than those obtained from the compactification of the Heterotic string theory on K3. As we shall describe shortly in the next section, the Heterotic string compactified on K3 can allow for enhanced gauge groups that arise from the Gepner points of the orbifold realization of K3 [107] or from small instantons [114, 115, 117]. In the dual F-theory on Calabi-Yau 3-fold construction, one can allow for even more complicated geometrical engineering [125] mechanisms. The vacua of consistent supergravity theories which are not obtained from the string theory compactification is called the "swampland". In what follows, we shall review the geometric compactification of the Heterotic string on K3 and next compare the non-geometric compactification with aid of NS5 branes to open up new possible gauge groups in D=6, N=(1,0) supergravity theory.

## 6.2 Heterotic string on K3: geometric compactification

We have underlined the properties of the K3 surface in section 3.5.3. Here we state once more the main topological quantities of K3. These are

$$\text{Chern classes } \; c_2 = \frac{1}{4(2\pi)^2} trR^2, \;\; c_1 = \frac{i}{2\pi} trR = 0, \tag{6.17a}$$
$$\text{Euler characteristic } \; \chi = \int_{K3} c_2 = 24, \tag{6.17b}$$
$$\text{Pontrjagin classe } \; p_1 = -\frac{1}{2(2\pi)^2} trR^2, \tag{6.17c}$$
$$\text{Dirac genus } \; \widehat{A(R)} = \frac{1}{48(2\pi)^2} trR^2. \tag{6.17d}$$

To these we add the well-known adjunction formula (5.231) for K3

$$c_2(K3) = 12 c_1(\mathbb{CP}^1) \wedge \omega. \tag{6.18}$$

From these topological data, we can determine the zero-mode spectrum of the wave operator on K3 3.5.3. The number of the harmonic p-forms is given by the Betti numbers

$$b_0 = 1 = b_4, \;\; b_1 = b_3 = 0, \;\; b_2^- = 19, \;\; b_2^+ = 3 \tag{6.19}$$

with $b_2^-$ and $b_2^+$ are respectively the number of (anti-)selfdual two forms. The number of zero modes of the Lichnerowicz operator is

$$n_L = 1 + b_2^- b_2^+ = 58. \tag{6.20}$$

Thus the 10D graviton gives to the 6D graviton and $n_L = 58$ scalars while the 10D $B_2$ field gives $b_0 = 1$ 2-form $B_2$ and $b_2 = b_2^- + b_2^+ = 22$ scalars upon compactification on K3. The dilaton reduces trivially. Thus the gravity multiplet of 10D Heterotic theory gives rise to the gravity multiplet and one tensor multiplet in 6D theory.

To analyze the reduction of 10D fermions and the 10D gauge group we note that the net number of positive chirality zero-modes of the Dirac and Rarita-Schwinger operator (in the presence of gauge fields) are given by the Atiyah-Singer index theorems

$$\begin{aligned} n_{1/2}(R,F) &= \int_{K3} \widehat{A(R)} ch(-iF) = \int_{K3} \left( \frac{dim(G)}{48(2\pi)^2} trR^2 + \frac{1}{2(2\pi)^2} TrF^2 \right) \\ &= \left( \frac{dim(G)}{48(2\pi)^2} V + 48 \frac{\int_{K3} TrF^2}{\int_{K3} trR^2} \right), \end{aligned} \tag{6.21}$$

$$n_{1/2}(R) = \int_{K3} \widehat{A(R)} = 2, \tag{6.22}$$

$$n_{3/2} = \int_{K3} \widehat{A(R)} \left( tr \left[ e^{\frac{iR}{2\pi}} \right] \right) = -40. \tag{6.23}$$

In the above we have introduced the volume integral $\int_{K3} trR^2 = V = 96(2\pi)^2$ for the purpose of clarifying the inclusion of instanton modes in the gauge group which we shall illustrate shortly. We also note that the 10D chirality operator $\Gamma^{11}$ is decomposed into the same for 6D and 4D projectors as $\Gamma^{11} = \Gamma^7 \Gamma^5$. The 10D fermions are Majorana-Weyl while the 6D ones are Weyl, thus the number of positive chirality fermions $n_{1/2}$ and $n_{3/2}$ must be divided by two. Thus the reduction of 10D positive chirality gravitino gives 1 negative chirality 6D gravitino plus 20 6D hyperinos of positive chirality and the negative chirality 10D dilatino reduces to a 6D positive chiral tensorino.

Let us now analyze the reduction of the 10D gauge group and the vector multiplet. The modified Bianchi identity (5.15) due to the Green-Schwarz mechanism of anomaly cancellation is

$$dH_3 = -\frac{\alpha'}{4}(trR^2 + trF^2). \tag{6.24}$$

The requirement that $H_3$ to be defined globally [126, 127, 128] implies that $dH$ should integrate to zero over any four-cycle in 10D space. Hence, assuming that we turn on the background fields $R_0$ and $F_0$ only inside K3, we should get

$$\int_{K3} (trR_0^2 + trF_0^2) = 0. \tag{6.25}$$

This is satisfied if one embeds the SU(2) holonomy group of K3 in the gauge group of the theory (SO(32) or $E_8 \times E_8$). The instanton number of the resulting configuration is

$$n = \frac{1}{16\pi^2} \int_{K3} trF_0^2 = -24. \tag{6.26}$$

Due to this embedding, the 10D gauge group decomposes as

$$G_{10} \to G_6 \times SU(2) \tag{6.27}$$

to give rise to the 6D gauge group $G_6$. The adjoint representation of $G_{10}$ is then gives rise to the sum of the $R_i$ and $S_i$ representation of $G_6$ and SU(2) respectively as

$$Dim(G_{10}) \to \sum_i Dim(R_i, S_i). \tag{6.28}$$

The net number of positive chirality zero-modes of fermions transforming under representation $S_i$ of SU(2) is given by

$$\begin{aligned} n_i &= \left( \frac{dim(S_i)}{48(2\pi)^2} V + 48 \frac{\int_{K3} tr_i F^2_{S_i}}{\int_{K3} tr R^2} \right) & (6.29)\\ &= 2dim(S_i) - 12c_i & (6.30) \end{aligned}$$

where $c_i$ is the trace transformation constant $tr_i F^2 = c_i tr F_0^2$ for SU(2).

Thus for each i the 10D positive chirality gaugino gives $n_i$ 6D spinors transforming in $R_i$ of $G_6$ according to the branching (6.28). If $n_i > 0$ these are negative chirality gaugini and if $n_i < 0$ these are negative chirality hyperini. Below we exemplify the reductions of SO(32) and $E_8 \times E_8$.

Embedding of the K3 instanton charge into SO(32) gives rise to the following decomposition

$$SO(32) \to (SO(28) \times SU(2)) \times SU(2). \tag{6.31}$$

The branching of the adjoint representation of SO(32) according to the above decomposition is

$$\mathbf{496} \to (\mathbf{378,1;1}) + (\mathbf{1,3;1}) + (\mathbf{28,2;2}) + (\mathbf{1,1;3}). \tag{6.32}$$

The index number $n_i$ for the dimension 1, 2 and 3 representation of SU(2) are $n_1 = 1$, $n_2 = -10$ and $n_3 = -45$ respectively using the formula (6.29) with $c_i = 0, 1, 4$ respectively. Thus we have 1 negative chirality gaugino in **(378,1)+(1,3)** of $SO(28) \times SU(2)$, 10 hyperini in **(28,2)**, 45 singlet hyperini along with their bosonic superpartners. Combining these with the reduction of the 10D gravity multiplet gives rise to 1 gravity multiplet, 1 tensor milultiplet, one vector multiplet in **(378,1)+(1,3)** of $SO(28) \times SU(2)$, 10 hypermultiplets in **(28,2)** and 65 neutral hypermultiplets. This spectrum is anomaly free as can be checked by inserting the numbers of different multiplets in the formuli (6.9):

$$\begin{aligned} n_H - n_V + 29 n_T &= (10 \times 28 \times 2 + 65) - (378 + 3) + 29 = 273, & (6.33)\\ TrF^4_{SO(28)} &= 20 trF^4 + 3(trF^2)^2 \Rightarrow & \\ a_{SO(28)} &= 20 = 10 \times \mathbf{2} \text{ from 10 hypers, etc.} & (6.34) \end{aligned}$$

In the case above, the gauge symmetry can be enhanced further by a non-perturbative mechanism involving instantons collapsing to zero size [114, 115, 117]. These instantons can be interpreted as solitonic 5-branes (to be discussed further in case of non-geometric compactification) which carry an extra gauge symmetry factor $USp(2)$ in case of SO(32). The N coincident instantons give rise to an extra $USp(2N)$ symmetry. For N=24, one thus obtain an enhancement $USp(48)$ while for $N < 24$ the full symmetry group is $SO(N+8) \times USp(2N)$ in which one embeds 24-N instanton charges within SO(32).

Next we consider the case where 10D gauge group is $E_8 \times E_8$. In this case we can incorporate the 24 instantons in either of the two $E_8$s provided the total number remains 24. For example we can embed all of them into only one $E_8$ with the resulting decomposition

$$E_8 \times E_8 \to (E_8 \times E_7) \times SU(2) \tag{6.35}$$

with the adjoint branching according to

$$(\mathbf{248,1)}) + (\mathbf{1,248}) \to (\mathbf{248,1;1)}) + (\mathbf{1,133;1}) + (\mathbf{1,56;2}) + (\mathbf{1,1;3}). \tag{6.36}$$

The multiplicities $n_i$ are as before and we get in total one gravity multiplet, one tensor multiplet, one vector multiplet in (248,1)+(1,133) of $E_8 \times E_7$, 10 hypermultiplets in (1,56) and 65 neutral hypermultiplets. All the hypermultiplets are neutral under the $E_8$. The anomaly polynomial factorizes as [107]

$$16(2\pi)^3 I_8 = (trR^2 - \frac{1}{6}trF_{E_7}^2 - \frac{1}{30}trF_{E_8}^2)(trR^2 - trF_{E_7}^2 + \frac{1}{5}trF_{E_8}^2), \tag{6.37}$$

where we have used $TrF_{E_8}^2 = 30trF_{E_8}^2$ and $TrF_{E_7}^2 = 3trF_{E_7}^2$.

The above case provides the ground for an interesting example of gauged supergravity. One considers the $T^4/\mathbb{Z}_8$ orbifold limit of K3 and reduces the $E_8 \times E_8$ heterotic theory on $T^4/\mathbb{Z}_8$ (this is however very much different from the models considered above as there is now no SU(2) instanton to be incorporated). The resulting gauge group is $E_8 \times E_7 \times U(1)$. The (1,56) hypermultiplets are now charged under U(1) and hence the mixed gauge-gravity anomaly with gauge group $E_8 \times E_7 \times U(1)$ occurs giving rise to the following factorisation of the anomaly 8-form [107]

$$16(2\pi)^3 I_8 = (trR^2 - \frac{1}{6}trF_{E_7}^2 - \frac{1}{30}trF_{E_8}^2 - F_{U(1)}^2)(trR^2 - trF_{E_7}^2 + \frac{1}{5}trF_{E_8}^2 - 6F_{U(1)}^2), \tag{6.38}$$

This example, which contains a U(1) factor in the gauge group will be of crucial importance in classification of compactification schemes according to geometric or non-geometric ones which we shall discuss in the next section.

## 6.3 Heterotic string on K3: Non-geometric compactification

What we call the non-geometric compactification of Heterotic on an elliptically fibered K3 is a two step process: first compactify Heterotic theory on $T^2$ with complex structure $U = U_1 + iU_2$ and Kähler structure $T = T_1 + iT_2 = B_{89} + iV_{T^2}$ then compactify this theory on a base $\mathbb{CP}^1$ with 24 punctures which denote the positions of 24 NS5 branes. Note that this procedure is same as that of F-theory compactified on an elliptically fibered K3. There the 24 punctures on the $\mathbb{CP}^1$ represented the positions of the 7-branes. The number 24 is again due to the fact that the $c_1$ of $\mathbb{CP}^1$ is non-vanishing and upon integration gives 2; thus to provide for a supersymmetry preserving background, one has to include codimension-2 branes, each of which provides -1/12 for the deficite angle and therefore a set of 24 cancels the tadpole of $c_1(\mathbb{CP}^1)$. In the perspective of 10D type IIB theory (the geometrization of the $SL(2,\mathbb{Z})$ symmetry of which was the foundation of F-theory) the codimension-2 brane is the 7-brane whereas in the perspective of 8D theory the codimension-2 brane is the NS5 brane. To include the NS5 branes in the compactification scenario, we need a kind of tadpole condition which relates the $c_1(\mathbb{CP}^1)$ to the composite connection field strengths $F(t,\bar{T})$ and $F(U,\bar{U})$ as in the case of 10D type IIB theory (see discussion following (4.24) in section 4.1.1). In case of 8D N=1 theory there are in fact 2 such conditions. First, let us start with the 10D heterotic theory compactified on a $T^2$ with complex structure $U$. Suppose that the resulting 8D N=1 theory couples to a gauge group $E = U(1)^{n+2} \times G$ where G is a product of the semisimple groups and is of rank =16-n. The factor 2 in n+2 is due to the photons obtained from the reduction of the metric and $B_2$ field of 10D Heterotic theory: the rest $U(1)^n \times G$ is obtained from the reduction of the 10D gauge group. The scalar coset is $SO(2,n+2)/SO(n+2) \times U(1)$ and we denote the curvature of the composite U(1) connection by $F^Q = F(z_i, \bar{Z}_i)$ with $z_i$ $(i = 1, \cdots, n)$ being the complex moduli of the n abelian vector multiplets. The general expression of $F^Q$ is complicated and does not split into the sum of individual terms of the schematic form $dz_i \wedge d\bar{z}_i/(Im(z_i))^2$. Now the effective action shall contain the 8D Ricci scalar along with the kinetic term for the U moduli

$$S_8 \sim \int_{\mathcal{M}_8} R *_8 1 - \frac{dU \wedge *d\bar{U}}{4U_2^2} \tag{6.39}$$

where once more, for the sake of simplicity we have not included other contributions from the $B_2$ field and vector multiplets. Taking the metric ansatz for the compactification on a $\mathcal{M}_8 = \mathcal{M}_6 \times \mathbb{CP}^1$ as

$$ds^2 = -dt^2 + dx_1^2 + \cdots + dx_5^2 + \underbrace{e^{\phi(z,\bar{z})} dz d\bar{z}}_{\mathbb{CP}^1} \tag{6.40}$$

and redoing the steps as in the discussion following (4.24) in section 4.1.1 we find the first tadpole condition

$$\begin{aligned} \frac{i}{2\pi} trR &= -\frac{1}{8\pi}\frac{dU \wedge d\bar{U}}{U_2^2} \Rightarrow \\ c_1(\mathbb{CP}^1) &= \frac{i}{2\pi} F(U, \bar{U}). \end{aligned} \tag{6.41}$$

To find the tadpole condition relating the composite connection engendered by the Kähler moduli we start with the 10D Heterotic Bianchi identity

$$dH = -\frac{\alpha'}{4}(trR^2 + trF^2). \tag{6.42}$$

The NS5 branes are the magnetic sources of the 3-form $H$ according to (we are absorbing the string scale factor $\frac{\alpha'}{4}$ inside $B_2$ for convenience)

$$dH_3 = \eta(NS5) \tag{6.43}$$

where $\eta(NS5) = \sum_{24} \delta(\mathbb{CP}^1) dz \wedge d\bar{z} \wedge \omega$ is the class of 24 NS5-branes with $\omega$ being the top form of the elliptic fiber and $z$, $\bar{z}$ the complex coordinates of $\mathbb{CP}^1$. We can write $trR^2$ in terms of the Pontrjagin class of the tangent bundle of K3 as $p_1(TM)$ and the $trF^2$ as the top Chern class of the gauge bundle $c_2(E)$ where G is the 8D gauge group obtained from the 10D gauge group (SO(32) or $E_8 \times E_8$ with or without Wilson lines while compactifying on the $T^2$). Thus the form of the 10D Bianchi identity can be recast as

$$\eta(NS5) = \frac{1}{2} p_1(TM) - c_2(E) \tag{6.44}$$

Now we use the adjunction formula for the elliptically fibered K3: the adjuction for the tangent bundle gives rise to $p_1 = c_1(\mathbb{CP}^1) \wedge \omega$ where $\omega$ is a top form on the fiber. The adjunction on the gauge bundle E chooses the Abelian part of the gauge group i.e. $c_2(E) = c_1(\tilde{E}) \wedge \omega$ with $\tilde{E} = U(1)^n$ being the Abelian part of the gauge group E.

Finally the class of the NS5 brane $\eta(NS5) = \sum_{24} \delta(\mathbb{CP}^1) dz \wedge d\bar{z} \wedge d\tau \wedge d\bar{\tau}$ is reduced to $12F(T,\bar{T}) \wedge \omega$, this is because the reduction of $H_3 = dB_2 - \frac{\alpha'}{4}(Q_3^{grav} + Q_3^{YM})$ will give the connection 1-form $Q(T) = \frac{dB_{89}}{2V_{T^2}} = \frac{dT_1}{2T_2}$ in 6D which couples magnetically to the codimension-2 NS5 brane by

$$dQ(T) = F(T,\bar{T}) = \frac{dT \wedge d\bar{T}}{4iT_2^2} = \delta_{NS5} \tag{6.45}$$

with $\delta_{NS5} = \delta(\mathbb{CP}^1) dz \wedge d\bar{z}$ being the class of NS5 brane swith two legs on the transverse direction i.e. on the direction of $\mathbb{CP}^1$. Accounting for everything in (6.44) with adjuction reduction we find the second tadpole equation

$$c_1(\mathbb{CP}^1) + \frac{1}{12} c_1(\tilde{E}) = \frac{i}{2\pi}[F(T,\bar{T})]. \tag{6.46}$$

These two tadpole condition (6.41) and (6.46) puts stringent condition on the stringy construction of D=6, N=(1,0) supergravity theories. For n=0 cases, the Abelian factor $\tilde{E}$ is absent in E and hence (6.46) dictates non-geometric compactification to quench $c_1(\mathbb{CP}^1)$ with $F(T,\bar{T})$. Equation (6.41) allows to build a K3 space ove $\mathbb{CP}^1$ base whereas (6.46) tells that only NS5-branes participate in cancelling the curvature contribution in the Bianchi identity.

In order to have geometric Heterotic realization of D=8, N=1 theory on a $\mathbb{CP}^1$ base, i.e. a K3 compactification, $n \geq 1$ is required. Indeed, taking $[F(T,\bar{T})] = 0$ in (6.46) one needs non-trivial $\tilde{E}$. The geometric compactification then should correspond only to $[F(U,\bar{U})] \neq 0$. An example of this case is the compactification of 10D $E_8 \times E_8$ Heterotic theory on the $T^4/\mathbb{Z}_8$ orbifold limit of K3 which we have discussed previously ($E_8 \times E_8 \to E_8 \times E_7 \times U(1)$). We have also mentioned before that the compactification on K3 with the embedding of SU(2) instantons inside the 10D gauge group is also geometric though in flavour it is nearer to the non-geometric one. In the former there are 24 instantons arising from the global definition of $H_3$ (6.26) while in the latter, there are 24 NS5-branes acting for the magnetic sources of $H_3$ in the Bianchi identity (5.15).

The usefulness of the non-geometric compactification is that it allows for the realization of the gauge groups which are otherwise not possible in geometric compactification. This is due to the fact that the NS5-branes each contains a tensor multiplet and a hypermultiplet [114, 127] which can be used to quench the anomaly cancelling conditions (6.9). Let us take for example the $E_8 \times E_8$ 10D Heterotic theory compactified on a $T^2$ without any Wilson line and then compactify on a $\mathbb{CP}^1$ with 24 NS5-Branes each having a tensor and hypermultiplet. We see then that the cancellation condition for $trR^4$ term in (6.9) is satisfied

$$\underbrace{(20+24)}_{n_H} - \underbrace{(496)}_{n_V} + 29 \times \underbrace{(1+24)}_{n_T} = 273 \tag{6.47}$$

for the $dim(E_8 \times E_8) = 496$ and that the total number of tensor multiplet $n_T = 25 = N(NS5) + 1$ where $N(NS5) = 24$ is the number of NS5-branes. The factor 20 in $n_H$ are 20 neutral hypermultiplets obtained from the K3 moduli. The complete anomaly polynomial is

$$16(2\pi)^3 I_8 = 2(trR^2)^2 + 5trR^2(trF_1^2 + trF_2^2) + 6((F_1^2)^2 + (F_2^2)^2) \tag{6.48}$$

which can then be written as a sum of 25 factorized terms due to 25 $B_2^\alpha$ as required by the Green-Schwarz-Sagnotti mechanism [123] as in (6.12). Thus we get the D=6, N=(1,0) theory coupled to $E_8 \times E_8$ Yang-Mills theory which is otherwise impossible with geometric compactification.

Next let us take the case of $SO(32)$ gauge group. In this case however, we proceed as before except for the fact that each NS5-brane brings along an $USp(2)$ factor. 24 NS5 branes collapse together to give rise to the hypermultiplet in bi-fundamental (**32,24**) representation of $SO(32) \times Sp(24)$ [115]. As before, the $trR^4$ term vanishes using (6.47) while the 24 USp(2) factors combines with the non-vanishing 4th order invariant of SO(32) to cancel the $trF^4$ term as in the trace decomposition of $SO(32)$

$$TrF^4_{SO(32)} = 24trF^4_{SO(32)} + 3(trF^2_{SO(32)})^2 \tag{6.49}$$

the $trF^4_{SO(32)}$ part comes with a factor 24 and according to $a_i = \sum_R x^i_R$ relation in (6.9) this factor is cancelled by $24 = x^i_R$ from the $Sp(24)$.

Note that in the above two case, that is, 10D Heterotic string theories with gauge group $SO(32)$ and $E_8 \times E_8$ compactified on $T^2$ give in the threshold computation the composite anomaly cancelling term along with world-sheet instanton terms. In the presence of Wilson lines, that is 8D N=1 theory with $SO(16) \times SO(16)$ and $SO(8)^4$ gauge groups, the string amplitude brings coupling due to massive vectors in the spinor or bi-fundamental representations according to the branching of the respective gauge groups according to (5.151), (5.179). A similar result follows in case of $E_8 \times E_7 \times U(1)$ which can be obtained by switching only one Wilson line along one of the 1-cycle of the compactification torus in case of compactification of $E_8 \times E_8$ Heterotic theory. For the latter case, we have seen, the resulting 6D theory is anomaly free (6.38) by properly taking into account the charged and neutral hyper-multiplets arising from the compactification over $\mathbb{CP}^1$ with the tadpole condition (6.41). These hyper-multiplets are otherwise massive in 8D perspective. For the case of $SO(16) \times SO(16)$ and $SO(8)^4$ in 8D, we see that they give rise to anomaly free 6D theory upon non-geometric compactification in accordance with (6.46) with trivial $\tilde{E}$ only if we include the vectors in the bi-fundamental representations as in the branching rules (5.151), (5.179). Hence we see that the correct i.e. string theoretic massive completion in 8D is necessary in order to obtain an anomaly free 6D compactification. Massive states that appear in $T^2$ compactification to 8D must become massless when the torus degenerates to yield an elliptically fibered K3. We shall endeavour to bring more clarification to this mechanism in future studies. It appears that $G = SO(16)^2$ and $G = SO(8)^4$ should admit a double realization, either with 25 tensor multiplets or with 24 $USp(2)$ gauge fields and hypers in bi-fundamentals (i.e. from compactification of either $E_8 \times E_8$ or $SO(32)$). Of course the two versions match once these theories are put on a circle. Eight-dimensional N=1 theories hold keys to large classes of interesting string backgrounds, most of which cannot be seen as ordinary compactifications of ten-dimensional string theories. We have argued here, that the study of composite connections and their anomalies may provide useful insights and constraints in constructing these backgrounds. In the next section, we shall discuss the (non-)geometric compactification to 6D N=2 theories.

## 6.4 Type II string on K3: (Non-)Geometric compactification

We shall end this chapter with a brief review of the geometric and non-geometric compactification of type II theories on K3 which shall complement the case of the

Heterotic compactification discussed above. The compactification of type IIA and type IIB theories on K3 gives rise to two different supergravities in D=6 with 16 supercharges:

**Type IIA on K3:** This is the D=6, N=(1,1) supergravity containing the following multiplets

Gravity multiplet: 1 graviton $g_{\mu\nu}$, two Weyl gravitini $\psi^i_\mu$ of opposite chirality, 1 antisymmetric 2-form $B_2$, 4 vectors $A^i_\mu$, 4 gaugini $\psi^j$ 2 of which are positive chiral and the rest negative, 1 real scalar $\phi$.

20 Vector multiplets: each with 1 vector $A_\mu$, 2 Weyl gaugini $\chi$ of opposite chirality and 4 real scalars.

The 80 scalars parametrize the coset space $\frac{SO(4,20)}{SO(4)\times SO(20)}$ and this theory is in fact dual to 10D Heterotic string compactified on $T^4$ [129, 130, 131, 116].

Next **Type IIB on K3:** This is the D=6, N=(2,0) supergravity containing the following multiplets

Gravity multiplet: 1 graviton $g_{\mu\nu}$, 5 self-dual anti-symmetric 2 forms $B^i_2$ ($i = 1,\cdots,5$), two negative chirality Weyl gravitini $\psi^-_\mu$.

21 tensor multiplets: each having 1 anti-self-dual anti-symmetric two form $B_2$, 5 scalars $\varphi^j$ ($j = 1,\cdots,5$) and two positive chirality Weyl tensorini $\psi^+_{1,2}$.

The $5\times 21$ scalars parametrize the coset space $\frac{SO(5,21)}{SO(5)\times SO(21)}$. The theory might seem to suffer from gravity anomaly because of the gravitini and (anti-)self-dual tensor fields. However,the combination of 4 self-dual tensor in gravity multiplet with 21 anti-self-dual tensor in tensor multiplet is what needed to cancel the contribution of gravitini and other fermions exactly. Thus the theory obtained from the compactification of type IIB on K3 is anomaly free.

It is intriguing to note that both type II theories compactified on a $T^2$ give rise to the same supergravity theory in D=8 i.e. the N=2 supergravity theory in 8D due to the T-duality. However, further compactification on a base $\mathbb{CP}^1$ with 24 singularities, i.e. the non-geometric compactification on K3 should give rise to such different supergravity theories in 6D. We intend to discuss this point in the light of tadpole conditions similar to the one in 10D type IIB (4.30) or the the ones in case of 8D Heterotic theory (6.41) and (6.46).

To derive the tadpole condition for 8D N=2 case, consider the compactification of 10D type IIB theory on a $T^2$ of complex structure $U = U_1 + iU_2$ and Kähler structure $T = B_{89} + iV_{T^2}$. The full U-duality group is $SL(2,\mathbb{R})\times SL(3,\mathbb{R})$ but we consider only the perturbative or the T-duality group $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$. In fact the scalar coset of the theory is $\frac{SL(2,\mathbb{R})}{U(1)}\times\frac{SL(3,\mathbb{R})}{SO(3)}$. The fermions are charged under the composite U(1) as we have discussed in section 4.2. If however one considers only the the T-duality group one finds the $\frac{SL(3,\mathbb{R})}{SO(3)}$ replaced by another $\frac{SL(2,\mathbb{R})}{U(1)}$. the fermions are not chiraly charged under this accidental $SL(2,\mathbb{R})$ as can be seen from the supersymmetry transformation of the fermions (4.35), e.g. for the gravitini

$$\mathcal{D}_\mu\psi_\nu = D_\mu\psi_\nu - \frac{i}{2}Q^{ab}_\mu T_{ab}\psi_\nu - \frac{i}{2}Q_\mu\gamma^9\psi_\nu. \tag{6.50}$$

The $SO(3)$ composite connection $Q^{ab}_\mu$ can be reduced to find a U(1) connection but is not chiraly switched due to the lack of $\gamma^9$ term.

The 8D effective action in case of N=2 supersymmetry shall contain the following terms (once again we are neglecting other bosonic and fermionic contributions for the sake of simplicity)

$$S_8 \sim \int_{\mathcal{M}_8} R *_8 1 - \frac{dU \wedge *d\bar{U}}{4U_2^2} - \frac{dT \wedge *d\bar{T}}{4T_2^2}. \tag{6.51}$$

The difference of (6.51) above with that of the N=1 case (6.39) is that due to the supersymmetric constraints, both T and U moduli kinetic terms are present in (6.51). This thing is a result of the scalar coset structure ($\frac{SO(2,n)}{SO(2)\times SO(n)}$ in N=1 case whereas $\frac{SO(2)_T}{U(1)} \times \frac{SO(2)_U}{U(1)}$ in N=2 case) dictated by the supersymmetry as discussed in section 2.4 equation (2.38).

Following the same procedure as the derivation of 10D type IIB tadpole (4.30) or the 8D Heterotic tadpole (6.41) we propose the metric ansatz for the compactification on a $\mathbb{CP}^1$ base using

$$ds^2 = -dt^2 + dx_1^2 + \cdots + dx_5^2 + \underbrace{e^{\phi(z,\bar{z})}dzd\bar{z}}_{\mathbb{CP}^1} \tag{6.52}$$

from which the Einstein equation gives the tadpole condition

$$c_1(\mathbb{CP}^1) = -\frac{dT \wedge d\bar{T}}{8\pi T_2^2} - \frac{dU \wedge d\bar{U}}{8\pi U_2^2} = \frac{i}{2\pi}[F(T,\bar{T}) + F(U,\bar{U})]. \tag{6.53}$$

Contrary to the 8 supercharge case, we now have both $F(T)$ and $F(U)$ present in the tadpole condition (6.53). To construct the K3, we need to set either $F(T)$ or $F(U)$ to zero as both of them integrates to -2 ($\frac{i}{2\pi}\int_{K3} F(T,\bar{T}) = -2$, $\frac{i}{2\pi}\int_{K3} F(U,\bar{U}) = -2$) separately. The classification of type IIB compactified on K3 or IIA compactified on K3 is done by considering which of the composite field strength is set trivial. Type IIB compactification corresponds to the case of trivial U and type IIA compactification corresponds to the trivial T [116].

Let us consider the above two cases separately. First let us examine the 8D N=2 supergravity compactified on $\mathbb{CP}^1$ with $F(T) = 0$ and the 24 singularities on $\mathbb{CP}^1$ corresponding to the positions of co-dimension-2 NS5-branes. These 24 branes however yield only 20 dimensional vectors in 6D due to the Crammer-Sherck mechanism elaborated in Douglas, Park & Schnell [132]. These vectors transform under $SO(2,18)$. 4 more vector fields comes from the reduction of the 4 neutral vectors of 8D theory (two graviphotons and two Abelian vectors in the vector multiplet coming from the $T^2$ reduction of metric $g_{\mu\nu}$ and $B_2$) which transform under $SO(2,2)$. Thus the symmetry group $SO(4,20)$ of N=(1,1) theory is recovered from the vectors in 6D.

In the case of D=8 N=2 supergravity compactified on $\mathbb{CP}^1$ with $F(U) = 0$, one finds instead tensors transforming in $SO(2,18)$ [132]. The three neutral tensor fields $B_2$, $C_2$ and $C_4|_{T^2} \to C_{\mu\nu 89} = C_2'$ in 8D supergravity give rise to 3 pairs of self-dual and anti-self-dual tensors in 6D. Thus we recover the symmetry group $SO(5,21)$ of N=(2,0) theory.

It is interesting to consider the case where $\frac{i}{2\pi}\int_{K3} F(T,\bar{T}) + F(U,\bar{U}) = -2$ i.e. both the T and U moduli are non-trivial. We intend to study this case in future.

# Chapter 7

# Two loop string amplitude and $D^2R^4$ terms in Heterotic string

In this final chapter of this thesis we shall consider the calculation of two-loop Heterotic string amplitude in 10D. This computation shall complement the computation of 10D and 8D Heterotic string theory considered in sections 5.2 and 5.3. The difference however is that here we shall compute the amplitude in the hyper-elliptic formalism [91] contrary to the elliptic formalism of the one-loop cases. The computation in the elliptic formalism have been carried out for type II theories in [133, 134, 135, 71, 136, 90, 137] giving rise to $D^4R^4$ terms and in Heterotic theory to give $D^2R^4$ terms in [138, 90, 139, 140, 141, 142, 137, 143]. In the latter references, the emphasize was however given to the fact that $R^4$, $F^4$ terms in Heterotic theory do not receive any renormalization from two loop computations corroborating the fact that these terms are related to anomaly cancellation and hence should not receive corrections beyond one-loop. The $D^4R^4$ terms in type II theories have been studied extensively for its direct relation with the Eisenstein-series related to the U-duality group $SL(2) \times SL(3)$ and hence providing windows for the relation between string amplitude, instanton corrections and number theory [144]. The case of 10D Heterotic $D^2R^4$ terms are less explored due to the lack of such connection with number theory. The interest of studying such terms are mostly academic. However, at 2-loops, these terms include genus-two elliptic functions i.e. Siegel modular forms as compared to the Eisenstein series in case of one-loop elliptic genus 5.2 which are of immense mathematical interest yet quite abstract from string or supergravity physics point of view. For example, in 8D, as we shall see, it is quite difficult to complete the modular integral on the moduli space $Sp(4)$ while for one loop the integral over $SL(2,\mathbb{Z})$ world-sheet moduli space was quite easy. Thus we take the course of hyperelliptic formalism to evaluate the multi-loop integral, a formalism which was studied in late 80s [145, 146, 85, 86, 87, 88]. The problem with multi-loop amplitude is that the process for the gauge fixing the conformal symmetries of the world-sheet Riemann surface is quite complicated and the standard process of inserting ghost state vertex operators along with vertex operators of the scattering states does not use up these residual symmetries completely [90]: one has to check for the unitarity of the S-matrix at the conclusion of computation which can prove to be a very tedious process. A neat method of gauge fixing method was proposed by D'Hocker & Phong in [90, 139, 140, 141, 142, 137, 143] which takes into account all these ambiguities and can be used for loop computations beyond 2-loops. Original calculation of 4 point amplitudes in type II and Heterotic theories implementing this chiral gauge fixed measure has been carried out in [137] where the emphasize was once more on the demonstration of the fact that $R^4$, $F^4$ terms do not receive higher loop corrections in case of Heterotic theory. We carry out the 2-loop computation in the hyper-elliptic language using these chiral measures, a study originally initiated by Stieberger &

Taylor [83, 84] to study $F^6$ type of coupling at two loops in Heterotic theory. We thus endeavour to complement these references with the computation of $D^2R^4$ terms in case of Heterotic theory.

We shall briefly describe the problem of fixing path-integral measure in string multi-loop amplitude and demonstrate the usefulness of chiral measures in such situations. Next we shall describe the genus-2 CFT using the hyperelliptic language and finally describe the computation of $D^2R^4$ and $D^2F^4$ amplitudes in 10D Heterotic theory with comments on such amplitudes in 8D Heterotic theory.

## 7.1 Generalities of higher genus amplitude and method of chiral splitting

Recall from our discussion of moduli space of world-sheet topology for the string amplitude in section 3.3.1 that the operator that plays the key role in the gauge fixing process is $\widehat{P}$ (3.33) which defines the infinitesimal diff $\times$ Weyl transformation of the metric $h_{\mu\nu}$ according to

$$\delta h_{\mu\nu} = (\widehat{P}\xi)_{\mu\nu} + (2\Lambda(\sigma) - \nabla^{\alpha}\xi_{\alpha})h_{\mu\nu}. \tag{7.1}$$

The space $Ker\widehat{P}$ is the space of all CKVs while the space $Ker\widehat{P}^{\dagger}$ is the space of all moduli deformations. Let us also recall that the Riemann-Roch formula on the world-sheet tells us that $dimKer\widehat{P} - dimKer\widehat{P}^{\dagger} = 3\chi$ for $\chi = 2 - 2g$ being the Euler characteristic of the world-sheet topology with genus (handle) g. It happens to be that if $\chi > 0$, e.g. for a sphere, $dimKer\widehat{P}^{\dagger} = 0$ which means that there are no metric moduli and 6 CKVs so that every metric is diff $\times$ Weyl equivalent to the round metric of the sphere $\mathbb{CP}^1$. In case of genus one interaction, that is a torus world-sheet, there are 2 metric moduli and 2 CKVs. The presence of CKVs needs the application of BRST ghost insertions [29] so that the final amplitude be BRST invariant. One thus inserts $N = -\chi + n_B = n_B$ numbers of picture changing operators along with $n_B$ bosonic vertices which participate in the amplitude and with the use of 2 metric deformations, coincides the PCOs with the physical vertex positions to have a BRST invariant amplitude. In case $\chi < 0$, $dimKer\widehat{P} = 0$ and there are no CKVs and only metric moduli. This leads into a very subtle trouble in finding BRST invariant amplitude. If one takes up the paradigm of PCO insertion, then the number required for a genus $g \geq 2$ interaction with $n_B$ bosons is $N = -\chi + n_B > n_B$. However as there are no CKV, we are à priori not at liberty to coincide the positions of the PCOs with that of the physical vertices. One can however proceed to do the coincidence of PCOs with vertices but has to be extremely careful that the final amplitude does not loose its unitarity because of this ad-hoc application of metric deformation [145, 146]. Let us give an example taken from the reference [83] where the authors took the course of the 2-loop closed 10D Heterotic string amplitude to prove the non-renormalization of $F^4$ terms, which we have seen in section 5.2 is related to the Green-Schwarz anomaly cancelling mechanism and thus should remain protected from higher loop corrections. The 4-point Green's function for the amplitude would look like

$$\mathcal{A} = \prod_{i=1}^{4}\langle \int \frac{d^2 z_i}{\pi} \underbrace{e^{i}_{\alpha_i\beta_i} V^{\alpha_i\beta_i}(p_i, z_i, \bar{z}_i)}_{\text{4 gauge vertices}} \underbrace{Y(x_1)Y(x_2)}_{\text{two rem. PCO}} \rangle \tag{7.2}$$

where in $\langle\rangle$ one have to include the 10D Heterotic partition function (5.27) in genus two format. Notice also that in this situation, one starts with the picture changing ansatz with all 4 gauge vertices in -1 picture. One next finds that for two-loop amplitude one has to insert $N = 2g - 2 + 4 = 6$ PCOs, 4 of which one makes coincide with 4 gauge vertices to put all the vertices in zero picture (5.32)

$$V_0^{a,\nu}(z,\bar{z}) = \frac{2g_{closed}}{l_s^2}\left[\partial X^\nu(z,\bar{z}) + ip.\psi(z)\psi^\nu(z)\right]\bar{J}^a(z,\bar{z})e^{ip.X(z,\bar{z})}. \tag{7.3}$$

There however remain two extra PCOs in arbitrary positions $x_1,\ x_2$. This ambiguity in the remaining PCOs need special care.

There are two ways to deal with this ambiguity. First is to use the hyper-elliptic formalism of string amplitude [83, 85, 86, 87] and second is the chiral splitting method of string partition function [91, 90, 139]. Let us outline both of them briefly.

Let us first start with the definition of hyper-elliptic surfaces. Suppose that a genus g compact orientable world-sheet topology $\Sigma$ has a double cover of spheres $S^1 \times S^1$ where z be the complex-coordinates on the first $S^1$ and y being the coordinates on the second. Then $\Sigma$ is the two-dimensional sub-manifold of $S^1 \times S^1$ given by the equation

$$y^2 = \prod_{i=1}^{2g+2}(z - Z_i) \tag{7.4}$$

for complex parameters $Z_i$. Notice that for g=1 & 2, such a hyper-elliptic definition (7.4) is available. In fact for genus 1 torus case, this gives the familiar Wierstass form (3.123) $y^2 = x^3 + fx + g$ where f and g are parameters (one would normally have 2g+2= 4th order polynomial in x but one can set one branch point $Z_i$ to infinity hence getting a cubic polynomial). In case $g \geq 3$, not all topologies allow for such a description, only the class called hyper-elliptic surfaces avail the privilege of such (7.4) description. The trick used in this format to tackle the position ambiguity in (7.2) is to coincide the extra PCO positions $x_1,\ x_2$ at a point x giving rise to the cumulative effect $I(x)$ which we shall state explicitly in due course. Then the integration over gauge vertex positions $\prod_{i=1}^4 \int \frac{d^2 z_i}{\pi}$ is cast in the following form using the hyper-elliptic form (7.4)

$$\int \prod_{i=1}^{4} \frac{(x - z_i)d^2 z_i}{y(z_i)} \tag{7.5}$$

thereby including the ambiguity in the position x inside the hyper-elliptic definition. The form of the $p^8$ order amplitude is then

$$\mathcal{A}_{TrF^4} = \int d\mu \prod_{i=1}^{4} \frac{(x - z_i)d^2 z_i}{y(z_i)} I(x)V^i(p_i, z_i, \bar{z}_i) \tag{7.6}$$

with $d\mu$ being the 10D string path-integral measure containing the partition $Z_{10}$ and gauge lattice $\Gamma_{16}$ (see for example (5.27)). One can then use the correlators in hyper-elliptic formalism to evaluate the amplitude.

The next ingenious method resolving the ambiguity is to use the chiral splitting method. At the heart of this method lies the complexity of the structure of the supermoduli space. We remind ourselves that the moduli space for a bosonic string interaction topology $\Sigma$ is (3.28)

$$\mathcal{M} = \frac{\mathcal{G}}{\text{Diff} \times \text{Weyl}} \tag{7.7}$$

with $\mathcal{G}$ being the space of all possible metrices $h_{\mu\nu}$ on $\Sigma$. In case we are considering the superstring interaction, the moduli space becomes the super-moduli space. To understand the essence of it, consider the Polyakov action for superstring (3.9)

$$S_P = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\eta_{\mu\nu}\left[\partial_\alpha X^\mu\partial_\beta X^\nu + \frac{i}{2}\bar{\psi}^\mu\partial\!\!\!/\psi^\nu\right] \tag{7.8}$$

which can be written in a more compact and supersymmetry coherent form using the following super-target-space coordinates

$$\mathbf{X}^\mu(\tau,\sigma) = X^\mu + i\theta\psi^\mu(\tau,\sigma) + i\bar{\theta}\tilde{\psi}^\mu(\tau,\sigma) + \theta\bar{\theta}F \tag{7.9}$$

with $\theta$, $\bar{\theta}$ being anti-commuting Grassmann variables which also define the super-derivatives

$$D_\theta = \partial_\theta + \theta\partial_\sigma, \tag{7.10}$$

$$D_{\bar{\theta}} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{\sigma}}. \tag{7.11}$$

The Polyakov action then becomes

$$S = \frac{1}{2\pi\alpha'}\int d^2\sigma d^2\theta D_{\bar{\theta}}\mathbf{X}^\mu D_\theta\mathbf{X}_\mu. \tag{7.12}$$

The equivalence of the two forms (7.8) and (7.12) is made evident using the fact that anti-commutation property of Grassmann integration $d^2\theta$ picks out the $\theta\bar{\theta}$ term in the expansion of the super-derivatives. In the question of diffeomorphism and Weyl invariance of the Polyakov action, one now has to include the transformation properties of the odd variables $\theta$, $\bar{\theta}$ too which are called the odd supermoduli. For the clarification of the discussion, let us note the transformation properties of $\sigma_i = (\tau, \sigma$ and $\theta$ coordinates on the world-sheet topology $\Sigma$: if $\sigma_m$, $\theta_m$ be coordinates on a patch m of the supermoduli-space and this patch has intersection, say with another patch n then in the intersection, the superconforal transformations are

$$\sigma_m = f_{mn}(\sigma_n) + \theta_n g_{mn}(\sigma_n)\omega_{mn}(\sigma_n), \tag{7.13}$$

$$\theta_m = g_{mn}(\sigma_n) + \theta_n\omega_{mn}(\sigma_n), \tag{7.14}$$

$$\omega^2_{mn}(\sigma_n) = \partial_\sigma f_{mn}(\sigma_n) + g_{mn}(\sigma_n)\partial_\sigma g_{mn}(\sigma_n) \tag{7.15}$$

where $\omega_{mn}(\sigma_n)$ is the spin-structure of $\Sigma$ and $f_{mn}$, $g_{mn}$ are even and odd transition functions respectively. The Faddeev-Popov gauge fixing method, in this formalism, means to choose a gauge slice such that $g_{mn}$ is set to zero. This leads to the transformation rules

$$\sigma_m = f_{mn}(\sigma_n), \;\; \theta_m = \theta_n\omega_{mn}(\sigma_n), \;\; \omega^2_{mn}(\sigma_n) = \partial_\sigma f_{mn}(\sigma_n). \tag{7.16}$$

The argument by D'Hoker & Phong [90, 139] states that in the absence of CKV, this gauge choice leads to the ambiguity of PCO paradigm as the loss of odd transition function $g_{mn}$ leads in turn to the non-integrability of the odd supermoduli $\theta$ which now shares equal footing in the super-moduli measure in the string path integral $d\mu$ in (7.6). The authors thus propose to construct the string amplitude from the chiral parts of both the path-integral measure and vertex operators, thereby retaining the the odd transition function for a given spin-structure on the world-sheet. This is the so-called chiral-splitting of the integral measure. In this formalism, one thus need not consider the insertion of BRST picture changing operators as the selection of chiral piece from the Green's function automatically restores the BRST invariance in addition to retaining its unitarity [140]. The method of chiral splitting is comparable in principle with the GSO projection in super-string spectrum construction which requires a systematic truncation of string excitations in order to have target space supersymmetry.

In our study of two-loop amplitude of Heterotic string theory we shall use the method of chiral splitting in the hyper-elliptic formalism discussed for example in the context of amplitude (7.6) so that we do not have to deal with the extraneous PCO contribution $I(x)$ which, though not ambiguous any more in hyper-elliptic formalism, yet does not guarantee the unitarity of the amplitude result. The admixture of chiral splitting method in hyper-elliptic language has been recorded for example by Stieberger & Taylor [83], Zheng, Wu & Zhu [147, 148, 149] to prove principally the non-renormalization of $R^4$ and $F^4$ terms. We complete their analysis by computing the 10 derivative correction terms $D^2R^4$, $D^2F^4$ terms in 10D Heterotic string theory which have been accomplished by D'Hoker & Phong [137] in the language of elliptic genus or Siegel modular forms in a quite abstract structure. We endeavour to propose such terms in hyper-elliptic language giving in a relatively simpler form.

## 7.2 Geometry and CFT on 2-Torus

The principal difficulty for the study of two-loop amplitude in chiral splitting formalism is to be accustomed with a new yet more severe plethora of definitions. We thus summarize the necessary definitions and CFT propagators in this section before we use the in the next section. We do not mean to be explicit in derivation of all the expressions which can be found in the standard references of Polchinki [27], Lerche, Schellekens & Warner [79] ,Morozov [85], D'Hoker & Phong [91]. For the necessary definitions of genus two elliptic functions, we refer to Mumford [150], Fay [151] and also D'Hoker & Phong [141]. We start our exploration by defining the so-called period matrix for a genus g closed orientable topology $\Sigma$. This is the direct generalization of the complex structure $\tau$ of the genus-1 torus. First, the hyper-elliptic description of 2-torus in terms of spherical double-cover yields

$$y(z)^2 = (z-u_1)(z-u_2)(z-u_3)(z-u_4)(z-u_5)(z-u_6) \tag{7.17}$$

with $u_i$ $i = 1, \cdots 6$ are the branch points on the double cover. One can split the total cover in terms of union of two branches A and B such that

$$r_A(z) = (z-u_1)(z-u_2)(z-u_3), \tag{7.18}$$
$$r_B(z) = (z-u_4)(z-u_5)(z-u_6). \tag{7.19}$$

An interesting fact of chiral splitting method is that the 6 picture changing operators could be inserted in the positions of these branch points to fix the ambiguity of the PCO positions. However, as this method does not appeal to the picture changing ansatz, one inserts 3 ghost vertices in say $r_A$ branch factors while gauge slice are chosen at $r_B$ branch points. We denote (according to the original reference [90] ) the first set of points as $p_i$, $i = 1, \cdots 3$ and the latter $q_i$.

Next, notice that one can chose a basis of 1-cycles on any genus g surfaces denoted by $A_i$ and $B_i$ (see figure 7.1 for the $a_1$, $a_2$, $b_1$ and $b_2$ cycles on genus-2 case). One can also define g Abelian differentials or holomorphic 1-forms $\Omega_i(z)$ (with $z = \tau + i\sigma$) such that

$$\partial_{\bar{z}}\Omega_i(z) = 0, \;\; \Omega_i(\bar{z}) = 0. \tag{7.20}$$

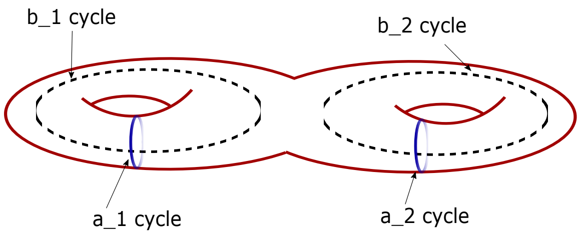


FIGURE 7.1: One cycles of 2-torus

The Riemann-Roch theorem (3.37) on the surface dictates that there are exactly g such 1-forms. The basis 1-cycles $A_i$ are chosen to be Poincaré dual to these 1-forms

$$\oint_{A_i} dz\Omega_j(z) = \delta_{ij}. \tag{7.21}$$

One then defines the period matrix $\tau_{ij}$

$$\tau_{ij} = \oint_{B_i} \Omega_j(z). \tag{7.22}$$

The period matrix $\tau_{ij}$ characterizes the complex structure of the closed oriented genus g topology. Just as the fundamental domain of $SL(2,\mathbb{Z})$ $\mathcal{F}$ was the space of all inequivalent complex structure of genus 1 case, in genus two case, this space is $Sp(4,\mathbb{Z})$. In genus two case, the 2 Abelian 1-forms are defined in the hyper-elliptic formalism (7.4) by

$$\Omega_1(z) = \frac{dz}{y(z)}, \quad \Omega_1(z) = \frac{zdz}{y(z)} \tag{7.23}$$

which are however not normalized according to (7.21). We thus make the change of basis with the matrix $K_{ij}$

$$K_{ij} = \oint_{a_i} \Omega_j(z) \tag{7.24}$$

which gives

$$\omega_i = K^{-1}_{ij}\Omega_j(z) \tag{7.25}$$

and now these $\omega_i(z)$ one-forms are the normalized Abelian differentials.

Next thing we explore is the spin-structure which are related to the spin-connection in (7.13). In our discussion of 1-loop string amplitude in 10D Heterotic theory in section 5.2 we have briefly described the spin-structure character (5.25). We now detail that description in order to use it also in genus-2 world-sheet topology. Recall that, in genus-1 torus case, the even and odd spin structures were according to the world-sheet Majorana-Weyl fermions are periodic (denote periodic case by 1 and P) or anti-periodic (denote it by 0 and A) over the basis 1-cycles $A_i$ and $B_i$ and the spin-structure character was represented by the genus-1 theta functions (see appendix B for more details)

$$\theta \left[ {a \atop b} \right] (\nu|\tau) = \sum_{n\in\mathbb{Z}} q^{(1/2)(n-a/2)^2} e^{2\pi i(\nu-b/2)(n-a/2)}. \tag{7.26}$$

Remember that in genus-1 case the (P,P) spin-structure was called odd because the character $\theta \left[ {1 \atop 1} \right] = 0$ identically.

The above can be easily generalized to the genus g case. We demonstrate the case for g=2 where the genus-2 $\theta$ functions are described by

$$\theta \left[ {a_1 \; a_2 \atop b_1 \; b_2} \right] (\nu_1, \nu_2|\tau_{ij}) = \sum_{n_i\in\mathbb{Z}^2} exp\left(i\pi(n_i+a_i)\tau_{ij}(n_j+a_j) + 2\pi i(n_i+a_i)(\nu_i+b_i)\right). \tag{7.27}$$

As demonstrated in [141], one can construct the whole elliptic genus as in section 5.2 with aid of these genus-2 elliptic functions (7.27). The difficulty however is that such functions does not allow for the integration of multiple correlators to some genus-2 modular functions (contrary to the genus-1 case where such integration yields Eisenstein series (5.47), (5.51)) and thus it is difficult to write down all the pieces of $D^2R^4$, $D^2(R^2)^2$, $D^2R^2F^2$, $D^2F^4$ terms once in a single generator. The hyper-elliptic formulation does not help either in this case as we shall see in the next section. To conclude the discussion of spin structure, we not that there are 10 even spin structures according to $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ periodic (1) and anti-periodic (0) combinations for $a_i$ and $b_i$ cycles

$$\begin{aligned} &\delta_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; \delta_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \; \delta_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; \delta_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ &\delta_5 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; \delta_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; \delta_7 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; \delta_8 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \\ &\qquad\qquad\qquad\qquad \delta_9 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; \delta_{10} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned} \tag{7.28}$$

There are also 6 odd spin-structures

$$\begin{aligned} &\epsilon_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \; \epsilon_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; \epsilon_3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \\ &\epsilon_4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; \epsilon_5 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; \epsilon_6 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned} \tag{7.29}$$

The reason behind of giving such details for spin-structure is that the chirally split measure and the fermionic propagators on the genus-2 surface depends upon these spin structures. However we shall only consider CP-even amplitude and shall not need the odd spin-structures $\epsilon_i$ much. The even spin-structure is however very important for our analysis and can be shown to be related to the branching (7.18) splitting of the double cover. One can associate a spin structure $\delta_i$ to the branch point separation $(u_{i_1}, u_{i_2}, u_{i_3}|u_{j_1}, u_{j_2}, u_{j_3})$ where $i_1, j_1, i_2, j_2, i_3, j_3$ are permutations of $u_1 \cdots u_6$. For the sake of completeness we provide the explicit form of this relation

$$\delta_1 \to (u_1, u_2, u_3|u_4, u_5, u_6), \quad \delta_2 \to (u_1, u_2, u_4|u_3, u_5, u_6), \quad \delta_3 \to (u_1, u_2, u_5|u_3, u_4, u_6), \tag{7.30}$$
$$\delta_4 \to (u_1, u_2, u_6|u_3, u_5, u_5), \quad \delta_5 \to (u_1, u_3, u_4|u_2, u_5, u_6), \quad \delta_6 \to (u_1, u_3, u_5|u_2, u_4, u_6), \tag{7.31}$$
$$\delta_7 \to (u_1, u_3, u_6|u_2, u_4, u_5), \quad \delta_8 \to (u_1, u_4, u_5|u_2, u_3, u_6), \quad \delta_9 \to (u_1, u_4, u_6|u_2, u_3, u_5), \tag{7.32}$$
$$\delta_{10} \to (u_1, u_5, u_6|u_2, u_3, u_4). \tag{7.33}$$

Our next step is to define the two-point Green's functions on the genus-2 surface in hyper-elliptic format. We remind ourselves once more that the superstring Polyakov action we are considering, is supplemented by a bc-ghost system and and a $\beta\gamma$- ghost system to apply the Faddeev-Popov gauge fixing process. The total CFT thus looks

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \left(\partial X^\mu \bar{\partial} X_\mu + \psi\bar{\partial}\psi + b\bar{\partial}c + \beta\bar{\partial}\gamma\right). \tag{7.34}$$

We thus need to consider the bosonic, fermionic as well as ghost correlators.

The spin-dependent fermionic correlator or the Szegö kernel takes the form

$$< \psi^\mu(z_1)\psi^\nu(z_2) >_\delta = -\eta^{\mu\nu} S_\delta(z_1, z_2) = -\eta^{\mu\nu} \frac{1}{2} \frac{1}{z_1 - z_2} \frac{r_\delta(z_1) + r_\delta(z_2)}{\sqrt{r_\delta(z_1) r_\delta(z_2)}}, \tag{7.35}$$

where

$$r_\delta(z) = \frac{(z - u_{i_1})(z - u_{i_2})(z - u_{i_3})}{y(z)} \tag{7.36}$$

using (7.17) and (7.18).

Next comes the scalar correlator: in the hyper-elliptic form on torus-2, it is of the form [152]

$$\begin{aligned} &< \partial X^\mu(z_1)\partial X^\nu(z_2) > = -\eta^{\mu\nu}\partial z_1 \partial z_2 ln E(z_1, z_2), \qquad (7.37)\\ &= -\eta^{\mu\nu}\left(\frac{1}{4(z_1 - z_2)^2} + \frac{1}{4T}\frac{\partial}{\partial z_2}\left[\frac{y(z_2)}{y(z_1)}\frac{1}{z_1 - z_2}\int_\Sigma d^2z d^2w \frac{(z_1 - z)(z_1 - w)|z - w|^2}{(z_2 - z)(z_2 - w)|y(z)y(w)|^2}\right]\right)\\ &+(z_1 \leftrightarrow z_2). \end{aligned}$$

In the above $E(z_1, z_2)$ (7.37) is known as the prime-form. We shall however seldom use the above correlator. Instead, we shall use the following important contraction which can be derived from above

$$< \partial X^\mu(z_1) k_\nu X^\nu(z_2) > = -\frac{k^\mu}{z_1 - z_2}. \tag{7.38}$$

Next we write down the bc-ghost propagator

$$< b(z_1)c(z_2) >_\delta = G_2[\delta](z_1, z_2) = -H(z_1, z_2) = -\frac{1}{2(z_1 - z_2)}(1 + \frac{y(z_1)}{y(z_2)})\frac{y(z_1)}{y(z_2)} \tag{7.39}$$

and $\beta\gamma-$ghost propagator

$$< \beta(z_1)\gamma(z_2) >_\delta = -G_{3/2}[\delta](z_1, z_2) = P(z_1, z_2) = \frac{\Omega(z_2)}{\Omega(z_1)} S_\delta(z_1, z_2). \tag{7.40}$$

It should be noted that the chiral splitting of integral measure and insertions of ghost vertices at 3 branch points $p_i$ along with choosing the gauge slice on the other 3 branch points $q_i$ modify the form of the bc and $\beta\gamma$ propagators (7.39), (7.40). We shall discuss those modifications after we discuss the gauge fixed integral measure which we are going to do in the next section.

## 7.3 Chiral measure for two-loop amplitude

Now we shall give the daunting expressions of the chiral measure following [90]. One particularity of the gauge fixing process in this chiral splitting formalism is that one chooses the six branch points $u_i$ of the hyperelliptic double cover of two-torus (7.17) to insert the PCO or equivalently one separates out $y^2 = r_A(p_i)r_B(q_i)$ into two disjoint branches and put the b-ghost operators on 3 points $p_i$ and $\beta-$ghosts on points $q_i$. However, using the gauge slice selection, one can coincide one of the three $q_i$ branch points with any of the other two and make the remaining $q_\alpha$, $\alpha = 1, 2$ points to live on two separate $S^1$ of the double cover of world-sheet $\Sigma$. From this process one gets the correlator $\mathcal{Z}$ between the chiral matter and superghosts

$$\mathcal{Z} = \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha))\rangle}{\det \omega_I \omega_J(p_a)} \tag{7.41}$$

where $\delta(\beta(q_\alpha))$ is the Dirac-delta functions to select the positions of $\beta$ ghosts on the gauge slice. We now give the expressions of the $G_2$ and $G_{3/2}$ propagators (7.39), (7.40) in case of particular choice of branch point gauge selection as discussed above. We find

$$G_2(z_1, z_2) = -H(z_1, z_2) + \sum_{\alpha=1,2} \sum_{a=1}^{3} H(z_2, p_a)\, \varpi_a(q_\alpha), \tag{7.42}$$

$$G_{3/2}[\delta](z_1, z_2) = -P(z_1, z_2) + P(z_2, q_1)\psi_1^*(z_1) + P(z_2, q_2)\psi_2^*(z_1), \tag{7.43}$$

with

$$\varpi_1(z) = -\frac{y^2(p_1)}{y^2(z)} \frac{(z-p_2)(z-p_3)}{(p_1-p_2)(p_1-p_3)}, \tag{7.44}$$

$$\varpi_2(z) = -\frac{y^2(p_2)}{y^2(z)} \frac{(z-p_1)(z-p_3)}{(p_2-p_1)(p_2-p_3)}, \tag{7.45}$$

$$\varpi_3(z) = -\frac{y^2(p_3)}{y^2(z)} \frac{(z-p_1)(z-p_2)}{(p_3-p_1)(p_3-p_2)}. \tag{7.46}$$

$\psi_\alpha^*(z)$'s are holomorphic $\frac{3}{2}$-differentials (similar to 1-forms $\Omega$ but constructed for the fermions)

$$\psi_\alpha^*(z) = (z - q_\alpha) S(z, q_\alpha) \frac{y(q_\alpha)}{y(z)}, \qquad \alpha = 1, 2. \tag{7.47}$$

A few more quantities are needed to define for the sake of the chiral measure. They are

$$f_{3/2}^{(1)}(q_2) = -\frac{\partial_{q_2} S(q_1, q_2)}{S(q_1, q_2)} + \partial\psi_2^*(q_2), \tag{7.48}$$

$$f_{3/2}^{(2)}(q_1) = \frac{\partial_{q_1} S(q_2, q_1)}{S(q_1, q_2)} + \partial\psi_1^*(q_1) = f_{3/2}^{(1)}(q_2)|_{q_1 \leftrightarrow q_2}, \tag{7.49}$$

and

$$\begin{aligned}\varpi_1^*(z) &= \frac{y(p_1)}{y(z)} \frac{(zp_1 - \frac{1}{2}(z+p_1)(p_2+p_3) + p_2 p_3)}{(p_1-p_2)(p_1-p_3)} \\ &= \frac{y(p_1)}{y(z)} \left[1 + \frac{1}{2}(z-p_1)\left(\frac{1}{p_1-p_2} + \frac{1}{p_1-p_3}\right)\right],\end{aligned} \tag{7.50}$$

$$\varpi_2^*(z) = \frac{y(p_2)}{y(z)} \frac{(zp_2 - \frac{1}{2}(z+p_2)(p_3+p_1) + p_1 p_3)}{(p_2-p_3)(p_2-p_1)}, \tag{7.51}$$

$$\varpi_3^*(z) = \frac{y(p_3)}{y(z)} \frac{(zp_3 - \frac{1}{2}(z+p_3)(p_1+p_2) + p_1 p_2)}{(p_3-p_1)(p_3-p_2)}. \tag{7.52}$$

With all these expressions behind we find the expression of the chiral measure

$$d\mu[\delta](\tau_{ij}) = \prod_{i\leq j} d\tau_{ij} \int_\Sigma d^2z \mathcal{Z}\mathcal{A}[\delta](\tau_{ij}) \tag{7.53}$$

which is spin-structure $\delta$ and the period matrix $\tau_{ij}$ dependent. The form of $\mathcal{A}[\delta](\tau_{ij})$ is given by

$$\mathcal{A}[\delta](\tau_{ij}) = 1 + \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 + \mathcal{X}_4 + \mathcal{X}_5 + \mathcal{X}_6 \tag{7.54}$$

with

$$\begin{aligned}
\mathcal{X}_1 + \mathcal{X}_6 &= \Big[ -\langle \psi(q_1)\cdot\partial X(q_1)\,\psi(q_2)\cdot\partial X(q_2)\rangle \\
&\quad -\partial_{q_1} G_2(q_1,q_2)\partial\psi_1^*(q_2) + \partial_{q_2} G_2(q_2,q_1)\partial\psi_2^*(q_1) \\
&\quad +2G_2(q_1,q_2)\partial\psi_1^*(q_2) f^{(1)}_{3/2}(q_2) - 2G_2(q_2,q_1)\partial\psi_2^*(q_1) f^{(2)}_{3/2}(q_1)\Big]\,, && (7.55)\\
\mathcal{X}_2 &= S_\delta(q_1,q_2)\sum_{a=1}^{3}\sum_{\alpha=1,2}\varpi_a(q_\alpha)\langle T(p_a)\rangle, && (7.56)\\
\mathcal{X}_3 &= 2S_\delta(q_1,q_2)\times\sum_{a=1}^{3}\sum_{\alpha=1,2}\varpi_a(q_\alpha)\Big[B_2(p_a) + B_{3/2}(p_a)\Big]\,, && (7.57)\\
\mathcal{X}_4 &= 2S_\delta(q_1,q_2)\sum_{a=1}^{3}\Big[\partial_{p_a}\partial_{q_1}\ln E(p_a,q_1)\varpi_a^*(q_2) \\
&\qquad +\partial_{p_a}\partial_{q_2}\ln E(p_a,q_2)\varpi_a^*(q_1)\Big]\,, && (7.58)\\
\mathcal{X}_5 &= \sum_{a=1}^{3}\sum_{\alpha=1,2}\Big[S_\delta(p_a,q_1)\partial_{p_a}S_\delta(p_a,q_2) \\
&\qquad -S_\delta(p_a,q_2)\partial_{p_a}S_\delta(p_a,q_1)\Big]\varpi_a(q_\alpha)\,. && (7.59)
\end{aligned}$$

In the above we also used the expressions

$$\begin{aligned}
B_2(w) &= -2\sum_{a=1}^{3}\partial_{p_a}\partial_w \ln E(p_a,w)\varpi_a^*(w)\,, && (7.60)\\
B_{3/2}(w) &= \sum_{\alpha=1}^{2}\Big(G_2(w,q_\alpha)\partial_{q_\alpha}\psi_\alpha^*(q_\alpha) + \frac{3}{2}\partial_{q_\alpha}G_2(w,q_\alpha)\psi_\alpha^*(q_\alpha)\Big)\,. && (7.61)
\end{aligned}$$

The expression $T(z)$ in (7.56) is the total stress tensor [27]

$$\begin{aligned}
T(z) &= -\frac{1}{2} : \partial_z X(z)\cdot\partial_z X(z) : +\frac{1}{2} : \psi(z)\cdot\partial_z\psi(z) : \\
&\quad - : (\partial bc + 2b\partial c + \frac{1}{2}\partial\beta\gamma + \frac{3}{2}\beta\partial\gamma)(z) : \\
&\equiv T_X(z) + T_\psi(z) + T_{bc}(z) + T_{\beta\gamma}(z)\,. && (7.62)
\end{aligned}$$

It may be noted that the form of the chiral measure can be written in terms of elliptic genus-2 modular forms according to

$$d\mu[\delta](\tau_{ij}) = d\mu_0[\delta](\tau_{ij}) + d\mu_2[\delta](\tau_{ij}) = \prod_{i\leq j} d\tau_{ij} \int_\Sigma d^2z \mathcal{Z}\mathcal{A}[\delta](\tau_{ij}) \tag{7.63}$$

which is spin-structure $\delta$ and the period matrix $\tau_{ij}$ dependent. The two distinct pieces $d\mu_0$ and $d\mu_2$ contribute differently to the amplitude. Their expressions in genus-2 elliptic format are

$$d\mu_0[\delta](\tau_{ij}) = \mathcal{Z}\prod_{i\leq j} d\tau_{ij}, \tag{7.64}$$

$$d\mu_2[\delta](\tau_{ij}) = \frac{\theta[\delta](0,\tau_{ij})^4 \Xi_6[\delta](\tau_{ij})}{16\pi^6 \chi_{10}(\tau_{ij})} \prod_{i\leq j} d\tau_{ij}. \tag{7.65}$$

In the above $\theta[\delta](0,\tau_{ij})$ is the genus-two theta functions defined in (7.27), $\chi_{10}(\tau_{ij}) = \prod_\delta \theta[\delta](0,\tau_{ij})^2$ is the weight 10 $Sp(4,\mathbb{Z})$ modular Siegel form (equivalent of j-function in genus 1 case). $\Xi_6[\delta](\tau_{ij})$ is a complicated form of weight 6 which satisfies the identity

$$\sum_\delta \theta[\delta](0,\tau_{ij})^4 \Xi_6[\delta](\tau_{ij}) = 2\sum_\delta \theta[\delta](0,\tau_{ij})^{16} - \frac{1}{2}(\sum_\delta \theta[\delta](0,\tau_{ij})^8)^2. \tag{7.66}$$

We shall leave the expression of the measure in the elliptic-genus-two format as this shall make the final form of the amplitude in terms of integration on the fundamental domain $\mathcal{F}_2$ of $Sp(4,\mathbb{Z})$ evident. Furthermore, this form is particularly suitable for deriving different vanishing identities similar to the Riemann identities in genus-1 case.
in the next section, we shall construct the 4-particle amplitude.

## 7.4 Ten-derivative terms from two-loop Heterotic string amplitude

The needed ingredient for the amplitude computation are the vertex functions which we take in zero picture

$$\text{Gravity:} \quad V_0^{\mu\nu}(z,\bar{z}) = \frac{2g_{closed}}{l_s^2}\left[\partial X^\mu(z,\bar{z}) + ik.\psi(z)\psi^\mu(z)\right]\bar{\partial}X^\nu(z,\bar{z})e^{ik.X(z,\bar{z})} \tag{7.67}$$

$$\text{Gauge:} \quad V_0^{a,\nu}(z,\bar{z}) = \frac{2g_{closed}}{l_s^2}\left[\partial X^\nu(z,\bar{z}) + ik.\psi(z)\psi^\nu(z)\right]\bar{J}^a(z,\bar{z})e^{ik.X(z,\bar{z})}. \tag{7.68}$$

It should be noted that the chiral projection of measure and partition function also bring up modifications to the vertex functions which we are neglecting in this analysis. Before discussing the functional form of the amplitude in functional form, let us discuss the kinematical factor for the 10 derivative amplitude. The 4-vertices are contracted according to

$$< \prod_{i=1}^4 e^i_{\alpha_i\beta_i} V^{\alpha_i\beta_i}(k_i, z_i, \bar{z}_i) > \tag{7.69}$$

with $e^i_{\alpha_i\beta_i}$ the polarization states and $k_i$ the momentum to reconstruct the Riemann tensor for the metric fluctuation or gauge fluctuation according to (5.34)

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left[e_{\alpha\gamma}k_\beta k_\delta - (\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) + \{(\alpha,\gamma) \leftrightarrow (\beta,\delta)\}\right]. \tag{7.70}$$

We wish to find terms of the form $D^2R^4$, $D^2(R^2)^2$, $D^2R^2F^2$, $D^2F^4$ from these amplitudes which are of 10 derivative level. Thus we need to extract momentum order $k^{10}$. From the structure of the vertex operators we find that order $k^4$ can be found from the $k.\psi(z)\psi^\mu(z)\bar\partial X^\nu(z,\bar z)e^{ik.X(z,\bar z)}$ piece from four vertices. The rest $k^4$ order can be retrieved from (compare (5.47) of section 5.2)

$$\begin{aligned}\prod_{i=1}^4\langle\int\frac{d^2z_i}{\pi}e_{\alpha_i\beta_i}V^{\alpha_i\beta_i}(p_i,z_i,\bar z_i)\rangle &= \int\prod_{i=1}^4\frac{d^2z_i}{\pi}\langle e_{\alpha_i\beta_i}\partial X^{\alpha_i}\bar\partial X^{\beta_i}e^{ip.X(z_i)}\rangle \\ &= \int\frac{d^2z_i}{\pi}\prod_{1\le i<j}^4 e_{\alpha_i\beta_i}g^{\alpha_i\beta_i}e^{-p_i.p_j lnE(i,j)/2}.\end{aligned} \tag{7.71}$$

We get the familiar $t_8$ tensor structure from the $e_{\alpha\gamma}k_\beta k_\delta$ contractions

$$\begin{aligned}t^{ijklmnpq} &= -\frac{1}{2}\epsilon^{ijklmnpq} - \frac{1}{2}[(g^{ik}g^{jl} - g^{il}g^{jk})(g^{mp}g^{nq} - g^{mq}g^{np}) \\ &+(g^{km}g^{ln} - g^{kn}g^{lm})(g^{pi}g^{qj} - g^{pj}g^{qi}) \\ &+(g^{im}g^{jn} - g^{in}g^{jm})(g^{kp}g^{lq} - g^{kq}g^{lp})] \\ &+\frac{1}{2}[g^{jk}g^{lm}g^{np}g^{qi} + g^{jm}g^{kn}g^{lp}g^{qi} + g^{jm}g^{np}g^{kq}g^{il} \\ &+45 \text{ more terms by anti-symmetrizing on}(ij),(kl),(mn),(pq)]\end{aligned} \tag{7.72}$$

and thus the $k^8$ order gives rise to the $t_8R^4$, $t_8F^4$ factors. The rest $k^2$ order is to be extracted from the exponential part $e^{ik.X(z,\bar z)}$ expanding upto the first order (the zero order expansion is in fact the two-loop $t_8R^4$ terms which are zero and thus gets no quantum correction at two loops). Thus we need to include the contraction

$$\sum_{i=1}^4\partial X^\nu(z_1,\bar z_1)k_i.X(z_2,\bar z_2) = -\sum_{i=1}^4\frac{k_i^\mu}{z_1 - z_2}. \tag{7.73}$$

From the $\mathcal{X}_i$ part in the chiral measure (7.55), there is a part $\partial X(q_1)\partial X(q_2)$ which contracts in fact with $k_i.X(z_2,\bar z_2)$ part of the vertices. Thus we shall get the modified version of (7.73)

$$\sum_{i=1}^4\partial X^\nu(q)k_i.X(z_i,\bar z_i) = -\sum_{i=1}^4\sum_{\alpha=1,2}\frac{k_i^\mu}{q_\alpha - z_i} \tag{7.74}$$

Thus for two $q_{1,2}$ one gets for two $\partial X^\nu(q)$ in (7.73) the following $k_2$ term

$$\sum_{\alpha=1,2}\sum_{i<j}^4\frac{k_i^\mu.k_j^\mu}{(q_\alpha - z_i)(q_\alpha - z_j)} = \frac{s(z_1z_2 + z_3z_4) + t(z_1z_3 + z_2z_4) + u(z_1z_4 + z_2z_3)}{\prod_{i=1}^4(q_1 - z_i)(q_2 - z_i)} \tag{7.75}$$

where we have used the standard definitions of the Mandelstam variable s, t, u

$$s = -2k_1.k_2 = -2k_3.k_4,\quad t = -2k_1.k_3 = -2k_2.k_4,\quad u = -2k_1.k_4 = -2k_2.k_3. \tag{7.76}$$

The 2-derivative s, t, u terms with 8-derivative $R^4$ terms are what called $D^2R^4$ terms in literature.

After having described the kinetic form of the amplitude, we shall deal with functional form of the amplitude. To this end, shall need to use various vanishing identities which are known as Fay's triscent identities [151] and can be proved in the elliptic formalism. We shall however take the expressions for these identities from [137] without proving them.

$$\begin{aligned} I_1 &= \sum_\delta \mathcal{Z}[\delta] S_\delta(q_1,z_1)S_\delta(z_1,z_2)S_\delta(z_2,z_3)S_\delta(z_3,z_4)S_\delta(z_4,q_2) = 0, &(7.77)\\ I_2 &= \sum_\delta \mathcal{Z}[\delta] S_\delta(q_1,z_1)S_\delta(z_1,z_2)S_\delta(z_2,q_2)S_\delta(z_3,z_4)^2 = 0, &(7.78)\\ I_3 &= \sum_\delta \mathcal{Z}[\delta] S_\delta(q_1,z_1)S_\delta(z_1,q_2)S_\delta(z_2,z_3)S_\delta(z_3,z_4)S_\delta(z_4,z_2) = 0, &(7.79)\\ I_4 &= \sum_\delta \mathcal{Z}[\delta] S_\delta(q_1,q_2)S_\delta(z_1,z_2)^2S_\delta(z_3,z_4)^2 = -2\prod_{i=1}^4 \varpi(z_i), &(7.80)\\ I_5 &= \sum_\delta \mathcal{Z}[\delta] S_\delta(q_1,q_2)S_\delta(z_1,z_2)S_\delta(z_2,z_3)S_\delta(z_3,z_4)S_\delta(z_4,z_1)\\ &= -2\prod_{i=1}^4 \varpi(z_i). &(7.81)\end{aligned}$$

An important corollary of the above identities is that the following contraction in $\mathcal{X}_1+\mathcal{X}_6$ in (7.55) vanishes

$$< \psi(q_1)\psi(q_2)\prod_{i=1}^4 \psi(z_i)\psi(z_i) >= 0. \qquad (7.82)$$

Another important quantity in the amplitudes will be the fermion stress tensor

$$\begin{aligned}\varphi[\delta](w;z_1,z_2) &= S_\delta(z_1,w)\partial_w S_\delta(w,z_2) - S_\delta(z_2,w)\partial_w S_\delta(w,z_1)\\ &= \frac{i}{4(w-z_1)(w-z_2)}\sum_{a=1}^3 \frac{1}{w-A_a} - \frac{1}{w-B_a} \qquad (7.83)\end{aligned}$$

according to the partition $(A_1,A_2,A_3|B_1,B_2,B_3)$ of branch points corresponding to the spin-structure (7.30).

For the gauge field $SO(32)$ or $E_8\times E_8$ we need the corresponding lattice structure in hyper-elliptic language. In elliptic format, the lattice structure is given by

$$\begin{aligned}\Gamma_{SO(32)} &= \frac{1}{2}\sum_\delta \theta[\delta]^{16}(0,\tau_{ij}), &(7.84)\\ \Gamma_{E_8\times E_8} &= \frac{1}{2}\left(\sum_\delta \theta[\delta]^8(0,\tau_{ij})\right)\left(\sum_\delta \theta[\delta]^8(0,\tau_{ij})\right). &(7.85)\end{aligned}$$

In hyper-elliptic format, genus-2 $\theta$ functions are given by

$$\theta[\delta](0,\tau_{ij}) = \prod_{i<j}(u_i-u_j)(v_i-v_j) = Q_\delta^{1/4} \qquad (7.86)$$

where $(u_1, u_2, u_3)$ and $(v_1, v_2, v_3)$ is a partition of the hyperelliptic branch point according to spin-structure (7.30). Thus one can use for lattice structure the following in hyper-elliptic format

$$\Gamma_{SO(32)} = \frac{1}{2}\sum_{\delta} Q_\delta^4, \tag{7.87}$$

$$\Gamma_{E_8\times E_8} = \frac{1}{2}(\sum_{\delta} Q_\delta^2)(\sum_{\delta} Q_\delta^2). \tag{7.88}$$

In the following subsections we shall evaluate the terms $t_8 trR^4$, $t_8 trR^2 trF^2$ and $t_8 trF^4$ terms separately from the 8 fermion correlators $< \prod_{i=1}^4 e^i_{\alpha_i} k.\psi(z)\psi^{\alpha_i}(z) >$ incorporating the pieces $\mathcal{A}[\delta](\tau_{ij}) = 1 + \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 + \mathcal{X}_4 + \mathcal{X}_5 + \mathcal{X}_6$ in (7.64) and the matter-ghost correlator $\mathcal{Z}$ (7.41) and simplify the terms using the identities (7.77), (7.78), (7.79), (7.80), (7.81), (7.82) and (7.83) etc. Finally we shall multiply them with the $k^2$ term (7.75)

$$\sum_{i<j}^{4} \frac{k_i^\mu . k_j^\mu}{(q-z_i)(q-z_j)} = \frac{s(z_1z_2+z_3z_4)+t(z_1z_3+z_2z_4)+u(z_1z_4+z_2z_3)}{\prod_{i=1}^4 (q_1-z_i)(q_2-z_i)} \tag{7.89}$$

along with $k^4$ term from $e^{-k_i.k_j lnE(i,j)/2}$ to get the respective $t_8D^2trR^4$, $t_8D^2trR^2trF^2$ and $t_8D^2trF^4$ terms.

### 7.4.1 $D^2t_8R^4$ term:

Let us begin by the $\mathcal{X}_1+\mathcal{X}_6$ term (7.55)

$$\begin{aligned}\mathcal{X}_1+\mathcal{X}_6 &= \frac{1}{16\pi^2}\Big[-\langle\psi(q_1)\cdot\partial X(q_1)\,\psi(q_2)\cdot\partial X(q_2)\rangle \\ &\quad -\partial_{q_1}G_2(q_1,q_2)\partial\psi_1^*(q_2)+\partial_{q_2}G_2(q_2,q_1)\partial\psi_2^*(q_1) \\ &\quad +2G_2(q_1,q_2)\partial\psi_1^*(q_2)f^{(1)}_{3/2}(q_2)-2G_2(q_2,q_1)\partial\psi_2^*(q_1)f^{(2)}_{3/2}(q_1)\Big].\end{aligned} \tag{7.90}$$

The total correlator with 8 fermionic correlators $< \prod_{i=1}^4 \psi(z_i)\psi(z_i) >$ from the vertex operators gives us

$$\begin{aligned}\langle\prod_{i=1}^4 k_i\cdot\psi(z_i)\epsilon_i\cdot\psi(z_i)\rangle &= K_1(S(z_1,z_2)S_\delta(z_3,z_4))^2 \\ &\quad +K_1(S_\delta(z_1,z_3)S(z_2,z_4))^2 \\ &\quad +K_1(S_\delta(z_1,z_4)S_\delta(z_2,z_3))^2 \\ &\quad +K_2S_\delta(z_1,z_2)S_\delta(z_2,z_3)S_\delta(z_3,z_4)S_\delta(z_4,z_1) \\ &\quad +K_2S_\delta(z_1,z_3)S_\delta(z_3,z_4)S_\delta(z_4,z_2)S_\delta(z_2,z_1) \\ &\quad +K_2S_\delta(z_1,z_4)S_\delta(z_4,z_2)S_\delta(z_2,z_3)S_\delta(z_3,z_1)\end{aligned} \tag{7.91}$$

where $K_1 = \frac{1}{4}K_2$ and $K_1$, $K_2$ are both constants.
Using this with $\mathcal{X}_1 + \mathcal{X}_6$ we get

$$\sum_\delta \mathcal{Z}_\delta \langle (\mathcal{X}_1 + \mathcal{X}_6) \prod_{i=1}^{4} e^i_{\alpha_i} k.\psi(z)\psi^{\alpha_i}(z)\rangle \tag{7.92}$$

$$= -\sum_\delta \mathcal{Z}_\delta \langle \psi(q_1)\psi(q_2) \prod_{i=1}^{4} e^i_{\alpha_i} k.\psi(z)\psi^{\alpha_i}(z)\rangle\langle \partial X(q_1)\partial X(q_2)\rangle$$
$$-(\partial_{q_1} G_2(q_1,q_2) + \partial_{q_2} G_2(q_2,q_1))$$
$$\times \sum_\delta \mathcal{Z}_\delta K_2 S_\delta(z_1,z_2)S_\delta(z_2,z_3)S_\delta(z_3,z_4)S_\delta(z_4,z_1) + \text{permutations}$$
$$-(\partial_{q_1} G_2(q_1,q_2) + \partial_{q_2} G_2(q_2,q_1))$$
$$\times \sum_\delta \mathcal{Z}_\delta K_1 (S_\delta(z_1,z_3)S(z_2,z_4))^2 + \text{permutations}$$
$$+2(G_2(q_1,q_2) + G_2(q_2,q_1))$$
$$\times \sum_\delta \mathcal{Z}_\delta (\partial\psi_1^*(q_1)S_\delta(q_1,q_2) - \partial_{q_2} S_\delta(q_1,q_2))$$
$$\times K_2 S_\delta(z_1,z_2)S_\delta(z_2,z_3)S_\delta(z_3,z_4)S_\delta(z_4,z_1) + \text{permutations}$$
$$+2(G_2(q_1,q_2) + G_2(q_2,q_1))$$
$$\times \sum_\delta \mathcal{Z}_\delta (\partial\psi_1^*(q_1)S_\delta(q_1,q_2) - \partial_{q_2} S_\delta(q_1,q_2)) K_1 (S_\delta(z_1,z_3)S(z_2,z_4))^2 + \text{permutations.}$$

The first term in the above correlator is zero according to (7.82)

$$< \psi(q_1)\psi(q_2) \prod_{i=1}^{4} \psi(z_i)\psi(z_i) >= 0. \tag{7.93}$$

The next lines can be simplified using (7.80) and (7.81) and we get the final expression for the $\mathcal{X}_1 + \mathcal{X}_6$ part of the $k^4$ amplitude factor

$$\Delta_1(z_1,z_2,z_3,z_4,q_\alpha,p_a) + \Delta_6(z_1,z_2,z_3,z_4,q_\alpha,p_a) \tag{7.94}$$
$$= K_2 \left[ \sum_{\alpha=1,2} \frac{1}{q_\alpha - p_1} \frac{(q_\alpha - p_2)(q_\alpha - p_3)}{(p_1 - p_2)(p_1 - p_3)} + \text{permutations of } p_i \right] \prod_{i=1}^{4} \varpi(z_i)$$
$$\times t_8 \left( trR^4 + \frac{1}{4}(trR^2)^2 \right). \tag{7.95}$$

We next consider the $\mathcal{X}_5$ piece in the chiral measure. Its correlation with 8 fermionic correlators $< \prod_{i=1}^4 \psi(z_i)\psi(z_i) >$gives

$$\mathcal{X}_5 < \prod_{i=1}^{4} e^i_{\alpha_i} k.\psi(z)\psi^{\alpha_i}(z) > \tag{7.96}$$

$$\begin{aligned} = \quad & \sum_\delta \sum_{a=1}^{3} \mathcal{Z}_\delta \varphi[\delta](p_a; q_1, q_2)\varpi_a(q_1, q_2) \\ & \times K_2 S_\delta(z_1, z_2)S_\delta(z_2, z_3)S_\delta(z_3, z_4)S_\delta(z_4, z_1) + \text{permutations} \\ & + \sum_\delta \sum_{a=1}^{3} \mathcal{Z}_\delta \varphi[\delta](p_a; q_1, q_2)\varpi_a(q_1, q_2) \\ & \times K_1(S_\delta(z_1, z_3)S(z_2, z_4))^2 + \text{permutations}. \end{aligned} \tag{7.97}$$

Using the expression for the fermionic stress tensor (7.83) however taking care of the poles as $A_i \to p_a$ and $B_i \to q_\alpha$ we get the $\mathcal{X}_5$ amplitude factor

$$\Delta_5(z_1, z_2, z_3, z_4, q_\alpha, p_a) = K_2 \sum_{a=1}^{3} \frac{\varpi(q_1)\varpi(q_2)\prod_{i=1}^4 \varpi(z_i)}{(q_1 - p_a)(q_2 - p_a)} \times t_8 \left( trR^4 + \frac{1}{4}(trR^2)^2 \right). \tag{7.98}$$

We next look at the $\mathcal{X}_4$ term (7.58)

$$\mathcal{X}_4 = 2S_\delta(q_1, q_2) \sum_{a=1}^{3} \Big[ \partial_{p_a}\partial_{q_1} \ln E(p_a, q_1)\varpi^*_a(q_2) + \partial_{p_a}\partial_{q_2} \ln E(p_a, q_2)\varpi^*_a(q_1) \Big] \tag{7.99}$$

which contains $S_\delta(q_1, q_2)$ term inside. The contraction with the 8 fermion term $< \prod_{i=1}^4 \psi(z_i)\psi(z_i) >$ then easily gives us the following by the application of (7.80) and (7.81)

$$\Delta_4(z_1, z_2, z_3, z_4, q_\alpha, p_a) \tag{7.100}$$

$$\begin{aligned} &= 2K_2 \sum_{a=1}^{3} \Big[ \partial_{p_a}\partial_{q_1} \ln E(p_a, q_1)\varpi^*_a(q_2) + \partial_{p_a}\partial_{q_2} \ln E(p_a, q_2)\varpi^*_a(q_1) \Big] \\ & \quad \prod_{i=1}^{4} \varpi(z_i) \times t_8 \left( trR^4 + \frac{1}{4}(trR^2)^2 \right). \end{aligned} \tag{7.101}$$

For the purpose of simplifying the algebraic computations, we shall use the leading singular terms in the expression of $\partial_{p_a}\partial_{q_1} \ln E(p_a, q_1)\varpi^*_a(q_2)$ etc which shall give

$$\partial_{p_a}\partial_{q_1} \ln E(p_a, q_1)\varpi^*_a(q_2) \sim \frac{\varpi^*_a(q_2)}{4(q_1 - p_a)^2}. \tag{7.102}$$

Thus the expression for the $\mathcal{X}_4$ part can be recast in the form

$$\Delta_4(z_1, z_2, z_3, z_4, q_\alpha, p_a) \tag{7.103}$$

$$\begin{aligned} &= 2K_2 \sum_{a=1}^{3} \left[ \frac{\varpi^*_a(q_2)}{4(q_1 - p_a)^2} + \frac{\varpi^*_a(q_1)}{4(q_2 - p_a)^2} \right] \\ & \quad \prod_{i=1}^{4} \varpi(z_i) \times t_8 \left( trR^4 + \frac{1}{4}(trR^2)^2 \right). \end{aligned}$$

Next piece comes from the $\mathcal{X}_3$ part of the chiral measure (7.57) given by

$$\mathcal{X}_3 = 2S_\delta(q_1, q_2) \times \sum_{a=1}^{3} \varpi_a(q) \Big[ B_2(p_a) + B_{3/2}(p_a) \Big] \tag{7.104}$$

where the expressions for $B_2$ and $B_{3/2}$ are given by

$$B_2(w) = -2\sum_{a=1}^{3} \partial_{p_a}\partial_w \ln E(p_a, w)\varpi_a^*(w)\,, \tag{7.105}$$

$$B_{3/2}(w) = \sum_{\alpha=1}^{2}\Big( G_2(w, q_\alpha)\partial_{q_\alpha}\psi_\alpha^*(q_\alpha) + \frac{3}{2}\partial_{q_\alpha} G_2(w, q_\alpha)\psi_\alpha^*(q_\alpha) \Big)\,. \tag{7.106}$$

For the sake of simplicity of the algebra we again use the leading order singularity of $\partial_{p_a}\partial_w \ln E(p_a, w)\varpi_a^*(w)$ (note that we cannot directly use the approximation (7.102) as the $p_a$s are coincident)

$$\partial_{p_a}\partial_{p_b} \ln E(p_a, p_b)\varpi_a^*(p_b) \sim \frac{y(p_b)}{y(p_a)}. \tag{7.107}$$

Note that, apparent singularity arising from $\partial_{p_a}\partial_{p_a}\ln E(p_a, p_a)$ is in fact an artefact of the hyper-elliptic formalism which sets the branch points as ghost insertion point. One can in principle renormalize those singularities by using appropriate delta function manipulation. In the elliptic formalism such singularities does not occur and in fact can be shown in terms of measures $d\mu_0[\delta](\tau_{ij})$ and $d\mu_2[\delta](\tau_{ij})$ (7.64) that the term in question comes about with the measure $d\mu_0[\delta](\tau_{ij})$ which can be recombined with pieces coming from $\mathcal{X}_2$ and can be recast in terms of $d\mu_2[\delta](\tau_{ij})$ measure which allows for further summation identities in terms of genus-two modular functions. In a similar manner, we can expand $B_{3/2}(p_a)$ in the vicinity $p_a \to q_{1,2}$ to get

$$\begin{aligned} B_{3/2} = &\sum_{a=1}^{3}\sum_{\alpha=1,2}\left[ \frac{3/2}{(p_a - q_\alpha)^2} + \frac{\partial_q \varpi_a^*(q_\alpha)}{(p_a - q_\alpha)} \right] \\ &- \sum_{\alpha=1,2}\left( \frac{3}{2(p_1 - q_\alpha)^2}\frac{(q_\alpha - p_2)(q_\alpha - p_3)}{(p_1 - p_2)(p_1 - p_3)} + \text{ permutations of } p_a \right) \end{aligned} \tag{7.108}$$

Combining the results from (7.107), (7.108) and using the combination of $S_\delta(q_1, q_2)$ term of $\mathcal{X}_3$ with the 8-fermion correlator and use identities (7.80) and (7.81) to get

$$\begin{aligned} &\Delta_3(z_1, z_2, z_3, z_4, q_\alpha, p_a) \\ &= 2K_2 \Bigg\{ \sum_{a=1}^{3}\sum_{\alpha=1,2}(-\frac{3}{8})\frac{1}{(p_a - q_\alpha)^2} - \frac{1}{16}(\sum_{a=1}^{3}\sum_{\alpha=1,2}\frac{\varpi_a^*(q_\alpha)}{(p_a - q_\alpha)})^2 \\ &\quad - \sum_{\alpha=1,2}\frac{1}{2}\left\{ \frac{1}{(q_\alpha - p_1)^2}\left[ \frac{y(p_2)}{y(p_3)} + 3(q_\alpha - p_1)\left( \frac{1}{q_\alpha - p_2} + \frac{1}{q_\alpha - p_3} \right) \right] \right. \\ &\quad \left. \times \frac{(q_\alpha - p_2)(q_\alpha - p_3)}{(p_1 - p_2)(p_1 - p_3)} + \text{ permutations of } p_a \right\} \Big] \Bigg\} \\ &\quad \prod_{i=1}^{4}\varpi(z_i) \times t_8 \left( trR^4 + \frac{1}{4}(trR^2)^2 \right) \end{aligned} \tag{7.109}$$

Finally we compute the $\mathcal{X}_2$ piece whose expression we rewrite for convenience (7.56)

$$\mathcal{X}_2 = S_\delta(q_1,q_2)\sum_{a=1}^{3}\varpi_a(q)\langle T(p_a)\rangle \tag{7.110}$$

with the total stress tensor

$$\begin{aligned} T(z) &= -\frac{1}{2}:\partial_z X(z)\cdot\partial_z X(z):+\frac{1}{2}:\psi(z)\cdot\partial_z\psi(z): \\ &\quad -:(\partial bc+2b\partial c+\frac{1}{2}\partial\beta\gamma+\frac{3}{2}\beta\partial\gamma)(z): \\ &\equiv T_X(z)+T_\psi(z)+T_{bc}(z)+T_{\beta\gamma}(z)\,. \end{aligned} \tag{7.111}$$

Similar to the case of $\mathcal{X}_3$ and $\mathcal{X}_4$ we also expand the individual stress tensors in the limit $p_a\to q_{1,2}$ to get the following (hiding once more the singular terms)

$$T_X(p_a)\sim\sum_{\alpha=1,2}\frac{-1}{4(p_a-q_\alpha)^2}, \tag{7.112}$$

$$T_\psi(p_a)\sim\sum_{\alpha=1,2}\frac{5}{8(p_a-q_\alpha)^2}-\frac{5}{32}\frac{\partial\psi_1^*(q_\alpha)}{p_a-q_\alpha}, \tag{7.113}$$

$$T_{bc}(p_a)\sim\frac{5}{16}\left(\sum_{\alpha=1,2}\frac{1}{(p_a-q_\alpha)}\right)^2-\frac{9}{8}\sum_{\alpha=1,2}\frac{1}{(p_a-q_\alpha)^2}+\sum_{\alpha=1,2}\frac{\varpi_a^*(q_\alpha)}{(q_\alpha-p_a)^2}, \tag{7.114}$$

$$T_{\beta\gamma}(p_a)\sim-\sum_{\alpha=1,2}\frac{3/2}{(p_a-q_\alpha)^2}-\frac{\partial\psi_1^*(q_\alpha)}{p_a-q_\alpha}-\frac{1}{8}\left(\sum_{\alpha=1,2}\frac{1}{(p_a-q_\alpha)}\right)^2. \tag{7.115}$$

Once again combining $S_\delta(q_1,q_2)$ term of $\mathcal{X}_3$ with the 8-fermion correlator and use identities (7.80) and (7.81) with the expressions of stress tensors (7.112) above we get

$$\begin{aligned} &\Delta_2(z_1,z_2,z_3,z_4,q_\alpha,p_a) \\ &=2K_2\Bigg\{\sum_{a=1}^{3}\sum_{\alpha=1,2}(-\frac{33}{16})\frac{1}{(p_a-q_\alpha)^2}+(\sum_{a=1}^{3}\sum_{\alpha=1,2}\frac{\varpi_a^*(q_\alpha)}{(p_a-q_\alpha)})^2-\frac{37}{32}\frac{\partial\psi_1^*(q_\alpha)}{p_a-q_\alpha} \\ &\quad+\sum_{\alpha=1,2}\frac{3}{16}\Bigg\{\frac{1}{(q_\alpha-p_1)^2}\Bigg[1+3(q_\alpha-p_1)\left(\frac{1}{q_\alpha-p_2}+\frac{1}{q_\alpha-p_3}\right)\Bigg] \\ &\quad\times\frac{(q_\alpha-p_2)(q_\alpha-p_3)}{(p_1-p_2)(p_1-p_3)}+\text{permutations of }p_a\Bigg\}\Bigg]\Bigg\} \\ &\quad\prod_{i=1}^{4}\varpi(z_i)\times t_8\left(trR^4+\frac{1}{4}(trR^2)^2\right) \end{aligned} \tag{7.116}$$

The final amplitude factor will be the sum of the factors $\sum_{j=1}^{6}\Delta_j$ from (7.94), (7.116), (7.109), (7.103) and (7.98) times the $k^4$ factor from the term $\prod_{i<j=1}^{4}e^{-k_i.k_j lnE(z_i,z_j)}$ times $\frac{s(z_1z_2+z_3z_4)+t(z_1z_3+z_2z_4)+u(z_1z_4+z_2z_3)}{\prod_{i=1}^{4}(q_1-z_i)(q_2-z_i)}$ term to get

$$\mathcal{A}(R)=K_2\frac{s(z_1z_2+z_3z_4)+t(z_1z_3+z_2z_4)+u(z_1z_4+z_2z_3)}{\prod_{i=1}^{4}(q_1-z_i)(q_2-z_i)}\prod_{i<j=1}^{4}e^{-k_i.k_j lnE(z_i,z_j)}\sum_{j=1}^{6}\Delta_j. \tag{7.117}$$

This term should then be integrated over all vertex positions $z_i$, $i = 1\cdots 4$ and then incorporating the integration over $Sp(4,\mathbb{Z})$ fundamental domain $\mathcal{F}_2$ and the integration over genus-two world-sheet $\Sigma$ with measure

$$d\mu = \frac{\prod_{a=1}^{3} d^2p_a|q_1-q_2|^2}{|(p_1-p_2)(p_2-p_3)(p_1-p_3)|^2\prod_{a=1}^{3}\prod_{\alpha=1}^{2}|p_a-q_\alpha|^2} \tag{7.118}$$

to get the final amplitude

$$\begin{aligned}
&\Delta(D^2R^4)\\
&= K_2\int_{\mathcal{F}_2}|\prod_{I<J} d\tau_{IJ}|^2\int_{\Sigma} d\mu\prod_{i=1}^{4} d^2z_i\frac{s(z_1z_2+z_3z_4)+t(z_1z_3+z_2z_4)+u(z_1z_4+z_2z_3)}{\prod_{i=1}^{4}(q_1-z_i)(q_2-z_i)}\\
&\qquad e^{-\sum_{i<j}k_i.k_j lnE(z_i,z_j)}\sum_{j=1}^{6}\Delta_j\times t_8\left(trR^4+\frac{1}{4}(trR^2)^2\right). 
\end{aligned}\tag{7.119}$$

In the above we have written the tensor structure $t_8\left(trR^4+\frac{1}{4}(trR^2)^2\right)$ explicitly from $\Delta_i$.

### 7.4.2 $D^2trF^4$ term:

We shall closely follow the expressions of various pieces of $\mathcal{X}_i$ from the previous subsection now to calculate the pure gauge amplitude for which we have to use the vertex

$$V_0^{a,\nu}(z,\bar z) = \frac{2g_{closed}}{l_s^2}\left[\partial X^\nu(z,\bar z)+ik.\psi(z)\psi^\nu(z)\right]\bar J^a(z,\bar z)e^{ik.X(z,\bar z)}. \tag{7.120}$$

Once again, we shall try to evaluate the amplitude on 10 order of momentum the $k^4$ order of which is found from the 8 fermion correlator $<\prod_{i=1}^{4}\psi(z_i)\psi(z_i)\bar J(z_i)>$. This correlator can be expanded in terms of Szegö kernel as

$$\begin{aligned}
\langle\prod_{i=1}^{4} k_i\cdot\psi(z_i)\epsilon_i\cdot\psi(z_i)\bar J(z_i)\rangle \;=\; & C_1(1,2)C_1(3,4)(S(z_1,z_2)S_\delta(z_3,z_4))^2 \\
&+C_1(1,3)C_1(2,4)(S_\delta(z_1,z_3)S(z_2,z_4))^2\\
&+C_1(1,4)C_1(2,3)(S_\delta(z_1,z_4)S_\delta(z_2,z_3))^2\\
&+C_2S_\delta(z_1,z_2)S_\delta(z_2,z_3)S_\delta(z_3,z_4)S_\delta(z_4,z_1)\\
&+C_2S_\delta(z_1,z_3)S_\delta(z_3,z_4)S_\delta(z_4,z_2)S_\delta(z_2,z_1)\\
&+C_2S_\delta(z_1,z_4)S_\delta(z_4,z_2)S_\delta(z_2,z_3)S_\delta(z_3,z_1)
\end{aligned}\tag{7.121}$$

where

$$C_1(1,2)=\frac{1}{2}tr(T^{a_1}T^{a_2}),\;\; C_2=-tr(T^{a_1}T^{a_2}T^{a_3}T^{a_4}), \tag{7.122}$$

with $T^{a_1}$ being the representation matrices of either $SO(32)$ or $E_8\times E_8$. Note that we also have to use the gauge lattices $\Gamma_{16}$ (7.87), (7.88) into the amplitude factors

$$\Gamma_{SO(32)} \;=\; \frac{1}{2}\sum_{\delta} Q_\delta^4, \tag{7.123}$$

$$\Gamma_{E_8\times E_8} \;=\; \frac{1}{2}(\sum_{\delta} Q_\delta^2)(\sum_{\delta} Q_\delta^2). \tag{7.124}$$

Their incorporation hinders the application of the identities (7.80) and (7.81). We thus need the following functions (according to [137])

$$F_4^{2,2}(z_1,z_2;z_3,z_4) = \sum_\delta S_\delta(q_1,q_2)Q_\delta^4(S_\delta(z_1,z_2)S_\delta(z_3,z_4))^2, \tag{7.125}$$

$$F_4^4(z_1,z_2,z_3,z_4) = \sum_\delta S_\delta(q_1,q_2)Q_\delta^4 S_\delta(z_1,z_2)S_\delta(z_2,z_3)S_\delta(z_3,z_4)S_\delta(z_4,z_1), \tag{7.126}$$

$$F_2^2(z_1,z_2) = \sum_\delta S_\delta(q_1,q_2)Q_\delta^2 S(z_1,z_2)^2. \tag{7.127}$$

The first two functions (7.125) and (7.126) are to be used for SO(32) gauge theory where as the last one (7.127) for the $E_8\times E_8$ gauge group. Thus in the expressions of the amplitude factors (7.94), (7.116), (7.109), (7.103) and (7.98) of the previous subsection, we need to replace $\prod_{i=1}^4 \varpi(z_i)\times t_8\left(trR^4+\frac{1}{4}(trR^2)^2\right)$ factor by

$$F^4 = \left(C_1(1,2)C_1(3,4)F_4^{2,2}(z_1,z_2;z_3,z_4) + \text{ permutations } z_i\right)(trF^2)^2 + \left(C_2F_4^4\right)trF^4 \tag{7.128}$$

for SO(32) case and by

$$F^4 = \left(C_1(1,2)C_1(3,4)F_2^2(z_1,z_2)F_2^2(z_3,z_4) + \text{ permutations } z_i\right)[(trF_1^2)^2+(trF_2^2)^2] + \left(C_1(1,2)C_1(3,4)F_2^2(z_1,z_2)F_2^2(z_3,z_4) + \text{ permutations } z_i\right)(trF_1^2)(trF_2^2) \tag{7.129}$$

for $E_8\times E_8$. We shall call these terms $F^4$ collectively. The rest $k^4\times k^2$ order momentum can be retrieved from the exponential factor $\prod_{i<j=1}^4 e^{-k_i.k_j lnE(z_i,z_j)}$. The total $k^8$ order shall be used for the $t_8F^4$ factor. For convenience, we repeat here the $\Delta_i$ amplitude factors following the previous subsection.

$$\Delta_1(z_1,z_2,z_3,z_4,q_\alpha,p_a)+\Delta_6(z_1,z_2,z_3,z_4,q_\alpha,p_a) \tag{7.130}$$

$$= \left[\sum_{\alpha=1,2}\frac{1}{q_\alpha-p_1}\frac{(q_\alpha-p_2)(q_\alpha-p_3)}{(p_1-p_2)(p_1-p_3)} + \text{permutations of } p_i\right] t_8F^4. \tag{7.131}$$

$$\Delta_2(z_1,z_2,z_3,z_4,q_\alpha,p_a) \tag{7.132}$$

$$= 2\Bigg\{\sum_{a=1}^{3}\sum_{\alpha=1,2}(-\frac{33}{16})\frac{1}{(p_a-q_\alpha)^2} + (\sum_{a=1}^{3}\sum_{\alpha=1,2}\frac{\varpi_a^*(q_\alpha)}{(p_a-q_\alpha)})^2 - \frac{37}{32}\frac{\partial\psi_1^*(q_\alpha)}{p_a-q_\alpha} + \sum_{\alpha=1,2}\frac{3}{16}\left\{\frac{1}{(q_\alpha-p_1)^2}\left[1+3(q_\alpha-p_1)\left(\frac{1}{q_\alpha-p_2}+\frac{1}{q_\alpha-p_3}\right)\right] \times \frac{(q_\alpha-p_2)(q_\alpha-p_3)}{(p_1-p_2)(p_1-p_3)} + \text{ permutations of } p_a\right\}\Bigg]\Bigg\} t_8F^4.$$

$$\Delta_3(z_1, z_2, z_3, z_4, q_\alpha, p_a) \tag{7.133}$$
$$= 2\left\{\sum_{a=1}^{3}\sum_{\alpha=1,2}(-\frac{3}{8})\frac{1}{(p_a-q_\alpha)^2} - \frac{1}{16}(\sum_{a=1}^{3}\sum_{\alpha=1,2}\frac{\varpi_a^*(q_\alpha)}{(p_a-q_\alpha)})^2\right.$$
$$-\sum_{\alpha=1,2}\frac{1}{2}\left\{\frac{1}{(q_\alpha-p_1)^2}\left[\frac{y(p_2)}{y(p_3)}+3(q_\alpha-p_1)\left(\frac{1}{q_\alpha-p_2}+\frac{1}{q_\alpha-p_3}\right)\right]\right.$$
$$\left.\left.\times\ \frac{(q_\alpha-p_2)(q_\alpha-p_3)}{(p_1-p_2)(p_1-p_3)} + \text{ permutations of } p_a\right\}\Bigg]\right\} t_8F^4.$$

$$\Delta_4(z_1, z_2, z_3, z_4, q_\alpha, p_a) = 2\sum_{a=1}^{3}\left[\frac{\varpi_a^*(q_2)}{4(q_1-p_a)^2}+\frac{\varpi_a^*(q_1)}{4(q_2-p_a)^2}\right]t_8F^4. \tag{7.134}$$

$$\Delta_5(z_1, z_2, z_3, z_4, q_\alpha, p_a) = \sum_{a=1}^{3}\frac{\varpi(q_1)\varpi(q_2)}{(q_1-p_a)(q_2-p_a)}t_8F^4. \tag{7.135}$$

The final amplitude will be

$$\Delta(D^2F^4)$$
$$= \int_{\mathcal{F}_2}|\prod_{I<J}d\tau_{IJ}|^2\int_\Sigma d\mu\prod_{i=1}^{4}d^2z_i\frac{s(z_1z_2+z_3z_4)+t(z_1z_3+z_2z_4)+u(z_1z_4+z_2z_3)}{\prod_{i=1}^4(q_1-z_i)(q_2-z_i)}$$
$$e^{-\sum_{i<j}k_i.k_j lnE(z_i,z_j)}\sum_{j=1}^{6}\Delta_j t_8F^4. \tag{7.136}$$

Once again, we have written the tensor structure $t_8F^4$ explicitly from $\Delta_i$.

### 7.4.3 $D^2R^2F^2$ term:

The last thing we need to compute is the mixed gauge-gravity part $D^2R^2F^2$. The analysis of the two previous subsections will be used except for the fact that the 8 fermion contraction term $< \prod_{i=1}^4 \psi(z_i)\psi(z_i) >$ shall now contain 4 fermions coming from 2 graviton vertices and the rest comes from 2 gauge vertices. We shall thus have the $k^4$ order 8 fermion contraction

$$< \prod_{i=1}^{2}\psi(z_i)\psi(z_i) >< \prod_{j=1}^{2}\psi(z_j)\psi(z_j)\bar{J}(z_j) >= \tag{7.137}$$
$$K_1C_1(z_3,z_4)S_\delta(z_1,z_2)^2S_\delta(z_3,z_4)^2 + \text{ permutations of } z_i$$

This contraction shall give us $k^4$ order in momentum. Combining the $S_\delta(z_1,z_2)^2S_\delta(z_3,z_4)^2$ factor of the Szegö kernel, we shall use the function $F_4^{2,2}$ (7.125)

$$F_4^{2,2}(z_1,z_2;z_3,z_4) = \sum_\delta S_\delta(q_1,q_2)Q_\delta^4(S_\delta(z_1,z_2)S_\delta(z_3,z_4))^2 \tag{7.138}$$

for $trR^2trF^2$ term in case of $SO(32)$ gauge group. For the $E_8 \times E_8$ gauge group we shall use function $F_2^2$ (7.127) for each of the $E_8$ factor

$$F_2^2(z_1, z_2) = \sum_\delta S_\delta(q_1, q_2)Q_\delta^2 S(z_1, z_2)^2. \tag{7.139}$$

From these we shall have the following mix gauge-gravity term

$$R^2F^2 = \Big(C_1(1,2)C_1(3,4)F_4^{2,2}(z_1, z_2; z_3, z_4) + \text{ permutations } z_i\Big)\,(trR^2trF^2) \tag{7.140}$$

for SO(32) case and

$$R^2F^2 = \big(C_1(1,2)C_1(3,4)F_2^2(z_1, z_2)F_2^2(z_3, z_4) + \text{ permutations } z_i\big)\,[(trF_1^2)+(trF_2^2)]trR^2. \tag{7.141}$$

We shall call the above terms collectively $R^2F^2$. One point of interest needs special mention. One can in principle, combine one $S(z_1, z_2)^2$ piece with $Q_\delta^2$ or $Q_\delta^4$ and multiply the rest $S_\delta(z_3, z_4)^2$ factor with $\mathcal{Z}[\delta]$ to get the spin-structure sum $\sum_\delta \mathcal{Z}[\delta]S_\delta(z_3, z_4)^2 \times \sum_\delta Q_\delta$ which is identically zero. Thus the above (7.140) and (7.141) are the only non-trivial $R^2F^2$ forms. This is however different in genus one case.

From the above contractions we get $k^4$ order in momentum. The rest $k^4 \times k^2$ order momentum can be retrieved from the exponential factor $\prod_{i<j=1}^4 e^{-k_i.k_j lnE(z_i,z_j)}$. The total $k^8$ order shall be used for the $t_8trR^2trF^2$ factor. As in case of $F^4$, the amplitude factors (7.94), (7.116), (7.109), (7.103) and (7.98) of the pure gauge amplitude, we need to replace $\prod_{i=1}^4 \varpi(z_i) \times t_8\left(trR^4 + \frac{1}{4}(trR^2)^2\right)$ factor by $R^2F^2$. We give the amplitude factors once again for convenience.

$$\Delta_1(z_1, z_2, z_3, z_4, q_\alpha, p_a) + \Delta_6(z_1, z_2, z_3, z_4, q_\alpha, p_a) \tag{7.142}$$

$$= \left[\sum_{\alpha=1,2} \frac{1}{q_\alpha - p_1}\frac{(q_\alpha - p_2)(q_\alpha - p_3)}{(p_1 - p_2)(p_1 - p_3)} + \text{permutations of } p_i\right] t_8R^2F^2. \tag{7.143}$$

$$\Delta_2(z_1, z_2, z_3, z_4, q_\alpha, p_a) \tag{7.144}$$

$$= 2\Bigg\{\sum_{a=1}^3\sum_{\alpha=1,2}(-\frac{33}{16})\frac{1}{(p_a - q_\alpha)^2} + (\sum_{a=1}^3\sum_{\alpha=1,2}\frac{\varpi_a^*(q_\alpha)}{(p_a - q_\alpha)})^2 - \frac{37}{32}\frac{\partial\psi_1^*(q_\alpha)}{p_a - q_\alpha}$$
$$+ \sum_{\alpha=1,2}\frac{3}{16}\Bigg\{\frac{1}{(q_\alpha - p_1)^2}\Bigg[1 + 3(q_\alpha - p_1)\left(\frac{1}{q_\alpha - p_2} + \frac{1}{q_\alpha - p_3}\right)\Bigg]$$
$$\times\ \frac{(q_\alpha - p_2)(q_\alpha - p_3)}{(p_1 - p_2)(p_1 - p_3)} + \text{ permutations of } p_a\Bigg\}\Bigg]\Bigg\}\, t_8R^2F^2.$$

$$\begin{aligned}
&\Delta_3(z_1,z_2,z_3,z_4,q_\alpha,p_a) \qquad (7.145)\\
&=2\Bigg\{\sum_{a=1}^{3}\sum_{\alpha=1,2}(-\frac{3}{8})\frac{1}{(p_a-q_\alpha)^2}-\frac{1}{16}(\sum_{a=1}^{3}\sum_{\alpha=1,2}\frac{\varpi_a^*(q_\alpha)}{(p_a-q_\alpha)})^2\\
&\quad-\sum_{\alpha=1,2}\frac{1}{2}\left\{\frac{1}{(q_\alpha-p_1)^2}\left[\frac{y(p_2)}{y(p_3)}+3(q_\alpha-p_1)\left(\frac{1}{q_\alpha-p_2}+\frac{1}{q_\alpha-p_3}\right)\right]\right.\\
&\quad\times\left.\frac{(q_\alpha-p_2)(q_\alpha-p_3)}{(p_1-p_2)(p_1-p_3)}+\text{ permutations of }p_a\right\}\Bigg]\Bigg\}\,t_8R^2F^2.
\end{aligned}$$

$$\Delta_4(z_1,z_2,z_3,z_4,q_\alpha,p_a)=2\sum_{a=1}^{3}\left[\frac{\varpi_a^*(q_2)}{4(q_1-p_a)^2}+\frac{\varpi_a^*(q_1)}{4(q_2-p_a)^2}\right]t_8R^2F^2. \qquad (7.146)$$

$$\Delta_5(z_1,z_2,z_3,z_4,q_\alpha,p_a)=\sum_{a=1}^{3}\frac{\varpi(q_1)\varpi(q_2)}{(q_1-p_a)(q_2-p_a)}t_8R^2F^2. \qquad (7.147)$$

The final amplitude will be

$$\begin{aligned}
&\Delta(D^2F^4)\\
&=\int_{\mathcal{F}_2}|\prod_{I<J}d\tau_{IJ}|^2\int_\Sigma d\mu\prod_{i=1}^{4}d^2z_i\frac{s(z_1z_2+z_3z_4)+t(z_1z_3+z_2z_4)+u(z_1z_4+z_2z_3)}{\prod_{i=1}^4(q_1-z_i)(q_2-z_i)}\\
&\qquad e^{-\sum_{i<j}k_i.k_jlnE(z_i,z_j)}\sum_{j=1}^{6}\Delta_jt_8R^2F^2 \qquad (7.148)
\end{aligned}$$

where we have written the tensor structure $t_8R^2F^2$ explicitly from $\Delta_i$.

## 7.5 Discussion of the result

We would like to compare our results with the one from elliptic formalism [137]. Below we give those expressions. For pure gravity part we have

$$\mathcal{A}_{R^4}=K\int_{\mathcal{F}_2}\frac{|\prod_{I<J}d\tau_{IJ}|^2}{(detIm\tau)^5\chi_{10}(\tau_{ij})}\int_\Sigma W_{R^4}Y_Se^{-\sum_{i<j}k_i.k_jG(z_i,z_j)} \qquad (7.149)$$

where $W_{R^4}=\frac{<\prod_{i=1}^4\partial X(z_i)e^{ik.X(z_i)}>}{\prod_{i=1}^4 e^{ik.X(z_i)}}$ and $G(i,j)$ the scalar Green's function $< X(z_i)X(z_j) >\sim ln|E(z_i,z_j)|^2$. The expression for $Y_S$ is given by

$$\begin{aligned}
3Y_S &= (k_1-k_2).(k_3-k_4)\Delta(z_1,z_2)\Delta(z_1,z_2) \qquad (7.150)\\
&+ z_i\text{permutations},
\end{aligned}$$

with $\Delta(x,y)=\omega_1(x)\omega_2(y)-\omega_1(y)\omega_2(x)$ , the $\omega_i$ being the canonical Abelian 1-form defined in (7.25).

For the pure gauge part, we find

$$\mathcal{A}_{F^4}=K\int_{\mathcal{F}_2}\frac{|\prod_{I<J}d\tau_{IJ}|^2}{(detIm\tau)^5\chi_{10}(\tau_{ij})}\int_\Sigma W_{F^4}Y_Se^{-\sum_{i<j}k_i.k_jG(z_i,z_j)} \qquad (7.151)$$

where the $W_{F^4}$ term is similar to $F^4$ terms in our convention (7.128), (7.129). Finally the mixed gauge-gravity part is

$$\mathcal{A}_{R^2F^2} = K\int_{\mathcal{F}_2}\frac{|\prod_{I<J}d\tau_{IJ}|^2}{(detIm\tau)^5\chi_{10}(\tau_{ij})}\int_{\Sigma}W_{R^2}W_{F^2}Y_S e^{-\sum_{i<j}k_i.k_jG(z_i,z_j)} \tag{7.152}$$

with $W_{F^2} = C_1F_2^2$ in our convention (7.127) and $W_{R^2} \sim \partial_z\partial_w E(z,w)$. Comparing these results with our computation, we see that the factor $Y_S$ corresponds to the factor $\frac{s(z_1z_2+z_3z_4)+t(z_1z_3+z_2z_4)+u(z_1z_4+z_2z_3)}{\prod_{i=1}^4(q_1-z_i)(q_2-z_i)}$ of our computations. Also the integral measures are corresponding. The difference is in the exponential factor $e^{-\sum_{i<j}k_i.k_jG(z_i,z_j)}$ which is different from our $e^{-\sum_{i<j}k_i.k_jlnE(z_i,z_j)}$. However this term is accompanied by $\sum_{i=1}^6\Delta_i$ and one might expect that the expansion of $e^{-\sum_{i<j}k_i.k_jG(z_i,z_j)}$ might correspond to our computations. Such a comparison is however very difficult and has been shown to be a little different for type II two-loop amplitudes. We thus also expect to find differences with our expression from that of the elliptic formalism. Plus we remind ourselves that the hyper-elliptic formalism needs to select the branch points as the ghost insertion points which give rise to certain artificial singularities which we have neglected for the sake of simplicity of algebra. This thing should account for the difference in elliptic and hyper-elliptic results.

## Chapter 8

# Conclusion

In the core of this thesis, we have explored the intricacies of the string amplitude computations in order to find the quantum corrections to the super-string effective action. Such corrections prove to be moduli stabilizing elements of the effective actions, that is, they provide for terms which generate mass for the massless scalars of the string dynamics in semi-compact space-time. Our point of view to study such corrections was to find their correlation with the supergravity anomaly cancelling terms. Our principal discovery in this work is to unravel the potential of composite connection in supergravity and curing the anomaly it generates due to chirally charged fermions of the theory. Composite connection engenders the class of discrete anomaly, the counter-term of which is essentially furnished by string theory when the supergravity theory in question does have stringy origin, or in other words, if it is not stuck in the swampland. Such conditions are mathematically characterized by tadpole constraints, in which, the composite connection of the moduli coset space plays the key role. These quantum corrections are promising candidates for delving in deep into the non-perturbative structure of F- and M-theory which we intend to study in future.

We have also ventured into the (much)uncharted waters of two-loop string amplitude in hyper-elliptic format using the novel method of chiral projection of string measure. Though our result does not allow for much simplification and is not quite illuminating in form, it paves the way for study of order 5 ($F^5$) and order 6 ($F^6$) gauge amplitude which might provide for correction terms to D-brane Born-Infeld action. We thus intend to carry these calculations in near future having the works presented here as a prelude.

# Appendix A

# Characteristic classes and Index polynomials

In this appendix we summarise the necessary definitions of the characteristic classes and index polynomials which have been used extensively for the purpose of computing anomaly and determining the massless spectrum in Kaluza-Klein reduction.

Let us first start by defining the Chern class and Chern character for a complex holomorphic vector bundle $GL(k,\mathbb{C})$ with $k \times k$ matrix curvature form F. By a suitable $GL(k,\mathbb{C})$ transformation, F may be brought to the diagonal form

$$\frac{i}{2\pi}F = \text{diag}(x_1, \cdots, x_k). \tag{A.1}$$

Then the total Chern class $c = c_1 + c_2 + \cdots + c_k$ is defined as the cohomology class $c \in H^0 + H^1 + H^2 + \cdots + H^{2k}$ as

$$c(V) = det\left(1 + \frac{i}{2\pi}F\right) = \prod_{i=1}^{k}(1 + x_k). \tag{A.2}$$

Thus we get for example

$$\begin{aligned}
c_1(V) &= \frac{i}{2\pi}Tr_cF, && \text{(A.3)}\\
c_2(V) &= \frac{1}{8\pi^2}\left(Tr_cF^2 - (Tr_cF)^2\right), && \text{(A.4)}\\
c_3(V) &= \frac{i}{48\pi^3}\left(-2Tr_cF^3 + 3Tr_cFTr_cF^2 - (Tr_cF)^3\right), && \text{(A.5)}\\
&\cdots && \text{(A.6)}
\end{aligned}$$

The Chern character is defined as

$$ch(F) = Tr_c\, exp\left(\frac{i}{2\pi}F\right) = \sum_{i=1}^{k} e^{x_i} = \sum_{r=1}^{k}\frac{(i)^r}{r!(2\pi)^r}Tr_cF^r. \tag{A.7}$$

In all of the above, the $Tr_c$ denotes the complex trace. We now consider the case of real vector bundle E with curvature R which is a real anti-symmetric matrix $\in$

$GL(n, \mathbb{R})$ valued form. R can be brought to a skew-diagonal form

$$\frac{R}{2\pi} = \begin{pmatrix} 0 & y_1 & 0 & \cdots & 0 \\ -y_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & y_n \\ 0 & 0 & \cdots & -y_n & 0 \end{pmatrix}. \tag{A.8}$$

Now we define the total Pontrjagin class

$$p(R) = det\left(\mathbb{1} + \frac{R}{2\pi}\right) = \prod_i^n (1 + y_i^2) = \sum_{i=1}^n p_n. \tag{A.9}$$

From this we find

$$p_1(E) = -\frac{1}{8\pi^2} TrR^2, \tag{A.10}$$

$$p_2(E) = \frac{1}{(2\pi)^4}\left(\frac{1}{8}(TrR^2)^2 - TrR^4\right), \tag{A.11}$$

$$\cdots \tag{A.12}$$

For the real bundle E, one can define its complexification by

$$E_{\mathbb{C}} = E \otimes \mathbb{C} \equiv V. \tag{A.13}$$

Then one can write the Pontrjagin classes in terms of Chern classes as

$$(-1)^j p_j(E) = \sum_{l=0}^{2j} (-1)^l c_l(V) c_{2j-l}(V), \tag{A.14}$$

that is,

$$p_1(E) = c_1^2(V) - 2c_2(V), \tag{A.15}$$

$$p_2(E) = c_2^2(V) - 2c_1(V)c_3(V) + 2c_4(V), \tag{A.16}$$

$$\cdots \tag{A.17}$$

Note that in case of real bundle "Tr" are real traces. Now we define following polynomials on real bundles

1. Hirzebruch L-sequence:

$$\widehat{L}(E) = \prod_{j=1}^n \frac{y_j}{tanh y_j} \tag{A.18}$$

$$= 1 + \frac{1}{3}p_1 + (-\frac{1}{45}p_1^2 + \frac{7}{45}p_2) + (\frac{2}{945}p_1^3 - \frac{13}{945}p_1 p_2 + \frac{62}{945}p_3) + \cdots \tag{A.19}$$

$$= 1 - \frac{1}{6\,(2\pi)^2} trR^2 + \frac{1}{(2\pi)^4}\left(-\frac{7}{180}trR^4 + \frac{1}{72}(trR^2)^2\right) + \frac{1}{(2\pi)^6}\left(-\frac{31}{2835}trR^6 + \frac{7}{1080}trR^4\,trR^2 - \frac{1}{1296}(trR^2)^3\right) + \cdots \tag{A.20}$$

2. A-roof genus:

$$\begin{aligned}
\widehat{A}(E) &= \prod_{j=1}^{n} \frac{y_j/2}{sinh(y_j/2)} && \text{(A.21)}\\
&= 1 - \frac{1}{24}p_1 + \frac{1}{2^4}(\frac{7}{360}p_1^2 + \frac{1}{90}p_2) && \text{(A.22)}\\
&\quad + \frac{1}{2^6}(\frac{-31}{15120}p_1^3 + \frac{11}{3780}p_1 p_2 - \frac{1}{945}p_3) + \cdots\\
&= 1 + \frac{1}{12\,(4\pi)^2} trR^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{360}trR^4 + \frac{1}{288}(trR^2)^2\right] && \text{(A.23)}\\
&\quad + \frac{1}{(4\pi)^6}\left[\frac{1}{5670}trR^6 + \frac{1}{4320}trR^4\, trR^2 + \frac{1}{10368}(trR^2)^3\right] + \cdots
\end{aligned}$$

# Appendix B

# Modular functions

In this appendix we provide the definitions of modular functions used through out this work e.g. Jacobi $\theta$ functions, Dedekind eta function and Eisenstein series along with useful identities relating them that we have used in the calculations.
Our convention for the $\theta$ function is

$$\theta \left[ {a \atop b} \right] (\nu|\tau) = \sum_{n\in\mathbb{Z}} q^{(1/2)(n-a/2)^2} e^{2\pi i(\nu-b/2)(n-a/2)}, \tag{B.1}$$

where a,b are real and $q = e^{2\pi i\tau}$.
We note $\theta_1 = \theta \left[ {1 \atop 1} \right]$, $\theta_2 = \theta \left[ {1 \atop 0} \right]$, $\theta_3 = \theta \left[ {0 \atop 0} \right]$ and $\theta_4 = \theta \left[ {0 \atop 1} \right]$.
Next we list different periodicity properties and modular transformations of the $\theta$ functions ($a, b \in \mathbb{Z}$):

$$\begin{aligned}
\theta \left[ {a+2 \atop b} \right] (\nu|\tau) &= \theta \left[ {a \atop b} \right] (\nu|\tau), \\
\theta \left[ {a \atop b+2} \right] (\nu|\tau) &= e^{i\pi a} \theta \left[ {a \atop b} \right] (\nu|\tau), \\
\theta \left[ {-a \atop -b} \right] (\nu|\tau) &= \theta \left[ {a \atop b} \right] (-\nu|\tau), \\
\theta \left[ {a \atop b} \right] (-\nu|\tau) &= e^{i\pi ab} \theta \left[ {a \atop b} \right] (\nu|\tau), \\
\theta \left[ {a \atop b} \right] (\nu|\tau+1) &= e^{-(i\pi/4)a(a-2)} \theta \left[ {a \atop a+b-1} \right] (\nu|\tau), \\
\theta \left[ {a \atop b} \right] (\nu/\tau|-1/\tau) &= \sqrt{-i\tau} e^{(i\pi/2)ab + i\pi\nu^2/\tau} \theta \left[ {b \atop -a} \right] (\nu|\tau).
\end{aligned} \tag{B.2}$$

We are now in position to define the Dedekind $\eta$-function:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{B.3}$$

satisfying

$$\eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau). \tag{B.4}$$

Some useful relations between the Jacobi $\theta$-functions and the $\eta$-function are

$$\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau) = 2\eta^3, \tag{B.5a}$$

$$\theta_3^{12} - \theta_2^{12} - \theta_4^{12} = 48\eta^{12}, \tag{B.5b}$$

$$\theta_2^4 + \theta_4^4 - \theta_3^4 = 0. \tag{B.5c}$$

Now we summarise the definitions of the Eisenstein series and Leech $j$ function

$$\hat{E}_2 = 1 - \frac{3}{\pi\tau_2} - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{B.6a}$$

$$E_4 = \frac{1}{2}\sum_{a=2}^{4}\theta_a^8 = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \tag{B.6b}$$

$$E_8 = E_4^2 = \frac{1}{2}\sum_{a=2}^{4}\theta_a^{16} = 1 + 480\sum_{n=1}^{\infty}\frac{n^7q^n}{1-q^n}, \tag{B.6c}$$

$$E_6 = \frac{1}{2}(\theta_2^4+\theta_3^4)(\theta_3^4+\theta_4^4)(\theta_4^4-\theta_2^4) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}, \tag{B.6d}$$

$$j = \frac{E_4^3}{\eta^{24}} = \frac{1}{q} + 744 + \cdots \tag{B.6e}$$

In the process of "gauging" the elliptic genus, we shall extensively use the following identities

$$\frac{\theta_2(\nu|\tau)}{\theta_2(0|\tau)} = exp\left\{\sum_{k=1}^{\infty}\frac{(2\pi i)^{2k}B_{2k}\nu^{2k}}{(2k+1)!-(2k)!}\left[E_{2k}(q) - 2^{2k}E_{2k}(q^2)\right]\right\} \tag{B.7a}$$

$$\frac{\theta_3(\nu|\tau)}{\theta_3(0|\tau)} = exp\left\{\sum_{k=1}^{\infty}\frac{(2\pi i)^{2k}B_{2k}\nu^{2k}}{(2k+1)!-(2k)!}\left[E_{2k}(q) - E_{2k}(-\sqrt{q})\right]\right\} \tag{B.7b}$$

$$\frac{\theta_4(\nu|\tau)}{\theta_4(0|\tau)} = exp\left\{\sum_{k=1}^{\infty}\frac{(2\pi i)^{2k}B_{2k}\nu^{2k}}{(2k+1)!-(2k)!}\left[E_{2k}(q) - E_{2k}(\sqrt{q})\right]\right\} \tag{B.7c}$$

where $B_k$ are the Bernoulli numbers: $B_2 = 1/6,\ B_4 = -1/30,\ B_6 = 1/42$ and we shall use the following combinations $f_1$, $f_2$, $f_3$ in the elliptic genus

$$f_1 = 4E_2(q^2) - 2E_2(q) = \theta_3^4 + \theta_4^4, \tag{B.8a}$$

$$f_2 = E_2(-\sqrt{q}) - 2E_2(q) = \theta_2^4 - \theta_4^4, \tag{B.8b}$$

$$f_3 = E_2(\sqrt{q}) - 2E_2(q) = -\theta_2^4 - \theta_3^4. \tag{B.8c}$$

$$E_4(q) - 16E_4(q^2) = 5(E_4(q)^2 - f_1^2) = -15\theta_3^4\theta_4^4, \tag{B.9a}$$

$$E_4(q) - E_4(-\sqrt{q}) = 5(E_4(q)^2 - f_2^2) = 15\theta_2^4\theta_4^4, \tag{B.9b}$$

$$E_4(q) - E_4(\sqrt{q}) = 5(E_4(q)^2 - f_3^2) = -15\theta_2^4\theta_3^4. \tag{B.9c}$$

There are various summation identities involving the Eisenstein series and $f_1$, $f_2$, $f_3$ which will be useful in the computation of the partition function

$$\frac{1}{2}\sum_{a=2}^{4}\theta_a(0|\tau)^{16}f_{a-1} = -E_4E_6, \tag{B.10a}$$

$$\frac{1}{2}\sum_{a=2}^{4}\theta_a(0|\tau)^{16}f_{a-1}^2 = E_4^3 - 2^7 3^2\eta^{24} = 2E_6^2 - E_4^3, \tag{B.10b}$$

$$\frac{1}{2}\sum_{a=2}^{4}\theta_a(0|\tau)^{8}f_{a-1} = -E_6, \tag{B.10c}$$

$$\frac{1}{2}\sum_{a=2}^{4}\theta_a(0|\tau)^{8}f_{a-1}^2 = E_4^2, \tag{B.10d}$$

$$\sum_{a=2}^{4}\theta_a(0|\tau)^{16} = 2\theta_3^8\theta_4^8 + 2\theta_2^8\theta_4^8 + 2\theta_2^8\theta_3^8, \tag{B.11a}$$

$$\begin{aligned}\sum_{a=2}^{4}\theta_a(0|\tau)^{16}(\hat{E}_2 + f_{a-1}) &= \theta_3^8\theta_4^8(2\hat{E}_2 + f_2 + f_3) \\ &+ \theta_2^8\theta_4^8(2\hat{E}_2 + f_1 + f_3) + \theta_2^8\theta_3^8(2\hat{E}_2 + f_1 + f_2),\end{aligned} \tag{B.11b}$$

$$\begin{aligned}\sum_{a=2}^{4}\theta_a(0|\tau)^{16}(\hat{E}_2 + f_{a-1})^2 &= 2\theta_3^8\theta_4^8(\hat{E}_2 + f_2)(\hat{E}_2 + f_3) \\ &+ 2\theta_2^8\theta_4^8(\hat{E}_2 + f_1)(\hat{E}_2 + f_3) \\ &+ 2\theta_2^8\theta_3^8(\hat{E}_2 + f_1)(\hat{E}_2 + f_2) + 2^8 3^2\eta^{24},\end{aligned} \tag{B.11c}$$

$$\begin{aligned}&= \theta_3^8\theta_4^8\left((\hat{E}_2 + f_2)^2 + (\hat{E}_2 + f_3)^2\right) \\ &+ \theta_2^8\theta_4^8\left((\hat{E}_2 + f_1)^2 + (\hat{E}_2 + f_3)^2\right) \\ &+ \theta_2^8\theta_3^8\left((\hat{E}_2 + f_1)^2 + (\hat{E}_2 + f_2)^2\right) - 2^9 3^2\eta^{24}.\end{aligned} \tag{B.11d}$$

For the pure gauge part there are very remarkable trivial identities

$$\begin{aligned}\frac{1}{2^8 3\eta^{24}}\left(-\theta_2^{16}\theta_3^4\theta_4^4 + \theta_3^{16}\theta_2^4\theta_4^4 - \theta_4^{16}\theta_2^4\theta_3^4\right) &= 1, \\ \frac{\theta_2^8\theta_3^8}{2^8 3\eta^{24}}\left(-\theta_3^4\theta_4^4 + \theta_2^4\theta_4^4\right) &= -\frac{1}{3}, \\ \frac{\theta_3^8\theta_4^8}{2^8 3\eta^{24}}\left(+\theta_2^4\theta_4^4 - \theta_2^4\theta_3^4\right) &= -\frac{1}{3}, \\ \frac{\theta_2^8\theta_4^8}{2^8 3\eta^{24}}\left(-\theta_3^4\theta_4^4 - \theta_2^4\theta_3^4\right) &= -\frac{1}{3}.\end{aligned} \tag{B.12}$$

Then we enlist the q-expansions of the different modular functions used in the elliptic genus

$$\frac{E_4}{\eta^{24}}\sum_{a=2}^{4}\theta_a(0|\tau)^{16} = \frac{2}{q} + 1488 + \mathcal{O}(q), \qquad \frac{E_4}{\eta^{24}}\theta_3^8\theta_4^8 = \frac{1}{q} + 232 + \mathcal{O}(q) \tag{B.13}$$

$$\frac{E_4}{\eta^{24}}\theta_2^8\theta_3^8 = 256 + \mathcal{O}(\sqrt{q}), \qquad \frac{E_4}{\eta^{24}}\theta_2^8\theta_4^8 = 256 + \mathcal{O}(\sqrt{q}),$$

$$\frac{E_2^2}{\eta^{24}}\sum_{a=2}^{4}\theta_a(0|\tau)^{16} = \frac{2}{q} + 912 + \mathcal{O}(q), \qquad \frac{E_2^2}{\eta^{24}}\theta_3^8\theta_4^8 = \frac{1}{q} - 56 + \mathcal{O}(q)$$

$$\frac{E_4}{\eta^{24}}\theta_2^8\theta_3^8 = 256 + \mathcal{O}(\sqrt{q}), \qquad \frac{E_4}{\eta^{24}}\theta_2^8\theta_4^8 = 256 + \mathcal{O}(\sqrt{q}),$$

$$\frac{1}{\eta^{24}}\sum_{a=2}^{4}\theta_a(0|\tau)^{16}(\hat{E}_2 + f_{a-1})^2 = 1152 + \mathcal{O}(q), \qquad \frac{\theta_3^8\theta_4^8}{\eta^{24}}\Big((\hat{E}_2+f_2)^2 + (\hat{E}_2+f_3)^2\Big) = 1152 + \mathcal{O}(q),$$

$$\frac{\theta_2^8\theta_4^8}{\eta^{24}}\Big((\hat{E}_2+f_1)^2 + (\hat{E}_2+f_3)^2\Big) = 2304 + \mathcal{O}(\sqrt{q}), \qquad \frac{\theta_2^8\theta_3^8}{\eta^{24}}\Big((\hat{E}_2+f_1)^2 + (\hat{E}_2+f_2)^2\Big) = 2304 + \mathcal{O}(\sqrt{q}),$$

$$\frac{E_2}{\eta^{24}}\sum_{a=2}^{4}\theta_a(0|\tau)^{16}(E_2 + f_{a-1}) = 1440 + \mathcal{O}(q), \qquad \frac{E_2}{\eta^{24}}\theta_3^8\theta_4^8(2\hat{E}_2 + f_2 + f_3) = -96 + \mathcal{O}(q),$$

$$\frac{E_2}{\eta^{24}}\theta_2^8\theta_3^8(2\hat{E}_2 + f_1 + f_2) = \mathcal{O}(\sqrt{q}), \qquad \frac{E_2}{\eta^{24}}\theta_2^8\theta_4^8(2\hat{E}_2 + f_1 + f_3) = \mathcal{O}(\sqrt{q}).$$

We now briefly discuss the large volume and decompactification limits. The large volume limit in case of a $T^2$ compactification means taking the torus volume $V_{T^2} \to \infty$. However the complex structure $U = U_1 + iU_2$ remains fixed. We recall that the compact space-time torus is formed by compactifying the 8th and 9th space dimensions for which we have the following metric

$$G_{ij} = \begin{pmatrix} g_{88} & g_{89} \\ g_{89} & g_{99} \end{pmatrix} = \frac{V}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}. \tag{B.14}$$

In the decompactification limit, we will take $V_{T^2} \to \infty$ and moreover impose orthonormality of the 8th and the 9th directions, i.e.

$$U_2 = \frac{V}{g_{88}} \to 1\,, \quad U_1 = \frac{g_{89}}{g_{88}} \to 0\,. \tag{B.15}$$

A useful summary of the q-expansion and the relevant limits of the different modular functions of $T$ and $U$ that have been used in expressions for the higher-derivative couplings given below:

$$log|\eta(T)|^2 = -\frac{\pi T_2}{6} - [\theta(T) + \bar{\theta}(\bar{T})], \tag{B.16}$$

$$\theta(\tau) = q + \frac{3q^2}{2} + \cdots\,, \quad q = e^{2i\pi\tau}, \tag{B.17}$$

$$\lim_{\tau\to i\infty}\theta(\tau) = 0, \quad \lim_{V\to\infty} log|\eta(T)|^2 = -\frac{\pi T_2}{6}, \tag{B.18}$$

$$log\frac{\eta(T)}{\bar{\eta}(\bar{T})} = \frac{i\pi T_1}{6} - [\theta(T) - \bar{\theta}(\bar{T})], \tag{B.19}$$

$$\lim_{V\to\infty}\left(log\frac{\eta(T)}{\bar{\eta}(\bar{T})} - \frac{i\pi T_1}{6}\right) = 0. \tag{B.20}$$

# Bibliography


[1] J. Wess and J. Bagger. "Supersymmetry and supergravity". In: *Princeton, USA: Univ. Pr. (1992) 259 p* (1992).

[2] Peter C. West. "Introduction to supersymmetry and supergravity". In: *Singapore, Singapore: World Scientific (1990) 425 p* (1990).

[3] Sidney R. Coleman and J. Mandula. "All Possible Symmetries of the S Matrix". In: *Phys. Rev.* 159 (1967), pp. 1251–1256. DOI: 10.1103/PhysRev.159.1251.

[4] E. Cremmer, B. Julia, and Joel Scherk. "Supergravity Theory in Eleven-Dimensions". In: *Phys. Lett.* B76 (1978), pp. 409–412. DOI: 10.1016/0370-2693(78)90894-8.

[5] S. Ferrara. "ELEVEN-DIMENSIONAL SUPERGRAVITY IN SUPERSPACE." In: *Europhysics Study Conference on Unification of the Fundamental Interactions Erice, Italy, March 17-24, 1980*. 1980, pp. 119–136.

[6] M. J. Duff, James T. Liu, and R. Minasian. "Eleven-dimensional origin of string-string duality: A One loop test". In: *Nucl. Phys.* B452 (1995), pp. 261–282. DOI: 10.1016/0550-3213(95)00368-3. arXiv: hep-th/9506126 [hep-th].

[7] Michael B. Green, Michael Gutperle, and Pierre Vanhove. "One loop in eleven-dimensions". In: *Phys. Lett.* B409 (1997), pp. 177–184. DOI: 10.1016/S0370-2693(97)00931-3. arXiv: hep-th/9706175 [hep-th].

[8] John H. Schwarz. "Covariant Field Equations of Chiral N=2 D=10 Supergravity". In: *Nucl. Phys.* B226 (1983), p. 269. DOI: 10.1016/0550-3213(83)90192-X.

[9] Paul S. Howe and Peter C. West. "The Complete N=2, D=10 Supergravity". In: *Nucl. Phys.* B238 (1984), pp. 181–220. DOI: 10.1016/0550-3213(84)90472-3.

[10] John H. Schwarz. "An SL(2,Z) multiplet of type IIB superstrings". In: *Phys. Lett.* B360 (1995). [Erratum: Phys. Lett.B364,252(1995)], pp. 13–18. DOI: 10.1016/0370-2693(95)01405-5, 10.1016/0370-2693(95)01138-G. arXiv: hep-th/9508143 [hep-th].

[11] Michael B. Green and John H. Schwarz. "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory". In: *Phys. Lett.* B149 (1984), pp. 117–122. DOI: 10.1016/0370-2693(84)91565-X.

[12] Abdus Salam and J. A. Strathdee. "On Kaluza-Klein Theory". In: *Annals Phys.* 141 (1982), pp. 316–352. DOI: 10.1016/0003-4916(82)90291-3.

[13] Robert Gilmore. *Lie groups, physics, and geometry: An introduction for physicists, engineers and chemists*. 2008. URL: http://www.cambridge.org/catalogue/catalogue.asp?isbn=9780521884006.

[14] Neil Marcus. "Composite Anomalies in Supergravity". In: *Phys. Lett.* B157 (1985), pp. 383–388. DOI: 10.1016/0370-2693(85)90385-5.

[15] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. 1995. ISBN: 9780201503975, 0201503972. URL: http://www.slac.stanford.edu/spires/find/books/www?cl=QC174.45%3AP4.

[16] Luis Alvarez-Gaume and Paul H. Ginsparg. "The Structure of Gauge and Gravitational Anomalies". In: *Annals Phys.* 161 (1985). [Erratum: Annals Phys.171,233(1986)], p. 423. DOI: 10.1016/0003-4916(85)90087-9.

[17] Luis Alvarez-Gaume and Edward Witten. "Gravitational Anomalies". In: *Nucl. Phys.* B234 (1984), p. 269. DOI: 10.1016/0550-3213(84)90066-X.

[18] Adel Bilal. "Lectures on Anomalies". In: (2008). arXiv: 0802.0634 [hep-th].

[19] Kazuo Fujikawa. "Path Integral for Gauge Theories with Fermions". In: *Phys. Rev.* D21 (1980). [Erratum: Phys. Rev.D22,1499(1980)], p. 2848. DOI: 10.1103/PhysRevD.21.2848, 10.1103/PhysRevD.22.1499.

[20] Luis Alvarez-Gaume. "Supersymmetry and the Atiyah-Singer Index Theorem". In: *Commun. Math. Phys.* 90 (1983), p. 161. DOI: 10.1007/BF01205500.

[21] Orlando Alvarez, I. M. Singer, and Bruno Zumino. "Gravitational Anomalies and the Family's Index Theorem". In: *Commun. Math. Phys.* 96 (1984), p. 409. DOI: 10.1007/BF01214584.

[22] Bruno Zumino, Yong-Shi Wu, and A. Zee. "Chiral Anomalies, Higher Dimensions, and Differential Geometry". In: *Nucl. Phys.* B239 (1984), pp. 477–507. DOI: 10.1016/0550-3213(84)90259-1.

[23] Luis Alvarez-Gaume and Paul H. Ginsparg. "The Topological Meaning of Nonabelian Anomalies". In: *Nucl. Phys.* B243 (1984), pp. 449–474. DOI: 10.1016/0550-3213(84)90487-5.

[24] D. Friedan and Paul Windey. "Supersymmetric Derivation of the Atiyah-Singer Index and the Chiral Anomaly". In: *Nucl. Phys.* B235 (1984), pp. 395–416. DOI: 10.1016/0550-3213(84)90506-6.

[25] William A. Bardeen and Bruno Zumino. "Consistent and Covariant Anomalies in Gauge and Gravitational Theories". In: *Nucl. Phys.* B244 (1984), pp. 421–453. DOI: 10.1016/0550-3213(84)90322-5.

[26] Michael B. Green, John H. Schwarz, and Edward Witten. *Superstring Theory vol 1 and 2*. Cambridge University Press, 2012. ISBN: 9781139534789, 9781107029132.

[27] J. Polchinski. *String theory. Vol. 1 and 2: An introduction to the bosonic string and Superstring and Beyond*. Cambridge University Press, 2007. ISBN: 9780511252273, 9780521672276, 9780521633031.

[28] Elias Kiritsis. *String theory in a nutshell*. Princeton, USA: Univ. Pr. (2007) 588 p, 2007.

[29] Daniel Friedan, Emil J. Martinec, and Stephen H. Shenker. "Conformal Invariance, Supersymmetry and String Theory". In: *Nucl. Phys.* B271 (1986), pp. 93–165. DOI: 10.1016/0550-3213(86)90356-1, 10.1016/S0550-3213(86)80006-2.

[30] Joseph Polchinski. "Tasi lectures on D-branes". In: *Fields, strings and duality. Proceedings, Summer School, Theoretical Advanced Study Institute in Elementary Particle Physics, TASI'96, Boulder, USA, June 2-28, 1996*. 1996, pp. 293–356. arXiv: hep-th/9611050 [hep-th]. URL: https://inspirehep.net/record/425724/files/arXiv:hep-th_9611050.pdf.

[31] Constantin P. Bachas. "Lectures on D-branes". In: *Duality and supersymmetric theories. Proceedings, Easter School, Newton Institute, Euroconference, Cambridge, UK, April 7-18, 1997*. 1998, pp. 414–473. arXiv: hep-th/9806199 [hep-th].

[32] Michael B. Green, Jeffrey A. Harvey, and Gregory W. Moore. "I-brane inflow and anomalous couplings on d-branes". In: *Class. Quant. Grav.* 14 (1997), pp. 47–52. DOI: 10.1088/0264-9381/14/1/008. arXiv: hep-th/9605033 [hep-th].

[33] Ruben Minasian and Gregory W. Moore. "K theory and Ramond-Ramond charge". In: *JHEP* 11 (1997), p. 002. DOI: 10.1088/1126-6708/1997/11/002. arXiv: hep-th/9710230 [hep-th].

[34] Jose F. Morales, Claudio A. Scrucca, and Marco Serone. "Anomalous couplings for D-branes and O-planes". In: *Nucl. Phys.* B552 (1999), pp. 291–315. DOI: 10.1016/S0550-3213(99)00217-5. arXiv: hep-th/9812071 [hep-th].

[35] Bogdan Stefanski Jr. "Gravitational couplings of D-branes and O-planes". In: *Nucl. Phys.* B548 (1999), pp. 275–290. DOI: 10.1016/S0550-3213(99)00147-9. arXiv: hep-th/9812088 [hep-th].

[36] Carlo Angelantonj and Augusto Sagnotti. "Open strings". In: *Phys. Rept.* 371 (2002). [Erratum: Phys. Rept.376,no.6,407(2003)], pp. 1–150. DOI: 10.1016/S0370-1573(02)00273-9, 10.1016/S0370-1573(03)00006-1. arXiv: hep-th/0204089 [hep-th].

[37] D. Bailin and A. Love. "KALUZA-KLEIN THEORIES". In: *Rept. Prog. Phys.* 50 (1987), pp. 1087–1170. DOI: 10.1088/0034-4885/50/9/001.

[38] Tohru Eguchi, Peter B. Gilkey, and Andrew J. Hanson. "Gravitation, Gauge Theories and Differential Geometry". In: *Phys. Rept.* 66 (1980), p. 213. DOI: 10.1016/0370-1573(80)90130-1.

[39] M. Nakahara. *Geometry, topology and physics*. 2003.

[40] C. Nash and S. Sen. *TOPOLOGY AND GEOMETRY FOR PHYSICISTS*. 1983.

[41] Matthias R. Gaberdiel, Tamas Hauer, and Barton Zwiebach. "Open string-string junction transitions". In: *Nucl. Phys.* B525 (1998), pp. 117–145. DOI: 10.1016/S0550-3213(98)00290-9. arXiv: hep-th/9801205 [hep-th].

[42] Mirjam Cvetic and James Halverson. "TASI Lectures: Particle Physics from Perturbative and Non-perturbative Effects in D-braneworlds". In: *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2010). String Theory and Its Applications: From meV to the Planck Scale: Boulder, Colorado, USA, June 1-25, 2010*. 2011, pp. 245–292. arXiv: 1101.2907 [hep-th]. URL: http://inspirehep.net/record/884475/files/arXiv:1101.2907.pdf.

[43] Mirjam Cvetic, Inaki Garcia Etxebarria, and James Halverson. "Three Looks at Instantons in F-theory – New Insights from Anomaly Inflow, String Junctions and Heterotic Duality". In: *JHEP* 11 (2011), p. 101. DOI: 10.1007/JHEP11(2011)101. arXiv: 1107.2388 [hep-th].

[44] Tom Banks et al. "M theory as a matrix model: A Conjecture". In: *Phys. Rev.* D55 (1997), pp. 5112–5128. DOI: 10.1103/PhysRevD.55.5112. arXiv: hep-th/9610043 [hep-th].

[45] Edward Witten. "On flux quantization in M theory and the effective action". In: *J. Geom. Phys.* 22 (1997), pp. 1–13. DOI: 10.1016/S0393-0440(96)00042-3. arXiv: hep-th/9609122 [hep-th].

[46] Cumrun Vafa and Edward Witten. "A One loop test of string duality". In: *Nucl. Phys.* B447 (1995), pp. 261–270. DOI: 10.1016/0550-3213(95)00280-6. arXiv: hep-th/9505053 [hep-th].

[47] Frederik Denef. "Les Houches Lectures on Constructing String Vacua". In: *String theory and the real world: From particle physics to astrophysics. Proceedings, Summer School in Theoretical Physics, 87th Session, Les Houches, France, July 2-27, 2007*. 2008, pp. 483–610. arXiv: 0803.1194 [hep-th]. URL: http://inspirehep.net/record/780946/files/arXiv:0803.1194.pdf.

[48] Petr Horava and Edward Witten. "Heterotic and type I string dynamics from eleven-dimensions". In: *Nucl. Phys.* B460 (1996), pp. 506–524. DOI: 10.1016/0550-3213(95)00621-4. arXiv: hep-th/9510209 [hep-th].

[49] Petr Horava and Edward Witten. "Eleven-dimensional supergravity on a manifold with boundary". In: *Nucl. Phys.* B475 (1996), pp. 94–114. DOI: 10.1016/0550-3213(96)00308-2. arXiv: hep-th/9603142 [hep-th].

[50] Michael B. Green and John H. Schwarz. "The Hexagon Gauge Anomaly in Type I Superstring Theory". In: *Nucl. Phys.* B255 (1985), pp. 93–114. DOI: 10.1016/0550-3213(85)90130-0.

[51] Adel Bilal, Jean-Pierre Derendinger, and Roger Sauser. "M theory on S**1 / Z(2): Facts and fakes". In: *Nucl. Phys.* B576 (2000), pp. 347–374. DOI: 10.1016/S0550-3213(00)00167-X. arXiv: hep-th/9912150 [hep-th].

[52] Adel Bilal and Steffen Metzger. "Anomaly cancellation in M theory: A Critical review". In: *Nucl. Phys.* B675 (2003), pp. 416–446. DOI: 10.1016/j.nuclphysb.2003.10.002. arXiv: hep-th/0307152 [hep-th].

[53] Cumrun Vafa. "Evidence for F theory". In: *Nucl. Phys.* B469 (1996), pp. 403–418. DOI: 10.1016/0550-3213(96)00172-1. arXiv: hep-th/9602022 [hep-th].

[54] Ashoke Sen. "An Introduction to nonperturbative string theory". In: *Duality and supersymmetric theories. Proceedings, Easter School, Newton Institute, Euroconference, Cambridge, UK, April 7-18, 1997*. 1998, pp. 297–413. arXiv: hep-th/9802051 [hep-th]. URL: http://alice.cern.ch/format/showfull?sysnb=0269224.

[55] Ruben Minasian, Tom G. Pugh, and Raffaele Savelli. "F-theory at order $\alpha'^3$". In: *JHEP* 10 (2015), p. 050. DOI: 10.1007/JHEP10(2015)050. arXiv: 1506.06756 [hep-th].

[56] Brian R. Greene et al. "Stringy Cosmic Strings and Noncompact Calabi-Yau Manifolds". In: *Nucl. Phys.* B337 (1990), pp. 1–36. DOI: 10.1016/0550-3213(90)90248-C.

[57] W. Lerche. "On the heterotic / F theory duality in eight-dimensions". In: *Proceedings, NATO Advanced Study Institute on Progress in String Theory and M-Theory: Cargese, France, May 24-June 5, 1999*. 1999, pp. 15–46. arXiv: hep-th/9910207 [hep-th]. URL: http://inspirehep.net/record/509128/files/arXiv:hep-th_9910207.pdf.

[58] Washington Taylor. "TASI Lectures on Supergravity and String Vacua in Various Dimensions". In: (2011). arXiv: 1104.2051 [hep-th].

[59] A. Johansen. "A Comment on BPS states in F theory in eight-dimensions". In: *Phys. Lett.* B395 (1997), pp. 36–41. DOI: 10.1016/S0370-2693(97)00053-1. arXiv: hep-th/9608186 [hep-th].

[60] Eric A. Bergshoeff and Jelle Hartong. "BPS Open Strings and A-D-E-singularities in F-theory on K3". In: *JHEP* 02 (2010), p. 007. DOI: 10.1007/JHEP02(2010)007. arXiv: 0910.3594 [hep-th].

[61] Michael R. Douglas and Miao Li. "D-brane realization of N=2 superYang-Mills theory in four-dimensions". In: (1996). arXiv: hep-th/9604041 [hep-th].

[62] Ashoke Sen. "F theory and orientifolds". In: *Nucl. Phys.* B475 (1996), pp. 562–578. DOI: 10.1016/0550-3213(96)00347-1. arXiv: hep-th/9605150 [hep-th].

[63] Matthias R. Gaberdiel and Michael B. Green. "An SL(2, Z) anomaly in IIB supergravity and its F theory interpretation". In: *JHEP* 11 (1998), p. 026. DOI: 10.1088/1126-6708/1998/11/026. arXiv: hep-th/9810153 [hep-th].

[64] Constantin P. Bachas, Pascal Bain, and Michael B. Green. "Curvature terms in D-brane actions and their M theory origin". In: *JHEP* 05 (1999), p. 011. DOI: 10.1088/1126-6708/1999/05/011. arXiv: hep-th/9903210 [hep-th].

[65] C. Montonen and David I. Olive. "Magnetic Monopoles as Gauge Particles?" In: *Phys. Lett.* 72B (1977), pp. 117–120. DOI: 10.1016/0370-2693(77)90076-4.

[66] Cumrun Vafa and Edward Witten. "A Strong coupling test of S duality". In: *Nucl. Phys.* B431 (1994), pp. 3–77. DOI: 10.1016/0550-3213(94)90097-3. arXiv: hep-th/9408074 [hep-th].

[67] Edward Witten. "On S duality in Abelian gauge theory". In: *Selecta Math.* 1 (1995), p. 383. DOI: 10.1007/BF01671570. arXiv: hep-th/9505186 [hep-th].

[68] Abdus Salam and E. Sezgin. "d = 8 SUPERGRAVITY". In: *Nucl. Phys.* B258 (1985), p. 284. DOI: 10.1016/0550-3213(85)90613-3.

[69] Anirban Basu. "Supersymmetry constraints on the R$^4 multiplet in type IIB on T^2$". In: *Class. Quant. Grav.* 28 (2011), p. 225018. DOI: 10.1088/0264-9381/28/22/225018. arXiv: 1107.3353 [hep-th].

[70] James T. Liu and Ruben Minasian. "U-branes and T**3 fibrations". In: *Nucl. Phys.* B510 (1998), pp. 538–554. DOI: 10.1016/S0550-3213(97)00732-3. arXiv: hep-th/9707125 [hep-th].

[71] Guillaume Bossard and Valentin Verschinin. "Minimal unitary representations from supersymmetry". In: *JHEP* 10 (2014), p. 008. DOI: 10.1007/JHEP10(2014)008. arXiv: 1406.5527 [hep-th].

[72] Elias Kiritsis and Boris Pioline. "On R**4 threshold corrections in IIb string theory and (p, q) string instantons". In: *Nucl. Phys.* B508 (1997), pp. 509–534. DOI: 10.1016/S0550-3213(97)00645-7. arXiv: hep-th/9707018 [hep-th].

[73] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. New York: Springer-Verlag, 1997. ISBN: 9780387947853, 9781461274759. DOI: 10.1007/978-1-4612-2256-9. URL: http://www-spires.fnal.gov/spires/find/books/www?cl=QC174.52.C66D5::1997.

[74] E. Kiritsis and N. A. Obers. “Heterotic type I duality in D < 10-dimensions, threshold corrections and D instantons”. In: *JHEP* 10 (1997), p. 004. DOI: 10.1088/1126-6708/1997/10/004. arXiv: hep-th/9709058 [hep-th].

[75] Lance J. Dixon, Vadim Kaplunovsky, and Jan Louis. “Moduli dependence of string loop corrections to gauge coupling constants”. In: *Nucl. Phys.* B355 (1991), pp. 649–688. DOI: 10.1016/0550-3213(91)90490-O.

[76] Jeffrey A. Harvey and Gregory W. Moore. “Algebras, BPS states, and strings”. In: *Nucl. Phys.* B463 (1996), pp. 315–368. DOI: 10.1016/0550-3213(95)00605-2. arXiv: hep-th/9510182 [hep-th].

[77] Michael Gutperle. “Heterotic / type I duality, D instantons and a N=2 AdS / CFT correspondence”. In: *Phys. Rev.* D60 (1999), p. 126001. DOI: 10.1103/PhysRevD.60.126001. arXiv: hep-th/9905173 [hep-th].

[78] W. Lerche et al. “Anomaly Cancelling Terms From the Elliptic Genus”. In: *Nucl. Phys.* B299 (1988), pp. 91–116. DOI: 10.1016/0550-3213(88)90468-3.

[79] W. Lerche, A. N. Schellekens, and N. P. Warner. “Lattices and Strings”. In: *Phys. Rept.* 177 (1989), p. 1. DOI: 10.1016/0370-1573(89)90077-X.

[80] Michael B. Green and Arnab Rudra. “Type I/heterotic duality and M-theory amplitudes”. In: *JHEP* 12 (2016), p. 060. DOI: 10.1007/JHEP12(2016)060. arXiv: 1604.00324 [hep-th].

[81] John R. Ellis, Philippe Jetzer, and Leah Mizrachi. “One Loop String Corrections to the Effective Field Theory”. In: *Nucl. Phys.* B303 (1988), pp. 1–35. DOI: 10.1016/0550-3213(88)90214-3.

[82] M. Abe, H. Kubota, and N. Sakai. “Loop Corrections to the $E(8)$ X $E(8)$ Heterotic String Effective Lagrangian”. In: *Nucl. Phys.* B306 (1988), pp. 405–424. DOI: 10.1016/0550-3213(88)90699-2.

[83] S. Stieberger and T. R. Taylor. “NonAbelian Born-Infeld action and type 1. - heterotic duality 2: Nonrenormalization theorems”. In: *Nucl. Phys.* B648 (2003), pp. 3–34. DOI: 10.1016/S0550-3213(02)00979-3. arXiv: hep-th/0209064 [hep-th].

[84] S. Stieberger and T. R. Taylor. “NonAbelian Born-Infeld action and type I. Heterotic duality (1): Heterotic F**6 terms at two loops”. In: *Nucl. Phys.* B647 (2002), pp. 49–68. DOI: 10.1016/S0550-3213(02)00895-7. arXiv: hep-th/0207026 [hep-th].

[85] A. Morozov and A. Perelomov. “A Note on Multiloop Calculations for Superstrings in the Nsr Formalism”. In: *Int. J. Mod. Phys.* A4 (1989). [Zh. Eksp. Teor. Fiz.95,1153(1989)], p. 1773. DOI: 10.1142/S0217751X89000716.

[86] E. Gava, R. Jengo, and G. Sotkov. “Modular Invariance and the Two Loop Vanishing of the Cosmological Constant”. In: *Phys. Lett.* B207 (1988), p. 283. DOI: 10.1016/0370-2693(88)90576-X.

[87] A. Morozov. “On Two Loop Contribution to Four Point Function for Superstring”. In: *Phys. Lett.* B209 (1988). [JETP Lett.47,219(1988)], p. 473. DOI: 10.1016/0370-2693(88)91176-8.

[88] Roberto Jengo, Galen M. Sotkov, and Chuan-Jie Zhu. “Two Loop Vacuum Amplitude in Four-dimensional Heterotic String Models”. In: *Phys. Lett.* B211 (1988), p. 425. DOI: 10.1016/0370-2693(88)91886-2.

[89] Alexander Sevrin, Jan Troost, and Walter Troost. "The nonAbelian Born-Infeld action at order F**6". In: *Nucl. Phys.* B603 (2001), pp. 389–412. DOI: 10.1016/S0550-3213(01)00104-3. arXiv: hep-th/0101192 [hep-th].

[90] Eric D'Hoker and D. H. Phong. "Two loop superstrings. 1. Main formulas". In: *Phys. Lett.* B529 (2002), pp. 241–255. DOI: 10.1016/S0370-2693(02)01255-8. arXiv: hep-th/0110247 [hep-th].

[91] Eric D'Hoker and D. H. Phong. "The Geometry of String Perturbation Theory". In: *Rev. Mod. Phys.* 60 (1988), p. 917. DOI: 10.1103/RevModPhys.60.917.

[92] A. N. Schellekens and N. P. Warner. "Anomalies, Characters and Strings". In: *Nucl. Phys.* B287 (1987), p. 317. DOI: 10.1016/0550-3213(87)90108-8.

[93] Milton Abramowitz. *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables,* Dover Publications, Incorporated, 1974. ISBN: 0486612724.

[94] Serge Lang. *Introduction to Modular Forms*. Vol. 222. Springer-Verlag Berlin Heidelberg, 1987. ISBN: 978-3-642-05716-8. DOI: 10.1007/978-3-642-51447-0.

[95] E. A. Bergshoeff and M. de Roo. "The Quartic Effective Action of the Heterotic String and Supersymmetry". In: *Nucl. Phys.* B328 (1989), pp. 439–468. DOI: 10.1016/0550-3213(89)90336-2.

[96] Arkady A. Tseytlin. "Heterotic type I superstring duality and low-energy effective actions". In: *Nucl. Phys.* B467 (1996), pp. 383–398. DOI: 10.1016/0550-3213(96)00080-6. arXiv: hep-th/9512081 [hep-th].

[97] Elias Kiritsis. "Duality and instantons in string theory". In: *Superstrings and related matters. Proceedings, Spring Workshop, Trieste, Italy, March 22-30, 1999.* 1999, pp. 127–205. arXiv: hep-th/9906018 [hep-th]. URL: http://alice.cern.ch/format/showfull?sysnb=0315238.

[98] Abdus Salam and E. Sezgin. "$d = 8$ Supergravity: Matter Couplings, Gauging and Minkowski Compactification". In: *Phys. Lett.* B154 (1985), p. 37. DOI: 10.1016/0370-2693(85)91563-1.

[99] D. J. Gross. "THE HETEROTIC STRING THEORY". In: *Phys. Scripta* T15 (1987), pp. 34–40. DOI: 10.1088/0031-8949/1987/T15/004.

[100] Ignatios Antoniadis et al. "Topological amplitudes in string theory". In: *Nucl. Phys.* B413 (1994), pp. 162–184. DOI: 10.1016/0550-3213(94)90617-3. arXiv: hep-th/9307158 [hep-th].

[101] Vadim S. Kaplunovsky. "One loop threshold effects in string unification". In: (1992). arXiv: hep-th/9205070 [hep-th].

[102] A. Gregori et al. "R**2 corrections and nonperturbative dualities of N=4 string ground states". In: *Nucl. Phys.* B510 (1998), pp. 423–476. DOI: 10.1016/S0550-3213(97)00635-4,10.1016/S0550-3213(98)81020-1. arXiv: hep-th/9708062 [hep-th].

[103] Ioannis Florakis and Boris Pioline. "On the Rankin-Selberg method for higher genus string amplitudes". In: (2016). arXiv: 1602.00308 [hep-th].

[104] Carlo Angelantonj, Ioannis Florakis, and Boris Pioline. "Rankin-Selberg methods for closed strings on orbifolds". In: *JHEP* 07 (2013), p. 181. DOI: 10.1007/JHEP07(2013)181. arXiv: 1304.4271 [hep-th].

[105] Carlo Angelantonj, Ioannis Florakis, and Boris Pioline. "One-Loop BPS amplitudes as BPS-state sums". In: *JHEP* 06 (2012), p. 070. DOI: 10.1007/JHEP06(2012)070. arXiv: 1203.0566 [hep-th].

[106] P. van Nieuwenhuizen. *Anomalies in quantum field theory: Cancellation of anomalies in d = 10 supergravity*. Vol. B3. Leuven notes in mathematical and theoretical physics. Leuven, Belgium: Leuven Univ. Pr., 1989.

[107] Jens Erler. "Anomaly cancellation in six-dimensions". In: *J. Math. Phys.* 35 (1994), pp. 1819–1833. DOI: 10.1063/1.530885. arXiv: hep-th/9304104 [hep-th].

[108] Michael Gutperle. "A Note on heterotic / type I-prime duality and D0-brane quantum mechanics". In: *JHEP* 05 (1999), p. 007. DOI: 10.1088/1126-6708/1999/05/007. arXiv: hep-th/9903010 [hep-th].

[109] Elias Kiritsis, Niels A. Obers, and Boris Pioline. "Heterotic / type II triality and instantons on K(3)". In: *JHEP* 01 (2000), p. 029. DOI: 10.1088/1126-6708/2000/01/029. arXiv: hep-th/0001083 [hep-th].

[110] W. Lerche and S. Stieberger. "Prepotential, mirror map and F theory on K3". In: *Adv. Theor. Math. Phys.* 2 (1998). [Erratum: Adv. Theor. Math. Phys.3,1199(1999)], pp. 1105–1140. arXiv: hep-th/9804176 [hep-th].

[111] Philip C. Argyres and Michael R. Douglas. "New phenomena in SU(3) supersymmetric gauge theory". In: *Nucl. Phys.* B448 (1995), pp. 93–126. DOI: 10.1016/0550-3213(95)00281-V. arXiv: hep-th/9505062 [hep-th].

[112] David R. Morrison and Cumrun Vafa. "Compactifications of F theory on Calabi-Yau threefolds. 2." In: *Nucl. Phys.* B476 (1996), pp. 437–469. DOI: 10.1016/0550-3213(96)00369-0. arXiv: hep-th/9603161 [hep-th].

[113] Alfred D. Shapere and Cumrun Vafa. "BPS structure of Argyres-Douglas superconformal theories". In: (1999). arXiv: hep-th/9910182 [hep-th].

[114] N. Seiberg and Edward Witten. "Comments on string dynamics in six-dimensions". In: *Nucl. Phys.* B471 (1996), pp. 121–134. DOI: 10.1016/0550-3213(96)00189-7. arXiv: hep-th/9603003 [hep-th].

[115] Paul S. Aspinwall and Mark Gross. "The SO(32) heterotic string on a K3 surface". In: *Phys. Lett.* B387 (1996), pp. 735–742. DOI: 10.1016/0370-2693(96)01095-7. arXiv: hep-th/9605131 [hep-th].

[116] Simeon Hellerman, John McGreevy, and Brook Williams. "Geometric constructions of nongeometric string theories". In: *JHEP* 01 (2004), p. 024. DOI: 10.1088/1126-6708/2004/01/024. arXiv: hep-th/0208174 [hep-th].

[117] Andreas Malmendier and David R. Morrison. "K3 surfaces, modular forms, and non-geometric heterotic compactifications". In: *Lett. Math. Phys.* 105.8 (2015), pp. 1085–1118. DOI: 10.1007/s11005-015-0773-y. arXiv: 1406.4873 [hep-th].

[118] H. Nishino and E. Sezgin. "Matter and Gauge Couplings of N=2 Supergravity in Six-Dimensions". In: *Phys. Lett.* B144 (1984), pp. 187–192. DOI: 10.1016/0370-2693(84)91800-8.

[119] H. Nishino and E. Sezgin. "The Complete $N = 2$, $d = 6$ Supergravity With Matter and Yang-Mills Couplings". In: *Nucl. Phys.* B278 (1986), pp. 353–379. DOI: 10.1016/0550-3213(86)90218-X.

[120] Hitoshi Nishino and Ergin Sezgin. "New couplings of six-dimensional supergravity". In: *Nucl. Phys.* B505 (1997), pp. 497–516. DOI: 10.1016/S0550-3213(97)00357-X. arXiv: hep-th/9703075 [hep-th].

[121] Sergio Ferrara, Fabio Riccioni, and Augusto Sagnotti. "Tensor and vector multiplets in six-dimensional supergravity". In: *Nucl. Phys.* B519 (1998), pp. 115–140. DOI: 10.1016/S0550-3213(97)00837-7. arXiv: hep-th/9711059 [hep-th].

[122] Fabio Riccioni. "All couplings of minimal six-dimensional supergravity". In: *Nucl. Phys.* B605 (2001), pp. 245–265. DOI: 10.1016/S0550-3213(01)00199-7. arXiv: hep-th/0101074 [hep-th].

[123] Augusto Sagnotti. "A Note on the Green-Schwarz mechanism in open string theories". In: *Phys. Lett.* B294 (1992), pp. 196–203. DOI: 10.1016/0370-2693(92)90682-T. arXiv: hep-th/9210127 [hep-th].

[124] Edward Witten. "An SU(2) Anomaly". In: *Phys. Lett.* B117 (1982), pp. 324–328. DOI: 10.1016/0370-2693(82)90728-6.

[125] M. Bershadsky et al. "F theory, geometric engineering and N=1 dualities". In: *Nucl. Phys.* B505 (1997), pp. 153–164. DOI: 10.1016/S0550-3213(97)00392-1. arXiv: hep-th/9612052 [hep-th].

[126] Edward Witten. "Small instantons in string theory". In: *Nucl. Phys.* B460 (1996), pp. 541–559. DOI: 10.1016/0550-3213(95)00625-7. arXiv: hep-th/9511030 [hep-th].

[127] M. J. Duff, R. Minasian, and Edward Witten. "Evidence for heterotic / heterotic duality". In: *Nucl. Phys.* B465 (1996), pp. 413–438. DOI: 10.1016/0550-3213(96)00059-4. arXiv: hep-th/9601036 [hep-th].

[128] M. J. Duff and R. Minasian. "Putting string / string duality to the test". In: *Nucl. Phys.* B436 (1995), pp. 507–528. DOI: 10.1016/0550-3213(94)00538-P. arXiv: hep-th/9406198 [hep-th].

[129] F. Giani, M. Pernici, and P. van Nieuwenhuizen. "GAUGED N=4 d = 6 SUPERGRAVITY". In: *Phys. Rev.* D30 (1984), p. 1680. DOI: 10.1103/PhysRevD.30.1680.

[130] L. J. Romans. "The F(4) Gauged Supergravity in Six-dimensions". In: *Nucl. Phys.* B269 (1986), p. 691. DOI: 10.1016/0550-3213(86)90517-1.

[131] Mirjam Cvetic, Hong Lu, and C. N. Pope. "Gauged six-dimensional supergravity from massive type IIA". In: *Phys. Rev. Lett.* 83 (1999), pp. 5226–5229. DOI: 10.1103/PhysRevLett.83.5226. arXiv: hep-th/9906221 [hep-th].

[132] Michael R. Douglas, Daniel S. Park, and Christian Schnell. "The Cremmer-Scherk Mechanism in F-theory Compactifications on K3 Manifolds". In: *JHEP* 05 (2014), p. 135. DOI: 10.1007/JHEP05(2014)135. arXiv: 1403.1595 [hep-th].

[133] Michael B. Green, Hwang-h. Kwon, and Pierre Vanhove. "Two loops in eleven-dimensions". In: *Phys. Rev.* D61 (2000), p. 104010. DOI: 10.1103/PhysRevD.61.104010. arXiv: hep-th/9910055 [hep-th].

[134] Anirban Basu. "The D**4 R**4 term in type IIB string theory on T**2 and U-duality". In: *Phys. Rev.* D77 (2008), p. 106003. DOI: 10.1103/PhysRevD.77.106003. arXiv: 0708.2950 [hep-th].

[135] Michael B. Green, Jorge G. Russo, and Pierre Vanhove. "Modular properties of two-loop maximal supergravity and connections with string theory". In: *JHEP* 07 (2008), p. 126. DOI: 10.1088/1126-6708/2008/07/126. arXiv: 0807.0389 [hep-th].

[136] Eric D'Hoker, Michael Gutperle, and D. H. Phong. "Two-loop superstrings and S-duality". In: *Nucl. Phys.* B722 (2005), pp. 81–118. DOI: 10.1016/j.nuclphysb.2005.06.010. arXiv: hep-th/0503180 [hep-th].

[137] Eric D'Hoker and D. H. Phong. "Two-loop superstrings VI: Non-renormalization theorems and the 4-point function". In: *Nucl. Phys.* B715 (2005), pp. 3–90. DOI: 10.1016/j.nuclphysb.2005.02.043. arXiv: hep-th/0501197 [hep-th].

[138] Eric D'Hoker. "Topics in Two-Loop Superstring Perturbation Theory". In: (2014). arXiv: 1403.5494 [hep-th].

[139] Eric D'Hoker and D. H. Phong. "Two loop superstrings. 2. The Chiral measure on moduli space". In: *Nucl. Phys.* B636 (2002), pp. 3–60. DOI: 10.1016/S0550-3213(02)00431-5. arXiv: hep-th/0110283 [hep-th].

[140] Eric D'Hoker and D. H. Phong. "Two loop superstrings. 3. Slice independence and absence of ambiguities". In: *Nucl. Phys.* B636 (2002), pp. 61–79. DOI: 10.1016/S0550-3213(02)00432-7. arXiv: hep-th/0111016 [hep-th].

[141] Eric D'Hoker and D. H. Phong. "Two loop superstrings 4: The Cosmological constant and modular forms". In: *Nucl. Phys.* B639 (2002), pp. 129–181. DOI: 10.1016/S0550-3213(02)00516-3. arXiv: hep-th/0111040 [hep-th].

[142] Eric D'Hoker and D. H. Phong. "Two-loop superstrings. V. Gauge slice independence of the N-point function". In: *Nucl. Phys.* B715 (2005), pp. 91–119. DOI: 10.1016/j.nuclphysb.2005.02.042. arXiv: hep-th/0501196 [hep-th].

[143] Eric D'Hoker and D. H. Phong. "Two-Loop Superstrings. VII. Cohomology of Chiral Amplitudes". In: *Nucl. Phys.* B804 (2008), pp. 421–506. DOI: 10.1016/j.nuclphysb.2008.04.030. arXiv: 0711.4314 [hep-th].

[144] Michael B. Green et al. "Eisenstein series for higher-rank groups and string theory amplitudes". In: *Commun. Num. Theor. Phys.* 4 (2010), pp. 551–596. DOI: 10.4310/CNTP.2010.v4.n3.a2. arXiv: 1004.0163 [hep-th].

[145] Erik P. Verlinde and Herman L. Verlinde. "Chiral Bosonization, Determinants and the String Partition Function". In: *Nucl. Phys.* B288 (1987), p. 357. DOI: 10.1016/0550-3213(87)90219-7.

[146] Erik P. Verlinde and Herman L. Verlinde. "Multiloop Calculations in Covariant Superstring Theory". In: *Phys. Lett.* B192 (1987), pp. 95–102. DOI: 10.1016/0370-2693(87)91148-8.

[147] Zhu-Jun Zheng, Jun-Bao Wu, and Chuan-Jie Zhu. "Two loop superstrings in hyperelliptic language 1: The Main results". In: *Phys. Lett.* B559 (2003), pp. 89–98. DOI: 10.1016/S0370-2693(03)00310-1. arXiv: hep-th/0212191 [hep-th].

[148] Zhu-Jun Zheng, Jun-Bao Wu, and Chuan-Jie Zhu. "Two loop superstrings in hyperelliptic language 2.: The Vanishing of the cosmological constant and the nonrenormalization theorem". In: *Nucl. Phys.* B663 (2003), pp. 79–94. DOI: 10.1016/S0550-3213(03)00380-8. arXiv: hep-th/0212198 [hep-th].

[149] Zhu-Jun Zheng, Jun-Bao Wu, and Chuan-Jie Zhu. "Two loop superstrings in hyperelliptic language 3: The Four particle amplitude". In: *Nucl. Phys.* B663 (2003), pp. 95–122. DOI: 10.1016/S0550-3213(03)00381-X. arXiv: hep-th/0212219 [hep-th].

[150] David Mumford. *Tata lectures on theta vol 1, 2 & 3*. Springer Birkhäuser Boston. ISBN: 978-0-8176-4569-4. DOI: 10.1007/978-0-8176-4578-6.

[151] John D. Fay. *Theta Functions on Riemann Surfaces*. Springer Berlin Heidelberg. ISBN: 9783540065173. DOI: 10.1007/bfb0060090.

[152] V. G. Knizhnik. "Explicit Expression for the Two Loop Measure in the Heterotic String Theory". In: *Phys. Lett.* B196 (1987). [JETP Lett.46,7(1987)], p. 473. DOI: 10.1016/0370-2693(87)90804-5.

ÉCOLE DOCTORALE
Physique
en île-de-France
(EDPIF)

**Titre :** Autour de Supergravité par l'Anomalie Composée et l'Amplitude en Théorie de Cordes

**Mots clefs :** QFT Anomalie, Supergravité, Amplitude en théorie de cordes

**Résumé :** Dans ce projet de thèse, nous étudions le rôle joué par l'anomalie dû à la connexion composée dans les théories de supergravités dans l'espace-temps à huit dimensions. Ce genre d'anomalie est en effet engendrée par la structure quotient d'espace des champs moduli de la supergravité là où le nombre des supercharges posent des contraintes rigoureux. Notre accomplissement principal est de proposer des termes supplémentaires pour annuler cette anomalie dans la théorie de supergravité en huit dimensions avec seize supercharges. Ces termes, en outre, peuvent être considérés comme une manifestation des corrections provenant de la théorie de super-cordes et nous montrons par des calculs explicites qu'une amplitude sur une boucle dans la théorie de cordes correspondante reproduit ces termes. Motivés par cette démonstration de la cohérence entre la supergravité et la théorie de cordes, nous proposons un seuil mathématique pour la compactification de ces théories dans huit dimensions vers six dimensions sur une sphère en présence des branes de co-dimension 2. Ceci est une simulation de compactification sur une surface K3 à l'aide des branes. Nous montrons que la présence d'anomalie composée ne peut être justifiée que par des branes de co-dimensions deux. Nous discutons la dualité entre la théorie Heterotic et la théorie-F sous la lumière de 7-branes et puis la compactification des supergravités de dix dimensions sur K3 en présence des 5-branes. Tous cela nous ouvrent nouvelles voies pour étudier des aspects non-perturbatives de la théorie de cordes. Nous concluons avec un calcul sur deux boucles dans la théorie de cordes Heterotic de dix dimensions qui n'était pas beaucoup exploré dans la littérature.

**Title :** Composite Anomaly in Supergravity and String Amplitude Comparison

**Keywords :** QFT Anomaly, Supergravity, String Amplitude

**Abstract :** We examine the structure of composite anomaly in maximal and half-maximal supergravity theories especially in eight space-time dimensions. The number of super-charges dictates the structure of the coset space of the moduli fields of the theory which in turn engenders the composite anomaly in such theories. Our main achievement lies in proposing counter-terms for such anomalies. These terms are of stringy nature and we show by explicit 1-loop amplitude calculations in corresponding string theories that those counter-terms are consistently provided by string amplitude. In the light of non-perturbative higher dimensional theories like F-theory, the anomaly cancelling counter-terms are seen to be related to co-dimension two branes e.g. 7-branes. We then use these results of 8-dimensional theories to provide for supergravity theories in six-dimensions by compactifying on a sphere in the presence of 5-branes. This is in fact a simulation of K3 compactification and our knowledge of composite connection provide us with threshold conditions to achieve such compactifications. All these analysis provide for greater insight into the non-perturbative regime of string theory. We then conclude with a calculation of 2-loop Heterotic string amplitude which has been very less explored in the literature.

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