Studies on classical observables from scattering amplitudes

By

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List of Publications arising from the thesis

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To my Parents:

Amma and Abba

And

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every destination is shaped by those who guide, question, and believe.

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Contents

Summary

The advent of gravitational-wave astronomy, marked by the LIGO-VIRGO-KAGRA (LVK) collaboration's detection of neutron star mergers and binary black holes [1–5], has stimulated the development of novel approaches to address the relativistic two-body problem in gravitational physics. Besides direct numerical integration of the Einstein field equations, the two primary perturbative methods for analyzing classical gravitational scattering of two objects are the post-Minkowskian (PM) and post-Newtonian (PN) schemes [6–24]. In the PM framework, one constructs classical observables, such as the scattering angle or the gravitational waveform, as a series expansion in the gravitational constant G. In contrast, the PN expansion is used when the incoming velocities of scattering particles or bound orbiting objects are non-relativistic. While PN is applicable in the inspiral phase* of the binary black hole dynamics, PM is applicable in the context of hyperbolic scattering involving relativistic bodies.

The array of tools available at our disposal in computing the gravitational observables to high orders in PN and PM expansions has seen a remarkable increase in the last few years. A key driving factor behind this progress is the realization that the scattering amplitudes in gauge theories and gravity can be leveraged to compute classical observables ranging from scattering angle to radiative flux in PM scattering. A particularly powerful formalism for deriving classical observables from the quantum S-matrix was introduced in a seminal work by Kosower, Maybee, and O'Connell (KMOC) [25]. The KMOC form-

*The inspiral phase in LIGO refers to the early stage of a compact binary merger when the two objects are orbiting each other and slowly losing energy due to gravitational wave emission.

alism employs scattering amplitudes to compute a set of asymptotic quantities, whose classical limits are the observables of interest.

Alongside the methodological advances achieved over the past few decades such as unitarity cut methods, integration-by-parts (IBP) reduction, double copy relations, etc. for perturbative amplitude calculations in gauge theory and gravity, there are a few non-perturbative results in the study of amplitudes, such as soft factorization theorems which can be used to significantly simplify the computation of the same. The late-time gravitational field emitted during the classical scattering can be derived using the soft factorization theorems, which offer remarkable insights into universal modes of gravitational radiation.

Among the non-perturbative category of tools, a technique was proposed in [26] which "spins" the external states in scalar QED or scalar-GR amplitudes from scalars to massive infinite spin particles that are minimally coupled to photons or gravitons. The minimal coupling of these particles to the gravitational or Maxwell field is equivalent to the classical coupling of the Kerr black hole with linearized gravity or the so-called $\sqrt{\text{Kerr}}$ charged state with the electromagnetic field. The action of the NJ mapping on scattering amplitudes was then used to compute the observable—linear impulse (the change in the momenta)—of the objects in the Schwarzschild-Kerr black hole scattering at 1PM order via the KMOC formalism.

This thesis is dedicated to the computation of classical observables beyond the linear impulse, including the change in angular momentum (angular impulse) and the radiative field, within the context of 2-2 electromagnetic scattering of scalar-$\sqrt{\text{Kerr}}$ at first post-Lorentzian (PL) expansion[†]. The analysis employs the Newman–Janis (NJ) algorithm within the scattering amplitude framework, highlighting its effectiveness in the non-conservative sector. We demonstrate that, for tree-level amplitudes, the Newman-Janis action can be reinterpreted as a dressing of the photon propagator, providing an efficient method for com-

[†]The expansion in the fine structure constant $\alpha \sim e^2$ in electromagnetism is referred to as the PL expansion, which is the analogue of the PM expansion in GR.

puting these classical observables. We derive the radiation emitted by the scalar particle to all orders in spin and find perfect agreement with its calculation using the equations of motion. Additionally, we compute the leading-order orbital angular impulse of the scalar particle to all orders in the spin of the $\sqrt{\text{Kerr}}$ particle and present a closed-form expression for the same. Furthermore, we provide a closed-form expression for the total angular impulse of the $\sqrt{\text{Kerr}}$ particle to leading order in spin. Along the way, we highlight a subtlety that arises in proving the conservation of angular momentum for scalar $-\sqrt{\text{Kerr}}$ scattering, taking into account the contribution to angular momentum stored in the late-time Coulombic modes, referred to as the "electromagnetic scoot."

We also investigate the significance of the infinite hierarchy of soft factorization theorems, beyond sub-leading order, for tree-level amplitudes in classical gravitational-scattering processes in four spacetime dimensions. The existence of these theorems has been recently linked with the existence of an infinite tower of asymptotic symmetries. For two massive scalar fields minimally coupled to gravity, we show that the infinite impact parameter limit (or the vanishing deflection limit) of the late-time gravitational field emitted during a classical scattering can be derived using these factorization theorems. The classical field obtained in this regime admits an expansion in the detector's frequency, with the modes scaling as $\omega^n \log \omega$, and exhibits a vanishing memory effect.

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Chapter 1

Introduction

The LIGO-VIRGO-KAGRA (LVK) collaboration's detection of neutron star mergers and binary black holes [1–5] has profoundly impacted multiple areas of astronomy, cosmology, and particle physics. However, these advancements are simply a glimpse of the field's future potential [27–30]. Next-generation space and ground-based observatories will have higher sensitivity than existing LIGO/Virgo detectors, allowing for the detection and categorization of a broader variety of merger events. The development of precise gravitational waveforms by theoretical modeling of compact binaries is a major problem in this endeavor. As a result, there has been a lot of interest in new gravitational physics approaches that try to solve the relativistic two-body problem. In addition to direct numerical integration of the Einstein field equations, the two primary perturbative procedures for exploring classical gravitational scattering between two objects are the post-Minkowskian (PM) and post-Newtonian (PN) expansions [6–24]. In the PM framework one constructs classical observables—such as the scattering angle and gravitational waveform—as a perturbative series in the gravitational constant G, making it ideal for studying relativistic scattering processes. In contrast, the PN expansion is designed for cases in which the interacting bodies' velocities are non-relativistic, making it useful for bound systems and quasi-circular inspirals. While the PN formalism is particularly relevant during the inspiral phase* of binary black hole dynamics, the PM approach is more appropriate for hyperbolic encounters with relativistic bodies.

Theoretical modeling of gravitational wave (GW) sources is intrinsically difficult due to the existence of numerous physical scales that are nonlinearly coupled via general relativity. Such complications have driven tremendous progress in recent decades in Quantum Field Theory (QFT), culminating in the development of the modern scattering amplitudes program. This paradigm uncovers deep mathematical features in gauge theory and gravity, resulting in highly efficient computational techniques and novel physical insights.

Techniques from theoretical particle physics have been effectively applied to GW physics, yielding significant advantages. In particular, the use of on-shell methods, Lorentz invariance, and the double copy construction streamlines perturbation theory, leading to compact expressions that expose deeper theoretical structures. Second, the substantial foundation for loop integrations in QFT, which was created primarily for collider physics, applies immediately to GW physics. This includes more advanced methods such as integration-by-parts (IBP) identities and the method of differential equations. Finally, effective field theory (EFT) provides a systematic and efficient framework for extracting relevant contributions to classical observables across a wide range of processes in the classical limit.

In recent years, there has been a surge of activity in applying double copy and the on-shell methods, along with EFT and advanced multi-loop integration techniques, to develop novel tools for high-precision predictions of GW signals [31]. These new approaches, rooted in theoretical high-energy physics, are intended to complement—and have significantly benefited from—decades of research achieved using established frameworks to handle the relativistic two-body problem [15, 32–35]. These proven methods include the PN [6–14] and PM [16–24, 36, 37] approximations, the self-force formalism [38, 39], the effective one-body (EOB) approach [40, 41], numerical relativity (NR) [42–44], and the

*The inspiral phase in LIGO denotes the early stage of a compact binary merger, wherein the constituent objects gradually spiral toward one another while emitting gravitational radiation.

non-relativistic general relativity (NRGR) [45] formalism.

A significant advantage of this method is that scattering amplitudes have proven to be highly effective tools for understanding and precisely modeling GW sources, just as they have been instrumental in elucidating fundamental particle interactions. Recent progress in this program has focused on generating precise predictions for gravitational radiation and the dynamics of compact binary systems, including black holes and neutron stars, by integrating analytic methods with numerical relativity. These efforts account for effects such as spin, tidal deformations, and radiation. A parallel objective is to advance the theoretical tools of high-energy physics for application to gravitational wave phenomena, by both refining existing methods and developing new techniques suited to binary dynamics. Additionally, the program investigates the classical limit of scattering amplitudes, emphasizing non-perturbative relationships with classical physics, the universality of high-energy scattering, and connections to spacetime geometry [31].

A landmark example is the Parke-Taylor formula [46] that condenses extensive Feynman diagrammatic computations for gluon scattering into a remarkably compact expression. This breakthrough emphasizes the importance of determining the underlying theoretical structures of scattering amplitudes. Two significant developments have driven advancement in recent years. The first involves the development of novel techniques for formulating QFT without the use of explicit field operators, instead focusing on directly physical quantities. These "on-shell methods" are effective for tree-level and loop-level calculations in gauge and gravity theories [47–49]. On-shell methods bypass the complexities of traditional Lagrangian-based approaches and Feynman diagrams by directly computing scattering amplitudes, which are physical observables (S-matrix elements) that describe the probabilities of particles interacting. These powerful techniques, including recursion relations and unitarity cuts, leverage the analytic properties and symmetries of quantum field theories, offering significant computational efficiency and clarity in complex calculations. The second development is the discovery of a novel perspective on gravity, wherein gravitational amplitudes is understood as a "double copy" of gauge theory amplitudes, revealing profound connections between the two theories [50, 51]. The double copy relates gravity amplitudes, which are typically much more intricate, to products of simpler gauge theory amplitudes, often based on a deep underlying color-kinematics duality. This remarkable principle provides a surprising and highly efficient bridge between fundamentally different quantum field theories, allowing gravitational computations to benefit from well-developed tools and insights from gauge theory.

The array of tools available at our disposal in computing the gravitational observables to high orders in PN and PM expansions has seen a remarkable increase in the last few years. A key driving factor behind this progress is the realization that the scattering amplitudes in gauge theories and gravity can be leveraged to compute classical observables ranging from scattering angle to radiative flux in PM scattering [52–83]. A particularly powerful formalism for deriving classical observables from the quantum S-matrix was introduced in a seminal work by Kosower, Maybee, and O'Connell (KMOC) [25, 84–87]. The KMOC formalism employs scattering amplitudes to compute a set of asymptotic quantities, whose classical limits are the observables of interest. The major advantage of this formalism is that it significantly simplifies the quantum computation in the desired classical limit. Additionally, the radiation reaction effects [81, 88] are naturally encoded in the framework.

Alongside the methodological advances achieved over the past few decades such as unitarity cut methods, IBP reduction, double copy relations, etc. for perturbative amplitude calculations in gauge theory and gravity, there are a few non-perturbative results in the study of amplitudes, such as soft factorization theorems [89–98] which can be used to significantly simplify the computation of the same. These theorems reveal the extent to which a gravitational or electromagnetic amplitude factorizes when one of the gravitons or photons becomes soft as compared to other external momenta. The late-time gravitational field emitted during the classical scattering can be derived using the soft factorization theorems, which offers remarkable insights into universal modes of gravitational radiation [89–103].

The two perturbative schemes PN and PM are intricately tied to each other in the case of large impact parameter, as explained in [53, 55] and are the most potent tools in analyzing the relationship between quantum amplitudes and classical scattering. However, a complementary perturbative expansion leads to different insights for gravitational radiation emitted in a scattering process. This expansion is not in terms of parameters intrinsic to the scattering process, but the characteristic frequency of the detector placed at null infinity. It is known as the soft expansion of gravitational radiation. At any given order in soft expansion, the radiative field is exact to all orders in PM and PN expansion. It is hence a non-perturbative probe to gravitational scattering and offers remarkable insights into universal modes of gravitational radiation in classical scattering. A correspondence between the soft limit in quantum scattering amplitudes and classical memory effects has been established [104]. The KMOC formalism offers a framework to study this relationship through the computation of the radiative field. In the soft-frequency regime, the scattering amplitude for radiation simplifies to a soft factor multiplied by a lower-point amplitude, resulting in the leading memory effect characterized by a shift in the field encoded in its low-frequency Fourier components [105]. The KMOC formalism connects classical physics, low-frequency radiation, and scattering amplitudes [92–94, 106, 107]. Tree-level gravitational amplitudes satisfy an infinite hierarchy of soft factorization theorems. The existence of these theorems has been recently linked with the existence of an infinite tower of asymptotic symmetries [108–111]. The outcome of these soft theorems for tree-level amplitudes in the context of classical gravitational scattering is one of the subjects of investigation in our thesis.

Waveform models for the binary inspiral phase are constructed from the conservative and dissipative aspects of two-body dynamics. Consequently, in modeling systems involving neutron stars and black holes [27, 30, 32, 112], scattering amplitudes play an increasingly significant role when accurate two-body potentials—including spin and tidal deformation effects—are incorporated. These potentials can be extracted from four-point scattering amplitudes through techniques such as amplitude matching [113], analytic continuation [114–116], and direct mapping to an EOB parameterization [117, 118]. But, at $\mathcal{O}(G^4)$, the Hamiltonian ceases to be universal across both unbound and bound trajectories. This breakdown arises due to the influence of radiation-reaction effects—commonly referred to as tail effects—which introduce non-local-in-time contributions to the Hamiltonian that depend explicitly on the trajectory [119–121]. Establishing a precise correspondence between bound and unbound orbits in the presence of radiation is therefore essential for leveraging the full potential of scattering amplitudes in classical bound-state dynamics, which have been addressed recently in [122, 123].

While there has been significant progress in computing observables at higher orders in the PM and PN expansions, as well as in modeling waveforms for spinless black holes, most astrophysical black holes are in fact spinning and are well described by the Kerr solution of general relativity. Incorporating spin and tidal deformations for such spinning black holes remains a theoretical challenge. In particular, including spin in amplitude-based approaches is difficult due to the need for fields with arbitrary spins. However, no-go theorems [124–127] demonstrate that such field theories have unphysical properties under certain assumptions. Multiple proposals [55, 128–131] have been made to derive classical binary Hamiltonians, with results that agree with those derived via traditional general relativity techniques as they become available. Notably, the higher-spin interactions in [132] lead to the derivation of the stress-energy tensor of a Kerr black hole and the stress tensor for more general extended spinning bodies was obtained in [55], building on earlier works in [15, 133, 134]. In [135], the scattering angle at $\mathcal{O}(G)$ was computed to all orders in spin. Furthermore, a number of studies have extended these results by calculating the scattering angle, linear impulse, and spin change up to $\mathcal{O}(G^3)$ and including contributions up to quartic order in spin [55, 135–138].

While these results are perturbative in spin, incorporating all orders in the exact spin of the Kerr black Hole is non-trivial. Beyond $\mathcal{O}(G^2)$, the construction of the stress tensor of the Kerr black hole to all orders of spin is a challenging task, as one needs to account for the tidal deformations, which in turn induce various spin multiple moments that depend on the tidal field. As a result, in an EFT treatment, the spinning black holes can no longer be treated as a point particle having infinite multipole moments, following the no-hair theorem. One needs to account for the change in the rigidity and shape of the body through tidal deformations by using higher-dimensional operators in the EFT that incorporate the response of the extended object to the tidal field.

An alternative approach involves the computation of the $\mathcal{O}(G)$ linearized stress-energy tensor for a stationary Kerr black hole [139]. This result was used to extract the gravitational couplings in the EFT action for a spinning black hole at linear order in the Riemann tensor. However, certain additional operators identified in [140] encode the "dynamical" multipole moments, which capture the response of the object to an external gravitational field. These operators do not contribute to the three-point amplitude at $\mathcal{O}(G)$ and hence do not affect the linearized stress-energy tensor of a stationary black hole. Nevertheless, [140] demonstrated that these dynamical multipole moments contribute to the Compton amplitude at $\mathcal{O}(G^2)$, thereby enabling the derivation of an all-orders-in-spin expression at this order within the EFT framework.

Recently, several attempts and approaches have been developed to analyze these spinning objects in the context of gravity and electromagnetism. Among the non-perturbative category of tools, a technique was proposed in [26] which "spins" the external states in scalar QED or scalar-GR amplitudes from scalars to massive infinite spin particles that are minimally coupled to photons or gravitons. The minimal coupling of these particles to the gravitational or Maxwell field is equivalent to the classical coupling of the Kerr black hole with linearized gravity or the so-called $\sqrt{\text{Kerr}}$ charged state with the electromagnetic field. The spinning technique discovered in [26] was inspired by the well-known

Newman-Janis (NJ) algorithm in classical general relativity and electromagnetism. As Newman and Janis showed, a complex coordinate transformation (a complex shift of the radial coordinate involving the spin parameter a) can be used to derive the Kerr metric from the Schwarzschild solution [141]. Remarkably, a similar mapping exists between solutions of the free Maxwell's equations. In electrodynamics, the NJ algorithm generates the so-called $\sqrt{\text{Kerr}}$ field from the Coulomb field of a charged point particle [142]. This $\sqrt{\text{Kerr}}$ field represents the electromagnetic field of a rotating charge distribution with radius a, defining an object with infinite multipole moments expressed solely in terms of its charge, mass, and angular momentum, analogous to the no-hair theorem for black holes. This field can also be understood as the flat-spacetime electromagnetic limit of a Kerr-Newman black hole. Its recent interpretation within the EFT framework shows that the $\sqrt{\text{Kerr}}$ object emerges as the classical limit of the three-point amplitude of a massive spin-S particle coupled to a photon, effectively being generated by a conserved current that defines a classical point particle with infinite multipole moments [143, 144]. Note that the Kerr solutions and the $\sqrt{\text{Kerr}}$ ones are related via the double copy, with the $\sqrt{\text{Kerr}}$ solution being the single copy of the Kerr solution, hence its name. The action of the NJ mapping on scattering amplitudes was then used to compute the observable—linear impulse (the change in the momenta) of the objects in the Schwarzschild-Kerr black hole scattering at 1PM order via the KMOC formalism. Thereafter, various kinds of Newman-Janis shifts have been explored at great length [145]- providing specific ways to implement such complex deformation to derive classical solutions for different spinning objects. The utilization of the Newman-Janis algorithm in amplitudes to compute observables is the subject of investigation in our thesis.

This thesis is dedicated to the computation of classical observables beyond the linear impulse, including the change in angular momentum (angular impulse) and the radiative field, within the context of 2-2 electromagnetic scattering of scalar-$\sqrt{\text{Kerr}}$ at first post-Lorentzian (PL) expansion[†]. The analysis employs the NJ algorithm within the scat-

[†]The expansion in the fine structure constant $\alpha \sim e^2$ in electromagnetism is referred to as the PL

tering amplitude framework, showing its power in the non-conservative sector. We also investigate the relevance of the infinite hierarchy of the soft factorization theorems in the context of such classical (gravitational) scattering processes in four spacetime dimensions. Chapter 2 reviews the background materials needed for this thesis. This includes the KMOC formalism, spinor-helicity variables, and the NJ algorithm for amplitudes. In chapter 3, we introduce the spin-dressed photon propagator, which is a reformulation of the NJ exponentiation in amplitudes and use it to compute classical observables for the scalar-$\sqrt{\text{Kerr}}$ scattering, such as the radiative gauge field and the angular impulse at 1PL. Chapter 4 begins with a review of the soft graviton theorems for tree-level amplitudes and is devoted to the investigation of the infinite hierarchy of soft factorization theorems in the context of classical gravitational radiation in $D = 4$. We conclude in chapter 5 by discussing the power of the NJ algorithm and the soft factorization theorems in amplitudes, along with a short outlook. We highlight some of the key results of our works [146, 147] arising from the thesis below.

- We demonstrate that, for tree-level amplitudes, the NJ action can be reinterpreted as a dressing of the photon propagator, providing an efficient method for computing the classical observables for the 2-2 electromagnetic scattering of scalar-$\sqrt{\text{Kerr}}$.
- We derive the radiation emitted by the scalar particle to all orders in spin. We also show that just as for the linear impulse, the radiation emitted by the scalar particle can also be obtained via complexification of the impact parameter.
- Additionally, we compute the leading-order orbital angular impulse of the scalar particle to all orders in the spin of the $\sqrt{\text{Kerr}}$ particle and present a closed-form expression for the same. Furthermore, we provide a closed-form expression for the total angular impulse of the $\sqrt{\text{Kerr}}$ particle to leading order in spin.
- Along the way, we highlight a subtlety that arises in proving the conservation of angular momentum for scalar $-\sqrt{\text{Kerr}}$ scattering, taking into account the contribution

expansion, which is the analogue of the PM expansion in GR.

to angular momentum stored in the late-time Coulombic modes, referred to as the "electromagnetic scoot."

- For two massive scalar fields minimally coupled to gravity, we show that the infinite impact parameter limit (or the vanishing deflection limit) of the late-time gravitational field emitted during a classical scattering can be derived using the soft factorization theorems for tree-level amplitudes.
- The classical field obtained in the infinite impact parameter regime admits an expansion in the detector's frequency, with the modes scaling as $\omega^n \log \omega$, and exhibits a vanishing memory effect.

Chapter 2

Background

In this chapter, we review the background material needed for this thesis. We start with reviewing the KMOC Formalism that is used to compute classical observables from on-shell scattering amplitudes. We then give a brief review of the Newman-Janis algorithm used as a classical solution-generating technique from static to spinning solutions, and its manifestation at the level of minimally coupled three-point amplitudes in electromagnetism. We also review the massive spinor-helicity variables used in writing these amplitudes. The soft factorization theorems for tree-level amplitudes will be reviewed in Chapter 4.

2.1 The KMOC Formalism

Scattering amplitudes have been utilized to investigate the general relativistic two-body problem. Prior to the development of the KMOC formalism [25], earlier works treating gravity as an effective field theory had already highlighted the significance of scattering amplitudes—particularly loop amplitudes—in determining classical gravitational potentials between two masses [148–151]. This perspective was notably emphasized by Donoghue and Holstein [151]. To fully utilize amplitude approaches in the gravitational-wave problem, it's important to grasp how to derive classical results from on-shell quantum scattering amplitudes.

We concentrate on observables that can be measured and extract classical quantities from a completely relativistic quantum computation. Specifically, we focus on three key observables. The first is the linear impulse, that is, the change in momentum of the particles during the scattering. The second is the change in the orbital angular momentum, i.e., orbital angular impulse, and the change in the spin, i.e, spin kick, during the scattering. The third is the radiated field during the event. The two observables, linear impulse and radiated momentum, are not entirely independent. Indeed, the relationship between these quantities lies at the heart of a key challenge in traditional approaches to point sources in classical field theory.

Let us begin by understanding the three observables purely within the framework of relativistic classical electrodynamics. We consider a 2-2 scattering event between two point-charged particles in spacetime:

- Particle 1: Charge Q_1, rest mass m_1. Its trajectory is $X_1^\mu(\tau_1)$, and its four-momentum is $P_1^\mu(\tau_1)$.
- Particle 2: Charge Q_2, rest mass m_2. Its trajectory is $X_2^\mu(\tau_2)$, and its four-momentum is $P_2^\mu(\tau_2)$.

The interaction occurs via the electromagnetic field, and we explicitly account for the emission of radiation during the acceleration of these charges.

1. **Linear Impulse:** In relativistic classical electrodynamics, linear momentum is naturally incorporated into the four-momentum vector P^μ. The concept of "linear impulse" for a particle in a scattering event is defined as the total change in its four-momentum due to the interaction with the other particle's electromagnetic field. For a particle with rest mass m and four-velocity $U^\mu = dX^\mu/d\tau$ (where τ is its proper time), its four-momentum is $P^\mu = mU^\mu$. In a given inertial frame, $P^\mu = (E/c, \mathbf{p})$,

where E is the relativistic total energy and $\mathbf{p}$ is the relativistic three-momentum. The interaction between charged particles is mediated by the electromagnetic field. The four-force acting on a charged particle Q with four-velocity U^ν due to an electromagnetic field $F^{\mu\nu}$ is given by

$$F^\mu = qF^{\mu\nu}U_\nu\,. \tag{2.1}$$

The linear impulse on particle 1 (ΔP_1^μ) represents the cumulative effect of the four-force exerted by particle 2's electromagnetic field on particle 1 over the entire scattering duration. It is the difference between particle 1's four-momentum far after the scattering and far before the scattering:

$$\Delta P_1^\mu = \int_{-\infty}^{\infty} F_{12}^\mu(\tau_1)\, d\tau_1 = P_{1,f}^\mu - P_{1,i}^\mu\,, \tag{2.2}$$

where $F_{12}^\mu(\tau_1)$ is the four-force exerted by particle 2's field on particle 1 as a function of particle 1's proper time. Similarly, for particle 2, it's the change in its four-momentum due to the four-force exerted by particle 1's electromagnetic field:

$$\Delta P_2^\mu = \int_{-\infty}^{\infty} F_{21}^\mu(\tau_2)\, d\tau_2 = P_{2,f}^\mu - P_{2,i}^\mu\,, \tag{2.3}$$

where $F_{21}^\mu(\tau_2)$ is the four-force exerted by particle 1's field on particle 2. The total four-momentum of the interacting system (particles + the electromagnetic field) is conserved. Since accelerating charges radiate, electromagnetic four-momentum is carried away by the emitted field. Therefore, for the particles alone:

$$P_{1,i}^\mu + P_{2,i}^\mu = P_{1,f}^\mu + P_{2,f}^\mu + P_{\text{rad}}^\mu\,. \tag{2.4}$$

Here, P_{rad}^μ is the total four-momentum (energy-momentum) carried away by the radiated electromagnetic field (Bremsstrahlung). This implies that the sum of the four-momentum transfers to the particles is precisely balanced by the four-momentum of the radiation:

$$\Delta P_1^\mu + \Delta P_2^\mu = -P_{\text{rad}}^\mu \,. \tag{2.5}$$

2. **Orbital Angular Impulse:** In relativistic mechanics, angular momentum is represented by an antisymmetric angular momentum tensor $L^{\mu\nu}$. This tensor describes both the spatial components of angular momentum and spatiotemporal components related to the center of energy. For a particle with four-position X^μ and four-momentum P^ν, its angular momentum tensor relative to the chosen origin is defined as:

$$L^{\mu\nu} = X^\mu P^\nu - X^\nu P^\mu \,. \tag{2.6}$$

The relativistic generalization of torque is an antisymmetric tensor, $\tau^{\mu\nu}$, which describes the rate of change of the angular momentum tensor with respect to proper time: $\tau^{\mu\nu} = dL^{\mu\nu}/d\tau$. It can also be expressed in terms of the four-position and four-force:

$$\tau^{\mu\nu} = X^\mu F^\nu - X^\nu F^\mu \,. \tag{2.7}$$

The Orbital Angular Impulse on particle 1 ($\Delta L_1^{\mu\nu}$) is the total change in the angular momentum tensor of Particle 1 due to the relativistic torques exerted by particle 2's field over the scattering process:

$$\Delta L_1^{\mu\nu} = \int_{-\infty}^{\infty} \tau_{12}^{\mu\nu}(\tau_1)\, d\tau_1 = L_{1,f}^{\mu\nu} - L_{1,i}^{\mu\nu} \,. \tag{2.8}$$

Similarly for particle 2,

$$\Delta L_2^{\mu\nu} = \int_{-\infty}^{\infty} \tau_{21}^{\mu\nu}(\tau_2)\, d\tau_2 = L_{2,f}^{\mu\nu} - L_{2,i}^{\mu\nu}\,. \tag{2.9}$$

The total angular momentum tensor of the entire system (particles + the electromagnetic field) is conserved. When charged particles accelerate and radiate, the emitted electromagnetic field carries away angular momentum. Thus, the sum of the angular momentum tensors of the particles alone is generally not conserved:

$$L_{1,i}^{\mu\nu} + L_{2,i}^{\mu\nu} = L_{1,f}^{\mu\nu} + L_{2,f}^{\mu\nu} + L_{\text{rad}}^{\mu\nu}\,. \tag{2.10}$$

Here, $L_{\text{rad}}^{\mu\nu}$ is the total angular momentum tensor carried away by the radiated electromagnetic field.

3. **Radiated Field:** In relativistic classical electrodynamics, the electromagnetic field itself is fundamentally described by the antisymmetric electromagnetic field strength tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{2.11}$$

The radiated field ($F_{\text{rad}}^{\mu\nu}$) is the specific component of this field tensor that describes the electromagnetic waves which propagate outwards from the accelerating charges, carrying energy and momentum away from the scattering event. These are the "radiation field" that fall off as $1/r$ in the far-field (radiation zone). Any acceleration of a charged particle ($A^\mu = dU^\mu/d\tau \neq 0$) in a relativistic setting necessarily leads to the emission of radiation. In the 2-2 scattering event, both particles undergo continuous acceleration due to their mutual electromagnetic interaction, particularly during their closest approach. The general solution for the electromagnetic fields of a point charge moving arbitrarily is derived from the Liénard-Wiechert four-potential. From this, one can extract the field tensor $F^{\mu\nu}$. The radiation component, $F^{\mu\nu}_{\text{rad}}$, is specifically the part of this field that carries energy to infinity. It is proportional to the four-acceleration of the charge. The radiated field produced during the scattering of charged particles is known as Bremsstrahlung. The characteristics of this radiation (its angular distribution, spectrum, and polarization) are determined by the relativistic kinematics of the colliding particles. For highly relativistic particles, the radiation is strongly "beamed" in the direction of the particle's instantaneous velocity. The energy and momentum carried by the electromagnetic field are described by its electromagnetic stress-energy tensor $T^{\mu\nu}_{\text{EM}}$:

$$T^{\mu\nu}_{\text{EM}} = \frac{1}{\mu_0}\left(F^{\mu\alpha}F^{\nu}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right). \tag{2.12}$$

The integral of the appropriate components of $T^{\mu\nu}_{\text{EM}}$ over a spacelike hypersurface containing the emitted radiation gives the total four-momentum P^{μ}_{rad} (and similarly for $L^{\mu\nu}_{\text{rad}}$) carried away by the radiated field. The "radiation flux" is quantified by the components of the Poynting vector, which are spatial components of $T^{\mu\nu}_{\text{EM}}$. Specifically, the energy flux (power per unit area) is given by $S^i = cT^{0i}$, where the spatial components of $T^{\mu\nu}_{\text{EM}}$ are proportional to the squared magnitude of the radiated fields (e.g., $|\mathbf{E}_{\text{rad}}|^2$).

Just to summarize, in relativistic classical electrodynamics, linear impulse and angular impulse are understood as changes in the particle's four-momentum and angular momentum tensor, respectively, directly caused by the four-force and relativistic torque tensors. The radiated field is the specific part of the electromagnetic field tensor that describes the emitted radiation, carrying away energy and momentum from the scattering particles. The Lorentz force describes the transfer of momentum between particles via the electromagnetic field. However, the Lorentz force does not account for the loss of energy-momentum by point particles due to radiation. The Abraham-Lorentz-Dirac (ALD) force contributes to momentum conservation. The addition of this radiation reaction force comes at a cost: it causes causality violations or runaway solutions in the classical electrodynamics of point sources. These problems are solved by using the quantum-mechanical model of charged-particle scattering. Indeed, the quantum-mechanical model conserves energy and momentum during particle scattering.

The KMOC formalism [25, 84] is a framework that is used to compute classical observables from on-shell quantum scattering amplitudes for large impact parameter scattering*. The formalism applies to both electrodynamics and gravity, but we will restrict ourselves to electrodynamics in this section. The fundamental method for computing classical observables involves preparing wave packets for the incoming particles, evolving them using the S-matrix operator, and evaluating the change in the expectation value of a self-adjoint quantum mechanical operator corresponding to the observable of interest in the final state. The classical result is subsequently obtained by taking the appropriate classical limit. The formalism's major characteristic is that the classical limit is taken at the loop integrand level before evaluating the full amplitude, which significantly simplifies the computation. Additionally, radiation reaction effects are naturally inbuilt within the framework. These advantages will be elaborated upon in the subsequent discussion. For a short sample of the results obtained with the formalism, we refer the reader to [59, 107, 152]. In this section, we shall highlight some of the features of the formalism relevant to us.

We analyze the scattering of two massive particles with momenta p_1 and p_2, separated by an impact parameter b^μ, transversal to the initial momenta. At the quantum level, the particles are modeled as wave packets. The point-particle approximation holds as long as the separation between the particles is large compared to their respective Compton

---

*The scattering setups in which the particles don't deviate much from the initial trajectories are known as large impact parameter scattering. Naturally, the characteristic length scale is set by the impact parameter. Recently, in [84], the formalism has been extended beyond this regime for incoming waves scattering off massive particles.

wavelengths ($l_c^{(i)} \equiv \frac{\hbar}{m_i}$), resulting in an accurate description if

$$\sqrt{-b^2} \gg l_c^{(1,2)} . \tag{2.13}$$

The wavefunctions possess an intrinsic length scale set by the spread of the wave packets, denoted l_w. This spread must be sufficiently narrow to ensure that the interaction does not probe the internal structure of the wave packet, yet not so small that quantum effects dominate its behavior. Only when the wave packet spread lies within this 'Goldilocks' regime—neither too wide nor too narrow—do quantum-mechanical expectation values reliably reproduce their classical counterparts for physical observables.

$$l_c \ll l_w \ll \sqrt{-b^2} . \tag{2.14}$$

We consider the 2-2 scattering of particles which are prepared in the distant past, $t \to -\infty$, and are directed toward each other with an impact parameter b^μ. As the particles are prepared in the asymptotic past, the relevant quantum states are the incoming states, denoted by $|\Psi\rangle_{\text{in}}$. These states are described in terms of wavefunctions $\phi_i(p_i)$, where p_i denotes the momentum of the i-th particle. Since our primary interest lies in the scattering of point-like classical particles, we consider wavefunctions that are sufficiently localized in both position and momentum space to admit a clear semiclassical interpretation. The initial state is then [25, 85],

$$|\Psi\rangle_{in} = \int \prod_{i=1}^{2} d\Phi(p_i) e^{ip_2 \cdot b/\hbar} \phi_i(p_i) \zeta_i^{a_i} |\vec{p}_1, a_1; \vec{p}_2, a_2\rangle , \tag{2.15}$$

where

$$d\Phi(p) = \frac{d^4 p}{(2\pi)^4} \hat{\delta}(p^2 - m^2) \Theta(p^0), \quad \int d\Phi(p) \, |\phi(p)|^2 = 1, \tag{2.16}$$

The functions $\phi_i(p_i)$ are taken to be minimum-uncertainty wave packets, modeled as relativistic momentum-space wavefunctions of the form

$$\phi(p) = \frac{1}{m}\left[\frac{8\pi^2}{\xi K_1\left(\frac{2}{\xi}\right)}\right]^{1/2} \exp\left(-\frac{p\cdot u}{\xi m}\right), \tag{2.17}$$

where the normalization factor involves the modified Bessel function of the second kind, K_1. The dimensionless parameter $\xi = \left(\frac{l_c}{l_w}\right)^2$ characterizes the wave packet width, with l_c the Compton wavelength and l_w the width of the wave packet. The four-vector u^μ denotes the classical four-velocity of the particle, normalized such that $u^2 = 1$. The classical limit is obtained by examining the behavior of physical observables in the limit $\xi \to 0$, corresponding to sharply peaked wave packets in both position and momentum space.

The spin wavefunctions $\zeta_i^{a_i}$ represent coherent spin states for the particles, with the little group indices denoted by a_i. These states are particularly well-suited for describing classical angular momentum [153–155]. The coherent spin states are defined by

$$|\zeta^a\rangle = \exp\left(-\frac{\tilde{\alpha}_a\alpha^a}{2}\right)\exp\left(\alpha^a a_a^\dagger\right)|0\rangle, \tag{2.18}$$

where $|0\rangle$ denotes the zero-spin (vacuum) state, and the definite-spin states are generated through the action of the creation operators $a_a^\dagger$. The quantity $\tilde{\alpha}_a$ represents the complex conjugate of the SU(2) spinor α^a. These coherent states can be expanded in terms of definite-spin states via Schwinger's construction [156]. A key property of the coherent spin states is their behavior under expectation values of the angular momentum operator [86]. In the classical limit, their one-particle expectation values reproduce classical spin vectors $\langle\zeta^a|S|\zeta^a\rangle \xrightarrow{\hbar\to 0} s_{\text{cl}}$, where the expectation value of the angular momentum operator takes the form $\langle S^i\rangle_\alpha = \frac{\hbar}{2}\left(\tilde{\alpha}\sigma^i\alpha\right)$, with σ^i being the Pauli matrices.

The wavepacket of the second particle is translated, with respect to the first particle's wavepacket, by a distance of b - the impact parameter. Since the initial particles are described by coherent states, we have

$$\langle \mathbb{P}_i^\mu \rangle = m_i u_i^\mu + \mathcal{O}(\hbar), \qquad \frac{\sigma_i^2}{m_i^2} = \frac{\left(\langle \mathbb{P}_i^2 \rangle - \langle \mathbb{P}_i \rangle^2\right)}{m_i^2} \xrightarrow{\hbar \to 0} 0, \tag{2.19}$$

where σ_i^2 is the variance and m_is are the masses of the particles. Here, the expectation value of the momentum operator is taken with respect to the initial state in equation (2.15).

The spin of a particle in QFT is given by the expectation value of the Pauli-Lubanski vector [85],

$$\mathbb{W}^\mu = \frac{1}{m} \epsilon^{\mu\nu\rho\sigma} \mathbb{P}_\nu \mathbb{S}_{\rho\sigma} \, . \tag{2.20}$$

Hence, it is the expectation value of the above operator which gives the classical spin pseudovector,

$$\langle \mathbb{W}_i^\mu \rangle = s_i^\mu + \mathcal{O}(\hbar). \tag{2.21}$$

The variance in spin is small, analogous to the behavior observed for momentum, as a consequence of the properties of the coherent spin states discussed earlier. This behavior is ensured by taking the spin quantum number to be large, $|S| \gg \hbar$, so that the spin effectively behaves as a classical vector. The particle wavefunctions are taken to be relativistic generalizations of Gaussian wavepackets, constructed so that their integrals are sharply peaked around classical momentum values.

We now move on to describe the construction of the classical observables. The basic idea is to compute the change in the expectation value of a quantum mechanical operator, as this is what is relevant from a classical perspective. So we write

$$\langle \Delta \mathbb{O} \rangle = {}_{in}\langle \Psi | S^\dagger \hat{O} S | \Psi \rangle_{in} - {}_{in}\langle \Psi | \hat{O} | \Psi \rangle_{in}. \tag{2.22}$$

where $S = I + iT$ is the S-matrix. Using unitarity, we get

$$\langle \Delta\mathbb{O}\rangle = {}_{in}\langle\Psi|i[\hat{O},T]|\Psi\rangle_{in} + {}_{in}\langle\Psi|T^{\dagger}[\hat{O},T]|\Psi\rangle_{in}\,. \tag{2.23}$$

In relation to conventional scattering amplitudes, it is worth noting that the first term is linear in the amplitude, while the second term corresponds to a weighted cut of an amplitude, which contributes only at one-loop order and beyond. The S-matrix-governed evolution sums over all possible out states and thus characterizes this framework as an "in-in" formalism. For the linear impulse, $\mathbb{O} = \mathbb{P}^{\mu}$, the momentum operator, $\mathbb{O} = \mathbb{L}^{\mu\nu}$ for orbital angular impulse, $\mathbb{O} = \mathbb{W}^{\mu}/m$ for the spin kick and $\mathbb{O} = \mathbb{A}^{\mu}(x)$, the gauge field operator, from which we read off the radiation kernel.

The expectation value of the radiated momentum is not entirely independent of the impulse; rather, the relationship between these two observables encodes essential physical information. In classical electrodynamics, for example, the total impulse imparted to a system of point charges arises from a combination of the Lorentz force—mediating the mutual exchange of momentum between particles—and the Abraham–Lorentz–Dirac (ALD) radiation reaction force, which accounts for the momentum carried away by the emitted radiation. In contrast, in the quantum theory, there is no ambiguity in how momentum conservation is implemented. Within the KMOC framework, quantum observables are defined in a way that inherently includes all contributions—both conservative and radiative—and respects exact momentum conservation. Notably, this feature holds to all orders in coupling and can be demonstrated formally within the formalism.

To see this explicitly, consider a scattering process involving two incoming particles. The expectation value of the total change in momentum is given by

$$\begin{aligned}\langle \Delta p_1^{\mu}\rangle + \langle \Delta p_2^{\mu}\rangle &= {}_{\rm in}\langle\Psi|i[\mathbb{P}_1^{\mu}+\mathbb{P}_2^{\mu},T]|\Psi\rangle_{\rm in} + {}_{\rm in}\langle\Psi|T^{\dagger}[\mathbb{P}_1^{\mu}+\mathbb{P}_2^{\mu},T]|\Psi\rangle_{\rm in}\\ &= {}_{\rm in}\langle\Psi|i\left[\sum_i \mathbb{P}_i^{\mu},T\right]|\Psi\rangle_{\rm in} + {}_{\rm in}\langle\Psi|T^{\dagger}[\mathbb{P}_1^{\mu}+\mathbb{P}_2^{\mu},T]|\Psi\rangle_{\rm in}\,,\end{aligned} \tag{2.24}$$

where the sum $\sum \mathbb{P}_i^\mu$ runs over all momentum operators in the theory. The second equality follows from the fact that the incoming state $|\Psi\rangle_{\text{in}}$ contains only particles 1 and 2, so $\mathbb{P}_i^\mu|\Psi\rangle_{\text{in}} = 0$ for all $i \neq 1, 2$.

The total momentum is conserved by the S-matrix, implying $[\sum_i \mathbb{P}_i^\mu, T] = 0$. Consequently,

$$
{}_{\text{in}}\langle\Psi|i[\mathbb{P}_1^\mu + \mathbb{P}_2^\mu, T]|\Psi\rangle_{\text{in}} = {}_{\text{in}}\langle\Psi|i\left[\sum_i \mathbb{P}_i^\mu, T\right]|\Psi\rangle_{\text{in}} = 0\,. \tag{2.25}
$$

Thus, the first term ${}_{\text{in}}\langle\Psi|i[\hat{\mathcal{O}}, T]|\Psi\rangle_{\text{in}}$ appearing in the general expression for the impulse, equation (2.23), accounts only for the momentum exchange between particles 1 and 2. It therefore corresponds to the classical Lorentz force. In contrast, the second term—which involves $T^\dagger[\hat{\mathcal{O}}, T]$—contains contributions from radiation.

To make this structure more transparent, let us restrict our attention to a theory in which the only other momentum operator corresponds to the radiated field and is denoted by $\mathbb{K}^\mu$. Total momentum conservation then implies

$$
[\mathbb{P}_1^\mu + \mathbb{P}_2^\mu + \mathbb{K}^\mu, T] = 0\,. \tag{2.26}
$$

Using this relation, the conservation of momentum at the level of expectation values follows immediately:

$$
\begin{aligned}
\langle\Delta p_1^\mu\rangle + \langle\Delta p_2^\mu\rangle &= -{}_{\text{in}}\langle\Psi|T^\dagger[\mathbb{K}^\mu, T]|\Psi\rangle_{\text{in}} = -{}_{\text{in}}\langle\Psi|T^\dagger\mathbb{K}^\mu T|\Psi\rangle_{\text{in}} \\
&= -\langle k^\mu\rangle = -P^\mu_{\text{rad}}\,.
\end{aligned} \tag{2.27}
$$

This confirms that the total impulse is equal and opposite to the expectation value of the radiated momentum, as required by momentum conservation.

The expression in eq. (2.23) can be simplified using the on-shell completeness relation and unitarity of the S-matrix [25]. Plugging in the expression for the initial state in eq.

(2.15), for the linear impulse, we get

$$\langle \Delta p_1^\mu \rangle = i \int \hat{d}^4 q \, \hat{\delta}(2p_1 \cdot q + q^2) \hat{\delta}(2p_2 \cdot q - q^2) \, e^{iq \cdot b/\hbar} q^\mu \mathcal{A}_4(p_1, p_2 \to p_1 + q, p_2 - q)$$
$$+ \int \hat{d}^4 q \, \hat{\delta}(2p_1 \cdot q + q^2) \hat{\delta}(2p_2 \cdot q - q^2) \, e^{iq \cdot b/\hbar} \sum_X \int \prod_{i=1,2} \hat{d}^4 w_i \, \hat{\delta}(2p_i \cdot w_i + w_i^2)$$
$$\times w_1^\mu \hat{\delta}^{(4)}(w_1 + w_2 + r_X)$$
$$\times \mathcal{A}(p_1, p_2 \to p_1 + w_1, p_2 + w_2, r_X) \times \mathcal{A}^*(p_1 + q, p_2 - q \to p_1 + w_1, p_2 + w_2, r_X), \tag{2.28}$$

where we have used $\mathbb{P}^\mu \, |\vec{p}_1, \vec{p}_2\rangle = p^\mu \, |\vec{p}_1, \vec{p}_2\rangle$ and

$$A_4(p_1, p_2 \to p_1 + q, p_2 - q) := \langle \vec{p}_1 + \vec{q}, \vec{p}_2 - \vec{q} | \, \mathbb{T} \, | \vec{p}_1, \vec{p}_2 \rangle \, . \tag{2.29}$$

Here, r_X is the total momentum carried by the particles in the intermediate state X. At leading order in the coupling, the only contribution comes from the tree-level four-point amplitude in the first term. At next-to-leading order (NLO), both terms contribute. The contribution from the first term is from the one-loop amplitude, while that from the second term has $X = \varnothing$, so that both the amplitude and conjugate inside the integral are tree-level four-point amplitudes. We are interested only in the leading order in the coupling, $\mathcal{O}(e^2)$ term for this thesis and hence, we concentrate only on the first term in eq. (2.23).

Plugging in the expression for the initial state in eq.(2.15), we get for the orbital angular impulse

$$\Delta L_i^{\mu\nu} = i \int \prod_{i=1}^{2} \, d\Phi(r_i) d\Phi(p_i) \, e^{-ir_2 \cdot b/\hbar} e^{ip_2 \cdot b/\hbar} \, \phi^*(r_i) \phi(p_i)$$
$$(\langle \vec{r}_1, \vec{r}_2 | \, \mathbb{L}_i^{\mu\nu} \mathbb{T} \, | \vec{p}_1, \vec{p}_2 \rangle - \langle \vec{r}_1, \vec{r}_2 | \, \mathbb{T} \mathbb{L}_i^{\mu\nu} \, | \vec{p}_1, \vec{p}_2 \rangle) + \mathcal{O}(T^\dagger T) \, . \tag{2.30}$$

Here we have suppressed the little group indices a_i. Now, we use

$$\langle \vec{r}_1, \vec{r}_2 | \mathbb{T}\mathbb{L}_i^{\mu\nu} | \vec{p}_1, \vec{p}_2 \rangle := i\hbar \left(p_i \wedge \frac{\partial}{\partial p_i} \right)^{\mu\nu} \langle \vec{r}_1, \vec{r}_2 | \mathbb{T} | \vec{p}_1, \vec{p}_2 \rangle \,, \tag{2.31}$$

to write the first term in terms of the differential operator, we use the hermiticity of the orbital angular momentum operator i.e

$$\langle \vec{r}_1, \vec{r}_2 | \mathbb{L}_i^{\mu\nu} \mathbb{T} | \vec{p}_1, \vec{p}_2 \rangle = \left(\langle \vec{p}_1, \vec{p}_2 | \mathbb{T}^\dagger \mathbb{L}_i^{\mu\nu} | \vec{r}_1, \vec{r}_2 \rangle \right)^\dagger \tag{2.32}$$

$$= -i\hbar \left(r_i \wedge \frac{\partial}{\partial r_i} \right)^{\mu\nu} \left(\langle \vec{p}_1, \vec{p}_2 | \mathbb{T}^\dagger | \vec{r}_1, \vec{r}_2 \rangle \right)^\dagger \tag{2.33}$$

$$= -i\hbar \left(r_i \wedge \frac{\partial}{\partial r_i} \right)^{\mu\nu} \langle \vec{r}_1, \vec{r}_2 | \mathbb{T} | \vec{p}_1, \vec{p}_2 \rangle \tag{2.34}$$

Plugging the last equation and eq.(2.31) into eq.(2.30), we get

$$\Delta L_i^{\mu\nu} = \hbar \int \prod_{i=1}^{2} d\Phi(r_i) d\Phi(p_i)\; e^{-ir_2\cdot b/\hbar} e^{ip_2\cdot b/\hbar}\; \phi^*(r_i)\phi(p_i) \left[\left(p_i \wedge \frac{\partial}{\partial p_i} \right)^{\mu\nu} + \left(r_i \wedge \frac{\partial}{\partial r_i} \right)^{\mu\nu} \right] \langle \vec{r}_1, \vec{r}_2 | \mathbb{T} | \vec{p}_1, \vec{p}_2 \rangle \tag{2.35}$$

By relabelling from $r_i = p_i + q_i$, we get

$$\langle \Delta L_i^{\mu\nu} \rangle = \hbar \int \prod_{i=1}^{2} \hat{d}^4 q_i\; \hat{\delta}(2p_i \cdot q_i + q_i^2)\; e^{-iq_2\cdot b/\hbar} \left(\left(p_i \wedge \frac{\partial}{\partial p_i} \right)^{\mu\nu} + \left((p_i + q_i) \wedge \frac{\partial}{\partial (p_i + q_i)} \right)^{\mu\nu} \right) A_4(p_1, p_2 \to p_1 + q_1, p_2 + q_2) \tag{2.36}$$

where $\langle \Delta L_i^{\mu\nu} \rangle$ denotes the integration over the wave functions. Here we have substituted the definition of the 4-point scattering amplitude

$$A_4(p_1, p_2 \to p_1 + q_1, p_2 + q_2) := \langle \vec{p}_1 + \vec{q}_1, \vec{p}_2 + \vec{q}_2 | \mathbb{T} | \vec{p}_1, \vec{p}_2 \rangle \,, \tag{2.37}$$

where we write the full 4-point amplitude as

$$A_4(p_1, p_2 \to p_1 + q_1, p_2 + q_2) := \mathcal{A}_4(p_1, p_2 \to p_1 + q_1, p_2 + q_2)\hat{\delta}^{(4)}(q_1 + q_2)\,. \tag{2.38}$$

We do not display the higher-order contributions, as in this thesis, since we will only be interested in calculating the observables to leading order in coupling. To compute the radiation flux emitted during a scattering event, the KMOC formalism introduces a key quantity known as the *radiation kernel* $\mathcal{R}^\mu(k)$. This object encodes the gauge field radiated with momentum $k^\mu = \omega(1, \hat{n})$, where ω is the frequency and $\hat{n}$ the direction of emission, and it originates from the inelastic nature of the scattering process. The total radiated flux is then obtained by integrating the modulus squared of the radiation kernel over the on-shell phase space of the emitted gauge boson. At leading order in the gauge coupling, the classical radiation kernel in the electromagnetic case takes on a particularly compact and elegant form [84]:

$$\begin{aligned}\langle \mathcal{R}^\mu(k)\rangle = \hbar^{3/2} \int \prod_{i=1}^{2} \hat{d}^4 q_i\, \hat{\delta}(2p_i \cdot q_i + q_i^2)\, e^{-iq_2\cdot b/\hbar} \\ \hat{\delta}^{(4)}(q_1 + q_2 - k)\, \mathcal{A}_5^\mu(p_1 + q_1, p_2 + q_2 \to p_1, p_{2,}, k) \;+\; \mathcal{O}(T^\dagger T)\,.\end{aligned} \tag{2.39}$$

The spin kick, to leading order in the coupling, is given by [85]

$$\begin{aligned}\langle \Delta a^\mu\rangle = \int \hat{d}^4 q\, \hat{\delta}(2p_1 \cdot q + q^2)\hat{\delta}(2p_2 \cdot q - q^2)\, e^{iq\cdot b/\hbar} \\ \Big(a^\mu(p+q)\mathcal{A}_4(p_1, p_2 \to p_1 + q, p_2 - q) - \mathcal{A}_4(p_1, p_2 \to p_1 + q, p_2 - q)\, a^\mu(p)\Big)\,.\end{aligned} \tag{2.40}$$

During a small-angle (large impact parameter) scattering, the semi-classical in-state evolves to a semi-classical out-state where the momenta are peaked around $p_i + \mathcal{O}(\frac{1}{b})$ as $b \to \infty$. As b and q are conjugate variables, $q \to 0$. This is naturally incorporated in the KMOC formalism by re-scaling all the massless momenta and replacing them with their wave numbers, i.e, the exchange momenta ($q_i = \hbar\bar{q}_i$) such that $\bar{q}$ is fixed in the classical limit $\hbar \to 0$. The dimensionful couplings are also rescaled appropriately, and then we extract the leading order in $\hbar$ contribution ($\mathcal{O}(\hbar^0)$) of the expectation value of the observable. In QED, the dimensionful coupling is obtained by[†] $e \to e/\sqrt{\hbar}$. This originates from the fundamental principle that, in the classical limit, physical quantities such as the action remain finite. The connection between quantum amplitudes and classical physics is encoded in the exponential factor $e^{iS_{\text{cl}}/\hbar}$, where S_{cl} denotes the classical action (or the effective potential). For this phase to admit a meaningful classical interpretation as $\hbar \to 0$, the classical action S_{cl} must remain finite and $\hbar$-independent. To illustrate this, consider the tree-level scattering amplitude in scalar QED for two charged particles exchanging a single photon. This amplitude scales with the coupling as e^2. If one attempts to extract an effective potential or S_{cl} from this amplitude, the combination $\hbar e^2$ must remain finite. This implies the classical scaling behavior $e^2 \propto \hbar^{-1}$. For finite classical contributions from loops, the massless momenta q scaling also forces the same scaling for the loop momenta automatically, i.e. ($l_i = \hbar\bar{l}_i$) such that $\bar{l}$ is fixed, as the $l \gg q$ regime leads to quantum contributions.

For spinning external states, in the classical limit, the final state spin pseudovector can be written as

$$
\begin{aligned}
a_i^\mu(p_i + \hbar\bar{q}_i) &= a_i^\mu(p_i) + \Delta a_i^\mu(p_i)\,, \\
\Delta a_i^\mu(p_i) &= \omega^\mu{}_\nu(p_i;\bar{q}_i)a^\nu(p)\,, \\
\omega^\mu{}_\nu(p_i;\bar{q}_i) &= -\frac{\hbar}{m^2}(p_i \wedge \bar{q})^\mu{}_\nu\,,
\end{aligned}
\tag{2.41}
$$

where $\omega^{\mu\nu}(p;\bar{q})$ is the infinitesimal boost parameter. With these expressions in hand, we can write down the classical limit of the linear impulse, the radiation kernel, and the spin kick, at leading order in the coupling. The expression for the former is

$$
\Delta p^\mu = \left\langle\!\!\left\langle \int \hat{d}^4\bar{q}\; \hat{\delta}(2p_1\cdot\bar{q})\hat{\delta}(2p_2\cdot\bar{q})\; e^{i\bar{q}\cdot b}\; \bar{q}^\mu\; \left(\hbar^2\mathcal{A}_4(p_1,p_2 \to p_1+\hbar\bar{q}, p_2-\hbar\bar{q})\right)\right\rangle\!\!\right\rangle. \tag{2.42}
$$

[†]The dimensionless coupling is the fine structure constant $\alpha = \frac{e^2}{\hbar}$.

Here $\left\langle\!\!\left\langle f(p_1, p_2, q \ldots) \right\rangle\!\!\right\rangle$ denotes the integration over the minimum uncertainty wave packets which localizes the momenta and spin onto their classical values. Similarly, for the radiation kernel, we get

$$\mathcal{R}^\mu(\bar{k}) = \left\langle\!\!\left\langle \int \prod_{i=1}^{2} \hat{d}^4\bar{q}_i\, \hat{\delta}(2p_i \cdot \bar{q}_i)\, e^{-i\bar{q}_2\cdot b}\, \hat{\delta}^{(4)}(\bar{q}_1 + \bar{q}_2 - \bar{k}) \right.\right.$$
$$\left.\left. \left(\hbar^2 \mathcal{A}_5(p_1 + \hbar\bar{q}_1, p_2 + \hbar\bar{q}_2 \to p_1, p_{2,}, \hbar\bar{k})\right) \right\rangle\!\!\right\rangle , \tag{2.43}$$

And for the spin kick, we get

$$\Delta a^\mu = \left\langle\!\!\left\langle i\hbar^2 \int \hat{d}^4\bar{q}\hat{\delta}(2p_1 \cdot \bar{q})\hat{\delta}(2p_2 \cdot \bar{q})e^{i\bar{q}\cdot b}\left\{ [a^\mu(p), \mathcal{A}_4] + \frac{\hbar}{m}(a \cdot \bar{q})u^\mu \mathcal{A}_4 \right\} \right\rangle\!\!\right\rangle . \tag{2.44}$$

In all of the above expressions, we have taken out the $\hbar-$ scaling of the coupling constant. For the orbital angular impulse, we shall derive the corresponding expression in Section 3.3 as it is slightly more detailed.

A key strength of the formalism is the ability to take the classical limit early in the process—specifically, at the level of loop integrands—before evaluating the full amplitude. This greatly simplifies the computation, as only a limited number of Feynman diagrams contribute to the classical limit. For example, in the context of the linear impulse for scalar particles in electromagnetism, the relevant contributions arise from the four-point scalar QED amplitudes. At the one-loop level, for each diagram, we systematically count powers of $\hbar$ by implementing the rescaling rules described earlier in this section, namely $l \to \hbar\bar{l}$, $q \to \hbar\bar{q}$ and $e \to e/\sqrt{\hbar}$.

Consider, for instance, the double-seagull diagram shown in Fig. 2.1: it contributes four powers of $\hbar$ from the loop measure, which are exactly canceled by four inverse powers from the two internal photon propagators. As a result, the overall $\hbar$-scaling is insufficient to compensate for the explicit power of $\hbar$ in front of the integral in eq. (2.28), and the dia-

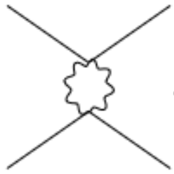

Figure 2.1: Double-seagull diagram in scalar QED

gram therefore vanishes in the classical limit. The remaining diagrams, such as triangles and boxes, will contribute in the classical limit.

A more general conclusion can be drawn from this: for a diagram to contribute classically, it must contain at least one internal matter line. This criterion follows directly from analyzing the classical limit and effectively rules out a large class of diagrams that yield purely quantum contributions. The KMOC formalism further emphasizes that the classical limit $\hbar \to 0$ does not uniformly influence the entire loop integral in a quantum scattering amplitude. Instead, only particular regions of the loop momentum space give rise to classical physics. KMOC systematically identifies these so-called "classical regions" prior to performing the integration. In such regions, loop momenta exhibit characteristic scalings with respect to powers of $\hbar$, which dictate their relevance to classical observables. Rather than computing the full quantum loop integrals, the KMOC approach reorganizes the integrand to isolate precisely those terms that survive in the classical limit. This targeted extraction leads to considerably simpler integrals, focused solely on the relevant classical phase space. Consequently, it avoids the need for many conventional quantum field theory techniques—such as elaborate regularization schemes and extensive algebraic computations—yielding a far more efficient pathway to extracting classical results from quantum amplitudes.

For 1-loop contributions, there exist "superclassical" divergences ($\mathcal{O}(1/\hbar)$) which arise from the contribution that is linear in amplitude. However, such superclassical terms cancel once the weighted cut term is included in eq. (2.23). Thus, the classical limit is smooth in the KMOC formalism. This smoothness of the classical limit has been proved very recently [157] to all orders in coupling, using the exponential representation of the S-matrix [158, 159]. This illustrates that the KMOC formalism enables the computation of classical observables by isolating all terms that scale as $\hbar^0$, since contributions with negative powers of $\hbar$ vanish in the classical limit. The KMOC formalism has been generalized to describe different types of scattering. For instance, in [84] the formalism has been extended to include incoming waves in the initial state. It has also been extended to include additional internal degrees of freedom like color charges in [160]. Finally, it has been generalized to describe scattering in curved backgrounds [87, 161].

2.2 Spinor helicity formalism

Scattering amplitudes are Lorentz-invariant quantities that transform covariantly under the little group. In four dimensions, the little group for massless particles is ISO(2), whereas for massive particles it is SU(2). As a result, external massless and massive states are labeled by helicity (h) and SU(2) indices, respectively. For massive spin-S states, it is convenient to represent them as symmetric SU(2) tensors of rank $2S$, since the standard SU(2) representation requires a preferred spin axis—an artifact that breaks the manifest rotational invariance of the S-matrix. Then the little group transformation of scattering amplitude involving massive as well as massless particles takes the following form [128]

$$\mathcal{A}^{h_j}_{I_1 I_2 \ldots I_{2S}}(p_i, p_j, \cdots) \to t^{-2h_j} W_{i,I_1}{}^{J_1} W_{i,I_2}{}^{J_2} \cdots W_{i,I_{2S}}{}^{J_{2S}} \mathcal{A}^{h}_{J_1 J_2 \ldots J_{2S}}(p_i, p_j, \cdots)\,, \tag{2.45}$$

where one massless j−th particle with helicity h_j and one massive i−th particle are transformed under their respective little groups. The factor t^{-2h_j} represents the ISO(2) ~ U(1) scaling, while the W_i's are SU(2) matrices in the fundamental representation. As scattering amplitudes are covariant under the little group, it is advantageous to express them in terms of the so-called "spinor-helicity variables," which inherently incorporate the transformation laws of the little group.

Particle with zero mass

We consider the SL(2, $\mathbb{C}$) representation of the momentum 4-vector, which is given by the 2×2 Hermitian matrix $p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = p_{\alpha\dot{\alpha}}$. For massless particles, the determinant condition $\det(p_{\alpha\dot{\alpha}}) = 0$ implies that the rank of the $p_{\alpha\dot{\alpha}}$ matrix is 1 and can be written as

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}\,, \tag{2.46}$$

where λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ are two-component Weyl spinors, often referred to as massless spinor-helicity variables. As

$$\lambda_\alpha \longrightarrow t\lambda_\alpha\,, \qquad \tilde{\lambda}_{\dot{\alpha}} \longrightarrow t^{-1}\tilde{\lambda}_{\dot{\alpha}}\,, \tag{2.47}$$

there is no unique way to express $p_{\alpha\dot{\alpha}}$ in terms of the spinor-helicity variables. However, this scaling corresponds precisely to the little group scaling for a massless particle. Hence we identify λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ as objects having little group weight ±1 respectively. Using spinor-helicity variables, we define Lorentz-invariant but little group-covariant angle and square brackets as follows:

$$\langle ij\rangle := \lambda_i^\alpha \lambda_{j\alpha}\,, \qquad [ij] := \tilde{\lambda}_{i\dot{\alpha}}\tilde{\lambda}_j^{\dot{\alpha}}\,, \qquad 2p\cdot q = \langle pq\rangle[qp]\,. \tag{2.48}$$

The above variables satisfy the Weyl equation:

$$p_i|i\rangle = p_i|i] = 0\,. \tag{2.49}$$

Particle with non-zero mass

For massive particles, the hermitian matrix $p_{\alpha\dot{\alpha}}$ has rank 2 as its determinant is non-zero ($\det(p_{\alpha\dot{\alpha}}) = m^2$). Thus, $p_{\alpha\dot{\alpha}}$ can be written as a linear combination of two rank-1 objects [128]

$$p_{\alpha\dot{\alpha}} = \sum_{I,J=1}^{2} \epsilon_{IJ} \lambda^I_\alpha \tilde{\lambda}^J_{\dot{\alpha}} \,. \tag{2.50}$$

Here, (I, J) are the SU(2) little group indices associated with the massive particle. The variables λ^I_α and $\tilde{\lambda}^J_{\dot{\alpha}}$ are referred to as massive spinor-helicity variables. Since,

$$\lambda^I_\alpha \longrightarrow W^I{}_J \lambda^J_\alpha \qquad \tilde{\lambda}^J_{\dot{\alpha}} \longrightarrow (W^{-1})^J{}_K \tilde{\lambda}^K_{\dot{\alpha}} \,, \tag{2.51}$$

there is no unique way to assign these spinors similar to the massless case. But for real momenta, it can be shown that W's are indeed SU(2) matrices with $\det(\lambda^I_\alpha) = \det\,(\tilde{\lambda}^I_{\dot{\alpha}}) = m$, identifying (2.51) as correct little group transformation for massive spinor helicity variables.

Unlike the massless spinor-helicity variables $\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}$, the massive spinor-helicity variables are related to each other via the Dirac equation

$$p_{\alpha\dot{\alpha}} \lambda^\alpha_I = -m \tilde{\lambda}_{I\dot{\alpha}} \;; \quad p_{\alpha\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}}_I = m \lambda_{I\alpha} \,. \tag{2.52}$$

Thus, for scattering involving massive particles, the amplitude can be expressed in terms of either λ^α_I or $\tilde{\lambda}_{I\dot{\alpha}}$, rather than both, in contrast to the purely massless case. This property of the amplitude is particularly advantageous for classifying all possible three-particle amplitudes involving both massive and massless particles [128].

Three particle amplitudes

The fundamental goal of the on-shell recursion relation is to construct higher-point amplitudes from lower-point (e.g., three-particle) amplitudes. We briefly review the necessary three-point amplitudes that form the building blocks for constructing four- and higher-point amplitudes. These three-point amplitudes are crucial in setting up the recursive structure, allowing for the systematic generation of more complex scattering amplitudes from simpler, lower-order ones [162, 163].

Massless amplitude

For three massless particles, if all are on-shell and momentum is conserved, a crucial kinematic constraint emerges: their momenta must be collinear. When working with real momenta, it is impossible to simultaneously satisfy the on-shell condition, momentum conservation, and the requirement that the particles be physically distinct and non-collinear. To circumvent this limitation and construct non-trivial three-point amplitudes for massless particles with non-collinear momenta, one typically employs complexified momenta. In this complexified setting, the on-shell condition and momentum conservation can be satisfied simultaneously for non-collinear configurations. As a result, the spinor variables $|i\rangle$ and $|i]$ are no longer related by complex conjugation. The three particle kinematics of massless particles constrain the structure of the amplitude: it can be a function of either λ_α or $\tilde{\lambda}_{\dot{\alpha}}$. Apart from an overall coupling, the rest of the structure of an amplitude involving particles with helicities (h_1, h_2, h_3) gets constrained by the little group scaling

$$\begin{aligned}\mathcal{A}_3^{h_1 h_2 h_3}[i, j, k] &= g[ij]^{h_1+h_2-h_3}[jk]^{h_2+h_3-h_1}[ki]^{h_3+h_1-h_2}\,; \quad h_1+h_2+h_3>0 \\ &= g'\langle ij\rangle^{h_3-h_1-h_2}\langle jk\rangle^{h_1-h_2-h_3}\langle ki\rangle^{h_1-h_2-h_3}\,; \quad h_1+h_2+h_3<0\,. \end{aligned} \tag{2.53}$$

It is important to note that, based solely on symmetry considerations, one could employ either of the two expressions for the three-particle amplitude, regardless of the sign of $h_1 + h_2 + h_3$. However, we further impose the condition that the amplitudes exhibit a smooth vanishing limit in Minkowski signature, where the spinor brackets vanish for real momenta. Under this requirement, and up to the overall coupling constants g and g', the three-particle amplitudes are entirely determined by Poincaré symmetry. The (in)consistency of theories involving three-particle amplitudes with helicities satisfying $h_1 + h_2 + h_3 = 0$ is particularly instructive [128]. Aside from the trivial case where all external states are scalars ($h_1 = h_2 = h_3 = 0$), such amplitudes typically involve singular phase-like structures in the couplings. For example, in a scenario where particle 1 has spin zero and particles 2 and 3 have helicities $\pm 1/2$, the amplitude might feature terms like $\langle 13\rangle/\langle 12\rangle$ or $[12]/[13]$. These forms of interaction are unfamiliar from the perspective of standard Lagrangian field theory and lack an interpretation in terms of conventional local couplings. Moreover, such structures fail to yield consistent four-particle amplitudes that factorize correctly. This breakdown of factorization implies that the original three-particle coupling must vanish, enforcing the condition that the corresponding three-point amplitude is identically zero.

Massive amplitude

There are two classes of three-point amplitudes involving both massive and massless particles: two massless-one massive and two massive-one massless. In this thesis, we consider the "minimally coupled"[‡] three-particle amplitudes involving a massless particle of helicity $|h|$ and a pair of massive particles of mass m and spin S [128]:

$$\mathcal{A}^{+h}_{3,\text{min}}(\mathbf{1},\mathbf{2},3^h) = g x_{12}^h \frac{\langle \mathbf{12}\rangle^{2S}}{m^{2S-1}} \;; \quad \mathcal{A}^{-h}_{3,\text{min}}(\mathbf{1},\mathbf{2},3^{-h}) = g x_{12}^{-h} \frac{[\mathbf{12}]^{2S}}{m^{2S-1}} \,, \tag{2.54}$$

where the factor x_{12} is defined as

$$x_{12} = \frac{\langle \zeta | p_1 | 3]}{m\langle \zeta 3\rangle} \quad \text{or} \quad x_{12}^{-1} = \frac{\langle 3 | p_1 | \zeta]}{m[3\zeta]} \tag{2.55}$$

where ζ is a reference spinor. For convenience, we omit the SU(2) little group indices of massive spinor helicity variables. Instead, we will be using products of boldface spinor-helicity variables, which are defined as the symmetric product of usual spinor helicity

[‡]In this thesis, we adopt the definition of "minimal coupling" from [128]. Here, a "minimally coupled" amplitude involving massive particles is defined as the three-particle amplitude that behaves well in the high-energy limit, meaning that the leading contribution in this limit is dominated by opposite-helicity massless particles. It is important to note that this definition of "minimal coupling" differs from the conventional terminology typically found in the literature.

brackets. For example,

$$\langle \mathbf{12}\rangle^2 = \langle 1^{I_1}2^{J_1}\rangle\langle 1^{I_2}2^{J_2}\rangle + \langle 1^{I_2}2^{J_1}\rangle\langle 1^{I_1}2^{J_2}\rangle\,, \tag{2.56}$$

$$\langle 3\mathbf{4}\rangle^2 = \langle 34^{J_1}\rangle\langle 34^{J_2}\rangle\,. \tag{2.57}$$

2.3 The Newman-Janis algorithm for three-point amplitudes

The Newman-Janis (NJ) algorithm has been known for a long time as a classical solution-generating technique, primarily used in the context of General relativity. In their original work [141], Newman and Janis demonstrated that the Kerr metric can be "derived" from the Schwarzschild solution when expressed in the so-called Kerr-Schild coordinates by doing a complex transformation of the radial coordinate (r), with the parameter (a), by which it transforms interpreted as the spin of the Kerr black hole solution. Writing the metric in Kerr-Schild form:

$$g_{\mu\nu}^{KS} = \eta_{\mu\nu} + \phi\, l_\mu l_\nu\,, \tag{2.58}$$

where l_μ is null w.r.t. to both $g^{\mu\nu}$ and $\eta^{\mu\nu}$. Here

$$\phi_{Schw}(r) = \frac{2GM}{r}\,, \qquad \phi_{Kerr}(r) = \frac{2GMr}{r^2 + a^2\cos^2\theta}\,, \qquad l^\mu\partial_\mu = \frac{\partial}{\partial t} - \frac{\partial}{\partial r} \tag{2.59}$$

In the Kerr metric, the coordinates (r, θ) do not correspond to the standard spherical polar coordinates of flat space. Instead, they are defined through oblate spheroidal coordinates adapted to the axial symmetry and rotation of the Kerr geometry. Specifically, the coordinates (r, θ) in Boyer–Lindquist form are related to (x, y, z) coordinates by:

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\phi\,, \quad y = \sqrt{r^2 + a^2}\sin\theta\sin\phi\,, \quad z = r\cos\theta. \tag{2.60}$$

In particular, for the Kerr black hole, r is the solution to the equation

$$\frac{x^2+y^2}{r^2+a^2}+\frac{z^2}{r^2}=1\,. \tag{2.61}$$

Specifically, the potential $\phi_{Kerr}(r)$ can be derived from the spherically symmetric Schwarzschild potential $\phi_{Schw}(r)$ via the complex shift $z \to z+ia$ [26]. Noting that $r^2 = x^2+y^2+z^2$ transforms under this shift as $x^2+y^2+z^2-a^2+2iaz = (r+ia\cos\theta)^2$, we see that the substitution $z \to z+ia$ is equivalent to replacing $r \to r+ia\cos\theta$. Under this shift,

$$\phi_{Schw}(r)|_{r\to r+ia\cos\theta} = \frac{2GM}{2}\left(\frac{1}{r}+\frac{1}{\bar{r}}\right)\Bigg|_{r\to r+ia\cos\theta} = \frac{2GMr}{r^2+a^2\cos^2\theta} = \phi_{Kerr}(r). \tag{2.62}$$

Interestingly, they observed that there exists a similar mapping between solutions of the free Maxwell's equations as well. In electrodynamics, the NJ algorithm generates the so-called $\sqrt{\text{Kerr}}$ field, the electromagnetic field of a rotating charge distribution with radius a, from the Coulomb field of a charged point particle sitting at the origin [142]. It can also be understood as the electromagnetic field on flat-spacetime obtained from Kerr-Newman black hole as $GM \to 0$ at fixed a and the charge Q.

For static electromagnetic fields in vacuum, one can define the magnetostatic potential exactly as done for electrostatic solutions, since $\vec{\nabla}\times\vec{B}=0 \;\Rightarrow B_i = \partial_i\chi$. For a static point charge at the origin, then we have

$$\Phi(\vec{x}) = \phi + i\chi = \frac{Q}{r}, \tag{2.63}$$

where ϕ and χ are the electrostatic and magnetostatic potential, respectively. Now, just as was done for the Kerr solution in GR, we do a complex transformation on the radial coordinate. We get

$$\Phi(\vec{x}) = \frac{Q}{r} \to \frac{Q}{\sqrt{(\vec{x}-i\vec{a})^2}} = \phi + i\chi. \tag{2.64}$$

Here $\vec{a}$ is to be interpreted as the ring radius of the field, the radius at which there is a ring singularity. Without loss of generality, we align the z-axis along $\vec{a}$. Setting $z = r\cos\theta = r\mu$, the potential becomes

$$\Phi(r) = \frac{Q}{(r^2 - 2iar\mu - a^2)^{1/2}} = \frac{Q}{r}\sum_{n=0}^{\infty}\left(\frac{ia}{r}\right)^n P_n(\mu)\,, \qquad r \geq a\,, \tag{2.65}$$

where $P_n(\mu)$ are the Legendre polynomials. The function $\Phi(r)$ exhibits a ring singularity at $\mu = 0$ and $r = a$. To define $\Phi(r)$ unambiguously, a branch cut must be specified for the square root. The expansion above, valid for $r \geq a$, ensures regularity at infinity. Choosing the branch cut across the circle defined by the ring singularity allows for a consistent analytic continuation of $\Phi(r)$ everywhere.

From the above expression, we can compute the $\sqrt{\text{Kerr}}$ electromagnetic field [142],

$$\vec{F} = -\vec{\nabla}\Phi = \vec{E} + i\vec{B} = Q\frac{\vec{r} - i\vec{a}}{[(\vec{r} - i\vec{a})^2]^{3/2}}\,. \tag{2.66}$$

By construction, $\nabla^2\Phi = 0$ except on the singular ring and the cut, indicating that the sources of the field are localized there. The expansion in eq. (2.65) shows that the field carries electric charge Q, no magnetic monopole moment, but does have a magnetic dipole moment Qa. Specifically, the electric field contains only even multipole moments, while the magnetic field contains only odd ones.

On the equatorial plane ($z = 0$), where $\vec{a} \cdot \vec{r} = 0$, the denominator in eq. (2.66) is real for $r \geq a$ and purely imaginary for $r < a$. Thus, close to the plane of symmetry and for $r < a$, the fields become

$$\vec{E} = -\frac{Q\vec{a}}{(a^2 - r^2)^{3/2}}, \tag{2.67}$$

$$\vec{B} = -\frac{Q\vec{r}}{(a^2 - r^2)^{3/2}}, \tag{2.68}$$

for $z > 0$, and are reversed for $z < 0$. This configuration results in an electric field directed vertically downward into the disk and a magnetic field parallel to the disk surface within

$r < a$, mimicking a Meissner-like effect.

From the electric field, one can compute the surface charge density on the disk as [142, 164]

$$\sigma = -\frac{Qa}{2\pi(a^2 - r^2)^{3/2}} . \tag{2.69}$$

Although this yields a divergent total negative charge, the divergence is exactly canceled by a ring of opposite-sign line charge at the edge, resulting in a net total charge of $+Q$. Specifically, the charge on the cut within radius R is given by

$$Q' = -Q\left[\frac{a}{\sqrt{a^2 - r^2}} - 1\right], \quad R < a , \tag{2.70}$$

which diverges with a negative sign as $R \to a$, but the total charge becomes $+Q$ for $R > a$.

From the discontinuity in the magnetic field $\vec{B}$ across the cut, the surface current density is found to be

$$J_\phi = -Q\frac{r}{2\pi(a^2 - r^2)^{3/2}} , \tag{2.71}$$

which corresponds precisely to the surface current that would result if the charge density were rotating rigidly with angular velocity $\Omega = c/a$. The total current enclosed within radius $R < a$ is $Q'\Omega/(2\pi c)$. As before, the magnetic effects of this distributed current are dominated by those of the current concentrated around the singular ring, which has the opposite sign.

In the recent past, there have been investigations in understanding the Newman-Janis algorithm in effective field theory (EFT). As shown in [139], the linearized Kerr metric

$$\bar{h}^{\mu\nu}_{Kerr} = u^\rho u^{(\mu} \exp\left(a * \partial\right)^{\nu)}_{\rho} \frac{4GM}{r} , \quad \text{where} \left(a * \partial\right)^{\mu}_{\nu} = \epsilon^{\mu}_{\nu\rho\sigma} a^\rho \partial^\sigma ,$$
$$\text{momentum} : p^\mu = m u^\mu \left(u^2 = 1\right) , \quad \text{spin vector} : S^\mu = m a^\mu , \tag{2.72}$$

where $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}h_\rho{}^\rho\eta^{\mu\nu}$, in EFT, can be thought of as being generated by a conserved stress energy tensor. The conserved stress tensor of the linearized Kerr

$$T^{\mu\nu}_{Kerr}(x) = \frac{1}{m}\int d\tau\left[p^\mu p^\nu + p^{(\mu}S^{\nu)\rho}\partial_\rho + \sum_{n\geq 2} C_n^{\mu\nu}(p,a,\partial)\right]\hat{\delta}^4(x - r(\tau))$$
$$= \frac{1}{m}\int d\tau p^{(\mu}p^\rho \exp{(a * \partial)^{\nu)}}_\rho \hat{\delta}^4(x - r(\tau)) \tag{2.73}$$

then defines a point particle with an infinite number of multipole moments described solely in terms of its mass (m), charge (q) and spin (a) that sources this metric, consistent with the no-hair theorem for Black holes. The first term is the usual monopole term associated to the spinless solution. The second term is the spin-orbit coupling/dipole moment and $C_n^{\mu\nu}$ are the higher-order multipole moments. An effective action can be constructed to reproduce the leading-order interactions characteristic of a Kerr black hole [15, 133, 134]. Similarly, as shown in [143], the $\sqrt{\text{Kerr}}$ field in EFT can be thought of as being generated by a conserved current

$$J^\mu(x) = \frac{Q}{m}\int d\tau\left[p^\mu + S^{\mu\rho}\partial_\rho + \sum_{n\geq 2} D_n^\mu(p,a,\partial)\right]\hat{\delta}^4(x - r(\tau))$$
$$= \frac{Q}{m}\int d\tau\, p^\rho(\tau)\exp{(a * \partial)^\mu}_\rho\hat{\delta}^4(x - r(\tau))\,. \tag{2.74}$$

The conserved current then defines a classical $\sqrt{\text{Kerr}}$ point particle with an infinite number of multipole moments described solely in terms of its mass (m), charge (q) and spin (a) [144]. From the conserved current, we can compute the gauge field created by this configuration,

$$A_\mu(x) = \int d^4x'\, G_r(x,x')\, J_\mu(x') = \sum_{n\geq 0} D_{\mu,n}(m,q,a)\frac{1}{r^n} = u_\nu \exp{(a * \partial)^\nu}_\mu \frac{Q}{r} \tag{2.75}$$

where $D_n^\mu(m,q,a)$ are the multipole moments. This represents the electromagnetic analog of the Kerr black hole solution. Note that the two solutions of equations (2.72) and (2.75) are related via double copy. Essentially, $\sqrt{\text{Kerr}}$ solution is the single copy of the

Kerr solution and hence the name "$\sqrt{\text{Kerr}}$". Remarkably, the recent interpretation of the $\sqrt{\text{Kerr}}$ field as a particle admits a natural description within the effective field theory (EFT) framework [143]. In this context, it emerges as the classical limit of the three-point amplitude consisting of a massive spin-S particle coupled to a photon. We now proceed to review this formulation.

We consider the three-point amplitude involving a generic massive spin-S particle of mass m_2, which is "minimally" coupled to a photon. In the massive spinor helicity formalism [128], the amplitude is given by

$$\begin{aligned}\mathcal{A}_3[\mathbf{2}^S,\mathbf{2}'^S,q^+] &= i\sqrt{2}Q_2\, x\frac{\langle\mathbf{2}\mathbf{2}'\rangle^{2S}}{m_2^{2S-1}}\,,\\ \mathcal{A}_3[\mathbf{2}^S,\mathbf{2}'^S,q^-] &= i\sqrt{2}Q_2 x^{-1}\frac{[\mathbf{2}\mathbf{2}']^{2S}}{m_2^{2S-1}}\,.\end{aligned} \tag{2.76}$$

Here the massive spinor helicity variables $|\mathbf{2}\rangle$ and $|\mathbf{2}'\rangle$ are defined w.r.t incoming momentum p_2 and outgoing momentum p_2', respectively. The x−factor, which is a hallmark of a minimally coupled amplitude is defined via the photon polarization: $x = \frac{1}{m_2}(\varepsilon^+(q)\cdot p_2)$. We now take the classical limit as described in the previous section. For the above amplitude, we replace $q = \hbar\bar{q}$ and $Q_2 \to Q_2/\sqrt{\hbar}$. From the three particle kinematics, $2p_2\cdot\bar{q} = -\hbar\bar{q}^2$, keeping the leading term in $\hbar$, we obtain

$$\frac{1}{m_2}\langle\mathbf{2}\mathbf{2}'\rangle = \mathbb{I} + \frac{1}{2S\,m_2}\bar{q}\cdot s_2\,, \tag{2.77}$$

where we have suppressed all the SU(2) indices.Here, s_2^μ denotes the Pauli-Lubanski pseudovector corresponding to the spin-S particle.

$$s_2^\mu = \frac{S\hbar}{m_2}\langle\mathbf{2}|\sigma^\mu|\mathbf{2}]\,. \tag{2.78}$$

It was shown in [26] that if we take $S\to\infty, \hbar\to 0$ such that $S\hbar$ = constant, then the above three-point amplitude exponentiates

$$\mathcal{A}^{\pm}_{3,\sqrt{\mathrm{Kerr}}} = iQ_2\sqrt{2}m_2\, x^{\pm 1} e^{\pm \bar{q}\cdot a_2} = \mathcal{A}^{\pm}_{3,\mathrm{scalar}}\, e^{\pm \bar{q}\cdot a_2}, \tag{2.79}$$

where $a_2^\mu = \frac{s_2^\mu}{m_2}$ is the rescaled spin of the $\sqrt{\mathrm{Kerr}}$ particle. We note that the classical limit of the massive spin - S particle has thus "spun" the three-point amplitude of a minimally coupled scalar (in the classical limit). This exponentiation is the realization of the Newman - Janis algorithm for three-point amplitudes, for scalars minimally coupled to the photon. Similarly, starting from the three-point amplitude describing a generic spin-S particle minimally coupled to a graviton, one obtains the following exponentiated form of the amplitude in the classical limit, defined by $S \to \infty$ and $\hbar \to 0$:

$$\mathcal{A}^{\pm}_{3,Kerr} = \frac{\kappa}{2} m_2\, x^{\pm 2} e^{\pm \bar{q}\cdot a_2} = \left(\mathcal{A}^{\pm}_{3,\mathrm{scalar}}\right)^2 e^{\pm \bar{q}\cdot a_2}, \tag{2.80}$$

where $\kappa = \sqrt{32\pi G}$. The double-copy relation between the two amplitudes is apparent.

We see that the NJ algorithm on the space of classical solutions in GR and EM could be used in the space of scattering amplitudes to map an amplitude with external scalar particles to an amplitude associated with the scattering of "infinite spin particles." The minimal coupling of these particles to the gravitational or Maxwell field is equivalent to the classical coupling of the Kerr black hole with linearized gravity or the $\sqrt{\mathrm{Kerr}}$ charged object with the electromagnetic field [135].

Chapter 3

Radiation kernel and Angular Impulse using NJ algorithm

To compute radiation flux emitted during the scattering, an important quantity defined by KMOC is the so-called radiation kernel $R^{\mu}(k)$. The radiation kernel represents the gauge field emitted with momentum $k^{\mu} = \omega(1, \hat{n})$, and arises as a result of inelastic scattering. The radiation flux is then obtained by integrating the modulus squared of the radiation kernel over the on-shell phase space of the photon momentum. To leading order in coupling, the classical radiation kernel in the context of electromagnetism has the following compact expression [84]

$$\mathcal{R}^{\mu}(\bar{k}) = \int \prod_{i=1}^{2} \hat{d}^4\bar{q}_i \, \hat{\delta}(2p_i \cdot \bar{q}_i) \, e^{-i\bar{q}_2 \cdot b} \, \hat{\delta}^{(4)}(\bar{q}_1 + \bar{q}_2 - \bar{k}) \\ \left(\hbar^2 \mathcal{A}_5^{\mu}(p_1 + \hbar\bar{q}_1, p_2 + \hbar\bar{q}_2 \to p_1, p_2, \hbar\bar{k})\right). \quad (3.1)$$

where $\mathcal{A}_5^{\mu}$ is the tree-level five-point amplitude. In this chapter, we propose a "spin dressed" photon propagator to compute the following observables in a scattering process involving a scalar and a $\sqrt{\text{Kerr}}$ particle. All our computations are at leading order in the coupling.*

- The electromagnetic field emitted by the scalar particle. This computation shows the power of the NJ algorithm even in the non-conservative sector. We show that just as for the linear impulse, the radiation emitted by scalar particle can also be obtained via complexification of the impact parameter $\vec{b} \to \vec{b} + i\vec{a}$.

- We use the NJ shift to compute the angular impulse for the scalar and $\sqrt{\text{Kerr}}$ particles. We also highlight an important subtlety that underlies the computation of angular impulse for the spinning particle. This subtlety is crucially tied to the use of spin tensor ($S^{\mu\nu}$) as opposed to the spin pseudovector (a^{μ}) as the fundamental variable. These are related via the following duality relation

$$a^{\mu} = \frac{1}{2m^2}\epsilon^{\mu\nu\rho\sigma} p_{\nu} S_{\rho\sigma} \,. \tag{3.2}$$

- We show that to linear order in the initial spin parameter, the total angular impulse (of the scalar-$\sqrt{\text{Kerr}}$ system) is consistent with classical results [165] so long as the initial coherent state of the spinning particle is parametrized in terms of $S_{\mu\nu}$ and p^{μ}. We also account for the contribution to angular momentum stored in the late-time Coulombic modes, called the "electromagnetic scoot".

This chapter is organized as follows. We begin by reformulating the Newman–Janis (NJ) algorithm in the context of scattering amplitudes, through a specific deformation of the photon polarization data in Section 3.1 and refer to it as the "spin dressed" photon propagator. We use this to compute the leading order radiative gauge field emitted by the scalar particle in the background of a $\sqrt{\text{Kerr}}$ particle in Section 3.2 and show that the radiation kernel can also be implemented as a shift in the impact parameter space. We also compute the net change in the angular momentum of the particles in scalar-$\sqrt{\text{Kerr}}$ scattering in Section 3.3, where we include the electromagnetic scoot contribution.

*This is the analog of leading order (LO) post-Minkowskian expansion in gravity.

3.1 Spin dressing of the photon propagator

In this section, we interpret the exponentiation of the three-point amplitude, in eq.(2.79), as a "spin dressing" of the photon propagator. This is motivated by the simple observation that the three-point amplitude in eq.(2.79) can be written as

$$\begin{aligned}\mathcal{A}^{\pm}_{3,\sqrt{\text{Kerr}}} &= iQ_2 m_2 \left(\varepsilon^{\pm}(\bar{q}) \cdot p_2\right) e^{\pm \bar{q} \cdot a_2}, \\ &= iQ_2 m_2 (\varepsilon'^{\pm}(\bar{q}, a_2) \cdot p_2)\end{aligned} \tag{3.3}$$

where $\varepsilon'^{\mu\pm}(\bar{q}, a_2) = \varepsilon^{\mu\pm}(\bar{q}) e^{\pm a_2 \cdot \bar{q}}$. This was first observed in [166]. Building on this observation, we move on to the construction of the four-point scattering amplitude involving a scalar particle of charge Q_1 and mass m_1 and a $\sqrt{\text{Kerr}}$ particle, mediated by photon. The incoming momenta for the particles are (p_1, p_2) and the outgoing momenta are $(p_1 + q, p_2 - q)$. Diagramatically, the four-point amplitude is represented in Figure 3.1. The amplitude is then

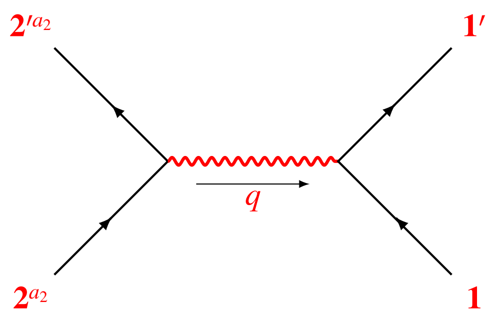

Figure 3.1: The four-point scalar - $\sqrt{\text{Kerr}}$ amplitude with photon exchange. Here 'a_2' denotes the rescaled spin of the $\sqrt{\text{Kerr}}$ particle.

$$\mathcal{A}_4 \left[p_1, p_2 \to p_1', p_2'\right] = \mathcal{A}^{\mu}_{3,\sqrt{\text{Kerr}}} \left[p_2', p_2, \hbar\bar{q}\right] \frac{\mathcal{P}_{\mu\nu}}{\hbar^2 \bar{q}^2} \mathcal{A}^{\nu}_{3,\text{scalar}} \left[p_1', p_1, -\hbar\bar{q}\right], \tag{3.4}$$

where $p_1' = p_1 + \hbar\bar{q}$, $p_2' = p_2 - \hbar\bar{q}$ and $\mathcal{P}^{\mu\nu} := \sum_{h=\pm} \varepsilon_h^{\mu}\, \varepsilon_{-h}^{\nu}$. Using the three-point amplitudes in eq.(2.79), we note that eq.(3.4) can be written as

$$\mathcal{A}_4\left[p_1, p_2 \to p_1', p_2'\right] = \mathcal{A}^{\mu}_{3,\text{scalar}}\left[p_2', p_2, \hbar\bar{q}\right] \frac{\widetilde{\mathcal{P}}_{\mu\nu}(\bar{q})}{\hbar^2 \bar{q}^2} \mathcal{A}^{\nu}_{3,\text{scalar}}\left[p_1', p_1, \hbar\bar{q}\right], \tag{3.5}$$

where $\mathcal{A}^{\mu}_{3,\text{scalar}}$ is the three-point minially coupled scalar amplitude. We now deform the internal photon projector $\mathcal{P}^{\mu\nu}$ as follows

$$\begin{aligned} \mathcal{P}^{\mu\nu} \to \widetilde{\mathcal{P}}^{\mu\nu}(\bar{q}) &:= e^{\bar{q}\cdot a_2} \varepsilon_+^{\mu}(\bar{q}) \varepsilon_-^{\nu}(\bar{q}) + e^{-\bar{q}\cdot a_2} \varepsilon_+^{\nu}(\bar{q}) \varepsilon_-^{\mu}(\bar{q}) \\ &= \cosh(a_2 \cdot \bar{q}) \eta^{\mu\nu} + \sinh(a_2 \cdot \bar{q}) \Pi^{\mu\nu}(\bar{q})\,. \end{aligned} \tag{3.6}$$

Here we have used $\varepsilon_+^{(\mu}\varepsilon_-^{\nu)} = \eta^{\mu\nu}$ and define the anti-symmetric part of the projector as $\Pi^{\mu\nu}(\bar{q}) := \varepsilon_+^{[\mu}\varepsilon_-^{\nu]}(\bar{q})$. Since the anti-symmetric part[†] of the projector is ambiguous up to a residual gauge, we shall choose an expression for $\Pi^{\mu\nu}(\bar{q})$ that can be used in the computation of all the physical observables. We choose[‡]

$$\Pi^{\mu\nu}(\bar{q}) = \frac{i}{(a_2 \cdot \bar{q})} \epsilon^{\mu\nu\rho\sigma} a_{2\rho} \bar{q}_{\sigma}\,, \tag{3.7}$$

and substitute in equation eq.(3.5) to obtain the amplitude [§]

$$\mathcal{A}_{4,\text{scalar}-\sqrt{\text{Kerr}}} = \frac{4Q_1Q_2}{\hbar^2\bar{q}^2}\left[(p_1\cdot p_2)\cosh(\bar{q}\cdot a_2) + i\frac{\sinh(\bar{q}\cdot a_2)}{(\bar{q}\cdot a_2)}\epsilon(p_1, p_2, a_2, \bar{q})\right], \tag{3.8}$$

[†]Usually the anti-symmetric part does not appear in the projector. In this case, we get it due to the helicity dependence of the exponentiation of the massive spin - S amplitude in eq.(2.79).

[‡]This particular choice comes from demanding that we get the correct classical 4-pt scalar-$\sqrt{\text{Kerr}}$ amplitude which will be then valid for any classical observable. The correct amplitude is the just the one we obtain by using the standard NJ prescription of eq. (2.79). However, in this matching, we note that the $\Pi^{\mu\nu}$ is ambiguous up to a correction $t^{\mu\nu}$:

$$\Pi^{\mu\nu} = \frac{i}{(a_2\cdot\bar{q})}\epsilon^{\mu\nu\rho\sigma}a_{2\rho}\bar{q}_{\sigma} + t^{\mu\nu}\,,$$

such that $t^{\mu\nu}p_{1\mu}p_{2\nu} = 0$. Any choice which satisfies this condition will also be a valid propagator.

[§]A similar construction of spin dressed photon propagator was obtained in [166].

with $\epsilon(p_1, p_2, a_2, \bar{q}) := \epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu a_2^\rho \bar{q}^\sigma$. The amplitude depends on the external momenta of the scattering particles as well as the (classical) spin vector a^μ. It is related to the spin tensor $S^{\mu\nu}$ via the dual relation, eq.(3.2). It is rather natural to interpret $S^{\mu\nu}$ as the independent spin tensor which can be thought of as an "intrinsic" spin angular momentum of a classical particle. In this case

$$a_2^\mu = a_2^\mu(S_2, p_2)\,. \tag{3.9}$$

We will denote the projection of $S_2^{\mu\nu}$ orthogonal to the time-like vector p_2^μ as $S_2^{\perp\mu\nu}$. Thus we will interpret the spin pseudovector as a function of $S_2^{\perp\mu\nu}$ and p_2^μ. We will not explicitly indicate the dependence of a_2^μ on $S_2^{\mu\nu}$ except in section 3.3, when we derive the angular impulse.¶

Using the above amplitude, the linear impulse for the scalar particle is

$$\begin{aligned}\Delta p_1^\mu = iQ_1Q_2 \int \hat{d}^4\bar{q}\; \hat{\delta}(\bar{q}\cdot p_1)\; \hat{\delta}(\bar{q}\cdot p_2) \frac{e^{i\bar{q}\cdot b}}{\bar{q}^2}\; \bar{q}^\mu \\ \left[\cosh(a_2\cdot\bar{q})\; (p_1\cdot p_2) + i\frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})}\epsilon(p_1, p_2, a_2, \bar{q})\right]\end{aligned} \tag{3.10}$$

Using the identities

$$\bar{q}_\mu \sinh w = i\epsilon_{\mu\nu\rho\sigma} u_1^\nu u_2^\rho \bar{q}^\sigma\,, \quad \sinh w = \sqrt{\gamma^2 - 1} = \gamma\beta\,, \quad \gamma = (u_1\cdot u_2) = \cosh w\,, \tag{3.11}$$

and rewriting the $\cosh(a_2\cdot\bar{q})$ and $\sinh(a_2\cdot\bar{q})$ terms as exponential functions, we obtain

$$\Delta p^\mu_{1,\sqrt{\mathrm{Kerr}}} = \frac{Q_1Q_2}{2\pi\gamma\beta}\,\mathrm{Re}\left[\frac{\gamma(b + i\Pi a_2)^\mu - i\epsilon^\mu(b + i\Pi a_2, u_1, u_2)}{(b + i\Pi a_2)^2}\right]. \tag{3.12}$$

This is the expression obtained in [26]. Hence, we see that the NJ algorithm, within the EFT for a $\sqrt{\mathrm{Kerr}}$ particle, can also be interpreted as a deformation on the photon data rather than on the impact parameter as shown in [26].

¶A moment of reflection reveals that for orbital angular impulse of $\sqrt{\mathrm{Kerr}}$ particle, the choice of a_2^μ versus $S_2^{\mu\nu}$ as independent variable in $\mathcal{A}_4$ will produce inequivalent results as $L_2^{\mu\nu} = (p_2 \wedge \partial_{p_2})^{\mu\nu}$.

The result for the Kerr black hole scattering can be obtained via the double copy method. Using the double copy relation at the level of 3-point amplitudes in eq. (2.80), all one needs is the following replacement: $i\sqrt{2}Q_i \to \frac{\kappa}{2}$, where $\kappa = \sqrt{32\pi G}$, $\cosh w \to \cosh 2w = 2\gamma^2 - 1$, and $\sinh w \to \sinh 2w = 2\gamma \sinh w$. We get

$$\Delta p^{\mu}_{1,Kerr} = -\frac{2m_1 m_2 G}{\gamma\beta} \,\mathrm{Re}\left[\frac{(2\gamma^2-1)(b+i\Pi a_2)^{\mu} - 2i\gamma\,\epsilon^{\mu}(b+i\Pi a_2, u_1, u_2)}{(b+i\Pi a_2)^2}\right], \tag{3.13}$$

where a_2 is interpreted as the ring radius of the Kerr black hole. Note that the double copy is manifest here directly at the level of the observable, in contrast to the conventional double copy applied at the level of amplitudes. This arises because the linear impulse is obtained from the four-point amplitude integrated over on-shell momentum exchange, and the four-point amplitude itself is built from three-point amplitudes where the double copy already holds. However, such a double copy prescription does not straightforwardly extend to observables involving derivatives, such as the expectation value of the change in orbital angular momentum, where momentum derivatives act on the amplitude and modify its structure.

3.2 Radiation Kernel to all order in spin

In this section, we use the spin-dressed photon propagator (3.6) to compute the leading order radiative gauge field emitted by a scalar particle as it scatters in the background of a $\sqrt{\text{Kerr}}$ particle. The basic ingredient for computing the radiative field via KMOC formalism is the inelastic five-point amplitude $\mathcal{A}_5(\sqrt{\text{Kerr}}\text{+scalar} \to \sqrt{\text{Kerr}}\text{+scalar+}\gamma)$. We will

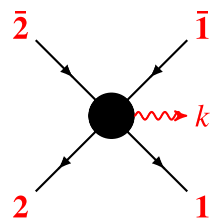

Figure 3.2: The five-point amplitude appearing in the radiation kernel at leading order.

only compute the radiation emitted from the scalar particle using the NJ algorithm. It is straightforward to compute the five-point amplitude when the photon is emitted from the scalar particle since, in this case, the complexity due to the spin is completely contained within the three-point amplitude involving the $\sqrt{\text{Kerr}}$ particle. The other sub-amplitude needed to obtain the full amplitude is then the ordinary scalar-Compton amplitude as indicated in Figure 3.3. Following the construction of the four-point massive amplitude in Section 3.1, we employ the deformed internal photon projector given in eq. (3.6).

$$\widetilde{\mathcal{P}}^{\mu\nu}(\bar{q}_2) = \cosh(a_2\cdot\bar{q}_2)\eta_{\mu\nu} + i\frac{\sinh(a_2\cdot\bar{q}_2)}{a_2\cdot\bar{q}_2}\epsilon^{\mu\nu\rho\sigma}a_{2\rho}\bar{q}_{2\sigma}\,. \tag{3.14}$$

to obtain the five-point amplitude as follows

$$\begin{aligned}\mathcal{A}_5^{\delta} &= \frac{1}{q_2^2}\mathcal{A}^{\mu}_{3,\sqrt{\text{Kerr}}}\mathcal{P}_{\mu\nu}\mathcal{A}^{\nu\delta}_{4,\text{Scalar-Compton}}\\ &= \frac{1}{q_2^2}\mathcal{A}^{\mu}_{3,\text{Scalar}}\widetilde{\mathcal{P}}_{\mu\nu}\mathcal{A}^{\nu\delta}_{4,\text{Scalar-Compton}}\,.\end{aligned} \tag{3.15}$$

Here $\mathcal{A}^{\mu}_{3,\text{Scalar}}$ and $\mathcal{A}^{\nu\delta}_{4,\text{Scalar-Compton}}$ are the three-point massive scalar-photon and the scalar-Compton amplitude in scalar QED.

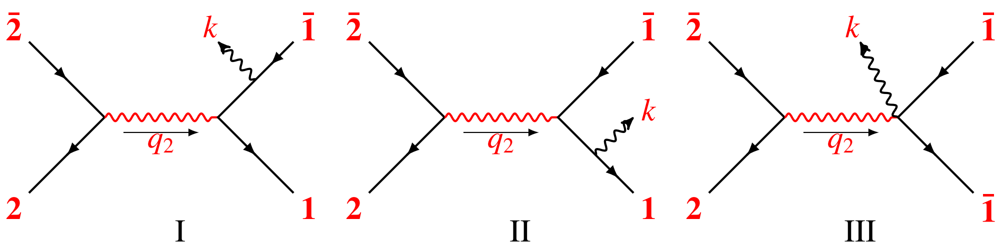

Figure 3.3: Diagrams contributing to the tree level five-particle amplitude with a photon emitted from the scalar particle.

To compute the amplitude, we'll be using the following momentum convention

$$(\bar{p}_1,\bar{p}_2)\to(p_1,p_2,k)\,,\quad \text{with}\quad \bar{p}_1 = p_1+q_1\,,\quad \bar{p}_2 = p_2+q_2\,,\quad k=(q_1+q_2)\,. \tag{3.16}$$

The contribution of diagram I in Figure 3.3 is given by

$$
\begin{aligned}
\mathcal{A}^{\delta}_{5,I} &= 4Q_1^2 Q_2 \frac{(2p_2+q_2)^{\mu} p_1^{\nu} (p_1-q_2)^{\delta}}{q_2^2[(p_1-q_2)^2-m_1^2]} \left[\cosh(a_2\cdot\bar{q}_2)\eta_{\mu\nu} + \frac{i\sinh(a_2\cdot\bar{q}_2)}{a_2\cdot\bar{q}_2}\epsilon^{\mu\nu\rho\sigma} a_{2\rho}\bar{q}_{2\sigma}\right] \\
&= \frac{4Q_1^2 Q_2 (p_1-q_2)^{\delta}}{q_2^2[(p_1-q_2)^2-m_1^2]} \left[\cosh(a_2\cdot\bar{q}_2)(2p_1\cdot p_2 + k\cdot p_1) + 2i\frac{\sinh(a_2\cdot\bar{q}_2)}{a_2\cdot\bar{q}_2}\epsilon(p_2,p_1,a_2,\bar{q}_2)\right].
\end{aligned}
\tag{3.17}
$$

Similarly, from diagram II, we obtain

$$
\begin{aligned}
\mathcal{A}^{\delta}_{5,II} = \frac{4Q_1^2 Q_2}{q_2^2(k\cdot p_1)} p_1^{\delta}\Big[\cosh(a_2\cdot\bar{q}_2)\left(p_1\cdot p_2 + k\cdot p_2 + \frac{k\cdot p_1}{2} + \frac{k\cdot q_2}{2}\right) \\
+ \frac{i\sinh(a_2\cdot\bar{q}_2)}{a_2\cdot\bar{q}_2}\epsilon(p_2,p_1+k,a_2,\bar{q}_2)\Big].
\end{aligned}
\tag{3.18}
$$

We use the usual Feynman rule for the scalar-photon four-point vertex in scalar QED theory to obtain the contribution from diagram III in Figure 3.3

$$
\begin{aligned}
\mathcal{A}^{\mu}_{5,III} &= -2Q_1^2 Q_2 \eta^{\rho\mu}\frac{1}{q_2^2}(2p_2+q_2)^{\nu}\left[\cosh(a_2\cdot\bar{q}_2)\eta_{\rho\nu} + \frac{i\sinh(a_2\cdot\bar{q}_2)}{a_2\cdot\bar{q}_2}\epsilon_{\nu\rho\sigma\delta}a_2^{\sigma}\bar{q}_2^{\delta}\right], \\
&= -4Q_1^2 Q_2 \frac{1}{q_2^2}\left[\cosh(a_2\cdot\bar{q}_2)\left(p_2+\frac{q_2}{2}\right)^{\mu} - \frac{i\sinh(a_2\cdot\bar{q}_2)}{a_2\cdot\bar{q}_2}\epsilon^{\mu}(p_2,a_2,\bar{q}_2)\right].
\end{aligned}
\tag{3.19}
$$

Next, we scale the massless momentum q_2^{μ} as $\hbar\bar{q}_2^{\mu}$ and collect the terms of $\mathcal{O}(\hbar^{-2})$ needed to compute the leading order radiation kernel. But unlike the four-point case, individual diagrams in this tree-level amplitude contain superclassical terms. As expected, they cancel after summing up all the diagrams. To see this first of all we rewrite the massive propagator in diagram I as

$$
[(p_1-q_2)^2-m_1^2]^{-1} = -\frac{1}{2\hbar\bar{k}\cdot p_1}\left(1-\hbar\frac{\bar{k}\cdot\bar{q}_2}{\bar{k}\cdot p_1}\right)^{-1}, \tag{3.20}
$$

where we set $k^2 = 0$ and $p_2\cdot q_2 = -\frac{q_2^2}{2}$ from the on-shell delta function appearing in the radiation kernel. Expanding the contribution from diagram I upto $\mathcal{O}(\hbar^{-2})$, we find

$$\mathcal{A}^{\delta}_{5,I,\mathcal{O}(\hbar^{-2})} = \frac{4Q_1^2 Q_2}{\hbar^2 \bar{q}_2^2 (\bar{k}\cdot p_1)}\left[\cosh(a_2\cdot\bar{q}_2)(p_1\cdot p_2) + i\frac{\sinh(a_2\cdot\bar{q}_2)}{a_2\cdot\bar{q}_2}\epsilon(p_2,p_1,a_2,\bar{q}_2)\right] \\ \times\left(\bar{q}_2^{\delta} - \frac{p_1^{\delta}(\bar{k}\cdot\bar{q}_2)}{\bar{k}\cdot p_1}\right) - \frac{2Q_1^2 Q_2}{\hbar^2\bar{q}_2^2}\cosh(a_2\cdot\bar{q}_2)p_1^{\delta}\,, \quad (3.21)$$

and

$$\mathcal{A}^{\delta}_{5,I,\mathcal{O}(\hbar^{-3})} = \frac{4Q_1^2 Q_2 p_1^{\delta}}{\hbar^3 \bar{q}_2^2 (\bar{k}\cdot p_1)}\left[(-p_1\cdot p_2)\cosh(a_2\cdot\bar{q}_2) - i\frac{\sinh(a_2\cdot\bar{q}_2)}{a_2\cdot\bar{q}_2}\epsilon(p_2,p_1,a_2,\bar{q}_2)\right] \quad (3.22)$$

Similarly, from diagram II, we get both $\mathcal{O}(\hbar^{-2})$ and $\mathcal{O}(\hbar^{-3})$ terms but the latter cancels with the contribution from I.

$$\mathcal{A}^{\delta}_{5,II,\mathcal{O}(\hbar^{-2})} = \frac{4Q_1^2 Q_2 p_1^{\delta}}{\hbar^2 \bar{q}_2^2 (\bar{k}\cdot p_1)}\left[\cosh(a_2\cdot\bar{q}_2)\left(\bar{k}\cdot p_2 + \frac{\bar{k}\cdot p_1}{2}\right) + i\frac{\sinh(a_2\cdot\bar{q}_2)}{a_2\cdot\bar{q}_2}\epsilon(p_2,\bar{k},a_2,\bar{q}_2)\right], \\ \mathcal{A}^{\delta}_{5,II,\mathcal{O}(\hbar^{-3})} = \frac{4Q_1^2 Q_2 p_1^{\delta}}{\hbar^3 \bar{q}_2^2 (\bar{k}\cdot p_1)}\left[(p_1\cdot p_2)\cosh(a_2\cdot\bar{q}_2) + i\frac{\sinh(a_2\cdot\bar{q}_2)}{a_2\cdot\bar{q}_2}\epsilon(p_2,p_1,a_2,\bar{q}_2)\right] \quad (3.23)$$

The $\mathcal{O}(\hbar^{-2})$ terms from diagram III can be found trivially. We collect all the terms of $\mathcal{O}(\hbar^{-2})$ below

$$\mathcal{A}^{\delta}_{5,\mathcal{O}(\hbar^{-2})} = \frac{4Q_1^2 Q_2}{\hbar^2\bar{q}_2^2}\frac{m_1 m_2}{\bar{k}\cdot p_1}\Big[\cosh(a_2\cdot\bar{q}_2)\Big\{\gamma\bar{q}_2^{\delta} - u_2^{\delta}(\bar{k}\cdot u_1) - \frac{p_1^{\delta}}{\bar{k}\cdot p_1}\Big(\gamma(\bar{k}\cdot\bar{q}_2) - (\bar{k}\cdot u_2)(\bar{k}\cdot u_1)\Big)\Big\} \\ + i\sinh(a_2\cdot\bar{q}_2)\Big\{\epsilon^{\delta}(u_2,u_1,\bar{q}_2) - \frac{p_1^{\delta}}{\bar{k}\cdot p_1}\epsilon(\bar{k},u_2,u_1,\bar{q}_2)\Big\}\Big]. \quad (3.24)$$

Using the formula of eq. (2.43), we obtain the radiation kernel as [146]

$$\mathcal{R}_1^{\mu}(\bar{k},a_2) = Q_1^2 Q_2 \int \hat{d}^4\bar{q}\,\hat{\delta}[u_1\cdot(\bar{q}-\bar{k})]\hat{\delta}(u_2\cdot\bar{q})\frac{e^{-i\bar{q}\cdot b}}{\bar{q}^2}\frac{1}{\bar{k}\cdot p_1} \\ \times\Big[\cosh(a_2\cdot\bar{q})\{\gamma\bar{q}^{\mu} - u_2^{\mu}(u_1\cdot\bar{k})\} + i\sinh(a_2\cdot\bar{q})\epsilon^{\mu}(u_2,u_1,\bar{q})$$

$$-\frac{p_1^\mu}{\bar{k}\cdot p_1}\Big\{\cosh(a_2\cdot\bar{q})\left(\gamma(\bar{k}\cdot\bar{q})-(\bar{k}\cdot u_2)(u_1\cdot\bar{k})\right)+i\sinh(a_2\cdot\bar{q})\epsilon(\bar{k},u_2,u_1,\bar{q})\Big\}\Big]. \tag{3.25}$$

This expression agrees with the result in (C.24), which is obtained using classical equations of motion.

3.2.1 Radiation kernel as a shift in the impact parameter space

The radiation kernel for the scalar can also be implemented as a complex shift in the impact parameter space, just as it was done for linear impulse in [26]. Consider the radiation kernel for scalar-scalar scattering in electrodynamics

$$\begin{aligned}\mathcal{R}_1^\mu(\bar{k}) = Q_1^2 Q_2 \int \hat{d}^4\bar{q}\hat{\delta}(u_2\cdot\bar{q})\hat{\delta}[u_1\cdot(\bar{q}-\bar{k})]\frac{e^{-i\bar{q}\cdot b}}{\bar{q}^2}\frac{1}{\bar{k}\cdot u_1}\\ \times\left[\bar{k}_\alpha(u_1\wedge\partial_{u_1})^{\mu\alpha}(u_1\cdot u_2)+(u_1\cdot u_2)\left\{\bar{q}^\mu-\frac{(\bar{k}\cdot\bar{q})u_1^\mu}{\bar{k}\cdot u_1}\right\}\right].\end{aligned} \tag{3.26}$$

Inside this expression for the radiation kernel, we complexify $\mathfrak{b} = b + ia_2$ and re-write the expression as follows.

$$\begin{aligned}\mathcal{R}_1^\mu(\bar{k},a_2) = Q_1^2 Q_2 \int \hat{d}^4\bar{q}\hat{\delta}(u_2\cdot\bar{q})\hat{\delta}[u_1\cdot(\bar{q}-\bar{k})]\frac{1}{\bar{q}^2(\bar{k}\cdot u_1)}\\ \times\left[k_\alpha(u_1\wedge\partial_{u_1})^{\mu\alpha}(e^{-i\bar{q}\cdot\mathfrak{b}^\star}e^{w}+e^{-i\bar{q}\cdot\mathfrak{b}}e^{-w})+(e^{-i\bar{q}\cdot\mathfrak{b}^\star}e^{w}+e^{-i\bar{q}\cdot\mathfrak{b}}e^{-w})\left\{\bar{q}^\mu-\frac{(\bar{k}\cdot\bar{q})u_1^\mu}{\bar{k}\cdot u_1}\right\}\right],\end{aligned} \tag{3.27}$$

with $u_1\cdot u_2 = \cosh w = \gamma$. We can now factor out the overall $e^{-i\bar{q}\cdot b}$ in the second line

$$(e^{-i\bar{q}\cdot\mathfrak{b}^\star}e^{w}+e^{-i\bar{q}\cdot\mathfrak{b}}e^{-w}) = e^{-i\bar{q}\cdot b}\left[\gamma\cosh(\bar{q}\cdot a_2)-\sqrt{\gamma^2-1}\sinh(\bar{q}\cdot a_2)\right], \tag{3.28}$$

where we use $\gamma = \cosh w$ and $\sqrt{\gamma^2 - 1} = \sinh w$. We can now evaluate the derivatives with respect to initial velocity u_1,

$$
\begin{aligned}
&(u_1 \wedge \partial_{u_1})^{\mu\nu} \left[\gamma \cosh(\bar{q} \cdot a_2) - \sqrt{\gamma^2 - 1} \sinh(\bar{q} \cdot a_2) \right] \\
&= (u_1 \wedge u_2)^{\mu\nu} \left[\cosh(a_2 \cdot \bar{q}) - \frac{1}{\beta} \sinh(a_2 \cdot \bar{q}) \right] ,
\end{aligned}
\tag{3.29}
$$

with $\sinh w = \beta\gamma$. Using the following identity, consistent with the two on-shell delta function constraints: $u_2 \cdot \bar{q} = 0$ and $u_1 \cdot \bar{q} = \bar{k} \cdot u_1$, we find that

$$
\bar{q}^\mu \sinh w = \frac{\bar{k} \cdot u_1}{\sinh w} (\gamma u_2^\mu - u_1^\mu) + i\epsilon^\mu(\bar{q}, u_1, u_2) ,
\tag{3.30}
$$

and we recover the radiation kernel in scalar-$\sqrt{\text{Kerr}}$ scattering when the photon is emitted from the scalar particle in (3.25). For the radiation corresponding to the scalar in a scattering involving a Kerr black hole, one needs the double copy of the non-abelian counterpart of the $\sqrt{\text{Kerr}}$ solution.

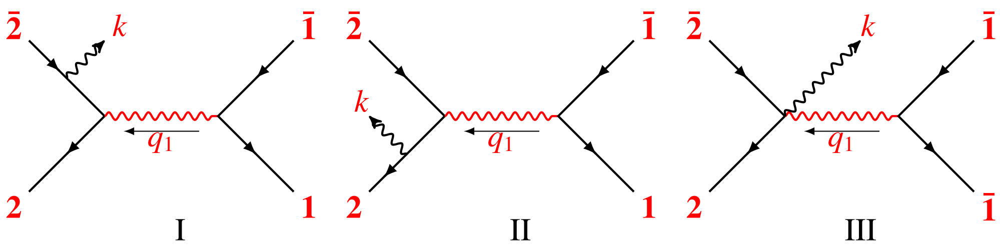

Figure 3.4: Diagrams contributing to the tree level five-point amplitude with a photon emitted from the $\sqrt{\text{Kerr}}$ particle.

We note that to compute the total radiative flux (which includes the radiation emitted by the $\sqrt{\text{Kerr}}$ particle), we require the expression for the five-point amplitude where the incoming $\sqrt{\text{Kerr}}$ and scalar states are scattered into $\sqrt{\text{Kerr}}$, scalar and a photon. A diagrammatic representation of this amplitude is in Fig. 3.4. This requires the expression for the $\sqrt{\text{Kerr}}$ Compton sub-amplitude. The four-point Compton amplitude for a minimally coupled spin-S particle in scalar QED develops spurious, unphysical poles for

$S > 1$ [128], rendering the expression non-local. Although one can rewrite it in local form by introducing additional inverse powers of m, all terms then scale with $1/m$ and become singular in the $m \to 0$ limit. This reflects the absence of a consistent massless high-energy limit.

One avenue to compute the inelastic amplitude $\mathcal{A}_5(\sqrt{\text{Kerr}} + \text{scalar} \to \sqrt{\text{Kerr}} + \text{scalar} + \gamma)$ is via an EFT computation of the Compton sub-amplitude which has been pursued extensively in the literature recently [60, 129, 130, 135, 140, 165–180]. Hence a possible strategy to compute the leading order radiation kernel in the present case would be to use the Compton sub-amplitude to evaluate all the diagrams in Fig. 3.4.

However, a more natural route is to start with a gauge invariant bare Lagrangian of QED with $\sqrt{\text{Kerr}}$ charged matter and compute the five-point amplitude directly using the resulting Feynman rules. Given that the three-point coupling of $\sqrt{\text{Kerr}}$ with a photon is known, one can in principle use gauge invariance to fix all the higher point couplings. In [166], it was shown that just as in the case of scalars and fermions, a gauge-invariant Lagrangian which describes the minimal coupling of a $\sqrt{\text{Kerr}}$ particle with the electromagnetic field is simply the scalar QED lagrangian in which the gauge covariant derivative is replaced by a twisted covariant derivative, $D^{(a)}_\mu = \partial_\mu - ie \exp\{\epsilon^\nu{}_\mu(a, \partial)\} A_\nu$. It would be rather natural to simply use this Lagrangian and compute the radiative field emitted during scalar$-\sqrt{\text{Kerr}}$ scattering. However as the author emphasizes in [166], such a Lagrangian is not consistent with all the Compton amplitudes, and as a result, it is unclear how it would lead to the correct answer for classical radiation. We stress that the computation of the complete radiation kernel emitted during scalar$-\sqrt{\text{Kerr}}$ scattering using the KMOC formalism has the potential to unravel the full power of the NJ algorithm. This will be pursued elsewhere [181].

3.3 Leading order angular impulse

In this section, we use the NJ algorithm to compute the leading order angular impulse $\Delta J^{\mu\nu}$ for the scalar and $\sqrt{\text{Kerr}}$ particles. Angular impulse is an important observable in $D = 4$ and it has some intriguing features. Already for the scattering of scalar particles, it was shown in [182] that the net angular impulse of the particles (in a $2 \to 2$ scattering) does not add up to zero even at leading order in the coupling. The missing contribution is due to angular momentum stored in the late-time Coloumbic modes. In [182] this contribution was called electromagnetic scoot. We will denote the scoot as $\delta^{\mu\nu}_{\text{scalar-scoot}}$, where the subscript indicates that the scoot has been computed for the case of scalar-scalar scattering.

The NJ algorithm offers a powerful tool to compute angular impulse for the scalar-$\sqrt{\text{Kerr}}$ system. We will denote the angular impulse for the scalar particle as $\Delta L^{\mu\nu}_1$. It can be computed to all orders in spin using the NJ algorithm and the final result is given in eq.(3.47). The computation of total angular impulse (i.e. change in orbital angular momentum, $\Delta L^{\mu\nu}_2$ plus the change in spin angular momentum, $\Delta S^{\perp\mu\nu}_2$) for the $\sqrt{\text{Kerr}}$ particle can also be done using the NJ algorithm. We will denote it as $\Delta J^{\mu\nu}_2$. A result (in the integral form) for $\Delta L^{\mu\nu}_2$ and $\Delta S^{\perp\mu\nu}_2$ appears in eq.(3.53) and (3.71), respectively. In principle, this completes the computation of angular impulse for the scalar-$\sqrt{\text{Kerr}}$ system to leading order in coupling.

To test our results, we compute the net angular impulse of the scattering particles and subject it to the conservation law.

Based on [182] we deduce that to leading order in the coupling,

$$\begin{aligned} \Delta J^{\mu\nu}_{\text{net}} &= \left(\Delta L^{\mu\nu}_1 + \Delta J^{\mu\nu}_2\right) + \delta^{\mu\nu}_{\text{scoot}} \\ &= \Delta J^{\mu\nu}_{\text{particles}} + \delta^{\mu\nu}_{\text{scoot}} = 0 \end{aligned} \tag{3.31}$$

On general grounds, we expect that the entire contribution to the electromagnetic scoot is independent of the spin of the particles as it simply arises due to the late-time Coloumbic effects which do not depend on the spin,

$$\delta^{\mu\nu}_{\text{scoot}} = \delta^{\mu\nu}_{\text{scalar-scoot}} \,. \tag{3.32}$$

This spin-independent behavior arises because the late-time Coulombic fields, responsible for the scoot, fall off at large distances as $1/R^2$ for a point charge in motion. Including linear-in-spin terms introduces a magnetic dipole moment due to the particle's intrinsic rotation, whose contribution decays faster, as $1/R^3$. Higher spin multipoles fall off even more rapidly (e.g., $1/R^4$ for an electric quadrupole). Thus, the leading asymptotic behavior remains $1/R^2$, governed by the dominant monopole term.

We thus expect that

$$\Delta J^{\mu\nu}_{\text{particles}} + \delta^{\mu\nu}_{\text{scalar-scoot}} = 0\,. \tag{3.33}$$

We verify the conservation to next to leading (i.e. linear) order in $S_2^{\mu\nu}$ in a perturbative expansion which is valid when $|a_2| \ll |b|$. As we will argue, verifying conservation for finite spin $|a_2| \sim |b|$ is rather subtle and will be pursued elsewhere [181].

We start with the computation of the orbital angular impulse for the scalar - $\sqrt{\text{Kerr}}$ scattering, to leading order in coupling.

3.3.1 Orbital Angular Impulse

The leading order orbital angular impulse in the KMOC formalism is given by

$$\Delta L_i^{\mu\nu} = \frac{\hbar^2}{4}\int \hat{d}^4\bar{q}_1\hat{d}^4\bar{q}_2\hat{\delta}(p_1\cdot\bar{q}_1)\hat{\delta}(p_2\cdot\bar{q}_2)e^{-i(b\cdot\bar{q}_2)}\left[\left(\tilde{p}_i\wedge\frac{\partial}{\partial\tilde{p}_i}\right)^{\mu\nu}+\left(p_i\wedge\frac{\partial}{\partial p_i}\right)^{\mu\nu}\right]$$
$$\hat{\delta}^{(4)}(\bar{q}_1+\bar{q}_2)\,\mathcal{A}_4(p_1,p_2\to\tilde{p}_1,\tilde{p}_2)\,, \tag{3.34}$$

where p_i's are initial momenta and we denote the final momenta as $\tilde{p}_i = p_i + \hbar\bar{q}_i$. Here $\mathcal{A}_4(p_1, p_2 \to \tilde{p}_1, \tilde{p}_2)$ is the four-point scalar$-\sqrt{\text{Kerr}}$ scattering amplitude given in (3.8). Since we express the amplitude as a function of $(p_i, \bar{q}_i)$, we shall treat them as independent variables and consider the transformation $(p_i, \tilde{p}_i) \to (p'_i, q'_i)$ and then set $p'_i = p_i$ to obtain the correct differential operator for the angular impulse. With

$$p'_i = p_i\,, \qquad q'_i = \tilde{p}_i - p_i\,, \tag{3.35}$$

we obtain the differential operators in new variables by treating $p_i = p_i(p'_i, q'_i)$ and $\tilde{p}_i = \tilde{p}_i(p'_i, q'_i)$. We find

$$\partial^\mu_{p_i} := \frac{\partial}{\partial p_{i\mu}} = \partial^\mu_{p'_i} - \partial^\mu_{q'_i}\,, \qquad \partial^\mu_{\tilde{p}_i} := \frac{\partial}{\partial \tilde{p}_{i\mu}} = \partial^\mu_{q'_i}\,. \tag{3.36}$$

Using these transformations, we write the orbital angular impulse where we treat $p'_i = p_i$ and q'_i as independent variables[‖]

$$\Delta L^{\mu\nu}_i = \frac{\hbar^2}{4}\int \hat{d}^4\bar{q}_1\hat{d}^4\bar{q}_2\hat{\delta}(p_1\cdot\bar{q}_1)\hat{\delta}(p_2\cdot\bar{q}_2)e^{-ib\cdot\bar{q}_2}\left[(p_i\wedge\partial_{p_i})^{\mu\nu} + (\bar{q}_i\wedge\partial_{\bar{q}_i})^{\mu\nu}\right]$$
$$\left\{\hat{\delta}^{(4)}(\bar{q}_1+\bar{q}_2)\mathcal{A}_4(p_1, p_2 \to p_1 + \hbar\bar{q}_1, p_2 + \hbar\bar{q}_2)\right\}\,. \tag{3.37}$$

Orbital angular impulse of the scalar particle

For the scalar particle, we do integration by parts on the second term in the first line in eq.(3.37) and integrate over $\bar{q}_2$ to obtain

$$\Delta L^{\mu\nu}_1 = \Delta L^{\mu\nu}_{1,I} + \Delta L^{\mu\nu}_{1,II}\,, \tag{3.38}$$

‖We have for convenience replaced $p'_i = p_i$ and $q'_i = q_i$ from now on.

with

$$\begin{aligned} \Delta L^{\mu\nu}_{1,I} &= \frac{\hbar^2}{4} \int \hat{d}^4\bar{q} e^{i\bar{q}\cdot b} \hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q}) \left(p_1 \wedge \frac{\partial}{\partial p_1}\right)^{\mu\nu} \mathcal{A}_4(p_1,p_2 \to p_1+\hbar\bar{q}, p_2-\hbar\bar{q}) \\ \Delta L^{\mu\nu}_{1,II} &= -\frac{\hbar^2}{4} \int \hat{d}^4\bar{q} e^{i\bar{q}\cdot b}\, \hat{\delta}'(p_1\cdot\bar{q})\, \hat{\delta}(p_2\cdot\bar{q}) (\bar{q}\wedge p_1)^{\mu\nu}\, \mathcal{A}_4(p_1,p_2 \to p_1+\hbar\bar{q}, p_2-\hbar\bar{q})\,. \end{aligned} \tag{3.39}$$

where $\mathcal{A}_4$ is given in eq. (3.8). As

$$\frac{\partial}{\partial p^{\mu}_j} p^{\alpha}_i = \delta^j_i \delta^{\alpha}_{\mu}, \qquad \partial_{p^{\mu}_1} a^{\alpha}_2 = 0\,,$$

we will suppress the explicit dependence of a^{μ}_2 on p^{μ}_2 as in the radiation kernel derivation.

It is straightforward to evaluate the expression in $\Delta L^{\mu\nu}_{1,I}$. Since p_1 and $\bar{q}$ are independent, we get

$$\Delta L^{\mu\nu}_{1,I} = Q_1 Q_2 \int \hat{d}^4\bar{q} e^{i\bar{q}\cdot b}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q}) \tfrac{1}{\bar{q}^2} \left[(p_1\wedge p_2)^{\mu\nu}\cosh(a_2\cdot\bar{q}) + i\tfrac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})} p_1^{[\mu}\epsilon^{\nu]}(p_2,a_2,\bar{q})\right]. \tag{3.40}$$

Evaluation of $\Delta L^{\mu\nu}_{1,II}$ on the other hand needs a more careful analysis since it involves derivative of the on-shell delta function. In order to simplify this, we shall decompose the momentum $\bar{q}^{\mu}$ along $p_{1,2}$ and in the transverse direction

$$\bar{q}^{\mu} = \alpha_1 p^{\mu}_1 + \alpha_2 p^{\mu}_2 + \bar{q}^{\mu}_{\perp}\,, \qquad p_i\cdot\bar{q}_{\perp} = 0\,, \tag{3.41}$$

where the coefficients are given by

$$\alpha_1 = \frac{1}{\mathcal{D}}[(p_1\cdot p_2)x_2 - m^2_2 x_1]\,, \quad \alpha_2 = \frac{1}{\mathcal{D}}[(p_1\cdot p_2)x_1 - m^2_1 x_2]\,, \tag{3.42}$$

with $x_{1,2} := (p_{1,2} \cdot \bar{q})$. Due to this change of variables, the measure transforms as follows

$$\hat{d}^4\bar{q} = \frac{1}{\sqrt{\mathcal{D}}}\hat{d}^2\bar{q}_\perp dx_1 dx_2\,, \qquad \mathcal{D} = (p_1 \cdot p_2)^2 - m_1^2 m_2^2\,. \tag{3.43}$$

In terms of $x_{1,2}$ and $\bar{q}_\perp$ variables, we rewrite

$$\varDelta L^{\mu\nu}_{1,II} = -\frac{Q_1 Q_2}{\sqrt{\mathcal{D}}}\int \hat{d}^2\bar{q}_\perp dx_1 \hat{\delta}'(x_1)\frac{e^{i\bar{q}_\perp\cdot b}}{\bar{q}^2}(\bar{q}\wedge p_1)^{\mu\nu}\Big\{(p_1\cdot p_2)\cosh(a_2\cdot\bar{q}) + i\frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})}\epsilon(p_1,p_2,a_2,\bar{q})\Big\}\,, \tag{3.44}$$

where we have done the x_2 integral and used $b \cdot p_{1,2} = 0$. Next, we perform an integration by parts in x_1. At this stage, we note that from eq. (B.14) we can write

$$q^2 = \alpha_1^2 p_1^2 + \alpha_2^2 p_2^2 + q_\perp^2 + 2\alpha_1\alpha_2(p_1\cdot p_2) \tag{3.45}$$

Therefore any first order derivative of $\frac{1}{\bar{q}^2}$ w.r.t x_1 or x_2 vanishes due to the on-shell delta function constraints: $x_1 = x_2 = 0$. Using $\partial_{x_1}(a_2\cdot\bar{q}) = -\frac{m_2^2}{\mathcal{D}}(a_2\cdot p_1)$, we get

$$\begin{aligned}\varDelta L^{\mu\nu}_{1,II} = \frac{Q_1Q_2}{\sqrt{\mathcal{D}}}\int \hat{d}^2\bar{q}_\perp e^{i\bar{q}_\perp\cdot b}\frac{1}{\bar{q}_\perp^2}\Big[\frac{p_1\cdot p_2}{\mathcal{D}}(p_2\wedge p_1)^{\mu\nu}\Big\{(p_1\cdot p_2)\cosh(a_2\cdot\bar{q}_\perp) + i\frac{\sinh(a_2\cdot\bar{q}_\perp)}{(a_2\cdot\bar{q}_\perp)}\epsilon(p_1,p_2,a_2,\bar{q}_\perp)\Big\}\\ +\frac{m_2^2}{\mathcal{D}}(a_2\cdot p_1)(p_1\wedge\bar{q}_\perp)^{\mu\nu}\left\{(p_1\cdot p_2)\sinh(a_2\cdot\bar{q}_\perp) + i\mathcal{Y}\epsilon(p_1,p_2,a_2,\bar{q}_\perp)\right\}\Big]\,,\end{aligned} \tag{3.46}$$

where $\mathcal{Y} = \Big[\frac{\cosh(a_2\cdot\bar{q}_\perp)}{(a_2\cdot\bar{q}_\perp)} - \frac{\sinh(a_2\cdot\bar{q}_\perp)}{(a_2\cdot\bar{q}_\perp)^2}\Big]$. Summing the two expressions $\varDelta L^{\mu\nu}_{1,I}$ and $\varDelta L^{\mu\nu}_{1,II}$, we obtain the orbital angular impulse of the scalar particle

$$\begin{aligned}\varDelta L^{\mu\nu}_1 = Q_1Q_2\int \hat{d}^4\bar{q}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\frac{e^{i\bar{q}\cdot b}}{\bar{q}^2}\Big[(p_1\wedge p_2)^{\mu\nu}\cosh(a_2\cdot\bar{q})\left(1-\frac{(p_1\cdot p_2)^2}{\mathcal{D}}\right)\\ + i\frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})}\left(p_1^{[\mu}\epsilon^{\nu]}(p_2,a_2,\bar{q}) + \frac{p_1\cdot p_2}{\mathcal{D}}\epsilon(p_1,p_2,a_2,\bar{q})(p_2\wedge p_1)^{\mu\nu}\right)\\ + \frac{m_2^2}{\mathcal{D}}(a_2\cdot p_1)(p_1\wedge\bar{q})^{\mu\nu}\left\{(p_1\cdot p_2)\sinh(a_2\cdot\bar{q}) + i\mathcal{Y}\epsilon(p_1,p_2,a_2,\bar{q})\right\}\Big]\,.\end{aligned} \tag{3.47}$$

In Appendix C.2 we have verified the result in classical theory.

We use the results of the integrals from appendix B and obtain the orbital angular impulse as follows

$$\begin{aligned}\mathit{\Delta}L_1^{\mu\nu} = \frac{Q_1Q_2}{2\pi\sqrt{\mathcal{D}}}\,\mathrm{Re}\Bigg[&\frac{1}{\gamma^2\beta^2}(p_2\wedge p_1)^{\mu\nu}\Big\{\Big(1+\sum_{n=1}\frac{(-a_2\cdot i\partial_b)^{2n}}{(2n)!}\Big)\log|\mu_1 b|\Big\}\\&+\Big(p_1^{[\mu}\epsilon^{\nu]\sigma\alpha\beta}p_{2\alpha}a_{2\beta}+\frac{p_1\cdot p_2}{\mathcal{D}}\epsilon^{\rho\sigma\alpha\beta}p_{1\rho}p_{2\alpha}a_{2\beta}(p_2\wedge p_1)^{\mu\nu}\Big)\mathcal{X}_\sigma\\&+\frac{m_2^2}{\mathcal{D}}(a_2\cdot p_1)\Big\{i(p_1\wedge(b+i\mathit{\Pi}a_2))^{\mu\nu}\frac{(p_1\cdot p_2)}{(b+i\mathit{\Pi}a_2)^2}+\epsilon^{\rho\sigma\alpha\beta}p_{1\rho}p_{2\alpha}a_{2\beta}\Big(p_1\wedge\frac{\partial}{\partial a_2}\Big)^{\mu\nu}\mathcal{X}_\sigma\Big\}\Bigg].\end{aligned} \tag{3.48}$$

Here we have defined

$$\mathcal{X}_\sigma := \frac{b_\sigma}{b^2+i(b\cdot a_2)}+i\frac{(\mathit{\Pi}a_2)_\sigma}{(\mathit{\Pi}a_2)^2}\Big[\frac{\mathit{\Pi}a_2}{\mathit{\Pi}a_2-ib}+\log\Big|\frac{b}{b+i\mathit{\Pi}a_2}\Big|\Big], \tag{3.49}$$

where $\mathit{\Pi}^\nu{}_\rho$ is the projector into the plane orthogonal to both u_1 and u_2 [85],

$$\mathit{\Pi}^\nu{}_\rho = \delta^\nu{}_\rho + \frac{1}{\gamma^2\beta^2}\left[u_1^\nu(u_{1\rho}-\gamma u_{2\rho})+u_2^\nu(u_{2\rho}-\gamma u_{1\rho})\right], \tag{3.50}$$

with $\mathit{\Pi}a_2 = \sqrt{\mathit{\Pi}a_2\cdot\mathit{\Pi}a_2}$ and $b = \sqrt{-b^2}$. In eq.(3.48), μ_1 is the infrared (IR) cut-off. Note that the spin dependent terms in first line are not IR divergent as it involves derivative over b and the scalar term contributes to the electromagnetic scoot.

The angular impulse for the scalar particle to linear order in spin written in terms of $S_2^{\perp\mu\nu}$ is

$$\mathit{\Delta}L_1^{\mu\nu} = \frac{Q_1Q_2}{2\pi\sqrt{\mathcal{D}}}\Big[\frac{1}{\beta^2\gamma^2}(p_2\wedge p_1)^{\mu\nu}\log|\mu_1 b|+\frac{1}{b^2}\Big(p_1^{[\mu}S_2^{\perp\nu]\rho}b_\rho+(p_2\wedge p_1)^{\mu\nu}\frac{(p_1\cdot p_2)}{\mathcal{D}}S_2^{\perp\rho\sigma}p_{1\rho}b_\sigma\Big)\Big] + \mathcal{O}(S_2^{\perp 2})\,. \tag{3.51}$$

Orbital angular impulse of the $\sqrt{\text{Kerr}}$ particle

The integral expression for the leading order orbital angular impulse of $\sqrt{\text{Kerr}}$ particle can be written as,

$$\Delta L_2^{\mu\nu} = \frac{\hbar^2}{4}\int \hat{d}^4\bar{q}e^{i\bar{q}\cdot b}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\left[\left(p_2\wedge\frac{\partial}{\partial p_2}\right)^{\mu\nu} - i(\bar{q}\wedge b)^{\mu\nu}\right]\mathcal{A}_4(p_1,p_2\rightarrow p_1+\hbar\bar{q},p_2-\hbar\bar{q})$$
$$-\frac{\hbar^2}{4}\int \hat{d}^4\bar{q}e^{i\bar{q}\cdot b}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}'(p_2\cdot\bar{q})\,(\bar{q}\wedge p_2)^{\mu\nu}\,\mathcal{A}_4(p_1,p_2\rightarrow p_1+\hbar\bar{q},p_2-\hbar\bar{q}). \tag{3.52}$$

We use the formula for linear impulse to rewrite the above integral as follows

$$\Delta L_2^{\mu\nu} = -(b\wedge\Delta p_2)^{\mu\nu} + \Delta L_{2,I}^{\mu\nu} + \Delta L_{2,II}^{\mu\nu}\,, \tag{3.53}$$

where

$$\begin{aligned}
\Delta L_{2,I}^{\mu\nu} &= \frac{\hbar^2}{4}\int \hat{d}^4\bar{q}e^{i\bar{q}\cdot b}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\left(p_2\wedge\frac{\partial}{\partial p_2}\right)^{\mu\nu}\mathcal{A}_4(p_1,p_2\rightarrow p_1+\hbar\bar{q},p_2-\hbar\bar{q})\,,\\
\Delta L_{2,II}^{\mu\nu} &= -\frac{\hbar^2}{4}\int \hat{d}^4\bar{q}e^{i\bar{q}\cdot b}\,\hat{\delta}(p_1\cdot\bar{q})\,\hat{\delta}'(p_2\cdot\bar{q})\,(\bar{q}\wedge p_2)^{\mu\nu}\,\mathcal{A}_4(p_1,p_2\rightarrow p_1+\hbar\bar{q},p_2-\hbar\bar{q})\,.\\
\Delta p_2^{\mu} &= \frac{\hbar^2}{4}\int \hat{d}^4\bar{q}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\,e^{i\bar{q}\cdot b}\,(-i\bar{q}^{\mu})\,\mathcal{A}_4(p_1,p_2\rightarrow p_1+\hbar\bar{q},p_2-\hbar\bar{q})
\end{aligned} \tag{3.54}$$

The evaluation of $\Delta L_{2,I}^{\mu\nu}$ is rather subtle as for any function $f(a_2,\bar{q})$ we obtain terms involving $\frac{\partial}{\partial p_2^{\mu}} f|_{a_2(S_2^{\perp},p_2)}$.

$$\frac{\partial}{\partial p_2^{\mu}} f(a_2,\bar{q}) \;=\; -\frac{1}{2m_2^2}\frac{\partial f(a_2,\bar{q})}{\partial a_2^{\alpha}}\epsilon_{\mu}{}^{\alpha\rho\sigma}S^{\perp}_{2\rho\sigma}\,, \tag{3.55}$$

where we have used

$$\frac{\partial a_2^{\alpha}}{\partial p_2^{\mu}} = \frac{1}{2m_2^2}\epsilon^{\alpha\beta\rho\sigma}S^{\perp}_{2\rho\sigma}\delta_{\mu\beta}\,,$$

using the dual relation (3.2). The derivation of $\varDelta L^{\mu\nu}_{2,I}$ (and hence $\varDelta L^{\mu\nu}_{2}$) (for $|a_2| \sim |b|$) will be pursued elsewhere [181]. In this thesis, we simply evaluate the orbital angular impulse to linear order in $S_2^{\perp\mu\nu}$.

Using the dual relation (3.2), we obtain

$$\begin{aligned}\varDelta L_2^{\mu\nu} = Q_1 Q_2 \int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}\cdot p_1)\hat{\delta}(\bar{q}\cdot p_2)e^{i\bar{q}\cdot b}\frac{1}{\bar{q}^2}\Big[\frac{1}{\beta^2\gamma^2}(p_1\wedge p_2)^{\mu\nu} - (b\wedge\bar{q})^{\mu\nu}\,(S_2^{\perp\rho\sigma}p_{1\rho}\bar{q}_\sigma) \\ + i(p_1\wedge p_2)^{\mu\nu}\frac{(p_1\cdot p_2)}{\mathcal{D}}S_2^{\perp\rho\sigma}p_{1\rho}\bar{q}_\sigma\Big] \\ = \frac{Q_1 Q_2}{2\pi\sqrt{\mathcal{D}}}\Big[\frac{1}{\beta^2\gamma^2}(p_1\wedge p_2)^{\mu\nu}\log|\mu_2 b| - \frac{1}{b^2}\Big(b^{[\mu}S_2^{\perp\nu]\rho}p_{1\rho} - (p_1\wedge p_2)^{\mu\nu}\frac{(p_1\cdot p_2)}{\mathcal{D}}S_2^{\perp\rho\sigma}p_{1\rho}b_\sigma\Big)\Big],\end{aligned} \tag{3.56}$$

where μ_2 is the IR cutoff. This matches with the result obtained in classical theory given in Appendix C.3.

3.3.2 Spin Angular Impulse

The computation of the spin angular impulse $\varDelta S_2^{\perp\mu\nu}$ is rather straightforward via the NJ algorithm. Using the inverse of the dual relation in eq.(3.2),

$$S^{\perp\mu\nu} = \epsilon^{\mu\nu\rho\sigma}p_\rho a_\sigma \tag{3.57}$$

we obtain [183]

$$\varDelta S_2^{\perp\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\varDelta p_{2\rho}a_{2\sigma} + \epsilon^{\mu\nu\rho\sigma}p_{2\rho}\varDelta a_{2\sigma}\,, \tag{3.58}$$

where $\varDelta a_2^\mu$ is known as the spin kick. We note that although $S_2^{\perp\mu\nu}$ is the fundamental spin degree of freedom, the NJ algorithm lets us directly compute the spin kick which can then be used to deduce $\varDelta S_2^{\perp\mu\nu}$.

Since the linear impulse doesn't receive any radiative contribution at leading order in the coupling, the expression for the linear impulse Δp_2^μ is exactly opposite to Δp_1^μ, derived in section 3.1 and it is given by

$$\Delta p_2^\mu = -iQ_1Q_2\int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}\cdot u_1)\hat{\delta}(\bar{q}\cdot u_2)\frac{e^{i\bar{q}\cdot b}}{\bar{q}^2}\Big[\gamma\cosh(a_2\cdot\bar{q})\bar{q}^\mu + i\sinh(a_2\cdot\bar{q})\epsilon^\mu(\bar{q},u_1,u_2)\Big], \tag{3.59}$$

where we rewrite the $\sinh(a_2\cdot\bar{q})$ term using the identity

$$(a_2\cdot\bar{q})\epsilon_\nu(u_1,u_2,\bar{q}) = \bar{q}_\nu\epsilon(u_1,u_2,a_2,\bar{q})\,. \tag{3.60}$$

We study the leading order spin kick using the following formula [85]

$$\Delta a_2^\mu = \left\langle\!\!\left\langle \frac{i\hbar^2}{4}\int \hat{d}^4\bar{q}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})e^{i\bar{q}\cdot b}\Big\{[a_2^\mu(p_2),\mathcal{A}_4] + \frac{\hbar}{m_2}(a_2\cdot\bar{q})u_2^\mu\mathcal{A}_4\Big\}\right\rangle\!\!\right\rangle. \tag{3.61}$$

The commutator in the first term is defined in the SU(2) little group space. The SU(2) indices are left implicit under the double angle bracket notation, explained in section 2.1. Note that, the formula (3.61) appears to be non-uniform in the order of $\hbar$, however, the commutator term also includes an additional factor of $\hbar$ and it is given by

$$[a^\mu_{2,IK}, a^\nu_{2,KJ}] = \frac{i\hbar}{m_2}\epsilon^{\mu\nu\rho\sigma}u_{2\rho}a_{2\sigma,IJ}\,, \tag{3.62}$$

where we display the SU(2) indices (I,J,K). From hereafter, we shall drop the SU(2) indices and the double angle bracket notation altogether. Using (3.62), we get

$$[a_2^\mu, \cosh(a_2\cdot\bar{q})] = \frac{i\hbar}{m_2}\sinh(a_2\cdot\bar{q})\epsilon^\mu(\bar{q},u_2,a_2)\,. \tag{3.63}$$

Next we consider the following commutator

$$\begin{aligned}\left[a_2^\mu, \frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})}\epsilon(u_1,u_2,a_2,\bar{q})\right] &= \frac{i\hbar}{m_2}\mathcal{Y}\epsilon^\mu(\bar{q},u_2,a_2)\epsilon(u_1,u_2,a_2,\bar{q}) \\ &+ \frac{i\hbar}{m_2}\frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})}\epsilon^{\mu\nu}(u_2,a_2)\epsilon_\nu(u_1,u_2,\bar{q})\,.\end{aligned} \tag{3.64}$$

where we defined $\mathcal{Y} := \left(\frac{\cosh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})} - \frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})^2}\right)$. Using eq. A.5 it can be shown that,

$$(a_2\cdot\bar{q})\epsilon_\nu(u_1,u_2,\bar{q}) = \bar{q}_\nu\epsilon(u_1,u_2,a_2,\bar{q})\,, \tag{3.65}$$

to get

$$\left[a_2^\mu, \frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})}\epsilon(u_1,u_2,a_2,\bar{q})\right] = \frac{i\hbar}{m_2}\frac{\cosh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})}\epsilon^\mu(\bar{q},u_2,a_2)\epsilon(u_1,u_2,a_2,\bar{q})\,. \tag{3.66}$$

Next, we use the identity

$$\epsilon^\mu(\bar{q},u_2,a_2)\epsilon(u_1,u_2,a_2,\bar{q}) = (a_2\cdot\bar{q})\left[(a_2\cdot\bar{q})(u_1^\mu-\gamma u_2^\mu) - \bar{q}^\mu(a_2\cdot u_1)\right]\,, \tag{3.67}$$

where we have set $(a_2\cdot u_2) = 0 = (u_{1,2}\cdot\bar{q})$ and $(u_1\cdot u_2) = \gamma$, to finally get

$$\left[a_2^\mu, \frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})}\epsilon(u_1,u_2,a_2,\bar{q})\right] = \frac{i\hbar}{m_2}\cosh(a_2\cdot\bar{q})[-\gamma u_2^\mu(a_2\cdot\bar{q}) + u_1^\mu(a_2\cdot\bar{q}) - \bar{q}^\mu(a_2\cdot u_1)]\,. \tag{3.68}$$

Substituting various commutator expressions, we obtain the leading order spin kick as

$$\Delta a_2^\mu = \frac{iQ_1Q_2}{m_2}\int \hat{d}^4\bar{q}\frac{1}{\bar{q}^2}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})e^{i\bar{q}\cdot b}\Big\{\cosh(a_2\cdot\bar{q})[u_1^\mu(a_2\cdot\bar{q}) - \bar{q}^\mu(a_2\cdot u_1)] \\ -i\sinh(a_2\cdot\bar{q})\epsilon^\mu(u_1,a_2,\bar{q})\Big\}\,, \tag{3.69}$$

where we make use of the identity

$$u_2^\mu\epsilon(u_1,u_2,a_2,\bar{q}) = \gamma\epsilon^\mu(u_2,a_2,\bar{q}) - \epsilon^\mu(u_1,a_2,\bar{q})\,. \tag{3.70}$$

This matches with the spin kick obtained in [143] using classical equations of motion. Plugging these expressions into (3.58), we obtain the following expression for spin angular impulse

$$\Delta S_2^{\perp\mu\nu} = -Q_1 Q_2 \int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}\cdot u_1)\hat{\delta}(\bar{q}\cdot u_2)\frac{e^{i\bar{q}\cdot b}}{\bar{q}^2}\Big[\cosh(a_2\cdot\bar{q})\Big\{iu_1^{[\mu}\epsilon^{\nu]}(\bar{q},u_2,a_2) - i\bar{q}^{[\mu}\epsilon^{\nu]}(u_1,u_2,a_2)\Big\} + \frac{1}{m_2}\sinh(a_2\cdot\bar{q})\Big\{u_2^{[\mu}u_1^{\nu]}(a_2\cdot\bar{q}) - u_2^{[\mu}\bar{q}^{\nu]}(a_2\cdot u_1) + \gamma a_2^{[\mu}\bar{q}^{\nu]}\Big\}\Big]. \tag{3.71}$$

Using the integral results, we get

$$\Delta S_2^{\perp\mu\nu} = \frac{Q_1 Q_2}{2\pi\gamma\beta}\mathrm{Re}\Big[\frac{1}{(b+i\Pi a_2)^2}\Big\{b^{[\mu}\epsilon^{\nu]}(u_1,u_2,a_2) - u_1^{[\mu}\epsilon^{\nu]}(u_2,a_2,b) + u_2^{[\mu}u_1^{\nu]}a_2^2 + \frac{1}{\gamma^2\beta^2}u_2^{[\mu}a_2^{\nu]}(a_2\cdot u_1) + \frac{(a_2\cdot u_1)}{\gamma\beta^2}a_2^{[\mu}u_1^{\nu]} + i\Big(a_2^{[\mu}\epsilon^{\nu]}(u_1,u_2,a_2) - \frac{(a_2\cdot u_1)}{\gamma\beta^2}u_2^{[\mu}\epsilon^{\nu]}(u_1,u_2,a_2) - u_2^{[\mu}u_1^{\nu]}(a_2\cdot b) + u_2^{[\mu}b^{\nu]}(a_2\cdot u_1) + \gamma a_2^{[\mu}b^{\nu]}\Big)\Big\}\Big]. \tag{3.72}$$

It can be verified that the RHS of eqn.(3.72) is $\Delta S_2^{\perp\mu\nu}$ as it satisfies the SSC constraint to leading order in the coupling.

$$\Delta S_2^{\perp\mu\nu}p_{2\nu} + S_2^{\perp\mu\nu}\Delta p_{2\nu} = 0\,. \tag{3.73}$$

At linear order in $S_2^{\perp\mu\nu}$, the spin angular impulse can be evaluated.

$$\Delta S_2^{\perp\mu\nu} = -iQ_1 Q_2 \int \hat{d}^4\bar{q}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})e^{i\bar{q}\cdot b}\frac{1}{\bar{q}^2}\Big[p_1^{[\mu}S_2^{\perp\nu]\sigma}\bar{q}_\sigma - \bar{q}^{[\mu}S_2^{\perp\nu]\sigma}p_{1\sigma}\Big] + \mathcal{O}(S_2^{\perp 2}) \tag{3.74}$$

$$\Delta S_2^{\perp\mu\nu} = \frac{Q_1 Q_2}{2\pi\sqrt{\mathcal{D}}\,b^2}\Big(b^{[\mu}S^{\perp\nu]\alpha}p_{1\alpha} - p_1^{[\mu}S^{\perp\nu]\alpha}b_\alpha\Big). \tag{3.75}$$

This expression is in agreement with the result of [165], which was derived using the classical equations of motion.

We now have all the expressions to compute the total angular impulse, $\Delta J^{\mu\nu}$ in eq.(3.31) for the scalar-$\sqrt{\text{Kerr}}$ scattering to linear order in spin. This is given by the sum of eqs.(3.51),(3.56) and (3.74). We obtain the following result [146]

$$
\begin{aligned}
\Delta J^{\mu\nu} &= \Delta J^{\mu\nu}_{\text{particles}} + \delta^{\mu\nu}_{\text{scalar-scoot}} \\
&= \frac{Q_1 Q_2}{2\pi\sqrt{\mathcal{D}}} \frac{1}{\beta^2\gamma^2} (p_1 \wedge p_2)^{\mu\nu} \log\left|\frac{\mu_2}{\mu_1}\right| + \delta^{\mu\nu}_{\text{scalar-scoot}} = 0 \,,
\end{aligned}
\tag{3.76}
$$

where

$$
\delta^{\mu\nu}_{\text{scalar-scoot}} = -\frac{Q_1 Q_2}{2\pi\sqrt{\mathcal{D}}} \frac{1}{\beta^2\gamma^2} (p_1 \wedge p_2)^{\mu\nu} \log\left|\frac{\tau_1}{\tau_2}\right| . \tag{3.77}
$$

It is worth noting that the spatial component $\Delta J^{ij}_{\text{particles}}$ is conserved independently, while the boost component $\Delta J^{0i}_{\text{particles}}$ is non-zero. The latter corresponds to the mechanical *mass moment*, which is precisely balanced by an equal and opposite contribution from the mass moment carried by the electromagnetic field—an effect responsible for the scoot [182]. In scattering processes involving relativistic particles, the proper times of the two worldlines generally differ. One way to account for this within amplitude-based computations is to introduce distinct infrared (IR) regulators for each particle. These regulators are related to the respective proper times via the relation $\frac{\mu_2}{\mu_1} = \frac{\tau_1}{\tau_2}$ [70, 184]. As a consequence, the total angular momentum for scalar - $\sqrt{\text{Kerr}}$ scattering is conserved, to linear order in spin. The conservation equation also shows that the contribution to the electromagnetic scoot is independent of the spin of the particles as it simply arises due to the late-time Coulombic effects which do not depend on the spin.

We also note that in our analysis, we adopted the spin tensor $S^{\mu\nu}$ rather than the spin vector a^μ as the fundamental spin degree of freedom. With this choice, we found that the result for the angular impulse of the $\sqrt{\text{Kerr}}$ object is consistent with angular momentum conservation at linear order in spin, and also aligns with the results of [165], where the equations of motion are derived in terms of $S^{\mu\nu}$. Parametrizing spin using either a^μ or $S^{\mu\nu}$ can lead to different outcomes. In particular, even starting from the classical equations of motion formulated in terms of a^μ, as done by the authors in [143], one observes violations of angular momentum conservation at linear order in spin and beyond. To maintain consistency at higher orders in spin, it becomes necessary to formulate the equations of motion directly in terms of $S^{\mu\nu}$.

However, this approach introduces additional complexities. For instance, the covariant spin supplementary condition (SSC), $S^{\mu\nu}p_\nu = 0$, may need to be relaxed to accommodate additional "boost" degrees of freedom. This relaxation results in the non-conservation of the spin vector's magnitude [165]. Moreover, beyond linear order in spin, the momentum is no longer parallel to the velocity. These intertwined challenges render the problem both interesting and nontrivial, and they point toward a broader foundational question: what constitutes a complete classical description of a spinning body? Should spin be modeled as a definite-spin classical field, or as a superposition of fields with different spin quantum numbers that can transition between them, thereby allowing variations in the spin vector magnitude even within the conservative sector? The study of conservation of angular momentum to all orders in spin is under investigation [181].

Chapter 4

Soft factorization theorems in $D = 4$

Soft factorization theorems reveal the extent to which a gravitational or electromagnetic amplitude factorizes when one of the gravitons or photons becomes soft as compared to other external momenta i.e. when its energy is very small compared to the energies of the other external momenta (or particles) involved in the scattering process. The late-time gravitational field emitted during the classical scattering can be derived using the soft factorization theorems, which offers remarkable insights into universal modes of gravitational radiation. Computing the gravitational waveform at a finite retarded time is a challenging problem that requires detailed information about the full scattering or collision process. However, the behavior of the waveform at asymptotically early and late retarded times—i.e., long before and long after the main burst of radiation reaches the detector—admits universal expressions. These depend solely on the momenta and spin of the incoming and outgoing objects, without any reference to the detailed dynamics during the interaction. Such asymptotic features of the waveform are governed by the low-frequency behavior of its time Fourier transform, which in turn is determined by soft theorems. These theorems relate amplitudes involving low-frequency graviton emission to those without such emissions. The universality of these results stems from the universality of the soft graviton theorem itself, which is a consequence of general coordinate invariance and holds independently of the specific interactions present in the system [102].

Tree-level gravitational amplitudes satisfy an infinite hierarchy of soft factorization theorems. The existence of these theorems has been recently linked with the existence of an infinite tower of asymptotic symmetries. In this chapter, we analyze the relevance of the soft graviton theorems beyond sub-leading order in the context of classical gravitational scattering in four dimensions. More in detail, we show that the infinite impact parameter limit (or the deflectionless scattering limit) of the late-time gravitational field emitted during a classical scattering can be derived using these factorization theorems. The classical field obtained in this (infinite impact parameter) regime has an expansion in the frequency of the detector where the modes scale as $\omega^n \log\omega$ with a vanishing memory.

This chapter is organized as follows. In Section 4.1, we begin by reviewing classical soft theorems, with particular emphasis on the logarithmic soft factors. Then we review the infinite hierarchy of soft graviton theorems for tree-level amplitudes in Section 4.2. We write down the factorized and the remainder terms for the five-point amplitude for consideration. In Section 4.3, we compute the radiation kernel for gravitational scattering of two massive spinless particles at leading order in the coupling, extending the analysis to (sub)^n-leading orders in frequency. The primary input for this computation is the five-point amplitude. We isolate the logarithmic contributions to the radiation and write the final expression for the same. In Section 4.4, we prove our main result. Given a (sub)^n-leading soft graviton theorem of a tree-level gravitational amplitude, the so-called remainder terms never contribute to the leading log contribution that arises in the classical limit. We also highlight some important results for the soft radiation to (sub)^3-leading order in frequency in Section 4.5.

4.1 A brief review of classical soft theorems

When a set of massive objects collide in space and fragment, the process generates gravitational radiation. Computing the full gravitational waveform or the radiative field is challenging due to the non-linearity of gravity and the possibility of complex non-gravitational interactions during the collision and fragmentation [102]. Nevertheless, the classical soft graviton theorem governs the leading behavior of the waveform at early and late retarded times, including logarithmic corrections, in terms of only the momenta and spin of the incoming and outgoing objects, independent of the details of the interaction.

For a generic gravitational scattering in $D = 4$, the radiative field has the following form under soft expansion [93, 94, 96–98, 100, 104, 185, 186],

$$h_{\mu\nu}(\omega, r, \hat{n}) = \frac{1}{r} e^{i\omega r} \sum_{N=-1}^{\infty} \omega^N h^N_{\mu\nu}(\hat{n}) + \sum_{m=0}^{\infty} \omega^m (\log\omega)^{m+1} h^{\log m}_{\mu\nu}(\hat{n}) + \sum_{N,M |\, M-N>-1} \omega^M (\log\omega)^N h^{\log(N,M)}_{\mu\nu}(\hat{n}) + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{4.1}$$

where the leading term (memory), the leading log term (tail to the memory), and the $\omega \log\omega$ term (spin-dependent tail memory) for $2 \to 2$ scattering are given by

$$\begin{aligned}
h^{-1}_{\mu\nu}(\hat{n}) &= \frac{\kappa}{4}\left(S^{(0)}_{\mu\nu}(\{p_a\}, \hat{n}) - S^{(0)}_{\mu\nu}(\{p'_a\}, \hat{n})\right), \\
h^{\log}_{\mu\nu}(\hat{n}) &= \frac{\kappa^3}{16\pi}\Bigg(\sum_{a,b=1|b\neq a}^{2} S^{(1)}_{\mu\nu}(\{p_a\}, \hat{n}) + \sum_{a,b=1|b\neq a}^{2} S^{(1)}_{\mu\nu}(\{p'_a\}, \hat{n}) \\
&\quad + \sum_{b=1}^{2}(p'_b \cdot \hat{n}) S^{(0)}_{\mu\nu}(\{p'_a\}, \hat{n}) - \sum_{b=1}^{2}(p_b \cdot \hat{n}) S^{(0)}_{\mu\nu}(\{p_a\}, \hat{n})\Bigg), \\
h^{\log(1,1)}_{\mu\nu}(\hat{n}) &= \frac{\kappa^3}{32\pi}\Bigg(\sum_{a,b=1|b\neq a}^{2} S^{(2)}_{\mu\nu}(\{p_a\}, \{\mathcal{S}_a\}, \{r_a\}, \hat{n}) + \sum_{a,b=1|b\neq a}^{2} S^{(2)}_{\mu\nu}(\{p'_a\}, \{\mathcal{S}'_a\}, \{r'_a\}, \hat{n}) \\
&\quad + 2\sum_{b=1}^{2}(p'_b \cdot \hat{n})\Bigg(\sum_{a=1}^{2} \frac{p'_{a,(\mu}\hat{n}^\rho}{p'_a \cdot \hat{n}}\left[(r'_a \wedge p'_a)_{\nu)\rho} + \mathcal{S}'_{a,\nu)\rho}\right] \\
&\quad - \sum_{a=1}^{2} \frac{p_{a,(\mu}\hat{n}^\rho}{p_a \cdot \hat{n}}\left[(r_a \wedge p_a)_{\nu)\rho} + \mathcal{S}_{a,\nu)\rho}\right]\Bigg)\Bigg),
\end{aligned} \tag{4.2}$$

where

$$
\begin{aligned}
S^{(0),\mu\nu}(\{p_a\},\hat{n}) &= \sum_{a=1}^{2} \frac{p_a^\mu p_a^\nu}{p_a \cdot \hat{n}}\,, \\
S^{(1),\mu\nu}(\{p_a\},\hat{n}) &= (p_1 \cdot p_2)\frac{(2(p_1 \cdot p_2)^2 - 3m_1^2 m_2^2)}{[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{3/2}} \frac{\hat{n}_\rho}{p_a \cdot \hat{n}} p_a^{(\mu} \left(p_a \wedge p_b\right)^{\nu)\rho}\,, \\
S^{(2),\mu\nu}(\{p_a\},\{\mathcal{S}_a\},\{r_a\},\hat{n}) &= (p_1 \cdot p_2)\frac{(2(p_1 \cdot p_2)^2 - 3m_1^2 m_2^2)}{[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{3/2}} \frac{\hat{n}_\rho \hat{n}_\sigma}{p_a \cdot \hat{n}} \Big((p_a \wedge p_b)^{\mu\rho} (r_a \wedge p_a + \mathcal{S}_a)^{\nu\sigma} \\
&\quad + (p_a \wedge p_b)^{\nu\sigma} (r_a \wedge p_a + \mathcal{S}_a)^{\mu\rho} \Big)\,. \qquad (4.3)
\end{aligned}
$$

Here, $\{p_a, \mathcal{S}_a, r_a\}$ and $\{p'_a, \mathcal{S}'_a, r'_a\}$ denote the initial and final {momenta, spin tensors and the unperturbed trajectories} of the particles. $\kappa = \sqrt{32\pi G}$ and $\hat{n}$ is the unit vector on the celestial sphere and all the terms subleading in $\frac{1}{r}$ do not contribute to the radiative flux at null infinity [187–189].

The classification of soft terms into three distinct families in equation 4.1 is as follows. The first one is a Laurent expansion in the detector frequency. The second class of terms, scaling as $\omega^N \ln \omega^{N+1}$, represent the infinite tower of leading logarithmic corrections at each order in the soft expansion. These were discovered in a series of papers [93, 94, 96–98, 100], where the integer N corresponds to the fall-off behavior of the low-frequency waveform's at very late and early times. In the retarded time (u) domain, the leading log terms decay as $\frac{1}{u}, \frac{\log u}{u^2}, \ldots, \frac{(\log u)^N}{u^{N+1}}$ as $u \to \infty$. Finally, the third family encapsulates the logarithmic terms which (for a given N) are sub-leading relative to the corresponding leading logarithmic soft factors.

The reason for such a classification is intricately tied to the idea of universality inherent in soft expansion. Given a set of incoming and outgoing momenta and spins of the scattering particles, a specific term in the soft expansion is referred to as universal if it is independent of the detailed dynamics of the underlying equations of motion and the specifics of the scattering process. It has been known since the early 60s that in a generic classical gravitational scattering in which massive objects with arbitrary multipole moments emit gravitational radiation, then the leading soft factor $h^{-1}_{\mu\nu}(\hat{n})$ is universal and only depends on the incoming and outgoing momenta of the scattering particles*. Over the past decade, B. Sahoo, A. Sen, and their collaborators have demonstrated that the universality of gravitational radiation in the soft expansion extends beyond just the leading soft factor. It has now been rigorously established that the leading logarithmic terms, with $N \leq 2$ are universal and depend solely on the momenta of the incoming and outgoing massive objects [89, 93, 94, 96–98, 100, 196]. It has been conjectured that the universality extends to all N and a specific formula for the coefficient of $\omega^N \ln \omega^{N+1}$ has been put forward by Heissenberg et al. in the case of $2-2$ scattering [196]. Very recently, an all-order classical electromagnetic waveform was obtained [197].

Classical soft theorems characterize the non-analytic behavior of gravitational and electromagnetic waveforms in the soft-frequency regime observed by detectors. A key manifestation of the classical soft graviton theorem is the gravitational memory effect [190–193], which predicts a permanent shift in the asymptotic metric fluctuations induced by the gravitational wave. Memory effect is an observable of classical scattering and is simply the coefficient of the leading term in the soft expansion of the radiative field [104, 185, 186]. As stated before, in recent years, a hierarchy of soft theorems in four dimensions have been discovered [93, 94, 96–98, 100]. They are universal and depend solely on the incoming and outgoing momenta, as well as the angular momenta of the scattering objects, including black holes. The table in [97] summarizes the different orders of the low-frequency gravitational waveform in the $\omega \to 0$ limit, highlighting their connections to the PM expansion. It also illustrates the late- and early-time behavior of the gravitational waveform at large retarded time u.

An interesting aspect of the leading log soft factor $h^{\log}_{\mu\nu}(\hat{n})$ is that, unlike $h^{-1}_{\mu\nu}$, it is sourced even in the limit when n particles which undergo scattering are mutually so far apart that each of them experiences a vanishing deflection! The source of such a radiative mode

*We here assume that massless particles model finite energy gravitational radiation that can also emit soft radiation. In the literature, the contribution of massive states to the leading soft factor is known as linear memory and the contribution of massless particles, including gravitons, to the leading soft factor is known as non-linear or null memory [190–195].

then is the asymptotic interaction between the incoming or outgoing states, leading to the emission of gravitational radiation only from $t \to \pm\infty$. In this chapter, we analyze the classical gravitational radiation in $D = 4$ dimensions where two massive objects with momenta p_1, p_2 scatter at finite impact parameter. We then analyze the soft expansion of gravitational radiation in $|b| \to \infty$ limit such that $\omega|b|$ is fixed. As we show, in this regime, all the terms of the form $\omega^m \log\omega | m \geq 1$ survive and can be completely determined by the so-called (sub)^n-leading soft graviton theorems for tree-level gravitational amplitudes.

Soft graviton theorems provide exact descriptions of gravitational radiation in the soft frequency expansion. We consider a $2 \to 2$ scattering process with a large impact parameter. Let p'_a denote the final momentum of a particle, with p_a representing its initial momentum. Thus, we have the relation:

$$p'^{\mu}_a = p^{\mu}_a + \sum_{n=1}^{\infty} \kappa^{2n} \Delta p^{(n)\mu}_a \,, \tag{4.4}$$

where $\Delta p^{(1)\mu}_a$ is the LO linear impulse and κ^{2n}-th term is the N^{n-1}LO impulse. Plugging the equation (4.4) in the radiative field (4.1), it is evident that both $\log\omega$ and $\omega\log\omega$ survive even at leading order in the coupling. This then connects the appearance of $\omega^n \log\omega$ from tree-level amplitudes which we will compute in the subsequent sections.

For example, in 2-2 scattering the classical log soft graviton factor is given by

$$h^{\mu\nu}(\omega, \hat{n})|_{\log\omega} = \frac{\kappa^3}{16\pi} \log(\omega) \left(\sum_{a,b=1}^{2} S^{(1),\mu\nu}(\{p_a\}, k) + \sum_{a,b=1}^{2} S^{(1),\mu\nu}(\{p'_a\}, k) \right) + \mathcal{O}(\kappa^5) \,, \tag{4.5}$$

where

$$S^{(1),\mu\nu}(\{p_a\}, k) = (p_1 \cdot p_2) \frac{(2(p_1 \cdot p_2)^2 - 3m_1^2 m_2^2)}{[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{3/2}} \frac{k_\rho}{p_a \cdot k} p^{(\mu}_a \left(p^{\nu)}_a p^{\rho}_b - p^{\nu)}_b p^{\rho}_a \right) . \tag{4.6}$$

Using equation (4.4), the classical logarithmic soft graviton factor at leading order in the coupling is expressed as follows:

$$h^{\mu\nu}(\omega,\hat{n})|_{\log\omega} = -\frac{i\kappa}{4}\log(\omega)\sum_i \frac{1}{p_i\cdot k}p_i^{(\mu}\hat{J}_i^{\nu)\rho}k_\rho\left[\frac{\kappa^2}{2\pi}\frac{(p_1\cdot p_2)^2-\frac{1}{2}m_1^2m_2^2}{\sqrt{(p_1\cdot p_2)^2-m_1^2m_2^2}}\right], \tag{4.7}$$

which arises from the action on gravitational tree-level four-point amplitude. Here $\hat{J}_i^{\mu\nu} = i(p_i\wedge\frac{\partial}{\partial p_i})^{\mu\nu}$. Thus one can use the soft graviton theorems for tree-level amplitudes in computing the radiation kernel, which generates the $\omega^n\log\omega$ terms.

4.2 Review of soft graviton theorems for tree-level amplitudes

In this section, we review the infinite hierarchy of soft graviton theorems for tree-level amplitudes. We consider massive scalars coupled to gravity and analyze five-point scattering amplitude in which two scalar particles with momenta $\tilde{p}_1, \tilde{p}_2$ scatter into p_1, p_2 and a graviton. Our kinematics satisfy the following on-shell conditions

$$p_i^2 = \tilde{p}_i^2 = m_i^2\,. \tag{4.8}$$

In a generic theory of gravity with arbitrary matter couplings, gauge invariance constrains the form of the (sub)^n-leading soft graviton theorem, which can be expressed as [198,199]:

$$\lim_{\omega\to 0}\partial_\omega^n\omega\mathcal{A}_5(\tilde{p}_1,\tilde{p}_2\to p_1,p_2,k) = \hat{S}^n\mathcal{A}_4 + \mathcal{B}_n(p_1,p_2,\tilde{p}_1,\tilde{p}_2,\hat{k})\,\mathcal{A}_4 + \mathcal{R}_n(p_1,p_2,\tilde{p}_1,\tilde{p}_2,\hat{k}) \tag{4.9}$$

where the (sub)^n-leading soft factor is defined as

$$\hat{S}^n = \begin{cases} \kappa \sum_{i=1,2} \left[\frac{1}{p_i \cdot k} p_i^{(\mu} p_i^{\nu)} - \frac{1}{\tilde{p}_i \cdot k} \tilde{p}_i^{(\mu} \tilde{p}_i^{\nu)} \right], & \text{if } \ n = 0 \\ i\frac{\kappa}{2} \sum_{i=1,2} \epsilon_{\mu\nu} \left[\frac{1}{p_i \cdot k} p_i^{(\mu} \hat{J}_i^{\nu)\rho} k_\rho - \frac{1}{\tilde{p}_i \cdot k} \tilde{p}_i^{(\mu} \hat{\tilde{J}}_i^{\nu)\rho} k_\rho \right], & \text{if } \ n = 1 \\ \frac{\kappa}{2} \sum_{i=1,2} \epsilon_{\mu\nu} \left[\frac{\hat{J}_i^{\mu\rho} k_\rho \hat{J}_i^{\nu\sigma} k_\sigma}{p_i \cdot k} \left(k \cdot \frac{\partial}{\partial p_i} \right)^{n-2} + \frac{\hat{\tilde{J}}_i^{\mu\rho} k_\rho \hat{\tilde{J}}_i^{\nu\sigma} k_\sigma}{\tilde{p}_i \cdot k} \left(k \cdot \frac{\partial}{\partial \tilde{p}_i} \right)^{n-2} \right] & \text{if } \ n \geq 2 \,. \end{cases} \tag{4.10}$$

$\mathcal{B}_n$ is the non-universal part of the factorization formula which depends on the irrelevant three-point couplings in the theory [90, 200]. In this work, we examine the gravitational scattering of two minimally coupled scalar fields ϕ_1 and ϕ_2 with respective masses m_1 and m_2. For this setup, one finds that $\mathcal{B}_n = 0 \ \forall \ n$, while $\mathcal{R}^n \neq 0 \ \forall \ n \geq 3$. Consequently, the soft expansion of the tree-level amplitude does not factorize beyond (sub)^2-leading order. The obstruction to factorization is encapsulated in the so-called *remainder term*,

$$\mathcal{R}_n = \epsilon_{\mu\nu} \hat{k}_{\alpha_1} \hat{k}_{\alpha_2} \cdots \hat{k}_{\alpha_{n-1}} A^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{n-1}} \,, \tag{4.11}$$

which vanishes for $n < 3$ and spoils factorization for $n \geq 3$. The tensor $A^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{n-1}}$ is antisymmetric in any pair of indices among μ and the α_i, but the hard momenta are otherwise unconstrained by gauge invariance.

At first glance, this suggests the absence of a factorization theorem beyond (sub)^2-leading order. However, as shown in [198], the remainder term can be systematically projected out, leading to a formulation of the so-called *sub-n soft theorem* for all $n \geq 1$. To elucidate this result, we restrict attention to a soft graviton with negative helicity. The soft momentum direction $\hat{k}^\mu$ is parametrized in terms of stereographic coordinates as $\hat{k}^\mu = (1, \hat{n}(z, \bar{z}))$, with the polarization vector given by

$$\epsilon^{-\mu} = \frac{1}{\sqrt{2}}(z, 1, i, -z) \,, \tag{4.12}$$

following the conventions of [201]. It is straightforward to verify that

$$D_z^2 \hat{k}_\mu = 0\,, \qquad D_z^2 \left[(1+|z|^2)^{-1} \epsilon_\mu^- \right] = 0\,. \tag{4.13}$$

As a result, the remainder term always satisfies the following constraint, arising purely from the tensor structure of the amplitude (which is linear in the soft graviton polarization tensor $\epsilon_{\mu\nu}$) [202]:

$$D_z^{n+1} \left[(1+|z|^2)^{-1} \mathcal{R}_n \right] = 0\,. \tag{4.14}$$

This constraint enables the construction of a "factorization formula" for tree-level amplitudes valid to all orders in the soft expansion:

$$\lim_{\omega\to 0} \partial_\omega^n \omega \varPi_{\hat{n}}^- \mathcal{A}_5(\tilde{p}_1, \tilde{p}_2 \to p_1, p_2, k) = \varPi_{\hat{n}}^- \hat{S}^n \mathcal{A}_4(\tilde{p}_1, \tilde{p}_2 \to p_1, p_2)\,, \tag{4.15}$$

where $\varPi_{\hat{n}}^- := D_z^{n+1}(1+|z|^2)^{-1}$ is the projection operator. We thus see that higher-order soft theorems appear to be rather limited in their ability to constrain the gravitational scattering as they are only a component of the radiative field which is "orthogonal" to the remainder term. However, we prove that despite their limitations, the (sub)^n-leading soft theorems can capture all the logarithmic soft factors in the limit of vanishing deflection.

The gravitational radiation at leading order in coupling arises from the tree-level five-point amplitude, which in this case is the sum of seven Feynman diagrams, as illustrated in Figure 4.1. The unstripped five-point amplitude is given by

$$\bar{\mathcal{M}}_5^{\mu\nu}[p_1 + \hbar\bar{q}_1, p_2 + \hbar\bar{q}_2 \to p_1, p_{2,}, \hbar\bar{k}] = \hat{\delta}^{(4)}(\bar{q}_1 + \bar{q}_2 - \bar{k}) \bar{\mathcal{A}}_5^{\mu\nu}[p_1 + \hbar\bar{q}_1, p_2 + \hbar\bar{q}_2 \to p_1, p_2, \hbar\bar{k}]\,. \tag{4.16}$$

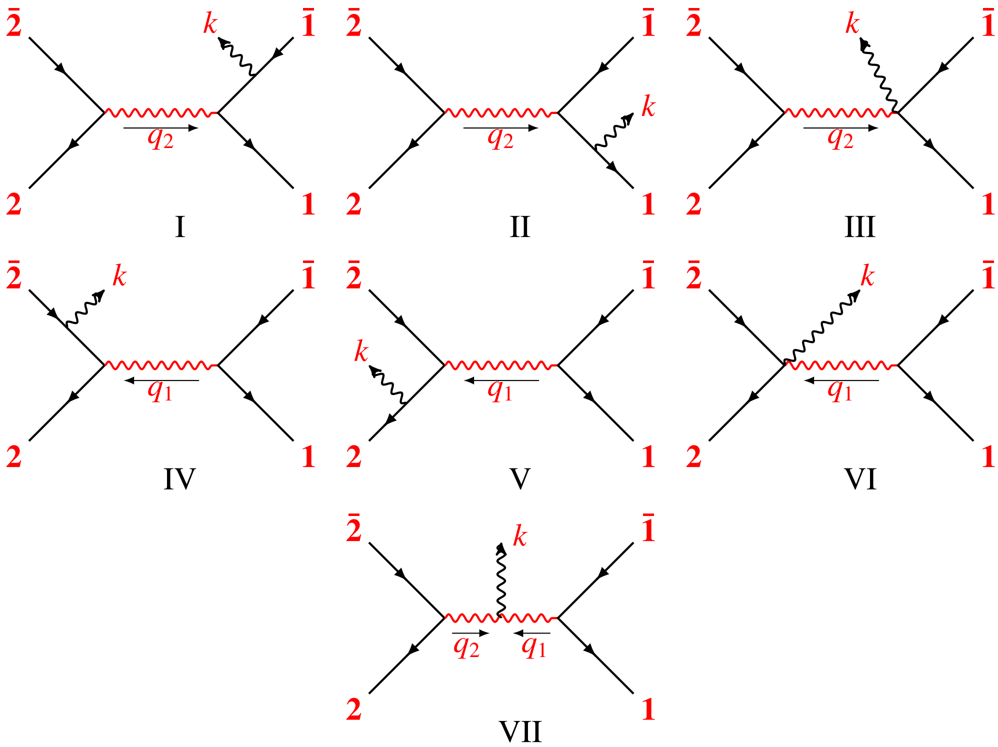

Figure 4.1: Tree-level five-point amplitudes for gravitational scattering of massive particles

Here the stripped amplitude $\bar{\mathcal{A}}_5^{\mu\nu}$ is given by [203, 204]:

$$\bar{\mathcal{A}}_5^{\mu\nu}[p_1+\hbar\bar{q}_1, p_2+\hbar\bar{q}_2 \to p_1, p_{2,}, \hbar\bar{k}] = -\frac{\kappa^3 m_1^2 m_2^2}{\hbar^2}\Big[\frac{4P^\mu P^\nu}{\bar{q}_1^2\bar{q}_2^2} + \frac{2\gamma}{\bar{q}_1^2\bar{q}_2^2}\Big(Q^\mu P^\nu + Q^\nu P^\mu\Big) + \Big(\gamma^2 - \frac{1}{2}\Big)\Big(\frac{Q^\mu Q^\nu}{\bar{q}_1^2\bar{q}_2^2} - \frac{P^\mu P^\nu}{\omega_1^2\omega_2^2}\Big)\Big], \tag{4.17}$$

where $\kappa = \sqrt{32\pi G}$, $\gamma = u_1 \cdot u_2$ and

$$\begin{aligned} P^\mu &= -\omega_1 u_2^\mu + \omega_2 u_1^\mu \\ Q^\mu &= (\bar{q}_1 - \bar{q}_2)^\mu + \frac{\bar{q}_1^2}{\omega_1}u_1^\mu - \frac{\bar{q}_2^2}{\omega_2}u_2^\mu, \ \omega_1 = -\bar{k}\cdot u_1, \ \omega_2 = -\bar{k}\cdot u_2 . \end{aligned} \tag{4.18}$$

As $k_\mu \bar{\mathcal{A}}_5^{\mu\nu} = k_\nu \bar{\mathcal{A}}_5^{\mu\nu} = 0$, the amplitude is gauge-invariant. The (sub)^n-leading soft graviton theorem for the tree-level amplitude can be restated as follows

$$\begin{aligned}&\hat{\delta}^{(4)}(q_1+q_2-k)\mathcal{A}_5^{\mu\nu}[p_1+q_1,p_2+q_2\to p_1,p_2,k]\\&=\sum_{r=0}^{n}\frac{(-1)^{n-r}}{(n-r)!}S^{(r),\mu\nu}(k\cdot\partial)^{n-r}(\hat{\delta}^{(4)}(q_1+q_2))\mathcal{A}_4[p_1+q_1,p_2+q_2\to p_1,p_2]+\mathcal{X}^{\mu\nu}\,.\end{aligned}\tag{4.19}$$

$\mathcal{R}_n=\epsilon_{\mu\nu}\mathcal{X}^{\mu\nu}$ is the remainder term that spoils the factorization beyond the sub-subleading order in the soft expansion. For the amplitude (4.17), it is given by the following expression

$$\mathcal{X}^{\mu\nu}=\frac{\kappa^3m_1^2m_2^2}{4}\sum_{r=3}^{n}\frac{(-1)^{n-r}}{(n-r)!}(k\cdot\partial)^{n-r}(\hat{\delta}^{(4)}(q_1+q_2))\Lambda_{r-1}^{\mu\nu}+(1\leftrightarrow2)\,,\tag{4.20}$$

where the polynomial $\Lambda_n^{\mu\nu}$ is defined as

$$\Lambda_{n\geq2}^{\mu\nu}=H_2^{\mu\nu}\frac{2^{n-2}(\bar{q}\cdot\bar{k})^{n-2}}{(\bar{q}^2)^{n-2}}\,.\tag{4.21}$$

Here

$$H_2^{\mu\nu}=-\frac{4}{(\bar{q}^2)^2}\Big(\omega_2^2u_1^\mu u_1^\nu-\frac{\omega_1\omega_2}{2}(u_2^\mu u_1^\nu+u_2^\nu u_1^\mu)\Big)\,.\tag{4.22}$$

As can be explicitly checked $\mathcal{R}_n$ satisfies the constraint equation (4.14) and the resulting A is antisymmetric in the indices as stated in equation (4.11). The factorized terms that were obtained via the soft graviton factors on the four-point amplitude in equation (4.19) can be written as follows

$$\frac{\kappa^3m_1^2m_2^2}{4}\sum_{r=0}^{n}\frac{(-1)^{n-r}}{(n-r)!}(k\cdot\partial)^{n-r}(\hat{\delta}^{(4)}(q_1+q_2))K_{r-1}^{\mu\nu}+(1\leftrightarrow2)\,,\tag{4.23}$$

where the polynomials $K_n^{\mu\nu}$ are defined as

$$
\begin{aligned}
K_{-1}^{\mu\nu} &= -\frac{1}{\bar{q}^2}\Big(\gamma^2 - \frac{1}{2}\Big)\Big\{ - \frac{1}{\omega_1}\bar{q}^{(\mu}u_1^{\nu)} - \frac{1}{\omega_1^2}(\bar{k}\cdot\bar{q})u_1^\mu u_1^\nu\Big\} \\
K_0^{\mu\nu} &= -\frac{2\gamma}{\bar{q}^2}\Big(- u_1^{(\mu}u_2^{\nu)} + \frac{2\omega_2}{\omega_1}u_1^\mu u_1^\nu\Big) + H_0^{\mu\nu} \\
K_{+1}^{\mu\nu} &= H_1^{\mu\nu} + H_0^{\mu\nu}\frac{2(\bar{q}\cdot\bar{k})}{\bar{q}^2} \\
K_{n\geq 2}^{\mu\nu} &= H_1^{\mu\nu}\frac{2^{n-1}(\bar{q}\cdot\bar{k})^{n-1}}{(\bar{q}^2)^{n-1}} + H_0^{\mu\nu}\frac{2^n(\bar{q}\cdot\bar{k})^n}{(\bar{q}^2)^n}\,,
\end{aligned}
\tag{4.24}
$$

where

$$
H_1^{\mu\nu} = \frac{4\gamma}{\bar{q}^2}\frac{\omega_2}{\bar{q}^2}\bar{q}^{(\mu}u_1^{\nu)} \quad \text{and} \quad H_0^{\mu\nu} = -\frac{2}{\bar{q}^2}\Big(\gamma^2 - \frac{1}{2}\Big)\frac{\bar{q}^\mu\bar{q}^\nu}{\bar{q}^2}\,. \tag{4.25}
$$

4.3 Gravitational scattering of massive spinless particles

We consider gravitational scattering of two scalars ϕ_1, ϕ_2 with masses m_1, m_2 respectively. KMOC formalism provides us with a formula to compute the gravitational radiative field. In a scattering process where two scalar particles with initial momenta p_1, p_2 gravitationally scatter at impact parameter distance b, the radiative field is given by the following equation

$$
\begin{aligned}
h_{\mu\nu}(\omega,\hat{n}) \;=\; \lim_{\hbar\to 0}\hbar^{3/2}\int\prod_{i=1}^{2}\hat{d}^4\bar{q}_i\;\hat{\delta}(2p_i\cdot\bar{q}_i)\;e^{i\bar{q}_1\cdot b}\;\hat{\delta}^{(4)}(\bar{q}_1+\bar{q}_2-\bar{k}) \\
\Big(\mathcal{A}_5^{\mu\nu}(p_1+\hbar\bar{q}_1, p_2+\hbar\bar{q}_2 \to p_1, p_2, \hbar\bar{k}\Big) + \cdots + \mathcal{O}(\kappa^5)
\end{aligned}
\tag{4.26}
$$

where $\mathcal{A}_5^{\mu\nu}$ is the five point amplitude. The final momenta are integrated over and this integration is written in terms of the momentum exchange $q_i = \tilde{p}_i - p_i$. The leading order contribution to the gravitational radiation arises from the tree-level amplitude (4.17) and it is given by

$$\begin{aligned}\mathcal{R}^{\mu\nu}(\omega,\hat{n}) = &-\frac{\kappa^3 m_1 m_2}{4}\int \hat{d}^4\bar{q}\hat{\delta}(u_1\cdot\bar{k}-u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})e^{ib\cdot(\bar{k}-\bar{q})}\\ &\times\Big[\frac{4}{(\bar{k}-\bar{q})^2\bar{q}^2}\Big(\omega_2^2 u_1^\mu u_1^\nu - \frac{\omega_1\omega_2}{2}(u_2^\mu u_1^\nu + u_2^\nu u_1^\mu)\Big)\\ &+\frac{2\gamma}{\bar{q}^2}\Big(-2\frac{\omega_2}{(\bar{k}-\bar{q})^2}\bar{q}^{(\mu}u_1^{\nu)} - u_1^{(\mu}u_2^{\nu)} + \frac{2\omega_2}{\omega_1}u_1^\mu u_1^\nu\Big)\\ &+\frac{2}{\bar{q}^2}\Big(\gamma^2-\frac{1}{2}\Big)\Big\{\frac{1}{(\bar{k}-\bar{q})^2}\bar{q}^\mu\bar{q}^\nu - \frac{1}{\omega_1}\bar{q}^{(\mu}u_1^{\nu)} - \frac{1}{\omega_1^2}(\bar{k}\cdot\bar{q})u_1^\mu u_1^\nu\Big\}\Big]\\ &-\frac{\kappa^3 m_1 m_2}{4}\int \hat{d}^4\bar{q}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{k}-u_2\cdot\bar{q})e^{ib\cdot\bar{q}}\\ &\times\Big[\frac{4}{(\bar{k}-\bar{q})^2\bar{q}^2}\Big(\omega_1^2 u_2^\mu u_2^\nu - \frac{\omega_1\omega_2}{2}(u_2^\mu u_1^\nu + u_2^\nu u_1^\mu)\Big)\\ &+\frac{2\gamma}{\bar{q}^2}\Big(-2\frac{\omega_1}{(\bar{k}-\bar{q})^2}\bar{q}^{(\mu}u_2^{\nu)} - u_2^{(\mu}u_1^{\nu)} + \frac{2\omega_1}{\omega_2}u_2^\mu u_2^\nu\Big)\\ &+\frac{2}{\bar{q}^2}\Big(\gamma^2-\frac{1}{2}\Big)\Big\{\frac{1}{(\bar{k}-\bar{q})^2}\bar{q}^\mu\bar{q}^\nu - \frac{1}{\omega_2}\bar{q}^{(\mu}u_2^{\nu)} - \frac{1}{\omega_2^2}(\bar{k}\cdot\bar{q})u_2^\mu u_2^\nu\Big\}\Big]. \quad (4.27)\end{aligned}$$

Soft expansion of the radiation kernel is obtained via the following two expansions:

$$\begin{aligned}\hat{\delta}(u_1\cdot\bar{k}-u_1\cdot\bar{q}) = \hat{\delta}(u_1\cdot\bar{q}) - (u_1\cdot\bar{k})\hat{\delta}'(u_1\cdot\bar{q}) + \frac{(u_1\cdot\bar{k})^2}{2!}\hat{\delta}''(u_1\cdot\bar{q})\\ + \cdots + (-1)^n\frac{(u_1\cdot\bar{k})^n}{n!}\hat{\delta}^{(n)}(u_1\cdot\bar{q}) \quad (4.28)\end{aligned}$$

and

$$\frac{1}{(\bar{k}-\bar{q})^2} = \frac{1}{\bar{q}^2-2(\bar{q}\cdot\bar{k})} = \frac{1}{\bar{q}^2}\Big(1+\frac{2(\bar{q}\cdot\bar{k})}{\bar{q}^2}+\frac{4(\bar{q}\cdot\bar{k})^2}{(\bar{q}^2)^2}+\frac{8(\bar{q}\cdot\bar{k})^3}{(\bar{q}^2)^3}+\cdots\Big) \quad (4.29)$$

Note that the integration range over the exchange momentum is $\omega < |q| < b^{-1}$, where $\omega = |k|$, ensuring that the integral of the terms from the expansion in equation (4.29) is infrared (IR) finite. The integration region is naturally restricted to $|q| \geq \omega$. The soft expansion of the unstripped amplitude is derived by performing a series expansion of the delta function, implicitly assuming that $\omega \ll |q|$. The upper limit of the integration region in the soft limit is determined by the phase term in the integrand, with the scattering length given by $1/|q| \sim b$. The leading order soft radiation can be written as follows

$$\begin{aligned}
\mathcal{R}^{(0),\mu\nu} &= \frac{\kappa^3 m_1 m_2}{4} \int \hat{d}^4\bar{q}\Big\{e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\Big(K^{\mu\nu}_{-1}\Big) + e^{ib\cdot\bar{q}}\Big(1\leftrightarrow 2\Big)\Big\} \\
&= \frac{\kappa^3 m_1 m_2}{4}\Big(\gamma^2 - \frac{1}{2}\Big)\int \frac{\hat{d}^4\bar{q}}{\bar{q}^2} e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\Big(\frac{1}{\omega_1}\bar{q}^{(\mu}u_1^{\nu)} + \frac{(\bar{k}\cdot\bar{q})}{\omega_1^2}u_1^\mu u_1^\nu\Big) + (1\leftrightarrow 2) \\
&= -\frac{\kappa^3 m_1 m_2}{8\pi\gamma\beta b^2}\Big(\gamma^2 - \frac{1}{2}\Big)\Big(\frac{1}{\omega_1}b^{(\mu}u_1^{\nu)} + \frac{(\bar{k}\cdot b)}{\omega_1^2}u_1^\mu u_1^\nu\Big) + (1\leftrightarrow 2)\,. \qquad (4.30)
\end{aligned}$$

Note that in the $|b| \to \infty$ limit, this term doesn't contribute to the kernel. This is expected as the particles are so far apart in this limit that they experience no deflection, and hence the memory effect vanishes! However, we show below that, unlike the memory effect, the leading log soft factor survives in this vanishing deflection limit.

Using the expansion of the delta functions and the five-point amplitude, the sub-leading order soft radiation can be expressed as follows

$$\begin{aligned}
\mathcal{R}^{(1),\mu\nu} = \frac{\kappa^3 m_1 m_2}{4} \int \hat{d}^4\bar{q}\Big[&\Big\{e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\Big(K^{\mu\nu}_{0}\Big) + e^{ib\cdot\bar{q}}\Big(1\leftrightarrow 2\Big)\Big\} \\
&- \Big\{e^{-ib\cdot\bar{q}}\hat{\delta}'(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(u_1\cdot k)\Big(K^{\mu\nu}_{-1}\Big) + e^{ib\cdot\bar{q}}\Big(1\leftrightarrow 2\Big)\Big\} \\
&+ e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(ib\cdot\bar{k})K^{\mu\nu}_{-1}\Big]. \qquad (4.31)
\end{aligned}$$

Here, the superscript in $\mathcal{R}^{(1),\mu\nu}$ denotes the truncation of the radiation kernel to $\mathcal{O}(\omega^0)$.

We can re-write $\mathcal{R}^{(1),\mu\nu}$ as

$$\mathcal{R}^{(1),\mu\nu} = \frac{\kappa^3 m_1 m_2}{4}\,(I_1^{\mu\nu} + I_2^{\mu\nu} + I_3^{\mu\nu}) \qquad (4.32)$$

All three terms are evaluated in Appendix B and the final result is summarised below.

$$\begin{aligned}
I_1^{\mu\nu} &= \int \hat{d}^4\bar{q} e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\Big[\frac{2\gamma}{\bar{q}^2}\Big(-u_1^{(\mu}u_2^{\nu)} + \frac{2\omega_2}{\omega_1}u_1^\mu u_1^\nu\Big) + \frac{2}{\bar{q}^2}\Big(\gamma^2 - \frac{1}{2}\Big)\frac{\bar{q}^\mu\bar{q}^\nu}{\bar{q}^2}\Big] \\
&= \frac{\gamma}{\pi\gamma\beta}\log(\omega b)\Big(u_1^{(\mu}u_2^{\nu)} - \frac{2(u_2\cdot\bar{k})}{(u_1\cdot\bar{k})}u_1^\mu u_1^\nu\Big) + O(\omega^0) \qquad (4.33)
\end{aligned}$$

$$
\begin{aligned}
I_2^{\mu\nu} &= -\int \frac{\hat{d}^4\bar{q}}{\bar{q}^2} e^{-ib\cdot\bar{q}} \hat{\delta}'(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\Big(\gamma^2-\frac{1}{2}\Big)(u_1\cdot\bar{k})\Big(\frac{1}{\omega_1}\bar{q}^{(\mu}u_1^{\nu)}+\frac{(\bar{k}\cdot\bar{q})}{\omega_1^2}u_1^\mu u_1^\nu\Big) \\
&= \frac{1}{2\pi\gamma^3\beta^3}\Big(\gamma^2-\frac{1}{2}\Big)\log(\omega b)\Big\{-(\gamma u_2-u_1)^{(\mu}u_1^{\nu)}+\Big(\gamma\frac{(u_2\cdot\bar{k})}{u_1\cdot\bar{k}}-1\Big)u_1^\mu u_1^\nu\Big\}, \qquad (4.34)
\end{aligned}
$$

and finally,

$$
\begin{aligned}
I_3^{\mu\nu} &= \int \frac{\hat{d}^4\bar{q}}{\bar{q}^2} e^{-ib\cdot\bar{q}} \hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(ib\cdot\bar{k})\Big(\gamma^2-\frac{1}{2}\Big)\Big(\frac{1}{\omega_1}\bar{q}^{(\mu}u_1^{\nu)}+\frac{(\bar{k}\cdot\bar{q})}{\omega_1^2}u_1^\mu u_1^\nu\Big) \\
&= -\frac{(b\cdot k)}{2\pi\gamma\beta b^2}\Big(\gamma^2-\frac{1}{2}\Big)\Big(\frac{1}{\omega_1}b^{(\mu}u_1^{\nu)}+\frac{(\bar{k}\cdot b)}{\omega_1^2}u_1^\mu u_1^\nu\Big), \qquad (4.35)
\end{aligned}
$$

We note that $I_3^{\mu\nu}$ has no logarithmic terms unlike $I_1^{\mu\nu}$ and $I_2^{\mu\nu}$. Therefore the logarithmic contribution of the first particle with initial momentum p_1 to the gravitational radiation is given by

$$
\mathcal{R}_1^{\log\omega,\mu\nu} = \frac{\kappa^3 m_1 m_2}{4\pi\gamma^3\beta^3}\log(\omega b)\gamma(2\gamma^2-3)\Big(u_1^{(\mu}u_2^{\nu)}-\frac{(u_2\cdot\bar{k})}{(u_1\cdot\bar{k})}u_1^{(\mu}u_1^{\nu)}\Big). \qquad (4.36)
$$

This expression matches with the classical result for sub-leading log soft graviton factor [93, 94] and with the result obtained using quantum soft theorems [106]. As stated before, in the deflection less limit ($|b| \to \infty$) such that ωb is fixed, the leading logarithmic contribution survives! In fact, all the log terms of the form $\omega^m \log\omega | m \geq 1$ survive and can be completely determined by the (sub)^n-leading soft graviton theorems for tree-level gravitational amplitudes.

We can similarly derive the radiation kernel at (sub)^2-leading order in the soft expansion, at leading order in the coupling, as given in Appendix D, and the result matches the existing results in the literature.

Similarly, using the expansion of the delta functions and the five-point amplitude given in equations (4.28) and (4.29) respectively, the (sub)^n-leading order soft radiation is given by

$$\mathcal{R}^{(n),\mu\nu}(\bar{k}) = \frac{\kappa^3 m_1 m_2}{4}\int \hat{d}^4\bar{q}\Big[\sum_{r=0}^{n}\frac{1}{(n-r)!}e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(ib\cdot\bar{k})^{n-r}K^{\mu\nu}_{r-1}$$
$$+\sum_{r=0}^{n-1}\frac{(-1)^{n-r}}{(n-r)!}\Big\{e^{-ib\cdot\bar{q}}\hat{\delta}^{(n-r)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(u_1\cdot\bar{k})^{n-r}\Big(K^{\mu\nu}_{r-1}\Big)+e^{ib\cdot\bar{q}}\Big(1\leftrightarrow 2\Big)\Big\}$$
$$+\sum_{r=0}^{n-2}\sum_{\substack{t,s\geq 1\\ \ni(t+s)=n-r}}\frac{(-1)^s}{t!s!}e^{-ib\cdot\bar{q}}(ib\cdot\bar{k})^t(u_1\cdot\bar{k})^s\hat{\delta}^{(s)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})K^{\mu\nu}_{r-1}\Big], \quad (4.37)$$

where the polynomial $K^{\mu\nu}_n$ is defined in Section 4.2. Here, the superscript in $\mathcal{R}^{(n),\mu\nu}$ denotes the truncation of the radiation kernel to $\mathcal{O}(\omega^{n-1})$.

Using the integrals evaluated in Appendix B, we will isolate the logarithmic contributions ($\omega^{n-1}\log(\omega b)$) that come only from the following two types of integrals:

$$I^{\mu\nu}_1 = \int \hat{d}^4\bar{q}\frac{1}{(n-1)!}e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(ib\cdot\bar{k})^{n-1}K^{\mu\nu}_0$$
$$= -\frac{1}{(n-1)!}\int \hat{d}^4\bar{q}e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(ib\cdot\bar{k})^{n-1}\Big[\frac{2\gamma}{\bar{q}^2}\Big(-u_1^{(\mu}u_2^{\nu)}+\frac{2\omega_2}{\omega_1}u_1^{\mu}u_1^{\nu}\Big)$$
$$+\frac{2}{\bar{q}^2}\Big(\gamma^2-\frac{1}{2}\Big)\frac{\bar{q}^\mu\bar{q}^\nu}{\bar{q}^2}\Big]$$
$$= \frac{i^{n-1}\gamma}{(n-1)!\pi\gamma\beta}(\omega b)^{n-1}\log(\omega b)\Big(u_1^{(\mu}u_2^{\nu)}-\frac{(u_2\cdot\bar{k})}{(u_1\cdot\bar{k})}u_1^{(\mu}u_1^{\nu)}\Big)+\mathcal{O}(\omega^{n-1}), \quad (4.38)$$

where we have used the integral result of equation (B.11) and

$$I^{\mu\nu}_2 = -\frac{1}{(n-1)!}e^{-ib\cdot\bar{q}}(ib\cdot\bar{k})^{n-1}(u_1\cdot\bar{k})\hat{\delta}'(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})K^{\mu\nu}_{-1}$$
$$= \frac{1}{(n-1)!}\int \hat{d}^4\bar{q}e^{-ib\cdot\bar{q}}\hat{\delta}'(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(u_1\cdot\bar{k})(ib\cdot\bar{k})^{n-1}\frac{1}{\bar{q}^2}\Big(\gamma^2-\frac{1}{2}\Big)\Big\{-\frac{1}{\omega_1}\bar{q}^{(\mu}u_1^{\nu)}$$
$$-\frac{1}{\omega_1^2}(\bar{k}\cdot\bar{q})u_1^{\mu}u_1^{\nu}\Big\}$$
$$= -\frac{i^{n-1}}{2(n-1)!\pi\gamma^3\beta^3}(\omega b)^{n-1}\log(\omega b)\Big(\gamma^2-\frac{1}{2}\Big)\Big\{(\gamma u_2-u_1)^{(\mu}u_1^{\nu)}$$
$$-\frac{1}{(u_1\cdot\bar{k})}(\bar{k}\cdot(\gamma u_2-u_1))u_1^{\mu}u_1^{\nu}\Big\}, \quad (4.39)$$

where we have used the integral result of equation (B.19).

Therefore, upon simplifying, the log term in (sub)^n-leading radiation kernel corresponding to particle 1 is obtained as [147]

$$\mathcal{R}_1^{(\omega b)^{n-1}\log(\omega b),\mu\nu} = \frac{i^{n-1} m_1 m_2 \kappa^3}{4\pi(n-1)!\gamma^3\beta^3}\gamma(2\gamma^2-3)(\omega b)^{n-1}\log(\omega b)\Big(u_1^{(\mu}u_2^{\nu)} - \frac{(u_2\cdot\bar{k})}{(u_1\cdot\bar{k})}u_1^{(\mu}u_1^{\nu)}\Big)\,. \tag{4.40}$$

Note that in the deflection less limit ($|b|\to\infty$) such that ωb is fixed, the logarithmic contributions survive. The rest of the terms constitute integrals similar to the ones described in section 4.4.1 and they do not give any logarithmic contributions. Also, the terms obtained using quantum soft theorems and then taking the classical limit, match with the counterparts in the soft expansion of the radiation kernel in equation (4.37).

4.4 Radiation kernel to (sub)^n-leading order in frequency

In this section, we prove our main result. Given a (sub)^n-leading soft graviton theorem of a tree-level gravitational amplitude, the so-called remainder terms never contribute to the leading log contribution that arises in the classical limit.

4.4.1 (sub)^n-leading order soft radiation from quantum soft theorems

We will now compute the soft radiation by applying the (sub)^n-leading soft graviton operator to the quantum four-point amplitude and then taking the classical limit. We will observe that not all terms in the soft expansion of the classical radiation kernel can be recovered using the quantum soft theorems, and the remaining terms do not correspond to any logarithmic contribution.

As $\omega^{n-1}\log\omega$ is more dominant than the ω^{n-1} terms, the low-frequency classical radiation during a scattering process is simply obtained from the soft theorems. One can discard the remainder terms then.

From quantum soft theorems, the (sub)^n-leading radiation kernel is given by

$$\mathcal{R}^{\mu\nu}_{\omega^{(n-1)}} = \frac{1}{4m_1 m_2}\int \hat{d}^4 q_1 \hat{d}^4 q_2 e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2) \times \frac{\kappa}{2}\sum_{i=1,2}\Big[\frac{J_i^{\mu\rho}k_\rho J_i^{\nu\sigma}k_\sigma}{p_i\cdot k}\Big(k\cdot\frac{\partial}{\partial p_i}\Big)^{n-2} + \frac{\tilde{J}_i^{\mu\rho}k_\rho \tilde{J}_i^{\nu\sigma}k_\sigma}{\tilde{p}_i\cdot k}\Big(k\cdot\frac{\partial}{\partial \tilde{p}_i}\Big)^{n-2}\Big] \Big(\hat{\delta}^{(4)}(q_1+q_2)\mathcal{A}_4\Big), \quad (4.41)$$

where

$$\mathcal{A}_4[p_1,\tilde{p}_1,p_2,\tilde{p}_2] = \frac{\kappa^2}{2q_2^2}\Big[(p_2\cdot\tilde{p}_2)(m_1^2 - p_1\cdot\tilde{p}_1) + m_2^2(p_1\cdot\tilde{p}_1 - 2m_1^2) + (p_1\cdot\tilde{p}_2)(p_2\cdot\tilde{p}_1) + (p_1\cdot p_2)(\tilde{p}_1\cdot\tilde{p}_2)\Big]. \quad (4.42)$$

From now on, we will consider the contribution from particle 1. First, let us evaluate the soft operators' action on $\mathcal{A}_4$. The classical contribution to the soft radiation in this case comes from the action of the soft operators on the denominator of the amplitude and is given by

$$\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},A} = (-1)^{n+1}\frac{\kappa^3 m_1 m_2}{4}\int \hat{d}^4\bar{q} e^{-i\bar{q}\cdot b}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\frac{2^{n-1}}{(\bar{q}^2)^{n+1}}(\bar{q}\cdot\bar{k})^{n-1} \times\left(\bar{q}^\mu\bar{q}^\nu\Big(\gamma^2-\frac{1}{2}\Big) + \bar{q}^2\frac{(u_2\cdot\bar{k})}{(\bar{q}\cdot\bar{k})}\bar{q}^{(\mu}u_1^{\nu)}\right). \quad (4.43)$$

This term doesn't give any logarithmic contributions using the integral results of Appendix B. Let us evaluate the soft operators' action on the delta function now where we use the following distributional identity:

$$S^{(n),\mu\nu}\hat{\delta}^{(4)}(q_1+q_2) = \hat{\delta}^{(4)}(q_1+q_2)S^{(n),\mu\nu} - (k\cdot\partial)\hat{\delta}^{(4)}(q_1+q_2)S^{(n-1),\mu\nu} + \frac{1}{2}(k\cdot\partial)^2\hat{\delta}^{(4)}(q_1+q_2)S^{(n-2),\mu\nu} + \cdots + \frac{(-1)^n}{n!}(k\cdot\partial)^n\hat{\delta}^{(4)}(q_1+q_2)S^{(0),\mu\nu}.$$

$$ \tag{4.44} $$

The soft radiation in this case is given by

$$
\begin{aligned}
\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},D} = \frac{1}{m_1 m_2}\int \hat{d}^4 q_1 \hat{d}^4 q_2 e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\Big[&- (k\cdot\partial)\hat{\delta}^{(4)}(q_1+q_2)S^{(n-1),\mu\nu} \\
&+ \frac{1}{2}(k\cdot\partial)^2\hat{\delta}^{(4)}(q_1+q_2)S^{(n-2),\mu\nu} + \cdots + \frac{(-1)^{n-2}}{(n-2)!}(k\cdot\partial)^{n-2}\hat{\delta}^{(4)}(q_1+q_2)S^{(2),\mu\nu} \\
&+ \frac{(-1)^{n-1}}{(n-1)!}(k\cdot\partial)^{n-1}\hat{\delta}^{(4)}(q_1+q_2)S^{(1),\mu\nu} + \frac{(-1)^n}{n!}(k\cdot\partial)^n\hat{\delta}^{(4)}(q_1+q_2)S^{(0),\mu\nu}\Big]\mathcal{A}_4 \,.
\end{aligned}
\tag{4.45}
$$

We provide a detailed derivation of the computation in Appendix E. Here, we isolate the logarithmic contributions to the classical soft radiation, which arise from the following two terms:

$$
\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},4} = \frac{1}{m_1 m_2}\int \hat{d}^4 q_1 \hat{d}^4 q_2 e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\frac{(-1)^{n-1}(k\cdot\partial)^{n-1}}{(n-1)!}\hat{\delta}^{(4)}(q_1+q_2)S^{(1),\mu\nu}\mathcal{A}_4 \,.
\tag{4.46}
$$

Integrating q_1 and relabelling $q_2 \to q$ and keeping only $\mathcal{O}(\omega^{n-1})$ terms, we have,

$$
\begin{aligned}
\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},4} &= \frac{1}{m_1 m_2}\int \hat{d}^4 q e^{-iq\cdot b}\hat{\delta}(u_2\cdot q)\Big\{\frac{(ik\cdot b)^{n-1}}{(n-1)!}\hat{\delta}(u_1\cdot q) \\
&\qquad + \sum_{\substack{r,s\geq 1 \\ \ni (r+s)=n-1}} \frac{(-1)^s}{r!s!}(ib\cdot k)^r (u_1\cdot k)^s \hat{\delta}^{(s)}(u_1\cdot q) \\
&\qquad + \frac{(-1)^{n-1}}{(n-1)!}(u_1\cdot k)^{n-1}\hat{\delta}^{(n-1)}(u_1\cdot q)\Big\}S^{(1),\mu\nu}\mathcal{A}_4 \\
&= \frac{i^{n-1} m_1 m_2 \kappa^3 \gamma}{(n-1)!\pi\gamma\beta}(\omega b)^{n-1}\log(\omega b) u_1^{(\mu}\Big(u_2^{\nu)} - u_1^{\nu)}\frac{(\bar{k}\cdot u_2)}{(\bar{k}\cdot u_1)}\Big) + \mathcal{O}(\omega^{n-1}) \,.
\end{aligned}
\tag{4.47}
$$

where we have used the integral result of equation (B.11), and

$$\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},5} = \frac{1}{m_1 m_2}\int \hat{d}^4 q_1 \hat{d}^4 q_2 e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\frac{(-1)^n (k\cdot\partial)^n}{n!}\hat{\delta}^{(4)}(q_1+q_2)S^{(0),\mu\nu}\mathcal{A}_4\,. \tag{4.48}$$

Integrating q_1 and relabelling $q_2 \to q$ and keeping only $\mathcal{O}(\omega^{n-1})$ terms, we have,

$$\begin{aligned}
\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},5} &= \frac{1}{m_1 m_2}\int \hat{d}^4 q e^{-iq\cdot b}\hat{\delta}(u_2\cdot q)\Big\{\frac{(ik\cdot b)^n}{n!}\hat{\delta}(u_1\cdot q) \\
&\quad + \sum_{\substack{r,s\\ \ni (r+s)=n}} \frac{(-1)^s}{r!s!}(ib\cdot k)^r (u_1\cdot k)^s \hat{\delta}^{(s)}(u_1\cdot q) + \frac{(-1)^n}{n!}(u_1\cdot k)^n \hat{\delta}^{(n)}(u_1\cdot q)\Big\} S^{(0),\mu\nu}\mathcal{A}_4 \\
&= -\kappa^3 (2(p_1\cdot p_2)^2 - m_1^2 m_2^2)\frac{i^{n-1}}{4\pi(n-1)!\gamma^3\beta^3}(\omega b)^{n-1}\log(\omega b) \\
&\qquad \times \Big((\gamma u_2 - u_1)^{(\mu}u_1^{\nu)} - \frac{((\gamma u_2 - u_1)\cdot\bar{k})u_1^{(\mu}u_1^{\nu)}}{(u_1\cdot\bar{k})}\Big) + \mathcal{O}(\omega^{n-1})\,,
\end{aligned} \tag{4.49}$$

where we have used the integral result of equation (B.19). The detailed derivations of the above steps are given in Appendix E. We collect the log terms and upon simplifying, the $\omega^{n-1}\log\omega$ terms of radiation kernel w.r.t particle 1 from the quantum soft theorems is given by

$$\mathcal{R}^{\mu\nu}_{\omega^{n-1}\log\omega} = \frac{i^{n-1}m_1 m_2 \kappa^3}{4\pi(n-1)!\gamma^3\beta^3}\gamma(2\gamma^2-3)(\omega b)^{n-1}\log(\omega b)\Big(u_1^{(\mu}u_2^{\nu)} - \frac{(u_2\cdot\bar{k})}{(u_1\cdot\bar{k})}u_1^{(\mu}u_1^{\nu)}\Big)\,, \tag{4.50}$$

which matches with the log terms of (sub)^n-leading order soft expansion of the radiation kernel in equation (4.40). One can also see that the rest of the terms obtained using quantum soft theorems also match with the counterparts in the soft expansion of the radiation kernel in equation (4.37).

4.4.2 Remainder terms in (sub)^n-leading order soft radiation

Comparing the soft expansion of the radiation kernel and the radiation kernel obtained using quantum soft theorems to (sub)^n-leading order in frequency, we see that not all the terms in the soft expansion of the radiation kernel in equation (4.37) are recovered by applying the soft theorems. Such terms in the (unstripped) five-point amplitude do not factorize as soft factors times the four-point amplitude. These are known as the "Remainder terms." We have identified such terms at (sub)^n-leading order for $n \geq 3$ in the soft radiation kernel obtained as [147]

$$\begin{aligned}\mathcal{X}^{\mu\nu}_{\mathcal{R},\omega^{(n-1)}} = \frac{\kappa^3 m_1 m_2}{4}\int \hat{d}^4\bar{q}\Big[&\sum_{r=3}^{n}\frac{1}{(n-r)!}e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(ib\cdot\bar{k})^{n-r}\Lambda^{\mu\nu}_{r-1}\\ &+\sum_{r=3}^{n-1}\frac{(-1)^{n-r}}{(n-r)!}\Big\{e^{-ib\cdot\bar{q}}\hat{\delta}^{(n-r)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(u_1\cdot\bar{k})^{n-r}\Big(\Lambda^{\mu\nu}_{r-1}\Big)+e^{ib\cdot\bar{q}}\Big(1\leftrightarrow 2\Big)\Big\}\\ &+\sum_{r=3}^{n-2}\sum_{\substack{t,s\geq 1\\ \ni(t+s)=n-r}}\frac{(-1)^s}{t!s!}e^{-ib\cdot\bar{q}}(ib\cdot\bar{k})^t(u_1\cdot\bar{k})^s\hat{\delta}^{(s)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\Lambda^{\mu\nu}_{r-1}\Big],\end{aligned} \tag{4.51}$$

where the polynomial $\Lambda^{\mu\nu}_n$ is defined in Section 4.2. Using the integral results of Appendix B, it is evident that these remainder terms do not give any logarithmic contributions. As $\omega^{n-1}\log\omega$ is more dominant than the ω^{n-1} terms, one can simply discard the remainder terms in computing the low-frequency classical radiation during a scattering process.

4.5 Radiation kernel to (sub)^3-leading order in frequency

In this section, we will compute the soft radiative gravitational field to (sub)^3-leading order in frequency. One can simply substitute $n = 3$ in the previous section for the analysis. We will only summarise the important results here.

- The leading logarithmic contribution to the radiation kernel in this order is given by

$$\mathcal{R}^{\mu\nu}_{\omega^2\log\omega} = -\frac{\kappa^3 m_1 m_2}{8\pi\gamma^3\beta^3}(\omega b)^2\log(\omega b)\gamma(2\gamma^2-3)\Big(u_1^{(\mu}u_2^{\nu)}-\frac{(u_2\cdot\bar{k})}{(u_1\cdot\bar{k})}u_1^{(\mu}u_1^{\nu)}\Big)+(1\leftrightarrow 2)\,. \tag{4.52}$$

As stated before, in the deflection less limit ($|b| \to \infty$) such that ωb is fixed, the logarithmic contribution survives.

- The factorized terms in the radiation kernel, obtained via the quantum soft graviton theorems, match the counterparts derived from the soft expansion of the classical radiation kernel at (sub)^3-leading order. However, at this order ($n \geq 3$), the radiation kernel is affected by the presence of non-factorizing remainder terms.

- We have identified such remainder terms at (sub)^3-leading order in the soft radiation kernel given by

$$\mathcal{X}^{\mu\nu}_{\mathcal{R},\omega^2} = \frac{\kappa^3 m_1 m_2}{4} \int \hat{d}^4\bar{q} \Big\{ e^{-ib\cdot\bar{q}} \hat{\delta}(u_1 \cdot \bar{q}) \hat{\delta}(u_2 \cdot \bar{q}) H_2^{\mu\nu} + e^{ib\cdot\bar{q}} \Big(1 \leftrightarrow 2 \Big) \Big\} , \qquad (4.53)$$

where

$$H_2^{\mu\nu} = -\frac{4}{(\bar{q}^2)^2} \Big(\omega_2^2 u_1^\mu u_1^\nu - \frac{\omega_1 \omega_2}{2} (u_2^\mu u_1^\nu + u_2^\nu u_1^\mu) \Big) . \qquad (4.54)$$

As expected, the remainder term does not give any logarithmic contributions. As $\omega^2 \log \omega$ is more dominant than the ω^2 terms, one can simply discard the remainder term in computing the low-frequency classical radiation during a scattering process.

Chapter 5

Conclusions

Building upon the synthesis of the Newman-Janis (NJ) algorithm with the KMOC formalism in [26], in this thesis, we have used the NJ algorithm to compute classical observables beyond the linear impulse for electromagnetic scattering involving $\sqrt{\text{Kerr}}$ particles. The results for the Kerr black hole can then be obtained via double-copy methods. As is well known, the real power of the NJ algorithm lies in all orders in spin results for classical observables. We hope that by combining the on-shell methods with the NJ algorithm along with the dynamical multipole moments [140], one can build "loop integrands" associated with the scattering of $\sqrt{\text{Kerr}}$ particles so that our analysis can be extended beyond leading order in the coupling. For some early attempts in this direction, see [78].

Our main idea was to re-interpret the three-point coupling involving $\sqrt{\text{Kerr}}$ as a "spin dressing" of the photon polarisation data while computing higher point amplitudes. Our work shows that the resulting "spin-dressed" photon propagator is particularly useful for constructing the five-point amplitude where the photon is emitted from the external scalar state. We used this five-point amplitude to derive the radiation emitted by the scalar to all orders in the spin and found perfect agreement with its calculation using equations of motion. This shows the power of the NJ algorithm even in the non-conservative sector.

We then used the four-point amplitude computed using the spin-dressed photon propagator to obtain the angular impulse of the scalar as well as $\sqrt{\text{Kerr}}$ particles. Here we encountered a subtlety. We used the spin tensor $S_2^{\mu\nu}$ instead of the spin-vector a_2^μ as the fundamental spin degree of freedom. With this choice, we found that the result for the angular impulse of the $\sqrt{\text{Kerr}}$ particle is consistent with the angular momentum conservation. As explained in Section 3.3, the reason for this choice becomes evident if one recalls that the calculation of the angular impulse via the KMOC formalism involves the expectation value of differential operators acting on amplitudes. Choosing the spin to be parametrized by either a_2 or S_2 leads to different results as a consequence. To leading order in S_2 (valid as long as $a_2 \ll |b|$) we have checked that our results are consistent with [165].

Using the NJ algorithm, we computed the leading-order orbital angular impulse of a scalar particle to all orders in the spin of the $\sqrt{\text{Kerr}}$ particle and provided a closed-form expression for it. In addition, we also gave a closed-form expression for the total angular impulse of the $\sqrt{\text{Kerr}}$ particle to leading order in spin. An all-order-in spin evaluation of the total angular momentum of the $\sqrt{\text{Kerr}}$ particle is beyond the scope of this thesis and will be pursued in [181]. We also account for the contribution to angular momentum stored in the late-time Coulombic modes, called the "electromagnetic scoot", where the conservation equation shows that the contribution to the scoot is independent of the spin of the particles, as it simply arises due to the late-time Coulombic effects which do not depend on the spin.

Our broader goal is to compute classical gravitational observables for Kerr black holes, using the double copy as a key tool [205]. For conservative observables at leading order, three-point amplitudes suffice, as shown in [26] for the 1PM linear impulse in scalar–Kerr scattering. Extending this to angular impulse would be interesting. Radiative observables, however, require a double copy of the non-Abelian analogue of the $\sqrt{\text{Kerr}}$ solution, which lacks a consistent bare Lagrangian. We thus leave Kerr radiation via amplitudes to future work. Alternatively, one may study radiation directly from equations of motion [140]. Recent progress in Black Hole Perturbation Theory (BHPT) has examined Kerr–Compton amplitudes using modern tools. These amplitudes are expressed via the Nekrasov–Shatashvili (NS) function [206, 207], encoding connection coefficients of the confluent Heun equation [208–212].

In the thesis, we also analyzed the relevance of the soft graviton factorization theorems beyond sub-leading order in the context of classical gravitational scattering in four dimensions. For two massive scalar fields minimally coupled to gravity, we have shown that the infinite impact parameter limit (or the vanishing deflection limit) of the late-time gravitational field emitted during such classical scattering can be derived using these factorization theorems. The classical field obtained in this regime has an expansion in the frequency of the detector where the modes scale as $\omega^n \log \omega$ with a vanishing memory.

In detail, we have shown that in the deflection less limit ($|b| \to \infty$) such that $\omega|b|$ is fixed, all the log terms of the form $(\omega b)^n \log(\omega b)$ survive and can be completely determined by the (sub)^n-leading soft graviton theorems for tree-level gravitational amplitudes. The source of such radiative modes is the asymptotic interaction between the incoming or outgoing states, leading to the emission of gravitational radiation only from $t \to \pm\infty$. However, there could be loop corrections to the result, and they would need to be investigated further. The universal log terms of the form $(\omega b)^n \log(\omega b)^{n+1}$, $n \geq 1$ arise from the higher-loop amplitudes which survive in the $\omega \to 0, |b| \to \infty$ limit such that ωb is fixed. The loop computations would be significantly simpler as we are interested only in the infinite impact parameter limit where $p_i' = p_i$. Here, p_i and p_i' are the initial and final momenta of the particles.

It would be intriguing to investigate the impact of non-universal terms in the soft factors arising from irrelevant terms in the Lagrangian. Even at the (sub)^2-leading order in the soft expansion, where the remainder terms vanish, the corresponding soft factor is altered by the inclusion of a finite set of higher derivative terms in the Lagrangian [200]. Higher derivative terms will certainly change the higher-order tree-level soft factors, but we believe they will not alter the leading logs in the deflection-less limit. However, this needs to be investigated further.

It would be valuable to compute soft gravitational radiation for $D > 4$ and study the resulting spectra. Unlike the $D = 4$ case, subleading contributions in higher dimensions arise from the region $|q| \sim 1/b$, consistent with higher-dimensional classical soft theorems [95]. These show that spacetime regions of size $\geq b$ contribute to subleading radiation. Notably, soft emission behavior reverses between $D = 4$ and $D > 4$: in higher D, ω^0 terms come from the UV region ($|q| \sim 1/b$), while ω^{D-4} terms arise from the IR ($|q| \geq \omega$); in $D = 4$, the IR yields $\log \omega$ and the UV gives ω^0. It would be interesting to explore possible logarithmic terms at (sub)^n-leading order in $D = 5$, the first non-trivial case, and analyze the remainder terms for a refined understanding of soft theorems in $D > 4$.

There are several directions that follow from the works presented in this thesis. We highlight some of them below.

- A natural extension is to compute classical observables for the $\sqrt{\text{Kerr}}$ object and Kerr black holes at higher PM orders, incorporating radiation-reaction effects. This requires Compton amplitudes, which have seen recent progress [170,208,213–217]. We are currently pursuing this using double-copy methods and tools from BHPT.

- Our calculation of the angular impulse indicates that the scoot is linked to the IR divergence of the S-matrix in QED. We aim to derive the electromagnetic scoot directly from the KMOC formalism using Faddeev-Kulish (FK) dressed states, wherein the angular momentum operator acts non-trivially on the states. Since FK states yield an IR-finite S-matrix, computing $\Delta J^{\mu\nu}$ directly in this framework may reflect angular momentum conservation, thereby capturing the asymptotic angular momentum responsible for the scoot.

- The existence of tree-level soft theorems has been recently linked to the discovery of the $w_{1+\infty}$ asymptotic algebra [110]. Our work may shed light on the connection between higher-spin asymptotic symmetries [108] and a subset of logarithmic terms in the soft expansion of gravitational radiation. The associated low-frequency observables—"tails to the memory"—signal a breakdown of the *peeling* property in asymptotically flat spacetimes. Building on [189], we aim to explore this further, examining the relation between polyhomogeneous evolution equations, the tower of universal logarithmic soft theorems, and the possible emergence of new symmetry structures such as $w_{1+\infty}$.

- In addition to the linear memory effect due to soft gravitons emitted by the initial and final massive states, there exists a non-linear memory effect sourced by radiation from the gravitons themselves. Similarly, beyond the $1/\omega$ pole, the subleading soft graviton theorem predicts the spin memory effect from the ω^0 term in the frequency expansion, which also includes both linear and non-linear components [218]. An infinite tower of higher-spin memory effects is likewise predicted by the (sub)^n-leading soft theorems. We aim to compute the non-linear contributions to these effects using the KMOC formalism.

Appendix A

Notations and Conventions

Throughout the paper, we will use the metric signature as $(+,-,-,-)$, unless otherwise stated. So, the on-shell condition is $p^2 = m^2$. Since the impact parameter is spacelike we have $-b^2 > 0$. The rescaled delta functions appearing in the main text are defined as

$$\hat{\delta}(p\cdot q) := 2\pi\delta(p\cdot q), \quad \hat{\delta}^{(4)}(p+q) := (2\pi)^4\delta^{(4)}(p+q). \tag{A.1}$$

where p^μ and q^μ are generic four vectors. We also absorb the 2π factor in the measure d^4q and define the rescaled measure as

$$\hat{d}^4 q := \frac{d^4 q}{(2\pi)^4}. \tag{A.2}$$

The anti-symmetric bracket in all the expressions are defined as

$$(A\wedge B)^{\mu\nu} = A^{[\mu}B^{\nu]} = A^\mu B^\nu - B^\mu A^\nu\,. \tag{A.3}$$

We use the following compact notations in the main text for convenience

$$\epsilon^{\mu\nu}(A,B) = \epsilon^{\mu\nu\alpha\beta}A_\alpha B_\beta$$

$$\epsilon^{\mu}(A,B,C) = \epsilon^{\mu\nu\alpha\beta}A_{\nu}B_{\alpha}C_{\beta}$$
$$\epsilon(A,B,C,D) = \epsilon^{\mu\nu\alpha\beta}A_{\mu}B_{\nu}C_{\alpha}D_{\beta}\,, \tag{A.4}$$

where $(A^{\mu}, B^{\mu}, C^{\mu}, D^{\mu})$ are generic 4-vectors. The following identity is used throughout this paper.

$$A^{[\mu}\epsilon^{\nu]}(B,C,D) = -(A\cdot B)\epsilon^{\mu\nu}(C,D) - (A\cdot C)\epsilon^{\mu\nu}(D,B) - (A.D)\epsilon^{\mu\nu}(B,C)\,. \tag{A.5}$$

Appendix B

Evaluation of Integrals

In this appendix, we perform the integrals required to calculate the angular impulse for the scalar$-\sqrt{\text{Kerr}}$ scattering, and to calculate the various terms of the soft radiation kernel that appear in the main text.

To obtain an expression for $\Delta L_1^{\mu\nu}$ from eq.(3.47), we need to evaluate a series of integrals which are discussed below.

- We start with the following integral

$$\begin{aligned} I_1 &= \int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}\cdot u_1)\hat{\delta}(\bar{q}\cdot u_2)\frac{e^{i\bar{q}\cdot b}}{\bar{q}^2}\cosh(a_2\cdot q) \\ &= \int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}\cdot u_1)\hat{\delta}(\bar{q}\cdot u_2)\frac{e^{i\bar{q}\cdot b}}{\bar{q}^2}\Big(1+\sum_{n=1}\frac{(a_2\cdot q)^{2n}}{(2n)!}\Big) \\ &= \frac{1}{2\pi\gamma\beta}\Big[1+\sum_{n=1}\frac{(-a_2\cdot i\partial_b)^{2n}}{(2n)!}\Big]\log|\mu b|\,, \end{aligned} \tag{B.1}$$

  where μ is the infrared (IR) cutoff. Note that the spin dependent terms are not IR divergent as it involves derivative over b.

- Next, we consider

$$I_{2,\sigma} = i\int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}\cdot u_1)\hat{\delta}(\bar{q}\cdot u_2)\frac{e^{i\bar{q}\cdot b}}{\bar{q}^2}\frac{\sinh(a_2\cdot q)}{(a_2\cdot q)}q_\sigma$$

$$
\begin{aligned}
&= \frac{i}{2}\int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}\cdot u_1)\hat{\delta}(\bar{q}\cdot u_2)\frac{e^{i\bar{q}\cdot b}}{\bar{q}^2}\int_0^1 d\lambda\Big(e^{\lambda(a_2\cdot q)} + e^{-\lambda(a_2\cdot q)}\Big)q_\sigma \\
&= \text{ Re }\int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}\cdot u_1)\hat{\delta}(\bar{q}\cdot u_2)\int_0^1 d\lambda e^{i\bar{q}\cdot(b+i\lambda a_2)}\frac{i}{\bar{q}^2}q_\sigma \\
&= \frac{1}{2\pi\gamma\beta}\text{ Re}\int_0^1 d\lambda\frac{(b+i\lambda\Pi a_2)_\sigma}{b^2+2i\lambda(b\cdot\Pi a_2)-\lambda^2(\Pi a_2)^2}\,.
\end{aligned} \tag{B.2}
$$

Now we do the λ integral as follows

$$
\begin{aligned}
&\int_0^1 d\lambda\frac{(b+i\lambda\Pi a_2)_\sigma}{b^2+2i\lambda(b\cdot\Pi a_2)-\lambda^2(\Pi a_2)^2} \\
&= b_\sigma\int_0^1\frac{d\lambda}{b^2+2i\lambda(b\cdot\Pi a_2)-\lambda^2(\Pi a_2)^2} + i(\Pi a_2)_\sigma\int_0^1\frac{\lambda d\lambda}{b^2+2i\lambda(b\cdot\Pi a_2)-\lambda^2(\Pi a_2)^2} \\
&= \frac{b_\sigma}{(b^2+i(b\cdot\Pi a_2))} + i\frac{(\Pi a_2)_\sigma}{(\Pi a_2)^2}\Big(\frac{\Pi a_2}{\Pi a_2 - ib} + \log\Big|\frac{b}{b+i\Pi a_2}\Big|\Big)\,.
\end{aligned} \tag{B.3}
$$

Therefore, we get

$$
I_{2,\sigma} = \frac{1}{2\pi\gamma\beta}\text{ Re}\Big[\frac{b_\sigma}{(b^2+i(b\cdot\Pi a_2))} + i\frac{(\Pi a_2)_\sigma}{(\Pi a_2)^2}\Big(\frac{\Pi a_2}{\Pi a_2 - ib} + \log\Big|\frac{b}{b+i\Pi a_2}\Big|\Big)\Big]\,. \tag{B.4}
$$

- We consider the integral

$$
\begin{aligned}
I_3^\alpha &= \int \hat{d}^4 q e^{iq\cdot b}\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot q)\frac{1}{q^2}\sinh(a_2\cdot q)q^\alpha \\
&= \frac{1}{2}\int \hat{d}^4 q e^{iq\cdot b}\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot q)\frac{1}{q^2}q^\alpha\Big(e^{(a_2\cdot q)} - e^{-(a_2\cdot q)}\Big) \\
&= \frac{1}{2\pi\gamma\beta}\text{ Re}\Big[\frac{i(b+i\Pi a_2)^\alpha}{(b+i\Pi a_2)^2}\Big]\,.
\end{aligned} \tag{B.5}
$$

- Lastly, we evaluate the integral

$$
\begin{aligned}
I_4^{\alpha\beta} &= i\int \hat{d}^4 q e^{iq\cdot b}\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot q)\frac{1}{q^2}\mathcal{Y}q^\alpha q^\beta \\
&= i\int \hat{d}^4 q e^{iq\cdot b}\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot q)\frac{1}{q^2}\Big(\frac{\cosh(a_2\cdot q)}{(a_2\cdot q)} - \frac{\sinh(a_2\cdot q)}{(a_2\cdot q)^2}\Big)q^\alpha q^\beta \\
&= i\frac{\partial}{\partial a_{2\alpha}}\int \hat{d}^4 q e^{iq\cdot b}\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot q)\frac{1}{q^2}\frac{\sinh(a_2\cdot q)}{(a_2\cdot q)}q^\beta\,.
\end{aligned} \tag{B.6}
$$

Using the result for $I_{2,\sigma}$ in (B.4), we find

$$I_4^{\alpha\beta} = \frac{1}{2\pi\gamma\beta}\frac{\partial}{\partial a_{2\alpha}}\mathrm{Re}\Big[\frac{b^\beta}{(b^2 + i(b\cdot\varPi a_2))} + i\frac{(\varPi a_2)^\beta}{(\varPi a_2)^2}\Big(\frac{\varPi a_2}{\varPi a_2 - ib} + \log\Big|\frac{b}{b + i\varPi a_2}\Big|\Big)\Big]. \tag{B.7}$$

We use the results of these integrals to derive the expression for $\varDelta L_1^{\mu\nu}$ in (3.48). For the spin angular impulse presented in eq.(3.71), we need the integral in (B.5) and evaluate the following integral

$$\begin{aligned} I_5^{\mu\nu} &= i\int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}\cdot u_1)\hat{\delta}(\bar{q}\cdot u_2)\frac{e^{i\bar{q}\cdot b}}{\bar{q}^2}\Big[q^{[\mu}S_2^{\nu]\rho}u_{1\rho} - u_1^{[\mu}S_2^{\nu]\sigma}q_\sigma\Big]\cosh(a_2\cdot q) \\ &= \mathrm{Re}\int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}\cdot u_1)\hat{\delta}(\bar{q}\cdot u_2)e^{i\bar{q}\cdot(b+ia_2)}\frac{i}{\bar{q}^2}\Big[q^{[\mu}S_2^{\nu]\rho}u_{1\rho} - u_1^{[\mu}S_2^{\nu]\sigma}q_\sigma\Big] \\ &= \frac{1}{2\pi\gamma\beta}\mathrm{Re}\Big[\frac{(b + i\varPi a_2)^{[\mu}S_2^{\nu]\rho}u_{1\rho} - u_1^{[\mu}S_2^{\nu]\sigma}(b + i\varPi a_2)_\sigma}{(b + i\varPi a_2)^2}\Big]. \end{aligned} \tag{B.8}$$

We use this and (B.5) to obtain the expression for $\varDelta S_2^{\mu\nu}$ in (3.72).

Note that in all of the above integrals, $\varPi^\nu{}_\rho$ is the projector into the plane orthogonal to both u_1 and u_2,

$$\varPi^\nu{}_\rho = \delta^\nu{}_\rho + \frac{1}{\gamma^2\beta^2}\left[u_1^\nu(u_{1\rho} - \gamma u_{2\rho}) + u_2^\nu(u_{2\rho} - \gamma u_{1\rho})\right], \tag{B.9}$$

with $\varPi a_2 = \sqrt{\varPi a_2\cdot\varPi a_2}$ and $b = \sqrt{-b^2}$.

Now we perform the integrals required to calculate the various terms of the soft radiation kernel that appear in the main text. The range of integration is $\omega < |q_\perp| < b^{-1}$ in all the integrals, where $k^\mu = \omega(1, \hat{n})$.

- We start with the following integral:

$$J_1 = \int \hat{d}^4\bar{q}e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\frac{1}{\bar{q}^2}$$

$$= -\frac{1}{\gamma\beta}\int \hat{d}^2\bar{q}_\perp \frac{e^{ib\cdot\bar{q}_\perp}}{\bar{q}_\perp^2} \,. \tag{B.10}$$

Let the magnitude of $\bar{q}_\perp$ be r and orient axes such that $b\cdot\bar{q}_\perp = |b| r\cos\theta$. The integral becomes:

$$J_1 = -\frac{1}{2\pi}\int dr \frac{J_0(b|r|)}{r} = \frac{1}{2\pi}\log(\omega b) \,. \tag{B.11}$$

- Next, we consider

$$\begin{aligned} J_2^\mu &= \int \hat{d}^4\bar{q} e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\frac{\bar{q}^\mu}{\bar{q}^2} \\ &= -\frac{1}{\gamma\beta}\int \hat{d}^2\bar{q}_\perp e^{ib\cdot\bar{q}_\perp}\frac{\bar{q}_\perp^\mu}{\bar{q}_\perp^2} \\ &= \frac{i}{\gamma\beta}\frac{\partial}{\partial b_\perp^\mu}\int \hat{d}^2\bar{q}_\perp \frac{e^{ib\cdot\bar{q}_\perp}}{\bar{q}_\perp^2} = \frac{i}{2\pi\gamma\beta}\frac{b^\mu}{b^2} \,, \end{aligned} \tag{B.12}$$

where equation (B.11) is used and $\frac{\hat{b}^\mu}{|b|} = -\frac{b^\mu}{b^2}$.

- We consider the integral

$$J_3^\mu = \int \hat{d}^4\bar{q} e^{-ib\cdot\bar{q}}\hat{\delta}^{(n)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\frac{\bar{q}^\mu}{\bar{q}^2} \,, \tag{B.13}$$

where (n) denotes the number of derivatives acting over the on-shell delta function. To simplify this, we shall decompose the momentum $\bar{q}^\mu$ along $u_{1,2}$ and in the transverse direction

$$\bar{q}^\mu = \alpha_1 u_1^\mu + \alpha_2 u_2^\mu + \bar{q}_\perp^\mu \,, \qquad u_i\cdot\bar{q}_\perp = 0 \,, \tag{B.14}$$

where the coefficients are given by

$$\alpha_1 = \frac{1}{\gamma^2\beta^2}[\gamma x_2 - x_1] \,, \quad \alpha_2 = \frac{1}{\gamma^2\beta^2}[\gamma x_1 - x_2] \,, \tag{B.15}$$

with $x_{1,2} := (u_{1,2}\cdot\bar{q})$. Due to this change of variables, the measure transforms as follows

$$\hat{d}^4\bar{q} = \frac{1}{\gamma\beta}\hat{d}^2\bar{q}_\perp dx_1 dx_2\,. \tag{B.16}$$

In terms of $x_{1,2}$ and $\bar{q}_\perp$ variables, we rewrite

$$J_3^\mu = \frac{1}{\gamma\beta}\int \hat{d}^2\bar{q}_\perp \hat{d}x_1\hat{d}x_2 e^{ib\cdot\bar{q}_\perp}\hat{\delta}^{(n)}(x_1)\hat{\delta}(x_2)\frac{\bar{q}^\mu}{\bar{q}^2}\,. \tag{B.17}$$

Integrating by parts, we have

$$J_3^\mu = (-1)^n\frac{1}{\gamma\beta}\int \hat{d}^2\bar{q}_\perp e^{ib\cdot\bar{q}_\perp}\frac{\partial^n}{\partial x_1^n}\left(\frac{\bar{q}^\mu}{\bar{q}^2}\right)\Big|_{x_1=x_2=0}\,. \tag{B.18}$$

$$J_3^\mu = \begin{cases} (-1)^n\frac{n!}{2\gamma\beta}\int \hat{d}^2\bar{q}_\perp e^{ib\cdot\bar{q}_\perp}\frac{\bar{q}_\perp^\mu}{\bar{q}_\perp^2}\left[\left(\frac{1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\right)^n + \left(\frac{-1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\right)^n\right], & \text{if } n \geq 2. \\ \frac{1}{2\pi\gamma^3\beta^3}\log(\omega b)(\gamma u_2 - u_1)^\mu, & \text{if } n = 1. \\ \frac{i}{2\pi\gamma\beta}\frac{b^\mu}{b^2}, & \text{if } n = 0. \end{cases} \tag{B.19}$$

where

$$\frac{\partial^n}{\partial x_1^n}\left(\frac{1}{\bar{q}^2}\right)\Big|_{x_1=x_2=0} = \frac{n!}{2\bar{q}_\perp^2}\left[\left(\frac{1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\right)^n + \left(\frac{-1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\right)^n\right] \tag{B.20}$$

and

$$\frac{\partial}{\partial x_1}\bar{q}^\mu\Big|_{x_1=x_2=0} = \frac{1}{\gamma^2\beta^2}(\gamma u_2 - u_1)^\mu \tag{B.21}$$

Therefore the first integral of equation (B.19) is evaluated as

$$\begin{aligned} J_{3,1}^\mu &= (-1)^n(-i)\frac{n!}{2\gamma\beta}\frac{\partial}{\partial b^\mu}\int \hat{d}^2\bar{q}_\perp e^{ib\cdot\bar{q}_\perp}\frac{1}{\bar{q}_\perp^2}\left[\left(\frac{1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\right)^n + \left(\frac{-1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\right)^n\right] \\ &= (-1)^n(-i)\frac{1}{4\pi\gamma\beta}\frac{\partial}{\partial b^\mu}\Bigg[-\omega^{-n}\Gamma(n)\left(\left(\frac{1}{\sqrt{\gamma^2\beta^2}}\right)^n + \left(\frac{-1}{\sqrt{\gamma^2\beta^2}}\right)^n\right) \end{aligned}$$

$$\times\left((\omega b)^n\, {}_1F_2\left(-\frac{n}{2};1,1-\frac{n}{2};-\frac{1}{4}\right)-1\right)\Bigg]$$

$$= (-1)^{n+1} i\frac{n!\, b^\mu}{4\pi\gamma\beta b^2}\Bigg[b^n\left(\left(\frac{1}{\sqrt{\gamma^2\beta^2}}\right)^n+\left(\frac{-1}{\sqrt{\gamma^2\beta^2}}\right)^n\right) \times {}_1F_2\left(-\frac{n}{2};1,1-\frac{n}{2};-\frac{1}{4}\right)\Bigg], \quad \text{(B.22)}$$

where ${}_pF_q(a;b;z)$ is the generalized hypergeometric function.

- Lastly, we evaluate the integral

$$J_4^\mu = \int \hat{d}^4\bar{q}e^{-ib\cdot\bar{q}}\hat{\delta}^{(n)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\frac{\bar{q}^\mu}{(\bar{q}^2)^m}, \quad \text{(B.23)}$$

where (n) denotes the number of derivatives acting over the on-shell delta function and $m \geq 2$. In terms of $x_{1,2}$ and $\bar{q}_\perp$ variables, we have the following integral

$$J_4^\mu = \frac{1}{\gamma\beta}\int \hat{d}^2\bar{q}_\perp\hat{d}x_1\hat{d}x_2 e^{ib\cdot\bar{q}_\perp}\hat{\delta}^{(n)}(x_1)\hat{\delta}(x_2)\frac{\bar{q}^\mu}{(\bar{q}^2)^m}, \quad m \geq 2. \quad \text{(B.24)}$$

Integrating by parts, we have

$$J_4^\mu = (-1)^n\frac{1}{\gamma\beta}\int \hat{d}^2\bar{q}_\perp e^{ib\cdot\bar{q}_\perp}\frac{\partial^n}{\partial x_1^n}\left(\frac{\bar{q}^\mu}{(\bar{q}^2)^m}\right)\Big|_{x_1=x_2=0}. \quad \text{(B.25)}$$

$$J_4^\mu = \begin{cases} (-1)^n\dfrac{n!\prod_{k=2}^{m}(n+2k-2)}{2\prod_{k=2}^{m}(2k-2)\gamma\beta}\int \hat{d}^2\bar{q}_\perp e^{ib\cdot\bar{q}_\perp}\frac{\bar{q}_\perp^\mu}{(\bar{q}_\perp^2)^m}\left[\left(\frac{1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\right)^n+\left(\frac{-1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\right)^n\right], & \text{if } n\geq 2. \\ \frac{b^{2m-2}}{4\pi(m-1)\gamma^3\beta^3}\, {}_1F_2\left(1-m;1,2-m;-\frac{1}{4}\right)(\gamma u_2-u_1)^\mu, & \text{if } n=1. \\ \frac{ib^{2m-4}}{2\pi\gamma\beta}\, {}_1F_2\left(1-m;1,2-m;-\frac{1}{4}\right)b^\mu, & \text{if } n=0. \end{cases} \quad \text{(B.26)}$$

where

$$\frac{\partial^n}{\partial x_1^n}\left(\frac{1}{(\bar{q}^2)^m}\right)\Big|_{x_1=x_2=0} = \frac{n!\prod_{k=2}^{m}(n+2k-2)}{2\prod_{k=2}^{m}(2k-2)(\bar{q}_\perp^2)^m}\left[\left(\frac{1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\right)^n + \left(\frac{-1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\right)^n\right]. \tag{B.27}$$

Therefore the first integral of equation (B.26) is evaluated as

$$\begin{aligned}
J^\mu_{4,1} &= (-1)^n(-i)\frac{n!\prod_{k=2}^{m}(n+2k-2)}{2\prod_{k=2}^{m}(2k-2)\gamma\beta}\frac{\partial}{\partial b^\mu}\int \hat{d}^2\bar{q}_\perp e^{ib\cdot\bar{q}_\perp}\frac{1}{(\bar{q}_\perp^2)^m}\left[\left(\frac{1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\right)^n + \left(\frac{-1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\right)^n\right] \\
&= (-1)^n(-i)\frac{n!\prod_{k=2}^{m}(n+2k-2)}{4\pi(2m+n-2)\prod_{k=2}^{m}(2k-2)\gamma\beta}\frac{\partial}{\partial b^\mu}\Bigg[-b^{2m+n-2}\left(\left(\frac{1}{\sqrt{\gamma^2\beta^2}}\right)^n + \left(\frac{-1}{\sqrt{\gamma^2\beta^2}}\right)^n\right) \\
&\qquad\qquad \times\, {}_1F_2\left(-m-\frac{n}{2}+1;1,1-m-\frac{n}{2}+2;-\frac{1}{4}\right)\Bigg] \\
&= (-1)^{n+1}i\frac{n!\prod_{k=2}^{m}(n+2k-2)\,b^\mu}{4\pi\prod_{k=2}^{m}(2k-2)\gamma\beta}\Bigg[b^{2m+n-4}\left(\left(\frac{1}{\sqrt{\gamma^2\beta^2}}\right)^n + \left(\frac{-1}{\sqrt{\gamma^2\beta^2}}\right)^n\right) \\
&\qquad\qquad \times\, {}_1F_2\left(-m-\frac{n}{2}+1;1,1-m-\frac{n}{2}+2;-\frac{1}{4}\right)\Bigg].
\end{aligned} \tag{B.28}$$

The main text also involves higher-rank integrals of the following form, which can be expressed in terms of derivatives with respect to b^μ:

$$\begin{aligned}
I_5^{\mu_1\mu_2\cdots\mu_r} &= \int \hat{d}^4\bar{q}e^{-ib\cdot\bar{q}}\hat{\delta}^{(n)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\frac{\bar{q}^{\mu_1}\bar{q}^{\mu_2}\cdots\bar{q}^{\mu_r}}{(\bar{q}^2)^m} \\
&= (-i\partial_b^{\mu_1})(-i\partial_b^{\mu_2})\cdots(-i\partial_b^{\mu_{r-1}})\int \hat{d}^4\bar{q}e^{-ib\cdot\bar{q}}\hat{\delta}^{(n)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\frac{\bar{q}^{\mu_r}}{(\bar{q}^2)^m}.
\end{aligned} \tag{B.29}$$

The results must lie in the plane orthogonal to both u_1 and u_2. Therefore we use the projected metric [80, 85, 139]

$$\frac{\partial}{\partial b_\mu} b^\nu = \Pi^{\mu\nu} = \eta^{\mu\nu} + \frac{1}{\gamma^2\beta^2}\left(u_1^\mu (u_1 - \gamma u_2)^\nu + u_2^\mu (u_2 - \gamma u_1)^\nu\right) \tag{B.30}$$

to generate the integrals of any rank. For example,

$$\begin{aligned} I_6^{\mu\nu\rho} &= \int \hat{d}^4\bar{q} e^{-ib\cdot\bar{q}} \hat{\delta}^{(n)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q}) \frac{\bar{q}^\mu \bar{q}^\nu \bar{q}^\rho}{(\bar{q}^2)^m} \\ &= (-i\partial_b^\mu)(-i\partial_b^\nu) I_4^\rho\,. \end{aligned} \tag{B.31}$$

Therefore,

$$I_6^{\mu\nu\rho} = \begin{cases} (-1)^{n+2} i(2m+n-4) \dfrac{n! \prod_{k=2}^{m}(n+2k-2)}{4\pi \prod_{k=2}^{m}(2k-2)\gamma\beta} \Bigg\{ b^{2m+n-8} \left(\left(\frac{1}{\sqrt{\gamma^2\beta^2}}\right)^n + \left(\frac{-1}{\sqrt{\gamma^2\beta^2}}\right)^n \right) \\ \times\, {}_1F_2\left(-m-\frac{n}{2}+1; 1, 1-m-\frac{n}{2}+2; -\frac{1}{4}\right) \Bigg\} \left[(2m+n-6) b^\mu b^\nu b^\rho + b^2 b^{(\mu}\Pi^{\nu\rho)}\right], & \text{if } n \geq 2. \\ \\ \frac{-b^{2m-6}}{2\pi\gamma^3\beta^3}\, {}_1F_2\left(1-m; 1, 2-m; -\frac{1}{4}\right) (\gamma u_2 - u_1)^{(\mu} \left[(2m-4) b^\nu b^{\rho)} + b^2 \Pi^{\nu\rho)}\right], & \text{if } n = 1. \\ \frac{-i(2m-4) b^{2m-8}}{2\pi\gamma\beta}\, {}_1F_2\left(1-m; 1, 2-m; -\frac{1}{4}\right) \left[(2m-6) b^\mu b^\nu b^\rho + b^2 b^{(\mu}\Pi^{\nu\rho)}\right], & \text{if } n = 0. \end{cases} \tag{B.32}$$

Appendix C

Classical calculations

In this appendix, we present the classical calculations of angular impulse and the radiation kernel for scalar in scalar-$\sqrt{\text{Kerr}}$ scattering to leading order in coupling.

C.1 Equations of motion

We present all the equations of motion that will be used to derive the physical observables discussed in the main text. The equation of motion for the scalar particle in scalar-$\sqrt{\text{Kerr}}$ scattering [143] is

$$\frac{dp_1^\mu}{d\tau} = Q_1 \text{Re} F_{2,+}^{\mu\nu}(x + ia_2) u_{1\nu} \, , \tag{C.1}$$

where $F_+^{\mu\nu}(x + ia_2)$ is the self-dual part of the electromagnetic field strength of the $\sqrt{\text{Kerr}}$ particle. The self-dual and anti-self-dual field strengths are defined with respect to the Minkowski metric as follows:

$$F_\pm^{\mu\nu}(x + ia_2) = F^{\mu\nu} \pm \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \, . \tag{C.2}$$

Note that, the self dual and anti-self dual fields are related to each other via complex conjugation *

$$F^{-}_{\mu\nu}(x - ia) = \left[F^{+}_{\mu\nu}(x + ia)\right]^{\dagger}\,. \tag{C.4}$$

Therefore we can rewrite the real part of the self dual field strength as

$$2\mathrm{Re}F^{\mu\nu}_{2,+}(x_1 + ia_2) = F^{\mu\nu}_{+}(x_1 + ia_2) + F^{\mu\nu}_{-}(x_1 - ia_2) \tag{C.5}$$

We then use the definition (C.2) to express the equation of motion as follows

$$\frac{dp^{\mu}_1}{d\tau} = Q_1\left[\cos(a_2\cdot\partial)F^{\mu\nu}(x) - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\sin(a_2\cdot\partial)F_{\rho\sigma}(x)\right]u_{1\nu}\,. \tag{C.6}$$

Note that, the field strength appearing here is due to a charged scalar particle! We now use the following expression for the field strength in momentum space

$$F^{\mu\nu}_2(\bar{q}) = (iQ_2)e^{-i\bar{q}\cdot b}\hat{\delta}(u_2\cdot\bar{q})\frac{1}{\bar{q}^2}(\bar{q}^{\mu}u^{\nu}_2 - \bar{q}^{\nu}u^{\mu}_2)\,, \tag{C.7}$$

to substitute in (C.6) to obtain an expression for $\dot{p}^{\mu}_1$ to all order in spin a^{μ}_2

$$\begin{aligned}\dot{p}^{\mu}_1(\tau) = iQ_1Q_2\int \hat{d}^4\bar{q}\hat{\delta}(u_2\cdot\bar{q})e^{-i\bar{q}\cdot x(\tau)}\frac{e^{-i\bar{q}\cdot b}}{\bar{q}^2}\Big[&\cosh(a_2\cdot\bar{q})\{\gamma\bar{q}^{\mu} - u^{\mu}_2(u_1\cdot\bar{q})\}\\ &+i\sinh(a_2\cdot\bar{q})\epsilon^{\mu\nu\rho\sigma}u_{1\nu}\bar{q}_{\rho}u_{2\sigma}\Big].\end{aligned} \tag{C.8}$$

*We assume that (for all c_n be real)

$$F_{\mu\nu}(x \pm ia) = \sum_n c_n(x \pm ia)^n \Rightarrow [F_{\mu\nu}(x \pm ia)]^{\dagger} = F_{\mu\nu}(x \mp ia)\,. \tag{C.3}$$

C.2 Orbital angular impulse for the scalar particle

Classically, the orbital angular impulse is defined as

$$\frac{dL^{\mu\nu}}{d\tau} = (x \wedge \dot{p})^{\mu\nu}\,, \tag{C.9}$$

where $p^\mu = m\dot{x}^\mu$. Now using the parametrization of the classical trajectory of particle 1 $x_1^\mu(\tau) = u_1^\mu \tau$, the orbital angular momentum impulse to leading order in coupling is

$$\frac{dL_1^{\mu\nu}}{d\tau} = \tau (u_1 \wedge \dot{p}_1)^{\mu\nu}\,. \tag{C.10}$$

We use the following identity from appendix A.5:

$$(a_2 \cdot \bar{q})\epsilon^\mu(p_1, p_2, \bar{q}) = \bar{q}^\mu \epsilon(p_1, p_2, a_2, \bar{q}) - (p_1 \cdot \bar{q})\epsilon^\mu(a_2, \bar{q}, p_2)\,, \tag{C.11}$$

to rewrite (C.8) as follows

$$\begin{aligned}\dot{p}_1^\mu(\tau) = \frac{iQ_1Q_2}{m_1}\int \hat{d}^4\bar{q}\hat{\delta}(p_2\cdot\bar{q})\frac{e^{+i\bar{q}\cdot b}}{\bar{q}^2}e^{-i(\bar{q}\cdot p_1)\frac{\tau}{m_1}}\Big[\cosh(a_2\cdot\bar{q})\{(p_1\cdot p_2)\bar{q}^\mu - (p_1\cdot\bar{q})p_2^\mu\} \\ + i\frac{\sinh(a_2\cdot\bar{q})}{a_2\cdot\bar{q}}\{(\bar{q}\cdot p_1)\epsilon^\mu(a_2, p_2, \bar{q}) + \bar{q}^\mu\epsilon(a_2, \bar{q}, p_1, p_2)\}\Big]\,,\end{aligned} \tag{C.12}$$

to derive the LO expression for the orbital angular impulse for the scalar particle

$$\begin{aligned}\Delta L_1^{\mu\nu} = \frac{iQ_1Q_2}{m_1}\int d\tau\hat{d}^4\bar{q}\hat{\delta}(p_2\cdot\bar{q})\frac{e^{+i\bar{q}\cdot b}}{\bar{q}^2}\left(i\partial_{p_1\cdot\bar{q}}e^{-i(\bar{q}\cdot p_1)\frac{\tau}{m_1}}\right)\Big[\cosh(a_2\cdot\bar{q})\{(p_1\cdot p_2)(p_1\wedge\bar{q})^{\mu\nu} \\ - (p_1\cdot\bar{q})(p_1\wedge p_2)^{\mu\nu}\} + i\frac{\sinh(a_2\cdot\bar{q})}{a_2\cdot\bar{q}}\{(\bar{q}\cdot p_1)p_1^{[\mu}\epsilon^{\nu]}(a_2, p_2, \bar{q}) + (p_1\wedge\bar{q})^{\mu\nu}\epsilon(a_2, \bar{q}, p_1, p_2)\}\Big]\,.\end{aligned} \tag{C.13}$$

Here we need the derivative w.r.t $u_1 \cdot \bar{q}$ due to the factor of τ in (C.10). Now in the classical computation, we shall replace

$$\bar{q}^\mu = \alpha_1 p_1^\mu + \alpha_2 p_2^\mu + \bar{q}_\perp^\mu \,, \tag{C.14}$$

where the coefficients are given in equation (B.14). Again, we can do the $x_2 = p_2 \cdot \bar{q}$ integral in the above integral and write

$$\begin{aligned} \Delta L_1^{\mu\nu} = -\frac{Q_1 Q_2}{m_1 \sqrt{\mathcal{D}}} \int d\tau d^2\bar{q}_\perp dx_1 \frac{e^{+i\bar{q}_\perp \cdot b}}{\bar{q}^2} \left(\partial_{x_1} e^{-i\frac{x_1 \tau}{m_1}}\right) \Big[\cosh(a_2 \cdot \bar{q}) \{ (p_1 \cdot p_2)(p_1 \wedge \bar{q})^{\mu\nu} \\ - (p_1 \cdot \bar{q})(p_1 \wedge p_2)^{\mu\nu} \} + i \frac{\sinh(a_2 \cdot \bar{q})}{a_2 \cdot \bar{q}} \{ x_1 p_1^{[\mu} \epsilon^{\nu]}(a_2, p_2, \bar{q}) + (p_1 \wedge \bar{q})^{\mu\nu} \epsilon(a_2, \bar{q}, p_1, p_2) \} \Big], \end{aligned} \tag{C.15}$$

where $x_1 = p_1 \cdot \bar{q}$. Integrating by parts in x_1 variable and then completing the τ integral, we obtain

$$\begin{aligned} \Delta L_1^{\mu\nu} = \frac{Q_1 Q_2}{\sqrt{\mathcal{D}}} \int d^2\bar{q}_\perp dx_1 \hat{\delta}(x_1) \frac{e^{i\bar{q}_\perp \cdot b}}{\bar{q}_\perp^2} \Big[\frac{1}{\beta^2 \gamma^2} \cosh(a_2 \cdot \bar{q})(p_1 \wedge p_2)^{\mu\nu} \\ - i \frac{\sinh(a_2 \cdot \bar{q})}{a_2 \cdot \bar{q}} \left\{ p_1^{[\mu} S_2^{\nu]\sigma} \bar{q}_\sigma + \frac{p_1 \cdot p_2}{\mathcal{D}} (p_2 \wedge p_1)^{\mu\nu} S_2^{\perp\rho\sigma} p_{1\rho} \bar{q}_{\perp\sigma} \right\} \\ - \frac{m_2^2 (a_2 \cdot p_1)}{\mathcal{D}} \left[\sinh(a_2 \cdot \bar{q})(p_1 \cdot p_2) + i \mathcal{Y} S_2^{\perp\rho\sigma} p_{1\rho} \bar{q}_{\perp\sigma} \right] (p_1 \wedge \bar{q}_\perp)^{\mu\nu} \Big] \end{aligned} \tag{C.16}$$

To linear order in spin, we find

$$\Delta L_{1,\mathcal{O}(S_2)}^{\mu\nu} = -\frac{i Q_1 Q_2}{\sqrt{\mathcal{D}}} \int d^2\bar{q}_\perp \frac{e^{+i\bar{q}_\perp \cdot b}}{\bar{q}_\perp^2} \left\{ p_1^{[\mu} S_2^{\nu]\sigma} \bar{q}_\sigma + \frac{p_1 \cdot p_2}{\mathcal{D}} (p_2 \wedge p_1)^{\mu\nu} S_2^{\perp\rho\sigma} p_{1\rho} \bar{q}_{\perp\sigma} \right\} \tag{C.17}$$

C.3 Angular impulse for $\sqrt{\text{Kerr}}$ particle

The orbital angular impulse for the $\sqrt{\text{Kerr}}$ particle is given by

$$\Delta L_2^{\mu\nu} = (b\wedge\Delta p_2)^{\mu\nu} + \int d\tau\tau(u_2\wedge\dot{p}_2)^{\mu\nu} = (b\wedge\Delta p_2)^{\mu\nu} + I^{\mu\nu}\,. \tag{C.18}$$

Using the equation of motion for the $\sqrt{\text{Kerr}}$ particle to linear order in spin

$$\dot{p}_2^\rho = Q_1 F_1^{\rho\sigma} p_{2\sigma} - \frac{Q_1}{2} S_2^{\perp\mu c}\partial^\rho F_{1,\mu c}\,, \tag{C.19}$$

we obtain the following integral expression for $\dot{p}_{2\rho}$

$$\dot{p}_{2\rho} = -\frac{iQ_1Q_2}{m_2}\int \hat{d}^4\bar{q}e^{i\bar{q}\cdot b}e^{i(p_2\cdot\bar{q})\frac{\tau}{m_2}}\frac{\hat{\delta}(p_1\cdot\bar{q})}{\bar{q}^2}\Big[(p_1\cdot p_2)\bar{q}_\rho - (p_2\cdot\bar{q})p_{1\rho} + i\bar{q}_\rho S_2^{\perp\rho\sigma}p_{1\rho}\bar{q}_\sigma\Big]. \tag{C.20}$$

We use this expression in (C.18) to obtain

$$\begin{aligned} I^{\mu\nu} = -\frac{Q_1Q_2}{m_2}\int \hat{d}^4\bar{q}d\tau e^{i\bar{q}\cdot b}\frac{\partial}{\partial(p_2\cdot\bar{q})}\Big(e^{i(p_2\cdot\bar{q})\frac{\tau}{m_2}}\Big)\frac{\hat{\delta}(p_1\cdot\bar{q})}{\bar{q}^2}\Big[(p_1\cdot p_2)(p_2\wedge\bar{q})^{\mu\nu} - (p_2\cdot\bar{q})(p_2\wedge p_1)^{\mu\nu} \\ + i(p_2\wedge\bar{q})^{\mu\nu}S_2^{\perp\rho\sigma}p_{1\rho}\bar{q}_\sigma\Big]. \end{aligned} \tag{C.21}$$

Again, in order to evaluate this integral we use the decomposition in (C.14) and following similar steps as we did in evaluating $\Delta L_1^{\mu\nu}$. Finally, we expand Δp_2^μ to $\mathcal{O}(S_2)$ and get the orbital angular impulse as

$$\begin{aligned} \Delta L_2^{\mu\nu} = \frac{Q_1Q_2}{\sqrt{\mathcal{D}}}\int \hat{d}^2\bar{q}_\perp e^{i\bar{q}_\perp\cdot b}\frac{1}{\bar{q}_\perp^2}\Big[\frac{1}{\beta^2\gamma^2} + i\frac{(p_1\cdot p_2)}{\mathcal{D}}S_2^{\perp\rho\sigma}p_{1\rho}\bar{q}_{\perp\sigma}\Big](p_2\wedge p_1)^{\mu\nu} \\ -i\frac{Q_1Q_2}{\sqrt{\mathcal{D}}}\int \hat{d}^2\bar{q}_\perp e^{i\bar{q}_\perp\cdot b}\frac{1}{\bar{q}_\perp^2}(b\wedge\bar{q}_\perp)^{\mu\nu}\Big[(p_1\cdot p_2) + iS_2^{\perp\rho\sigma}p_{1\rho}\bar{q}_{\perp\sigma}\Big]. \end{aligned} \tag{C.22}$$

C.4 Radiation from the scalar particle

The classical current from particle 1 in momentum space is given by

$$J_1^\mu(x) = Q_1 \int d\tau_1 e^{i\bar{k}\cdot x_1(\tau_1)} \frac{i}{\bar{k}\cdot p_1}\left[\dot{p}_1^\mu - \frac{\bar{k}\cdot\dot{p}_1}{\bar{k}\cdot p_1} p_1^\mu\right]. \tag{C.23}$$

Using the expression in (C.8), we obtain the current to all order in spin

$$\begin{aligned}
J_1^\mu(\bar{k},a_2) = Q_1^2 Q_2 \int \hat{d}^4\bar{q}\hat{\delta}(u_1\cdot\bar{q} - u_1\cdot\bar{k})\hat{\delta}(u_2\cdot\bar{q})\frac{e^{-i\bar{q}\cdot b}}{\bar{q}^2}\frac{1}{\bar{k}\cdot p_1} \\
\times\Big[\cosh(a_2\cdot\bar{q})\{\gamma\bar{q}^\mu - u_2^\mu(u_1\cdot\bar{q})\} + i\sinh(a_2\cdot\bar{q})\epsilon^{\mu\nu\rho\sigma}u_{1\nu}\bar{q}_\rho u_{2\sigma} \\
- \frac{p_1^\mu}{\bar{k}\cdot p_1}\big\{\cosh(a_2\cdot\bar{q})\big(\gamma(\bar{k}\cdot\bar{q}) - (\bar{k}\cdot u_2)(u_1\cdot\bar{q})\big) + i\sinh(a_2\cdot\bar{q})\epsilon(\bar{k},u_1,\bar{q},u_2)\big\}\Big].
\end{aligned} \tag{C.24}$$

We now take a soft limit of the current and show that, on comparing with the classical sub-leading soft factor, it reproduces the angular impulse for the scalar particle. We use the following identity from Appendix A.5:

$$(a_2\cdot\bar{q})\epsilon^\mu(u_1,u_2,\bar{q}) = \bar{q}^\mu\epsilon(u_1,u_2,a_2,\bar{q}) - (u_1\cdot\bar{k})\epsilon^\mu(a_2,\bar{q},u_2). \tag{C.25}$$

The classical current is then rewritten as

$$\begin{aligned}
J_1^\mu(\bar{k},a_2) = Q_1^2 Q_2 \int \hat{d}^4\bar{q}\hat{\delta}[u_1\cdot(\bar{q}-\bar{k})]\hat{\delta}(u_2\cdot\bar{q})\frac{e^{-i\bar{q}\cdot b}}{\bar{q}^2}\frac{1}{\bar{k}\cdot p_1} \\
\times\Big[\cosh(a_2\cdot\bar{q})\Big\{\gamma\bar{q}^\mu - u_2^\mu(u_1\cdot\bar{k}) - \frac{p_1^\mu}{p_1\cdot\bar{k}}\Big(\gamma(\bar{k}\cdot\bar{q}) - (\bar{k}\cdot u_2)(\bar{k}\cdot u_1)\Big)\Big\} \\
- i\frac{\sinh(a_2\cdot\bar{q})}{a_2\cdot\bar{q}}\Big\{\bar{q}^\mu\epsilon(u_1,u_2,a_2,\bar{q}) - (u_1\cdot\bar{k})\epsilon^\mu(a_2,\bar{q},u_2) \\
- \frac{p_1^\mu}{\bar{k}\cdot p_1}\Big((\bar{q}\cdot\bar{k})\epsilon(u_1,u_2,a_2,\bar{q}) - (u_1\cdot\bar{k})\epsilon(\bar{k},a_2,\bar{q},u_2)\Big)\Big\}\Big].
\end{aligned} \tag{C.26}$$

Next, we do a soft expansion in $\bar{k}$ where $k^\mu = \omega(1,\hat{n})$. We expand the delta function as follows

$$\hat{\delta}(p_1\cdot\bar{q}-p_1\cdot\bar{k}) = \hat{\delta}(p_1\cdot\bar{q}) - (p_1\cdot\bar{k})\hat{\delta}'(p_1\cdot\bar{q}) + \mathcal{O}(\bar{k}^2)\,, \tag{C.27}$$

and write the sub-leading soft terms of the classical current as follows

$$\begin{aligned}
J_1^\mu(\bar{k},a_2)|_{\mathcal{O}(\omega^0)} = Q_1^2Q_2\int \hat{d}^4\bar{q}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\frac{e^{-i\bar{q}\cdot b}}{\bar{q}^2}\Big[\cosh(a_2\cdot\bar{q})\Big\{-p_2^\mu + p_1^\mu\frac{(\bar{k}\cdot p_2)}{(\bar{k}\cdot p_1)}\Big\} \\
- i\frac{\sinh(a_2\cdot\bar{q})}{a_2\cdot q}\Big\{-\epsilon^\mu(a_2,\bar{q},p_2) + \frac{p_1^\mu}{(\bar{k}\cdot p_1)}\epsilon(\bar{k},a_2,\bar{q},p_2)\Big\}\Big] \\
- Q_1^2Q_2\int \hat{d}^4\bar{q}\hat{\delta}'(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\frac{e^{-i\bar{q}\cdot b}}{\bar{q}^2}\Big[\cosh(a_2\cdot\bar{q})(p_1\cdot p_2)\Big\{\bar{q}^\mu - \frac{p_1^\mu}{p_1\cdot\bar{k}}(\bar{k}\cdot\bar{q})\Big\} \\
- i\frac{\sinh(a_2\cdot\bar{q})}{a_2\cdot\bar{q}}\Big\{\bar{q}^\mu\epsilon(p_1,p_2,a_2,\bar{q}) - \frac{p_1^\mu}{\bar{k}\cdot p_1}(\bar{q}\cdot\bar{k})\epsilon(p_1,p_2,a_2,\bar{q})\Big\}\Big].
\end{aligned} \tag{C.28}$$

Comparing with the classical sub-leading soft factor [106],

$$S_1^{(1)\mu} = Q_1\Big[\frac{\varDelta J_1^{\mu\nu}\bar{k}_\nu}{(p_1\cdot\bar{k})} - \frac{(\varDelta p_1\cdot\bar{k})}{(p_1\cdot\bar{k})^2}J_{-1}^{\mu\nu}\bar{k}_\nu\Big], \tag{C.29}$$

where $\varDelta p_1^\mu$ is the LO linear impulse of the scalar particle, $\varDelta J_1^{\mu\nu}$ is the LO total angular impulse of the scalar particle and $J_{-1}^{\mu\nu}$ is the initial angular momentum tensor, we obtain

$$\begin{aligned}
\varDelta J_1^{\mu\nu} = Q_1Q_2\int \hat{d}^4\bar{q}e^{-i\bar{q}\cdot b}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\frac{1}{\bar{q}^2}\Big[(p_1\wedge p_2)^{\mu\nu}\cosh(a_2\cdot\bar{q}) - i\frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})}p_1^{[\mu}S_2^{\perp\nu]\sigma}\bar{q}_\sigma\Big] \\
- Q_1Q_2\int \hat{d}^4\bar{q}e^{-i\bar{q}\cdot b}\hat{\delta}'(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\frac{1}{\bar{q}^2}(\bar{q}\wedge p_1)^{\mu\nu}\Big[\cosh(a_2\cdot\bar{q})(p_1\cdot p_2) \\
- i\frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})}S_2^{\perp\rho\sigma}p_{1\rho}\bar{q}_\sigma\Big].
\end{aligned} \tag{C.30}$$

Replacing $\bar{q} \to -\bar{q}$, we get

$$\begin{aligned}
\Delta J_1^{\mu\nu} &= Q_1 Q_2 \int \hat{d}^4\bar{q} e^{i\bar{q}\cdot b} \hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q}) \frac{1}{\bar{q}^2}\Big[(p_1\wedge p_2)^{\mu\nu}\cosh(a_2\cdot\bar{q}) + i\frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})} p_1^{[\mu} S_2^{\perp\nu]\sigma}\bar{q}_\sigma\Big] \\
&\quad - Q_1 Q_2 \int \hat{d}^4\bar{q} e^{i\bar{q}\cdot b} \hat{\delta}'(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q}) \frac{1}{\bar{q}^2}(\bar{q}\wedge p_1)^{\mu\nu}\Big[\cosh(a_2\cdot\bar{q})(p_1\cdot p_2) \\
&\qquad\qquad + i\frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})} S_2^{\perp\rho\sigma} p_{1\rho}\bar{q}_\sigma\Big],
\end{aligned} \tag{C.31}$$

which matches with the computation of angular impulse of the scalar particle of eq. (3.39). Here $J_{-1}^{\mu\nu} = (b_1 \wedge p_1)^{\mu\nu} = 0$ in our setup as $b_1 = 0, b_2 = b$.

Appendix D

(sub)^2-leading order soft radiation from quantum soft theorems

In this appendix, we will review the computation of the soft radiation by applying (sub)^2-leading soft graviton operator on the quantum four-point amplitude and then take the classical limit.

From quantum soft theorems, the (sub)^2-leading radiation kernel is given by

$$\mathcal{R}^{\mu\nu}_{\omega} = \frac{1}{4m_1m_2}\int \hat{d}^4q_1\hat{d}^4q_2 e^{iq_1\cdot b/\hbar}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\ \kappa \sum_{i=1,2}\Big[\frac{J_i^{\mu\rho}k_\rho J_i^{\nu\sigma}k_\sigma}{p_i\cdot k} + \frac{\tilde{J}_i^{\mu\rho}k_\rho \tilde{J}_i^{\nu\sigma}k_\sigma}{\tilde{p}_i\cdot k}\Big] \Big(\hat{\delta}^{(4)}(q_1+q_2)\mathcal{A}_4\Big), \tag{D.1}$$

where

$$\mathcal{A}_4[p_1,\tilde{p}_1,p_2,\tilde{p}_2] = \frac{\kappa^2}{2q_2^2}\Big[(p_2\cdot\tilde{p}_2)(m_1^2-p_1\cdot\tilde{p}_1) + m_2^2(p_1\cdot\tilde{p}_1-2m_1^2) + (p_1\cdot\tilde{p}_2)(p_2\cdot\tilde{p}_1) + (p_1\cdot p_2)(\tilde{p}_1\cdot\tilde{p}_2)\Big] \tag{D.2}$$

First, let us evaluate the soft operators' action on $\mathcal{A}_4$. We consider the contribution from particle 1 for now. The action of the soft operators on the numerator of the amplitudes is given by

$$\begin{aligned}\kappa\frac{J_1^{\mu\rho}k_\rho J_1^{\nu\sigma}k_\sigma}{p_1\cdot k}[\mathcal{A}_4]_N &= -\kappa^3\frac{k_\rho k_\sigma}{2q_2^2(p_1\cdot k)}\Big(p_1\wedge\frac{\partial}{\partial p_1}\Big)^{\mu\rho}\Big(p_1\wedge\frac{\partial}{\partial p_1}\Big)^{\nu\sigma}\Big[(p_2\cdot\tilde{p}_2)(m_1^2-p_1\cdot\tilde{p}_1)\\&\qquad+m_2^2(p_1\cdot\tilde{p}_1-2m_1^2)+(p_1\cdot\tilde{p}_2)(p_2\cdot\tilde{p}_1)+(p_1\cdot p_2)(\tilde{p}_1\cdot\tilde{p}_2)\Big]\\&=0\,.\end{aligned}\tag{D.3}$$

and

$$\begin{aligned}\kappa\frac{\tilde{J}_1^{\mu\rho}k_\rho\tilde{J}_1^{\nu\sigma}k_\sigma}{\tilde{p}_1\cdot k}[\mathcal{A}_4]_N &= -\kappa^3\frac{k_\rho k_\sigma}{2q_2^2(\tilde{p}_1\cdot k)}\Big(\tilde{p}_1\wedge\frac{\partial}{\partial\tilde{p}_1}\Big)^{\mu\rho}\Big(\tilde{p}_1\wedge\frac{\partial}{\partial\tilde{p}_1}\Big)^{\nu\sigma}\Big[(p_2\cdot\tilde{p}_2)(m_1^2-p_1\cdot\tilde{p}_1)\\&\qquad+m_2^2(p_1\cdot\tilde{p}_1-2m_1^2)+(p_1\cdot\tilde{p}_2)(p_2\cdot\tilde{p}_1)+(p_1\cdot p_2)(\tilde{p}_1\cdot\tilde{p}_2)\Big]\\&=0\,.\end{aligned}\tag{D.4}$$

The classical contribution from the action of the soft operators on the denominator of the amplitude is given by

$$-\frac{\kappa^3}{2(\bar{q}^2)^3}(\bar{q}\cdot\bar{k})\left(\bar{q}^\mu\bar{q}^\nu((p_1\cdot p_2)^2-\frac{1}{2}m_1^2m_2^2)+\bar{q}^2\frac{(p_2\cdot\bar{k})}{(\bar{q}\cdot\bar{k})}\bar{q}^{(\mu}p_1^{\nu)}\right)\,.\tag{D.5}$$

Therefore the classical contribution to soft radiation from the action of the (sub)^2-leading soft operator ($S^{(2),\mu\nu}$) on the four-point amplitude alone is given by

$$\begin{aligned}\mathcal{R}^{\mu\nu}_{\omega,A} = &= -\frac{\kappa^3m_1m_2}{4}\int\hat{d}^4\bar{q}e^{-i\bar{q}\cdot b}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\frac{1}{2(\bar{q}^2)^3}(\bar{q}\cdot\bar{k})\\&\qquad\times\left(\bar{q}^\mu\bar{q}^\nu\Big(\gamma^2-\frac{1}{2}\Big)+\bar{q}^2\frac{(u_2\cdot\bar{k})}{(\bar{q}\cdot\bar{k})}\bar{q}^{(\mu}u_1^{\nu)}\right)\,.\end{aligned}\tag{D.6}$$

Let us evaluate the soft operators' action on the delta function now. Again we restrict to the contribution from particle 1. We use the distributional identity:

$$S^{(2),\mu\nu}\hat{\delta}^{(4)}(q_1+q_2) = \hat{\delta}^{(4)}(q_1+q_2)S^{(2),\mu\nu} - (k\cdot\partial)\hat{\delta}^{(4)}(q_1+q_2)S^{(1),\mu\nu} + \frac{1}{2}(k\cdot\partial)^2\hat{\delta}^{(4)}(q_1+q_2)S^{(0),\mu\nu} \tag{D.7}$$

Here, $S^{(0),\mu\nu}, S^{(1),\mu\nu}, S^{(2),\mu\nu}$ are the leading, sub-leading and (sub)^2-leading soft operators respectively. We have,

$$\mathcal{R}^{\mu\nu}_{\omega,D} = \frac{1}{m_1m_2}\int \hat{d}^4q_1\hat{d}^4q_2 e^{iq_1\cdot b/\hbar}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\Big[\hat{\delta}^{(4)}(q_1+q_2)S^{(2),\mu\nu} - (k\cdot\partial)\hat{\delta}^{(4)}(q_1+q_2)S^{(1),\mu\nu} + \frac{1}{2}(k\cdot\partial)^2\hat{\delta}^{(4)}(q_1+q_2)S^{(0),\mu\nu}\Big]\mathcal{A}_4 \tag{D.8}$$

From equation (D.5), we have the classical contribution of $S^{(2),\mu\nu}$ on the amplitude. Therefore, we compute the remaining two terms. We have

$$\begin{aligned}\mathcal{R}^{\mu\nu}_{\omega,1} &= -\frac{1}{m_1m_2}\int \hat{d}^4q_1\hat{d}^4q_2 e^{iq_1\cdot b/\hbar}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)(k\cdot\partial)\hat{\delta}^{(4)}(q_1+q_2)S^{(1),\mu\nu}\mathcal{A}_4\\ &= \frac{1}{m_1m_2}\int \hat{d}^4q_1\hat{d}^4q_2 e^{iq_1\cdot b/\hbar}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\Big\{\hat{\delta}^{(4)}(q_1+q_2-k)\\ &\qquad - \hat{\delta}^{(4)}(q_1+q_2)\Big\}S^{(1),\mu\nu}\mathcal{A}_4\,.\end{aligned} \tag{D.9}$$

Integrating q_1 and relabelling $q_2\to q$, we have

$$\mathcal{R}^{\mu\nu}_{\omega,1} = \int \hat{d}^4q e^{-iq\cdot b/\hbar}\hat{\delta}(u_2\cdot q)\Big\{\hat{\delta}(u_1\cdot(k-q))e^{ik\cdot b/\hbar} - \hat{\delta}(u_1\cdot q)\Big\}S^{(1),\mu\nu}\mathcal{A}_4\,. \tag{D.10}$$

Writing only the $\mathcal{O}(\omega)$ term,

$$\mathcal{R}^{\mu\nu}_{\omega,1} = \int \hat{d}^4q e^{-iq\cdot b/\hbar}\hat{\delta}(u_2\cdot q)\Big\{\hat{\delta}(u_1\cdot q)(ik\cdot\frac{b}{\hbar}) - (u_1\cdot k)\hat{\delta}'(u_1\cdot q)\Big\}S^{(1),\mu\nu}\mathcal{A}_4\,. \tag{D.11}$$

The sub-leading soft graviton operator is given by

$$S^{(1),\mu\nu}\mathcal{A}_4 = \frac{\kappa}{2}\Big[\frac{p_1^{(\mu}J_1^{\nu)\rho}k_\rho}{(p_1\cdot k)} - \frac{\tilde{p}_1^{(\mu}\tilde{J}_1^{\nu)\rho}k_\rho}{(\tilde{p}_1\cdot k)}\Big]\mathcal{A}_4\,. \tag{D.12}$$

The action of the soft operators on the amplitude is given by

$$\begin{aligned}
\kappa\frac{p_1^{(\mu}J_1^{\nu)\rho}k_\rho}{(p_1\cdot k)}\mathcal{A}_4 &= \frac{i\kappa}{p_1\cdot k}\Big(p_1^{(\mu}p_1^{\nu)}(k\cdot\frac{\partial}{\partial p_1}) - (p_1\cdot k)p_1^{(\mu}\frac{\partial}{\partial p_{1\nu)}}\Big)\mathcal{A}_4\\
&= -\frac{i\kappa^3}{2q^2(k\cdot p_1)}p_1^{\mu}\Big[(k\cdot p_1)(\tilde{p}_1^{\nu}(m_2^2 - p_2\cdot\tilde{p}_2) + p_2^{\nu}(\tilde{p}_1\cdot\tilde{p}_2))\\
&\qquad - p_1^{\nu}((k\cdot\tilde{p}_1)(m_2^2 - p_2\cdot\tilde{p}_2) + (k\cdot\tilde{p}_2)(\tilde{p}_1\cdot p_2)\\
&\qquad + (k\cdot p_2)(\tilde{p}_1\cdot\tilde{p}_2)) + \tilde{p}_2^{\nu}(k\cdot p_1)(\tilde{p}_1\cdot p_2)\\
&\quad - \frac{1}{2q^2}\Big(p_1^{\nu}(k\cdot\tilde{p}_1 + k\cdot p_2 - k\cdot\tilde{p}_2) - (p_1\cdot k)(\tilde{p}_1^{\nu} + p_2^{\nu} - \tilde{p}_2^{\nu})\Big)(2(p_1\cdot p_2)^2 - m_1^2m_2^2)\Big].
\end{aligned} \tag{D.13}$$

The classical contribution is given by

$$\kappa\frac{p_1^{(\mu}J_1^{\nu)\rho}k_\rho}{(p_1\cdot k)}\mathcal{A}_4 = -\frac{i\kappa^3}{\bar{q}^2(\bar{k}\cdot p_1)}p_1^{(\mu}\Big[p_2^{\nu)}(\bar{k}\cdot p_1)(p_1\cdot p_2) - p_1^{\nu)}(\bar{k}\cdot p_2)(p_1\cdot p_2)\Big]. \tag{D.14}$$

and

$$\begin{aligned}
\kappa\frac{\tilde{p}_1^{(\mu}\tilde{J}_1^{\nu)\rho}k_\rho}{(\tilde{p}_1\cdot k)}\mathcal{A}_4 &= \frac{-i\kappa}{\tilde{p}_1\cdot k}\Big(\tilde{p}_1^{(\mu}\tilde{p}_1^{\nu)}(k\cdot\frac{\partial}{\partial\tilde{p}_1}) - (\tilde{p}_1\cdot k)\tilde{p}_1^{(\mu}\frac{\partial}{\partial\tilde{p}_{1\nu)}}\Big)\mathcal{A}_4\\
&= \frac{i\kappa^3}{2q^2(k\cdot\tilde{p}_1)}\tilde{p}_1^{\mu}\Big[(k\cdot\tilde{p}_1)(p_1^{\nu}(m_2^2 - p_2\cdot\tilde{p}_2) + p_2^{\nu}(p_1\cdot\tilde{p}_2))\\
&\qquad - \tilde{p}_1^{\nu}((k\cdot p_1)(m_2^2 - p_2\cdot\tilde{p}_2) + (k\cdot\tilde{p}_2)(p_1\cdot p_2)\\
&\qquad + (k\cdot p_2)(p_1\cdot\tilde{p}_2)) + \tilde{p}_2^{\nu}(k\cdot\tilde{p}_1)(p_1\cdot p_2)\\
&\quad - \frac{1}{2q^2}\Big(\tilde{p}_1^{\nu}(k\cdot p_1 + k\cdot\tilde{p}_2 - k\cdot p_2) - (\tilde{p}_1\cdot k)(p_1^{\nu} + \tilde{p}_2^{\nu} - p_2^{\nu})\Big)(2(p_1\cdot p_2)^2 - m_1^2m_2^2)\Big].
\end{aligned} \tag{D.15}$$

The classical contribution is given by

$$\kappa\frac{\tilde{p}_1^{(\mu}\tilde{J}_1^{\nu)\rho}k_\rho}{(\tilde{p}_1\cdot k)}\mathcal{A}_4 = \frac{i\kappa^3}{\bar{q}^2(\bar{k}\cdot p_1)}\Big[p_1^{(\mu}p_2^{\nu)}(\bar{k}\cdot p_1)(p_1\cdot p_2) - p_1^{(\mu}p_1^{\nu)}(\bar{k}\cdot p_2)(p_1\cdot p_2) - \frac{1}{2\bar{q}^2}\bar{q}^\mu\bar{q}^\nu(\bar{k}\cdot p_1)(2(p_1\cdot p_2)^2 - m_1^2m_2^2)\Big]. \tag{D.16}$$

Therefore, substituting equations (D.14) and (D.16) in equation (D.12) the classical contribution of $\mathcal{R}^{\mu\nu}_{\omega,1}$ is given by

$$\begin{aligned}\mathcal{R}^{\mu\nu}_{\omega,1} &= \frac{1}{m_1m_2}\int\frac{\hat{d}^4\bar{q}}{\bar{q}^2}e^{-i\bar{q}\cdot b}\hat{\delta}(u_2\cdot\bar{q})\Big\{\hat{\delta}(u_1\cdot\bar{q})(i\bar{k}\cdot b) - (u_1\cdot\bar{k})\hat{\delta}'(u_1\cdot\bar{q})\Big\}\\ &\quad\times -\frac{2i\kappa^3}{(\bar{k}\cdot p_1)}\Big[p_1^{(\mu}p_2^{\nu)}(\bar{k}\cdot p_1)(p_1\cdot p_2) - p_1^{(\mu}p_1^{\nu)}(\bar{k}\cdot p_2)(p_1\cdot p_2) + \frac{1}{2\bar{q}^2}\bar{q}^\mu\bar{q}^\nu(\bar{k}\cdot p_1)\Big]\\ &= \frac{\kappa^3m_1m_2\gamma}{\pi\gamma\beta}u_1^{(\mu}\Big[u_2^{\nu)} - u_1^{\nu)}\frac{(\bar{k}\cdot u_2)}{(\bar{k}\cdot u_1)}\Big]\omega b\log(\omega b) + \mathcal{O}(\omega)\,,\end{aligned} \tag{D.17}$$

where we have used the integral result of equation (B.11). We are now left with computing one last term.

$$\begin{aligned}\mathcal{R}^{\mu\nu}_{\omega,2} &= \frac{1}{2m_1m_2}\int\hat{d}^4q_1\hat{d}^4q_2e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)(k\cdot\partial)^2\hat{\delta}^{(4)}(q_1+q_2)S^{(0),\mu\nu}\mathcal{A}_4\\ &= \frac{1}{m_1m_2}\int\hat{d}^4q_1\hat{d}^4q_2e^{iq_1\cdot b/\hbar}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\Big\{\hat{\delta}^{(4)}(q_1+q_2-k) - \hat{\delta}^{(4)}(q_1+q_2)\\ &\qquad + (k\cdot\partial)\hat{\delta}^{(4)}(q_1+q_2)\Big\}S^{(0),\mu\nu}\mathcal{A}_4\,.\end{aligned} \tag{D.18}$$

Integrating q_1 and relabelling $q_2\to q$, we have

$$\begin{aligned}\mathcal{R}^{\mu\nu}_{\omega,2} &= \frac{1}{m_1m_2}\int\hat{d}^4qe^{-iq\cdot b/\hbar}\hat{\delta}(u_2\cdot q)\Big\{\hat{\delta}(u_1\cdot(k-q))e^{ik\cdot b/\hbar} - \hat{\delta}(u_1\cdot q)\\ &\qquad + \hat{\delta}(u_1\cdot q) - \hat{\delta}(u_1\cdot(k-q))e^{ik\cdot b/\hbar}\Big\}S^{(0),\mu\nu}\mathcal{A}_4\,,\end{aligned} \tag{D.19}$$

where the $(k\cdot\partial)$ term is written from the sub-leading distributional identity. Therefore, one should be careful and expand the last term to sub-leading order only. The (sub)^2-

leading contribution arises from expanding the first term to quadratic order in frequency. Therefore,

$$\mathcal{R}^{\mu\nu}_{\omega,2} = \frac{1}{m_1 m_2}\int \hat{d}^4 q e^{-iq\cdot b/\hbar}\hat{\delta}(u_2\cdot q)\Big\{\hat{\delta}(u_1\cdot q)\frac{(ik\cdot b/\hbar)^2}{2} - (u_1\cdot k)(ik\cdot b/\hbar)\hat{\delta}'(u_1\cdot q) + \frac{1}{2}(u_1\cdot k)^2\hat{\delta}''(u_1\cdot q)\Big\}S^{(0),\mu\nu}\mathcal{A}_4\,. \tag{D.20}$$

We have, for particle 1

$$S^{(0),\mu\nu} = \frac{1}{p_1\cdot k}p_1^{(\mu}p_1^{\nu)} - \frac{1}{\tilde{p}_1\cdot k}\tilde{p}_1^{(\mu}\tilde{p}_1^{\nu)} = -\frac{\bar{q}^{(\mu}p_1^{\nu)}}{p_1\cdot\bar{k}} + \frac{(\bar{q}\cdot\bar{k})p_1^{(\mu}p_1^{\nu)}}{(p_1\cdot\bar{k})^2}\,. \tag{D.21}$$

We have the following integrals

$$\begin{aligned}\mathcal{I}^{\mu\nu}_1 &= -\kappa^3(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)\frac{(\bar{k}\cdot b)^2}{4}\int\frac{\hat{d}^4\bar{q}}{\bar{q}^2}e^{-i\bar{q}\cdot b}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\Big(-\frac{\bar{q}^{(\mu}p_1^{\nu)}}{p_1\cdot\bar{k}} + \frac{(\bar{q}\cdot\bar{k})p_1^{(\mu}p_1^{\nu)}}{(p_1\cdot\bar{k})^2}\Big)\\ &= i\kappa^3(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)\frac{(\bar{k}\cdot b)^2}{8\pi\gamma\beta}\Big(\frac{b^{(\mu}p_1^{\nu)}}{p_1\cdot\bar{k}} - \frac{(b\cdot\bar{k})p_1^{(\mu}p_1^{\nu)}}{(p_1\cdot\bar{k})^2}\Big),\end{aligned} \tag{D.22}$$

where we have used the integral result of equation (B.12). Next, we have

$$\begin{aligned}\mathcal{I}^{\mu\nu}_2 &= \kappa^3(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)\frac{(u_1\cdot\bar{k})(i\bar{k}\cdot b)}{2}\int\frac{\hat{d}^4\bar{q}}{\bar{q}^2}e^{-i\bar{q}\cdot b}\hat{\delta}'(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\Big(\frac{\bar{q}^{(\mu}p_1^{\nu)}}{p_1\cdot\bar{k}} - \frac{(\bar{q}\cdot\bar{k})p_1^{(\mu}p_1^{\nu)}}{(p_1\cdot\bar{k})^2}\Big)\\ &= -\frac{\kappa^3}{4\pi\gamma^3\beta^3}(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)\omega b\log(\omega b)\Big((\gamma u_2 - u_1)^{(\mu}u_1^{\nu)} - \frac{((\gamma u_2 - u_1)\cdot\bar{k})u_1^{(\mu}u_1^{\nu)}}{(u_1\cdot\bar{k})}\Big),\end{aligned} \tag{D.23}$$

where we have used the integral result of equation (B.19) and

$$\begin{aligned}\mathcal{I}^{\mu\nu}_3 &= -\kappa^3(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)\frac{(u_1\cdot\bar{k})^2}{4}\int\frac{\hat{d}^4\bar{q}}{\bar{q}^2}e^{-i\bar{q}\cdot b}\hat{\delta}''(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\Big(\frac{\bar{q}^{(\mu}p_1^{\nu)}}{p_1\cdot\bar{k}} - \frac{(\bar{q}\cdot\bar{k})p_1^{(\mu}p_1^{\nu)}}{(p_1\cdot\bar{k})^2}\Big)\\ &= -\kappa^3(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)\frac{(u_1\cdot\bar{k})}{2\gamma^2\beta^2}\int \hat{d}^2\bar{q}_\perp e^{-i\bar{q}_\perp\cdot b}\frac{1}{(q_\perp^2)^2}\Big(\bar{q}_\perp^{(\mu}u_1^{\nu)} - \frac{(\bar{q}_\perp\cdot\bar{k})u_1^{(\mu}u_1^{\nu)}}{(u_1\cdot\bar{k})}\Big)\end{aligned}$$

$$\to \mathcal{O}(\omega)\,, \tag{D.24}$$

using the integral result of equation (B.22). Therefore we collect the log terms and upon simplifying the $\omega \log \omega$ terms of radiation kernel w.r.t particle 1 from the quantum soft theorems is given by

$$\mathcal{R}^{\mu\nu}_{\omega \log \omega} = \frac{\kappa^3 m_1 m_2}{4\pi\gamma^3\beta^3}(\omega b) \log{(\omega b)} \gamma(2\gamma^2 - 3)\Big(u_1^{(\mu} u_2^{\nu)} - \frac{(u_2 \cdot \bar{k})}{(u_1 \cdot \bar{k})} u_1^{(\mu} u_1^{\nu)}\Big)\,. \tag{D.25}$$

This matches with the tree-level contribution to the $\omega \log \omega$ term and the logarithmic terms of the $(\text{sub})^2$-leading order soft expansion of the radiation kernel. The remaining terms derived from quantum soft theorems also align with the soft expansion of the radiation kernel in equation (4.27).

Appendix E

Detailed derivations of (sub)^n-leading order soft radiation

In this appendix, we give a detailed derivation of the steps in computing the classical (sub)^n-leading order soft radiation from the quantum soft theorems which appeared in Section 4.4. First, let us evaluate the soft operators' action on $\mathcal{A}_4$ that appeared in the main text. We consider the contribution from particle 1 for now. The action of the soft operators on the numerator of the amplitudes is given by

$$\begin{aligned} \kappa \frac{J_1^{\mu\rho} k_\rho J_1^{\nu\sigma} k_\sigma}{p_1 \cdot k} \Big(k \cdot \frac{\partial}{\partial p_i} \Big)^2 [\mathcal{A}_4]_N &= \kappa^3 \frac{k_\rho k_\sigma}{2 q_2^2 (p_1 \cdot k)} \Big(p_1 \wedge \frac{\partial}{\partial p_1} \Big)^{\mu\rho} \Big(p_1 \wedge \frac{\partial}{\partial p_1} \Big)^{\nu\sigma} \Big(k \cdot \frac{\partial}{\partial p_1} \Big) \\ &\qquad \Big[(k \cdot \tilde{p}_1)(m_2^2 - p_2 \cdot \tilde{p}_2) + (k \cdot \tilde{p}_2)(p_2 \cdot \tilde{p}_1) + (k \cdot p_2)(\tilde{p}_1 \cdot \tilde{p}_2) \Big] \\ &= 0 \,. \end{aligned} \tag{E.1}$$

and

$$\begin{aligned} \kappa \frac{\tilde{J}_1^{\mu\rho} k_\rho \tilde{J}_1^{\nu\sigma} k_\sigma}{\tilde{p}_1 \cdot k} \Big(k \cdot \frac{\partial}{\partial \tilde{p}_1} \Big)^2 [\mathcal{A}_4]_N &= \kappa^3 \frac{k_\rho k_\sigma}{2 q_2^2 (\tilde{p}_1 \cdot k)} \Big(\tilde{p}_1 \wedge \frac{\partial}{\partial \tilde{p}_1} \Big)^{\mu\rho} \Big(\tilde{p}_1 \wedge \frac{\partial}{\partial \tilde{p}_1} \Big)^{\nu\sigma} \Big(k \cdot \frac{\partial}{\partial \tilde{p}_1} \Big) \\ &\qquad \Big[(k \cdot p_1)(m_2^2 - p_2 \cdot \tilde{p}_2) + (k \cdot \tilde{p}_2)(p_1 \cdot p_2) + (k \cdot p_2)(p_1 \cdot \tilde{p}_2) \Big] \\ &= 0 \,. \end{aligned} \tag{E.2}$$

The classical contribution comes from the action of the soft operators on the denominator of the amplitude and it is given by

$$\frac{\kappa}{2}\frac{J_1^{\mu\rho}k_\rho J_1^{\nu\sigma}k_\sigma}{p_1\cdot k}\Big(k\cdot\frac{\partial}{\partial p_1}\Big)^{n-2}\mathcal{A}_4+\frac{\kappa}{2}\frac{\tilde{J}_1^{\mu\rho}k_\rho \tilde{J}_1^{\nu\sigma}k_\sigma}{\tilde{p}_1\cdot k}\Big(k\cdot\frac{\partial}{\partial p_1}\Big)^{n-2}\mathcal{A}_4$$
$$=(-1)^{n+1}\kappa^3\frac{2^{n-3}}{(\bar{q}^2)^{n+1}}(\bar{q}\cdot\bar{k})^{n-1}\left(\bar{q}^\mu\bar{q}^\nu((p_1\cdot p_2)^2-\frac{1}{2}m_1^2m_2^2)+\bar{q}^2\frac{(p_2\cdot\bar{k})}{(\bar{q}\cdot\bar{k})}\bar{q}^{(\mu}p_1^{\nu)}\right).\quad\text{(E.3)}$$

Therefore, the classical contribution to soft radiation from the action of (sub)^n-leading soft operator ($S^{(n),\mu\nu}$) on the four-point amplitude alone is given by

$$\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},A}=(-1)^{n+1}\frac{\kappa^3m_1m_2}{4}\int\hat{d}^4\bar{q}e^{-i\bar{q}\cdot b}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\frac{2^{n-1}}{(\bar{q}^2)^{n+1}}(\bar{q}\cdot\bar{k})^{n-1}$$
$$\times\left(\bar{q}^\mu\bar{q}^\nu\Big(\gamma^2-\frac{1}{2}\Big)+\bar{q}^2\frac{(u_2\cdot\bar{k})}{(\bar{q}\cdot\bar{k})}\bar{q}^{(\mu}u_1^{\nu)}\right).\quad\text{(E.4)}$$

This term doesn't give any logarithmic contributions using the integral results of Appendix B. Let us evaluate the soft operators' action on the delta function now. Again we restrict to the contribution from particle 1 where we use the distributional identity of equation (4.44). We have the following terms that contribute to the classical soft radiation:

$$\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},1}=\frac{1}{m_1m_2}\int\hat{d}^4q_1\hat{d}^4q_2e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\Big\{-(k\cdot\partial)\hat{\delta}^{(4)}(q_1+q_2)S^{(n-1),\mu\nu}\mathcal{A}_4\Big\}$$
$$=\frac{1}{m_1m_2}\int\hat{d}^4q_1\hat{d}^4q_2e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\Big\{\hat{\delta}^{(4)}(q_1+q_2-k)$$
$$-\hat{\delta}^{(4)}(q_1+q_2)\Big\}S^{(n-1),\mu\nu}\mathcal{A}_4.\quad\text{(E.5)}$$

Integrating q_1 and relabelling $q_2\to q$ and keeping only $\mathcal{O}(\omega^{n-1})$ terms, we have,

$$\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},1}=\frac{1}{m_1m_2}\int\hat{d}^4qe^{-iq\cdot b}\hat{\delta}(u_2\cdot q)\Big\{(ik\cdot b)\hat{\delta}(u_1\cdot q)-(u_1\cdot k)\hat{\delta}'(u_1\cdot q)\Big\}S^{(n-1),\mu\nu}\mathcal{A}_4$$
$$=\frac{1}{m_1m_2}\int\hat{d}^4\bar{q}e^{-i\bar{q}\cdot b}\hat{\delta}(u_2\cdot\bar{q})\Big\{(i\bar{k}\cdot b)\hat{\delta}(u_1\cdot\bar{q})-(u_1\cdot\bar{k})\hat{\delta}'(u_1\cdot\bar{q})\Big\}$$

$$\times(-1)^n\kappa^3\frac{2^{n-1}}{(\bar{q}^2)^n}(\bar{q}\cdot\bar{k})^{n-2}\left(\bar{q}^\mu\bar{q}^\nu((p_1\cdot p_2)^2-\frac{1}{2}m_1^2m_2^2)+\bar{q}^2\frac{(p_2\cdot\bar{k})}{(\bar{q}\cdot\bar{k})}\bar{q}^{(\mu}p_1^{\nu)}\right). \tag{E.6}$$

as the classical contribution of $S^{(n-1),\mu\nu}$ on the amplitude is given in equation (E.3). This doesn't give any logarithmic contributions using the integral results of Appendix B. Therefore, we compute the other terms.

$$\begin{aligned}\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},2}&=\frac{1}{m_1m_2}\int\hat{d}^4q_1\hat{d}^4q_2e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\frac{1}{2}(k\cdot\partial)^2\hat{\delta}^{(4)}(q_1+q_2)S^{(n-2),\mu\nu}\mathcal{A}_4\\&=\frac{1}{m_1m_2}\int\hat{d}^4q_1\hat{d}^4q_2e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\Big\{\hat{\delta}^{(4)}(q_1+q_2-k)-\hat{\delta}^{(4)}(q_1+q_2)\\&\qquad+(k\cdot\partial)\hat{\delta}^{(4)}(q_1+q_2)\Big\}S^{(n-2),\mu\nu}\mathcal{A}_4.\end{aligned} \tag{E.7}$$

Integrating q_1 and relabelling $q_2\to q$ and keeping only $\mathcal{O}(\omega^{n-1})$ terms, we have,

$$\begin{aligned}\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},2}=\frac{1}{m_1m_2}\int\hat{d}^4\bar{q}e^{-i\bar{q}\cdot b}\hat{\delta}(u_2\cdot\bar{q})\Big\{&\frac{(i\bar{k}\cdot b)^2}{2}\hat{\delta}(u_1\cdot\bar{q})-(ib\cdot\bar{k})(u_1\cdot\bar{k})\hat{\delta}'(u_1\cdot\bar{q})\\&+\frac{1}{2}(u_1\cdot\bar{k})^2\hat{\delta}''(u_1\cdot\bar{q})\Big\}S^{(n-2),\mu\nu}\mathcal{A}_4.\end{aligned} \tag{E.8}$$

This too doesn't give any logarithmic contributions. Similarly, all the other terms till the action of $S^{(2),\mu\nu}$ do not lead to any log terms using the integral results of Appendix B. For instance

$$\begin{aligned}\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},3}&=\frac{1}{m_1m_2}\int\hat{d}^4q_1\hat{d}^4q_2e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\frac{(-1)^{n-2}(k\cdot\partial)^{n-2}}{(n-2)!}\hat{\delta}^{(4)}(q_1+q_2)S^{(2),\mu\nu}\mathcal{A}_4\\&=\frac{1}{m_1m_2}\int\hat{d}^4q_1\hat{d}^4q_2e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\Big\{\hat{\delta}^{(4)}(q_1+q_2-k)-\hat{\delta}^{(4)}(q_1+q_2)\\&\qquad+(k\cdot\partial)\hat{\delta}^{(4)}(q_1+q_2)+\cdots-\frac{(-1)^{n-3}}{(n-3)!}(k\cdot\partial)^{n-3}\hat{\delta}^{(4)}(q_1+q_2)\Big\}S^{(2),\mu\nu}\mathcal{A}_4\end{aligned} \tag{E.9}$$

Integrating q_1 and relabelling $q_2 \to q$ and keeping only $\mathcal{O}(\omega^{n-1})$ terms, we have,

$$\begin{aligned}
\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},3} = \frac{1}{m_1 m_2} \int \hat{d}^4 q e^{-iq\cdot b} \hat{\delta}(u_2 \cdot q) \Big\{ & \frac{(ik\cdot b)^{n-2}}{(n-2)!} \hat{\delta}(u_1 \cdot q) \\
& + \sum_{\substack{r,s\geq 1 \\ \ni (r+s)=n-2}} \frac{(-1)^s}{r!s!} (ib\cdot k)^r (u_1\cdot k)^s \hat{\delta}^{(s)}(u_1\cdot q) \\
& + \frac{(-1)^{n-2}}{(n-2)!} (u_1\cdot k)^{n-2} \hat{\delta}^{(n-2)}(u_1\cdot q) \Big\} S^{(2),\mu\nu} \mathcal{A}_4 \\
= -\frac{1}{m_1 m_2} \kappa^3 \int \hat{d}^4 \bar{q} e^{-i\bar{q}\cdot b} \hat{\delta}(u_2\cdot \bar{q}) \Big\{ & \frac{(i\bar{k}\cdot b)^{n-2}}{(n-2)!} \hat{\delta}(u_1\cdot \bar{q}) \\
+ \sum_{\substack{r,s\geq 1 \\ \ni (r+s)=n-2}} \frac{(-1)^s}{r!s!} (ib\cdot \bar{k})^r (u_1\cdot \bar{k})^s \hat{\delta}^{(s)}(u_1\cdot \bar{q}) & \\
+ \frac{(-1)^{n-2}}{(n-2)!} (u_1\cdot \bar{k})^{n-2} \hat{\delta}^{(n-2)}(u_1\cdot \bar{q}) \Big\} \frac{1}{(\bar{q}^2)^3} (\bar{q}\cdot \bar{k}) & \\
\times \left(\bar{q}^\mu \bar{q}^\nu ((p_1\cdot p_2)^2 - \frac{1}{2} m_1^2 m_2^2) + \bar{q}^2 \frac{(p_2\cdot \bar{k})}{(\bar{q}\cdot \bar{k})} \bar{q}^{(\mu} p_1^{\nu)} \right) , &
\end{aligned} \tag{E.10}$$

as the classical contribution of $S^{(2),\mu\nu}$ on the amplitude is given in equation (D.5). Therefore, we compute the remaining two terms which should give logarithmic contributions.

$$\begin{aligned}
\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},4} &= \frac{1}{m_1 m_2} \int \hat{d}^4 q_1 \hat{d}^4 q_2 e^{iq_1\cdot b} \hat{\delta}(u_1\cdot q_1) \hat{\delta}(u_2\cdot q_2) \frac{(-1)^{n-1}(k\cdot\partial)^{n-1}}{(n-1)!} \hat{\delta}^{(4)}(q_1+q_2) S^{(1),\mu\nu} \mathcal{A}_4 \\
&= \frac{1}{m_1 m_2} \int \hat{d}^4 q_1 \hat{d}^4 q_2 e^{iq_1\cdot b} \hat{\delta}(u_1\cdot q_1) \hat{\delta}(u_2\cdot q_2) \Big\{ \hat{\delta}^{(4)}(q_1+q_2-k) - \hat{\delta}^{(4)}(q_1+q_2) \\
&\quad + (k\cdot\partial)\hat{\delta}^{(4)}(q_1+q_2) + \cdots - \frac{(-1)^{n-2}}{(n-2)!} (k\cdot\partial)^{n-2} \hat{\delta}^{(4)}(q_1+q_2) \Big\} S^{(1),\mu\nu} \mathcal{A}_4 .
\end{aligned} \tag{E.11}$$

Integrating q_1 and relabelling $q_2 \to q$ and keeping only $\mathcal{O}(\omega^{n-1})$ terms, we have,

$$\begin{aligned}
\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},4} = \frac{1}{m_1 m_2} \int \hat{d}^4 q e^{-iq\cdot b} \hat{\delta}(u_2\cdot q) \Big\{ & \frac{(ik\cdot b)^{n-1}}{(n-1)!} \hat{\delta}(u_1\cdot q) \\
& + \sum_{\substack{r,s\geq 1 \\ \ni (r+s)=n-1}} \frac{(-1)^s}{r!s!} (ib\cdot k)^r (u_1\cdot k)^s \hat{\delta}^{(s)}(u_1\cdot q)
\end{aligned}$$

$$+\frac{(-1)^{n-1}}{(n-1)!}(u_1\cdot k)^{n-1}\hat{\delta}^{(n-1)}(u_1\cdot q)\Big\}S^{(1),\mu\nu}\mathcal{A}_4\,. \tag{E.12}$$

From equation (D.17), the classical contribution of $S^{(1),\mu\nu}$ on the amplitude is given by

$$S^{(1),\mu\nu}\mathcal{A}_4=-\frac{2\kappa^3}{(\bar{k}\cdot p_1)}\Big[p_1^{(\mu}p_2^{\nu)}(\bar{k}\cdot p_1)(p_1\cdot p_2)-p_1^{(\mu}p_1^{\nu)}(\bar{k}\cdot p_2)(p_1\cdot p_2)+\frac{1}{2\bar{q}^2}\bar{q}^\mu\bar{q}^\nu(\bar{k}\cdot p_1)\Big]. \tag{E.13}$$

Therefore, by substituting the above expression we get

$$\begin{aligned}\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},4}=&\frac{1}{m_1m_2}\int\hat{d}^4\bar{q}e^{-i\bar{q}\cdot b}\hat{\delta}(u_2\cdot\bar{q})\Big\{\frac{(i\bar{k}\cdot b)^{n-1}}{(n-1)!}\hat{\delta}(u_1\cdot\bar{q})\\&+\sum_{\substack{r,s\geq1\\ \ni(r+s)=n-1}}\frac{(-1)^s}{r!s!}(ib\cdot\bar{k})^r(u_1\cdot\bar{k})^s\hat{\delta}^{(s)}(u_1\cdot\bar{q})\\&+\frac{(-1)^{n-1}}{(n-1)!}(u_1\cdot\bar{k})^{n-1}\hat{\delta}^{(n-1)}(u_1\cdot\bar{q})\Big\}\\&\times-\frac{2\kappa^3}{(\bar{k}\cdot p_1)\bar{q}^2}p_1^{(\mu}\Big[p_2^{\nu)}(\bar{k}\cdot p_1)(p_1\cdot p_2)-p_1^{\nu)}(\bar{k}\cdot p_2)(p_1\cdot p_2)\Big]\\&+\mathcal{O}(\omega^{n-1})\\=&\frac{i^{n-1}m_1m_2\kappa^3\gamma}{(n-1)!\pi\gamma\beta}(\omega b)^{n-1}\log(\omega b)u_1^{(\mu}\Big(u_2^{\nu)}-u_1^{\nu)}\frac{(\bar{k}\cdot u_2)}{(\bar{k}\cdot u_1)}\Big)+\mathcal{O}(\omega^{n-1})\,,\end{aligned} \tag{E.14}$$

where we have used the integral result of equation (B.11). The second integral and the third one do not give log terms following the result of equation (B.22). We are now left with computing one last term.

$$\begin{aligned}\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},5}&=\frac{1}{m_1m_2}\int\hat{d}^4q_1\hat{d}^4q_2e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\frac{(-1)^n(k\cdot\partial)^n}{n!}\hat{\delta}^{(4)}(q_1+q_2)S^{(0),\mu\nu}\mathcal{A}_4\\&=\frac{1}{m_1m_2}\int\hat{d}^4q_1\hat{d}^4q_2e^{iq_1\cdot b}\hat{\delta}(u_1\cdot q_1)\hat{\delta}(u_2\cdot q_2)\Big\{\hat{\delta}^{(4)}(q_1+q_2-k)-\hat{\delta}^{(4)}(q_1+q_2)\\&\quad+(k\cdot\partial)\hat{\delta}^{(4)}(q_1+q_2)+\cdots-\frac{(-1)^{n-1}}{(n-1)!}(k\cdot\partial)^{n-1}\hat{\delta}^{(4)}(q_1+q_2)\Big\}S^{(0),\mu\nu}\mathcal{A}_4\end{aligned} \tag{E.15}$$

Integrating q_1 and relabelling $q_2 \to q$ and keeping only $\mathcal{O}(\omega^{n-1})$ terms, we have,

$$\mathcal{R}^{\mu\nu}_{\omega^{(n-1)},5} = \frac{1}{m_1 m_2}\int \hat{d}^4 q e^{-iq\cdot b}\hat{\delta}(u_2\cdot q)\Big\{\frac{(ik\cdot b)^n}{n!}\hat{\delta}(u_1\cdot q) + \sum_{\substack{r,s\\ \ni(r+s)=n}}\frac{(-1)^s}{r!s!}(ib\cdot k)^r(u_1\cdot k)^s\hat{\delta}^{(s)}(u_1\cdot q) + \frac{(-1)^n}{n!}(u_1\cdot k)^n\hat{\delta}^{(n)}(u_1\cdot q)\Big\}S^{(0),\mu\nu}\mathcal{A}_4\,. \tag{E.16}$$

We have, for particle 1

$$S^{(0),\mu\nu} = \frac{1}{p_1\cdot k}p_1^{(\mu}p_1^{\nu)} - \frac{1}{\tilde{p}_1\cdot k}\tilde{p}_1^{(\mu}\tilde{p}_1^{\nu)} = -\frac{\bar{q}^{(\mu}p_1^{\nu)}}{p_1\cdot\bar{k}} + \frac{(\bar{q}\cdot\bar{k})p_1^{(\mu}p_1^{\nu)}}{(p_1\cdot\bar{k})^2}\,. \tag{E.17}$$

We have the following integrals

$$\begin{aligned}\mathcal{I}_1^{\mu\nu} &= -\kappa^3(2(p_1\cdot p_2)^2 - m_1^2m_2^2)\frac{(i\bar{k}\cdot b)^n}{2n!}\int\frac{\hat{d}^4\bar{q}}{\bar{q}^2}e^{-i\bar{q}\cdot b}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\Big(\frac{\bar{q}^{(\mu}p_1^{\nu)}}{p_1\cdot\bar{k}} - \frac{(\bar{q}\cdot\bar{k})p_1^{(\mu}p_1^{\nu)}}{(p_1\cdot\bar{k})^2}\Big)\\ &= -i\kappa^3(2(p_1\cdot p_2)^2 - m_1^2m_2^2)\frac{(i\bar{k}\cdot b)^n}{4n!\pi\gamma\beta}\Big(\frac{b^{(\mu}p_1^{\nu)}}{p_1\cdot\bar{k}} - \frac{(b\cdot\bar{k})p_1^{(\mu}p_1^{\nu)}}{(p_1\cdot\bar{k})^2}\Big),\end{aligned} \tag{E.18}$$

using the integral result of equation (B.12). Next, we have the integral

$$\begin{aligned}\mathcal{I}_2^{\mu\nu} &= -\kappa^3(2(p_1\cdot p_2)^2 - m_1^2m_2^2)\sum_{\substack{r,s\\ \ni(r+s)=n}}\frac{(-1)^s}{2r!s!}(ib\cdot\bar{k})^r(u_1\cdot\bar{k})^s\int\frac{\hat{d}^4\bar{q}}{\bar{q}^2}e^{-i\bar{q}\cdot b}\hat{\delta}^{(s)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\\ &\qquad\times\Big(\frac{\bar{q}^{(\mu}p_1^{\nu)}}{p_1\cdot\bar{k}} - \frac{(\bar{q}\cdot\bar{k})p_1^{(\mu}p_1^{\nu)}}{(p_1\cdot\bar{k})^2}\Big)\\ &= \kappa^3(2(p_1\cdot p_2)^2 - m_1^2m_2^2)\sum_{\substack{r,s\\ \ni(r+s)=n}}\frac{(-1)^s}{4r!\gamma\beta}(ib\cdot\bar{k})^r(u_1\cdot\bar{k})^{s-1}\\ &\qquad\int\hat{d}^2\bar{q}_\perp e^{-i\bar{q}_\perp\cdot b}\frac{1}{\bar{q}_\perp^2}\Big[\frac{1}{(\bar{q}_\perp^2\gamma^2\beta^2)^{s/2}} + \Big(\frac{-1}{\sqrt{\bar{q}_\perp^2\gamma^2\beta^2}}\Big)^s\Big]\Big(\bar{q}_\perp^{(\mu}u_1^{\nu)} - \frac{(\bar{q}_\perp\cdot\bar{k})u_1^{(\mu}u_1^{\nu)}}{(u_1\cdot\bar{k})}\Big)\\ &\quad-\kappa^3(2(p_1\cdot p_2)^2 - m_1^2m_2^2)\frac{1}{2(n-1)!\gamma\beta}(ib\cdot\bar{k})^{n-1}\int\hat{d}^2\bar{q}_\perp e^{-i\bar{q}_\perp\cdot b}\frac{1}{\bar{q}_\perp^2}\\ &\qquad\frac{\partial}{\partial(u_1\cdot\bar{q})}\Big(\bar{q}^{(\mu}u_1^{\nu)} - \frac{(\bar{q}\cdot\bar{k})u_1^{(\mu}u_1^{\nu)}}{(u_1\cdot\bar{k})}\Big)\end{aligned}$$

$$= -\kappa^3(2(p_1 \cdot p_2)^2 - m_1^2 m_2^2)\frac{i^{n-1}}{4\pi(n-1)!\gamma^3\beta^3}(\omega b)^{n-1}\log(\omega b)$$
$$\times\Big((\gamma u_2 - u_1)^{(\mu}u_1^{\nu)} - \frac{((\gamma u_2 - u_1)\cdot \bar{k})u_1^{(\mu}u_1^{\nu)}}{(u_1\cdot\bar{k})}\Big) + \mathcal{O}(\omega^{n-1}), \tag{E.19}$$

where we have used the integral result of equation (B.19), and lastly

$$\mathcal{I}_3^{\mu\nu} = -\kappa^3(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)\frac{(-1)^n(u_1\cdot\bar{k})^n}{2n!}\int \frac{\hat{d}^4\bar{q}}{\bar{q}^2}e^{-i\bar{q}\cdot b}\hat{\delta}^{(n)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})$$
$$\Big(\frac{\bar{q}^{(\mu}p_1^{\nu)}}{p_1\cdot\bar{k}} - \frac{(\bar{q}\cdot\bar{k})p_1^{(\mu}p_1^{\nu)}}{(p_1\cdot\bar{k})^2}\Big)$$
$$= \kappa^3(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)\frac{(-1)^n(u_1\cdot\bar{k})^n}{2n!\gamma\beta}\int \hat{d}^2\bar{q}_\perp e^{-i\bar{q}_\perp\cdot b}$$
$$\frac{\partial^n}{\partial(u_1\cdot\bar{q})^n}\frac{1}{\bar{q}^2}\Big(\frac{\bar{q}^{(\mu}p_1^{\nu)}}{p_1\cdot\bar{k}} - \frac{(\bar{q}\cdot\bar{k})p_1^{(\mu}p_1^{\nu)}}{(p_1\cdot\bar{k})^2}\Big). \tag{E.20}$$

Using the integral result of equation (B.19), we get

$$\mathcal{I}_3^{\mu\nu} = \begin{cases} \mathcal{O}(\omega^{n-1}), & \text{if } n \geq 2. \\ -\kappa^3(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)\frac{1}{4\pi\gamma^3\beta^3}\log(\omega b)\Big((\gamma u_2 - u_1)^{(\mu}u_1^{\nu)} - \frac{((\gamma u_2-u_1)\cdot\bar{k})u_1^{(\mu}u_1^{\nu)}}{(u_1\cdot\bar{k})}\Big), & \text{if } n = 1. \end{cases} \tag{E.21}$$

Here the $\log(\omega b)$ contribution comes only from $n = 1$.

Therefore we collect the log terms and upon simplifying the $\omega^{n-1}\log\omega$ terms of radiation kernel w.r.t particle 1 from the quantum soft theorems is given by

$$\mathcal{R}^{\mu\nu}_{\omega^{n-1}\log\omega} = \frac{i^{n-1}m_1 m_2\kappa^3}{4\pi(n-1)!\gamma^3\beta^3}\gamma(2\gamma^2-3)(\omega b)^{n-1}\log(\omega b)\Big(u_1^{(\mu}u_2^{\nu)} - \frac{(u_2\cdot\bar{k})}{(u_1\cdot\bar{k})}u_1^{(\mu}u_1^{\nu)}\Big). \tag{E.22}$$

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