

A $SO(1, 3)$ gauge theory of quantum gravity: quantization and renormalizability proof

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Abstract

A new $SO(1,3)$ gauge field theory classically equivalent to general relativity in a limiting case is quantized and the gauge-fixed path integral representation of the quantum effective action (QEA) is derived. Both the gauge-fixed classical action and the QEA are shown to be invariant under nilpotent BRST variations of the gauge, matter, ghost, antighost and Nakanishi–Lautrup fields defining the theory and a Zinn-Justin equation constraining the QEA is derived. Dimensional analysis and the various linear constraints put on the QEA plus the ones from the non-linear Zinn-Justin equation are deployed to demonstrate full renormalizability such that all infinities appearing in a perturbative expansion of the QEA can be absorbed into the gauge-fixed classical action solely by field renormalizations and coupling redefinitions—providing the third step in consistently quantizing the $SO(1,3)$ gauge field theory at hands, and with it potentially gravity.

Keywords: renormalizable gauge field theory of gravity, renormalizable, ghost-free and unitary quantum gravity, proof of renormalizability for a $SO(1,3)$ gauge theory of quantum gravity

1. Introduction

This is the third in a series of papers on a classical and quantum $SO(1,3)$ gauge field theory [1, 2] in which we step-by-step develop a programme aimed at quantizing gravity.

So far we have taken two steps.

The first step has been to formulate a gauge theory of the Lorentz group which is new due to treating the gauge symmetry under $SO(1,3)$ as purely *internal* [1] in complete analogy to



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the Yang–Mills case, hence separating the local gauge symmetry completely from the global spacetime symmetries under translations and Lorentz rotations which the theory incorporates separately. The theory (a) contains as the fundamental dynamical field the dimension-one Lorentz gauge field in terms of which all dynamical quantities can be expressed, (b) allows for actions at most quadratic in the first derivatives of the gauge field and hence for renormalizability by power-counting, and (c) is equivalent to general relativity (GR) in a limiting case. In other words it is a candidate theory of gravitation viable at the classical level which is not plagued by the well-known flaws preventing consistent perturbative quantization in the usual approaches.

Although this paper deals with the quantization and the renormalizability proof of the $\mathbf{SO}(1,3)$ gauge theory let us add here a few remarks on the classical theory developed in detail in [1].

The theory by its construction is invariant under local $\mathbf{SO}(1,3)$ gauge transformations treated as *internal* transformations, i.e. acting on fields only and leaving the underlying Minkowski spacetime and its coordinates invariant. For global $\mathbf{SO}(1,3)$ gauge transformations these internal transformations are equivalent to the usual combined Lorentz transformations of spacetime coordinates and of fields [1], at least infinitesimally. This does not hold true anymore for local gauge transformations which becomes clear looking at the underlying geometrical structure behind our approach which has been established in section 5 in [1]: the bundles in our approach are trivial with Minkowski space as the base space and *infinite* dimensional functional spaces as typical fibres in generalization of the flat spacetime Yang–Mills case, whereas in GR the bundles are not trivial with a non-Minkowskian space as base space and *finite* dimensional tensor spaces as typical fibres. By construction the general gauge field and matter actions are invariant under internal local $\mathbf{SO}(1,3)$ gauge transformations which leave spacetime coordinates untouched. But the theory is not diffeomorphism invariant, i.e. invariant under general coordinate transformations and their corresponding action on tensor fields exactly in the same way other gauge field theories of internal symmetry groups are not diffeomorphism invariant. This illuminates the difference to GR and the two approaches become equivalent only in an appropriate classical limit [1]. By construction the Lorentz gauge field couples to the angular momentum density tensor J for any matter field of arbitrary spin. J contains in general both orbital and spin terms. As the orbital contribution to $J \sim x \times \Theta$ is proportional to the energy momentum density Θ both energy-momentum and spin act as sources for the Lorentz gauge field. Finally we note that for a scalar field there are no spin terms and only orbital terms containing Θ act as gauge field source which proves crucial for the equivalence of the theory to GR in an appropriate limit at the classical level.

What makes our approach promising is exactly this limiting case in which the theory becomes equivalent to GR as defined by the Einstein–Hilbert action given in terms of the vierbeins as dynamical variables. In fact for Newton’s gravitational constant being small and spin neglected which is a good approximation in the classical macroscopic world the resulting gauge field action contains at most dimension-two gauge field terms and scalar matter coupled to the gauge field. In [1] we have shown that in this case and for a specific choice of the numerical parameters in the truncated theory the gauge field and scalar matter actions can be expressed solely in terms of what in GR would be the vierbeins and the scalar curvature belonging to the spin connection as function of the vierbeins. In this limit the $\mathbf{SO}(1,3)$ gauge field is completely shielded into an expression that formally resembles the vierbeins which emerge as the only relevant dynamical field variables. For the technically quite involved details of the demonstration we refer to sections 7 and 8 in [1]. As a result in addition to the local internal Lorentz invariance the theory in this limit allows for an additional and separate diffeomorphism invariance in the usual sense. And the matter current to which the analogue of the vierbeins couple reduces

to the energy momentum tensor which acts as the source of gravitation as in GR. Hence in this limit GR expressed in terms of vierbeins emerges as the classical limit of a more general renormalizable gauge field theory. The field equations derived from the resulting action are the Einstein equations for the vierbeins the vacuum solution of which is the Schwarzschild solution.

Turning back to our programme the second step has been to establish that the canonical quantization of the non-interacting gauge field in the $\mathbf{SO}(1,3)$ gauge field theory allows for the definition of positive-norm, positive-energy states and a corresponding relativistically-invariant physical Fock space for the quantum theory in spite of the non-compactness of the gauge group $\mathbf{SO}(1,3)$ and the corresponding indefinite Cartan metric on the gauge algebra [2]. This has been achieved by intertwining relativistic covariance with positivity of the norm and energy expectation values for physical states, and consequentially putting restrictions needed in establishing a physical Fock space on state vectors, and not on the algebra of creation and annihilation operators—generalizing the Gupta–Bleuler approach.

The third and current step of our programme is the proof of perturbative renormalizability for the full quantum theory including the demonstration that unphysical ghosts decouple which appear in the gauge-fixed path integral quantization of the classical theory, and establishing the pseudo-unitarity of the S -matrix on the naïve Fock space containing negative-norm, negative-energy states besides the physical ones.

The final step of our programme will be to establish the unitarity of the S -matrix on the physical Fock space constructed in [2].

To effectively establish perturbative renormalizability of the full quantum theory in this paper following [3] we start in section 2 revisiting some aspects of the gauge field theory of the Lorentz group $\mathbf{SO}(1,3)$ at the classical level to then derive gauge-fixed path integral expressions for the expectation values of physical observables and the quantum effective action (QEA). In section 3 we rewrite these expressions in terms of path integrals over additional ghost, antighost and Nakanishi–Lautrup fields allowing in section 4 in an elegant way to introduce nilpotent BRST field variations and to demonstrate the invariance of the gauge-fixed classical action and the QEA under these variations. In section 5 we derive the Zinn-Justin equation which puts crucial constraints on the QEA and its loop-wise expansion. In section 6 we finally demonstrate the perturbative renormalizability of the $\mathbf{SO}(1,3)$ gauge field theory which marks a further key step towards a potential consistent quantum theory of gravitation.

All fields in this paper are defined on Minkowski spacetime $\mathbf{M}^4 \equiv (\mathbf{R}^4, \eta)$ with points $x \in \mathbf{M}^4$ given in Cartesian coordinates. $\eta = \text{diag}(-1, 1, 1, 1)$ is the flat spacetime metric with which indices $\alpha, \beta, \gamma, \dots$ are raised and lowered. They appear in quantities defined on \mathbf{M}^4 which transform covariantly. All other notations deployed in the paper are explained wherever they appear first.

2. Path integral quantization of the $\mathbf{SO}(1,3)$ gauge field theory

In this section we review some key elements of the $\mathbf{SO}(1,3)$ gauge field theory equivalent to GR in a limiting case as developed in [1]. We then quantize the theory and derive gauge-fixed path-integral representations for gauge-invariant physical quantities applying the Faddeev–Popov–deWitt approach.

Let us start with the gauge-invariant path integral representing the expectation value of a physical observable $\mathcal{O}[B]$

$$\int \prod_{x;\alpha,\gamma,\delta} dB_\alpha^{\gamma\delta}(x) \mathcal{O}[B] \exp i \{S_G[B] + \varepsilon\text{-terms}\} . \quad (1)$$

Above $B_{\alpha}{}^{\gamma\delta}$ denotes the **SO(1,3)** gauge field antisymmetric in the indices γ, δ which we have introduced in [1], and $\mathcal{O}[B]$ a gauge-invariant observable which is a functional of $B_{\alpha}{}^{\gamma\delta}$.

$\prod_{x;\alpha,\gamma,\delta} dB_{\alpha}{}^{\gamma\delta}(x)$ is the integration measure over gauge field space which is invariant under the gauge transformations equation (17) below as demonstrated in appendix C.

To specify the notations needed we next recall some key results from [1]. We stress that this is not a self-contained re-derivation of the classical theory and we refer for all technical details to [1]. There the dynamics of the gauge field $B_{\alpha}{}^{\gamma\delta}$ has been properly developed and is governed by the most general action of dimension four or less for the $B_{\alpha}{}^{\gamma\delta}$

$$S_G[B] = S_G^{(0)}[B] + S_G^{(2)}[B] + S_G^{(4)}[B]. \quad (2)$$

Here

$$S_G^{(0)}[B] = \Lambda \int d^4x \det e^{-1}[B] \quad (3)$$

is the most general dimension-zero contribution with Λ a constant of dimension $[\Lambda] = 4$ and $\det e^{-1}[B]$ the determinant of a matrix $e_{\alpha}{}^{\beta}[B]$ which will be properly introduced in equation (7) below.

The most general dimension-two contribution reads

$$\begin{aligned} S_G^{(2)}[B] = & \frac{1}{\kappa} \int d^4x \det e^{-1}[B] \{ \alpha_1 R_{\alpha\beta}{}^{\alpha\beta}[B] \\ & + \alpha_2 T_{\alpha\beta\gamma}[B] T^{\alpha\beta\gamma}[B] + \alpha_3 T_{\alpha\beta\gamma}[B] T^{\gamma\beta\alpha}[B] \\ & + \alpha_4 T_{\alpha}{}^{\gamma\alpha}[B] T_{\beta\gamma}{}^{\beta}[B] + \alpha_5 \nabla_{\alpha}^B T_{\beta}{}^{\alpha\beta}[B] \}. \end{aligned} \quad (4)$$

The constant $\frac{1}{\kappa} = \frac{1}{16\pi\Gamma}$ has mass-dimension $[\frac{1}{\kappa}] = 2$ with Γ denoting the Newtonian gravitational constant. $T[B]$ and $R[B]$ are Lorentz tensors properly introduced in equations (15) and (16) below and the α_i above are constants of dimension $[\alpha_i] = 0$.

Finally the most general dimension-four contribution reads

$$\begin{aligned} S_G^{(4)}[B] = & \int d^4x \det e^{-1}[B] \{ \beta_1 R_{\alpha\beta}{}^{\gamma\delta}[B] R^{\alpha\beta}{}_{\gamma\delta}[B] \\ & + \beta_2 R_{\alpha\gamma}{}^{\alpha\delta}[B] R^{\beta\gamma}{}_{\beta\delta}[B] + \beta_3 R_{\alpha\beta}{}^{\alpha\beta}[B] R_{\gamma\delta}{}^{\gamma\delta}[B] \\ & + \beta_4 \nabla_B^{\gamma} \nabla_{\delta}^B R_{\alpha\gamma}{}^{\alpha\delta}[B] + \beta_5 \nabla_B^{\gamma} \nabla_{\gamma}^B R_{\alpha\beta}{}^{\alpha\beta}[B] \\ & + \dots \\ & + \gamma_1 \nabla_{\gamma}^B T_{\alpha\beta\delta}[B] \nabla_B^{\gamma} T^{\alpha\beta\delta}[B] + \gamma_2 \nabla_{\gamma}^B T_{\alpha\beta\delta}[B] \nabla_B^{\gamma} T^{\delta\beta\alpha}[B] \\ & + \dots \\ & + \gamma_j T^4 - \text{terms} \\ & + \dots \\ & + \delta_k R T^2 - \text{terms}, R \nabla^B T - \text{terms} \\ & + \dots \}. \end{aligned} \quad (5)$$

with $\beta_i, \gamma_j, \delta_k$ constants of dimension $[\beta_i] = [\gamma_j] = [\delta_k] = 0$.

Above

$$\begin{aligned}\nabla_\alpha^B &\equiv \partial_\alpha + \frac{i}{2}B_\alpha{}^{\gamma\delta}\vec{L}_{\gamma\delta} + \frac{i}{2}B_\alpha{}^{\gamma\delta}\Sigma_{\gamma\delta} \\ &= (\eta_\alpha{}^\gamma - B_\alpha{}^{\gamma\delta}x_\delta)\partial_\gamma + \frac{i}{2}B_\alpha{}^{\gamma\delta}\Sigma_{\gamma\delta} \\ &\equiv d_\alpha^B + \bar{B}_\alpha\end{aligned}\quad (6)$$

denotes the covariant derivative w.r.t. to the gauge group $\mathbf{SO}(1,3)$ as introduced in [1]. $\vec{L}_{\gamma\delta} = -i(x_\gamma\partial_\delta - x_\delta\partial_\gamma)$ are the generators of the *internal* $\mathbf{SO}(1,3)$ Lorentz algebra acting on spacetime coordinates and $\Sigma_{\gamma\delta}$ generic generators of the Lorentz algebra acting on spin degrees of freedom.

To simplify notations we have defined the matrix

$$e_\alpha{}^\vartheta[B] \equiv \eta_\alpha{}^\vartheta - B_\alpha{}^{\vartheta\zeta}x_\zeta \quad (7)$$

resembling a vierbein which, however, is solely a functional of the fundamental dynamical variable $B_\alpha{}^{\gamma\delta}$ in our theory, and have introduced

$$d_\alpha^B \equiv e_\alpha{}^\vartheta[B]\partial_\vartheta, \quad \bar{B}_\alpha \equiv \frac{i}{2}B_\alpha{}^{\gamma\delta}\Sigma_{\gamma\delta}. \quad (8)$$

We have elaborated in depth in [1] why $e_\alpha{}^\vartheta[B]$ not being a fundamental dynamical field in our approach is so crucial for the further development of the theory to be both equivalent to GR in a limiting case and renormalizable.

To define the covariant objects of the theory we next look at the field strength operator G acting on fields

$$G_{\alpha\beta}[B] \equiv [\nabla_\alpha^B, \nabla_\beta^B] \quad (9)$$

and express it in terms of the gauge field B

$$\begin{aligned}G_{\alpha\beta}[B] &= [d_\alpha^B, d_\beta^B] + d_\alpha^B\bar{B}_\beta - d_\beta^B\bar{B}_\alpha \\ &\quad + [\bar{B}_\alpha, \bar{B}_\beta] + (B_{\alpha\beta}{}^\eta - B_{\beta\alpha}{}^\eta)\nabla_\eta^B.\end{aligned}\quad (10)$$

To re-express

$$[d_\alpha^B, d_\beta^B] = (e_\alpha{}^\zeta[B]\partial_\zeta e_\beta{}^\eta[B] - e_\beta{}^\zeta[B]\partial_\zeta e_\alpha{}^\eta[B])\partial_\eta \quad (11)$$

we assume that the matrix $e_\alpha{}^\zeta[B]$ is non-singular, i.e. $\det e[B] \neq 0$. Hence there is an inverse $e^\gamma{}_\eta[B]$ with $e^\gamma{}_\eta[B]e_\gamma{}^\zeta[B] = \delta_\eta{}^\zeta$, and we can write

$$[d_\alpha^B, d_\beta^B] = H_{\alpha\beta}{}^\gamma[B]d_\gamma^B \quad (12)$$

introducing

$$H_{\alpha\beta}{}^\gamma[B] \equiv e^\gamma{}_\eta[B](e_\alpha{}^\zeta[B]\partial_\zeta e_\beta{}^\eta[B] - e_\beta{}^\zeta[B]\partial_\zeta e_\alpha{}^\eta[B]). \quad (13)$$

As a result we can rewrite

$$\begin{aligned}G_{\alpha\beta}[B] &= (H_{\alpha\beta}{}^\gamma[B] + B_{\alpha\beta}{}^\gamma - B_{\beta\alpha}{}^\gamma)\nabla_\gamma^B \\ &\quad + d_\alpha^B\bar{B}_\beta - d_\beta^B\bar{B}_\alpha + [\bar{B}_\alpha, \bar{B}_\beta] - H_{\alpha\beta}{}^\gamma[B]\bar{B}_\gamma\end{aligned}$$

$$\equiv -T_{\alpha\beta}{}^\gamma[B] \nabla_\gamma^B + R_{\alpha\beta}[B] \quad (14)$$

in terms of the covariant field strength components T

$$T_{\alpha\beta}{}^\gamma[B] \equiv -(B_{\alpha\beta}{}^\gamma - B_{\beta\alpha}{}^\gamma) - H_{\alpha\beta}{}^\gamma[B] \quad (15)$$

and R

$$\begin{aligned} R_{\alpha\beta}[B] &\equiv \frac{i}{2} R_{\alpha\beta}{}^{\gamma\delta}[B] \Sigma_{\gamma\delta} \\ R_{\alpha\beta}{}^{\gamma\delta}[B] &= d_\alpha^B B_\beta{}^{\gamma\delta} - d_\beta^B B_\alpha{}^{\gamma\delta} + B_\alpha{}^{\gamma\eta} B_{\beta\eta}{}^\delta \\ &\quad - B_\beta{}^{\gamma\eta} B_{\alpha\eta}{}^\delta - H_{\alpha\beta}{}^\eta[B] B_\eta{}^{\gamma\delta}. \end{aligned} \quad (16)$$

Under a local *internal* Lorentz variation

$$\delta_\omega B_\alpha{}^{\gamma\delta} = -\omega^{\eta\zeta} x_\zeta \partial_\eta B_\alpha{}^{\gamma\delta} - d_\alpha^B \omega^{\gamma\delta} + \omega_\alpha{}^\beta B_\beta{}^{\gamma\delta} + \omega^\gamma{}_\eta B_\alpha{}^{\eta\delta} + \omega^\delta{}_\eta B_\alpha{}^{\gamma\eta} \quad (17)$$

of the gauge field $B_\alpha{}^{\gamma\delta}$ assuring covariance of the derivative equation (6) as established in [1] we find the field strength components to display the homogenous variations

$$\begin{aligned} \delta_\omega T_{\alpha\beta}{}^\gamma[B] &= -\omega^{\eta\zeta} x_\zeta \partial_\eta T_{\alpha\beta}{}^\gamma[B] + \omega_\alpha{}^\eta T_{\eta\beta}{}^\gamma[B] \\ &\quad + \omega_\beta{}^\eta T_{\alpha\eta}{}^\gamma[B] + \omega^\gamma{}_\eta T_{\alpha\beta}{}^\eta[B] \end{aligned} \quad (18)$$

and

$$\begin{aligned} \delta_\omega R_{\alpha\beta}{}^{\gamma\delta}[B] &= -\omega^{\eta\zeta} x_\zeta \partial_\eta R_{\alpha\beta}{}^{\gamma\delta}[B] + \omega_\alpha{}^\eta R_{\eta\beta}{}^{\gamma\delta}[B] \\ &\quad + \omega_\beta{}^\eta R_{\alpha\eta}{}^{\gamma\delta}[B] + \omega^\gamma{}_\eta R_{\alpha\beta}{}^{\eta\delta}[B] + \omega^\delta{}_\eta R_{\alpha\beta}{}^{\gamma\eta}[B], \end{aligned} \quad (19)$$

where δ_ω denotes the variation under an infinitesimal gauge transformation. Note that the first terms $-\omega^{\eta\zeta} x_\zeta \partial_\eta \dots$ in all the variations above account for the coordinate change related to a local Lorentz transformation in our approach whilst $\delta_\omega x^\alpha = 0$ [1].

By construction the action $S_G[B]$ in equation (2) is the most general action of dimension ≤ 4 in the gauge fields $B_\alpha{}^{\gamma\delta}$ and their first and second derivatives $\partial_\beta B_\alpha{}^{\gamma\delta}$, $\partial_\eta \partial_\beta B_\alpha{}^{\gamma\delta}$ which is locally Lorentz invariant and renormalizable by power-counting.

The actual proof of renormalizability delivered in this paper requires the much more involved demonstration that counterterms needed to absorb infinite contributions to the perturbative expansion of the effective action of the full quantum theory are again of the form equation (2) plus gauge-fixing and ghost terms with possibly renormalized fields and coupling constants.

We finally note that for the choice

$$\alpha_1 = 1, \quad \alpha_2 = -\frac{1}{4}, \quad \alpha_3 = -\frac{1}{2}, \quad \alpha_4 = -1, \quad \alpha_5 = 2 \quad (20)$$

$S_G^{(0)}[B] + S_G^{(2)}[B]$ coupled to scalar matter is equivalent to GR with a cosmological constant term as demonstrated in [1].

Let us go back to the path integral in equation (1). It runs over all possible gauge-equivalent field configurations thence counting a physically relevant field configuration multiple times in the integration. In order to separate the part of the integration related to gauge-invariance from the physically relevant integration over gauge-non-equivalent field configurations we divide

the configuration space $\{B_\alpha^{\gamma\delta}\}$ into equivalence classes $[B_\alpha^{\gamma\delta}]$ of fields which are gauge-equivalent under the gauge transformation equation (17). The integrand in equation (1) is then constant over a given equivalence class, and the integral itself proportional to the infinite volume of the Lorentz gauge group. In itself this poses no insurmountable problem when calculating equation (1) non-perturbatively. However, the quadratic part of the action S_G in equation (2) as defined in [2] is not invertible due to zero eigenvalues related to the gauge symmetry. So in order to perturbatively deal with calculating integrals of the type of equation (1) we have to factor out the volume of the Lorentz gauge group in the integration.

Following the Faddeev–Popov–deWitt approach [3, 4] we introduce

$$1 = \Delta[B] \int_x \Pi dg(x) \delta(f^{\eta\zeta}[B^g](x)), \tag{21}$$

where g is an element of the gauge group $\mathbf{SO}(1,3)$, $\int_x \Pi dg(x)$ is a gauge-invariant measure over the gauge group and $f^{\eta\zeta}[B^g](x) = 0$ has exactly one solution and hence fixes a gauge. Note that $\Delta[B]$ is gauge-invariant.

Let us next insert the expression above into the path integral equation (1) and change the order of integration

$$\begin{aligned} & \int_x \Pi dg(x) \int_{x;\alpha,\gamma,\delta} \Pi dB_\alpha^{\gamma\delta}(x) \Delta[B] \delta(f^{\eta\zeta}[B^g](x)) \\ & \times \mathcal{O}[B] \exp i \{S_G[B] + \varepsilon\text{-terms}\}. \end{aligned} \tag{22}$$

The expression

$$\begin{aligned} & \int_{x;\alpha,\gamma,\delta} \Pi dB_\alpha^{\gamma\delta}(x) \Delta[B^g] \delta(f^{\eta\zeta}[B^g](x)) \\ & \times \mathcal{O}[B^g] \exp i \{S_G[B^g] + \varepsilon\text{-terms}\} \end{aligned} \tag{23}$$

turns out to be gauge-invariant which allows us to separate the group volume from the gauge-fixed remainder of the integral in the $f^{\eta\zeta}[B^g] = 0$ gauge

$$\begin{aligned} & \left(\int_x \Pi dg(x) \right) \int_{x;\alpha,\gamma,\delta} \Pi dB_\alpha^{\gamma\delta}(x) \Delta[B] \delta(f^{\eta\zeta}[B^g](x)) \\ & \times \mathcal{O}[B] \exp i \{S_G[B] + \varepsilon\text{-terms}\}. \end{aligned} \tag{24}$$

Next we calculate $\Delta[B]$ by changing variables

$$\Delta^{-1}[B] = \int_{x;\eta,\zeta} \Pi df^{\eta\zeta}(x) \left(\text{Det} \frac{\delta f^{\eta\zeta}[B^g]}{\delta g} \right)^{-1} \delta(f^{\eta\zeta}(x)) \tag{25}$$

and find

$$\Delta[B] = \text{Det} \frac{\delta f^{\eta\zeta}[B^g]}{\delta g} \Big|_{f^{\eta\zeta}[B^g]=0} = \text{Det} \frac{\delta f^{\eta\zeta}[B^\omega]}{\delta \omega^{\nu\kappa}} \Big|_{\omega=0}, \tag{26}$$

where the last equality relates to it being sufficient to calculate the value of the Jacobian related to infinitesimal variations.

The Faddeev–Popov–deWitt operator is defined as

$$\mathcal{F}^{\eta\zeta}{}_{\iota\kappa}[B; x, y] \equiv \frac{\delta f^{\eta\zeta}[B^\omega(x)]}{\delta \omega^{\iota\kappa}(y)} \Big|_{\omega=0}. \quad (27)$$

Choosing the axial gauge with the gauge fixing functional

$$f^{\gamma\delta}[B] = n^\alpha B_\alpha{}^{\gamma\delta}, \quad (28)$$

where n^α is a constant vector in tangent space with $\delta_\omega n^\alpha = -n^\beta \omega_{\beta}{}^\alpha$, we find

$$\begin{aligned} \int d^4y \mathcal{F}^{\gamma\delta}{}_{\alpha\beta}[B; x, y] \omega^{\alpha\beta}(y) &= n^\alpha (-\omega^{\eta\zeta} x_\zeta \partial_\eta B_\alpha{}^{\gamma\delta} \\ &\quad - d_\alpha^B \omega^{\gamma\delta} + \omega_\alpha{}^\beta B_\beta{}^{\gamma\delta} + \omega^\gamma{}_\eta B_\alpha{}^{\eta\delta} + \omega^\delta{}_\eta B_\alpha{}^{\gamma\eta}) \\ &\quad - n^\beta \omega_{\beta}{}^\alpha B_\alpha{}^{\gamma\delta} = -n^\alpha \partial_\alpha \omega^{\gamma\delta}. \end{aligned} \quad (29)$$

Note that we have used $n^\alpha B_\alpha{}^{\gamma\delta} = 0$. In this case the Faddeev–Popov–deWitt determinant $\text{Det} \frac{\delta f^{\eta\zeta}[B^\omega]}{\delta \omega^{\iota\kappa}} \Big|_{\omega=0} = \text{Det}(-n^\alpha \partial_\alpha)$ is field-independent and can be taken in front of the integral equation (24) which is generally not the case.

The existence of a gauge with this property guarantees the decoupling of ghosts and anti-ghosts from the real physics in our theory and the pseudo-unitarity of the S -matrix on the naïve Fock space of both positive-norm, positive-energy and negative-norm, negative-energy states related to the gauge field as introduced in [2].

To demonstrate the actual renormalizability we however choose the Lorentz gauge condition

$$f^{\gamma\delta}[B] = \partial^\alpha B_\alpha{}^{\gamma\delta} = 0 \quad (30)$$

with $\delta_\omega \vec{\partial}^\alpha = -\vec{\partial}^\beta \omega_{\beta}{}^\alpha$. Here we find the field-dependent Faddeev–Popov–deWitt operator

$$\begin{aligned} \int d^4y \mathcal{F}^{\gamma\delta}{}_{\alpha\beta}[B; x, y] \omega^{\alpha\beta}(y) &= \partial^\alpha (-\omega^{\eta\zeta} x_\zeta \partial_\eta B_\alpha{}^{\gamma\delta} \\ &\quad - d_\alpha^B \omega^{\gamma\delta} + \omega_\alpha{}^\beta B_\beta{}^{\gamma\delta} + \omega^\gamma{}_\eta B_\alpha{}^{\eta\delta} + \omega^\delta{}_\eta B_\alpha{}^{\gamma\eta}) \\ &\quad - \vec{\partial}^\beta \omega_{\beta}{}^\alpha B_\alpha{}^{\gamma\delta} = \partial^\alpha \left(-\omega^{\eta\zeta} x_\zeta \partial_\eta B_\alpha{}^{\gamma\delta} \right. \\ &\quad \left. - d_\alpha^B \omega^{\gamma\delta} + \frac{1}{2} C^{\gamma\delta}{}_{\iota\kappa}{}^{\eta\zeta} B_\alpha{}^{\iota\kappa} \omega^{\eta\zeta} \right) \end{aligned} \quad (31)$$

with the expression in brackets on the last line being the covariant derivative $\nabla_\alpha^B \omega^{\gamma\delta}$ of the infinitesimal gauge parameter $\omega^{\gamma\delta}$ as expected.

Note that in both cases above the term $\dots (-\omega^{\eta\zeta} x_\zeta \partial_\eta B_\alpha{}^{\gamma\delta} - \dots)$ relates to taking into account both the spacetime and spin degrees of freedom of the gauge group $\mathbf{SO}(1,3)$ in the Faddeev–Popov–deWitt approach.

Next we note that we can change the gauge fixing condition $f^{\eta\zeta}[B^g] = 0$ to $f^{\eta\zeta}[B^g] - C^{\eta\zeta} = 0$ in

$$\begin{aligned} &\left(\int_x \Pi dg(x) \right) \int_{x;\alpha,\gamma,\delta} \Pi dB_\alpha{}^{\gamma\delta}(x) \delta(f^{\eta\zeta}[B^g](x) - C^{\eta\zeta}(x)) \\ &\quad \times \mathcal{O}[B] \text{Det } \mathcal{F}[B] \exp i \{S_G[B] + \varepsilon\text{-terms}\}, \end{aligned} \quad (32)$$

and integrate over a field-independent weight function $\mathcal{G}[C]$

$$\int \prod_{x;\alpha,\gamma,\delta} dB_\alpha^{\gamma\delta}(x) \mathcal{O}[B] \text{Det } \mathcal{F}[B] \exp i \{S_G[B] + \varepsilon\text{-terms}\} \\ \times \int \prod_x dC(x) \delta(f^{\eta\zeta}[B^g](x) - C^{\eta\zeta}(x)) \mathcal{G}[C] \quad (33)$$

without altering the physics involved [3, 4]. A familiar choice compatible with renormalizability is

$$\mathcal{G}[C] = \exp -\frac{i}{2\xi} \int C_{\gamma\delta} C^{\gamma\delta}. \quad (34)$$

Leaving aside the infinite gauge group volume $\int \prod_x dg(x)$ this amounts to adding a gauge-fixing term

$$S_G[B] \longrightarrow S_G[B] + S_{GF}[B] \\ S_{GF}[B] \equiv -\frac{i}{2\xi} \int f_{\gamma\delta}[B] f^{\gamma\delta}[B] \quad (35)$$

to the gauge field action, destroying gauge invariance in the process as it better should if we want to use the combined action to perturbatively evaluate our path integrals. So finally we get the gauge-fixed expression for the path integral representing the expectation value of an observable $\mathcal{O}[B]$

$$\int \prod_{x;\alpha,\gamma,\delta} dB_\alpha^{\gamma\delta}(x) \mathcal{O}[B] \text{Det } \mathcal{F}[B] \exp i \{S_G[B] + S_{GF}[B] + \varepsilon\text{-terms}\}. \quad (36)$$

3. Ghosts, antighosts and Nakanishi–Lautrup fields

In this section we recast the Faddeev–Popov–deWitt determinant as a fermionic path integral over ghost and antighost fields and introduce the Nakanishi–Lautrup fields in preparation of the demonstration of BRST invariance of the gauge-fixed action.

Using the fact that Gaussian path integrals yield determinants we can re-express the Faddeev–Popov–deWitt determinant as a fermionic Gaussian path integral over anti-commuting ghost and antighost fields $\omega^{\iota\kappa}$ and $\omega_{\eta\zeta}^*$

$$\text{Det } \mathcal{F}[B] \propto \int \prod_{x;\eta,\zeta} d\omega_{\eta\zeta}^*(x) \int \prod_{x;\iota,\kappa} d\omega^{\iota\kappa}(x) \exp i S_{GH}. \quad (37)$$

Above $\omega^{\iota\kappa}$ and $\omega_{\eta\zeta}^*$ are antisymmetric tensors of integer spin and the ghost action S_{GH} is given by

$$S_{GH} \equiv \int d^4x \int d^4y \omega_{\eta\zeta}^*(x) \mathcal{F}^{\eta\zeta}_{\iota\kappa}[B; x, y] \omega^{\iota\kappa}(y) \\ = \int d^4x \omega_{\eta\zeta}^*(x) \Delta^{\eta\zeta}[B; x], \quad (38)$$

where we have introduced the shorthand notation

$$\Delta^{\eta\zeta}[B; x] \equiv \int d^4y \mathcal{F}^{\eta\zeta}_{\iota\kappa}[B; x, y] \omega^{\iota\kappa}(y) \quad (39)$$

for later use.

Finally we re-express $\exp -\frac{i}{2\xi} \int f_{\gamma\delta}[B] f^{\gamma\delta}[B]$ as a bosonic Gaussian path integral over the Nakanishi–Lautrup fields $h^{\eta\zeta}$

$$\exp -\frac{i}{2\xi} \int f_{\gamma\delta}[B] f^{\gamma\delta}[B] \propto \int \prod_{x;\eta,\zeta} dh^{\eta\zeta}(x) \exp i \left\{ \frac{\xi}{2} \int h_{\eta\zeta} h^{\eta\zeta} + \int h_{\eta\zeta} f^{\eta\zeta}[B] \right\} \quad (40)$$

to arrive at the form of the gauge-fixed expression for the path integral representing the expectation value of an observable $\mathcal{O}[B]$ which is most convenient for our purpose to demonstrate renormalizability

$$\int \prod_{x;\alpha,\gamma,\delta} dB_{\alpha}{}^{\gamma\delta}(x) \int \prod_{x;\eta,\zeta} d\omega_{\eta\zeta}^*(x) \int \prod_{x;\iota,\kappa} d\omega^{\iota\kappa}(x) \int \prod_{x;\rho,\sigma} dh^{\rho\sigma}(x) \times \mathcal{O}[B] \exp i \{S_{\text{NEW}} + \varepsilon\text{-terms}\}. \quad (41)$$

Here

$$S_{\text{NEW}} \equiv S_{\text{G}} + \int \omega_{\eta\zeta}^* \Delta^{\eta\zeta}[B] + \int h_{\eta\zeta} f^{\eta\zeta}[B] + \frac{\xi}{2} \int h_{\eta\zeta} h^{\eta\zeta} \quad (42)$$

is the gauge-fixed action for the gauge, ghost, antighost and Nakanishi–Lautrup fields which we will use as the starting point for the actual renormalizability proof.

Note the absence of the determinant $\det e^{-1}$ in all contributions to S_{NEW} apart from S_{G} which will prove crucial to rewrite $S_{\text{NEW}} - S_{\text{G}}$ as a BRST transformation in the next section.

The modified action above is not gauge invariant—indeed, it had better not be, if we want to be able to use it in perturbative calculations.

4. BRST invariance

In this section we introduce fermionic BRST field variations, demonstrate their nilpotence and based on this establish the invariance of S_{NEW} under those BRST transformations.

Let us write down the various BRST variations starting with the one for a generic matter field ψ

$$\delta_{\theta}\psi = \theta s\psi = \frac{i}{2}\theta \omega^{\gamma\delta} (\vec{L}_{\gamma\delta} \psi) + \frac{i}{2}\theta \omega^{\gamma\delta} \Sigma_{\gamma\delta} \psi, \quad (43)$$

where θ is a fermionic parameter and assures the right statistics for the various field variations above and $s \dots$ indicates the infinitesimal variation of a given field without the factor θ . We recall that $\vec{L}_{\gamma\delta} = -i(x_{\gamma}\partial_{\delta} - x_{\delta}\partial_{\gamma})$ denotes the generators of the $\mathbf{SO(1,3)}$ Lorentz algebra acting on spacetime coordinates and $\Sigma_{\gamma\delta}$ generators of the Lorentz algebra acting on spin degrees of freedom. Note that for a generic matter field the BRST variation is nothing but an infinitesimal gauge variation with gauge parameter $\theta\omega^{\gamma\delta}$.

The gauge field variation reads

$$\begin{aligned} \delta_{\theta}B_{\alpha}{}^{\gamma\delta} &= \theta sB_{\alpha}{}^{\gamma\delta} = \frac{i}{2}\theta \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} B_{\alpha}{}^{\gamma\delta}) - \theta \partial_{\alpha}\omega^{\gamma\delta} \\ &\quad - \frac{i}{2}\theta B_{\alpha}{}^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) + \frac{i}{2}\theta \omega^{\eta\zeta} (\Sigma_{\eta\zeta}^V)_{\alpha}{}^{\beta} B_{\beta}{}^{\gamma\delta} \\ &\quad + \frac{1}{2}\theta C^{\gamma\delta}{}_{\iota\kappa}{}_{\eta\zeta} B_{\alpha}{}^{\iota\kappa} \omega^{\eta\zeta} \end{aligned} \quad (44)$$

which is an infinitesimal gauge variation with gauge parameter $\theta\omega^{\gamma\delta}$. Above $(\Sigma_{\eta\zeta}^A)^{\gamma\delta}{}_{\iota\kappa} = iC^{\gamma\delta}{}_{\eta\zeta}{}_{\iota\kappa}$ denotes the generators of the Lorentz algebra in the adjoint representation.

Next the ghost field variation is defined by

$$\delta_\theta\omega^{\gamma\delta} = \theta s\omega^{\gamma\delta} = \frac{i}{2}\theta\omega^{\eta\zeta}(\vec{L}_{\eta\zeta}\omega^{\gamma\delta}) - \frac{1}{4}\theta C^{\gamma\delta}{}_{\alpha\beta}{}_{\eta\zeta}\omega^{\alpha\beta}\omega^{\eta\zeta} \quad (45)$$

and the antighost variation

$$\delta_\theta\omega_{\gamma\delta}^* = \theta s\omega_{\gamma\delta}^* = -\theta h_{\gamma\delta} \quad (46)$$

with both being perspicuously distinct from a regular infinitesimal gauge transformation. Finally the Nakanishi–Lautrup field is taken to be invariant

$$\delta_\theta h_{\gamma\delta} = \theta s h_{\gamma\delta} = 0 \quad (47)$$

under BRST variations. Note the absence of the spacetime-related part $\frac{i}{2}\theta\omega^{\eta\zeta}(\vec{L}_{\eta\zeta}\dots)$ in both the antighost and Nakanishi–Lautrup field variations.

For later use we also write down the BRST variation of $\det e^{-1}[B]$

$$\delta_\theta \det e^{-1}[B] = -\theta \partial_\eta (\omega^{\eta\zeta} x_\zeta \det e^{-1}[B]). \quad (48)$$

It is crucial for the sequel that all the BRST variations above are nilpotent, or $ss\dots = 0$, as some quite tedious algebra in appendix B demonstrates. This also holds true for any functional F of the fields above, or $ssF = 0$ [3].

Note that we have written the BRST variations above in terms of the Lorentz algebra generators which proves to be of enormous help to organize the lengthy algebra involved in proving nilpotence.

Let us turn to evaluate

$$\delta_\theta \left(\omega_{\eta\zeta}^* f^{\eta\zeta}[B] + \frac{\xi}{2} \omega_{\eta\zeta}^* h^{\eta\zeta} \right). \quad (49)$$

For the variation of $f^{\eta\zeta}[B]$ we find

$$\delta_\theta f^{\eta\zeta}[B] = \int \frac{\delta f^{\eta\zeta}[B]}{\delta B_{\alpha}{}^{\iota\kappa}} (\delta_\theta B_{\alpha}{}^{\iota\kappa}) + \int \left(\delta_\theta \frac{\delta f^{\eta\zeta}[B]}{\delta B_{\alpha}{}^{\iota\kappa}} \right) B_{\alpha}{}^{\iota\kappa} = \theta \Delta^{\eta\zeta}[B], \quad (50)$$

where the second term accounts for the non-trivial transformation of the α -index in $\frac{\delta f^{\eta\zeta}[B]}{\delta B_{\alpha}{}^{\iota\kappa}}$. Using this and taking into account that θ and ω^* anticommute we get

$$\delta_\theta \left(\omega_{\eta\zeta}^* f^{\eta\zeta}[B] + \frac{\xi}{2} \omega_{\eta\zeta}^* h^{\eta\zeta} \right) = -\theta \left(\omega_{\eta\zeta}^* \Delta^{\eta\zeta}[B] + h_{\eta\zeta} f^{\eta\zeta}[B] + \frac{\xi}{2} h_{\eta\zeta} h^{\eta\zeta} \right) \quad (51)$$

which allows us to rewrite

$$S_{\text{NEW}} = S_G - s \left(\omega_{\eta\zeta}^* f^{\eta\zeta}[B] + \frac{\xi}{2} \omega_{\eta\zeta}^* h^{\eta\zeta} \right). \quad (52)$$

Evoking nilpotence for the term $s(\dots)$ in brackets, or $ss(\dots) = 0$, and the fact that S_G is gauge-invariant we find that

$$\delta_\theta S_{\text{NEW}} = 0 \quad (53)$$

or that S_{NEW} is indeed BRST invariant—and so is the gauge-fixed expression for the path integral representing the expectation value of an observable $\mathcal{O}[B]$

$$\int \prod_x d\psi(x) \int \prod_{x;\alpha,\gamma,\delta} dB_\alpha{}^{\gamma\delta}(x) \int \prod_{x;\eta,\zeta} d\omega_{\eta\zeta}^*(x) \int \prod_{x;\iota,\kappa} d\omega^{\iota\kappa}(x) \int \prod_{x;\rho,\sigma} dh^{\rho\sigma}(x) \times \mathcal{O}[B] \exp i \{S_{\text{NEW}} + S_{\text{M}} + \varepsilon\text{-terms}\} \quad (54)$$

if the action $S_{\text{M}}[\psi, B]$ for a matter field ψ is gauge-invariant. We note that all the integration measures over field space are BRST invariant as demonstrated in appendix C.

5. Zinn-Justin equation

In this section we derive a fundamental property of the theory, the Zinn-Justin equation for the QEA related to the connected vacuum persistence amplitude $\mathcal{W}[J, K]$ in the presence of external currents J and K for the fundamental fields χ^n and their BRST variations $s\chi^n$ respectively [3].

Let us introduce the shorthand notation χ^n for the fundamental fields

$$\chi^n \sim \psi, B_\alpha, \omega, \omega^*, h \quad (55)$$

The BRST transformations in this notation read

$$\begin{aligned} \chi^n(x) &\rightarrow \chi^{n'}(x) = \chi^n(x) + \delta_\theta \chi^n[\chi^t; x] \\ \delta_\theta \chi^n[\chi^t; x] &= \theta s\chi^n[\chi^t; x] \equiv \theta \Delta^n[\chi^t; x] \end{aligned} \quad (56)$$

with

$$\begin{aligned} \Delta^\psi &= \frac{i}{2} \omega^{\gamma\delta} \vec{L}_{\gamma\delta} \psi + \frac{i}{2} \omega^{\gamma\delta} \Sigma_{\gamma\delta} \psi \\ \Delta_\alpha^{B\gamma\delta} &= \frac{i}{2} \omega^{\eta\zeta} \vec{L}_{\eta\zeta} B_\alpha{}^{\gamma\delta} - \partial_\alpha \omega^{\gamma\delta} - \frac{i}{2} B_\alpha{}^{\eta\zeta} \vec{L}_{\eta\zeta} \omega^{\gamma\delta} \\ &\quad + \frac{i}{2} \omega^{\eta\zeta} (\Sigma_{\eta\zeta}^V)_\alpha{}^\beta B_\beta{}^{\gamma\delta} + \frac{1}{2} C^{\gamma\delta}{}_{\iota\kappa}{}_{\eta\zeta} B_\alpha{}^{\iota\kappa} \omega^{\eta\zeta} \\ \Delta^{\omega\gamma\delta} &= \frac{i}{2} \omega^{\eta\zeta} \vec{L}_{\eta\zeta} \omega^{\gamma\delta} - \frac{1}{4} C^{\gamma\delta}{}_{\alpha\beta}{}_{\eta\zeta} \omega^{\alpha\beta} \omega^{\eta\zeta} \\ \Delta^{\omega^*}_{\gamma\delta} &= -h_{\gamma\delta} \\ \Delta^h_{\gamma\delta} &= 0. \end{aligned} \quad (57)$$

As demonstrated above we have

$$\begin{aligned} S_{\text{TOT}}[\chi^{n'}] &= S_{\text{TOT}}[\chi^n + \theta \Delta^n[\chi^t]] \\ &= S_{\text{TOT}}[\chi^n] + \delta_\theta S_{\text{TOT}}[\chi^n] \theta \Delta^n[\chi^t] = S_{\text{TOT}}[\chi^n] \end{aligned} \quad (58)$$

with $S_{\text{TOT}} = S_{\text{NEW}} + S_{\text{M}}$. In addition we have

$$\prod_{x;n} d(\chi^n(x) + \delta_\theta \chi^n[\chi^t; x]) = \prod_{x;m} d\chi^m(x) \mathcal{J} \quad (59)$$

with the Berezinian

$$\mathcal{J} = \text{Det} \left(\frac{\delta \chi^{n'}}{\delta \chi^m} \right) = 1 + \text{Tr} \log \left(\frac{\delta \chi^{n'}}{\delta \chi^m} \right) = 1 \quad (60)$$

being trivial as demonstrated in appendix C.

Next we introduce the connected vacuum persistence amplitude $\mathcal{W}[J, K]$ in the presence of external currents J and K for the fundamental fields χ^n and their BRST variations $s\chi^n$ respectively

$$\begin{aligned} \mathcal{Z}[J, K] &\equiv \exp i \mathcal{W}[J, K] \equiv \int \prod_{x;n} d\chi^n(x) \\ &\times \exp i \left\{ S_{\text{TOT}} + S_M + \int d^4x \Delta^n K_n + \int d^4x \chi^n J_n + \varepsilon\text{-terms} \right\}. \end{aligned} \quad (61)$$

This allows us to derive a condition on the QEA

$$\Gamma[\chi, K] \equiv \mathcal{W}[J_{\chi,K}, K] - \int \chi^n J_{n,\chi,K} \quad (62)$$

belonging to the connected vacuum persistence amplitude $\mathcal{W}[J, K]$.

Note that for $K = 0$ the functional $\mathcal{Z}[J, 0]$ reduces to the usual generating functional for the Green functions of the interacting theory which are equal to the vacuum expectation values of time-ordered products of interacting field operators from which the S -matrix is derived via the LSZ approach. Also, $\Gamma[\chi, 0]$ is the usual QEA which contains all connected one-particle irreducible graphs of the interacting theory in the presence of the current $J_{\chi,0}$.

The condition referred to above, a Slavnov–Taylor identity, follows from the BRST invariance of $\mathcal{W}[0, 0]$ for vanishing currents J, K which is easy to demonstrate on the basis of equations (58) and (59). To derive the Slavnov–Taylor identity we calculate

$$\begin{aligned} \mathcal{Z}[J, K] &= \int \prod_{x;n} d(\chi^n + \theta \Delta^n[\chi])(x) \\ &\times \exp i \left\{ S_{\text{TOT}}[\chi^n + \theta \Delta^n[\chi]] + \int d^4x \Delta^n[\chi^m + \theta \Delta^m[\chi]] K_n \right. \\ &\quad \left. + \int d^4x (\chi^n + \theta \Delta^n[\chi]) J_n \right\} \\ &= \mathcal{Z}[J, K] + i\theta \int \prod_{x;n} d\chi^n(x) \left(\int d^4y \Delta^m[\chi'; y] J_m(y) \right) \\ &\times \exp i \left\{ S_{\text{TOT}} + \int d^4x \Delta^n K_n + \int d^4x \chi^n J_n \right\}, \end{aligned} \quad (63)$$

where we have taken into account the nilpotence of the BRST variations.

Defining the quantum average

$$\begin{aligned} \langle \Delta^n[\chi'; y] \rangle_{J_{\chi,K}, K} &= \int \prod_{x;n} d\chi^n(x) \Delta^m[\chi'; y] \\ &\times \exp i \left\{ S_{\text{TOT}} + \int d^4x \Delta^n K_n + \int d^4x \chi^n J_n \right\}, \end{aligned} \quad (64)$$

in the presence of currents J and K we get

$$\int d^4x \langle \Delta^n [\chi^t; x] \rangle_{J_{\chi,K}, K} J_n(x) = 0. \quad (65)$$

Noting that

$$\frac{\delta_L \Gamma [\chi, K]}{\delta \chi^n(x)} = -J_{n, \chi, K}(x) \quad (66)$$

we can recast equation (65) in the more perspicuous form

$$\int d^4x \langle \Delta^n [\chi^t; x] \rangle_{J_{\chi,K}, K} \frac{\delta_L \Gamma [\chi, K]}{\delta \chi^n(x)} = 0. \quad (67)$$

In other words $\Gamma [\chi, K]$ is invariant under the infinitesimal transformations

$$\chi^n(x) \longrightarrow \chi^n(x) + \theta \langle \Delta^n [\chi^t; x] \rangle_{J_{\chi,K}, K} \quad (68)$$

establishing a Slavnov–Taylor identity which is the basis for the Zinn–Justin equation we derive next.

Noting that

$$\frac{\delta_R \Gamma [\chi, K]}{\delta K_n(x)} = \langle \Delta^n [\chi^t; x] \rangle_{J_{\chi,K}, K}, \quad (69)$$

where we have introduced the left and right derivatives δ_L and δ_R respectively taking the (anti-)commuting properties of the various fields into proper account, equation (67) can finally be rewritten as the Zinn–Justin equation

$$\int d^4x \frac{\delta_R \Gamma [\chi, K]}{\delta K_n(x)} \frac{\delta_L \Gamma [\chi, K]}{\delta \chi^n(x)} = 0. \quad (70)$$

Defining the antibracket of two functionals $F [\chi, K]$ and $G [\chi, K]$ w.r.t. to the fields χ^n and the currents K_n

$$(F, G) \equiv \int d^4x \left\{ \frac{\delta_R F [\chi, K]}{\delta \chi^n(x)} \frac{\delta_L G [\chi, K]}{\delta K_n(x)} - \frac{\delta_R F [\chi, K]}{\delta K_n(x)} \frac{\delta_L G [\chi, K]}{\delta \chi^n(x)} \right\} \quad (71)$$

we can recast the Zinn–Justin equation in its final form as

$$(\Gamma, \Gamma) = 0 \quad (72)$$

which is the starting point for the renormalizability proof for our theory in the next and final section of the paper.

6. Perturbative renormalizability of the QEA $\Gamma [\chi, K]$

In this section we prove the renormalizability of our theory closely following the approach outlined in [3]. First, we use renormalizability in the Dyson sense to derive the explicit K -dependence of $\Gamma_{N,\infty} [\chi, K]$ which contains the infinite contributions of order N to the loop expansion of the effective action $\Gamma [\chi, K] = \sum_{N=0}^{\infty} \Gamma_N [\chi, K]$. Second, evaluating the Zinn–Justin equation we find the combination $\Delta_N^{(\varepsilon)n}(x) \equiv \Delta^n(x) + \varepsilon \mathcal{D}_N^n(x)$ to be nilpotent with \mathcal{D}_N^n properly defined below, and the combination $\Gamma_N^{(\varepsilon)} [\chi] \equiv S_R [\chi] + \varepsilon \Gamma_{N,\infty} [\chi, 0]$ to be invariant

under the renormalized BRST field variations $\delta_{\theta(\epsilon)}\chi^n(x) = \theta \Delta_N^{(\epsilon)n}(x)$. This will finally allow us to prove the renormalizability of our theory.

6.1. K -dependence of $\Gamma_{N,\infty}[\chi, K]$

In this subsection we use the renormalizability of our theory in the Dyson sense to derive the explicit K -dependence of $\Gamma_{N,\infty}[\chi, K]$ which contains the infinite contributions of order N to the loop expansion of the effective action $\Gamma[\chi, K] = \sum_{N=0}^{\infty} \Gamma_N[\chi, K]$.

We start by noting that $S_{\text{TOT}}[\chi]$ is by construction a sum of integrals over Lagrangians of dimension four or less expressed in the fundamental fields χ^n —in fact it is the most general BRST-invariant action of dimension four or less in those fields. As a consequence power-counting allows to show that the corresponding QEA of the quantized theory only contains divergent contributions of dimension four or less in those fields—or is renormalizable in the Dyson sense [5]. They then can be cancelled by counterterms of dimensionality four or less.

However, there is more to full renormalizability. The action used in the path integral or canonical quantization of our gauge field theory is constrained by BRST invariance—in fact it is the most general BRST-invariant action of dimension four or less in all the dynamical fields. For the quantum theory to be renormalizable, i.e. all infinities to be absorbable solely by field renormalizations and coupling redefinitions, the infinite contributions to the QEA and the counterterms needed to cancel them have to satisfy the same BRST constraints up to such renormalizations of fields and couplings—which guarantees that the counterterms must be of the same form as the terms in the original action. In other words BRST invariance and the resulting Zinn-Justin equation are enough of an algebraic ‘straightjacket’ to assure renormalizability.

The first of a sequence of steps to prove full renormalizability is to determine the K -dependence of the infinite contributions to the effective action $\Gamma[\chi, K]$ deploying dimensional analysis.

Based on the Dyson renormalizability we can rewrite the action $S[\chi, K]$ in the presence of sources K as

$$\begin{aligned} S[\chi, K] &= S_{\text{TOT}}[\chi] + \int \left\{ \Delta^\psi K_\psi + \Delta_\alpha^B \gamma^\delta K_B^\alpha \gamma_\delta + \Delta^\omega \gamma^\delta K_\omega \gamma_\delta + \Delta^{\omega^*} \gamma_\delta K_{\omega^*} \gamma^\delta \right\} \\ &= S_R[\chi, K] + S_\infty[\chi, K], \end{aligned} \tag{73}$$

where masses and coupling constants in $S_R[\chi, K]$ are set to their renormalized values plus the correction $S_\infty[\chi, K]$ containing all the counterterms needed to cancel infinities from loop graphs in the perturbative loop expansion of the effective action $\Gamma[\chi, K]$

$$\Gamma[\chi, K] = \sum_{N=0}^{\infty} \Gamma_N[\chi, K]. \tag{74}$$

Above Γ_N contains all the diagrams with N loops, plus contributions from graphs with $N-M$ loops, $1 \leq M \leq N$, involving the counterterms in S_∞ introduced to cancel infinities in graphs with M loops. Note that no source term $K_h \gamma^\delta$ for $\Delta^h \gamma_\delta = 0$ appears.

The power-counting rules of renormalization theory imply that after all infinities in subgraphs of Γ_N have been cancelled the infinite part $\Gamma_{N,\infty}[\chi, K]$ of $\Gamma_N[\chi, K]$ must be an integral over a sum of local products of fields χ, K and their derivatives of dimension four or less [5].

Now it is possible to determine the K -dependence of the infinite contributions to the effective action $\Gamma[\chi, K]$. To that end we first establish the dimensions of the various fields. If the fields χ^n have dimensionality $[\chi^n] \equiv d_n$ then inspection of equation (57) shows that the dimensionality of Δ^n is $[\Delta^n] = d_n + 1$ and K_n has dimensionality $[K_n] = 4 - [\Delta^n] = 3 - d_n$.

So we find the dimensionalities for the various fields to be

$$\begin{aligned} [B] &= [\omega] = [\omega^*] = 1, \\ [h] &= [K_B] = [K_\omega] = [K_{\omega^*}] = 2, \\ [\psi] &= 3/2, [K_\psi] = 3/2, \end{aligned} \tag{75}$$

where we assume the matter field ψ to be a spin- $\frac{1}{2}$ Dirac fermion. The dimension four quantity $\Gamma_{N,\infty}[\chi, K]$ can then be at most quadratic in any of the K_n , and terms quadratic in any of the K_n cannot involve any other fields with the exception of a term quadratic in K_ψ which may contain one additional field of dimension one.

Using ghost number conservation we next demonstrate $\Gamma_{N,\infty}[\chi, K]$ to be at most linear in K_n . If the fields χ^n have ghost number $|\chi^n| \equiv g_n$ then inspection of equation (57) shows that the ghost number of Δ^n is $|\Delta^n| = g_n + 1$ and K_n has ghost number $|K_n| = -|\Delta^n| = -g_n - 1$.

So we find the ghost numbers for the various fields to be

$$\begin{aligned} |B| &= |\psi| = |h| = 0, \quad |K_B| = |K_\psi| = -1, \\ |\omega| &= 1, \quad |K_\omega| = -2, \\ |\omega^*| &= -1, \quad |K_{\omega^*}| = 0. \end{aligned} \tag{76}$$

This rules out all potential contributions to $\Gamma_{N,\infty}[\chi, K]$ of second order in K_n with the exception of a potential term of second order in K_{ω^*} . Now as

$$\frac{\delta_R \Gamma[\chi, K]}{\delta K_{\omega^*}{}^{\gamma\delta}} = \left\langle \Delta^{\omega^*}{}_{\gamma\delta} \right\rangle_{J_{\chi,K,K}} = -\langle h_{\gamma\delta} \rangle_{J_{\chi,K,K}} = -h_{\gamma\delta} \tag{77}$$

is independent of K_{ω^*} the effective action $\Gamma[\chi, K]$ is linear in K_{ω^*} through a term $-\int K_{\omega^*}{}^{\gamma\delta} h_{\gamma\delta}$ and the $\Gamma_{N,\infty}[\chi, K]$ are independent of K_{ω^*} for $N \geq 1$.

Note that the last equality in equation (77) above follows from the fact that for transformations

$$\chi^n(x) \rightarrow \chi^n(x) + \varepsilon F^n[\chi^m; x] \tag{78}$$

which are linear in the fields

$$F^n[\chi^m; x] = s^n(x) + \int t^n{}_m(x, y) \chi^m(y) \tag{79}$$

with ε infinitesimal and s and t field-independent, the quantum average of the field variation $\langle F^n[\chi^m] \rangle_{J_{\chi,K,K}}$ equals its classical value $F^n[\chi^m]$ as is easily shown in a calculation analogous to the one in equation (63).

In fact, if the effective action $\Gamma[\chi, K]$ is invariant under a variation with a general $\langle F^n[\chi^m] \rangle$

$$\delta_\varepsilon \Gamma = \int \langle F^n[\chi^m] \rangle_{J_{\chi,K,K}} \frac{\delta \Gamma[\chi, K]}{\delta \chi^n} = \int \langle F^n[\chi^m] \rangle_{J_{\chi,K,K}} J_n = 0 \tag{80}$$

then for linear transformations equation (79) we have

$$\begin{aligned} \langle F^n[\chi^m] \rangle_{J_{\chi,K,K}} &= s^n + \int t^n{}_m \langle \chi^m \rangle_{J_{\chi,K,K}} \\ &= s^n + \int t^n{}_m \chi^m = F^n[\chi^m] \end{aligned} \tag{81}$$

so the invariance becomes

$$\delta_\epsilon \Gamma = \int F^n [\chi^m] \frac{\delta \Gamma[\chi, K]}{\delta \chi^n} = 0 \tag{82}$$

and the full QEA is invariant under the same linear transformation under which the classical action is, assuming the integration measure is invariant as well.

Hence, finally we find the desired K -dependence of $\Gamma_{N,\infty}[\chi, K]$ to be

$$\Gamma_{N,\infty}[\chi, K] = \Gamma_{N,\infty}[\chi, 0] + \int d^4x \mathcal{D}_N^n[\chi; x] K_n(x) \tag{83}$$

defining $\mathcal{D}_N^n[\chi; x]$ in the process which is a functional of the fields χ^n only.

6.2. Invariance of $\Gamma_N^{(\epsilon)}[\chi]$ under nilpotent transformations $\Delta_N^{(\epsilon)n}$

In this subsection we evaluate the Zinn-Justin equation perturbatively allowing us to demonstrate the combination $\Gamma_N^{(\epsilon)}[\chi] \equiv S_R[\chi] + \epsilon \Gamma_{N,\infty}[\chi, 0]$ to be invariant under nilpotent renormalized BRST field variations $\delta_{\theta(\epsilon)} \chi^n(x) = \theta \Delta_N^{(\epsilon)n}(x)$ with $\Delta_N^{(\epsilon)n}(x) \equiv \Delta^n(x) + \epsilon \mathcal{D}_N^n(x)$ and ϵ infinitesimal.

Taking the Zinn-Justin equation $(\Gamma, \Gamma) = 0$ and inserting the perturbative expansion $\Gamma[\chi, K] = \sum_{N=0}^\infty \Gamma_N[\chi, K]$ we get for fixed N

$$\sum_{M=0}^N (\Gamma_M, \Gamma_{N-M}) = 0. \tag{84}$$

The leading term in the expansion equation (74) is

$$\Gamma_0[\chi, K] = S_R[\chi, K] \tag{85}$$

which is finite. Supposing that all infinities from loops for $M \leq N - 1$ have been absorbed by respective counterterms in $S_\infty[\chi, K]$ new infinities can only appear in the $M = 0$ and $M = N$ terms which are equal. Now the infinite part of the condition equation (84) is

$$(S_R, \Gamma_{N,\infty}) = 0. \tag{86}$$

Recalling that $S_R[\chi, K] = S_R[\chi] + \int \Delta^n[\chi; x] K_n(x)$ and inserting this together with $\Gamma_{N,\infty}[\chi, K] = \Gamma_{N,\infty}[\chi, 0] + \int \mathcal{D}_N^n[\chi; x] K_n(x)$ into the infinite part of the N th order contribution to the Zinn-Justin equation above we get to zeroth order in K

$$\int d^4x \left\{ \Delta^n[\chi; x] \frac{\delta_L \Gamma_{N,\infty}[\chi, 0]}{\delta \chi^n(x)} + \mathcal{D}_N^n[\chi; x] \frac{\delta_L S_R[\chi]}{\delta \chi^n(x)} \right\} = 0 \tag{87}$$

whilst terms linear in K yield

$$\int d^4y \left\{ \Delta^n[\chi; x] \frac{\delta_L \mathcal{D}_N^m[\chi; y]}{\delta \chi^n(x)} + \mathcal{D}_N^n[\chi; x] \frac{\delta_L \Delta^m[\chi; y]}{\delta \chi^n(x)} \right\} = 0. \tag{88}$$

To bring these two results into their most perspicuous form we define the N th order contribution to the corrected QEA

$$\Gamma_N^{(\epsilon)}[\chi] \equiv S_R[\chi] + \epsilon \Gamma_{N,\infty}[\chi, 0] \tag{89}$$

and

$$\Delta_N^{(\varepsilon)n}[\chi; x] \equiv \Delta^n[\chi; x] + \varepsilon \mathcal{D}_N^n[\chi; x] \quad (90)$$

with ε infinitesimal. Then equation (87) in conjunction with the BRST invariance of S_R tells us that to leading order in ε the expression $\Gamma_N^{(\varepsilon)}[\chi]$ is invariant under the field variations $\delta_{\theta(\varepsilon)}\chi^n(x)$

$$\chi^n(x) \longrightarrow \chi^n(x) + \theta \Delta_N^{(\varepsilon)n}[\chi; x] \quad (91)$$

or

$$\delta_{\theta(\varepsilon)}\Gamma_N^{(\varepsilon)}[\chi] = \int d^4x \Delta_N^{(\varepsilon)n}[\chi; x] \frac{\delta_L \Gamma_N^{(\varepsilon)}[\chi]}{\delta \chi^n(x)} = 0. \quad (92)$$

In addition equation (88) in conjunction with the BRST invariance of $\Delta^n[\chi; x]$ tells us that to leading order in ε the variations $\Delta_N^{(\varepsilon)n}[\chi; x]$ are nilpotent

$$\delta_{\theta(\varepsilon)}\Delta_N^{(\varepsilon)n}(x) = 0. \quad (93)$$

6.3. Nilpotence forcing the $\Delta_N^{(\varepsilon)n}$ to be renormalized BRST transformations

In this subsection we determine the most general form of the nilpotent field variations $\Delta_N^{(\varepsilon)n}[\chi; x]$.

Noting that $\Gamma_N^{(\varepsilon)}[\chi]$ is of dimensionality four or less, $\mathcal{D}_N^n[\chi; x]$ and hence $\Delta_N^{(\varepsilon)n}[\chi; x]$ have at most the dimension of the original BRST transformations $\Delta^n[\chi; x]$. In addition the $\mathcal{D}_N^n[\chi; x]$ must share their Lorentz transformation behaviours with those of the $\Delta^n[\chi; x]$.

The most general renormalized nilpotent BRST variations are then found to be for the Dirac field

$$\delta_{\theta(\varepsilon)}\psi = \theta \Delta_N^{(\varepsilon)\psi} = \frac{i}{2}\theta \omega^{\gamma\delta} \mathcal{Z}_N^{(\varepsilon)}(\vec{L}_{\gamma\delta} \psi) + \frac{i}{2}\theta \omega^{\gamma\delta} \mathcal{Z}_N^{(\varepsilon)} \Sigma_{\gamma\delta} \psi, \quad (94)$$

for the gauge field

$$\begin{aligned} \delta_{\theta(\varepsilon)}B_\alpha{}^{\gamma\delta} &= \theta \Delta_N^{(\varepsilon)B\gamma\delta} = \frac{i}{2}\theta \omega^{\eta\zeta} \mathcal{Z}_N^{(\varepsilon)}(\vec{L}_{\eta\zeta} B_\alpha{}^{\gamma\delta}) - \theta \mathcal{Z}_N^{(\varepsilon)} \mathcal{N}_N^{(\varepsilon)} \partial_\alpha \omega^{\gamma\delta} \\ &\quad - \frac{i}{2}\theta B_\alpha{}^{\eta\zeta} \mathcal{Z}_N^{(\varepsilon)}(\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) + \frac{i}{2}\theta \omega^{\eta\zeta} \mathcal{Z}_N^{(\varepsilon)}(\Sigma_{\eta\zeta}^V)_\alpha{}^\beta B_\beta{}^{\gamma\delta} \\ &\quad + \frac{1}{2}\theta \mathcal{Z}_N^{(\varepsilon)} C^{\gamma\delta}{}_{\nu\kappa} B_\alpha{}^{\nu\kappa} \omega^{\eta\zeta} \end{aligned} \quad (95)$$

and for the ghost field

$$\delta_{\theta(\varepsilon)}\omega^{\gamma\delta} = \theta \Delta_N^{(\varepsilon)\omega\gamma\delta} = \frac{i}{2}\theta \omega^{\eta\zeta} \mathcal{Z}_N^{(\varepsilon)}(\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) - \frac{1}{4}\theta \mathcal{Z}_N^{(\varepsilon)} C^{\gamma\delta}{}_{\alpha\beta} \omega^{\alpha\beta} \omega^{\eta\zeta}. \quad (96)$$

In comparison to the original BRST variations $\Delta^n[\chi; x]$ we find the various generators of the Lorentz algebra to be renormalized by a factor $\mathcal{Z}_N^{(\varepsilon)}$ whilst the derivative term in the gauge field transformation picks up a separate factor $\mathcal{Z}_N^{(\varepsilon)} \mathcal{N}_N^{(\varepsilon)}$. The renormalized BRST variations above are easily shown to be nilpotent by repetition of the calculations in appendix A.

Noting that both the BRST transformations for the antighost and Nakanishi–Lautrup fields are linear we get the original BRST variations back for the antighost field

$$\delta_{\theta(\varepsilon)}\omega_{\gamma\delta}^* = \theta \Delta_N^{(\varepsilon)\omega_{\gamma\delta}^*} = -\theta h_{\gamma\delta} \quad (97)$$

and for the Nakanishi–Lautrup field

$$\delta_{\theta^{(\varepsilon)}} h_{\gamma\delta} = \theta \Delta_N^{(\varepsilon)h} = 0. \tag{98}$$

6.4. Renormalized BRST invariance forcing $\Gamma_N^{(\varepsilon)}[\chi]$ to be of the form of the original action S_{TOT} up to possible field renormalizations and coupling redefinitions

In this subsection we determine the most general form of the renormalized action $\Gamma_N^{(\varepsilon)}[\chi]$ invariant (A) under all the linear symmetry operations under which the original action is invariant and (B) under the renormalized BRST variations $\Delta_N^{(\varepsilon)n}[\chi; x]$ determined in the preceding subsection.

Let us turn to the final step in our renormalization proof: the demonstration that the most general form of the renormalized action invariant under (i) all the linear symmetry operations under which the original action is invariant as well as under (ii) the renormalized BRST variations $\Delta_N^{(\varepsilon)n}[\chi; x]$ is of the form of our original BRST-invariant action S_{TOT} up to potential field and coupling constant renormalizations.

We start with the N th order contribution to the corrected renormalized action $\Gamma_N^{(\varepsilon)}[\chi] \equiv S_R[\chi] + \varepsilon \Gamma_{N,\infty}[\chi, 0]$ which contains the original renormalized action plus the infinite part of the N -loop contributions to the QEA. According to the general rules of renormalization theory it must be the integral over local terms in the dynamical fields and their derivatives of dimensionality equal or less than four [5]

$$\Gamma_N^{(\varepsilon)}[\chi] = \int \mathcal{L}_N^{(\varepsilon)}[\chi]. \tag{99}$$

The expression $\Gamma_N^{(\varepsilon)}[\chi]$ is invariant under all linearly realized symmetries of the original action as argued above. To identify them we explicitly write down the original action in the Lorentz gauge as given by equations (42), (30), (31), (8) and (44)

$$\begin{aligned} S_{\text{TOT}} &= S_M + S_{\text{NEW}} = S_M + S_G \\ &+ \int \omega_{\gamma\delta}^* \partial^\alpha \left(\frac{i}{2} \omega^{\eta\zeta} \vec{L}_{\eta\zeta} B_{\alpha\ \gamma\delta} - \partial_\alpha \omega^{\gamma\delta} \right. \\ &\left. - \frac{i}{2} B_{\alpha\ \eta\zeta} \vec{L}_{\eta\zeta} \omega^{\gamma\delta} + \frac{1}{2} C^{\gamma\delta\ \iota\kappa\ \eta\zeta} B_{\alpha\ \iota\kappa} \omega^{\eta\zeta} \right) \\ &+ \int h_{\gamma\delta} \partial^\alpha B_{\alpha\ \gamma\delta} + \frac{\xi}{2} \int h_{\gamma\delta} h^{\gamma\delta}. \end{aligned} \tag{100}$$

By inspection S_{TOT} is invariant under all the linearly realized symmetry operations which in particular are (A) global Lorentz transformations equalling global gauge transformations parametrized by the constant infinitesimal gauge parameter ρ .

Under the latter the Dirac field living in the spin- $\frac{1}{2}$ representation varies as

$$\delta_\rho \psi = \frac{i}{2} \rho^{\gamma\delta} (\vec{L}_{\gamma\delta} \psi) + \frac{i}{2} \rho^{\gamma\delta} \Sigma_{\gamma\delta} \psi, \tag{101}$$

the gauge field living in the vector-cum-adjoint representation varies as

$$\delta_\rho B_{\alpha\ \gamma\delta} = \frac{i}{2} \rho^{\eta\zeta} (\vec{L}_{\eta\zeta} B_{\alpha\ \gamma\delta}) + \frac{i}{2} \rho^{\eta\zeta} (\Sigma_{\eta\zeta}^V)_\alpha^\beta B_{\beta\ \gamma\delta} + \frac{1}{2} C^{\gamma\delta\ \iota\kappa\ \eta\zeta} B_{\alpha\ \iota\kappa} \rho^{\eta\zeta}, \tag{102}$$

the ghost field living in the adjoint representation varies as

$$\delta_\rho \omega^{\gamma\delta} = \frac{i}{2} \rho^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) + \frac{1}{2} C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} \omega^{\alpha\beta} \rho^{\eta\zeta}, \quad (103)$$

the antighost field living in the adjoint representation varies as

$$\delta_\rho \omega_{\gamma\delta}^* = \frac{i}{2} \rho^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega_{\gamma\delta}^*) + \frac{1}{2} C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} \omega_{\alpha\beta}^* \rho^{\eta\zeta} \quad (104)$$

and the Nakanishi–Lautrup field living in the adjoint representation varies as

$$\delta_\rho h_{\gamma\delta} = \frac{i}{2} \rho^{\eta\zeta} (\vec{L}_{\eta\zeta} h_{\gamma\delta}) + \frac{1}{2} C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} h_{\alpha\beta} \rho^{\eta\zeta}. \quad (105)$$

In addition S_{TOT} is invariant under (B) the linearly realized antighost translations

$$\omega_{\gamma\delta}^* \rightarrow \omega_{\gamma\delta}^* + c_{\gamma\delta} \quad (106)$$

with $c_{\gamma\delta}$ an arbitrary constant antisymmetric Lorentz tensor—which is a particular feature of the Lorentz gauge condition, and is subject to (C) ghost number conservation. The invariance under (B) is obvious as the $c_{\gamma\delta}$ -term adds nothing but a total divergence.

Next we turn to determine the most general form of the renormalized action invariant under all the linear symmetry operations (A) to (C) under which the original action is invariant. Recalling the dimensionalities and ghost numbers of the various fields from subsection 6.1 we first note that ghost number conservation requires that ω and ω^* come in pairs, and antighost invariance that ω^* comes together with a derivative $\omega^* \partial$ which we can always shuffle to the left of any other expression in the fields. Altogether any pair of ω and $\omega^* \partial$ carries dimension three, so there cannot be more than one such pair in $\Gamma_N^{(\varepsilon)}[\chi]$ and such a pair can come with only one more derivative ∂ or gauge field B . As a result the only remaining possibilities respecting the invariances under (A) above are linear combinations of $\int \omega_{\gamma\delta}^* \partial^\alpha (\omega^{\eta\zeta} \vec{L}_{\eta\zeta} B_{\alpha}{}^{\gamma\delta})$, $\int \omega_{\gamma\delta}^* \partial^\alpha (\partial_\alpha \omega^{\gamma\delta})$, $\int \omega_{\gamma\delta}^* \partial^\alpha (B_{\alpha}{}^{\eta\zeta} \vec{L}_{\eta\zeta} \omega^{\gamma\delta})$, $\int \omega_{\gamma\delta}^* \partial^\alpha (\omega^{\eta\zeta} (\Sigma_{\eta\zeta}^V)_{\alpha}{}^{\beta} B_{\beta}{}^{\gamma\delta})$ and $\int \omega_{\gamma\delta}^* \partial^\alpha (C^{\gamma\delta}{}_{\mu\kappa\eta\zeta} B_{\alpha}{}^{\mu\kappa} \omega^{\eta\zeta})$.

We turn to terms containing h and other fields but neither ω nor ω^* . As h has dimension two the condition (A) only allow for linear combinations of $\int h_{\gamma\delta} \partial^\alpha B_{\alpha}{}^{\gamma\delta}$, $\int h^{\gamma\delta} (B_{\alpha}{}^{\eta\zeta} \vec{L}_{\eta\zeta} B^{\alpha}{}_{\gamma\delta})$, $\int h^{\gamma\delta} (B_{\alpha}{}^{\eta\zeta} (\Sigma_{\eta\zeta}^V)_{\alpha}{}^{\beta} B_{\beta}{}^{\gamma\delta})$, $\int h^{\gamma\delta} (C^{\gamma\delta}{}_{\mu\kappa\eta\zeta} B_{\alpha}{}^{\mu\kappa} B^{\alpha}{}_{\eta\zeta})$ and $\int h_{\gamma\delta} h^{\gamma\delta}$.

Finally $\Gamma_N^{(\varepsilon)}[\chi]$ contains terms involving gauge and matter fields only of dimension four or less which we collect in the expression $S_{B,\psi}[\chi]$.

As a result the most general form of the renormalized action invariant under all the linear symmetry operations under which the original action is invariant—most notably under the global gauge transformations equations (101)–(105)—takes the form

$$\begin{aligned} \Gamma_N^{(\varepsilon)}[\chi] &= S_R[\chi] + \varepsilon \Gamma_{N,\infty}[\chi, 0] = S_{B,\psi}[\chi] \\ &+ Z_\omega^{N(\varepsilon)} \int \omega_{\gamma\delta}^* \partial^\alpha \left(\frac{i}{2} a_1^{N(\varepsilon)} \omega^{\eta\zeta} \vec{L}_{\eta\zeta} B_{\alpha}{}^{\gamma\delta} + \partial_\alpha \omega^{\gamma\delta} \right. \\ &+ \frac{i}{2} a_2^{N(\varepsilon)} B_{\alpha}{}^{\eta\zeta} \vec{L}_{\eta\zeta} \omega^{\gamma\delta} + \frac{i}{2} a_3^{N(\varepsilon)} \omega^{\eta\zeta} (\Sigma_{\eta\zeta}^V)_{\alpha}{}^{\beta} B_{\beta}{}^{\gamma\delta} \\ &\left. + \frac{1}{2} a_4^{N(\varepsilon)} C^{\gamma\delta}{}_{\mu\kappa\eta\zeta} B_{\alpha}{}^{\mu\kappa} \omega^{\eta\zeta} \right) \end{aligned}$$

$$\begin{aligned}
 &+ b_1^{N(\varepsilon)} \int h_{\gamma\delta} \left(\partial^\alpha B_\alpha{}^{\gamma\delta} + \frac{i}{2} b_2^{N(\varepsilon)} B_\alpha{}^{\eta\zeta} (\vec{L}_{\eta\zeta} B^\alpha{}_{\gamma\delta}) \right. \\
 &+ \frac{i}{2} b_3^{N(\varepsilon)} B_\alpha{}^{\eta\zeta} (\Sigma_{\eta\zeta}^V)^\alpha{}_\beta B^\beta{}_{\gamma\delta} + \frac{1}{2} b_4^{N(\varepsilon)} C^{\gamma\delta}{}_{\mu\kappa}{}_{\eta\zeta} B^\alpha{}_{\mu\kappa} B^\alpha{}_{\eta\zeta} \left. \right) \\
 &+ \frac{\xi_N^{(\varepsilon)}}{2} \int h_{\gamma\delta} h^{\gamma\delta}
 \end{aligned} \tag{107}$$

with $Z_\omega^{N(\varepsilon)}, a_1^{N(\varepsilon)}, a_2^{N(\varepsilon)}, a_3^{N(\varepsilon)}, a_4^{N(\varepsilon)}, b_1^{N(\varepsilon)}, b_2^{N(\varepsilon)}, b_3^{N(\varepsilon)}, b_4^{N(\varepsilon)}$ and $\xi_N^{(\varepsilon)}$ unknown constants.
 Imposing invariance under the renormalized BRST variations equations (94)–(98)

$$\begin{aligned}
 \delta_{\theta^{(\varepsilon)}} \Gamma_N^{(\varepsilon)}[\chi] &= \delta_{\theta^{(\varepsilon)}} S_{B,\psi}[\chi] \\
 &- Z_\omega^{N(\varepsilon)} \theta \int h_{\gamma\delta} \partial^\alpha \left(\frac{i}{2} a_1^{N(\varepsilon)} \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} B_\alpha{}^{\gamma\delta}) + \partial_\alpha \omega^{\gamma\delta} \right. \\
 &+ \frac{i}{2} a_2^{N(\varepsilon)} B_\alpha{}^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) + \frac{i}{2} a_3^{N(\varepsilon)} \omega^{\eta\zeta} (\Sigma_{\eta\zeta}^V)^\alpha{}_\beta B^\beta{}_{\gamma\delta} \\
 &+ \left. \frac{1}{2} a_4^{N(\varepsilon)} C^{\gamma\delta}{}_{\mu\kappa}{}_{\eta\zeta} B_\alpha{}^{\mu\kappa} \omega^{\eta\zeta} \right) \\
 &+ Z_\omega^{N(\varepsilon)} \int \omega_{\gamma\delta}^* \delta_{\theta^{(\varepsilon)}} \vec{\partial}^\alpha \left(\frac{i}{2} a_1^{N(\varepsilon)} \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} B_\alpha{}^{\gamma\delta}) + \partial_\alpha \omega^{\gamma\delta} \right. \\
 &+ \frac{i}{2} a_2^{N(\varepsilon)} B_\alpha{}^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) + \frac{i}{2} a_3^{N(\varepsilon)} \omega^{\eta\zeta} (\Sigma_{\eta\zeta}^V)^\alpha{}_\beta B^\beta{}_{\gamma\delta} \\
 &+ \left. \frac{1}{2} a_4^{N(\varepsilon)} C^{\gamma\delta}{}_{\mu\kappa}{}_{\eta\zeta} B_\alpha{}^{\mu\kappa} \omega^{\eta\zeta} \right) \\
 &+ b_1^{N(\varepsilon)} \theta \int h_{\gamma\delta} \partial^\alpha \left(\frac{i}{2} \omega^{\eta\zeta} \mathcal{Z}_N^{(\varepsilon)} (\vec{L}_{\eta\zeta} B_\alpha{}^{\gamma\delta}) - \mathcal{Z}_N^{(\varepsilon)} \mathcal{N}_N^{(\varepsilon)} \partial_\alpha \omega^{\gamma\delta} \right. \\
 &- \frac{i}{2} B_\alpha{}^{\eta\zeta} \mathcal{Z}_N^{(\varepsilon)} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) + \frac{i}{2} \omega^{\eta\zeta} \mathcal{Z}_N^{(\varepsilon)} (\Sigma_{\eta\zeta}^V)^\alpha{}_\beta B^\beta{}_{\gamma\delta} \\
 &+ \left. \frac{1}{2} \mathcal{Z}_N^{(\varepsilon)} C^{\gamma\delta}{}_{\mu\kappa}{}_{\eta\zeta} B_\alpha{}^{\mu\kappa} \omega^{\eta\zeta} \right) \\
 &- b_1^{N(\varepsilon)} \theta \int h_{\gamma\delta} \partial^\beta \left(\frac{i}{2} \omega^{\eta\zeta} \mathcal{Z}_N^{(\varepsilon)} (\Sigma_{\eta\zeta}^V)^\alpha{}_\beta B_\alpha{}^{\gamma\delta} \right) \\
 &+ b_1^{N(\varepsilon)} \int h_{\gamma\delta} \delta_{\theta^{(\varepsilon)}} \left(\frac{i}{2} b_2^{N(\varepsilon)} B_\alpha{}^{\eta\zeta} (\vec{L}_{\eta\zeta} B^\alpha{}_{\gamma\delta}) \right. \\
 &+ \left. \frac{i}{2} b_3^{N(\varepsilon)} B_\alpha{}^{\eta\zeta} (\Sigma_{\eta\zeta}^V)^\alpha{}_\beta B^\beta{}_{\gamma\delta} + \frac{1}{2} b_4^{N(\varepsilon)} C^{\gamma\delta}{}_{\mu\kappa}{}_{\eta\zeta} B^\alpha{}_{\mu\kappa} B^\alpha{}_{\eta\zeta} \right) \\
 &\stackrel{!}{=} 0
 \end{aligned} \tag{108}$$

forces the various constants to take the following values

$$b_1^{N(\varepsilon)} = - \frac{Z_\omega^{N(\varepsilon)}}{\mathcal{Z}_N^{(\varepsilon)} \mathcal{N}_N^{(\varepsilon)}}$$

$$\begin{aligned}
a_1^{N(\varepsilon)} &= -a_2^{N(\varepsilon)} = a_4^{N(\varepsilon)} = -\frac{1}{\mathcal{N}_N^{(\varepsilon)}} & (109) \\
a_3^{N(\varepsilon)} &= b_2^{N(\varepsilon)} = b_3^{N(\varepsilon)} = b_4^{N(\varepsilon)} = 0.
\end{aligned}$$

Setting the constants to the values as in equation (109) the variation of the term $\delta_{\theta(\varepsilon)} \vec{\partial}^\alpha \left(\frac{i}{2} a_1^{N(\varepsilon)} \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} B_\alpha{}^{\gamma\delta}) + \dots \right) = -Z_N^{(\varepsilon)} \mathcal{N}_N^{(\varepsilon)} \delta_{\theta(\varepsilon)} (s^{(\varepsilon)} \partial^\alpha B_\alpha{}^{\gamma\delta}) = 0$ must—and indeed does—vanish due to the nilpotence of the renormalized BRST transformation as the expression in brackets is nothing but the renormalized BRST variation $s^{(\varepsilon)}$ of $\partial^\alpha B_\alpha{}^{\gamma\delta}$, and the variation of the term $\delta_{\theta(\varepsilon)} \left(\frac{i}{2} b_2^{N(\varepsilon)} B_\alpha{}^{\eta\zeta} (\vec{L}_{\eta\zeta} B^\alpha{}_{\gamma\delta}) + \dots \right) = 0$ vanishes as per the values of the constants $b_2^{N(\varepsilon)} = b_3^{N(\varepsilon)} = b_4^{N(\varepsilon)} = 0$.

As a result we are just left with the new constants $Z_\omega^{N(\varepsilon)}$ and $\xi_N^{(\varepsilon)}$ and we get

$$\begin{aligned}
\Gamma_N^{(\varepsilon)}[\chi] &= S_R[\chi] + \varepsilon \Gamma_{N,\infty}[\chi, 0] = S_{B,\psi}[\chi] \\
&\quad - \frac{Z_\omega^{N(\varepsilon)}}{Z_N^{(\varepsilon)} \mathcal{N}_N^{(\varepsilon)}} \int \omega_{\gamma\delta}^* \partial^\alpha \left(\frac{i}{2} \omega^{\eta\zeta} Z_N^{(\varepsilon)} (\vec{L}_{\eta\zeta} B_\alpha{}^{\gamma\delta}) \right. \\
&\quad - Z_N^{(\varepsilon)} \mathcal{N}_N^{(\varepsilon)} \partial_\alpha \omega^{\gamma\delta} - \frac{i}{2} B_\alpha{}^{\eta\zeta} Z_N^{(\varepsilon)} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
&\quad \left. + \frac{1}{2} Z_N^{(\varepsilon)} C^{\gamma\delta}{}_{\nu\kappa}{}_{\eta\zeta} B_\alpha{}^{\nu\kappa} \omega^{\eta\zeta} \right) \\
&\quad - \frac{Z_\omega^{N(\varepsilon)}}{Z_N^{(\varepsilon)} \mathcal{N}_N^{(\varepsilon)}} \int h_{\gamma\delta} \partial^\alpha B_\alpha{}^{\gamma\delta} + \frac{\xi_N^{(\varepsilon)}}{2} \int h_{\gamma\delta} h^{\gamma\delta} & (110)
\end{aligned}$$

Turning to $\delta_{\theta(\varepsilon)} S_{B,\psi}[\chi]$ we first note that the renormalized BRST variations for the gauge and matter fields equations (95) and (94) are nothing but local gauge transformations with gauge parameter

$$\rho^{\gamma\delta}(x) = Z_N^{(\varepsilon)} \mathcal{N}_N^{(\varepsilon)} \theta \omega^{\gamma\delta}(x) \quad (111)$$

and with the generators $J_{\gamma\delta}$ of the **SO(1,3)** gauge algebra replaced by the rescaled generators $\tilde{J}_{\gamma\delta}$

$$J_{\gamma\delta} \rightarrow \tilde{J}_{\gamma\delta} = \frac{1}{\mathcal{N}_N^{(\varepsilon)}} J_{\gamma\delta}. \quad (112)$$

Because terms from $\delta_{\theta(\varepsilon)} S_{B,\psi}[\chi]$ will not contain any h or ω^* they cannot mix with the terms discussed above and we separately have to have

$$\delta_{\theta(\varepsilon)} S_{B,\psi}[\chi] = 0 \quad (113)$$

which means that $S_{B,\psi}[\chi]$ must be locally gauge-invariant under the renormalized gauge transformations defined by the BRST variations equations (94)–(98) for B and ψ with renormalized gauge parameter and gauge algebra generators as in equations (111) and (112).

As the gauge field action $S_G[B]$ in equation (2) is by construction the most general gauge-invariant action of dimension four or less we conclude that the most general $\Gamma_N^{(\varepsilon)}[\chi]$ compatible with renormalized BRST invariance is

$$\Gamma_N^{(\varepsilon)}[\chi] = \tilde{S}_G[B] + \tilde{S}_M[B, \psi]$$

$$\begin{aligned}
& - Z_\omega^{N(\varepsilon)} \int \omega_{\gamma\delta}^* \partial^\alpha \left(\frac{i}{2} \omega^{\eta\zeta} \vec{L}_{\eta\zeta} B_\alpha{}^{\gamma\delta} \right) - \partial_\alpha \omega^{\gamma\delta} \\
& - \frac{i}{2} B_\alpha{}^{\eta\zeta} \vec{L}_{\eta\zeta} \omega^{\gamma\delta} + \frac{1}{2} \tilde{C}^{\gamma\delta}{}_{\nu\kappa}{}_{\eta\zeta} B_\alpha{}^{\nu\kappa} \omega^{\eta\zeta} \\
& - \frac{Z_\omega^{N(\varepsilon)}}{Z_N^{(\varepsilon)} \mathcal{N}_N^{(\varepsilon)}} \int h_{\gamma\delta} \partial^\alpha B_\alpha{}^{\gamma\delta} + \frac{\xi_N^{(\varepsilon)}}{2} \int h_{\gamma\delta} h^{\gamma\delta},
\end{aligned} \tag{114}$$

where \tilde{S}_ω indicates that all gauge algebra generators J in S_ω have been replaced by the rescaled generators \tilde{J} as in equation (112). Above we have assumed that S_M and as a consequence \tilde{S}_M is the most general renormalizable Dirac matter action coupled to the gauge field.

Inspection of equation (114) shows that apart from the appearance of new constants $\Gamma_N^{(\varepsilon)}[\chi]$ is functionally the same expression in the dynamical fields as is the action S_{TOT} given by equation (100) with which we have started.

By adjusting the N th order terms in the corresponding constants in the original unrenormalized action all the new constants may be absorbed in S_R so that finally

$$\Gamma_N^{(\varepsilon)}[\chi] = S_R[\chi] + \varepsilon \Gamma_{N,\infty}[\chi, 0] = S_R[\chi]. \tag{115}$$

For this particular choice of renormalized constants in $S_R[\chi]$ we then have $\Gamma_{N,\infty}[\chi, 0] = 0$.

Q.E.D.

7. Conclusions

In two preceding papers [1, 2] we have developed a gauge field theory of the Lorentz group equivalent to GR in a limiting case, yet free from the well-known flaws of GR when it comes to quantization [1], and we have canonically quantized the non-interacting gauge field of that theory and defined the corresponding relativistically-invariant physical Fock space of positive-norm, positive-energy particle states [2].

In this paper we have given full proof of the renormalizability of the quantized theory. In fact we have proven the renormalizability of the perturbatively defined QEA in essence following the steps usually taken to prove the renormalizability of the QEA of the standard model of particle physics [3]. As in that case the ghosts and antighosts appearing as a byproduct of gauge-fixing the path integral expressions for the Green functions of the theory decouple, and the S -matrix is as a result unitary—in our case on the naïve Fock space of both positive-norm, positive-energy and negative-norm, negative-energy states related to only the gauge field, and on the physical Fock spaces related to possible other physical fields.

The last step to be taken in consistently quantizing the **SO(1,3)** gauge field theory at hands, and hence potentially gravitation, will be the demonstration of the unitarity of the S -matrix on the physical Fock space for the gauge field.

And then more work starts: what about asymptotic freedom versus the observability of the gravitational interaction—or the β -function of the theory determining the running of the gauge coupling? What about instantons which definitely exist in the Euclidean version of the theory given that **SO(4) = SU(2) × SU(2)**, and anomalies? And what about the interplay of $S_G^{(2)}[B]$ and $S_G^{(4)}[B]$ whereby the former dominates the gravitational interaction at long distances or in the realm of classical physics and the latter at the short distances governing quantum physics? And what about the gravitational quanta implied by the latter already in the non-interacting theory? Could they be at the origin of dark energy—forming a cosmological radiation background consisting of gravitational quanta in analogy to the CMB—and helping to resolve the mystery

surrounding 70% of the observed energy content of the Universe? And in that process nicely feeding back as a sort of cosmological constant into $S_G^{(0)}[B]$ in the current standard model of cosmology at a phenomenological level? And . . . ?

Finally let us take a step back from the more technical aspects and look at the potentially emerging holistic understanding of the four fundamental interactions.

Potentially it also seems fruitful in the case of gravitation to take the historically superbly successful approach to fundamental physics starting with a set of conservation laws based on observations, then evoking Noether's miraculous theorem linking such conservation laws to global internal symmetries of local field theories and finally uncovering a related force field and its dynamics by gauging the global internal symmetry.

So what the observed conservation of the electric, weak and colour charges have done for the formulation of the standard model the observed conservation of angular momentum and the observed uniformity of the speed of light across all Lorentz frames of motion might indeed do for the formulation of a consistent quantum theory of gravitation—and its seamless inclusion in the existing standard model of particle physics.

In that case both (A) working from the conservation of energy-momentum and (B) geometrizing gravitation might ultimately prove to have been optical illusions too close to reality to be easily recognized as such, but not close enough to provide the final keys to quantize gravitation.

Hence, there might be a consistent framing of all physics across the four observed fundamental interactions and it seems that a programme started long ago resulting in the standard model of particle physics might eventually come to its ultimate fruition by seamlessly including gravitation.

Appendix A. The SO(1,3) Lorentz gauge algebra

In this section we introduce notations and normalizations for the **SO(1,3)** Lorentz gauge algebra central to this work.

The **SO(1,3)** Lie or Lorentz gauge algebra is defined by the commutation relations

$$\begin{aligned} [J_{\alpha\beta}, J_{\gamma\delta}] &= i\{\eta_{\alpha\gamma}J_{\beta\delta} - \eta_{\beta\gamma}J_{\alpha\delta} + \eta_{\beta\delta}J_{\alpha\gamma} - \eta_{\alpha\delta}J_{\beta\gamma}\} \\ &\equiv i C^{\eta\zeta}_{\alpha\beta\gamma\delta} J_{\eta\zeta}, \end{aligned} \quad (116)$$

where $J_{\alpha\beta}$ denotes a generic set of the six Lie algebra generators and $C_{\alpha\beta\gamma\delta\eta\zeta}$ its structure constants

$$\begin{aligned} C_{\alpha\beta\gamma\delta\eta\zeta} &= \frac{1}{2} \{ \eta_{\gamma\eta} (\eta_{\delta\alpha}\eta_{\zeta\beta} - \eta_{\zeta\alpha}\eta_{\delta\beta}) - \eta_{\delta\eta} (\eta_{\gamma\alpha}\eta_{\zeta\beta} - \eta_{\zeta\alpha}\eta_{\gamma\beta}) \\ &\quad + \eta_{\delta\zeta} (\eta_{\gamma\alpha}\eta_{\eta\beta} - \eta_{\eta\alpha}\eta_{\gamma\beta}) - \eta_{\alpha\zeta} (\eta_{\delta\alpha}\eta_{\eta\beta} - \eta_{\eta\alpha}\eta_{\delta\beta}) \} \\ &= C_{\eta\zeta\alpha\beta\gamma\delta} = C_{\gamma\delta\eta\zeta\alpha\beta}. \end{aligned} \quad (117)$$

The $C_{\alpha\beta\gamma\delta\eta\zeta}$ are antisymmetric in all the pairs of indices $C_{\beta\alpha\gamma\delta\eta\zeta} = -C_{\alpha\beta\gamma\delta\eta\zeta} = \dots$, antisymmetric in exchanging two adjacent pairs of indices $C_{\gamma\delta\alpha\beta\eta\zeta} = -C_{\alpha\beta\gamma\delta\eta\zeta} = \dots$ and subject to the Jacobi identity

$$C^{\gamma\delta}_{\alpha\beta\eta\zeta} C^{\eta\zeta}_{\rho\sigma\tau\chi} + C^{\gamma\delta}_{\rho\sigma\eta\zeta} C^{\eta\zeta}_{\tau\chi\alpha\beta} + C^{\gamma\delta}_{\tau\chi\eta\zeta} C^{\eta\zeta}_{\alpha\beta\rho\sigma} = 0 \quad (118)$$

which follows from

$$\begin{aligned}
[J_{\alpha\beta}, [J_{\rho\sigma}, J_{\tau\chi}]] + \text{cycl. perm.} &= [J_{\alpha\beta}, i C^{\eta\zeta}_{\rho\sigma\tau\chi} J_{\eta\zeta}] + \text{cycl. perm.} \\
&= i^2 C^{\eta\zeta}_{\rho\sigma\tau\chi} C^{\gamma\delta}_{\alpha\beta\eta\zeta} J_{\gamma\delta} + \text{cycl. perm.} = 0.
\end{aligned} \tag{119}$$

Let us next display three sets of **SO(1,3)** generators regularly appearing throughout the paper, namely:

(A) The generators of infinitesimal gauge transformations acting on spacetime coordinates

$$\vec{L}_{\eta\zeta} = -i(x_\eta \partial_\zeta - x_\zeta \partial_\eta), \tag{120}$$

(B) The generators of infinitesimal gauge transformations in the vector representation

$$(\Sigma_{\eta\zeta}^V)^{\gamma\delta} = -i (\eta_\eta^\gamma \eta_\zeta^\delta - \eta_\zeta^\delta \eta_\eta^\gamma) \tag{121}$$

from which the generators of tensor representations are built and

(C) the generators of infinitesimal gauge transformations in the adjoint representation

$$(\Sigma_{\eta\zeta}^A)^{\gamma\delta}_{\iota\kappa} = i C^{\gamma\delta}_{\eta\zeta\iota\kappa}. \tag{122}$$

It is easy to show that they all obey the commutation relations equation (116).

Appendix B. Nilpotence of BRST transformations

In this section we demonstrate the nilpotence of the BRST transformations introduced in section 4.

B.1. Ghosts

Ghost BRST variation:

$$s\omega^{\gamma\delta} = \frac{i}{2} \omega^{\eta\zeta} \vec{L}_{\eta\zeta} \omega^{\gamma\delta} - \frac{1}{4} C^{\gamma\delta}_{\alpha\beta\eta\zeta} \omega^{\alpha\beta} \omega^{\eta\zeta} \tag{123}$$

Nilpotence of ghost BRST variation:

$$\begin{aligned}
\delta_\theta s\omega^{\gamma\delta} &= \frac{i}{2} \delta_\theta \omega^{\eta\zeta} \vec{L}_{\eta\zeta} \omega^{\gamma\delta} + \frac{i}{2} \omega^{\eta\zeta} \vec{L}_{\eta\zeta} \delta_\theta \omega^{\gamma\delta} \\
&\quad - \frac{1}{4} C^{\gamma\delta}_{\alpha\beta\eta\zeta} (\delta_\theta \omega^{\alpha\beta} \omega^{\eta\zeta} + \omega^{\alpha\beta} \delta_\theta \omega^{\eta\zeta}) \\
&= \frac{i}{2} \left\{ \frac{i}{2} \theta \omega^{\rho\sigma} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \right. \\
&\quad \left. - \frac{1}{4} \theta C^{\eta\zeta}_{\alpha\beta\rho\sigma} \omega^{\alpha\beta} \omega^{\rho\sigma} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \right\} \\
&\quad + \frac{i}{2} \omega^{\eta\zeta} \vec{L}_{\eta\zeta} \left\{ \frac{i}{2} \theta \omega^{\rho\sigma} (\vec{L}_{\rho\sigma} \omega^{\gamma\delta}) \right. \\
&\quad \left. - \frac{1}{4} \theta C^{\gamma\delta}_{\alpha\beta\rho\sigma} \omega^{\alpha\beta} \omega^{\rho\sigma} \right\} \\
&\quad - \frac{1}{2} C^{\gamma\delta}_{\alpha\beta\eta\zeta} \omega^{\alpha\beta} \left\{ \frac{i}{2} \theta \omega^{\rho\sigma} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{4} \theta C^{\eta\zeta}{}_{\rho\sigma\tau\chi} \omega^{\rho\sigma} \omega^{\tau\chi} \Big\} \\
& = \left(\frac{i}{2}\right)^2 \theta \omega^{\rho\sigma} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
& - \frac{i}{8} \theta C^{\eta\zeta}{}_{\alpha\beta\rho\sigma} \omega^{\alpha\beta} \omega^{\rho\sigma} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
& - \left(\frac{i}{2}\right)^2 \theta \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\rho\sigma}) (\vec{L}_{\rho\sigma} \omega^{\gamma\delta}) \\
& - \left(\frac{i}{2}\right)^2 \theta \omega^{\eta\zeta} \omega^{\rho\sigma} (\vec{L}_{\eta\zeta} \vec{L}_{\rho\sigma} \omega^{\gamma\delta}) \\
& + \frac{i}{8} \theta C^{\gamma\delta}{}_{\alpha\beta\rho\sigma} \omega^{\eta\zeta} \left\{ (\vec{L}_{\eta\zeta} \omega^{\alpha\beta}) \omega^{\rho\sigma} + \omega^{\alpha\beta} (\vec{L}_{\eta\zeta} \omega^{\rho\sigma}) \right\} \\
& + \frac{i}{4} \theta C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} \omega^{\alpha\beta} \omega^{\rho\sigma} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) \\
& - \frac{1}{8} \theta C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} C^{\eta\zeta}{}_{\rho\sigma\tau\chi} \omega^{\alpha\beta} \omega^{\rho\sigma} \omega^{\tau\chi} \\
& = \left(\frac{i}{2}\right)^2 \theta \omega^{\rho\sigma} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
& - \left(\frac{i}{2}\right)^2 \theta \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\rho\sigma}) (\vec{L}_{\rho\sigma} \omega^{\gamma\delta}) \\
& - \frac{i}{8} \theta C^{\eta\zeta}{}_{\alpha\beta\rho\sigma} \omega^{\alpha\beta} \omega^{\rho\sigma} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
& + \frac{1}{4} \theta \omega^{\alpha\beta} \omega^{\rho\sigma} \frac{1}{2} [\vec{L}_{\alpha\beta}, \vec{L}_{\rho\sigma}] \omega^{\gamma\delta} \\
& + \frac{i}{4} \theta C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} \omega^{\rho\sigma} \omega^{\alpha\beta} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) \\
& + \frac{i}{4} \theta C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} \omega^{\alpha\beta} \omega^{\rho\sigma} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) \\
& - \frac{1}{24} \theta \left\{ C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} C^{\eta\zeta}{}_{\rho\sigma\tau\chi} + C^{\gamma\delta}{}_{\rho\sigma\eta\zeta} C^{\eta\zeta}{}_{\tau\chi\alpha\beta} \right. \\
& \left. + C^{\gamma\delta}{}_{\tau\chi\eta\zeta} C^{\eta\zeta}{}_{\alpha\beta\rho\sigma} \right\} \omega^{\alpha\beta} \omega^{\rho\sigma} \omega^{\tau\chi} \\
& = 0 \tag{124}
\end{aligned}$$

B.2. Matter

Matter BRST variation:

$$s\psi = \frac{i}{2} \omega^{\gamma\delta} \vec{L}_{\gamma\delta} \psi + \frac{i}{2} \omega^{\gamma\delta} \Sigma_{\gamma\delta} \psi \tag{125}$$

Nilpotence of matter BRST variation:

$$\delta_\theta s\psi = \frac{i}{2} \delta_\theta \omega^{\gamma\delta} \vec{L}_{\gamma\delta} \psi + \frac{i}{2} \omega^{\gamma\delta} \vec{L}_{\gamma\delta} \delta_\theta \psi$$

$$\begin{aligned}
& + \frac{i}{2} \delta_\theta \omega^{\gamma\delta} \Sigma_{\gamma\delta} \psi + \frac{i}{2} \omega^{\gamma\delta} \Sigma_{\gamma\delta} \delta_\theta \psi \\
= \dots & = \left(\frac{i}{2}\right)^2 \theta \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) (\vec{L}_{\gamma\delta} \psi) \\
& - \frac{i}{8} \theta C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} \omega^{\alpha\beta} \omega^{\eta\zeta} (\vec{L}_{\gamma\delta} \psi) \\
& - \left(\frac{i}{2}\right)^2 \theta \omega^{\gamma\delta} (\vec{L}_{\gamma\delta} \omega^{\eta\zeta}) (\vec{L}_{\eta\zeta} \psi) \\
& - \left(\frac{i}{2}\right)^2 \theta \omega^{\gamma\delta} \omega^{\eta\zeta} (\vec{L}_{\gamma\delta} \vec{L}_{\eta\zeta} \psi) \\
& - \left(\frac{i}{2}\right)^2 \theta \omega^{\gamma\delta} (\vec{L}_{\gamma\delta} \omega^{\eta\zeta}) (\Sigma_{\eta\zeta} \psi) \\
& - \left(\frac{i}{2}\right)^2 \theta \omega^{\gamma\delta} \omega^{\eta\zeta} (\vec{L}_{\gamma\delta} \Sigma_{\eta\zeta} \psi) \\
& + \left(\frac{i}{2}\right)^2 \theta \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) (\Sigma_{\gamma\delta} \psi) \\
& - \frac{i}{8} \theta C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} \omega^{\alpha\beta} \omega^{\eta\zeta} (\Sigma_{\gamma\delta} \psi) \\
& - \left(\frac{i}{2}\right)^2 \theta \omega^{\gamma\delta} (\Sigma_{\gamma\delta} \omega^{\eta\zeta}) (\vec{L}_{\eta\zeta} \psi) \\
& - \left(\frac{i}{2}\right)^2 \theta \omega^{\gamma\delta} \omega^{\eta\zeta} (\Sigma_{\gamma\delta} \Sigma_{\eta\zeta} \psi) \\
= \dots & = 0
\end{aligned} \tag{126}$$

B.3. Gauge fields

Gauge field BRST variation:

$$\begin{aligned}
sB_\mu{}^{\gamma\delta} & = \frac{i}{2} \omega^{\eta\zeta} \vec{L}_{\eta\zeta} B_\mu{}^{\gamma\delta} - \partial_\mu \omega^{\gamma\delta} - \frac{i}{2} B_\mu{}^{\eta\zeta} \vec{L}_{\eta\zeta} \omega^{\gamma\delta} \\
& + \frac{i}{2} \omega^{\eta\zeta} (\Sigma_{\eta\zeta})_\mu{}^\nu B_\nu{}^{\gamma\delta} + \frac{1}{2} C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} B_\mu{}^{\alpha\beta} \omega^{\eta\zeta}
\end{aligned} \tag{127}$$

Nilpotence of gauge field BRST variation:

$$\begin{aligned}
\delta_\theta sB_\mu{}^{\gamma\delta} & = \frac{i}{2} \delta_\theta \omega^{\eta\zeta} \vec{L}_{\eta\zeta} B_\mu{}^{\gamma\delta} + \frac{i}{2} \omega^{\eta\zeta} \vec{L}_{\eta\zeta} \delta_\theta B_\mu{}^{\gamma\delta} \\
& - \partial_\mu \delta_\theta \omega^{\gamma\delta} - \frac{i}{2} \delta_\theta B_\mu{}^{\eta\zeta} \vec{L}_{\eta\zeta} \omega^{\gamma\delta} - \frac{i}{2} B_\mu{}^{\eta\zeta} \vec{L}_{\eta\zeta} \delta_\theta \omega^{\gamma\delta} \\
& + \frac{i}{2} \delta_\theta \omega^{\eta\zeta} (\Sigma_{\eta\zeta})_\mu{}^\nu B_\nu{}^{\gamma\delta} + \frac{i}{2} \omega^{\eta\zeta} (\Sigma_{\eta\zeta})_\mu{}^\nu \delta_\theta B_\nu{}^{\gamma\delta} \\
& + \frac{1}{2} C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} \delta_\theta B_\mu{}^{\alpha\beta} \omega^{\eta\zeta} + \frac{1}{2} C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} B_\mu{}^{\alpha\beta} \delta_\theta \omega^{\eta\zeta}
\end{aligned}$$

$$\begin{aligned}
&= \dots = \left(\frac{i}{2}\right)^2 \theta \omega^{\rho\sigma} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) (\vec{L}_{\eta\zeta} B_\mu^{\gamma\delta}) \\
&\quad - \frac{i}{8} \theta C^{\eta\zeta}_{\rho\sigma\tau\chi} \omega^{\rho\sigma} \omega^{\tau\chi} (\vec{L}_{\eta\zeta} B_\mu^{\gamma\delta}) \\
&\quad - \left(\frac{i}{2}\right)^2 \theta \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\rho\sigma}) (\vec{L}_{\rho\sigma} B_\mu^{\gamma\delta}) \\
&\quad - \left(\frac{i}{2}\right)^2 \theta \omega^{\eta\zeta} \omega^{\rho\sigma} (\vec{L}_{\eta\zeta} \vec{L}_{\rho\sigma} B_\mu^{\gamma\delta}) \\
&\quad + \frac{i}{2} \theta \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} \partial_\mu \omega^{\gamma\delta}) \\
&\quad + \left(\frac{i}{2}\right)^2 \theta \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} B_\mu^{\rho\sigma}) (\vec{L}_{\rho\sigma} \omega^{\gamma\delta}) \\
&\quad + \left(\frac{i}{2}\right)^2 \theta \omega^{\eta\zeta} B_\mu^{\rho\sigma} (\vec{L}_{\eta\zeta} \vec{L}_{\rho\sigma} \omega^{\gamma\delta}) \\
&\quad - \left(\frac{i}{2}\right)^2 \theta \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\rho\sigma}) (\Sigma_{\rho\sigma})_\mu{}^\nu B_\nu^{\gamma\delta} \\
&\quad - \left(\frac{i}{2}\right)^2 \theta \omega^{\eta\zeta} \omega^{\rho\sigma} (\Sigma_{\rho\sigma})_\mu{}^\nu (\vec{L}_{\eta\zeta} B_\nu^{\gamma\delta}) \\
&\quad - \frac{i}{4} \theta C^{\gamma\delta}_{\rho\sigma\tau\chi} \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} B_\mu^{\rho\sigma}) \omega^{\tau\chi} \\
&\quad - \frac{i}{4} \theta C^{\gamma\delta}_{\rho\sigma\tau\chi} \omega^{\eta\zeta} B_\mu^{\rho\sigma} (\vec{L}_{\eta\zeta} \omega^{\tau\chi}) \\
&\quad - \frac{i}{2} \theta \partial_\mu \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
&\quad - \frac{i}{2} \theta \omega^{\eta\zeta} \partial_\mu (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
&\quad + \frac{1}{4} \theta C^{\gamma\delta}_{\alpha\beta\eta\zeta} (\partial_\mu \omega^{\alpha\beta} \omega^{\eta\zeta} + \omega^{\alpha\beta} \partial_\mu \omega^{\eta\zeta}) \\
&\quad - \left(\frac{i}{2}\right)^2 \theta \omega^{\rho\sigma} (\vec{L}_{\rho\sigma} B_\mu^{\eta\zeta}) (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
&\quad + \frac{i}{2} \theta \partial_\mu \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
&\quad + \left(\frac{i}{2}\right)^2 \theta B_\mu^{\rho\sigma} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
&\quad - \left(\frac{i}{2}\right)^2 \theta \omega^{\rho\sigma} (\Sigma_{\rho\sigma})_\mu{}^\nu B_\nu^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
&\quad - \frac{i}{4} \theta C^{\eta\zeta}_{\rho\sigma\tau\chi} B_\mu^{\rho\sigma} \omega^{\tau\chi} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta})
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{i}{2}\right)^2 \theta B_{\mu}{}^{\rho\sigma} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
& - \left(\frac{i}{2}\right)^2 \theta B_{\mu}{}^{\rho\sigma} \omega^{\eta\zeta} (\vec{L}_{\rho\sigma} \vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
& + \frac{i}{4} \theta C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} B_{\mu}{}^{\rho\sigma} \omega^{\alpha\beta} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) \\
& + \left(\frac{i}{2}\right)^2 \theta \omega^{\rho\sigma} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) (\Sigma_{\eta\zeta})_{\mu}{}^{\nu} B_{\nu}{}^{\gamma\delta} \\
& - \frac{i}{8} \theta C^{\eta\zeta}{}_{\rho\sigma\tau\chi} \omega^{\rho\sigma} \omega^{\tau\chi} (\Sigma_{\eta\zeta})_{\mu}{}^{\nu} B_{\nu}{}^{\gamma\delta} \\
& - \left(\frac{i}{2}\right)^2 \theta \omega^{\rho\sigma} (\Sigma_{\rho\sigma})_{\mu}{}^{\nu} \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} B_{\nu}{}^{\gamma\delta}) \\
& + \frac{i}{2} \theta \omega^{\rho\sigma} (\Sigma_{\rho\sigma})_{\mu}{}^{\nu} \partial_{\nu} \omega^{\gamma\delta} \\
& + \left(\frac{i}{2}\right)^2 \theta \omega^{\rho\sigma} (\Sigma_{\rho\sigma})_{\mu}{}^{\nu} B_{\nu}{}^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\gamma\delta}) \\
& - \left(\frac{i}{2}\right)^2 \theta \omega^{\rho\sigma} \omega^{\eta\zeta} (\Sigma_{\rho\sigma})_{\mu}{}^{\kappa} (\Sigma_{\eta\zeta})_{\kappa}{}^{\nu} B_{\nu}{}^{\gamma\delta} \\
& - \frac{i}{4} \theta \omega^{\rho\sigma} (\Sigma_{\rho\sigma})_{\mu}{}^{\nu} C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} B_{\nu}{}^{\alpha\beta} \omega^{\eta\zeta} \\
& + \frac{i}{4} \theta C^{\gamma\delta}{}_{\alpha\beta\rho\sigma} \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} B_{\mu}{}^{\alpha\beta}) \omega^{\rho\sigma} \\
& - \frac{1}{2} \theta C^{\gamma\delta}{}_{\alpha\beta\rho\sigma} \partial_{\mu} \omega^{\alpha\beta} \omega^{\rho\sigma} \\
& - \frac{i}{4} \theta C^{\gamma\delta}{}_{\alpha\beta\rho\sigma} B_{\mu}{}^{\eta\zeta} (\vec{L}_{\eta\zeta} \omega^{\alpha\beta}) \omega^{\rho\sigma} \\
& + \frac{i}{4} \theta C^{\gamma\delta}{}_{\alpha\beta\rho\sigma} \omega^{\eta\zeta} \omega^{\rho\sigma} (\Sigma_{\eta\zeta})_{\mu}{}^{\nu} B_{\nu}{}^{\alpha\beta} \\
& + \frac{1}{4} \theta C^{\gamma\delta}{}_{\alpha\beta\rho\sigma} C^{\alpha\beta}{}_{\tau\chi\eta\zeta} B_{\mu}{}^{\tau\chi} \omega^{\eta\zeta} \omega^{\rho\sigma} \\
& + \frac{i}{4} \theta C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} B_{\mu}{}^{\alpha\beta} \omega^{\rho\sigma} (\vec{L}_{\rho\sigma} \omega^{\eta\zeta}) \\
& - \frac{1}{8} \theta C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} C^{\eta\zeta}{}_{\rho\sigma\tau\chi} B_{\mu}{}^{\alpha\beta} \omega^{\rho\sigma} \omega^{\tau\chi} \\
= \dots & = \frac{i}{2} \theta \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} \partial_{\mu} \omega^{\gamma\delta}) \\
& - \frac{i}{2} \theta \omega^{\eta\zeta} (\vec{L}_{\eta\zeta} \partial_{\mu} \omega^{\gamma\delta}) \\
& - \frac{i}{2} \theta \omega^{\eta\zeta} (\partial_{\mu} \vec{L}_{\eta\zeta}) \omega^{\gamma\delta}
\end{aligned}$$

$$\begin{aligned}
& + \frac{i}{2} \theta \omega^{\rho\sigma} (\Sigma_{\rho\sigma})_{\mu}{}^{\nu} \partial_{\nu} \omega^{\gamma\delta} \\
& - \frac{1}{8} \theta \left\{ C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} C^{\eta\zeta}{}_{\rho\sigma\tau\chi} + C^{\gamma\delta}{}_{\tau\chi\eta\zeta} C^{\eta\zeta}{}_{\alpha\beta\rho\sigma} \right. \\
& \left. + C^{\gamma\delta}{}_{\rho\sigma\eta\zeta} C^{\eta\zeta}{}_{\tau\chi\alpha\beta} \right\} B_{\mu}{}^{\alpha\beta} \omega^{\rho\sigma} \omega^{\tau\chi} \\
& = 0
\end{aligned} \tag{128}$$

using

$$\begin{aligned}
& \frac{i}{2} \theta \omega^{\eta\zeta} (\partial_{\mu} \vec{L}_{\eta\zeta}) \omega^{\gamma\delta} + \frac{i}{2} \theta \omega^{\rho\sigma} (\Sigma_{\rho\sigma})_{\mu}{}^{\nu} \partial_{\nu} \omega^{\gamma\delta} \\
& = \theta \omega^{\eta}{}_{\mu} \partial_{\eta} \omega^{\gamma\delta} + \theta \omega_{\mu}{}^{\nu} \partial_{\nu} \omega^{\gamma\delta} = 0
\end{aligned} \tag{129}$$

B.4. Antighosts

Antighost BRST variation:

$$s\omega_{\gamma\delta}^* = -h_{\gamma\delta} \tag{130}$$

Nilpotence of antighost BRST variation:

$$\delta_{\theta} s\omega_{\gamma\delta}^* = -\delta_{\theta} h_{\gamma\delta} = 0 \tag{131}$$

B.5. Nakanishi–Lautrup fields

Nakanishi–Lautrup fields BRST variation:

$$sh_{\gamma\delta} = 0 \tag{132}$$

Nilpotence of Nakanishi–Lautrup fields BRST variation:

$$\delta_{\theta} sh_{\gamma\delta} = 0 \tag{133}$$

Appendix C. Berezinian determinant

In this section we demonstrate the triviality of the Berezinian determinant introduced in section 5.

C.1. Ghosts

Jacobian matrix for ghosts:

$$\begin{aligned}
\frac{\delta \omega^{\gamma\delta}(x)}{\delta \omega^{\iota\kappa}(y)} & = \left(\eta^{\gamma}{}_{[\iota, \eta^{\delta}{}_{\kappa]}]} + \frac{i}{2} \theta \omega^{\eta\zeta}(x) \vec{L}_{\eta\zeta}{}^{\gamma}{}_{[\iota, \eta^{\delta}{}_{\kappa]}]} \right. \\
& \left. - \frac{1}{4} \theta C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} (\omega^{\eta\zeta}(x) \eta^{\alpha}{}_{[\iota, \eta^{\beta}{}_{\kappa]}]} - \omega^{\alpha\beta}(x) \eta^{\eta}{}_{[\iota, \eta^{\zeta}{}_{\kappa]}]} \right) \delta(x-y)
\end{aligned} \tag{134}$$

Above the square brackets with comma $\eta^{\eta}{}_{[\iota, \eta^{\zeta}{}_{\kappa]}]} \equiv \eta^{\eta}{}_{\iota} \eta^{\zeta}{}_{\kappa} - \eta^{\eta}{}_{\kappa} \eta^{\zeta}{}_{\iota}$ indicate antisymmetrization in the indices concerned

C.2. Matter

Jacobian matrix for matter:

$$\frac{\delta\psi'(x)}{\delta\psi(y)} = \left(1 + \frac{i}{2} \theta \omega^{\eta\zeta}(x) \vec{L}_{\eta\zeta} + \frac{i}{2} \theta \omega^{\eta\zeta}(x) \Sigma_{\eta\zeta} \right) \delta(x-y) \quad (135)$$

C.3. Gauge fields

Jacobian matrix for gauge fields:

$$\begin{aligned} \frac{\delta B_{\mu}^{\prime\ \gamma\delta}(x)}{\delta B_{\rho}^{\prime\ \kappa}(y)} = & \left(\eta_{\mu}^{\ \rho} \eta^{\gamma}{}_{[\iota, \eta^{\delta}{}_{\kappa]} + \frac{i}{2} \theta \omega^{\eta\zeta}(x) \vec{L}_{\eta\zeta} \eta_{\mu}^{\ \rho} \eta^{\gamma}{}_{[\iota, \eta^{\delta}{}_{\kappa]} \right. \\ & - \frac{i}{2} \theta \left(\vec{L}_{\eta\zeta} \omega^{\gamma\delta}(x) \right) \eta_{\mu}^{\ \rho} \eta^{\gamma}{}_{[\iota, \eta^{\delta}{}_{\kappa]} \\ & + \frac{i}{2} \theta \omega^{\eta\zeta}(x) (\Sigma_{\eta\zeta})_{\mu}{}^{\nu} \eta_{\nu}^{\ \rho} \eta^{\gamma}{}_{[\iota, \eta^{\delta}{}_{\kappa]} \\ & \left. + \frac{1}{2} \theta C^{\gamma\delta}{}_{\alpha\beta\eta\zeta} \omega^{\eta\zeta}(x) \eta_{\mu}^{\ \rho} \eta^{\alpha}{}_{[\iota, \eta^{\beta}{}_{\kappa]} \right) \delta(x-y) \end{aligned} \quad (136)$$

C.4. Antighosts

Jacobian matrix for antighosts:

$$\frac{\delta\omega_{\gamma\delta}^{\prime}(x)}{\delta\omega_{\iota\kappa}^*(y)} = \eta_{\gamma}{}^{[\iota, \eta_{\delta}{}_{\kappa]} \delta(x-y) \quad (137)$$

C.5. Nakanishi–Lautrup fields

Jacobian matrix for Nakanishi–Lautrup fields:

$$\frac{\delta h'_{\gamma\delta}(x)}{\delta h_{\iota\kappa}(y)} = \eta_{\gamma}{}^{[\iota, \eta_{\delta}{}_{\kappa]} \delta(x-y) \quad (138)$$

C.6. Berezinian determinant

Berezinian determinant:

$$\mathcal{J} = \text{Det} \left(\frac{\delta\chi^{n'}}{\delta\chi^m} \right) = 1 + \text{Tr} \log \left(\frac{\delta\chi^{n'}}{\delta\chi^m} \right) \quad (139)$$

Note that this is an exact expression as all higher terms on the rhs vanish due to the antisymmetric nature of θ to which all the non-trivial contributions to the Jacobian matrices above are proportional

Trace of the sum of logarithms of the Jacobians:

$$\begin{aligned} \text{Tr} \log \left(\frac{\delta\chi^{n'}}{\delta\chi^m} \right) &= -\theta \text{Tr}_{\psi} \left(\frac{i}{2} \theta \omega^{\eta\zeta} \vec{L}_{\eta\zeta} + \frac{i}{2} \theta \omega^{\eta\zeta} \Sigma_{\eta\zeta} \right) \\ &\quad + \theta \text{Tr}_B(\dots) - \theta \text{Tr}_{\omega}(\dots) \\ &= \theta \text{Tr} \left(\frac{i}{2} \omega^{\eta\zeta} \vec{L}_{\eta\zeta} \right) (-\dim_{\psi} + \dim_B - \dim_{\omega}) \end{aligned} \quad (140)$$

Functional trace of the infinitesimal algebra parameter $\omega^{\eta\zeta}(x) \vec{L}_{\eta\zeta}$:

$$\begin{aligned}
\text{Tr} \left(\frac{i}{2} \omega^{\eta\zeta}(x) \vec{L}_{\eta\zeta} \delta(x-y) \right) &= - \int d^4x \int d^4y \omega^{\eta\zeta}(x) x_\zeta \partial_\eta^x \delta(x-y) \\
&= \int d^4x \int d^4y \delta(x-y) \partial_\eta^x (\omega^{\eta\zeta}(x) x_\zeta \partial_\eta^x) \\
&= \int d^4x \partial_\eta^x (\omega^{\eta\zeta}(x) x_\zeta) = 0
\end{aligned} \tag{141}$$

Berezinian determinant:

$$\mathcal{J} = 1 \tag{142}$$

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