# CANCELLATION OF LASER NOISE IN AN UNEQUAL-ARM INTERFEROMETER DETECTOR OF GRAVITATIONAL RADIATION

M. TINTO
*Jet Propulsion Laboratory,*
*California Institute of Technology*
*4800 Oak Grove Drive,*
*Pasadena, California, 91109.*
*U.S.A.*

We present a technique for canceling the laser noise in a one-bounce, unequal-arm, Michelson interferometer detector of gravitational radiation. This method requires separate measurements of the phase difference in each arm, made by interfering the returning laser light in each arm with the outgoing light. Let these two time series of phase difference be $z_i$ , $i = 1, 2$. By forming the quantity $[z_1(t - 2L_2/c) - z_1(t)] - [z_2(t - 2L_1/c) - z_2(t)]$, where $L_i$ are the arm lengths, gravitational wave signals remain while the laser noise is cancelled. Unlike other proposed techniques, this procedure exactly cancels the laser noise if the arm lengths are known, it is direct in time, and allows for time-varying arm-lengths.

## 1 Introduction

Interferometric, non-resonant, detectors of gravitational radiation (with frequency content $0 < f < f_H$) use a coherent train of electromagnetic waves (of nominal frequency $\nu_0 \gg f_H$) folded into several beams, and at one or more points where these intersect, monitor relative fluctuations of frequency or phase (homodyne detection). The observed low frequency signals are due to frequency variations of the source of the electromagnetic signal about $\nu_0$, to relative motions of the source and the mirrors (or amplifying transponders) that do the folding, to temporal variations of the index of refraction along the beams, and, according to general relativity, to any time-variable gravitational fields present, such as the transverse traceless metric curvature of a passing plane gravitational wave train. To observe gravitational waves in this way, it is thus necessary to control, or monitor, the other sources of relative frequency fluctuations, and, in the data analysis, to use optimal algorithms based on the different characteristic interferometer responses to gravitational waves (the signal) and to the other sources (the noise) [1]. By comparing phases of split beams propagated along non-parallel equal-length arms, frequency fluctuations of the frequency reference can be removed and gravitational wave signals at levels many orders of magnitude lower can be detected. Especially for space-based interferometers, that may use lasers with a frequency stability at best of a few parts in $10^{-13}/\sqrt{\text{Hz}}$, it is essential to be able to remove these fluctuations when searching for gravitational waves of dimensionless amplitudes less than $10^{-20}/\sqrt{\text{Hz}}$ in the millihertz band [2].

Since the armlengths of these space-based interferometers can be different by several percent, the direct recombination of the two beams at a photo detector will not however effectively remove the laser noise. This is because the frequency fluctuations of the laser will be delayed by a different amount of time inside the two different-length arms. In order to solve this problem, we will show that it is possible to remove completely the frequency fluctuations of the laser by taking a suitable linear combination of the two Doppler time series after having time shifted them properly. This direct method achieves the exact cancellation of the laser frequency fluctuations, and does not require any Fourier transform of the data[3].

## 2 Statement of the problem

Let us consider three spacecraft flying in an equilateral triangle-like formation, each acting as a free falling test particle, and continuously tracking each other via coherent laser light. One spacecraft, which we will refer to as spacecraft $a$, transmits a laser beam of nominal frequency $\nu_0$ to the other spacecraft (spacecraft $b$ and $c$ at distances $L_1$ and $L_2$, respectively). The phase of the light received at spacecraft $b$ and $c$ is used by lasers on board spacecraft $b$ and $c$ for coherent transmission back to spacecraft $a$. The relative two *two-way* frequency (or phase) changes as functions of time are then independently measured at two photo detectors on board spacecraft $a$. When a gravitational wave crossing the solar system propagates through these electromagnetic links, it causes small perturbations in frequency (or phase), which are replicated three times in each arm's data[4].

To determine the response of an unequal arm interferometer to a gravitational wave pulse, let us introduce a set of Cartesian orthogonal coordinates $(X, Y, Z)$ centered on spacecraft $a$. The $X$ axis is assumed to be oriented along the bisector of the angle enclosed between the two arms, $Y$ is orthogonal to it in the plane containing the three spacecraft, and the $Z$ axis is chosen in such a way to form with $(X, Y)$ a right-handed, orthogonal triad of axes. In this coordinate system we can write the two two-way Doppler responses, measured by spacecraft $a$ at time $t$, as follows[4,5,6] (units in which the speed of light $c = 1$).

$$\left(\frac{\Delta\nu(t)}{\nu_0}\right)_1 \equiv y_1(t) = \left[-\frac{(1-\vec{k}\cdot\vec{\rho_1})}{2}\,\Psi_1(t) - \vec{k}\cdot\vec{\rho_1}\,\Psi_1(t-(1+\vec{k}\cdot\vec{\rho_1})L_1(t)) + \frac{(1+\vec{k}\cdot\vec{\rho_1})}{2}\,\Psi_1(t-2L_1(t))\right] + C(t-2L_1(t)) - C(t) + n_1(t)\ , \quad (1)$$

$$\left(\frac{\Delta\nu(t)}{\nu_0}\right)_2 \equiv y_2(t) = \left[-\frac{(1-\vec{k}\cdot\vec{\rho_2})}{2}\,\Psi_2(t) - \vec{k}\cdot\vec{\rho_2}\,\Psi_2(t-(1+\vec{k}\cdot\vec{\rho_2})L_2(t)) + \frac{(1+\vec{k}\cdot\vec{\rho_2})}{2}\,\Psi_2(t-2L_2(t))\right] + C(t-2L_2(t)) - C(t) + n_2(t)\ , \quad (2)$$

where $\vec{k}$ is the unit vector in the direction of propagation of the planar gravitational wave pulse. In Equations (1, 2) we have denoted by $\vec{\rho_1}$, $\vec{\rho_2}$, the unit vectors from spacecraft $a$ to spacecraft $b$ and $c$ respectively; $\Psi_{(1,2)}(t)$ are the following two scalar functions

$$\Psi_{(1,2)}(t) = \left[\frac{\rho^i_{(1,2)}\rho^j_{(1,2)}}{1-(\vec{k}\cdot\vec{\rho}_{(1,2)})^2}\right] h_{ij}(t)\ , \quad (3)$$

with $h_{ij}(t)$ being the rank-2 tensor associated with the gravitational wave pulse in the $(X, Y, Z)$ coordinate system[7], and the sum over the repeated space-like indices has been assumed. We have denoted by $C(t)$ the random process associated with the frequency fluctuations of the master laser on board spacecraft $a$, and $n_1(t)$, $n_2(t)$ are the remaining noise sources affecting the Doppler responses $y_1(t)$, $y_2(t)$ respectively.

From equations (1, 2) it is important to note the characteristic time signature of the random process $C(t)$ in the Doppler responses $y_1$, $y_2$. The time signature of the noise $C(t)$ in $y_1(t)$ for instance, can be understood by observing that the frequency of the signal received at time $t$ contains laser frequency fluctuations transmitted $2L_1$ seconds earlier. By subtracting from the frequency of the received signal the frequency of the signal transmitted at time $t$, we also subtract the frequency fluctuations $C(t)$ [8] with the net result shown in equation (1).

Among all the noise sources included in equation (1), the frequency fluctuations due to the laser are expected to be by far the largest. A space-qualified single-mode laser, such as a diode-pumped Nd:YAG ring laser of frequency $\nu_0 = 3.0 \times 10^{14}$ Hz and phase-locked to a Fabry-Perot optical cavity, is expected to have a spectral level of frequency fluctuations equal to about $1.0 \times 10^{-13}/\sqrt{Hz}$ in the millihertz band [2]. A single point frequency stability measurement performed on such a laser by McNamara *et al.* [9] indicates that a stability of about $1.0 \times 10^{-14}/\sqrt{Hz}$ might be achievable in the same frequency band. In this paper however we will assume the laser frequency stability mentioned in [2]. Laser noise is to be compared with the expected secondary noises, which will be $10^7$ or more times smaller.

If the armlengths are unequal by an amount $\Delta L = L_2 - L_1 \equiv \epsilon L_1$ (with $\epsilon \simeq 3 \times 10^{-2}$ for a space based interferometer [2]), the simple subtraction of the two Doppler data $y_1(t)$, $y_2(t)$ gives a new data set that is still affected by the laser fluctuations by an amount equal to

$$C(t - 2L_1) - C(t - 2L_2) \simeq 2\dot{C}(t - 2L_1)\epsilon L_1 \ . \tag{4}$$

As a numerical example of equation (4) we find that, at a frequency of $10^{-3}$ Hz and by using a laser of frequency stability equal to about $10^{-13}/\sqrt{Hz}$, the residual laser frequency fluctuations are equal to about $10^{-16}/\sqrt{Hz}$. Since the goal of proposed space-based interferometers [2] is to observe gravitational radiation at levels of $10^{-20}/\sqrt{Hz}$ or lower, it is crucial for the success of these missions to cancel laser frequency fluctuations by many more orders of magnitude.

## 3 Algorithm for unequal-arm interferometers

In what follows we will show that there exists an algorithm in the time domain for removing the frequency fluctuations of the laser from the two Doppler data $y_1(t)$, $y_2(t)$ at any time $t$. This approach does not require Fourier transforms on the Doppler data. As it will be shown below, this method relies only on a properly chosen linear combination of the two Doppler data in the time domain. In our derivation of the algorithm we will assume the two armlengths $L_1$, $L_2$ to be constant and known exactly. The reader is referred to Tinto and Armstrong [3] for the derivation of the armlength accuracy needed in order for the method described here to be still effective.

From equations (1, 2) we may notice that, by taking the difference of the two Doppler data $y_1(t)$, $y_2(t)$, the frequency fluctuations of the laser now enter into this new data set in the following way

$$\begin{aligned} \Lambda_1(t) \equiv y_1(t) - y_2(t) &= h_1(t) - h_2(t) + C(t - 2L_1) - C(t - 2L_2) \\ &+ n_1(t) - n_2(t) \ , \end{aligned} \tag{5}$$

where for simplicity of notation we have defined $h_1(t)$ and $h_2(t)$ to be the following functions

$$\begin{aligned} h_1(t) = & \left[ -\frac{(1 - \vec{k} \cdot \vec{\rho_1})}{2} \, \Psi_1(t) \ - \ \vec{k} \cdot \vec{\rho_1} \cdot \Psi_1(t - (1 + \vec{k} \cdot \vec{\rho_1})L_1) \right. \\ & \left. + \ \frac{(1 + \vec{k} \cdot \vec{\rho_1})}{2} \, \Psi_1(t - 2L_1) \right] \end{aligned} \tag{6}$$

$$h_2(t) = \left[ -\frac{(1 - \vec{k} \cdot \vec{\rho_2})}{2} \, \Psi_2(t) \ - \ \vec{k} \cdot \vec{\rho_2} \, \Psi_2(t - (1 + \vec{k} \cdot \vec{\rho_2})L_2) \right.$$

$$+ \quad \frac{(1+\vec{k}\cdot\vec{\rho_2})}{2} \; \Psi_2(t-2L_2)\Bigg] \; . \tag{7}$$

If we now compare how the laser frequency fluctuations enter into equation (5) against how they appear into equations (1, 2), we can further make the following observation. If we time-shift the data $y_1(t)$ by the round trip light time in arm 2, $y_1(t-2L_2)$, and subtract from it the data $y_2(t)$ after it has been time shifted by the round trip light time in arm 1, $y_2(t-2L_1)$, we obtain the following data set

$$\begin{aligned} \Lambda_2(t) \equiv y_1(t-2L_2) - y_2(t-2L_1) &= h_1(t-2L_2) - h_2(t-2L_1) + C(t-2L_1) \\ &- C(t-2L_2) + n_1(t-2L_2) - n_2(t-2L_1) \; . \end{aligned} \tag{8}$$

In other words, the laser frequency fluctuations enter into $\Lambda_1(t)$, and $\Lambda_2(t)$ with the same time-structure. This implies that, by subtracting equation (5) from equation (8), we can generate a new data set that does not contain the laser frequency fluctuations $C(t)$

$$\begin{aligned} \Sigma(t) \equiv \Lambda_2(t) - \Lambda_1(t) &= h_1(t-2L_2) - h_1(t) - h_2(t-2L_1) + h_2(t) \\ &+ n_1(t-2L_2) - n_1(t) - n_2(t-2L_1) + n_2(t) \; . \end{aligned} \tag{9}$$

From the expression of $\Lambda_2(t)$ given in equation (8), it is easy to see that the new data set $\Sigma(t)$ should be set to zero for the initial $MAX[2L_1, 2L_2]$ seconds. This is because some of the data from $y_1$ and $y_2$ entering into $\Lambda_2(t)$ "do not yet exist" during this time interval. Since the typical round trip light time for proposed space-based laser interferometer detectors of gravitational waves will never be greater than about 33 seconds[2], we conclude that the amount of data lost in the implementation of our method is negligible.

The unequal-arm interferometer response, $\Sigma(t)$, derived in equation (9), can be rewritten explicitly, in terms of the gravitational wave functions $\Psi_1$, $\Psi_2$, as follows

$$\begin{aligned} \Sigma(t) &= \left[\left(\frac{1-\vec{k}\cdot\vec{\rho_1}}{2}\right)\Psi_1(t) - \left(\frac{1-\vec{k}\cdot\vec{\rho_2}}{2}\right)\Psi_2(t)\right] \\ &+ \left[\left(\frac{1+\vec{k}\cdot\vec{\rho_2}}{2}\right)\Psi_2(t-2L_2) - \left(\frac{1-\vec{k}\cdot\vec{\rho_1}}{2}\right)\Psi_1(t-2L_2)\right] \\ &+ \left[\left(\frac{1-\vec{k}\cdot\vec{\rho_2}}{2}\right)\Psi_2(t-2L_1) - \left(\frac{1+\vec{k}\cdot\vec{\rho_1}}{2}\right)\Psi_1(t-2L_1)\right] \\ &+ \left[\left(\frac{1+\vec{k}\cdot\vec{\rho_1}}{2}\right)\Psi_1(t-2L_1-2L_2) - \left(\frac{1+\vec{k}\cdot\vec{\rho_2}}{2}\right)\Psi_2(t-2L_1-2L_2)\right] \\ &+ \vec{k}\cdot\vec{\rho_1}\;\Psi_1(t-(1+\vec{k}\cdot\vec{\rho_1})L_1) - \vec{k}\cdot\vec{\rho_2}\;\Psi_2(t-(1+\vec{k}\cdot\vec{\rho_2})L_2) \\ &+ \vec{k}\cdot\vec{\rho_2}\;\Psi_2(t-2L_1-(1+\vec{k}\cdot\vec{\rho_2})L_2) - \vec{k}\cdot\vec{\rho_1}\;\Psi_1(t-2L_2-(1+\vec{k}\cdot\vec{\rho_1})L_1) \\ &+ n_1(t-2L_2) - n_1(t) - n_2(t-2L_1) + n_2(t) \; . \end{aligned} \tag{10}$$

Equation (10) shows that the gravitational wave signal enters into the response of an unequal-arm interferometer at *eight* distinct times. In analogy with the terminology used for the Doppler tracking response to a gravitational wave pulse[4], we will refer to equation (10) as the *eight-pulse* response function.

It is important to point out that, as a consequence of the analytic form of the unequal-arm interferometer response given by equation (9), both the signal and the secondary noise sources will show a modulation of their power spectra. If we take the Fourier transform of equation (9), it is easy to derive the following expression for the one-sided power spectral density of $\Sigma(t)$

$$S_\Sigma(f) \;\equiv\; 4|\widetilde{h_1}(f)|^2\,\sin^2(2\pi f L_2) + 4|\widetilde{h_2}(f)|^2\,\sin^2(2\pi f L_1)$$

$$- \; 4\sin(2\pi f L_1)\sin(2\pi f L_2)\left[\widetilde{h_1}(f)\widetilde{h_2^*}(f)e^{2\pi i f(L_2-L_1)} + \widetilde{h_1^*}(f)\widetilde{h_2}(f)e^{-2\pi i f(L_2-L_1)}\right]$$
$$+ \; 4S_{n_1}(f)\ \sin^2(2\pi f L_2) + 4S_{n_2}(f)\ \sin^2(2\pi f L_1)\ , \tag{11}$$

where the symbol $^*$ denotes complex conjugation, the two random processes $n_1$, $n_2$ have been assumed to be uncorrelated, and $S_{n_1}(f)$, $S_{n_2}(f)$ are their respective one-sided power spectral densities. Since the proposed space-based interferometer detectors will have armlengths that will differ by up to a few percent [2], in the frequency band of interest equation (11) can be further simplified by neglecting terms of the order $f(L_2 - L_1)$ and higher

$$S_\Sigma(f) \simeq 4|\widetilde{h_1}(f) - \widetilde{h_2}(f)|^2\ \sin^2(2\pi f L_1) + 4\left[S_{n_1}(f) + S_{n_2}(f)\right]\sin^2(2\pi f L_1)\ . \tag{12}$$

Equation (12) shows that the one-sided power spectral densities of the signal and the noise display the same modulation in the Fourier domain. This result implies that the signal-to-noise ratio in an interferometer with arms that are different by a few percent is in principle equal to the signal-to-noise ratio achievable with an equal-arm detector.

By further expanding equation (12) in the long wavelength limit ($2\pi f L_1 \ll 1$ i.e. $f \ll 10^{-2}$ Hz for a five million kilometers arm length), and taking into account the expressions for $h_1$, $h_2$ given by equations (3, 6, 7), we obtain the following expression for the low-frequency response of the interferometer

$$S_\Sigma(f) \simeq 4\ |(\rho_1^i\rho_1^j - \rho_2^i\rho_2^j)\widetilde{h}_{ij}(f)|^2\ (2\pi f L_1)^4 + 4\left[S_{n_1}(f) + S_{n_2}(f)\right]\ (2\pi f L_1)^2\ , \tag{13}$$

which is the response of an equal-arm, one-bounce, Michelson interferometer detector of gravitational radiation [5,7] multiplied by the factor $16(2\pi f L_1)^2$. For $f \geq 10^{-3}$ Hz, most of the band to which LISA will be sensitive [2], the 8-pulse structure will be visible.

The real limitations on the procedure described above, however, come from the remaining noise sources affecting the two Doppler data, and the accuracy in the determination of the distances between the two pairs of spacecraft. Tinto and Armstrong [3] have performed a detailed error analysis, and found that an armlength's accuracy of about 30 meters is needed in order to reduce the magnitude of the remaining frequency fluctuations of the laser to a level below the level identified by the remaining noise sources entering into $\Sigma(t)$. Such a requirement on the armlength's accuracy can be met by using standard ranging capabilities [2].

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