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PAPER

Non-Gaussian quantum steering produced by quasi-phase-matching third-harmonic generation

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Abstract

With the rapid advancements in fields such as quantum entanglement distillation and quantum metrology, the limitations of Gaussian states in certain applications within quantum computing and information processing have come to the forefront. This has necessitated the development of methods to prepare non-Gaussian states, which exhibit negative Wigner values and are indispensable for enhancing the capabilities of quantum systems in these tasks. Wigner negativity, a renowned indicator of nonclassicality, is integral to quantum computing and the simulation of continuous-variable systems. It is also employed to discern non-Gaussian characteristics in optical fields. We demonstrate Gaussian Einstein–Podolsky–Rosen (EPR) steering between second and third harmonic generations prior to performing non-Gaussian operations. Inducing non-Gaussian attributes in the second harmonic is achieved by coupling the third harmonic with a vacuum state and subtracting photons via a beamsplitter. The Wigner stochastic trajectory approach is utilized to investigate the non-Gaussian properties of both the second and third harmonics. By varying the coupling parameter lambda λ and the ratio of nonlinear coupling constants κ_2/κ_1, symmetric and asymmetric non-Gaussian EPR steering can be observed. This proposed scheme for non-Gaussian EPR steering holds promise for applications in quantum computing and quantum information processing.

1. Introduction

In recent years, quantum steering as a phenomenon of quantum correlation has attracted widespread attention. Its origin can be traced back to the 1935 paradox proposed by Einstein, Podolsky, and Rosen (EPR paradox) [1]. Subsequently, Schrödinger further investigated this paradox, introducing the concept of quantum entanglement and proposing another nonclassical correlated state quantum steering [2, 3]. The early works of EPR and Schrödinger inspired subsequent research, especially after the proposal of Bell inequality [4], leading to quantum entanglement and steering becoming focal points of study. If the state of a composite quantum system can be approximated by a convex combination of product states, then it is called classical correlation, otherwise EPR correlation. Any classical correlated state can be modeled using implicit variable theory, thus satisfying all generalized Bell inequalities. Werner [5] demonstrated through an explicit example that the converse of this statement is false. Reid [6] proposed criteria for detecting quantum steering. Wiseman *et al* [7] provided a rigorous mathematical definition of quantum steering, identifying it as a form of quantum correlation that lies between entanglement and Bell nonlocality. Unlike entanglement and Bell nonlocality, quantum steering exhibits asymmetry: it allows one party (Alice) to influence the quantum state of another party (Bob), but not vice versa. This characteristic has been theoretically [8–10] and experimentally [11–13] verified. Recently, with the introduction of the concept of non-classical steering in two-mode Gaussian states [14], Frigerio *et al* provided a detailed explanation of the definition of quantum steering [15]. They also offered a comprehensive formulation of Gaussian steering and the theory of nonclassicality steering in two-mode Gaussian states. In their study of two-mode squeezed thermal states , they demonstrated the consistency between EPR steering and nonclassicality steering. Quantum steering, like entanglement and Bell nonlocality, has a wide range of applications in various fields, such as quantum secret sharing [16–20] and quantum key distribution [21]. He *et al* have demonstrated that bidirectional EPR steering is an important resource for achieving transmission fidelity beyond the unclonable threshold for coherent state safe stealth transmission [22]. Due to its unique asymmetry, EPR steering enables certain quantum information tasks that other forms of quantum correlation cannot, such as one-way quantum computing [23].

In the realm of continuous variables, quantum states can be classified into Gaussian and non-Gaussian states, which can be distinguished by the form of their Wigner functions. Gaussian states, due to their theoretical simplicity and ease of experimental preparation, have been widely applied in continuous variable quantum information processing schemes [24–26]. However, as quantum information research rapidly advances in areas such as quantum entanglement distillation [27, 28] and quantum metrology [29], the limitations of Gaussian states in certain tasks within quantum computing and quantum information processing have become apparent. Consequently, researchers have shifted their focus from Gaussian to non-Gaussian states. For instance, in quantum computing, non-Gaussian states with negative Wigner values are valuable for quantum error correction [30, 31]. Similarly, in quantum measurement, non-Gaussian states with negative Wigner values can enhance measurement precision [32, 33]. Xiang *et al* studied the EPR steering of continuous variable binary states through non-Gaussian pseudospin measurements [34]. They first derived the density matrix elements of dual-mode compressed thermal Gaussian states in the Fock basis to study the steering of these states. Through research, it has been found that non-Gaussian pseudospin measurements are always more effective in revealing their differentiability than Gaussian spin measurements. This result provides useful insights into the role of non-Gaussian measurements in describing the quantum correlations between Gaussian and non-Gaussian states in continuous variable quantum systems. Therefore, the preparation of non-Gaussian states with negative Wigner values is crucial.

There are typically two methods to generate non-Gaussian states with Wigner negativity: through strong higher-order interaction processes [35, 36] or by performing photon addition and subtraction operations on Gaussian modes [37–40]. Both methods require local preparation. In recent years, various non-Gaussian states have been experimentally prepared remotely, including single-photon states [41], squeezed cat states [42], and superpositions of coherent states [43]. However, these methods rely on utilizing existing non-Gaussian states or hybrid entangled states for remote non-Gaussian state preparation. In 2020, Walschaers *et al* [44] proposed a method to remotely generate non-Gaussian states with Wigner negativity by performing photon subtraction on the two-mode squeezed state. They concluded that Gaussian quantum steering from Bob to Alice is a necessary condition for the appearance of a Wigner-negative non-Gaussian state in the Bob mode. In 2022, Liu *et al* [45] demonstrated that quantum steering through local non-Gaussian operations and shared Gaussian entangled states can remotely prepare optical non-Gaussian states with negative Wigner functions at distant nodes. By performing photon subtraction on one mode, they generated Wigner negativity in the remote target mode. Their results showed that Wigner negativity is highly sensitive to losses in the target mode but robust to losses in the mode where photon subtraction is performed. This experiment confirmed the connection between remotely generated Wigner negativity and quantum manipulation. As an application, they suggested that the produced non-Gaussian states have significant metrological capabilities in quantum phase estimation. In the same year, Xiang *et al* [46] proved that for a two-mode squeezed state distributed between Alice and Bob, performing photon subtraction on the Alice mode is a necessary and sufficient condition for quantum steering from Alice to Bob to generate a Wigner-negative non-Gaussian state in the Bob mode, without requiring additional local Gaussian operations on the Alice mode before photon subtraction.

Although the preparation of non-Gaussian states has been extensively studied, reports on the entanglement and steering of non-Gaussian states remain relatively scarce. In 2013, Olsen and Corney [47] characterized the non-Gaussian properties of their output by calculating third-order cumulants for quadrature variables and using the Duan-Simon and Reid Einstein–Podolsky–Rosen criteria to predict potential entanglement levels. In 2017, Olsen proposed and analyzed pumped Bose–Hubbard dimers as a non-Gaussian statistical source for continuous variable EPR steering [48]. Using truncated Wigner representation, he calculated the third-order and fourth-order cumulants, discovered clear non-Gaussian signals, and calculated the product of inferred orthogonal variances. The results showed the existence of EPR paradox states. In 2020, a major breakthrough was achieved with the direct observation of strong non-Gaussian properties in three-photon spontaneous parametric down-conversion (SPDC) within a flux-pumped superconducting parametric cavity [49]. Recent studies have demonstrated that stable tripartite quantum steering can only be achieved in the situation where injected signals are present during the three-photon down-conversion process [50], while three-photon SPDC process may generate nonGaussian quantum steering. Other recent study introduced higher-order quadrature operators and constructed linear combinations of these operators to derive a genuine non-Gaussian entanglement criterion for complete inseparability in a threephoton SPDC [51]. They also studied the non- Gaussian time-energy entanglement generated by three photons in spontaneous six wave mixing both theoretically and experimentally [52, 53], making a significant contribution to the study of non-Gaussian entanglement. Currently, quantum entanglement and steering between Gaussian light fields generated by nonlinear processes have been extensively studied [54–58]. Gaussian quantum entangled and steered states can be prepared in optical superlattices by cascaded nonlinear processes [59–64]. With the development of phase matching technology, quasi-phase-matching (QPM) technology has been used in various frequency conversion processes. To achieve two cascaded nonlinear processes in a single optical superlattice, multiple QPM conditions can be achieved by designing the optical superlattice structure reasonably (such as multi period or quasi period structure), thus supporting two cascaded nonlinear processes. The experiment of generating third harmonic using two cascaded nonlinear processes has been reported. For example, Zhu *et al* [65] used QPM technology in 1997 to achieve QPM third harmonic generation in a single quasi-periodic optical superlattice (QOSL), and they proved that harmonic generation can be efficiently achieved by using the maximum nonlinear optical coefficient throughout the entire transparent range of the material. However, is it possible to obtain non-Gaussian quantum states by combining non-Gaussian operations with cascaded nonlinear processes? This is a topic worth investigating.

In this study, we present a scheme for the generation of non-Gaussian quantum steering through the nonlinear process of QPM third-harmonic generation. The second harmonic is generated via the initial nonlinear process of double-frequency generation from the pump signal. Subsequently, the third harmonic is produced through a cascaded nonlinear process involving the sum-frequency generation between the pump and the second harmonic. By coupling the third harmonic with a vacuum field at a beam splitter (BS), a photon is subtracted. We examine the non-Gaussian steering properties between the second and third harmonics, utilizing the steering criterion, and demonstrate that the non-Gaussian characteristics can be manipulated by modulating the nonlinear parameters.

2. Equations of motion

The setup diagram proposed by this scheme is shown in figure 1. We consider a pump with the frequency ω_0 is incident onto a QOSL. Second harmonic with a frequency of ω_1 is generated by a double-frequency process of the pump. By using the QPM technique, the third harmonic with a frequency of ω_2 is produced by a cascaded sum-frequency process between second harmonic and pump in the same optical superlattice. Ferraro *et al* investigated three-mode entanglement induced by two I-type, non-collinearly phase-matched, interconnected bilinear interactions in an optical $\chi^{(}2)$ medium [66]. They demonstrated the feasibility of the scheme through experimental results. Bondani *et al* [67] also proposed a compact experimental implementation of five field mode interactions in $\chi^{(}2)$ nonlinear crystals. Our scheme differs from theirs in that it is collinear, and the output beams require a prism to separate them. Compared to our approach, their choice of non-collinear phase matching provides significant flexibility for experimental setups. In this study, the first QPM condition is given by $\mathbf{k}_1 = 2\mathbf{k}_0 + \mathbf{G}_1$ and the second QPM condition for the cascaded sum-frequency process is $\mathbf{k}_2 = \mathbf{k}_0 + \mathbf{k}_1 + \mathbf{G}_2$, where $\mathbf{k}_0$, $\mathbf{k}_1$ and $\mathbf{k}_2$ are the wave vectors of pump, second-harmonic, and third-harmonic, respectively. $\mathbf{G}_i$ represents a reciprocal vector of the QOSL and ensures the phase-matching condition. The energy conservation relation can be written as $\omega_1 = 2\omega_0$, $\omega_2 = \omega_0 + \omega_1 = 3\omega_0$. The interaction Hamiltonian can be written as

$$H_I = i\hbar\kappa_1 \left(\hat{a}_0^2\hat{a}_1 - \hat{a}_0^{\dagger 2}\hat{a}_1\right) + i\hbar\kappa_2 \left(\hat{a}_0\hat{a}_1\hat{a}_2^\dagger - \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_2\right) + i\hbar\lambda \left(\hat{a}_2\hat{b}^\dagger - \hat{a}_2^\dagger\hat{b}\right), \tag{1}$$

where κ_i represents the nonlinear coupling coefficient which is closely related to the nonlinear susceptibility, structure parameters of the optical superlattice, and pump power density, and can be taken as real for simplicity. In practical experiments, due to the lower conversion rate of the second linear process in cascaded nonlinear processes, in order to improve conversion efficiency, the nonlinear coupling coefficient κ_2 of the second nonlinear process is usually slightly larger than that of the first nonlinear coupling constant κ_1. The third term represents the non-Gaussian operation process of coupling the third-harmonic light field with the vacuum field minus one photon. From figure 1, we can see that after the third harmonic is coupled with the vacuum field by the BS, the reflection part records the subtracted photon number with the detector D_1 that can resolve the photon number, and the transmission part is the subtracted photon third harmonic and detected by the detector D_2. A photon detector is capable of measuring at least one photon, and its quality is typically characterized by two basic parameters: quantum efficiency (QE) and loss. Generally, QE mainly depends on the wavelength, while loss are inherent to the detector. High detector losses result in reduced signal strength, reduced noise ratio (SNR), and limited dynamic range. It is non- unit QE that affects the

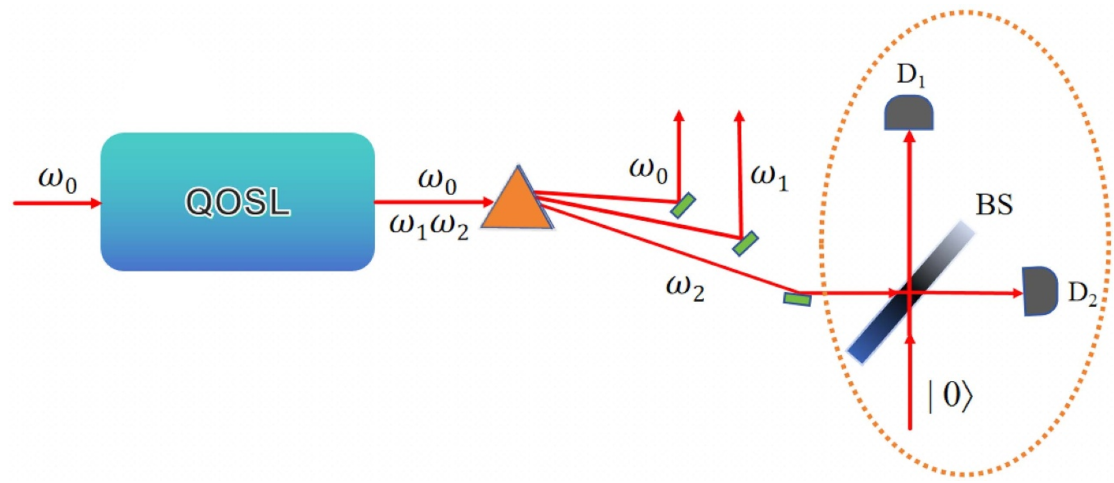

Figure 1. Sketch for the proposed scheme.

photon detection probability. So, a good photon detector should have high QE and low loss. For example, when the input state is the Fock state, $\rho = |n\rangle\langle n|$, the probabilities of Single photon detector is given by $P = T^{n-1}n(1-T)^2$. The coupling coefficient λ of the BS is related to its transmittance T. To guarantee a high probability of photon subtraction, the transmittance T is generally chosen such that $1 - T \ll 1$. $\hat{a}_i$ and $\hat{b}$ are bosonic annihilation operators of the cavity mode with the frequency ω_i ($i = 0, 1, 2$) and the vacuum field $|0\rangle$ with the frequency ω_b satisfying the Bose commutation relation $[\hat{a}_i(\hat{b}), \hat{a}_i^\dagger(\hat{b}^\dagger)] = 1$. Here, all the four optical fields are assumed to resonance in the system. The master equation of the system can be expressed as

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}[H_I, \hat{\rho}], \tag{2}$$

where ρ is density matrix of the system. In [48], the authors use and compare approximately truncated Wigner and precise positive-P representations for calculating and comparing predictions of strength, second-order quantum correlations, and fourth-order cumulants. The pump and damping Bose–Hubbard dimer are also analyzed as a source of the non-Gaussian statistical continuous variables EPR steering. Here, we primarily use the Wigner function representation to study the non-Gaussian steering properties between the second- and third-harmonic optical fields. By mapping the master equation equation (2) to the Wigner function representation, the Fokker–Planck equation can be expressed as

$$\begin{aligned}\frac{\mathrm{d}W(\alpha,\alpha^*,\beta,\beta^*)}{\mathrm{d}t} = &\left\{(-2\kappa_1\alpha_0^*\alpha_1 - \kappa_2\alpha_1^*\alpha_2)\frac{\partial}{\partial\alpha_0} + (-2\kappa_1\alpha_0\alpha_1^* - \kappa_2\alpha_1\alpha_2^*)\frac{\partial}{\partial\alpha_0^*}\right.\\ &+ (\kappa_1\alpha_0^2 - \kappa_2\alpha_0^*\alpha_2)\frac{\partial}{\partial\alpha_1} + \left(\kappa_1\alpha_0^{*2} - \kappa_2\alpha_0\alpha_2^*\right)\frac{\partial}{\partial\alpha_1^*}\\ &+ (\kappa_2\alpha_0\alpha_1 + \kappa_3\alpha_3)\frac{\partial}{\partial\alpha_2} + (\kappa_2\alpha_0^*\alpha_1^* + \lambda\beta^*)\frac{\partial}{\partial\alpha_2^*}\\ &\left. + (-\lambda\alpha_2)\frac{\partial}{\partial\beta} + (-\lambda\beta^*)\frac{\partial}{\partial\beta^*}\right\}W(\alpha,\alpha^*,\beta,\beta^*),\end{aligned} \tag{3}$$

where $\alpha_i(\beta)$ and $\alpha_i^*(\beta^*)$ are independent variables $\hat{a}_i(\hat{b})$ and $\hat{a}_i^\dagger(\hat{b}^\dagger)$ correspond to the operator. Our scheme is different from the one in [48], since there are no second-order or higher derivatives involved and truncation is not necessary here. The coupled differential equations of motion can be obtained from equation (3) as

$$\begin{aligned}\frac{\mathrm{d}\alpha_0}{\mathrm{d}t} &= -2\kappa_1\alpha_0^*\alpha_1 - \kappa_2\alpha_1^*\alpha_2,\\ \frac{\mathrm{d}\alpha_1}{\mathrm{d}t} &= \kappa_1\alpha_0^2 - \kappa_2\alpha_0^*\alpha_2,\\ \frac{\mathrm{d}\alpha_2}{\mathrm{d}t} &= \kappa_2\alpha_0\alpha_1 + \lambda\beta,\\ \frac{\mathrm{d}\beta}{\mathrm{d}t} &= -\beta\alpha_2.\end{aligned} \tag{4}$$

The key difference from the positive-P representation equation is that, in addition to multiplicative noise terms, the initial conditions for each stochastic trajectory must be drawn from the appropriate Wigner distribution of the desired quantum state. The relationship between the variables $\alpha_i(\beta)$ and the operators $\hat{a}_i((\hat{b}))$ is as follows: when we take the average of the product of Wigner variables over numerous stochastic trajectories, the outcome corresponds to the symmetrically ordered expectation values of the corresponding operators. In other words, $\overline{|\alpha_i|^2} = \frac{1}{2}\langle \hat{a}_i^\dagger \hat{a}_i + \hat{a}_i \hat{a}_i^\dagger \rangle$. The steady state solution can be obtained from the formula equation (4) and setting $\frac{\mathrm{d}\alpha_i(\beta)}{\mathrm{d}t} = 0$. If we discard the third term in equation (1), then the system is a simple Gaussian system whose differential equation can be written as

$$\begin{aligned}\frac{\mathrm{d}\alpha_0}{\mathrm{d}t} &= -2\kappa_1\alpha_0^*\alpha_1 - \kappa_2\alpha_1^*\alpha_2,\\ \frac{\mathrm{d}\alpha_1}{\mathrm{d}t} &= \kappa_1\alpha_0^2 - \kappa_2\alpha_0^*\alpha_2,\\ \frac{\mathrm{d}\alpha_2}{\mathrm{d}t} &= \kappa_2\alpha_0\alpha_1.\end{aligned} \tag{5}$$

3. Gaussian quantum steering

EPR steering has recently been demonstrated as a crucial resource for the remote generation of Wigner negativity on the steering mode by performing single-photon subtraction on the steered mode [44–46]. For our purposes, based on the Gaussian EPR steering criterion proposed by Cavalcanti *et al* [68], we can obtain the Gaussian EPR steering between the third harmonic and the second harmonic. The Gaussian EPR steering criterion formula is [68]

$$\begin{aligned}S_{i|j} &= \Delta\left(\hat{X}_i - g\hat{X}_j\right)\Delta\left(\hat{Y}_i + g\hat{Y}_j\right) \geqslant 1,\\ S_{j|i} &= \Delta\left(\hat{X}_j - g\hat{X}_1\right)\Delta\left(\hat{Y}_j + g\hat{Y}_i\right) \geqslant 1,\end{aligned} \tag{6}$$

where $\Delta(A) = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$. The quadratures are defined as $\hat{X}_i = (\hat{a}_i + \hat{a}_i^\dagger)/2$ and $\hat{Y}_i = (\hat{a}_i - \hat{a}_i^\dagger)/2i$. If the inequality $S_{i|j}(S_{j|i})$ is violated then mode $j(i)$ can steer mode $i(j)$.

Using equations (5) and (6), figure 2 plots the relative steering varies $S_{2|1}$ and $S_{1|2}$ of the second harmonic and third harmonic versus the dimensionless propagation length ξ with $\kappa_1 = 0.01$ for different coupling constant of κ_2: $\kappa_2 = 0.005, 0.01, 0.015$, and 0.02, respectively. One can see from figure 2 that when $\kappa_2 = 0.005$, $S_{2|1} < 1$ in the whole dimensionless propagation length ξ range, indicating that second harmonic can steer third harmonic. However, it is obvious that with the increase of the coupling constant κ_2, the ability of mode $\hat{a}_1$ steering mode $\hat{a}_2$ is weaker. Because the conversion efficiency of the cascaded sumfrequency process increases with the increase of κ_2 and more second harmonic are consumed to generate third harmonic. When κ_2 is fixed, we can see that $S_{2|1}$ increases and then decreases as ξ increases, and then increases again. Because the dimensionless length ξ, which is related to the crystal structure, can influence the conversion efficiency of nonlinear processes. With an increase in the dimensionless length ξ, conversion efficiency rises, generating more second and third harmonics. However, this also leads to a gradual decrease in quantum properties and weakening of quantum steering, ultimately resulting in an increase in the $S_{2|1}$ value. As the dimensionless length ξ continues to increase, the nonlinear process may experience backflow, leading to a reduction in the photon counts of the second and third harmonics. This, in turn, enhances quantum properties and strengthens quantum steering, which translates to a decrease in the $S_{2|1}$ value. When the dimensionless length ξ reaches a threshold, the pump beam may be completely consumed, causing the photon counts of the second and third harmonics to saturate. Consequently, quantum properties vanish, and quantum steering disappears as well. It can also be seen from figure 2(b) that with the increase of coupling constant κ_2, the steering ability of mode $\hat{a}_2$ steering mode $\hat{a}_1$ is weaker, and EPR steering exists in the whole parameter range. The nonlinear coupling coefficient κ_i is related to the nonlinear susceptibility and the structure parameters of the optical superlattice. Thus, one can adjust the nonlinear coupling coefficient κ_i by adjusting the structure parameters of the optical superlattice in order to obtain better EPR steering optical fields.

4. Result analysis of non-Gaussian quantum steering

In the previous section, we have shown that there is Gaussian EPR steering between the third harmonic and the second harmonic before the subtraction operation. In the category of continuous variables, quantum states can be divided into Gaussian states and non-Gaussian states, and the most obvious difference between these two states is the representation of their corresponding Wigner functions. The Wigner function of the

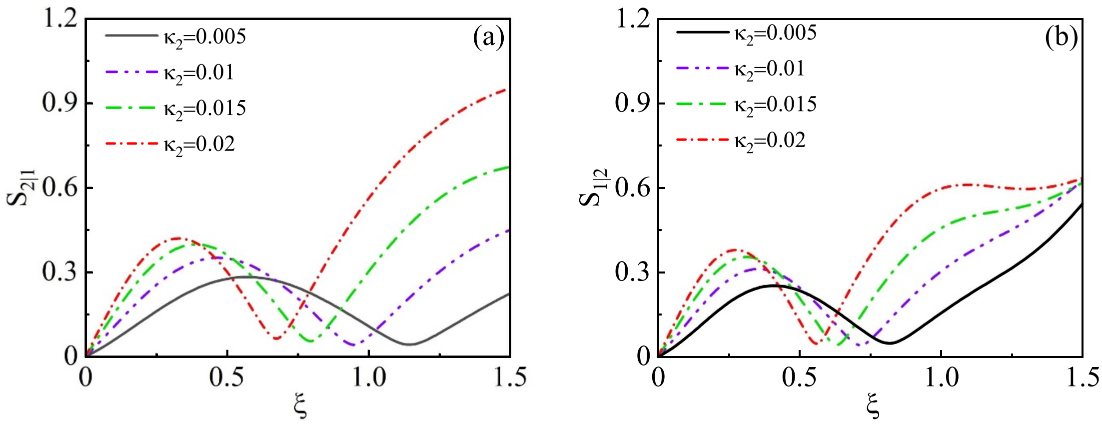

Figure 2. Evolution of (a) $S_{2|1}$ and (b) $S_{1|2}$ as a function of the dimensionless length $\xi = \kappa_1|\alpha_0|t$ with $\kappa_1 = 0.01$ on Gaussian EPR steering for different coupling constant κ_2.

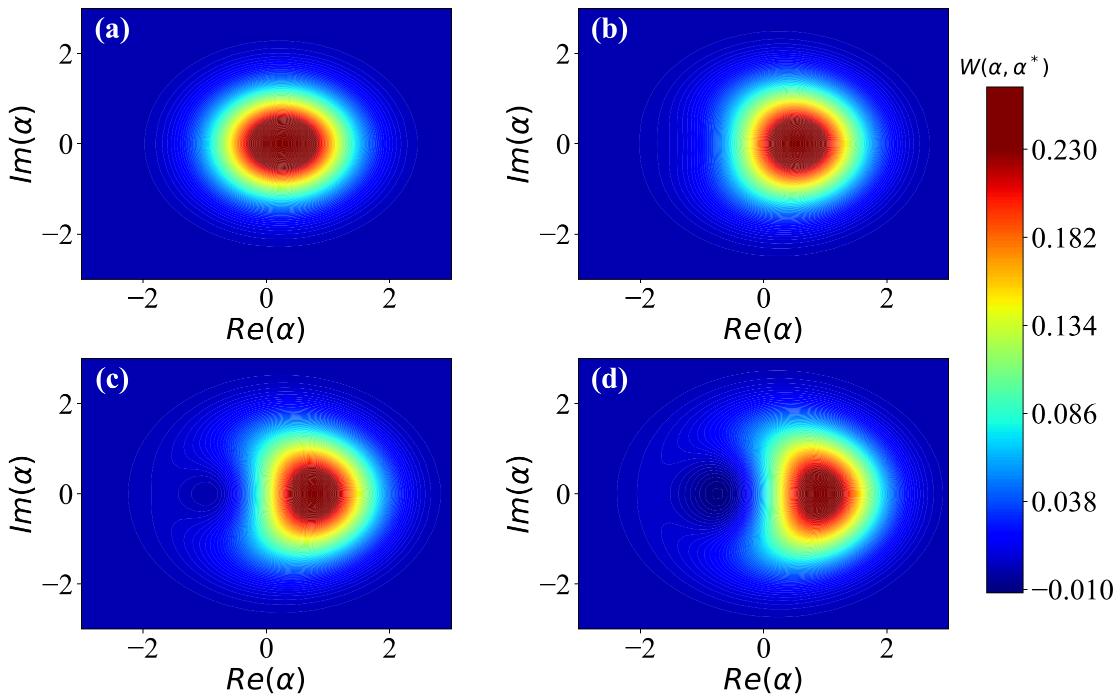

Figure 3. Wigner function of third harmonic $\hat{a}_2$. Coupling constant κ_2: (a) 0.005, (b) 0.01, (c) 0.015, and (d) 0.02, respectively.

non-Gaussian state has negative region, while the Gaussian state does not. The corresponding Wigner function is defined as [70]

$$W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \left[e^{\eta^*\alpha-\eta\alpha^*}\chi(\eta)\right]\mathrm{d}^2\eta, \tag{7}$$

with the characteristic function $\chi(\eta) = Tr[\rho e^{\eta^*\alpha-\eta\alpha^*}]$.

Figure 3 plots the Wigner function of the third harmonic $\hat{a}_2$ for differnet couping constant κ_2 with $\xi = 1$, $\lambda = 0.004$, and $\kappa_1 = 0.01$. It can be see from figure 3 that the non-Gaussian property of the third harmonic gradually appears and becomes stronger with the increase of the coupling constant κ_2. When the κ_2 of the cascaded nonlinear coefficient is relatively small, the conversion efficiency of the third harmonic is relatively low [68]. When κ_2/κ_1 is greater than 1, the conversion effi- ciency of the third harmonic is relatively high [68]. At this time, after non-Gaussian operation, the non-Gaussian characteristics of the third harmonic are revealed. Figure 4 plots the Wigner function of the second-harmonic $\hat{a}_1$ for different coupling constant κ_2 with $\xi = 1$, $\lambda = 0.01$, and $\kappa_1 = 0.01$. The initial values of the optical fields are chosen with the same values as in figure 2. One can see that Wigner function of the second-harmonic has negative region, which shows that

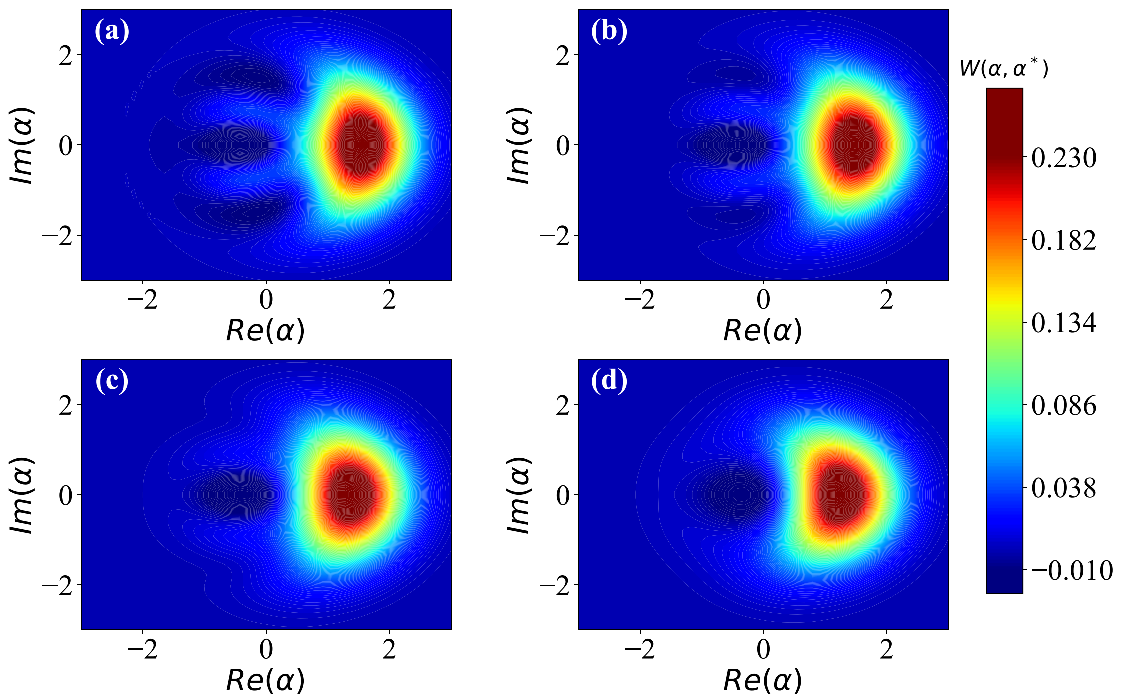

Figure 4. Wigner function of second harmonic $\hat{a}_1$. Coupling constant κ_2: (a) 0.005, (b) 0.01, (c) 0.015, and (d) 0.02, respectively.

the third harmonic can induce the second harmonic to produce non-Gaussian property after photon reduction operation of third harmonic, and as we can see from figure 4 that the non-Gaussian property of the second-harmonic becomes weaker as the increasing of the coupling constant κ_2. Combined with figure 2, it is evident that the stronger the Gaussian steering ability of the third harmonic to the second harmonic, the stronger the ability to induce the second harmonic to produce non-Gaussian properties after the third harmonic undergo photon reduction operation. Moreover, in order to study the relation between Wigner negativity of optical field and parameter variation, the non-Gaussian properties of the corresponding optical field modes can be quantified by Wigner negativities [70, 71]

$$\delta = \int \left[\left| W(\alpha,\alpha^*) \right| - W(\alpha,\alpha^*) \right] \mathrm{d}^2\alpha > 0, \tag{8}$$

where $W(\alpha,\alpha^*)$ is the Wigner function of optical field $\hat{a}$. If $\delta > 0$, optical mode $\hat{a}$ has non-Gaussian property. And the larger the value of δ, the stronger the non-Gaussian property of the optical field $\hat{a}$. In 1989, Reid [6] gave the criterion for EPR quantum steering, which manifests itself as follows

$$V_{\mathrm{inf}}\left(\hat{X}_{ij}\right) = V\left(\hat{X}_i\right) - \frac{\left[V\left(\hat{X}_i, \hat{X}_j\right)\right]^2}{V\left(\hat{X}_j\right)},$$
$$V_{\mathrm{inf}}\left(\hat{Y}_{ij}\right) = V\left(\hat{Y}_i\right) - \frac{\left[V\left(\hat{Y}_i, \hat{Y}_j\right)\right]^2}{V\left(\hat{Y}_j\right)}, \tag{9}$$

where $V(\hat{X},\hat{Y}) = \langle \hat{X}\hat{Y} \rangle - \langle \hat{X} \rangle \langle \hat{Y} \rangle$. $\mathrm{EPR}_{ij} = V_{\mathrm{inf}}(\hat{X}_{ij}) V_{\mathrm{inf}}(\hat{Y}_{ij})$ to express the product of these two inferred variances. If EPR_{ij} is less than 1, it means that optical field $\hat{a}_j$ can steer optical fisld $\hat{a}_i$. This condition is optimal for bipartite Gaussian systems, and at least sufficient for non-Gaussian systems. Therefore, we only need to prove that the optical fields $\hat{a}_i$ have non-Gaussian properties, and we believe that the detected quantum steering is non-Gaussian quantum steering. Figure 5 plots the Wigner negativity δ and EPR_{ij} of the second harmonic $\hat{a}_1$ and the third harmonic $\hat{a}_2$ versus the λ and κ_2/κ_1, respectively. One can see from figure 5(a) that the Wigner negativity δ of the second harmonic is greater than 0 throughout the parameter range and it gradually increases with the increase of the coupling constant λ, which indicates that the non-Gaussian property of the second harmonic gradually increases. Because the larger λ, the stronger interaction between the third harmonic and the vacuum field, the more photons are subtracted, resulting in the stronger the ability to induce the second harmonic to produce non-Gaussian properties. However, the third harmonic is just the opposite of the second harmonic. The Wigener negativity of the third harmonic

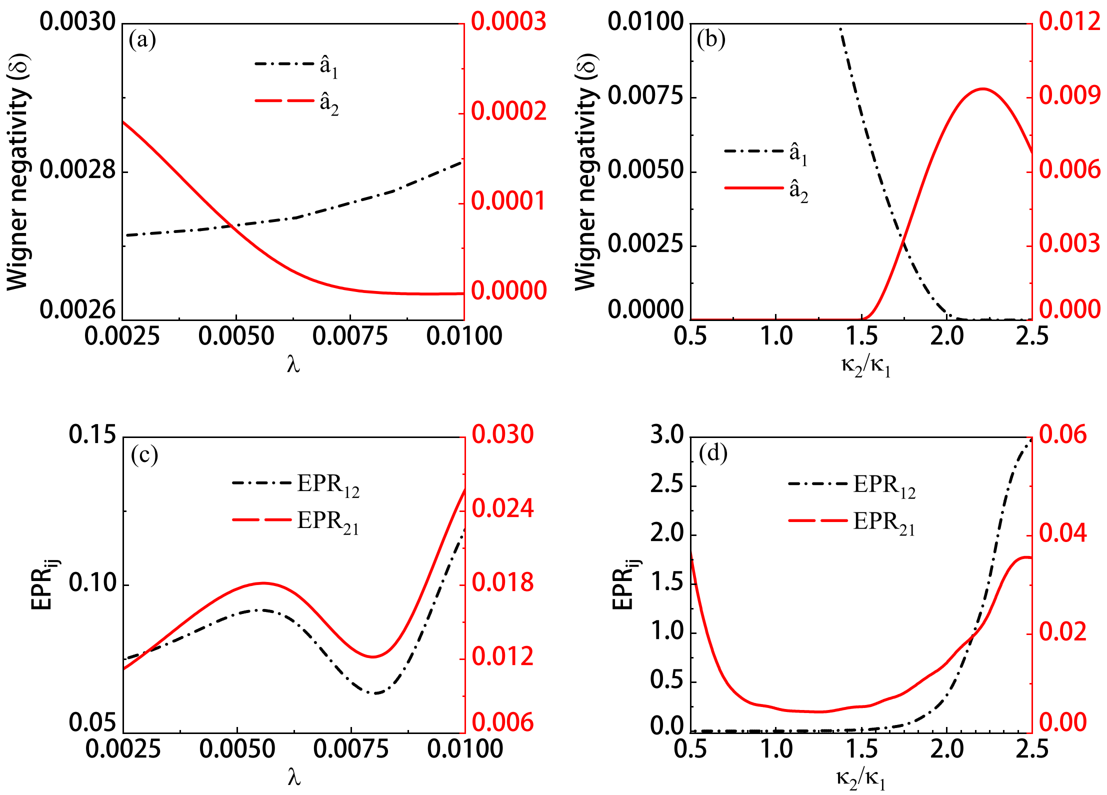

Figure 5. Wigner negativity versus (a) the coupling constant λ with $\kappa_1 = 0.01$ and $\kappa_2 = 1.5\kappa_1$ and (b) the ratio κ_2/κ_1 with $\kappa_1 = 0.01$ and $\lambda = 0.004$, respectively. EPR_{ij} versus (c) the coupling constant λ and (d) the ratio κ_2/κ_1, respectively.

decreases gradually with the increase of coupling strength λ. This is because as λ increases, the third harmonic becomes weaker, and the corresponding non-Gaussian characteristics also weaken. Interestingly, we can see that when $\lambda < 0.005$, the Wigner negative value of third harmonic $\hat{a}_2$ is greater than the second harmonic $\hat{a}_1$, because with the increase of λ, the number of photons subtracted from third harmonic $\hat{a}_2$ is large, and the ability of inducing second harmonic $\hat{a}_1$ to produce non-Gaussian is strong. The Wigner negative values of second harmonic $\hat{a}_1$ and third harmonic $\hat{a}_2$ are equal when about $\lambda = 0.005$, but the Wigner negative values are less than second harmonic $\hat{a}_1$ as $\lambda > 0.005$. We can appropriately choose $\lambda = 0.005$ as the optimal parameter for generating non-Gaussiality. Since a is related to the transmittance of the BS and the detection probability of the detector, we can control it by controlling the structure of the BS to improve the probability of photon reduction.

Figure 5(b) shows that the non-Gaussiality of the second harmonic gradually decreases with the increase of the coupling constant κ_2/κ_1 and disappears when $\kappa_2/\kappa_1 > 2$. Because larger nonlinear coefficient κ_2 in cascaded nonlinear process increase the conversion efficiency of third harmonic and weaken second harmonic, thereby weakening the non-Gaussian characteristics of second harmonic. This is consistent with the result in figure 4. The Wigner negativity of the third harmonic does not change significantly until around $\kappa_2/\kappa_1 = 1.5$, which is consistent with the results in figure 3. But from figure 5(b), it can be seen that the Wigner negativity of the third harmonic increases to the extreme value and then decreases, indicating that κ_2 cannot be too large. This is because as the conversion efficiency of the third harmonic increases, the third harmonic becomes stronger, and the non-Gaussian operation effect of photon reduction decreases, so the non-Gaussian characteristics of the third harmonic also weaken. In addition, within the range of κ_2/κ_1 greater than 1.5 and less than 2, both optical fields exhibit non-Gaussian characteristics.

Figures 5(c) and (d) depict the EPR steering between the third harmonic and the second harmonic after photon reduction operation. In connection with figure 5(a), we can see from the figure 5(c) that the EPR_{12} and EPR_{21} are all less than 1 which indicates that steering between the third harmonic and the second harmonic can be generated. It also shows that when both third and second harmonics are non-Gaussian, better symmetric non-Gaussian steering can be generated in this case. From figure 5(d), it can be observed that EPR_{21} is less than 1 across the entire parameter range, indicating that optical field $\hat{a}_1$ can non-Gaussian steer optical field $\hat{a}_2$ in the whole range. When about $0.5 < \kappa_2/\kappa_1 < 2.2$, EPR_{12} is less than 1 which shows that optical field $\hat{a}_2$ can non-Gaussian steer optical field $\hat{a}_1$. When about $\kappa_2/\kappa_1 > 2.2$, EPR_{12} is more than 1 which shows that optical field $\hat{a}_2$ cannot steer optical field $\hat{a}_1$ in this case. At this point, there exists an asymmetric non-Gaussian steering between second harmonic and third harmonic.

5. Conclusions

In this paper, we first investigate the generation of non-Gaussian quantum steering between the third harmonic and the second harmonic through cascaded sum-frequency processes. We demonstrate that the third harmonic can steer the state of the second harmonic and, through a non-Gaussian operation such as photon subtraction, can induce non-Gaussian properties in the second harmonic. Across a broad range of coupling constants λ, both the second and third harmonics exhibit strong non-Gaussian features, leading to symmetric non-Gaussian EPR steering between them. As the coupling parameter κ_2 increases, we observe an enhancement in the conversion efficiency of the third harmonic. Concurrently, the non-Gaussian characteristics of the second harmonic progressively diminish, while those of the third harmonic become more pronounced. This transition results in a shift from symmetric to asymmetric non-Gaussian EPR steering between the two optical fields. The production of non-Gaussian states via remote photon subtraction has been the subject of extensive theoretical and experimental research, as seen in [44–46]. Recently, it has been proven that non Gaussian entangled states are indispensable resources for CV entanglement distillation [72, 73], quantum enhanced sensing [34, 74], and quantum computers. Specifically, it has been demonstrated that the existence of sampling complexity in boson quantum computing is a necessary requirement for non Gaussian entanglement [75]. Our proposed scheme provides a promising route for experimental realization and has great potential for the development of the above application fields. In addition, by using quasi-phase-matching technology [65], multiple cascaded nonlinear processes can be achieved in an optical superlattice, thereby obtaining multiple output light fields of different frequencies. Our method provides a flexible approach for preparing multi-mode non-Gaussian EPR steering states.

Data availability statement

The data cannot be made publicly available upon publication because no suitable repository exists for hosting data in this field of study. The data that support the findings of this study are available upon reasonable request from the authors.

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