*Master's Thesis*

# Higgs Inflation


Christian van Es

Supervisor: Dr. Diederik Roest

Centrum voor Theoretische Natuurkunde
Rijks*universiteit* Groningen



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**Abstract**

In this thesis, we examine the possibility that the Standard Model Higgs boson played the role of the inflaton. A standard $\lambda\phi^4$ potential requires an unphysically small coupling constant ($\lambda \sim 10^{-13}$) to obtain density perturbations in agreement with observational data. A large coupling between the scalar field and the Ricci curvature scalar relaxes this condition, and the field $\phi$ might be identified with the Standard Model Higgs field. In such a model, the predicted values of the spectral index and the tensor-to-scalar ratio are also in agreement with current observational data. However, quantum corrections seem to break the theory down at the cut-off scale $\Lambda = M_{\rm Pl}/\xi$, which is below the energy scale where inflation takes place. Finally, we analyze the role of the Goldstone bosons. During inflation the first derivative of the Higgs potential doesn't vanish. This leads to a nonzero mass term for the the Goldstone bosons and hence they contribute to the Coleman-Weinberg potential.

# Contents

# Chapter 1

# Introduction

The idea that all matter around us consists of indivisible particles dates back to the ancient Greeks, although it should be said that this principle has always been more based on abstract reasoning or just pure speculation rather than empirical grounds. This changed at the beginning of the 18th century when experimental observations led to the development of atomic theory. Later it turned out that also atoms consist of smaller sub-particles. The first known particle still seen as elementary today is the electron, discovered by J.J. Thomson in 1897. In the middle of the 20th century, with particle accelerators reaching higher energies, more and more exotic particles were found. A careful analysis of all these experiments resulted in the conclusion that all matter is made up of three generations of quarks and leptons. The Standard Model, which was more more or less completed in the mid 1970's, successfully describes the dynamics of these elementary particles. A crucial ingredient is the Higgs mechanism, which is the way particles obtain their masses in this model. This predicts the existence of the Higgs boson, which was finally discovered at the Large Hadron Collider in 2012. Unfortunately, the Standard Model fails to describe gravitation.

Although gravity is negligibly small compared to the other forces in the Standard Model, at large scales it becomes the dominant force. The currently accepted description of gravitation, that suffices at least at large scales, is Einstein's General Theory of Relativity. The Einstein's equations, which are part of this theory, allow the possibility of a non-static universe. Indeed, in 1929 it was shown that we live in an expanding universe and in 1998 it turned out that it is even accelerating. Reversing this picture, we see the universe must originate from a singular point in spacetime, known as the Big Bang. According to this theory the universe was once much hotter than it is now. The 'thermal history' of our universe after $t \sim 10^{-10}$ seconds is quite well understood by a combination of different disciplines in physics and cosmology. Going back even further in time, the energy density was so high that the Standard Model can no longer be trusted. Interestingly, the very early universe has left its mark on the Cosmic Microwave Background. Thus a careful study of the CMB might tell us more about physics beyond the Standard Model.

The standard theory of an expanding universe has several shortcomings as was realized in the 1970's. First of all, the Cosmic Microwave Background looks almost exactly the same in every direction. However, at the time of decoupling only regions of the CMB observed over an angle of about one degree were in causal contact. Another problem is that the energy density of the universe seems to have an extremely fine-tuned value. If it were only slightly bigger or smaller, the universe would immediately have collapsed or ripped apart, yet we know the universe is more than 13 billion years old. This is of course related to the anthropic principle, but this doesn't give a satisfactory answer. In 1981, A. Guth proposed an inflationary epoch right after the Big Bang as a solution to these problems. An important confirmation for inflation was the observed scale invariance of the density perturbations.

Although we don't know what caused the inflationary epoch, it is well known that a scalar field slowly rolling down a potential can lead to inflation. This mechanism is known as slow-roll inflation. In this thesis we will investigate the possibility that inflation is caused by the only scalar field in the Standard Model, namely the Higgs field. As we will see, a crucial ingredient will be a coupling between the Ricci scalar and the Higgs field. But before discussing Higgs Inflation, we give an introduction to cosmology and particle physics in the first two chapters.

# Chapter 2

# Cosmology

## 2.1 The Expanding Universe

In 1929 Hubble, investigating spectral lines of distant galaxies, noted a shift towards the red end of the spectrum. If one assumes that distant galaxies move away with respect to our own galaxy, the redshift can be interpreted as a Doppler shift. The observed wavelength $\lambda'$ of a galaxy receding away with velocity $v$ is

$$\lambda' = \lambda(1+z) = \lambda\sqrt{\frac{1+\beta}{1-\beta}}, \tag{2.1}$$

where $\beta = v/c$ and the amount of redshift $z = \delta\lambda/\lambda$. For small $v$ and thus small redshifts [1]

$$\lambda' \approx \lambda(1+\beta), \tag{2.2}$$

so $v$ is proportional to $z$. There turns out to be a linear relation between the distance $d$ of a galaxy and the amount of redshift $z$, which yields the Hubble law

$$v = H_0 d. \tag{2.3}$$

In this equation $H_0$ is the Hubble constant, and while it was vastly overrated by Hubble himself, today's accepted value is [1]

$$H_0 = 70.4^{+1.3}_{-1.4}\ \mathrm{km\,s^{-1}\,Mpc^{-1}}. \tag{2.4}$$

In an accelerating or decelerating universe $H$ will not be constant, that's why the subscript is assigned when the value measured in our era is meant. As we will see, the possibility of a dynamic universe is allowed by general relativity.

The Cosmological Principle states that our universe is homogeneous and isotropic. A homogeneous universe is one that is translation invariant, while isotropy means that there is no preferred direction. This may seem strange, since this does certainly not hold for the universe as we see it. The Earth is clearly inhomogeneous and anisotropic and there is also structure on larger scales, e.g. planets, stars, galaxies etc. However, on very large scales the Cosmological

[1] For distant galaxies with redshift $z \gtrsim 1$ also gravitational redshifts become important, so for high redshifts the interpretation as a Doppler shift certainly fails.

Principle does hold and then the universe can be regarded as some kind of cosmic fluid, the dynamics of it described by the laws of gravity.

According to Einstein's Theory of General Relativity, gravity is not a force but is instead due to the fact that spacetime is curved, the source of the curvature being the stress-energy tensor. The geometry of spacetime is described by the metric, which determines the gravitational field. It can be shown that the only metric consistent with the Cosmological Principle is the Robertson-Walker metric [2]. This metric is an exact solution of the Einstein field equations and can be obtained by multiplying the spatial part of the metric for static space by a time-dependent scale factor $a(t)$. In terms of the spherical coordinates $(r, \theta, \phi)$ it has the following form:

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi\right)\right]. \qquad (2.5)$$

From now on we will often stick to the convention $c = 1$. The curvature of the space is defined by the constant $k$, with $k = 0$ corresponding to a flat universe. If $k$ is nonzero, it can always be set to $\pm 1$ by rescaling $a(t)$, with $+1$ corresponding to a closed universe and $-1$ to an open universe.

The coordinates $(r, \theta, \phi)$ are comoving coordinates, which means that even in a dynamic universe a particle initially in rest keeps the same coordinates . However, the physical separation between two points, one at the origin and the other at $r$, is given by the time-dependent quantity

$$d(r,t) = \int \mathrm{d}s = a(t)\int_0^r \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = a(t) \times \begin{cases} \sinh^{-1} r & \text{if } k = -1, \\ r & \text{if } k = 0, \\ \sin^{-1} r & \text{if } k = 1. \end{cases} \qquad (2.6)$$

Now consider a flat universe ($k = 0$). Because also wavelengths scale with $a$, taking the time derivative gives Hubble's law eq. (2.3)

$$\dot{d} = \frac{\dot{a}}{a} d \equiv Hd. \qquad (2.7)$$

The Hubble parameter $H$ turns out to be a very important quantity in cosmology, because it sets the characteristic scales of the FLRW spacetime. A rough estimate of age of the universe is given by the Hubble time $H_0^{-1}$, while the observable universe is of the order of the Hubble length $cH_0^{-1}$.

The time evolution of the scale factor is determined by the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \qquad (2.8)$$

In this equation $G_N = \hbar c/m_{\mathrm{Pl}}^2$ is Newton's constant and $R$ and $R_{\mu\nu}$ are the Ricci scalar and Ricci curvature tensor respectively. Assuming homogeneity and isotropy, the stress-energy tensor $T_{\mu\nu}$ can be written in such a way that it resembles the one for a perfect fluid:

$$T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p), \qquad (2.9)$$

with $p$ the pressure and $\rho$ the energy density. The energy density consists of a matter, radiation and vacuum component. With $T_{\mu\nu}$ in this form, Einstein's

| Type | Energy density | $w$ | Scale factor |
|---|---|---|---|
| Radiation | $\rho \propto a^{-4}$ | $1/3$ | $a \propto t^{1/2}$ |
| Matter | $\rho \propto a^{-3}$ | $0$ | $a \propto t^{2/3}$ |
| Vacuum energy | $\rho = \text{const.}$ | $-1$ | $a \propto e^{Ht}$ |

Table 2.1: The expansion of the universe for regimes dominated by radiation, matter or vacuum energy.

equations reduce to the two independent non-linear ordinary differential equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_{\rm Pl}^2} - \frac{k}{a^2} \tag{2.10}$$

and

$$\left(\frac{\ddot{a}}{a}\right)^2 = -\frac{1}{6M_{\rm Pl}^2}(\rho + 3p). \tag{2.11}$$

Here we have introduced the reduced Planck mass $M_{\rm Pl} = m_{\rm Pl}/\sqrt{8\pi} = 2.44 \times 10^{18}\,\text{GeV}$. The first equation is called the *Friedmann equation* and the second the *Raychaudhuri equation.*

Energy conservation or $\nabla_\nu T^{\mu\nu} = 0$ gives the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.12}$$

In general it is possible to relate the pressure $p$ and energy density $\rho$ by the equation of state

$$p \equiv w\rho, \tag{2.13}$$

with $w$ the equation of state parameter. Then integrating the continuity equation gives

$$\frac{\mathrm{d}\rho}{\rho} = -3(1+w)\frac{\mathrm{d}a}{a}, \tag{2.14}$$

which has solution $\rho \propto a^{-3(1+w)}$. Using the Friedmann equation and neglecting the curvature term gives

$$\frac{\mathrm{d}a}{\mathrm{d}t} \propto a^{-(1+3w)/2}. \tag{2.15}$$

This leads to

$$a \propto \begin{cases} t^{2/(3(1+\omega))} & \text{if} \quad \omega \neq -1, \\ e^{Ht} & \text{if} \quad \omega = -1. \end{cases} \tag{2.16}$$

The universe consists of a mixture of non-relativistic matter, radiation (or relativistic matter) and vacuum energy. For non-relativistic matter the equation of state parameter $w = 0$ and it follows that a matter dominated universe expands as $a \propto t^{2/3}$. The energy density of ordinary matter simply scales as $\rho_{\rm mat} \propto a^{-3}$. The energy density of radiation falls off faster since also momenta $p \propto 1/a$, thus $\rho_{\rm rad} \propto a^{-4}$. The equation of state for radiation is $p = \rho/3$, so $w = 1/3$ and it follows that a radiation dominated universe expands as $a \propto t^{1/2}$. The energy density of the vacuum $\rho_\Lambda$ is constant. The state parameter $w_\Lambda = -1$, so a universe dominated by vacuum energy expands exponentially.

## 2.2 History of our Universe

The Hubble law shows that the universe was once much denser than it is now. If we go back in time we see that all spacetime collapses into a single point. This is the Big Bang scenario. It should be stressed that the Big Bang should not be seen as an ordinary explosion, where matter is pushed radially outward from a single point in space, because this would imply an anisotropic universe. The Big Bang is a singularity where spacetime itself emerges and starts expanding everywhere at the same rate, resulting in a completely isotropic universe.

According to the Stefan-Boltzmann law $\rho_{\rm rad} \propto T^4$, which implies that an expanding universe cools down as $T \propto a^{-1}$. This causes various epochs in the history of the universe. Because it is not known to what energy scale the Standard Model is valid, what happened before $10^{-10}$ s (when the temperature was around 1 TeV, the energy reached by modern particle accelerators) is more or less speculation.

The electroweak symmetry broke down when the temperature dropped below $\sim 100$ GeV. The $W^{\pm}$ and $Z$ bosons became massive and hence the cross section for electroweak processes started to decrease. At $T \sim 3$ MeV the reaction rate of the process $\nu_i + \bar{\nu}_i \to e^+ + e^-$ dropped below the expansion rate of the universe and as a result the neutrinos froze out.

In the first minutes after the Big Bang the protons and neutrons combine into nucleons. This stage is called primordial or Big Bang nucleosynthesis (BBN). The process

$$n + p \longleftrightarrow {}^2\mathrm{H} + \gamma + Q, \tag{2.17}$$

with the binding energy $Q = 2.22$ MeV, remains in equilibrium until the temperature falls below $T \sim Q/40$. This is because the baryon over photon ratio $\eta \sim 10^{-10}$. The formation of stable helium nucleons is delayed until this so called deuterium bottleneck is passed. After $t \sim 180$ s, almost all the neutrons that have frozen out are found in bound states of $^4$He, leading to a helium mass fraction of around 25%. Also the abundances of the light elements D, $^3$He, $^4$He, and $^7$Li are more or less fixed at this time. The predicted deuterium mass abundance is 0.01%, while trace amounts of other isotopes and heavier elements are predicted. These predicted values only depend on the number of baryons per photon. That the observed abundances almost exactly matched the predicted ones was an early and very important confirmation of the Big Bang scenario [3].

In 1965 another major discovery was made by Penzias and Wilson [4]. Searching for cosmic radio waves, they noticed an isotropic background of microwave radiation. They first assumed that this was due to terrestrial sources or a problem in their experimental setup, it later turned out to be the Cosmic Microwave Background (CMB) predicted by Gamow, Alpher and Herman in the 1940's. The observed photons originate from the recombination epoch that took place around $10^5$ years after the Big Bang. Before this time the universe was opaque, because photons strongly interacted with charged particles. The equilibrium process

$$H + \gamma \longleftrightarrow p + e^- \tag{2.18}$$

requires the photons to have at least an energy of 13.6 eV. Due to the dominance of photons over baryons and the long tail of the photon spectrum, recombination happened not before the temperature dropped below $T \sim 4000$ K or $\sim 0.3$ eV. Suddenly the mean free path length of the photons increased enormously and

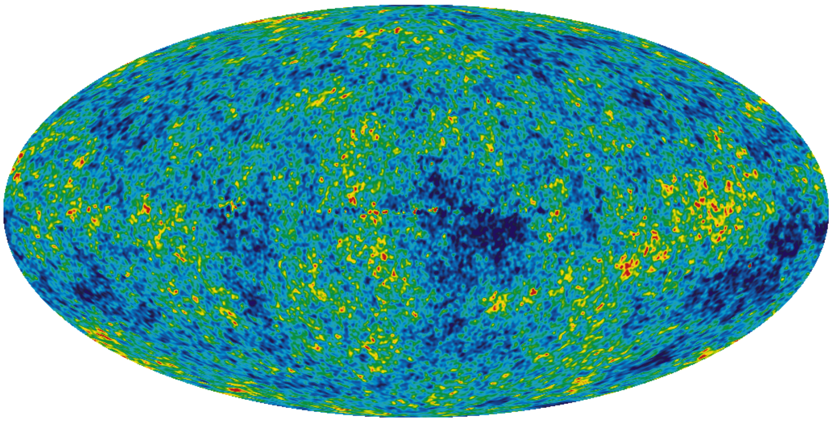

Figure 2.1: The Cosmic Microwave Background mapped by the Wilkinson Microwave Anisotropy Probe (WMAP). The fluctuations around the background temperature $T_0 = 2.7\,\mathrm{K}$ are at the $10^{-5}$ level and are represented as blue and red spots, corresponding to cooler and hotter regions respectively. [5]

they are more or less non-interacting until we observe them. This means that the CMB is some kind of blueprint of the universe when it was around 379,000 years old, see figure 2.1. Because of the expansion of the universe, the CMB is presently observed as a perfect blackbody radiating at $T = 2.73\,\mathrm{K}$. Although it is extremely isotropic, the most important feature of the CMB are actually the tiny perturbations at the $10^{-5}$ level. The regions that are slightly overdense have a lower temperature in the CMB (and underdense regions have a higher temperature). Although overdense regions should be hotter according to the Stefan-Boltzmann law, this is dominated by the fact that the photons in overdense regions have experienced a higher gravitational redshift (the Sachs-Wolfe effect).

The formation of gravitational bound objects started only $10^8$ years after the Big Bang (or $z \lesssim 10$). During the radiation dominant era, clustering was very slow due to the background radiation pressure. It became much more efficient during matter domination. The small density perturbations started to grow via gravitational instabilities. The background expansion counteracts this effect, which is why the instabilities grow as a power law rather than exponentially. In the end, the small scale fluctuations of the very early universe evolved into the large-scale structures (LSS) of the universe. The reionization epoch started when objects had formed that were energetic enough to ionize neutral hydrogen.

Finally after $10^9$ years, vacuum energy started to dominate the universe, resulting in an accelerated expansion. At present vacuum energy constitutes 73 % of the total energy density. The accelerated expansion of the universe was first observed in 1997 in the study of Type Ia supernovae [6], for which the Nobel Prize of Physics was awarded in 2011.

## 2.3 The Standard ΛCDM Model

The closure parameter Ω is the ratio of the actual density to the critical density

$$\Omega = \frac{\rho}{\rho_c}. \tag{2.19}$$

If it is larger than one, the energy density of the universe is larger than the critical density and the universe will eventually collapse. A closure parameter smaller than one will result in a big crunch. As seen in figure 2.2, the closure parameter is very close to one, corresponding to a flat universe.

An estimate of the total mass in a galaxy or a galaxy cluster can be obtained by analyzing the gravitational effects on its visible components. The assumption that most of the baryonic matter in a galaxy is X-ray luminous leads to the conclusion that not all matter can be baryonic of origin. The majority of matter is believed to be so-called cold dark matter (CDM). By definition, the particles that constitute CDM only weakly interact and move slow compared to the speed of light, however, their true nature is still unknown. Cold dark matter played an essential role in the forming of gravitational bound objects in the early universe.

A combination of CMB data, baryon acoustic oscillations (BAO) and supernovae observations, result in the sweet spots as seen in figures 2.2 and 2.3. For this reason the ΛCDM model is also called the concordance model. The different density parameters measured to be

$$\begin{aligned} &\Omega - 1 \leq 10^{-2}, \\ &\Omega_b \simeq 0.05, \\ &\Omega_{\mathrm{CDM}} \simeq 0.23, \\ &\Omega_\Lambda \simeq 0.73. \end{aligned} \tag{2.20}$$

The energy density of radiation $\Omega_r \simeq 5 \times 10^{-5}$ comes mainly from the redshifted CMB photons. The primordial neutrinos are presently non-relativistic and constitute sill 0.3 % of the total energy density. Only a small fraction of all baryonic matter in the universe is luminous, $\Omega_{b,\mathrm{lum}} \simeq 0.01$.

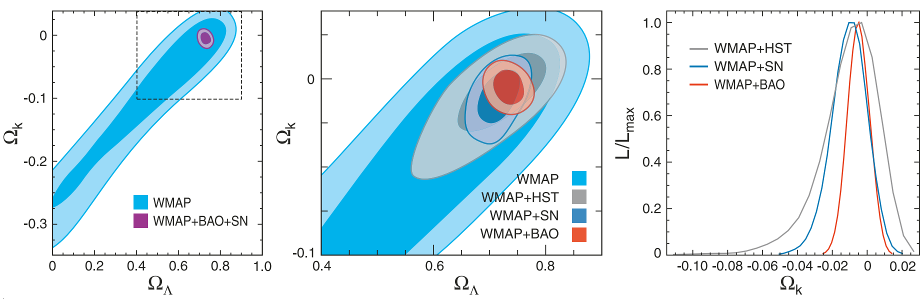


Figure 2.2: A combination of WMAP, baryon acoustic oscillations (BAO), Type 1a supernovae (SN) and the Hubble Space Telescope (HST) constraint the spatial curvature parameter and the vacuum energy density. [5]

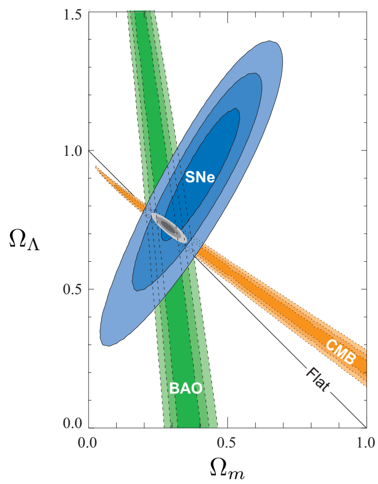


Figure 2.3: Observations of the cosmic microwave background (CMB), baryon acoustic oscillations (BAO) and supernovae (SNe) give a sweet spot indicating that our universe is largely dominated by vacuum energy (72 %), while the rest of the energy density comes from matter (28 %). [7]

## 2.4 Inflation

The ΛCDM-model seems to give a very convincing description about the history of our universe. The observed abundances of the light elements are in perfect agreement with predictions and only a hot Big Bang is consistent with the cosmic microwave background. However in the 1970's it was realized that it had several shortcomings, which we will discuss below. The solutions to the first two problems will be given in section 2.5.

**Horizon problem** The CMB is a nearly perfect black-body spectrum, the anisotropies are only at the $10^{-5}$ level. Thus two photons coming from opposite directions must have been in thermal equilibrium in the past. However at the time of last scattering the universe we observe today consisted of a large number of causally disconnected regions. Only regions within angle of about one degree were causally connected at the time of recombination.

**Flatness problem** The closure parameter was defined as $\Omega = \frac{\rho}{\rho_c}$. The critical density is the density that would just close the universe. From the Friedmann equation

$$H^2 = \frac{\rho}{3M_{\mathrm{Pl}}^2} - \frac{k}{a^2}, \tag{2.21}$$

we see that $\rho_c = 3M_{\mathrm{Pl}}^2 H^2$. The closure parameter can be written as

$$\Omega = \frac{\rho}{\rho_c} = \frac{k}{(aH)^2}. \tag{2.22}$$

A flat universe with $k = 0$ will have the time-independent value $\Omega_0 = 1$, which is an unstable fixed point. In standard cosmology the comoving Hubble radius increases with time, thus $|\Omega - 1|$ diverges with time. Experiments show that $\Omega$ is very close to $\Omega_0$, the observed value is $\Omega_{\text{obs}} = 1.00 \pm 0.01$ [1]. To have $\Omega$ still of order unity today, $\Omega$ must have been extremely fine-tuned to the value $\Omega_0 = 1$ in the past. This brings up the question why our universe is so flat or equivalently why is our universe is so old? A universe that would not immediately collapse or end up in a big crunch requires $\Omega$ to be extremely fine-tuned to zero.

**Unwanted relics/Magnetic-monopole problem** With unwanted relics we mean particles or topological defects created during the very early universe. They are unwanted in the sense that they have never been experimentally observed, yet such relics are predicted in a wide range of theories. In the original paper of Guth [8], inflation served as a solution to the magnetic-monopole problem. Magnetic-monopoles are topological defects which are created during the GUT phase transition. It is easy to see how inflation can serve as a solution to this problem: during inflation the number density drops by many orders of magnitude, while after inflation the temperature is too low to form new particles.

In 1980, Guth proposed inflation as a solution to these problems [8]. The inflation hypothesis states that the universe underwent a period of extremely rapid exponential expansion somewhere between $10^{-36}$ and $10^{-32}$ seconds after the Big Bang in which its volume increased by at least a factor $10^{78}$. Since then physicists have come up with hundreds of different inflation models (of which many have already been rejected). Up to now the mechanism behind inflation is unclear, but after more than 30 years the inflation model itself is still a working hypothesis about the very early universe. In the future the Planck satellite will come up with more precise observations, which hopefully enables us to discriminate between all the possible realizations of inflation.

Nowadays the most important reason to believe that there was a period of inflation are not the problems sketched in this section, but is the scale invariance of the fluctuations in the CMB which will be discussed in section 2.6.

## 2.5 The Shrinking Hubble Sphere

The causal or particle horizon is equal to the maximum distance a light ray coming from a particle could have travelled. In comoving coordinates the comoving particle horizon $\tau$ is given by

$$\tau \equiv \int_0^t \frac{\mathrm{d}t'}{a(t')} = \int_0^a \frac{\mathrm{d}a}{Ha^2} = \int_0^a \mathrm{d}\ln a \left(\frac{1}{aH}\right) \tag{2.23}$$

From eq. (2.15) we see that it scales with $a$ and $a^{1/2}$ for respectively a radiation and matter dominated universe. In general

$$\tau \propto a^{(1+3w)/2}. \tag{2.24}$$

Inflation is defined as a period with $\ddot{a} > 0$ or equivalently $(aH)^{-1} < 0$. Thus during inflation the comoving Hubble length, which is the characteristic length scale of the universe, decreases with time. In comoving coordinates the observable universe becomes smaller.

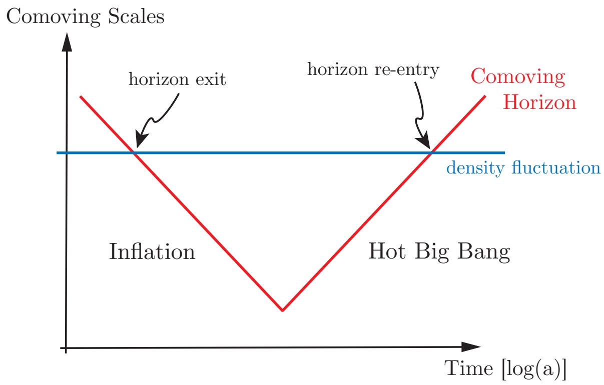


Figure 2.4: This figure shows how quantum fluctuations in the inflationary universe turn into classical perturbations. Since there can be no causal physics at super-horizon scales, perturbations freeze-in when they leave the horizon. After inflation the comoving Hubble radius increases and fluctuations re-enter the horizon when $k \gg aH$. [7]

What does this mean to quantum fluctuations during inflation? Particle and anti-particles pairs pop out of the vacuum all the time. Under normal circumstances they annihilate almost immediately. However, during inflation it is possible that they are ripped apart before they had time to annihilate again. If they are separated by a distance greater than the causal horizon, the quantum fluctuation has turned into a classical perturbation. Since no causal physics can happen on super-horizon scales, the amplitude of a fluctuation remains constant and we say that the perturbation is frozen in. Due to the expansion of the universe all fluctuations are stretched over enormously large distances. Since the fluctuations are constantly generated, they exist at all length scales $k^{-1}$. They leave the horizon when $k < aH$, i.e. they are larger than the characteristic length scale of the universe. After inflation the comoving Hubble length increases and eventually all fluctuations become physical again when $k \gg aH$. The small scales that lead to the formation of baryonic structure re-entered the horizon when the universe was about 100.000 yrs old.

Now the horizon problem can easily be understood. Although the CMB consists of many areas that are not connected now, they have been in causal contact in the past. Homogeneity on very small scales established by causal physics before the inflation era resulted in the homogeneity on very large scales today.

Also the flatness problem is solved by inflation, because during inflation

$$|\Omega(a) - 1| = \frac{k}{(aH)^2}. \tag{2.25}$$

is driven to zero. Remember that during the conventional expansion of the universe the comoving Hubble radius $(aH)^{-1}$ increases withe time and eq. (2.25) diverges. In contrast, in an inflationary era $(aH)^{-1}$ decreases by definition. Physically this means that inflation flattens the curvature of the universe.

## 2.6 Primordial Density Perturbations

Perturbations in one or more scalar fields during inflation are the seeds for the structure of our present day universe. If we call the perturbations existing at the time were nucleosynthesis is about to start, a bit below $T \sim 1\,\mathrm{MeV}$, the primordial perturbations, we have a simple initial condition to study the subsequent evolution of the universe. At this time all the positrons have annihilated with the electrons, so all that remains are baryons, cold dark matter, photons an neutrinos. So the universe can be regarded as a cosmic fluid consisting of 4 components

$$\rho = \rho_\nu + \rho_\gamma + \rho_B + \rho_c \tag{2.26}$$

The photons and neutrinos have already decoupled, while the baryons and photons still interact strongly by Compton scattering. The cold dark matter doesn't interact by definition. It is assumed that on cosmological scales the adiabatic condition holds. This means that the composition of the energy density is the same at every location in the universe. So every individual component can be written as a function of the total energy density, $\rho_a = \rho_a(\rho)$.

The primordial density perturbations have the striking property that they are nearly scale invariant. This is of course due to the fact that the universe inflated almost exponentially. The departure of scale invariance is described by the spectral index $n(k)$, defined by

$$\mathcal{P}_s(k) \propto k^{n(k)-1}. \tag{2.27}$$

Another feature of these primordial perturbations is that they seem to be Gaussian. This means that there is no correlation between the Fourier components of $\delta\rho_{\mathbf{k}}$ measured at two different points in space except the reality condition $\delta\rho^*_{\mathbf{k}} = \rho_{-\mathbf{k}}$. The observed non-Gaussianity is less than $10^{-3}$.

## 2.7 The Power Spectrum

Before recombination baryonic matter is strongly coupled to photons, resulting in oscillatory modes of the baryon-photon plasma. Because they move with the speed of sound, these modes are called baryon acoustic oscillations. The temperature fluctuations in the CMB are a direct result of these oscillations.

The CMB anisotropies can be expressed with the power spectrum, where the temperature fluctuations are shown as a function of the angular scale. In this way the million of pixels of the CMB map are represented in a much more compact form. The CMB temperature fluctuations $\delta T$ with respect to the background temperature $T_0 = 2.7\,\mathrm{K}$ can be expanded in terms of spherical harmonics

$$\frac{\delta T(\mathbf{n})}{T} = \sum_{lm} a_{lm} Y_{lm}(\mathbf{n}), \tag{2.28}$$

with $l$ the multipole number and $m = -l, \ldots, +l$. The vector $\mathbf{n}$ denotes the direction in the sky. Assuming the fluctuations are of Gaussian origin, all information is then included in the expression

$$\langle a^*_{lm} a_{lm} \rangle = \delta_{ll'} \delta_{mm'} C_l. \tag{2.29}$$

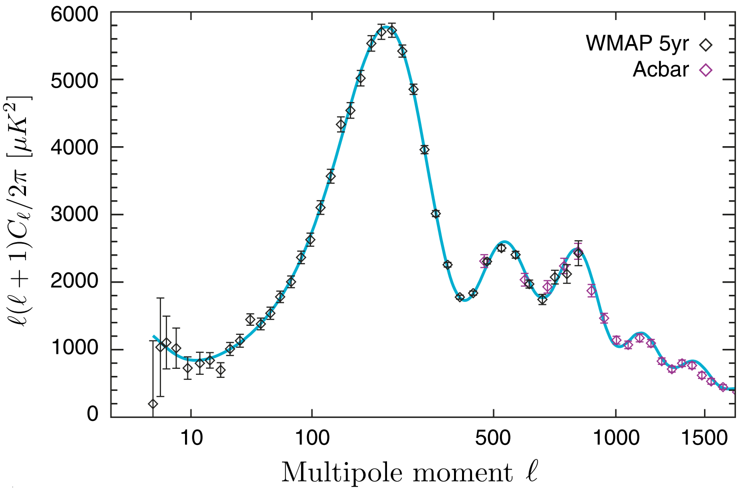


Figure 2.5: Power spectrum of the CMB temperature fluctuations in terms of angular scale. [5]

The power spectrum shown in figure 2.5 is obtained by plotting the quantity $l(l+1)C_l$ against $l$. The statistical variance for the low multipole moments is due to cosmic variance, which means that the sampling size is too small for the largest scales.

Remember that during inflation quantum fluctuations evolve into classical perturbations that are stretched over enormously large distances. These fluctuations are thus created on all possible length scales. Because the comoving wavenumbers are constant and $1/aH$ shrinks during inflation, the fluctuation leaves the horizon when $k < aH$. When they leave the horizon they are frozen in. After inflation the smallest scales will enter the horizon first ($k \ll aH$ means sub-horizon) and start to oscillate again.

Now the powers spectrum can easily be understood. The first peak corresponds to the wave that had just enough time to do a partial oscillation between horizon re-entry and recombination. The second peak had just enough time to do a full oscillation, etc. Similarly $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \ldots$ waves are dips in the power spectrum.

Another feature of the power spectrum is that the peaks rapidly start to decrease for higher multipole moment $l$. One reason for this effect is that recombination did not happen at a single point in time. This doesn't effect the low multipole moments, because the wavenumber of a wave that has just enough time to do one partial oscillation barely changes in this period. On the other hand this has of course a huge influence on the higher multipole moments. A second reason is due to the diffusion of photons before recombination took place. This effect is known as Silk damping and leads to a decrease of the CMB anisotropies for multipole moments $l \gtrsim 800$.

# Chapter 3

# Particle Physics

This chapter starts with a quick review of the Standard Model. After introducing Yang-Mills theory, we are ready to explain the strong and the electroweak interaction. Finally, we discuss the Faddeev-Popov procedure and the $R_\xi$ gauges.

## 3.1 Introduction to the Standard Model

### 3.1.1 Historical Background

The Standard Model provides us a with a theory about elementary particles and their interactions. Since the 1970's this theory has been in agreement with the results of nearly all particle physics experiments. In fact present day experiments confirm it to an even higher accuracy.

A major step towards the theory was made by Glashow in 1960. Based on the work on non-Abelian gauge theories by Yang and Mills, he showed how the weak and electromagnetic interaction could be combined into one force [9]. His electroweak theory was not complete until 1967, when Weinberg implemented the Higgs Mechanism into it [10]. This mechanism predicts the existence of the Higgs boson, which had long been the last missing particle of the Standard Model. It was finally found at the Large Hadron Collider in 2012 [11, 12]. Another important contribution to the development of the Standard Model came from 't Hooft in 1970. He proved that under certain conditions Yang-Mills theories, and the Standard Model in particular, are renormalizable.

The theory of the strong interaction started with the notion of the elementary particles called quarks that make up the proton and other hadrons. The quark model was independently proposed by Gell-Mann [13] and Zweig [14] in 1963 to explain the particle spectrum of the hadrons. In 1968 the quarks were first experimentally discovered in deep inelastic scattering experiments at the Stanford Linear Accelerator Center (SLAC) [15]. Already in 1972, Kobayashi and Maskawa had predicted three generations of quarks to explain CP violation in kaon decays [16]. However, it took until 1995 before the experimental discovery of the last missing quark, the top quark with a mass over $170\,\mathrm{GeV}/c^2$. After the discovery of asymptotic freedom Quantum Chromodynamics became the widely accepted theory for the strong interaction.

For a long time it was believed that neutrinos were massless particles and this is also assumed in the Standard Model. However, the experimental discovery of neutrino oscillations requires the neutrinos to have a nonzero mass [17]. A number of solutions to this problem have been proposed, e.g. the seesaw mechanism [18].

Unfortunately there is one force missing in the Standard Model, namely gravity. This is because it turns out to be very hard to combine general relativity with quantum mechanics. Up to now a quantum theory of gravity is still missing. The classical description presumably only fails at scales smaller than $10^{-33}$ cm, so under most circumstances the classical limit of this force can be applied. Only for the study of very massive objects or extremely high energetic events, like black holes or the very early universe, a quantum theory of gravity is required

It is believed that the Standard Model is only the low-energy limit of a more fundamental theory. Such a theory might fix some of the many free parameters of the Standard Model. For example, the underlying symmetry of the Standard Model is $SU(3) \times SU(2) \times U(1)$ with each symmetry group having a different coupling constant. In so-called Grand Unified Theories (GUT) the symmetry of the Standard Model is put into one larger group with only one coupling constant. This is motivated by the fact that the three Standard Model coupling constants seem to merge into a single value at a certain energy scale ($\Lambda_{\rm GUT} \sim 10^{15}$ GeV). It is then assumed that the higher symmetry is broken when the energy drops below this energy scale, causing the separation of the strong and electroweak interaction. A natural choice for the higher symmetry group would be $SU(5)$. However, since this is in conflict with the observed proton lifetime this theory has already been rejected.

String theory is another attempt to overcome the problems above. The identification of the spin-2 particles that appear in this theory with gravitons provides a natural way to incorporate gravity. Also string theory is unique in the sense that only one parameter (the length scale of the strings, $l_s \sim 10^{-33}$ cm) is needed to specify the theory, which makes it really attractive for aesthetic reasons. In the mid-eighties it was realized that string theory was able to reproduce the Standard Model. Moreover, the large number of extra fields that emerge in string theory might provide candidates for dark energy and dark matter.

### 3.1.2 Particle Content of the Standard Model

The Standard Model is based on Quantum Field Theory [19, 20, 21]. In such theories particles are represented as excitations of quantum fields. One can divide the particles of the Standard Model into fermions and gauge bosons. The fermions, which form the building blocks of all matter around us, can then be split into quarks and leptons. The interactions between these 'matter particles' are described by the exchange of gauge bosons, which can be seen as 'the force carriers'. Finally there is one spin-0 particle in the Standard Model, the Higgs boson. This particle is an excitation of the Higgs field above its ground state. The gauge bosons and fermions obtain their masses by interactions with the Higgs field.

The leptons come in three generations. The first one is well-known and consists of the electron and the electron neutrino, furthermore there are the muon and tau generations. Every generation has its own lepton number. For the electron family it is $L_e$, and this number is defined as one for $e^-$ and $\nu_e$ and zero for the other leptons. The other two lepton numbers, $L_\mu$ and $L_\tau$, are defined in the same way. The different lepton numbers are conserved quantities in all interactions. Every lepton has an corresponding anti-particle with an opposite sign for its charge and lepton number.

The quarks are the elementary particles that form the hadrons. They are sensitive to the strong interaction and hence they must have a color quantum number. Just like the leptons they come in three generations and for every quark there is an anti-quark with opposite charge and color. Due to color confinement, quarks can only be observed as bound states and never as free particles. The baryon number is defined as $B = (n_q - n_{\bar{q}})/3$ and is also conserved in all interactions. Based on this number the hadrons can be separated into the baryons with $B = 1$ and mesons with $B = 0$. The baryons are composed of three quarks, while the mesons are composed of one quark and one anti-quark. For example the proton is the bound state *uud*, while the neutron is denoted as *udd*. A big open question in physics is what causes the baryon-antibaryon asymmetry of the universe.

The six types of quarks (up, down, charm, strange, top, bottom) are also called flavors. Only in the strong interaction flavor is always conserved. So the heavier quarks can only decay to up and down quarks via weak interaction. Because particles of the first generation are the lightest, only these particles are stable. For this reason ordinary matter is almost exclusively made up of particles of the first generation, i.e. electrons, protons and neutrons.

In the Standard Model there are three different kinds of gauge bosons. First of all there is the photon, denoted as $\gamma$, which carries the electromagnetic force. The weak force is mediated by the $W^{\pm}$ and $Z^0$ bosons (the $W$ boson comes in a positively and negatively charged variant). Finally there are eight gluons, the force carriers of the strong interaction.

## 3.2 Non-Abelian Gauge Invariance

A gauge theory is a kind of field theory in which the Lagrangian is invariant under certain transformations that may vary from point to point in space and time. The prototype of a gauge theory is quantum electrodynamics (QED), where the Lagrangian is invariant under $U(1)$ transformations. The QED Lagrangian

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\gamma^{\mu}(\partial_{\mu} + ieA_{\mu})\psi - m\bar{\psi}\psi, \tag{3.1}$$

with the field strength tensor $F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, is indeed invariant under

$$\begin{aligned} \psi(x) &\rightarrow e^{-i\alpha(x)}\psi(x), \\ \bar{\psi}(x) &\rightarrow e^{i\alpha(x)}\bar{\psi}(x), \\ A_{\mu}(x) &\rightarrow A_{\mu}(x) + \frac{1}{e}\partial_{\mu}\alpha(x). \end{aligned} \tag{3.2}$$

The gauge symmetry group of the Standard Model is $SU(3) \times SU(2) \times U(1)$. The $SU(3)$ symmetry comes from the strong interaction and the $SU(2) \times U(1)$ from the electroweak part. Since the $SU(3)$ and $SU(2)$ groups are non-Abelian, we need a generalization of the gauge principle to non-Abelian groups. In 1954 Yang and Mills worked out a non-Abelian gauge theory to describe local $SU(2)$ isotopic spin [22]. Their formalism could easily be adjusted to arbitrary

| Symbol | Name | Mass (MeV/$c^2$) | Electric charge ($e$) |
|---|---|---|---|
| $e^-$ | electron | 0.511 | -1 |
| $\nu_e$ | electron neutrino | ? | 0 |
| $\mu^-$ | muon | 105.66 | -1 |
| $\nu_\mu$ | muon neutrino | ? | 0 |
| $\tau^-$ | tau | 1784.1 | -1 |
| $\nu_\tau$ | tau neutrino | ? | 0 |
| $u$ | up | 2.4 | 2/3 |
| $d$ | down | 4.8 | -1/3 |
| $c$ | charm | 1270 | 2/3 |
| $s$ | strange | 104 | -1/3 |
| $t$ | top | 171200 | 2/3 |
| $b$ | bottom | 4200 | -1/3 |
| $\gamma$ | photon | 0 | 0 |
| $W^\pm$ | W-boson | 80.4 GeV/$c^2$ | $\pm1$ |
| $Z$ | Z-boson | 91.2 GeV/$c^2$ | 0 |
| $g$ | gluon | 0 | 0 |
| $h$ | Higgs boson | 125 GeV/$c^2$ | 0 |

Table 3.1: Elementary particles with their properties. The spin-1/2 fermions have a corresponding anti-particle, denoted as $e^+, \bar{\nu}_e, \bar{u}, \bar{d}$ etc. The quarks come in an $r, g$ or $b$ color state. The gauge bosons have spin-1, the Higgs boson spin-0.

symmetry group and since then non-Abelian gauge theories are known as Yang-Mills theories. However, because of the emergence of seemingly unphysical massless particles, these Yang-Mills theories had long been neglected, .

First we will derive Yang-Mills theory with an arbitrary $SU(N)$ symmetry group. Let the complex scalar field $\varphi$ be $\varphi(x) = \{\varphi_1(x), \varphi_2(x), \ldots, \varphi_N(x)\}$. This field transforms as $\varphi(x) \to U\varphi(x)$, with $U$ an element of the group $SU(N)$. Obviously, the Lagrangian $\mathcal{L} = \partial\varphi^\dagger\partial\varphi - V(\partial\varphi^\dagger\varphi)$ is invariant under such transformations. But what happens if the transformations are local, which means that they may vary from point to point in spacetime, so $U = U(x)$? Clearly $\varphi^\dagger\varphi \to \varphi^\dagger U^\dagger U\varphi = \varphi^\dagger\varphi$, but note that $\partial\varphi^\dagger\partial\varphi$ is no longer invariant:

$$\begin{aligned} \partial_\mu\varphi \to \partial_\mu(U\varphi) =& U\partial_\mu\phi + (\partial_\mu U)\varphi \\ =& U\left[\partial_\mu\varphi + (U^\dagger\partial_\mu U)\varphi\right]. \end{aligned} \tag{3.3}$$

Suppose the transformations are infinitesimal, so they can be written as $U = I - i\epsilon^a T_a$. Here the $T_a$ are the generators of the symmetry group and the $\epsilon^a$ are infinitesimal parameters. We can get rid of the second term in eq. (3.3) if we replace the ordinary derivative by a so called covariant derivative defined by

$$D_\mu\varphi = (\partial_\mu - igA^a_\mu T_a)\varphi. \tag{3.4}$$

We have introduced additional vector fields $A^a_\mu$, which are called gauge fields. Note that there are as many gauge fields as there are generators of the group, which is $N^2 - 1$ for $SU(N)$. The covariant derivative of $\varphi$ should transform in the same way as $\varphi$ itself. This requires

$$\begin{aligned}(D_\mu\varphi)' =&(\partial_\mu - igA'_\mu\cdot T)U\varphi\\ =&(\partial_\mu U)\varphi + U(\partial_\mu\varphi) - igA'_\mu\cdot TU\varphi\\ =&U(\partial_\mu - igA_\mu\cdot T)\varphi,\end{aligned} \tag{3.5}$$

where we have used the notation $A_\mu\cdot T = A^a_\mu T_a$. To see how the fields $A^a_\mu$ should transform, subtract the third line from the second line:

$$\begin{aligned}0 =&(\partial_\mu U)\varphi - igA'_\mu\cdot TU\varphi + igUA_\mu\cdot T\varphi\\ =&(-igA'_\mu\cdot T + igUA_\mu\cdot TU^{-1} + (\partial_\mu U)U^{-1})U\varphi\end{aligned} \tag{3.6}$$

So for the transformed gauge fields we have the relation

$$\begin{aligned}A'_\mu\cdot T =&UA_\mu\cdot TU^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}\\ =&UA_\mu\cdot TU^{-1} + \frac{i}{g}U\partial_\mu U^{-1}.\end{aligned} \tag{3.7}$$

Writing out the infinitesimal transformation explicitly, we can see that the gauge fields transform as

$$\begin{aligned}\delta A'_\mu\cdot T =& -i\epsilon^a\left[T_a, A_\mu\cdot T\right] - \frac{1}{g}(\partial_\mu\epsilon^a)T_a\\ =&\epsilon^a A^b_\mu f_{ab}{}^c T_c - \frac{1}{g}(\partial_\mu\epsilon^c)T_c.\end{aligned} \tag{3.8}$$

Note that if the transformations are global, that is the parameters do not depend on spacetime, the fields $A^a_\mu$ transform as the adjoint representation of the group.

The next step is to find the kinetic terms for the gauge fields. In quantum electrodynamics the kinetic term of the gauge fields

$$\mathcal{L}_{A,\mathrm{kin}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3.9}$$

is clearly gauge invariant, because

$$\begin{aligned}F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu\\ &= \partial_\mu A_\nu + \frac{1}{e}\partial_\mu\partial_\nu\alpha(x) - \partial_\nu A_\mu - \frac{1}{e}\partial_\nu\partial_\mu\alpha(x)\\ &= F_{\mu\nu}.\end{aligned} \tag{3.10}$$

The generalization to the non-Abelian case is a little more involved, because there the combination $\partial_\mu A_\nu - \partial_\nu A_\mu$ transforms non-trivially:

$$\begin{aligned}\partial_\mu A'_\nu\cdot T - \partial_\nu A'_\mu\cdot T =&\partial_\mu(UA_\nu\cdot TU^{-1}) + \frac{i}{g}\partial_\mu(U\partial_\nu U^{-1}) - (\mu\leftrightarrow\nu)\\ =&U(\partial_\mu A_\nu\cdot T)U^{-1}\\ &+ (\partial_\mu U)A_\nu\cdot TU^{-1} + UA_\nu\cdot T\partial_\mu U^{-1}\\ &+ \frac{i}{g}(\partial_\mu U)(\partial_\nu U^{-1}) + \frac{i}{g}(U\partial_\mu\partial_\nu U^{-1})\\ &- (\mu\leftrightarrow\nu)\end{aligned} \tag{3.11}$$

As we will see later, the required transformation law for the field strength is

$$F'_{\mu\nu} \cdot T = UF_{\mu\nu} \cdot TU^{-1}, \tag{3.12}$$

so the undesired terms in eq. (3.11) are

$$\begin{aligned}
&(\partial_\mu U)A_\nu \cdot TU^{-1} + UA_\nu \cdot T\partial_\mu U^{-1} \\
&+ \frac{i}{g}(\partial_\mu U)(\partial_\nu U^{-1}) + \frac{i}{g}(U\partial_\mu\partial_\nu U^{-1}) \\
&- (\mu \leftrightarrow \nu)
\end{aligned} \tag{3.13}$$

The second term in the second line can be dropped, because derivatives commute. To get rid of the remaining terms, define the field strength tensor as

$$F_{\mu\nu} \cdot T \equiv \partial_\mu A_\nu \cdot T - \partial_\nu A_\mu \cdot T - ig[A_\mu \cdot T, A_\nu \cdot T]. \tag{3.14}$$

Explicitly, the extra term transforms as

$$\begin{aligned}
-ig[A'_\mu \cdot T, A'_\nu \cdot T] = &- igU[A_\mu \cdot T, A_\nu \cdot T]U^{-1} \\
&- ig[UA_\mu \cdot TU^{-1}, \frac{i}{g}U\partial_\nu U^{-1}] \\
&- ig[\frac{i}{g}U\partial_\mu U^{-1}, UA_\nu \cdot TU^{-1}] \\
&- ig[\frac{i}{g}U\partial_\mu U^{-1}, \frac{i}{g}U\partial_\nu U^{-1}]
\end{aligned} \tag{3.15}$$

Using the relation $0 = \partial(U^{-1}U) = (\partial_\nu U^{-1})U + U^{-1}\partial_\nu U$, the second and third line combine into

$$\begin{aligned}
&UA_\mu \cdot T\partial_\nu U^{-1} - U(\partial_\nu U^{-1})UA_\mu \cdot TU^{-1} - (\mu \leftrightarrow \nu) = \\
&UA_\mu \cdot T\partial_\nu U^{-1} + (\partial_\nu U)A_\mu \cdot TU^{-1} - (\mu \leftrightarrow \nu),
\end{aligned} \tag{3.16}$$

and the fourth line becomes

$$\begin{aligned}
&\frac{i}{g}(U(\partial_\mu U^{-1})U(\partial_\nu U^{-1}) - (\mu \leftrightarrow \nu) = \\
&-\frac{i}{g}(\partial_\mu U)(\partial_\nu U^{-1}) - (\mu \leftrightarrow \nu).
\end{aligned} \tag{3.17}$$

This cancels the remaining terms in eq. (3.13) and we can finally conclude that the defined field strength has the transformation property eq. (3.12) Writing out the components of the field strength, we obtain

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{abc}A^b_\mu A^c_\nu. \tag{3.18}$$

The gauge invariant combination

$$tr\ F'_{\mu\nu} \cdot TF'^{\mu\nu} \cdot T = tr\ UF'_{\mu\nu} \cdot TU^{-1}UF'^{\mu\nu} \cdot TU^{-1} = tr\ F_{\mu\nu} \cdot TF^{\mu\nu} \cdot T \tag{3.19}$$

serves as the proper kinetic term of the gauge fields in Yang-Mills theory. If the generators are chosen such that $tr[T_aT_b] \sim \delta_{ab}$, the kinetic term can be written as

$$\mathcal{L}_{A^a,\text{kin}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \tag{3.20}$$

The physical interpretation of the kinetic terms is as follows. Analogous to QED, the quadratic term describes the propagation of a massless vector boson. Moreover the cubic and quartic terms in eq. (3.20) describe the self-interactions of the gauge bosons, which arise because the gauge bosons are charged themselves.

As we have seen in this section, the existence of the gauge fields is a direct consequence of imposing symmetries. The interactions of the matter particles with these gauge fields are determined by the covariant derivative. Also the cubic and quartic self-interactions of the gauge field come from the kinetic term eq. (3.20) and are completely fixed by the gauge group structure.

## 3.3 The Strong Interaction

The strong interaction is responsible for the fact that quarks form bound states of mesons and baryons. On larger length scales, a residual effect of the strong interaction is the attractive force between protons and neutrons, which enables the formation of nucleons. In this sense it is known as the nuclear force or residual strong force. As we will see in this section, the strong interaction can be described by quantum chromodynamics. Remarkably, the strong interaction is much better understood at high-energy scales.

The original quark model had two serious problems. Firstly, one could not explain why particles with fractional charge could not be found in nature. On the other hand, particles were observed that seemed to be symmetric under the interchange of spin and flavor quantum number of the quarks. This is inconsistent with the fact that quarks are fermions which should obey Fermi-Dirac statistics. An example is the $\Delta^{++}$ resonance with zero orbital momentum. This spin-3/2 particle is a bound state of three up quarks with their spins parallel, apparently violating Pauli's exclusion principle.

These problems can be solved if one assumes that quarks carry an additional quantum number. The $e^+e^-$ gives a clear indication for the existence of this 'hidden' quantum number. One possible reaction in this process is $e^+e^- \to \mu^+\mu^-$ with cross section

$$\sigma_{\text{tot}} \sim \frac{e^4}{E^2}. \tag{3.21}$$

Another possibility is $e^+e^- \to q_f + \bar{q}_f$, where the subscript $f$ stands for flavor. The free quarks that are formed in this reaction will instantly form hadrons due to the strong interaction. If we assume that the energy is high enough that the quark masses can be neglected, we find the cross section

$$\sigma_{\text{tot}} \sim \frac{e^2}{E^2} \sum_f Q_f^2. \tag{3.22}$$

Here $Q = 2/3$ for the $u$, $c$ and $t$ quarks, while for $d$, $s$ and $b$ quarks $Q = -1/3$. Define the ratio

$$R(E) \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{\sum_f Q_f^2}{e^2}. \tag{3.23}$$

At an energy scale where the five lightest quarks $(u, d, s, c, b)$ can be formed, the experimental value of $R$ is $\frac{11}{3}$. This contradicts eq. (3.23) which equals

$$\frac{4}{9} + \frac{1}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9} = \frac{11}{9}, \tag{3.24}$$

so the experimental value seems to be a factor three too high. This problem is solved by assuming that the quarks carry an additional color quantum 'number' that can be either *red*, *green* or *blue*. Then we should not only sum over the flavor in eq. (3.23), but also over the three extra color degrees of freedom, giving the right ratio $R$. Later we will see why this new quantum number is called color.

We require that the $\Delta^{++}$ wave function is antisymmetric under the interchange of the color quantum number. In this way we get a totally antisymmetric overall wave function, because we saw that the wave function was totally symmetric in spin and flavor. The three up-quarks are represented by $u_i$, where $i = 1, 2, 3$ is the color index. Then the wave function is written as

$$\varepsilon^{ijk} u_i u_j u_k. \tag{3.25}$$

There is a global $SU(3)$ symmetry connected to the color degree of freedom. It is assumed that all hadron wave functions are invariant under $SU(3)$ transformations. This means that only color singlet states are physically observable, so the only allowed combinations are

$$\bar{q}^i q_i, \qquad \varepsilon^{ijk} q_i q_j q_k, \qquad \varepsilon_{ijk} \bar{q}_i \bar{q}_j \bar{q}_k, \tag{3.26}$$

which are respectively the mesons, baryons and anti-baryons. Note that the quarks in eq. (3.26) can have different flavor quantum numbers.

Experimentalists showed that quarks hardly interact at very high energies, which means that at very small distances they act as if they are free particles. This phenomenon is called asymptotic freedom. Theorists started looking for asymptotically free quantum field theories, which were not known at that time. Eventually it was realized that Yang-Mills theory can be asymptotically free. The next step was to identify the right symmetry group. At that time the global $SU(3)$ symmetry was a bit mysterious. One wondered where this symmetry came from and what physical mechanism assured that only color singlets were observable. A first guess of course was to gauge the global $SU(3)$ symmetry. Indeed Yang-Mills theory with the symmetry group $SU(3)$ and six quark flavors turns out to be asymptotically free [20]. These requirements lead to an almost unique theory for the strong interaction, which is known as quantum chromodynamics (QCD).

The QCD Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_f \bar{\psi}_{f,i} (iD_\mu - m_f)_{ij} \psi_{f,j}, \tag{3.27}$$

where the field strength is given by eq. (3.18),

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu, \tag{3.28}$$

and the covariant derivative is

$$(D_\mu \psi_f)_i = (\partial_\mu \delta^{ij} - ig \lambda^a_{ij} A^a_\mu) \psi_{f,j}. \tag{3.29}$$

Here the matrices $\lambda^a$ are traceless Hermitean $3 \times 3$ matrices which form a basis for the Lie algebra of SU(3). A proper basis is

$$\lambda_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda_4 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda_5 = \frac{1}{2}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda_6 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\lambda_7 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{\sqrt{3}}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{3.30}$$

The dimension of the SU(3) group is eight, so this model contains eight vector fields $A_\mu$. The quanta of these fields are called gluons and they let the quarks interact by exchanging color. Thus the gluons must have a color-anticolor charge. One might wonder why there are only eight gluons and and not just the following nine: $r\bar{r}$, $r\bar{g}$, $r\bar{b}$, $g\bar{r}$, $g\bar{g}$, $g\bar{b}$, $b\bar{r}$, $b\bar{g}$ and $b\bar{b}$. Well, with this set of eigenstates it is possible to form a colorless gluon ($r\bar{r} + b\bar{b} + g\bar{g}$), which would imply that also colorless particles can interact with each other. So the eight gluons in nature are linear combinations of the above nine, under the condition that color must be transferred, resulting in eight independent color states. With these eight states it is impossible to form the forbidden singlet state.

We have seen that hadrons are singlets under $SU(3)$, meaning that free quarks can never be observed. This is known as confinement, which is next to asymptotic freedom the other characteristic property of the strong interaction. Although the strong interaction disappears at very small scales, it becomes stronger and stronger at larger distances. This means that below some energy scale the perturbative approach no longer holds. Lattice QCD provides a non-perturbative tool to study the strong interaction. Indeed confinement has been shown in lattice QCD calculations, however an analytical proof of confinement is still missing. The scale at which the coupling becomes of order unity is called the QCD scale. The experimental value of this scale is $\Lambda_{\mathrm{QCD}} \sim 220\,\mathrm{MeV}$. At the very early stages of the universe, when the energy was far above the QCD scale, quarks and gluons could freely move, forming the so called quark-gluon plasma.

## 3.4 The Electroweak Interaction

### 3.4.1 Spontaneous Symmetry Breaking

Consider the Lagrangian for a set of $N$ interacting scalar fields

$$\mathcal{L} = \frac{1}{2}(\partial\phi_i)^2 - V(\phi_i^2), \tag{3.31}$$

with the potential

$$V(\phi_i) = \frac{m^2}{2}\phi_i^2 + \frac{\lambda}{4}\phi_i^4, \tag{3.32}$$

where $i = 1, \ldots, N$. This Lagrangian is invariant under the operation

$$\phi_i \to R_{ij}\phi_j, \tag{3.33}$$

with $R_{ij}$ an arbitrary $N \times N$ orthogonal matrix. These operations form the rotation group in $N$ dimensions, simply called the $O(N)$ group.

Clearly, adding terms like $\phi_1^2$ or $\phi_1^4$ breaks the symmetry down to $O(N-1)$. In fact we can break it down to $O(N-M)$ for any $M < N$ by adding more terms like these. Adding terms that reduce the symmetry of a system is called breaking the symmetry 'by hand'. A more interesting case of symmetry breaking happens when the system breaks its own original symmetry. This is called spontaneous symmetry breaking, which is a very common concept in modern physics. As we will see, it also plays a fundamental role in the Standard Model.

Let's start with a simple example of spontaneous symmetry breaking. Take the Lagrangian eq. (3.31) for a single field, and slightly adjust by replacing the mass term $m^2$ with a negative parameter $-\mu^2$:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi), \tag{3.34}$$

with the potential

$$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4. \tag{3.35}$$

Note that because the mass term is negative, the potential has two minima at $\phi = v \equiv \pm\sqrt{\frac{\mu^2}{\lambda}}$ instead of just a single minimum at $\phi = 0$. The Lagrangian has a discrete symmetry, because it is invariant under the reflection $\phi \to -\phi$.

Instead of describing the system in terms of the field $\phi$, we can analyze the system in terms of a perturbation around one of the ground states. As long as the fluctuations are small, all what is physically relevant is the behaviour of the potential around this ground state. In some sense this is a more natural choice and as we will see this gives a clear physical interpretation of the theory. Say we choose the ground state at $+v$, then we write $\phi = v + \sigma$. In terms of this new field $\sigma$ the Lagrangian becomes

$$\mathcal{L} = \frac{\mu^4}{4\lambda} + \frac{1}{2}(\partial\sigma)^2 - \mu^2\sigma^2 - 3\sqrt{\lambda}\mu\sigma^3 - \frac{3}{2}\lambda\sigma^4. \tag{3.36}$$

Note that the reflection symmetry is no longer present, by decaying to one of the ground states the system has broken this symmetry! The first term is just a constant shift of the potential. Looking at he last three terms, we see that the Lagrangian describes a scalar field with mass $\sqrt{2}\mu$ interacting via $\sigma^3$ and $\sigma^4$ interactions. The ground state is called the vacuum, since it is the state where no particles are present.

Now we will see what happens when the broken symmetry is continuous instead of discrete. As an example we analyze the Lagrangian eq. (3.31) for N=2, which is clearly $O(2)$ symmetric. The potential of this Lagrangian is shown in figure 3.1. Instead of a discrete number of different ground states, this model has infinitely many ground states related to each other by $O(2)$ rotations. Let us do exactly the same as before: consider fluctuations around one of the ground states. Say we choose the ground state to point in the 1-direction, so $\phi_1 = v$ and $\phi_2 = 0$. Then we can rewrite the fields as $\phi_1 = v + \sigma$ and $\phi_2 = \pi$, and express the Lagrangian in terms of the perturbations as

$$\mathcal{L} = \frac{\mu^4}{4\lambda} + \frac{1}{2}\left[(\partial\sigma)^2 + \partial\pi)^2\right] - \mu^2\sigma^2 - \ldots, \tag{3.37}$$

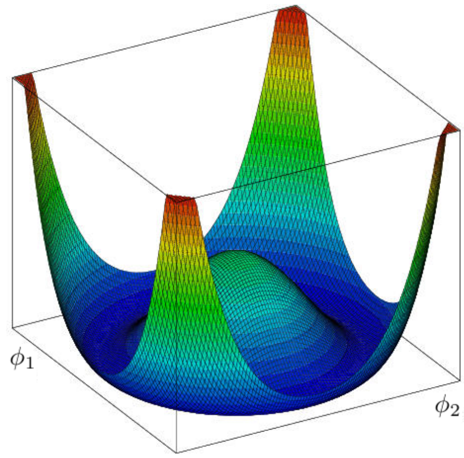


Figure 3.1: The potential of a massive scalar field with quartic interactions has the shape of a Mexican hat.

here the dots represent the terms of cubic and quartic order. The first term is exactly equal to the constant term we found in eq. (3.36). Just like before we also find a particle with mass $\sqrt{2}\mu$. The remarkable thing here is that there are no terms proportional to $\pi^2$, so we conclude that $\pi$ is a massless field. This can also be seen from the shape of the potential; the field $\pi$ describes excitations along the trough of the potential, which don't cost energy. What we have found in the previous example is in accordance with Goldstone's theorem. This theorem states that for every broken continuous symmetry, a massless scalar field known as a Nambu-Goldstone boson appears in the theory.

In our last example of spontaneous symmetry breaking we will see what happens if the broken symmetry is a local symmetry. Let's take the simple $U(1)$ symmetric Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^\mu\phi)^\dagger D_\mu\phi - V(\phi^\dagger\phi), \tag{3.38}$$

with

$$V(\phi^\dagger\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2. \tag{3.39}$$

The covariant derivative is

$$D_\mu\phi = (\partial - igA_\mu)\phi, \tag{3.40}$$

with $A_\mu$ the gauge field. This Lagrangian is invariant under the transformations

$$\phi \quad \rightarrow \quad e^{-i\alpha(x)}\phi, \tag{3.41}$$

$$A_\mu \quad \rightarrow \quad A_\mu - \frac{1}{g}\partial_\mu\alpha(x). \tag{3.42}$$

The potential has a minimum at $|\phi| = v/\sqrt{2}$. If we choose the ground state to point in the 1-direction, the field $\phi$ can be parametrized in terms of the perturbations $\phi_1$ and $\phi_2$ as

$$\phi = \frac{1}{\sqrt{2}}(v + \phi_1 + i\phi_2). \tag{3.43}$$

After writing out the kinetic term for $\phi$ we obtain

$$|D_\mu\phi|^2 = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - gvA_\mu\partial^\mu\phi_2 + \frac{g^2v^2}{2}A_\mu A^\mu + \ldots, \quad (3.44)$$

where the dots stand for terms cubic and quartic in the fields $\phi_1$, $\phi_2$ and $A_\mu$. Identifying the fourth term as a mass term for the field $A_\mu$, it can be seen that $M_A = gv$. At first sight it might seem that the sign of the mass term is incorrect, but note that the physical spacelike components of $A_\mu$ have exactly the right sign.

The Goldstone boson can be removed from the Lagrangian by going to the unitary gauge, in which the field $\phi$ is strictly real at every point in space. First parametrize the complex field $\phi$ in polar coordinates

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2}}\left[v+\eta(x)\right]\exp\left(i\xi(x)/v\right)\\ &= \frac{1}{\sqrt{2}}\left[v+\eta(x)+i\xi(x)+\ldots\right]. \end{aligned} \quad (3.45)$$

Then define the new fields

$$\phi'(x) = \exp\left(-i\xi/v\right)\phi(x) = \frac{1}{\sqrt{2}}(v+\eta(x)), \quad (3.46)$$

$$B_\mu(x) = A_\mu - \frac{1}{gv}\partial\xi(x). \quad (3.47)$$

Writing out the Lagrangian we obtain

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \mu^2\eta^2 - \frac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B)\\ &+ \frac{1}{2}g^2v^2B_\mu B^\mu + \frac{1}{2}g^2B_\mu B^\mu\eta(2v+\eta)\\ &- \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4. \end{aligned} \quad (3.48)$$

We see that the field $\xi(x)$ has disappeared from the Lagrangian. We say that its degree of freedom has been eaten by the massive field $B_\mu$. This phenomenon is called the Higgs mechanism.

### 3.4.2 The GWS Theory of Weak Interactions

A unified description of the weak and electromagnetic interactions is given by the GWS (Glashow-Weinberg-Salam) theory. This model has four spin-1 gauge fields, which are $A^i_\mu$ $(i = 1, 2, 3)$ transforming under $SU(2)_L$ and $B_\mu$ transforming under $U(1)_Y$. Based on the discussion of Yang-Mills theory, we expect these vector particles to be massless. Somehow they should become massive, since the only massless gauge boson in nature is the photon. If explicit mass terms like

$$\frac{1}{2}m^2A_\mu A^\mu \quad (3.49)$$

are added to the Lagrangian, we destroy the gauge invariance eq. (3.7). As we will see it is the Higgs mechanism that gives the vector particles their masses.

The $SU(2)_L \times U(1)_Y$ symmetry is spontaneously broken to $U(1)_Q$ by adding the complex doublet of scalar fields

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}. \tag{3.50}$$

Under a gauge transformation this doublet transforms as

$$\phi \to e^{i\alpha^a \tau^a} e^{i\beta Y} \phi, \tag{3.51}$$

with $\vec{\tau} = \vec{\sigma}/2$. Suppose the neutral part of the doublet acquires a nonzero vacuum expectation value $v$. By gauge transforming it can always be pointed in the real direction, so that the ground state simply becomes

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{3.52}$$

The original symmetry should not be completely broken, because we know that one of the gauge bosons should remain massless. Obviously a gauge transformation with $\alpha^1 = \alpha^2 = 0$ and $\alpha^3 = \beta$ leaves the ground state invariant, which means that we can construct the generator $Q = T^3 + T^Y = T^3 + \frac{1}{2}Y$ and have invariance under $U(1)_Q$.

The Lagrangian for the Higgs field and the gauge fields is

$$\mathcal{L} = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + (D_\mu \phi)^\dagger D^\mu \phi - V(\phi^\dagger \phi), \tag{3.53}$$

where

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g \varepsilon^{ijk} A^j_\mu A^k_\nu, \qquad i = 1,2,3 \tag{3.54}$$

and

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{3.55}$$

The potential given by

$$V(\phi^\dagger \phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \tag{3.56}$$

is of the most general form that is renormalizable and invariant under Lorentz and parity transformations. It has the shape of a Mexican hat, with a local maximum at zero and minima at $|\phi^0| = v = \sqrt{\mu^2/\lambda}$. The scale $v$ sets the electroweak scale, experimentally it is known that $v \simeq 246\,\mathrm{GeV}$.

The covariant derivative of $\phi$ is

$$D_\mu \phi = \left( \partial_\mu - i g \vec{\tau} \cdot \vec{A}_\mu - \frac{i}{2} g' B_\mu \right) \phi, \tag{3.57}$$

where $\vec{A}_\mu$ and $B_\mu$ are the gauge fields with their coupling constants g and g'. Writing out the kinetic term explicitly we obtain

$$\begin{aligned}(D_\mu \phi)^\dagger D^\mu \phi &= \frac{1}{2} (0 \;\; v) \left( g A^a_\mu \tau^a + \frac{1}{2} g' B_\mu \right) \left( g A^{b\mu} \tau^b + \frac{1}{2} g' B^\mu \right) \begin{pmatrix} 0 \\ v \end{pmatrix} \\ &= \frac{1}{2} \frac{v^2}{4} \left[ g^2 (A^1_\mu)^2 + g^2 (A^2_\mu)^2 + (-g A^3_\mu + g' B_\mu)^2 \right] \end{aligned} \tag{3.58}$$

Note that we have broken the original symmetry to a $U(1)_{\text{em}}$ symmetry that is generated by $Q = T^3 + T^Y$. The field that couples to this generator is the electromagnetic field, which can be obtained by making the change of basis from $\{A^3, B\}$ to $\{Z, A\}$:

$$Z_\mu^0 = A_\mu^3 \cos\theta_W - B_\mu \sin\theta_W \tag{3.59}$$

and

$$A_\mu = B_\mu \cos\theta_W + A_\mu^3 \sin\theta_W, \tag{3.60}$$

where the weak mixing angle is fixed by

$$\tan\theta_W = \frac{g'}{g}. \tag{3.61}$$

Then the $Z$ boson is the combination

$$Z_\mu^0 = \frac{1}{\sqrt{g^2 + g'^2}}(gA_\mu^3 - g'B_\mu) \tag{3.62}$$

which has mass

$$M_Z = \frac{v}{2}(g^2 + g'^2). \tag{3.63}$$

The electromagnetic field, corresponding to the orthogonal combination, remains massless

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g'A_\mu^3 + gB_\mu). \tag{3.64}$$

The electromagnetic gauge coupling is

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}}. \tag{3.65}$$

Writing out the kinetic terms of the gauge fields, we see that the combinations

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2), \tag{3.66}$$

couple to the electromagnetic field $A_\mu$. These fields are the $W$ bosons with electric charge $Q = \pm 1$ and mass

$$M_W = g\frac{v}{2}. \tag{3.67}$$

## 3.5 The Faddeev-Popov Procedure

What is the path integral for a theory with gauge fields $A_\mu^a$ and scalar fields $\phi$? Naively one could write it as

$$Z = \int \mathcal{D}A\,\mathcal{D}\phi\, e^{iS(A,\phi)}, \tag{3.68}$$

with

$$\mathcal{D}A = \prod_x \prod_{a,\mu} dA_\mu^a. \tag{3.69}$$

There is, however, a problem with this integral. Because the Lagrangian is invariant under gauge transformations, an infinite number of configurations are equivalent to each other. So the integral over $\mathcal{D}A$ runs over an infinite subset of configurations for every single physical state.

What we want is to integrate over all equivalent configurations only once. This can be done by constraining the gauge field at each point $x$ by a gauge fixing condition $G(A) = 0$. Note that $A$ stands for $A^a_\mu$. The Faddeev-Popov determinant $\Delta(A)$ for the gauge fixing condition $G(A)$ is defined by the equation

$$1 = \Delta(A) \int \mathcal{D}g\delta(G(A_g)). \tag{3.70}$$

In this equation $A_g$ means the gauge field transformed by the group element $g(x)$. Furthermore $\mathcal{D}g'g = \mathcal{D}g$. To see that the $\Delta(A)$ is gauge invariant, write it as

$$\Delta(A)^{-1} = \int \mathcal{D}g'\delta(G(A_{g'})), \tag{3.71}$$

then

$$\begin{aligned}\Delta(A_g)^{-1} &= \int \mathcal{D}g'\delta(G(A_{gg'})) \\ &= \int \mathcal{D}gg'\delta(G(A_{gg'})) \\ &= \int \mathcal{D}g''\delta(G(A_{g''})) \\ &= \Delta(A)^{-1}.\end{aligned} \tag{3.72}$$

After inserting eq. (3.70) into eq. (3.68) we have

$$\begin{aligned}Z &= \int \mathcal{D}A\,\mathcal{D}\phi\, e^{iS(A,\phi)} \\ &= \int \mathcal{D}A\,\mathcal{D}\phi\, e^{iS(A,\phi)}\Delta(A) \int \mathcal{D}g\delta(G(A_g)) \\ &= \int \mathcal{D}g \int \mathcal{D}A\,\mathcal{D}\phi\, e^{iS(A,\phi)}\Delta(A)\delta(G(A_g)).\end{aligned} \tag{3.73}$$

Now we change the integration variables from $A$ to $A_{g^{-1}}$. This is just a simple shift so $\mathcal{D}A_{g^{-1}} = \mathcal{D}A$. Moreover the action $S(A)$ and the FP determinant $\Delta(A)$ remain the same as they are gauge invariant. We finally arrive at

$$Z = \left(\int \mathcal{D}g\right) \int \mathcal{D}A\,\mathcal{D}\phi\, e^{iS(A,\phi)}\Delta(A)\delta(G(A)). \tag{3.74}$$

So we managed to factor out the group integration $\left(\int \mathcal{D}g\right)$, which although infinite for gauge theories is just an irrelevant overall constant. The Faddeev-Popov determinant can be written as

$$\Delta(A) = \frac{\delta G(A^\alpha)}{\delta\alpha} = \frac{1}{g}\partial^\mu D_\mu, \tag{3.75}$$

and the path integral becomes

$$Z = C \cdot \int \mathcal{D}A\,\mathcal{D}\phi\, e^{i\int d^4x\,\mathcal{L}(A,\phi) - \mathcal{L}_{gf}}\ \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right). \tag{3.76}$$

The next problem will be how to deal with the Faddeev-Popov determinant. If we introduce a new set of anti-commuting fields $c^a$ and $\bar{c}^a$, the determinant can be written as

$$\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \int \mathcal{D}c\,\mathcal{D}\bar{c}\,\exp\left[i\int d^4x\,\bar{c}(-\partial^\mu D_\mu)c\right]. \tag{3.77}$$

The fields $c^a$ and $\bar{c}^a$ violate the spin-statistics theorem, so they can't be real physical particles. For this reason such fields are known as ghost fields.

## 3.6 The $R_\xi$ Gauges

### 3.6.1 Abelian Case

In this section we introduce the $R_\xi$ gauge for the Abelian case, which suffices to show the most interesting features. The non-Abelian case is a bit more technical and is postponed to chapter 6.

Consider the Lagrangian eq. (3.38). Using the expression eq. (3.40) and expanding the potential up to second order in $\phi_1$ (the first order terms vanish since we are in a minimum and there is no term proportional to $\phi_2^2$ due to the $U(1)$ symmetry), the Lagrangian to quadratic order in fields is

$$\begin{aligned}\mathcal{L} = &-\frac{\mu^4}{4\lambda} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[\partial_\mu\phi_1\partial^\mu\phi_1 - 2\mu^2\phi_1^2\right] + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 \\ &- gvA_\mu\partial^\mu\phi_2 + \frac{g^2v^2}{2}A_\mu A^\mu \end{aligned} \tag{3.78}$$

By a right choice of the gauge-fixing function $G$ the effective Lagrangian can become considerably simpler. We choose as the gauge fixing function

$$G = \frac{1}{\sqrt{\xi}}(\partial_\mu A^\mu - \xi gv\phi_2). \tag{3.79}$$

and add

$$\mathcal{L}_{gf} = -\frac{1}{2}G^2, \tag{3.80}$$

to the total Lagrangian. Up to quadratic order in fields the Lagrangian becomes

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}A_\mu\left(-g^{\mu\nu}\partial^2 + (1-\frac{1}{\xi})\partial^\mu\partial^\nu - (gv)^2g^{\mu\nu}\right)A_\nu \\ &+ \frac{1}{2}\left[\partial_\mu\phi_1\partial^\mu\phi_1 - 2\mu^2\phi_1^2\right] + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{\xi}{2}(gv)^2\phi_2^2. \end{aligned} \tag{3.81}$$

We also have to add the Lagrangian of the ghosts. The gauge variation of $G$ is

$$\frac{\delta G}{\delta\alpha} = \frac{1}{\sqrt{\xi}}\big(-\frac{1}{g}\partial^2 - \xi gv(v+\phi_2)\big) \tag{3.82}$$

So

$$\mathcal{L}_{\text{FP}} = \bar{\eta}g\frac{\delta G}{\delta\alpha}\eta \tag{3.83}$$

Limiting ourselves to quadratic terms, we are left with

$$\mathcal{L}_{\text{FP}} = \bar{\eta}\left[-\partial^2\delta - \xi g^2v^2(1+\phi_1/v)\right]\eta + \ldots \tag{3.84}$$

Adding this to eq. (3.81) gives the Lagrangian in the $R_\xi$ gauge. In chapter 6.4.2, the gauge fixed Lagrangian of the Standard Model will be obtained.

### 3.6.2 Unitary Gauge vs. $R_\xi$ Gauge

From the gauge fixed Lagrangian, it can easily be seen that the propagators for the gauge bosons are

$$\frac{-i}{k^2 - m^2}\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi m^2}(1-\xi)\right], \tag{3.85}$$

where $m$ is $m_W$ or $m_Z$ for the massive gauge bosons and zero for the photon. The Goldstone boson propagators are given by

$$\frac{i}{k^2 - \xi m^2} \tag{3.86}$$

with $m$ the same as the mass of gauge boson by which they have been eaten. The Higgs field $h$ has not been eaten by a gauge boson - it is a physical field with no $\xi$ dependence in the propagator.

Clearly, if we take the limit $\xi \to \infty$ the propagators for the unphysical Goldstone fields (with masses proportional to $\sqrt{\xi}$ ) vanish. This gauge is called the unitary gauge. The problem with this gauge is that the gauge boson goes as $k^\mu k^\nu / k^2$ and one has a hard time proving renormalizability. In the $R_\xi$ gauges, with a finite value of $\xi$, all the propagators fall off as $1/k^2$ and renormalizability can easily be proved.

# Chapter 4

# Inflation Models

## 4.1 Slow-Roll Inflation

Inflation is an era of accelerated expansion, that is $\ddot{a} > 0$ during inflation. By looking what happens to the comoving Hubble length one finds that

$$\frac{d}{dt}\frac{1}{aH} < 0, \tag{4.1}$$

so during inflation the characteristic length scale of the universe decreases. The *Raychaudhuri equation*

$$\left(\frac{\ddot{a}}{a}\right)^2 = -\frac{1}{6M_{\mathrm{Pl}}^2}(\rho + 3p), \tag{4.2}$$

shows that inflation is equivalent to a period with $\rho + 3p < 0$. If we assume that the energy density can only be positive, the pressure needs to be negative during inflation, which corresponds to repulsive gravity. By combining the Friedmann equation

$$H = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_{\mathrm{Pl}}^2} - \frac{k}{a^2} \tag{4.3}$$

and the continuity equation

$$\frac{\mathrm{d}\rho}{\rho} = -3(1+w)\frac{\mathrm{d}a}{a}, \tag{4.4}$$

it can be seen that when the equation of state parameter $w = -1$ the universe expands exponentially as

$$a \propto e^{Ht}. \tag{4.5}$$

The total energy density $\rho$ includes the energy density of the vacuum $\rho_\Lambda$, for which the equation of state parameter is $w_\Lambda = -1$. A universe with $\rho_\Lambda$ dominating will grow almost exponentially. However, there is no way out since matter and radiation in it can only dilute. Eventually it will turn into a universe consisting completely of vacuum energy, also called a *de Sitter* universe. So vacuum energy could not have been the source of the inflationary epoch right after the Big Bang.

What is needed for inflation is some dynamical vacuum-like state. This can be obtained if we assume that the universe was once dominated by one or more scalar fields. Suppose the universe is dominated by the scalar field $\phi$, then the energy density of the universe is $\rho \approx \rho_\phi$. By comparing the stress-energy tensor of the scalar field with that of a perfect fluid (2.9), the energy density and pressure are found to be

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{4.6}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{4.7}$$

In general there is no unique relation between $\rho$ and $p$, but assuming the potential dominates the kinetic term gives $p_\phi \simeq -\rho_\phi$ and the universe expands exponentially. With the continuity equation (2.12) the equation of motion for the field $\phi$ is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{4.8}$$

where the prime denotes a derivative with respect to the field $\phi$. In the slow-roll approximation it is also assumed that

$$|\ddot{\phi}| \ll 3H|\dot{\phi}|, \tag{4.9}$$

which guarantees that exponential inflation holds for a prolonged period of time. By differentiating eq. (4.3) and then using eq. (4.8),

$$2H\dot{H} = \frac{1}{3M_{\rm Pl}^2}\left(\dot{\phi}\ddot{\phi} + \dot{\phi}V'(\phi)\right) = \frac{1}{3M_{\rm Pl}^2}\dot{\phi}(-3H\dot{\phi}), \tag{4.10}$$

which gives the following relation for $\dot{H}$:

$$\dot{H} = \frac{-1}{2M_{\rm Pl}^2}\dot{\phi}^2. \tag{4.11}$$

If the Hubble parameter is nearly constant during inflation, i.e. $|\dot{H}|/H^2 \ll 1$, the universe expands almost exponentially as $a \sim e^{Ht}$.

The slow-roll parameters $\epsilon$ and $\eta$ are given by

$$\epsilon = \frac{M_{\rm Pl}^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \tag{4.12}$$

and

$$\eta = M_{\rm Pl}^2 \frac{V''(\phi)}{V(\phi)}. \tag{4.13}$$

Necessary (but not sufficient) conditions for slow-roll inflation are $\epsilon \ll 1$ and $|\eta| \ll 1$. The first condition guarantees that the potential is sufficiently flat during inflation, making slow-roll inflation possible. The second condition is required to have a flat potential for a big enough range of $\phi$, so that the slow-roll inflation holds long enough. Similarly, a third slow-roll parameter useful for analyzing the curvature perturbation is given by

$$\zeta^2 = M_{\rm Pl}^4 \frac{(dV(\phi)/d\phi)d^3V(\phi)/d\phi^3}{V(\phi)^2}. \tag{4.14}$$

The number of $e$-folds between some point in time $t$ when the universe is inflating and the time at the end of inflation $t_{\rm end}$ is defined as

$$N \equiv \ln \frac{a(t_{\rm end})}{a(t)} = \int_t^{t_{\rm end}} H \, {\rm d}t = \int_\phi^{\phi_{\rm end}} \frac{H}{\dot{\phi}} {\rm d}\phi. \tag{4.15}$$

Assuming slow-roll inflation, so $H$ is constant during inflation, this becomes

$$N \simeq \frac{1}{M_{\rm Pl}^2} \int_{\phi_{\rm end}}^{\phi} \frac{V}{V'} \, {\rm d}\phi. \tag{4.16}$$

The amplitude of the density perturbations is often parametrized as

$$\mathcal{P}_s(k) = \Delta_{\mathcal{R}}^2 \left( \frac{k}{k_0} \right)^{n_s(k_0)-1}. \tag{4.17}$$

Here $\Delta_{\mathcal{R}}^2$ is the amplitude at some pivot scale $k_0$, which by using the slow-roll approximation becomes

$$\Delta_{\mathcal{R}}^2 \simeq \frac{1}{24\pi^2 M_{\rm Pl}^4} \frac{V}{\epsilon} \bigg|_{k_0}. \tag{4.18}$$

The scale dependence is determined by the spectral index $n(k)$ given by

$$n_s(k) - 1 = -6\epsilon + 2\eta. \tag{4.19}$$

A scale invariant spectrum is obtained when $n_s = 1$. Experimentally it is known that $\Delta_{\mathcal{R}}^2 = (2.44 \pm 0.01) \times 10^{-9}$ and $n_s = 0.96 \pm 0.01$ at the pivot scale $k_0 = 0.002\,{\rm Mpc}^{-1}$ [23].

Deviations from the parametrization $\mathcal{P}_s \propto k^{n_s(k_0)-1}$ are described by the running of the spectral index:

$$\alpha \equiv dn_s/d(\ln k) = -16\epsilon\eta + 24\epsilon^2 + 2\zeta^2. \tag{4.20}$$

Besides the density perturbations, fluctuations of the metric result in tensor perturbations. The tensor modes correspond to gravitational waves and just like the scalar modes they freeze-in when leaving the horizon. Inflation predicts for the tensor-to-scalar ratio

$$r \equiv \frac{\mathcal{P}_h}{\mathcal{P}_s} = 16\epsilon. \tag{4.21}$$

A combination of WMAP-7 data, baryon acoustic oscillations and supernovae observations show that $\alpha = -0.022 \pm 0.020$ and $r < 0.24$ (at 95 % confidence level) [1].

## 4.2 Modified Gravity

The corresponding action for the Einstein equation in vacuum (i.e. $T^{\mu\nu} = 0$) is the Einstein-Hilbert action

$$S_{\rm EH} = \int {\rm d}^4x \sqrt{-g} \left[ \frac{1}{2} M_{\rm Pl}^2 \mathcal{R} \right]. \tag{4.22}$$

In this formula $M_{\rm Pl} \equiv 1/\sqrt{8\pi G_{\rm N}} = 2.44 \times 10^{18}\,$GeV is the reduced Planck mass and $\mathcal{R}$ is the Ricci scalar. The factor $\sqrt{-g}$ with $g$ defined as the determinant of $g_{\mu\nu}$ is included to make the action invariant under general coordinate transformations. The simplest extension to this model would be the addition of some scalar matter fields to the vacuum:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm Pl}^2\mathcal{R} + \mathcal{L}_{\rm mat}\right]. \tag{4.23}$$

The Einstein field equations are obtained by the principle of least action,

$$T_{\mu\nu} = -2\frac{\partial\mathcal{L}_{\rm mat}}{\partial g_{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\rm mat}. \tag{4.24}$$

In the action given by eq. (4.23) it is assumed that there is no coupling between the Ricci scalar and the fields in $\mathcal{L}_{\rm mat}$. In the quantum theory the Lagrangian can contain any term not forbidden by some symmetry. This could be an indication that a non-minimal coupling exists. Moreover, a non-minimal coupling is required for renormalization purposes in theories of interacting scalar fields in curved spacetime [24]. Consider the case where $\mathcal{L}_{\rm mat}$ contains the field $\phi$ that couples to $\mathcal{R}$. The potential of this field is supposed to be of the form

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2, \tag{4.25}$$

with $v = \langle\phi\rangle$ the vacuum expectation value of the field $\phi$. If we choose to separate $\phi$ from $\mathcal{L}_{\rm mat}$, the action can be written as

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[f(\phi)\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + \mathcal{L}_{\rm mat}\right], \tag{4.26}$$

with

$$f(\phi) = \frac{1}{2}\left(M^2 + \xi\phi^2\right) \qquad \text{and} \qquad M_{\rm Pl}^2 = M^2 + \xi v^2. \tag{4.27}$$

Inflation models based on non-minimally coupled scalar fields were first studied by D.S. Salopek, J.R. Bond and J.M. Bardeen in 1989 [25].

Let's see what happens for different values of $\xi$:

**Minimal coupling:** In this case the parameter $\xi$ is set to zero and the system is said to be minimally coupled. Can this give rise to inflation? First assume that the field $\phi$ is large with respect to it's vacuum expectation value, so that the potential eq. (4.25) becomes

$$V(\phi) = \frac{\lambda}{4}\phi^4. \tag{4.28}$$

With eq. (4.12) we see that slow roll ends when

$$\epsilon = \frac{M_{\rm Pl}^2}{2}\left(4\phi^{-1}\right)^2 \simeq 1, \tag{4.29}$$

thus $\phi_{\rm end} = \sqrt{8}M_{\rm Pl}$. According to eq. (4.16), the number of $e$-folds between $\phi_0$ and $\phi_{\rm end}$ is

$$N \simeq \frac{1}{M_{\rm Pl}^2}\int_{\phi_{\rm end}}^{\phi_0}\frac{V(\phi)}{V'(\phi)}\,\mathrm{d}\phi = \frac{1}{M_{\rm Pl}^2}\int_{\sqrt{8}M_{\rm Pl}}^{\phi_0}\frac{1}{4}\phi\,\mathrm{d}\phi. \tag{4.30}$$

Thus the value of the field $\phi$ at $N$ $e$-folds before the end of inflation is

$$\phi_{\mathrm{N}} = \sqrt{8(N+1)}M_{\mathrm{Pl}}. \tag{4.31}$$

Now we have seen at which energy scales inflation takes place, we will analyze the cosmological implications of this model. According to the WMAP normalization $V/\epsilon = (0.027M_{\mathrm{Pl}})^4$ at 62 $e$-folds before the end of inflation. Evaluating $V(\phi)$ and $V'(\phi)$ at $\phi_{62}$ we get

$$V(\phi_{62}) = \frac{\lambda}{4}\left(\sqrt{8(62+1)}M_{\mathrm{Pl}}\right)^4 = \lambda \cdot 63504 \cdot M_{\mathrm{Pl}}^4 \tag{4.32}$$

and

$$V'(\phi_{62}) = \lambda\left(\sqrt{8(62+1)}M_{\mathrm{Pl}}\right)^3 = \lambda \cdot 504^{3/2} \cdot M_{\mathrm{Pl}}^3 \tag{4.33}$$

Thus we obtain

$$\epsilon = \frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{V'(\phi_{62})}{V(\phi_{62})}\right)^2 \approx 0.016 \tag{4.34}$$

and

$$\frac{V}{\epsilon} = 2\frac{(\lambda \cdot 63504)^3}{\left(\lambda \cdot 504^{3/2}\right)^2}M_{\mathrm{Pl}}^4 = (0.027M_{\mathrm{Pl}})^4. \tag{4.35}$$

The last equation gives $\lambda \approx 1.33 \times 10^{-13}$. Such an extremely fine-tuned coupling constant seems very unphysical. The tensor-to-scalar ratio is $r = 16\epsilon \approx 0.26$, which is also in conflict with the observed value of $r$.

**Induced gravity:** This is the other extreme of the minimally coupled system. Here the parameter $M$ is set to zero and it is assumed that the Planck scale is generated by the field $\phi$, analogous to the Higgs field generating the electroweak scale in the Standard Model [26]. So we have

$$f = \frac{1}{2}\xi\phi^2 \qquad \text{and} \qquad M_{\mathrm{Pl}}^2 = \xi v^2. \tag{4.36}$$

The Planck mass is completely generated by the vacuum expectation value of the field $\phi$. It starts to run at energies above the $v$.

First of all assume $\xi \gg 1$. As we will see in the next section, the action can be rewritten in terms of an ordinary Einstein-Hilbert term and a canonically normalized kinetic field $\chi$ which has the modified potential $V(\chi)$. Then for $\chi \gtrsim M_{\mathrm{Pl}}$, the region where inflation takes place, the potential can be approximated by

$$V(\chi) \simeq \frac{\lambda M_{\mathrm{Pl}}^4}{4\xi^2}\left(1 - \exp\left(-\frac{2\chi}{\sqrt{6}M_{\mathrm{Pl}}}\right)\right)^2 \tag{4.37}$$

The reheating takes place around the well of the potential. Assuming this happened at a reheat temperature $T_{\mathrm{reh}} \sim 10^{15}$ GeV, this gives

$$\left(\frac{\lambda M_{\mathrm{Pl}}^4}{4\xi^2}\right)^{1/4} \sim 10^{15}\,\mathrm{GeV}. \tag{4.38}$$

Then the field $\phi$ can be identified with a GUT Higgs field, with $v \sim 10^{16}$ GeV. This gives the physically reasonable values $\lambda \sim 10^{-2}$ and $\xi \sim 10^4$.

**Variable Planck mass:** In a variable Planck mass theory, the Planck scale is more or less set by choosing $M$, but a small part of it is still 'induced' by the field $\phi$. This corresponds to

$$f = \frac{1}{2}(M + \xi\phi^2), \qquad M_{\rm Pl}^2 = M^2 + \xi v^2. \tag{4.39}$$

For $1 \ll \sqrt{\xi} \lll M_{\rm Pl}/v$ the potential

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 \tag{4.40}$$

can still be approximated by eq. (4.37) in the frame with canonically normalized gravity. However, while $\xi$ is fixed by the relation $M_{\rm Pl}^2 = \xi v^2$ in the case of induced gravity, with a variable Planck mass we no longer have this constraint. As we will see in chapter 5, this makes it possible to identify the field $\phi$ with the Standard Model Higgs field .

## 4.3 The Conformal Transformation

In case of a non-minimal coupling $\xi$, gravity looks quite different for high values of the field $\phi$. Ordinary gravity can be obtained by making a conformal transformation form the original (Jordan) frame to the Einstein frame. It is a transformation that changes the curvature of spacetime, which mixes up the scalar and tensor degrees of freedom. This implies that phenomena have different origins depending on which frame is used.[1] The Einstein frame is the frame where the scalar and tensor degrees of freedom don't mix, which makes it often a more convenient frame to use. It is important to note that a conformal transformation is really different from a coordinate transformation. A coordinate transformation $x^\mu \to x'^\mu$ is just a relabeling of the coordinates on a manifold, while under a conformal transformation the manifold itself is stretched or shrunk.

If in some coordinates the metric can be written in the form $g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$ the spacetime is called conformally flat. Then this transformation reduces the action to that of a field in the flat Minkowski spacetime. Thus a field in a conformally flat spacetime is completely decoupled from gravity.

### 4.3.1 Transforming to the Einstein Frame

A conformal transformation corresponds to a change of the metric

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \tag{4.41}$$

with $\Omega^2(x)$ a continuous, non-vanishing, finite real function. Then clearly $\tilde{g}^{\mu\nu} = \Omega^{-2}(x)g^{\mu\nu}$. The determinant of the metric is defined as

$$g \equiv |\det g_{\mu\nu}| \tag{4.42}$$

In D-dimensional spacetime, the metric is diag$(-1, +1, +1, +1, \ldots)$ and it is easy to see that

$$\sqrt{-\tilde{g}} = \Omega^D \sqrt{-g} \tag{4.43}$$

[1] Whether or not the two frames are physically equivalent is considered as an open question, see for example [27].

Also the Ricci scalar changes with a conformal transformation, because the Ricci scalar in the transformed frame $\tilde{\mathcal{R}}$ is calculated using the new metric $\tilde{g}_{\mu\nu}$. The result is [24]

$$\tilde{\mathcal{R}} = \frac{1}{\Omega^2}\left[\mathcal{R} - \frac{2(D-1)}{\Omega}\Box\Omega - \frac{(D-1)(D-4)}{\Omega^2}g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega\right], \tag{4.44}$$

where the box operator is defined as

$$\Box\Omega = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Omega\right). \tag{4.45}$$

For $D = 4$ the transformed Ricci scalar simply becomes

$$\tilde{\mathcal{R}} = \frac{1}{\Omega^2}\left[\mathcal{R} - \frac{6}{\Omega}\Box\Omega\right]. \tag{4.46}$$

In what follows we show how a non-minimally coupled scalar field in $D$-dimensional spacetime can be written in the Einstein form. In this derivation we closely follow [28]. Start with the original Lagrangian written in the Jordan frame:

$$S_J = \int \mathrm{d}^{\mathrm{D}}x\sqrt{-g}\left[f(\phi)\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{4.47}$$

with

$$f(\phi) = \frac{1}{2}\left(M_0^{D-2} + \xi\phi^2\right), \tag{4.48}$$

and an arbitrary potential $V(\phi)$. Furthermore,

$$M_{(D)}^{D-2} = M_0^{D-2} + \xi\phi^2, \tag{4.49}$$

with

$$M_{(D)}^{D-2} \equiv \frac{1}{8\pi G_D}. \tag{4.50}$$

So in four-dimensional spacetime, $M_{\mathrm{Pl}} \equiv M_{(4)} = 1/\sqrt{8\pi G_4} = 2.44\times10^{18}\,\mathrm{GeV}$, where $G_4$ is Newton's constant.

The first term in eq. (4.47) becomes with eqs. (4.43) and (4.46)

$$\int \mathrm{d}^{\mathrm{D}}x\sqrt{-g}f(\phi)\mathcal{R} = \int \mathrm{d}^{\mathrm{D}}x\frac{\sqrt{-\tilde{g}}}{\Omega^D}f(\phi)\left[\Omega^2\tilde{\mathcal{R}} + \frac{2(D-1)}{\Omega}\Box\Omega + \frac{(D-1)(D-4)}{\Omega^2}\nabla_\mu\Omega\nabla_\nu\Omega\right] \tag{4.51}$$

In $D$-dimensional spacetime, to have ordinary gravity in the Einstein frame

$$\Omega^{D-2} = \frac{2}{M_{(D)}^{D-2}}f(\phi). \tag{4.52}$$

Using this relation we can write the second term under the integral in eq. (4.51) as

$$\int \mathrm{d}^{\mathrm{D}}x\sqrt{-\tilde{g}}f(\phi)\frac{2(D-1)}{\Omega^{D+1}}\Box\Omega = \int d^Dx\sqrt{-\tilde{g}}M_{(D)}^{D-2}\frac{(D-1)}{\Omega^2}\Box\Omega \tag{4.53}$$

Note that the box operator eq. (4.45) is given in terms of the original metric $g^{\mu\nu} = \Omega^2(x)\tilde{g}^{\mu\nu}$, using partial integration this becomes

$$\begin{aligned}
& \int d^D x\sqrt{-\tilde{g}} M_{(D)}^{D-2}(D-1)\Omega^{-3}\left[\Omega^D \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu\left(\Omega^{-D}\Omega^2\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\partial_\nu\Omega\right)\right] \\
= & \int d^D x\sqrt{-\tilde{g}} M_{(D)}^{D-2}(D-1)\Omega^{D-3}\tilde{g}_{\mu\nu}\partial_\mu\left(\Omega^{2-D}\partial_\nu\Omega\right) \\
= & -\int d^D x\sqrt{-\tilde{g}} M_{(D)}^{D-2}(D-1)(D-3)\Omega^{D-4}\tilde{g}_{\mu\nu}\partial_\mu\Omega\tilde{g}^{\mu\nu}\Omega^{2-D}\partial_\nu\Omega \\
= & -\int d^D x\sqrt{-\tilde{g}} M_{(D)}^{D-2}(D-1)(D-3)\Omega^{-2}\tilde{g}_{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega
\end{aligned} \tag{4.54}$$

Now we have finished writing the first term in the Lagrangian eq. (4.47) in the Einstein frame. The modifications to the kinetic term and the potential are much simpler. Using eq. (4.43) and $\tilde{g}^{\mu\nu} = \Omega^{-2}(x)g^{\mu\nu}$ it can easily be seen that the kinetic part becomes

$$\Omega^{2-D}\left(-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right). \tag{4.55}$$

The potential becomes with eq. (4.43)

$$\tilde{V}(\phi) = \frac{V(\phi)}{\Omega^D}. \tag{4.56}$$

For $D = 4$ the complete Lagrangian in the Einstein frame is

$$\int \mathrm{d}^4 x\sqrt{-g_E}\left[\frac{M_{\mathrm{Pl}}^2}{2}\mathcal{R}_E - \frac{3M_{\mathrm{Pl}}^2}{\Omega^2}g_E^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega - \Omega^{-2}\left(-\frac{1}{2}g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) - V_E(\phi)\right]. \tag{4.57}$$

This can be rewritten using eq. (4.52) in terms of $f(\phi)$

$$\int \mathrm{d}^4 x\sqrt{-g_E}\left[\frac{M_{\mathrm{Pl}}^2}{2}\mathcal{R}_E - \frac{3M_{\mathrm{Pl}}^2}{4f(\phi)^2}g_E^{\mu\nu}\partial_\mu f(\phi)\partial_\nu f(\phi) - \frac{M_{\mathrm{Pl}}^2}{4f(\phi)}g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V_E(\phi)\right]. \tag{4.58}$$

We can now try to make the kinetic term canonical by introducing a new field $\chi$ defined by

$$-\frac{1}{2}g_E^{\mu\nu}\partial_\mu\chi\partial_\nu\chi = -\frac{3M_{\mathrm{Pl}}^2}{4f(\phi)^2}g_E^{\mu\nu}\partial_\mu f(\phi)\partial_\nu f(\phi) - \frac{M_{\mathrm{Pl}}^2}{4f(\phi)}g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{4.59}$$

Then finally the action in the Einstein frame takes the form

$$S_{\mathrm{E}} = \int \mathrm{d}^4 x\sqrt{-g_E}\left[\frac{1}{2}M_{\mathrm{Pl}}^2\mathcal{R}_E - \frac{1}{2}(\partial_E\chi)^2 - V_E(\chi(\phi))\right]. \tag{4.60}$$

Unfortunately the field redefinition in eq. (4.59) is only possible for a single scalar field. In that case we have a one-to-one mapping between the field $\phi$ and the new field $\chi$ given by

$$\left(\frac{d\chi}{d\phi}\right) = M_{\mathrm{Pl}}\sqrt{\frac{f(\phi) + 3f(\phi)'^2}{2f(\phi)^2}}. \tag{4.61}$$

### 4.3.2 Conformal Invariance

Consider the following action for a scalar field with mass $m$ in $D$ dimensions non-minimally coupled to the Ricci scalar,

$$S = \int \mathrm{d}^{\mathrm{D}}x\sqrt{-g}\left[\frac{M_{(D)}^{D-2}+\xi\phi^2}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right]. \qquad (4.62)$$

Varying the action with respect to $\phi$ gives the equation of motion

$$(\Box - m^2 + \xi\mathcal{R})\phi = 0. \qquad (4.63)$$

For $m = 0$ there is a special value of $\xi$ for which the system gets an extra symmetry. This is the conformal symmetry and when a system has this symmetry it is said to be conformally coupled. The value of $\xi$ for which a system in $D$-dimensional spactime is conformally coupled is [29]

$$\xi_{\mathrm{conf}} = \frac{D-2}{4(D-1)}. \qquad (4.64)$$

So in $(1+1)$-dimensional spacetime the system is conformally invariant if $\xi = 0$, while for $D = 4$ the system in conformally coupled if $\xi = 1/6$.

If we transform the metric

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x), \qquad (4.65)$$

and we rescale the field $\tilde{\phi} = \Omega^{(2-D)/2}\phi$, the action and the equations of motion stay the same. Thus $S[\phi, g] = S[\tilde{\phi}, \tilde{g}]$ and the equations of motion are

$$(\Box + \xi\mathcal{R})\phi = 0 \qquad (4.66)$$

and

$$(\Box + \xi\tilde{\mathcal{R}})\tilde{\phi} = 0. \qquad (4.67)$$

For $D = 2$ the field $\tilde{\phi} = \phi$, and then also $S[\phi, g] = S[\phi, \tilde{g}]$. Note that the action is invariant for any $\xi$ if $\Omega$ is constant.

# Chapter 5

# Higgs Inflation

## 5.1 Introduction

Although the underlying physics are still unknown, inflation is an essential ingredient in the understanding of the very early universe. A large class of inflation models fall into the category of slow-roll inflation. In chapter 4 it was shown that a scalar field slowly rolling down a potential can lead to inflation. The disadvantage of most inflation models is that they are based on the existence of additional scalar fields. Another approach is to see if degrees of freedom already contained in the Standard Model can lead to inflation. An obvious candidate would be the Higgs field, because this is the only scalar field in the Standard Model. However, as seen in chapter 4, a scalar field with quartic self-coupling requires an extremely small coupling constant $\lambda = \mathcal{O}(10^{-13})$ to give the right size of the density perturbations.

In 2008, F.L. Bezrukov and M. Shaposhnikov showed that the Higgs boson could act as the inflaton [30]. They found that a non-minimal coupling between the Higgs field and gravity can lead to inflation with cosmological implications in agreement with WMAP-7 data [1]. The non-minimal coupling corresponds to adding a term of the form $\xi H^\dagger H \mathcal{R}$ to the usual Einstein-Hilbert action. Such a term with nonzero $\xi$ is required for renormalization purposes in theories of interacting scalar fields in curved spacetime [24]. A potential problem is that a large $\xi$ is unlikely from a particle physics point of view.

The action that plays a central role in this chapter is [1]

$$S_J = \int \mathrm{d}^4x \sqrt{-g} \left[ \frac{1}{2} M^2 \mathcal{R} + \xi H^\dagger H \mathcal{R} + \mathcal{L}_{\mathrm{SM}} \right]. \tag{5.1}$$

The mass parameter $M$ is nearly equal to the Plank mass. In this chapter the unitary gauge $H = h/\sqrt{2}$ will be used, with $h$ the real neutral component of the Higgs doublet being the only degree of freedom left after the Higgs mechanism. This model is very elegant in the sense that only a very simple extension is needed. Moreover, the new operator has mass dimension $\leq 4$ at the classical

[1] Limiting to mass dimension 4 operators, the action would be of the form $S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}M^2\mathcal{R} + \xi H^\dagger H \mathcal{R}\mathcal{R}^2 + \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\lambda\rho}\mathcal{R}^{\mu\nu\lambda\rho} + \Box\mathcal{R}\right]$. The last three operators require new degrees of freedom, since they lead to higher derivatives in the equations of motion.

level. However, as we will see in the next chapter, the new operator results in a tower of higher dimensional operators when the quantum nature of gravity becomes important.

Another drawback of the Higgs inflation model is that it requires that the Standard Model is valid up to at least a few orders of magnitude below the Planck scale. Unfortunately, the Standard Model has only been tested up to one TeV. Moreover, new physics are supposed to happen around the GUT-scale $\sim 10^{15}$ GeV.

In this chapter we will see that a non-minimal coupling of the Higgs field will naturally lead to inflation, because the potential becomes flat for large field values $h \gg M_{\rm Pl}/\sqrt{\xi}$. This is the regime where there is a running of the effective Planck mass in the Jordan frame. The parameter that determines the size of the CMB fluctuations is not the scalar self coupling, but the combination $\lambda/\xi^2$. By setting $\xi \sim 10^4$, the predicted values of the spectral index and the tensor-to-scalar ratio are in agreement with WMAP-7 data.

## 5.2 Cosmological Implications

As seen in chapter 4, gravity looks quite different for high field values in case of a nonzero coupling $\xi$. However, with a conformal transformation the action can be rewritten in Einstein form. If we suppose that $\mathcal{L}_{\rm SM}$ in eq. (5.1) is the Standard Model Lagrangian in the Jordan frame, then the transformed Lagrangian $\tilde{\mathcal{L}}_{\rm SM}$ in the Einstein frame will in principle be quite different. At low energies they coincide of course.

It is assumed that the only degree of freedom during inflation is the Higgs field, so in what follows all remaining Standard Model terms are dropped. The action written in the Jordan frame is

$$S_{\rm J} = \int {\rm d}^4x\sqrt{-g}\left[f(h)\mathcal{R} - \frac{1}{2}\partial_\mu h\partial^\mu h - V(h)\right], \tag{5.2}$$

with the potential $V(h) = \frac{\lambda}{4}\left(h^2 - v^2\right)^2$ and the function in front of the Ricci scalar given by

$$f(h) = \frac{1}{2}(M^2 + \xi h^2). \tag{5.3}$$

The inflationary dynamics are studied in the Einstein frame. This frame can be obtained by the conformal transformation

$$g^E_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{5.4}$$

with

$$\Omega^2 = \frac{2}{M_{\rm Pl}^2}f(h) \simeq 1 + \frac{\xi h^2}{M_{\rm Pl}^2}. \tag{5.5}$$

Then the Einstein frame action becomes

$$S_{\rm E} = \int {\rm d}^4x\sqrt{-g_E}\left[\frac{M_{\rm Pl}^2}{2}\mathcal{R}_E - \frac{3M_{\rm Pl}^2}{4f(h)^2}g_E^{\mu\nu}\partial_\mu f(h)\partial_\nu f(h)\right. \tag{5.6}$$

$$\left. - \frac{M_{\rm Pl}^2}{4f(h)}g_E^{\mu\nu}\partial_\mu h\partial_\nu h - V_E(h)\right]. \tag{5.7}$$

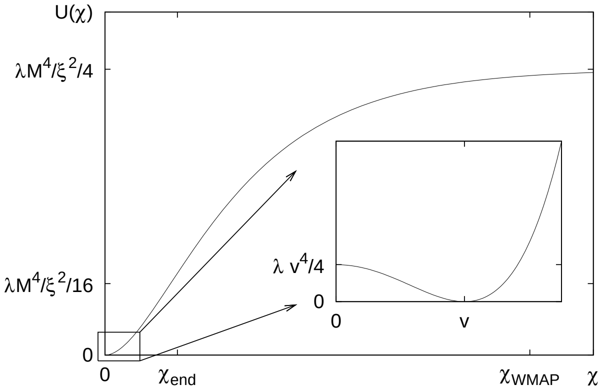


Figure 5.1: The potential of the Higgs field in the Einstein frame. [31]

The potential in the Einstein frame is

$$V_E(h) = \frac{V(h)}{\Omega^4} = \frac{\lambda}{4}\frac{\left(h^2 - v^2\right)^2}{\left(1 + \xi h^2/M_{\rm Pl}^2\right)^2}, \tag{5.8}$$

where in the last equality it is assumed that $h \gg v$. It can easily be seen that the potential becomes flat for $h \gg M_{\rm Pl}\sqrt{\xi}$, this makes slow-roll inflation possible.

To make the kinetic term canonical, we introduce the new field $\chi$ defined by

$$-\frac{1}{2} g_E^{\mu\nu}\partial_\mu\chi\partial_\nu\chi = -\frac{3M_{\rm Pl}^2}{4f(h)^2} g_E^{\mu\nu}\partial_\mu f(h)\partial_\nu f(h) - \frac{M_{\rm Pl}^2}{4f(h)} g_E^{\mu\nu}\partial_\mu h\partial_\nu h \tag{5.9}$$

or equivalently (eq. (4.61))

$$\left(\frac{d\chi}{dh}\right) = M_{\rm Pl}\sqrt{\frac{f(h) + 3f(h)'^2}{2f(h)^2}}. \tag{5.10}$$

Then the action in the Einstein frame takes the form

$$S_{\rm E} = \int {\rm d}^4x\sqrt{-g_E}\left[\frac{1}{2}M_{\rm Pl}^2\mathcal{R}_E - \frac{1}{2}(\partial_E\chi)^2 - V_E(\chi)\right], \tag{5.11}$$

which is the form that enables us to analyze cosmological implications. The Einstein frame potential as a function of the field $\chi$ is shown in figure 5.1. In the inflationary region, there is a simple analytic relation between the fields $h$ and $\chi$,

$$1 + \frac{\xi h^2}{M_{\rm Pl}^2} \simeq \exp\frac{2\chi}{\sqrt{6}M_{\rm Pl}}. \tag{5.12}$$

Substituting this into the potential eq. (5.8), we see that for large $\chi$ it is well approximated by

$$V_E(\chi) \simeq \frac{\lambda M_{\rm Pl}^4}{4\xi^2}\left(1 - \exp\left(-\frac{2\chi}{\sqrt{6}M_{\rm Pl}}\right)\right)^2. \tag{5.13}$$

During the inflationary epoch $h \gg M_{\rm Pl}/\sqrt{\xi}$ and the slow-roll parameters are approximately given by

$$\epsilon = \frac{M_{\rm Pl}^2}{2}\left(\frac{dV/d\chi}{V}\right)^2 \simeq \frac{4M_{\rm Pl}^4}{3\xi^2 h^4}, \tag{5.14a}$$

$$\eta = M_{\rm Pl}^2 \frac{d^2V/d\chi^2}{V} \simeq -\frac{4M_{\rm Pl}^2}{3\xi h^2}, \tag{5.14b}$$

$$\zeta^2 = M_{\rm Pl}^4 \frac{(d^3V/d\chi^3)dV/d\chi}{V^2} \simeq \frac{16M_{\rm Pl}^4}{9\xi^2 h^4}. \tag{5.14c}$$

Slow roll inflation ends when $\epsilon \simeq 1$, which enables us to determine the value of the Higgs field at the time inflation ends,

$$h_{\rm end} \simeq (4/3)^{1/4} M_{\rm Pl}/\sqrt{\xi}. \tag{5.15}$$

According to eq. (4.16), the number of $e$-folds is given by

$$N = \frac{1}{M_{\rm Pl}^2}\int_{\chi_{\rm end}}^{\chi_0} \frac{V}{dV/d\chi}\,{\rm d}\chi = \frac{1}{M_{\rm Pl}^2}\int_{h_{\rm end}}^{h_0} \frac{V}{dV/dh}\left(\frac{d\chi}{dh}\right)^2 dh \simeq \frac{6}{8}\frac{h_0^2 - h_{\rm end}^2}{M_{\rm Pl}^2/\xi}. \tag{5.16}$$

This gives us the value of the Higgs field $N_{\rm WMAP} \simeq 62$ e-folds before the end of inflation

$$h_{62} \simeq 9.4 M_{\rm Pl}/\sqrt{\xi}. \tag{5.17}$$

The WMAP normalization constraints $V/\epsilon = (0.027 M_{\rm Pl})^4$ at 62 e-folds before the end of inflation. Now the value of the coupling $\xi$ can be determined

$$\xi \simeq 49000\sqrt{\lambda} \simeq 18000. \tag{5.18}$$

Furthermore the spectral index and the tensor-to-scalar ratio are

$$n = 1 - 6\epsilon + 2\eta \simeq 0.97, \tag{5.19}$$

$$r = 16\epsilon \simeq 0.0033. \tag{5.20}$$

As seen in figure 5.2, these predicted values agree well with WMAP-5 observations.

## 5.3 Unitarity of Higgs Inflation

In the previous section we found that a non-minimal coupling causes a running of the effective Planck mass at energies $h \sim M_{\rm Pl}/\sqrt{\xi}$. In the in the Einstein frame, where the curvature coupling with the Higgs field is removed, the potential is given by

$$V_E(h) = \frac{\lambda}{4}\frac{\left(h^2 - v^2\right)^2}{\left(1 + \xi h^2/M_{\rm Pl}^2\right)^2}. \tag{5.21}$$

This potential has a plateau for $h \gg M_{\rm Pl}/\sqrt{\xi}$.

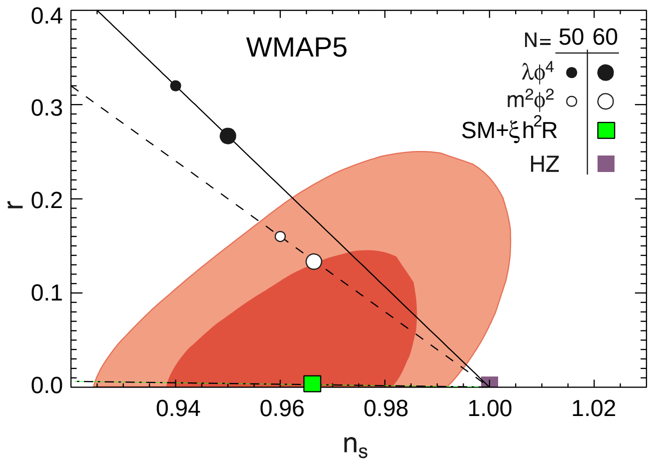


Figure 5.2: WMAP-5 constraints and predicted values of $n_s$ and $r$. [31]

Up to now, the analysis has been purely classical. In this section we will see what influence quantum corrections have on the flatness of the potential needed for slow-roll inflation. First of all we shall assume that the theory is valid up to some threshold below the Planck scale. The Higgs expectation value at $N$ $e$-foldings before the end of inflation is $\langle h \rangle \sim \sqrt{N} M_{\rm Pl}/\sqrt{\xi}$. This means that we can safely assume that for $N \sim 10^2$ all the Planck suppressed operators will be negligible, because $\xi$ is very large. However, we will see that in non-minimally coupled models operators suppressed by lower energy scales arise.

### 5.3.1 Single Field

Assume that the only degree of freedom is $h$, the other degrees of freedom being absorbed by the gauge fields. Consider then the $\frac{1}{2}\xi h^2 \mathcal{R}$ term in the Jordan frame. By making an expansion around flat space, the metric can be decomposed as

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{\gamma_{\mu\nu}}{M_{\rm Pl}}. \tag{5.22}$$

Here $\eta_{\mu\nu}$ is the Minkowski tensor and $\gamma_{\mu\nu}$ is a perturbation representing a graviton. The Ricci scalar, which is a function of the metric, becomes

$$\mathcal{R} \sim \frac{\Box \gamma_{\mu\nu}}{M_{\rm Pl}} + \dots \tag{5.23}$$

The leading order term is the mass dimension 5 operator

$$\frac{1}{2}\frac{\xi}{M_{\rm Pl}} h^2 \eta^{\mu\nu} \Box \gamma_{\mu\nu}. \tag{5.24}$$

At first sight it seems that this operator substantially contributes for $E \sim M_{\rm Pl}/\xi$, so it is tempting to say it has a cutoff at $\Lambda = M_{\rm Pl}/\xi$. However, this is incorrect as can be seen from the scattering process $2h \to 2h$. At high energies the mass of $h$ can be neglected and the tree-level process corresponds to the exchange of a single graviton (see figure 5.3). This gives the scattering amplitude

$$\mathcal{M}_c(2h \to 2h) \sim \frac{\xi^2 E^2}{M_{\rm Pl}^2}, \tag{5.25}$$

where a sum over the $s$, $t$ and $u$ channels is assumed. Again it appears that the cut-off is $\Lambda = M_{\rm Pl}/\xi$. However, it turns out that the three diagrams cancel each other exactly. The first nonzero contribution to the corresponding scattering amplitude is

$$\mathcal{M}_{\rm tot}(2h \to 2h) \sim \frac{E^2}{M_{\rm Pl}^2}. \tag{5.26}$$

So the true cut-off for this process is at the Planck scale.

A similar result can be obtained in the Einstein frame. The kinetic sector in this frame is

$$-\frac{1}{2}\frac{1}{1+\xi h^2/M_{\rm Pl}^2}(\partial h)^2 - \frac{3\xi^2}{M_{\rm Pl}^2}\frac{h^2}{\left(1+\xi h^2/M_{\rm Pl}^2\right)^2}(\partial h)^2. \tag{5.27}$$

Expanding for small $h$ gives

$$-\frac{1}{2}(\partial h)^2 - \frac{3\xi^2 h^2}{M_{\rm Pl}^2}(\partial h)^2 + \ldots \tag{5.28}$$

The second term looks like an operator with cut-off $\Lambda = m_{\rm Pl}/\xi$. Indeed if we compute the contribution of the 4-point vertex to the tree-level process, we find (see figure 5.3)

$$\mathcal{M} \sim \frac{\xi^2 E^2}{M_{\rm Pl}^2}. \tag{5.29}$$

However, when the external particles are on-shell the scattering amplitude vanishes. The reason for this is that for a non-minimally coupled system, we can make a field redefinition resulting in a minimally coupled system with a canonical kinetic term and modified potential:

$$S_{\rm E} = \int {\rm d}^4x\sqrt{-g_E}\left[\frac{1}{2}M_{\rm Pl}^2\mathcal{R}_E - \frac{1}{2}(\partial_E\chi)^2 - V_E(\chi)\right]. \tag{5.30}$$

In this theory quantum corrections coming from the kinetic part are suppressed by the Planck scale, resulting in the the cut-off $\Lambda = M_{\rm Pl}$.

Of course, since we have shifted to the Einstein frame, we have to deal with a modified potential. For small field values the Higgs field $h$ can be expressed in terms of the redefined field $\chi$ by

$$h = \chi\left[1 - (\xi\chi/M_{\rm Pl})^2\right] + \ldots. \tag{5.31}$$

Plugging this in the potential we find the dimension 6 operator

$$-\frac{\lambda\xi^2}{M_{\rm Pl}^2}\chi^6 \tag{5.32}$$

and we have to conclude that the theory already breaks down at $\Lambda \sim M_{\rm Pl}/\xi$.

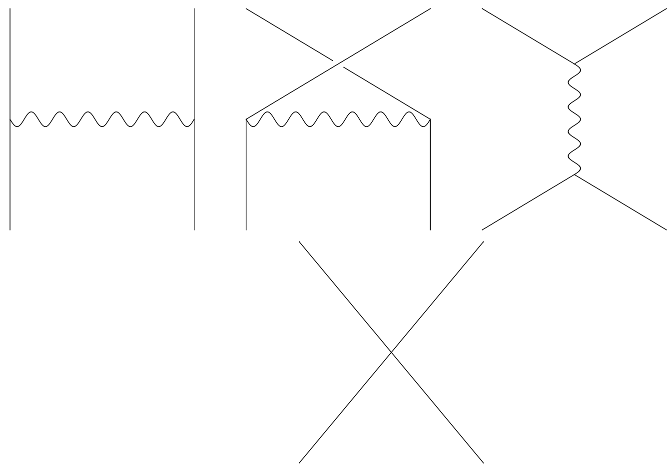

Figure 5.3: Tree level diagrams of the scattering process $2h \to 2h$. The upper panel shows graviton exchange through t, u, and s-channels in the Jordan frame. In the Einstein frame this is equivalent to a single 4-point vertex, as seen in the lower panel. [32]

### 5.3.2 Multiple Fields

The propagators for the gauge bosons are

$$\frac{-i}{k^2 - m^2}\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi m^2}(1-\xi)\right], \tag{5.33}$$

with $m$ the masses of the $W^\pm$ and $Z^0$ bosons. The unitary gauge can be obtained by taking $\xi \to \infty$. In the unitary gauge the Goldstone bosons completely disappear from our theory. However far above the EW scale the propagator goes as $k^\mu k^\nu / k^2$ and the non-renormalizability of this gauge comes into play. This means that if we want to study inflationary dynamics we have to describe the Higgs field as a complex doublet and not only the Higgs boson, but also the Goldstone bosons will be non-minimally coupled the Ricci scalar.

So we have to express the Higgs field as four scalars forming the doublet.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix}. \tag{5.34}$$

For multiple fields, however, the situation is quite different from the one sketched in the previous paragraph. Consider the scattering process $\phi_1 + \phi_2 \to \phi_1 + \phi_2$ at tree-level. Again this happens by a single graviton exchange. Now however we don't have to sum in eq. (5.25) over all the three channels, because this process only occurs through the $s$-channel. In this case there is no cancellation and the cut-off is at $\Lambda = m_{\text{Pl}}/\xi$.

The same cut-off is obtained in the Einstein frame. For a single field it was always possible to make a redefinition that brings the kinetic terms in canonical form. However this is not true in general for multiple fields. So the second term in eq. (5.28) will result in for example $\phi_1 + \phi_2 \to \phi_1 + \phi_2$ scattering via the

4-point vertex with amplitude

$$\mathcal{M} \sim \frac{\xi^2 E^2}{m_{\rm Pl}^2}. \tag{5.35}$$

So also in the Einstein frame the cut-off is at $\Lambda = m_{\rm Pl}/\xi$.

# Chapter 6

# Goldstone Bosons in Higgs Inflation

In this chapter we analyze the role Goldstone bosons play during inflation. It turns out they become massive and hence contribute to the Coleman-Weinberg potential. The analysis was done for a $U(1)$ symmetric toy model by S. Mooij and M. Postma in their article [34]. Here their results are generalized to symmetry group of the Standard Model. The notation introduced in section 6.1.1 is based on *Peskin and Schroeder (p. 739-740)* [19].

## 6.1 Higgs Mechanism and the Real Representation

As seen in section 3.4, the original $SU(2)_L \times U(1)_Y$ symmetry of the Standard Model to is spontaneously broken to $U(1)_Q$ by the Higgs field. The potential of this field is given by

$$V(\Phi^\dagger\Phi) = -\mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2, \tag{6.1}$$

where

$$\Phi = \begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix}. \tag{6.2}$$

Under a gauge transformation this doublet transforms as

$$\Phi \to e^{i\alpha^a\tau^a}e^{i\beta Y}\Phi, \tag{6.3}$$

with $\vec{\tau} = \vec{\sigma}/2$. Spontaneous symmetry breaking happens when $\lambda > 0$, in this case potential has a minimum at

$$\Phi^\dagger\Phi = \frac{1}{2}v^2, \qquad \text{with } v = \sqrt{\mu^2/\lambda}. \tag{6.4}$$

Denote the ground state as

$$\Phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 \\ v\end{pmatrix}. \tag{6.5}$$

A gauge transformation with $\alpha^1 = \alpha^2 = 0$ and $\alpha^3 = \beta$ leaves $\Phi_0$ invariant, which implies that we can construct the generator $Q = T^3 + T^Y = T^3 + \frac{1}{2}Y$ and have invariance under $U(1)_Q$.

The Lagrangian for the Higgs field and the gauge fields is

$$\mathcal{L} = -\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + (D_\mu\Phi)^\dagger D^\mu\Phi - V(\Phi^\dagger\Phi), \tag{6.6}$$

where

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk}A^j_\mu A^k_\nu, \qquad i = 1,2,3 \tag{6.7}$$

and

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{6.8}$$

The covariant derivative of $\Phi$ is given by

$$D_\mu\Phi = \left(\partial_\mu - ig\vec{\tau}\cdot\vec{A_\mu} - \frac{i}{2}g'B_\mu\right)\Phi. \tag{6.9}$$

### 6.1.1 The Real Representation

For the following discussion it will be more convenient to write the complex doublet as a multiplet of real-valued fields $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$. Under an infinitesimal gauge transformation the scalars $\phi_i$ transform as

$$\phi_i \to (1 + i\alpha^a t^a)_{ij}\phi_j, \tag{6.10}$$

so the generators $t^a$ are antisymmetric and strictly imaginary. Instead of the generators $t^a_{ij}$, we shall use the representation matrices $T^a_{ij} = -it^a_{ij}$, which are real and antisymmetric. Then the scalar fields transform as

$$\delta\phi_i = i\alpha^a t^a_{ij}\phi_j = -\alpha^a T^a_{ij}\phi_j, \tag{6.11}$$

while the gauge fields transform as

$$\delta A^a_\mu = \frac{1}{g}\partial_\mu\alpha^a - f^{abc}\alpha^b A^c_\mu. \tag{6.12}$$

In these equations the index $a = 1,2,3,Y$ and $A^Y_\mu = B_\mu$. Since the gauge group is not simple, the coupling constant for $a = Y$ should be read as $g'$. The Lagrangian invariant under these gauge transformations is

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2}(D_\mu\Phi)^2 - V(\Phi), \tag{6.13}$$

with the covariant derivative

$$D_\mu\phi_i = \partial_\mu\phi_i + gA^a_\mu T^a_{ij}\phi_j. \tag{6.14}$$

Explicitly, the representation matrices are

$$T^1\Phi = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \phi_4 \\ -\phi_3 \\ \phi_2 \\ -\phi_1 \end{pmatrix}, \tag{6.15a}$$

$$T^2\Phi = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\phi_3 \\ -\phi_4 \\ \phi_1 \\ \phi_2 \end{pmatrix}, \tag{6.15b}$$

$$T^3\Phi = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \phi_2 \\ -\phi_1 \\ -\phi_4 \\ \phi_3 \end{pmatrix}, \tag{6.15c}$$

$$T^Y\Phi = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \phi_2 \\ -\phi_1 \\ \phi_4 \\ -\phi_3 \end{pmatrix}. \tag{6.15d}$$

Note that we have the desired commutation relations $[T^a, T^b] = \epsilon^{abc}T^c$ and $[T^a, T^Y] = 0$.

Introduce the matrix

$$F^a_i \equiv T^a_{ij}\langle\phi_j\rangle, \tag{6.16}$$

with the indices $a = 1, 2, 3, Y$ denoting the rows and $i = 1, 2, 3, 4$ the columns. There is a sum over $j$, but in our case only $\langle\phi_3\rangle = v$ is nonzero. Filling in all the components we get

$$gF^a_i = \frac{v}{2}\begin{pmatrix} 0 & -g & 0 & 0 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & g \\ 0 & 0 & 0 & -g' \end{pmatrix}. \tag{6.17}$$

Again, the $g$ on the left-hand side should be read as $g'$ for $a = Y$.

For more convenience we use the parametrization

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i(\phi_1 - i\phi_2) \\ \phi_3 + i\phi_4 \end{pmatrix}, \tag{6.18}$$

with the ground state still

$$\Phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{6.19}$$

With this parametrization the real and antisymmetric representation matrices are

$$T^1\Phi = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \phi_3 \\ -\phi_4 \\ -\phi_1 \\ \phi_2 \end{pmatrix}, \tag{6.20a}$$

$$T^2\Phi = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \phi_4 \\ \phi_3 \\ -\phi_2 \\ -\phi_1 \end{pmatrix}, \tag{6.20b}$$

$$T^3\Phi = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\phi_2 \\ \phi_1 \\ -\phi_4 \\ \phi_3 \end{pmatrix}, \tag{6.20c}$$

$$T^Y\Phi = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\phi_2 \\ \phi_1 \\ \phi_4 \\ -\phi_3 \end{pmatrix}. \tag{6.20d}$$

Finally, after filling in all the components, we get the matrix

$$gF_i^a = \frac{v}{2}\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & 0 & g \\ 0 & 0 & 0 & -g' \end{pmatrix}. \tag{6.21}$$

Note that we can drop the third column; the elements of this column all equal zero, because the vectors $T^a\Phi_0 = T^a\phi_3$ are orthogonal to $\phi_3$ for all $a$.

### 6.1.2 The Global $SO(4)$ Symmetry

The generators in the previous section belong to the $SO(4)$ group. We wrote the spinor representation of $SU(2)$ in terms of $SO(4)$ generators, which was possible because $SU(2)$ is a subgroup of $SO(4)$.

The six generators of $SO(4)$ can be written as:

$$L_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$L_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad L_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$L_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad L_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{6.22}$$

Consider the Lagrangian

$$\mathcal{L} = (\partial_\mu\Phi)^T\partial^\mu\Phi - V(\Phi^T\Phi), \tag{6.23}$$

with the Higgs field written as a 4-vector

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}. \tag{6.24}$$

It can easily be seen that this Lagrangian is $SO(4)$ invariant. Suppose the Higgs field has a nonzero vacuum expectation value $v_3$ in the $\phi_3$ direction. Looking at the generators eq. (6.22), we see that three of the six generators don't leave the vacuum expectation value invariant. So we know that this model has three massless Goldstone bosons. The other three generators form an $SO(3)$ subgroup under which the expectation value is invariant.

Mathematically this is also easy to show. In eq. (6.35) we will see that

$$\frac{\partial^2 V}{\partial\phi_i\partial\phi_k} T^a_{ij}\phi_j + \frac{\partial V}{\partial\phi_i} T^a_{ik} = 0. \tag{6.25}$$

If we evaluate this equation at some value $\phi_i = v_i$ for which $V$ is a minimum, this equation becomes

$$\left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_k}\right|_{\phi_i=v_i} L_{ij}v_j = 0. \tag{6.26}$$

The factor $L_{ij}v_j$ is only nonzero for $j = 3$ and $i = 1, 2, 4$. Then we get

$$\begin{aligned}
\left.\frac{\partial^2 V}{\partial\phi_1\partial\phi_k}\right|_{\phi_1=0} \times v_3 &= 0, \\
\left.\frac{\partial^2 V}{\partial\phi_2\partial\phi_k}\right|_{\phi_2=0} \times v_3 &= 0, \\
\left.\frac{\partial^2 V}{\partial\phi_4\partial\phi_k}\right|_{\phi_4=0} \times (-v_3) &= 0.
\end{aligned} \tag{6.27}$$

(6.28)

We see that we have three massless Goldstone bosons. The remaining field $\phi_3$ is massive, its mass simply given by the second derivative of the potential.

The generators in eq. (6.15) and eq. (6.20) are linear combinations of the generators of the $SO(4)$ group given in eq. (6.22). Because they are linearly independent we know that they are four out of the six generators of SO(4). What about the remaining two? As noted before, the ground state is invariant under $SO(3)$, so there are three generators that leave the ground state invariant. We already know one, namely $Q = T^Y + T^3$. Noting the similar structure of $T^3$ and $T^Y$ in eq. (6.15), we construct two other matrices with the same structure as $T^1$ and $T^2$:

$$T^A = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{6.29}$$

and

$$T^B = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{6.30}$$

The combinations $A = T^1 + T^A$ and $B = T^2 + T^B$ leave the ground state invariant, so they form together with $Q$ the $SO(3)$ subgroup. The matrices $\{T^1, T^2, T^3, A, B, Q\}$ can be used as a set of generators for $SO(4)$.

The Lagrangian of the Standard Model is not $SO(4)$ invariant, since the Yukawa coupling between the Higgs field and the fermions breaks the symmetry down to $SU(2)$.

## 6.2 Massive Goldstone Bosons

During the inflationary epoch the energy density is many orders of magnitude higher than the electroweak scale $v \simeq 246\,\mathrm{GeV}$, which means that we can no longer assume that the Higgs field will be in its vacuum state. This has two important implications for the following analysis. First of all the Higgs field will not be in a minimum, so the first derivative of the potential won't vanish. Secondly, because the background field is not constant, we have to deal with a time-dependent expectation value instead of the constant vacuum expectation value $v$.

The ground state eq. (6.19) simply becomes

$$\Phi_0(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \phi_3(t) \end{pmatrix}. \tag{6.31}$$

In analogy with eq. (6.16) we define the new matrix

$$H_i^a \Phi_0 \equiv T_{ij}^a \langle \phi_j \rangle = \frac{\phi_3}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{6.32}$$

To make the following discussion more clear, the third column corresponding to $i = 3$ is still included.

Given that the potential is invariant under arbitrary gauge transformations, we have

$$0 = \delta V = \frac{\partial V}{\partial \phi_i}\delta\phi_i = -\alpha^a \frac{\partial V}{\partial \phi_i} T_{ij}^a \phi_j, \tag{6.33}$$

where we sum over $a$ and all $i, j$. Clearly,

$$\frac{\partial V}{\partial \phi_i} T_{ij}^a \phi_j = 0, \tag{6.34}$$

for each $a$ independently, as can be seen by taking only one infinitesimal parameter nonzero. Differentiating with respect to $\phi_k$ gives

$$\frac{\partial^2 V}{\partial \phi_i \partial \phi_k} T_{ij}^a \phi_j + \frac{\partial V}{\partial \phi_i} T_{ik}^a = 0. \tag{6.35}$$

If we evaluate this equation at the ground state, we get (see eq. (6.32))

$$\frac{\partial^2 V}{\partial \phi_i \partial \phi_k} H_i^a \phi_3 + \frac{\partial V}{\partial \phi_i} T_{ik}^a = 0. \tag{6.36}$$

The second derivative of the potential can be identified with a mass term, because

$$V(\phi_i) = V\big(\langle \phi_i \rangle\big) + \frac{1}{2}\frac{\partial^2 V}{\partial \phi_i \partial \phi_k}\bigg|_{\phi_i = \langle \phi_i \rangle} \big(\phi_i - \langle \phi_i \rangle\big)\big(\phi_k - \langle \phi_k \rangle\big) + \dots, \tag{6.37}$$

so that the mass matrix is

$$(M^2)_{ik} = \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_k}\right|_{\phi_i=\langle\phi_i\rangle} . \tag{6.38}$$

If the potential is a minimum at the classical background, then the second term in eq. (6.36) vanishes and the Goldstone bosons are massless. However, with the Higgs field displaced from its minimum, the masses will be nonzero.

As an example we calculate the mass term for $\phi_1$. Take $a = k = 1$, then the first term is only nonzero for $i = 1$ and the second only for $i = 3$. Thus we get

$$\left.\frac{\partial^2 V}{\partial\phi_1\partial\phi_1}\frac{1}{2}\phi_3 + \frac{\partial V}{\partial\phi_3}(-\frac{1}{2})\right|_{cl} = 0, \tag{6.39}$$

which we can rewrite as

$$\left.\frac{\partial^2 V}{\partial\phi_1\partial\phi_1}\right|_{cl} = \left.\frac{1}{\phi_3}\frac{\partial V}{\partial\phi_3}\right|_{cl} = -\left.\frac{\ddot{\phi}_3}{\phi_3}\right|_{cl} . \tag{6.40}$$

In the last equality we used the background equation of motion, $\ddot{\phi}_3 + \partial_{\phi_3}V = 0$. Mass terms for the fields $\phi_2$ and $\phi_4$ are obtained in the same way. To get the mass term for $\phi_4$ we can either set $a = 3$ or $a = 4$. Since $H_3^a = 0$ we don't get a non-trivial equation involving $V_{\phi_3\phi_3}$, so we can't make any statements about the mass of the field $\phi_3$. Furthermore, we obtain some trivial equations stating that mixed derivatives and first derivatives with respect to $\phi_1$, $\phi_2$ and $\phi_4$ are zero.

One can show in another way that Goldstone bosons rolling in a potential become massive [35]. This already becomes clear in the simple $U(1)$ symmetric model

$$\mathcal{L} = (\partial_\mu\phi)^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi. \tag{6.41}$$

We are of course free to parametrize the field $\phi$ any way we like. First we use Cartesian coordinates, and parametrize the field as $\phi = \frac{1}{\sqrt{2}}(\Phi + \phi_1 + i\phi_2)$. Consider an arbitrary point $\Phi$ in field space, such a point can always be aligned with the real direction. Then the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\left((\Phi + \phi_1)^2 + \phi_2^2\right). \tag{6.42}$$

We see that both the fields $\phi_1$ and $\phi_2$ have a mass $m$. If we use polar coordinates and write $\phi = \frac{1}{\sqrt{2}}(\Phi + \phi_1)e^{i\theta}$, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}(\Phi + \rho)^2\partial_\mu\theta\partial^\mu\theta - \frac{1}{2}m^2(\Phi + \rho)^2. \tag{6.43}$$

Now we have a non-canonical kinetic term for the field $\theta$. There are no terms proportional to $\theta^2$, which was to be expected since it doesn't cost energy to rotate along $\theta$. One might conclude that $\theta$ is a massless field.

Another possibility is to define the mass by the Laplacian, which is the sum of all second derivatives in Cartesian coordinates. Then the trace of the mass matrix eq. (6.38) is equal to the Laplacian. The Laplacian for a system written in polar coordinates is

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \theta^2}. \tag{6.44}$$

For the field $\rho$ the mass term is of course

$$m_\rho^2 = \frac{\partial^2 V}{\partial \rho^2}. \tag{6.45}$$

Then the nonzero remaining part of the Laplacian is assigned to the field $\theta$, so

$$m_\theta^2 = \frac{1}{\rho}\frac{\partial V}{\partial \rho} = -\frac{\ddot{\rho}}{\rho}, \tag{6.46}$$

where in the last equality the classical equation of motion is used.

## 6.3 Rolling Goldstone Bosons in $U(1)$ Theory

In this section, the idea of rolling Goldstone bosons is applied to a U(1) symmetric toy model. Consider the following Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\Phi)^\dagger D_\mu\Phi - V(\Phi^\dagger\Phi), \tag{6.47}$$

where

$$D_\mu\Phi = (\partial + igA_\mu)\Phi. \tag{6.48}$$

This Lagrangian is invariant under the gauge transformation

$$\Phi \quad\rightarrow\quad e^{i\alpha}\Phi, \tag{6.49}$$

$$A_\mu \quad\rightarrow\quad A_\mu - \frac{1}{g}\partial_\mu\alpha. \tag{6.50}$$

The Higgs field has a time-dependent expectation value $\Phi_{cl} = (\phi_R(t)+i\phi_I(t))/\sqrt{2}$. By gauge transforming we can always rotate the system and set $\phi_I = 0$. Since the potential is invariant under U(1), we have for an infinitesimal transformation $\Phi \rightarrow e^{i\alpha}\Phi$

$$0 = \delta V = \frac{\partial V}{\partial \phi_i}\delta\phi_i, \tag{6.51}$$

where $i = \{R, I\}$. Written out in terms of real fields we have $\delta\phi_R = -\alpha\phi_I$ and $\delta\phi_I = \alpha\phi_R$.

Differentiating eq. (6.51) with respect to $\phi_R$ gives

$$\frac{\partial}{\partial \phi_R}\left(\frac{\partial V}{\partial \phi_I}\phi_R - \frac{\partial V}{\partial \phi_R}\phi_I\right)\Bigg|_{cl} = 0. \tag{6.52}$$

The second term between brackets vanishes at at the classical background and the equation becomes trivial. Differentiation with respect to $\phi_I$ gives

$$\frac{\partial^2 V}{\partial \phi_I \partial \phi_I}\phi_R - \frac{\partial V}{\partial \phi_R}\Bigg|_{cl} = 0. \tag{6.53}$$

If the potential $V$ is a minimum at the classical background, then the second term in eq. (6.53) vanishes. So we conclude that we get a massless Goldstone boson. However during inflation the Higgs field is displaced from its minimum and the second term won't vanish, therefore the Goldstone boson gets the mass

$$m_I^2 \equiv \left.\frac{\partial^2 V}{\partial \phi_I^2}\right|_{cl} = \left.\frac{1}{\phi_R}\frac{\partial V}{\partial \phi_R}\right|_{cl} = -\left.\frac{\ddot{\phi}_R}{\phi_R}\right|_{cl}. \tag{6.54}$$

Since we used the equation of motion for the background field $\ddot{\phi}_R + \partial_{\phi_R} V = 0$, the last equality is only valid on-shell.

Let's see what happens if we promote the global symmetry to a local symmetry. This means that we have to introduce the gauge field $A_\mu(x,t)$. One might expect that the Higgs mechanism now fails, because the Goldstone boson is massive and hence its degree of freedom cannot be obtained by the gauge boson.

Consider perturbations around the classical background Higgs field

$$\Phi(x,t) = \frac{1}{\sqrt{2}}(\Phi_R(x,t) + i\Phi_I(x,t)) = \frac{1}{\sqrt{2}}[\phi_R(t) + h(x,t) + i\theta(x,t)]. \tag{6.55}$$

In terms of the perturbations $h$ and $\theta$ the potential expanded around the background is

$$V = V\Big|_{cl} + \frac{\partial V}{\partial \phi_R}\Big|_{cl} h + \frac{1}{2}\frac{\partial^2 V}{\partial\phi_R \partial\phi_R}\Big|_{cl} h^2 + \frac{1}{2}\frac{\partial^2 V}{\partial\phi_I\partial\phi_I}\Big|_{cl}\theta^2 + \ldots \tag{6.56}$$

The dots represent terms of cubic or higher order in the perturbations. From eq. (6.51) it can easily be seen that all the mixed terms are zero. The first derivative with respect to the field $\phi_I$ is zero at the classical background, due to the $U(1)$ symmetric shape of the potential.

The kinetic part of the Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{kin}} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + D_\mu\Phi(D^\mu\Phi)^\dagger \\ &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (\partial_\mu + igA_\mu)(\phi_R + h + i\theta)(\partial^\mu - igA^\mu)(\phi_R + h - i\theta).\end{aligned} \tag{6.57}$$

Writing out gives

$$\begin{aligned}\mathcal{L}_{\text{kin}} =& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial_\mu h\partial^\mu h + \partial_\mu\theta\partial^\mu\theta + g^2\phi_R^2 A_\mu A^\mu) + g\phi_R A_\mu\partial^\mu\theta \\ &- g\dot{\phi}_R\theta A_0 + \dot{h}\dot{\phi}_R + \frac{1}{2}\dot{\phi}_R^2 \\ &- gA^\mu\theta\partial_\mu h + \frac{1}{2}g^2 A_\mu A^\mu\theta^2 + gA^\mu h\partial_\mu\theta + g^2 A_\mu A^\mu\phi_R h + \frac{1}{2}g^2 A_\mu A^\mu h^2.\end{aligned} \tag{6.58}$$

The terms in the second an third line are absent when the Higgs is in a static minimum. Let's try to remove the Goldstone boson from our theory. Transform to the unitary gauge by redefining the gauge field

$$A_\mu = B_\mu - \frac{1}{g}\partial_\mu(\theta/\phi_R) = B_\mu - \frac{1}{g\phi_R}\partial_\mu\theta + \frac{\theta}{g\phi_R^2}\partial_\mu\phi_R. \tag{6.59}$$

When we write the kinetic part in terms of the field $B_\mu$ we get

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} + \frac{1}{2}(\partial_\mu h\partial^\mu h + g^2\phi_R^2 B_\mu B^\mu) - \frac{\theta^2\dot{\phi}_R^2}{2\phi_R^2} + \frac{\theta\dot{\theta}\dot{\phi}_R}{\phi_R} + \dot{h}\dot{\phi}_R + \frac{1}{2}\dot{\phi}_R^2 + \dots \tag{6.60}$$

We see that for $\dot{\phi}_R \neq 0$ the Goldstone boson is still present in both the potential and the kinetic part. However, it turns out that they are canceled by each other. Indeed,

$$\begin{aligned} -\frac{\theta^2\dot{\phi}_R^2}{2\phi_R^2} + \frac{\theta\dot{\theta}\dot{\phi}_R}{\phi_R} + \dot{h}\dot{\phi}_R + \frac{1}{2}\dot{\phi}_R^2 &= -\frac{\ddot{\phi}_R}{2}\left(\frac{\theta^2}{\phi_R} + 2h + \phi_R\right) \\ &= \frac{1}{2}\frac{\partial^2 V}{\partial\phi_I\partial\phi_I}\bigg|_{cl}(\theta^2 + 2\phi_R h + \phi_R^2), \end{aligned} \tag{6.61}$$

where the last equality is only valid on-shell. The first term in the last equality cancels the quadratic term in $\theta$ in the potential, while the second term cancels the tadpole in $h$,

$$\frac{1}{2}\frac{\partial^2 V}{\partial\phi_I\partial\phi_I}\bigg|_{cl} 2\phi_R h = \frac{1}{\phi_R}\frac{\partial V}{\partial\phi_R}\bigg|_{cl}\phi_R h = \frac{\partial V}{\partial\phi_R}h. \tag{6.62}$$

The Goldstone boson has disappeared from the Lagrangian. Its degree of freedom is still absorbed by the field $B_\mu$ to become a massive vector field.

## 6.4 Lagrangian in the $R_\xi$ Gauge

In section 6.4.1, the idea of the last section is applied to the SM. Next, we gauge fix the obtained Lagrangian in section 6.4.2, taking into account the time-dependent background Higgs field. Lastly, the mass terms needed for the Coleman-Weinberg potential are derived in section 6.4.3.

### 6.4.1 Non-Abelian Analysis

Write the parametrization eq. (6.18) as

$$\Phi(x,t) = \frac{1}{\sqrt{2}}\begin{pmatrix} -i(\theta_1 - i\theta_2) \\ \phi + (h + i\theta_3) \end{pmatrix}. \tag{6.63}$$

The notation makes clear that we have perturbations around the time-dependent ground state $\Phi_0(t) = \binom{0}{\phi}/\sqrt{2}$ described with the physical Higgs boson $h$ and the Goldstone bosons $\theta_i$ ($i = 1, 2, 3$). As we have seen in section 6.2, the masses for the Goldstone bosons are given by the diagonal entries of the mass matrix

$$(M^2)_{ij} = \frac{\partial^2 V}{\partial\theta_i\partial\theta_j}\bigg|_{cl} = -\frac{\ddot{\phi}}{\phi}\delta_{ij}. \tag{6.64}$$

Now work out the kinetic term,

$$\begin{aligned} (D_\mu\Phi)^2 =& \frac{1}{2}(\partial_\mu h\partial^\mu h + \partial_\mu\theta_i\partial^\mu\theta_i + g^2\phi^2 H_i^a H_i^b A_\mu^a A^{b\mu}) + g\partial^\mu\theta_i A_\mu^a\phi H_i^a \\ &+ g\partial_\mu\phi A^{a\mu}T_{0i}^a\theta_i + \dot{h}\dot{\phi} + \frac{1}{2}\dot{\phi}^2 + \dots, \end{aligned} \tag{6.65}$$

where the dots stand for terms cubic and quartic in $h$, $\theta_i$ and $A^\mu$. Since the representation matrices $T^a_{ij}$ are antisymmetric, we have

$$gT^a_{0i}\theta_i = -gT^a_{i0}\theta_i = -H^a_i\theta_i, \tag{6.66}$$

and the first term on the second line in eq. (6.65) can be written as $-g\partial_\mu\phi A^{a\mu}H^a_i\theta_i$. Assuming the background field is only *time*-dependent, this is, explicitly,

$$-g\partial_\mu\phi A^{a\mu}H^a_i\theta_i = -\frac{1}{2}gA^k_0\theta_k\partial_t\phi + \frac{1}{2}g'A^Y_0\theta_3\partial_t\phi, \tag{6.67}$$

with $k = 1, 2, 3$.

We expand the potential up to second order in perturbations

$$V_2(\Phi) = V\bigg|_{cl} + \frac{\partial V}{\partial\phi}\bigg|_{cl} h + \frac{1}{2}\frac{\partial^2 V}{\partial\phi\partial\phi}\bigg|_{cl} h^2 + \frac{1}{2}\frac{\partial^2 V}{\partial\theta_i\partial\theta_i}\bigg|_{cl}\theta_i^2, \tag{6.68}$$

where again $i = 1, 2, 3$. With our parametrization the $\theta_i$'s are identical with the Goldstone bosons up to first order, so there are no linear terms in $\theta_i$. In the unitary gauge the $\theta_i$'s are precisely the Goldstone bosons, so there also the higher order derivatives in $\theta$ are zero.

Finally, we can write the quadratic part of the Lagrangian as (note that this implies that drop the self-interactions of the non-Abelian gauge fields)

$$\begin{aligned}\mathcal{L}_2 = &-\frac{1}{2}A^a_\mu(-g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu)A^a_\nu + \frac{1}{2}(\partial_\mu\theta_i)^2 + \frac{1}{2}(\partial_\mu h)^2 \\ &+ g\partial^\mu\theta_i A^a_\mu\phi H^a_i + \frac{1}{2}(m^2_A)^{ab}A^a_\mu A^{\mu b} - \frac{1}{2}(M^2)_{ij}\theta_i\theta_j \\ &- g\partial_\mu\phi A^{a\mu}H^a_i\theta_i + \dot{h}\dot{\phi} + \frac{1}{2}\dot{\phi}^2 - V_2(\phi).\end{aligned} \tag{6.69}$$

Here the mass matrices for the gauge bosons and Goldstone bosons are respectively

$$(m^2_A)^{ab} = g^2H^a_iH^b_i\phi^2 \tag{6.70}$$

and

$$(M^2)_{ij} = -\frac{\ddot{\phi}}{\phi}\delta_{ij}. \tag{6.71}$$

### 6.4.2 Gauge Fixing the Lagrangian

In section 3.6 we calculated the Lagrangian in the $R_\xi$ gauge for the $U(1)$ symmetric case. Here we extend the analysis to the non-Abelian case. Moreover, the background Higgs field is now time-varying and not in a minimum.

The gauge fixing function for the $R_\xi$ gauge is

$$G^a = \frac{1}{\sqrt{\xi}}(\partial_\mu A^{a\mu} - \xi g\phi H^a_i\theta_i). \tag{6.72}$$

This means that we have to add

$$\mathcal{L}_{gf} = -\frac{1}{2}G^2 \tag{6.73}$$

to the Lagrangian. The terms up to second order are

$$\mathcal{L}_{gf} = \frac{1}{2}A^a_\mu(\frac{1}{\xi}\partial^\mu\partial^\nu)A^a_\nu + g\partial_\mu A^{a\mu}\phi H^a_i\theta_i - \frac{1}{2}\xi g^2(\phi H^a_i\theta_i)^2. \tag{6.74}$$

The second term can be rewritten as

$$g\partial_\mu A^{a\mu}\phi H^a_i\theta_i = g\partial_\mu(A^{a\mu}\phi H^a_i\theta_i) - g\partial_\mu\theta_i A^{a\mu}\phi H^a_i - g\partial_\mu\phi A^{a\mu}H^a_i\theta_i. \tag{6.75}$$

The first term is a total derivative so it can be neglected. The second term cancels $g\partial^\mu\theta_i A^a_\mu H^a_i\phi$ in $\mathcal{L}_2$. The third term vanishes in case of a time independent (constant) expectation value. Note that we already had such a term in eq. (6.69). So even after gauge fixing there remains a coupling between the Goldstone bosons and the gauge fields.

We also have to add the Lagrangian of the ghosts. The gauge variation of $G^a$ is

$$\frac{\delta G^a}{\delta\alpha^b} = \frac{1}{\sqrt{\xi}}\big(\frac{1}{g}(\partial_\mu D^\mu)^{ab} + \xi g(T^a\phi)\cdot T^b(\phi+\theta)\big), \tag{6.76}$$

thus

$$\mathcal{L}_{\mathrm{FP}} = \bar\eta^a g\frac{\delta G^a}{\delta\alpha^b}\eta^b = \bar\eta^a\left[-(\partial_\mu D^\mu)^{ab} - \xi g^2(T^a\phi)\cdot T^b(\phi+\theta)\right]\eta^b. \tag{6.77}$$

Limiting ourselves to quadratic terms, we are left with

$$\mathcal{L}_{\mathrm{FP}} = \bar\eta^a\left[-\partial^2\delta^{ab} - \xi g^2\phi^2 H^a_i H^b_i\right]\eta^b + \dots \tag{6.78}$$

### 6.4.3 Mass Terms in the $R_\xi$ Gauge

For practical purposes, decompose the Lagrangian we found so far as

$$\mathcal{L}_{\mathrm{tot}} = \mathcal{L} + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}} = \mathcal{L}_{\mathrm{cl}}(\phi) + \mathcal{L}_{\mathrm{free}} + \mathcal{L}_{\mathrm{int}}(t). \tag{6.79}$$

The classical Lagrangian contains the kinetic and potential terms of the field $\phi$. The free Lagrangian contains the time-independent terms quadratic in the fluctuations fields. From this expression it is easy to read of the free propagators. The interaction Lagrangian contains all other terms, which are treated as perturbations. Explicitly,

$$\mathcal{L}_{\mathrm{cl}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{6.80}$$

$$\begin{aligned}\mathcal{L}_{\mathrm{free}} =& -\frac{1}{2}A^a_\mu\big[\big(-g^{\mu\nu}\partial^2 + (1-\frac{1}{\xi})\partial^\mu\partial^\nu\big)\delta^{ab} - g^2\phi_0^2 H^a_i H^b_i g^{\mu\nu}\big]A^b_\nu \\ &+ \bar\eta^a\left[-\partial^2\delta^{ab} - \xi g^2\phi_0^2 H^a_i H^b_i\right]\eta^b - \frac{1}{2}h\big[\partial^2 + V_{hh}(0)\big]h \\ &- \frac{1}{2}\theta_i\big[\big(\partial^2 + V_{\theta\theta}(0)\big)\delta^{ij} + \xi g^2\phi_0^2 H^a_i H^a_j\big]\theta_j\end{aligned} \tag{6.81}$$

$$\begin{aligned}\mathcal{L}_{\mathrm{int}} =& -h\big[\partial^2\phi + V_\phi\big] + \frac{g^2}{2}(\phi^2-\phi_0^2)\big[A^a_\mu g^{\mu\nu}H^a_i H^b_i A^b_\nu - \xi H^a_i H^a_j\theta_i\theta_j - 2\xi\bar\eta_a H^a_i H^b_i\eta_b\big] \\ &- 2g\partial_\mu\phi A^{a\mu}H^a_i\theta_i - \frac{1}{2}(V_{hh}(t) - V_{hh}(0))h^2 - \frac{1}{2}(V_{\theta_i\theta_i}(t) - V_{\theta_i\theta_i}(0))\theta_i^2 + \dots\end{aligned} \tag{6.82}$$

Here the dots stand for all higher order terms. Define a mass matrix by

$$m^2_{\alpha\beta} = -\frac{\partial^2 \mathcal{L}}{\partial\theta_\alpha \partial\theta_\beta}, \qquad \text{with } \chi_\alpha = \{A^{a\mu}, \eta^a, h, \theta_i\}. \tag{6.83}$$

Then the nonzero elements of the mass matrix are (diagonal entries are represented with a single subscript)

$$\begin{aligned}
m^2_{A^{a\mu}A^{b\nu}} &= -g^2 H^a_i H^b_i \phi^2 g^{\mu\nu} \\
m_{\bar{\eta}^a \eta^b} &= \xi g^2 H^a_i H^b_i \phi^2 \\
m_h &= V_{hh} \\
m^2_{\theta_i\theta_j} &= V_{\theta\theta} + \xi g^2 H^a_i H^a_j \phi^2 \\
m^2_{\theta_i A^{a\mu}} &= 2g H^a_i \dot{\phi} \delta^\mu_0
\end{aligned} \tag{6.84}$$

The mass of the Goldstone bosons and the mass of the ghosts depend on the gauge parameter $\xi$. This is just an indication that they are not real physical particles. Such fields are called fictious fields and they can only exist as intermediate states.

Diagonalizing gives

$$\begin{aligned}
m^2_{W^\pm} &= \frac{1}{4} g^2 \phi^2 \\
m^2_{Z^0} &= \frac{1}{4}(g^2 + g'^2)\phi^2 \\
m^2_{\eta_{W^\pm}} &= \frac{1}{4}\xi g^2 \phi^2 \\
m^2_{\eta_{Z^0}} &= \frac{1}{4}\xi (g^2 + g'^2)\phi^2 \\
m^2_h &= V_{hh} \\
m^2_{\theta_1} &= V_{\theta\theta} + \frac{1}{4}\xi g^2 \phi^2 \\
m^2_{\theta_2} &= V_{\theta\theta} + \frac{1}{4}\xi g^2 \phi^2 \\
m^2_{\theta_3} &= V_{\theta\theta} + \frac{1}{4}\xi (g^2 + g'^2)\phi^2
\end{aligned} \tag{6.85}$$

From now on, we drop the superscripts of $W^\pm$ and $Z^0$. Work out the term $-2g\partial_\mu \phi A^{a\mu} H^a_i \theta_i$ in eq. (6.82):

$$\begin{aligned}
-2g\partial_\mu \phi A^{a\mu} H^a_i \theta_i &= -2g\dot{\phi} A^a_0 H^a_i \theta_i \\
&= -g\dot{\phi} A^1_0 \theta_1 - g\dot{\phi} A^2_0 \theta_2 - g\dot{\phi} A^3_0 \theta_3 + g'\dot{\phi} B_0 \theta_3 \\
&= -g\dot{\phi} A^1_0 \theta_1 - g\dot{\phi} A^2_0 \theta_2 - \sqrt{g^2 + g'^2}\dot{\phi} Z_0 \theta_3,
\end{aligned} \tag{6.86}$$

where we used

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g A^3_\mu - g' B^0_\mu). \tag{6.87}$$

Now it is easy to read of the remaining (off-diagonal) elements of the mass matrix:

$$\begin{aligned} m^2_{\theta_1 A^1_0} &= g\dot{\phi} \\ m^2_{\theta_2 A^2_0} &= g\dot{\phi} \\ m^2_{\theta_3 Z_0} &= \sqrt{g^2+g'^2}\dot{\phi} \end{aligned} \quad (6.88)$$

## 6.5 Corrections to the Coleman-Weinberg Potential

In this section we calculate the one-loop corrected equations of motion for the classical fields. The one-loop correction to the effective action is $\Gamma^{1-\text{loop}} = -\int \mathrm{d}^4x\, V_{CW}$, with the Coleman-Weinberg potential [36]

$$V_{\text{CW}} = \frac{1}{32\pi^2}\sum_i (-1)^{2J_i}(2J_i+1)m_i^2(\Lambda^2 - m_i^2\ln\Lambda).$$

Here there is a sum over all the fields in the theory, with $J_i$ and $m_i$ standing for the spin and the mass of the field. So we see that also the massive Goldstone bosons should add to the Coleman-Weinberg potential.

Using the Schwinger-Keldysh formalism the one-loop corrected equation of motion is [34]

$$0 = \Box\phi + V_\phi + \frac{1}{2}(\partial_\phi m^2_{\alpha\beta})G^{++}_{\alpha\beta}(0). \quad (6.89)$$

We first calculate the propagator for the field $h$,

$$\frac{1}{2}(\partial_\phi m_h^2)G_h^{++}(0) = \frac{\partial_\phi m_h^2}{16\pi^2}\left(\Lambda^2 - \frac{1}{2}m_h^2\ln\left(\frac{\Lambda^2}{m_h^2}\right)\right). \quad (6.90)$$

The propagators for the other fields that don't couple in the equations of motion are obtained in the same way. However, the fields $(\theta_1, A^1_0)$, $(\theta_2, A^2_0)$ and $(\theta_3, Z_0)$ do couple in the equations of motion due to the nonzero off-diaganol matrix elements eq. (6.88). In [34] also these propagators have been calculated, here we explicitly give the propagator for $(\theta_1, A^1_0)$,

$$\frac{\partial_\phi m^2_{\theta_1 A^1_0}}{4\pi^2} m^2_{\theta_1 A^1_0}\ln\left(\frac{\Lambda^2}{m^2_{\theta_1 A^1_0}}\right). \quad (6.91)$$

Calculating the propagators for the remaining fields and adding them all up, the one-loop equation of motion becomes

$$\begin{aligned} 0 = \Box\phi + V_\phi &+ \frac{\partial_\phi m^2_{\theta_1 A^1_0}}{4\pi^2} m^2_{\theta_1 A^1_0}\ln\left(\frac{\Lambda^2}{m^2_{\theta_1 A^1_0}}\right) \\ &+ \frac{\partial_\phi m^2_{\theta_2 A^2_0}}{4\pi^2} m^2_{\theta_2 A^2_0}\ln\left(\frac{\Lambda^2}{m^2_{\theta_2 A^2_0}}\right) \\ &+ \frac{\partial_\phi m^2_{\theta_3 Z_0}}{4\pi^2} m^2_{\theta_3 Z_0}\ln\left(\frac{\Lambda^2}{m^2_{\theta_3 Z_0}}\right) \\ &+ \sum_{\{h,\eta_W,\eta_Z,W^\pm,Z,\theta_i\}} \frac{S_i}{16\pi^2}\partial_\phi m_i^2\left(\Lambda^2 - \frac{1}{2}m_i^2\ln\left(\frac{\Lambda^2}{m_i^2}\right)\right). \end{aligned} \quad (6.92)$$

Here the degrees of freedom are $S_i = \{1, -4, -2, 8, 4, 3\}$ for $i = \{h, \eta_{W^\pm}, \eta_Z, W^\pm, Z^0, \theta_i\}$. The minus sign for the ghost fields is due to their anti-commuting nature.

Finally, the effective action up to a field-independent constant is obtained by integrating the equations of motion,

$$\begin{aligned}
\Gamma^{1-\text{loop}} &= \frac{-1}{16\pi^2}\int \mathrm{d}^4x \Bigg[\Lambda^2\Big(m_h^2 - 4m_{\eta_W}^2 - 2m_{\eta_Z}^2 + 4(2m_W^2 + m_Z^2) + m_{\theta_1}^2 + m_{\theta_2}^2 + m_{\theta_3}^2\Big) \\
&\quad - \frac{\ln\Lambda^2}{4}\Big(m_h^4 - 4m_{\eta_W}^4 - 2m_{\eta_Z}^4 + 4(2m_W^4 + m_Z^4) + m_{\theta_1}^4 + m_{\theta_2}^4 + m_{\theta_3}^4 \\
&\quad - 2(m_{\theta_1 A_0^1}^4 + m_{\theta_2 A_0^2}^4 + m_{\theta_3 Z_0}^4)\Big)\Bigg] + \text{finite} \\
&= \frac{-1}{16\pi^2}\int \mathrm{d}^4x \Bigg[\Lambda^2\Big(V_{hh} + 3V_{\theta\theta} + 6m_W^2 + 3m_Z^2\Big) \\
&\quad - \frac{\ln\Lambda^2}{4}\Big(V_{hh}^2 + 3V_{\theta\theta}^2 + 6m_W^4 + 3m_Z^4 - 12V_{\theta\theta}m_W^2 - 6V_{\theta\theta}m_Z^2\Big)\Bigg].
\end{aligned} \tag{6.93}$$

The second step follows almost immediately after using eq. (6.85). The only problems are the off-diagonal terms given by eq. (6.88), which using the background equation of motion can be rewritten as

$$\begin{aligned}
\int \mathrm{d}t\Big[-2\big(m_{\theta_1 A_0^1}^4 + m_{\theta_2 A_0^2}^4 + m_{\theta_3 Z_0}^4\big)\Big] &= -2\int \mathrm{d}t\Big[\big(g\dot{\phi}\big)^2 + \big(g\dot{\phi}\big)^2 + \big(\sqrt{g^2+g'^2}\dot{\phi}\big)^2\Big] \\
&= 2\int \mathrm{d}t\Big[g^2\phi\ddot{\phi} + g^2\phi\ddot{\phi} + \big(g^2+g'^2\big)\phi\ddot{\phi}\Big] \\
&= -2\int \mathrm{d}t\Big[2g^2\phi V_\phi + \big(g^2+g'^2\big)\phi V_\phi\Big] \\
&= -2\int \mathrm{d}t\Big[2g^2\phi^2 V_{\theta\theta} + \big(g^2+g'^2\big)\phi^2 V_{\theta\theta}\Big] \\
&= \int \mathrm{d}t\Big[-16V_{\theta\theta}m_W^2 - 8V_{\theta\theta}m_Z^2\Big]
\end{aligned} \tag{6.94}$$

Note that the gauge parameter $\xi$ is no longer present in the last equality of eq. (6.93).

# Chapter 7

# Conclusions

The inflationary epoch right after the Big Bang solves the horizon and flatness problems. It also explains the scale invariance of the density perturbations found in the Cosmic Microwave Background. We have seen that a minimally coupled Higgs field could not have caused inflation, since this doesn't give the right amplitude of density perturbations.

When a large coupling between the Higgs field and the Ricci scalar is introduced, the right size of the density perturbations can be obtained. This model has been proposed by Bezrukov and Shaposhnikov in 2008 [30]. Also the predicted values for the spectral index and the tensor-to-scalar ratio fall well within the limits of WMAP-7. This model has the advantage that no additional fields are needed. A potential problem is that a large coupling seems unlikely from a particle physics point of view. Another problem is that this model requires that the Standard Model is valid up to the inflationary scale $\sim 10^{15}$ GeV, while it has only been tested up to energies of about one TeV.

In the quantum theory the model seems to have the lower cut-off $M_{\rm Pl}/\xi$, which is below the scale where inflation takes place. For a single field, this is due to a mass dimension 6 operator coming from the potential in the Einstein frame. However for multiple fields, as will be the case for the Standard Model where the Higgs field is represented as a complex doublet, also the gravity sector in the Jordan frame and the kinetic sector in the Einstein frame seem to imply a lower cut-off at $M_{\rm Pl}/\xi$.

Finally, we have seen that Goldstone bosons become massive during inflation. For this reason they contribute to the Coleman-Weinberg potential. In [34] the one-loop effective action was calculated for a $U(1)$ symmetric toy model. In this thesis we have extended the analysis to the $SU(2) \times U(1)$ symmetry of the Standard Model.

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