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Quantum Fluctuations in Vacuum Energy: Cosmic Inflation as a Dynamical Phase Transition

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Abstract: A variety of models of inflation in the early Universe have been proposed and applied to describe successfully the origin of all possible structures in the Universe. On the other hand, inflation theory is still phenomenological and needs systematic physical foundations, including the relation to dark matter and dark energy. The essence of cosmic inflation would be a dynamical phase transition and the spontaneous symmetry breaking process, which are common in ordinary physics in the laboratory. At the beginning of the phase transition, the system is often in an adiabatic ground state and produces a squeezed state. This is widely interpreted as the generation of classical structures; however, it is not. The common notion of decoherence is not sufficient to describe the inflationary phase transition: a particular trajectory must be singled out in the dynamics. When an interaction turns on, dissipation or the energy flow/cascade is possible, and the c-number random field appears. The separation of these classical statistical fluctuations from the deterministic time evolution is indicated by the secular divergence or the infrared divergence of the system. We describe this phase transition based on the closed-time-path method and derive a quantum Langevin equation with classical noise, which sources the development of a coherent state. Introducing the effective action method to describe the evolution of the coherent state, we describe the order parameter that characterizes the phase transition and the associated spontaneous symmetry breaking. Since this phase transition process is common in physics, we discuss further applications of this formalism in other physical systems.

Keywords: inflation; vacuum fluctuations; phase transition; squeezed state; Langevin equation; coherent state; spontaneous symmetry breaking

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1. Introduction

The physics of dark energy (DE) and dark matter (DM) present a fundamental unsolved problem at present. However, this is not an isolated problem from many other cosmological and physical problems that are also unsolved thus far [1,2]. They are: (a) The nature of the vacuum energy or DE-like object in the early Universe that triggered the cosmic inflation. Both DE and DE-like objects cause exponential expansion of the Universe. Then, what is the relation between them? (b) If DE is the vacuum energy, how is it related to the vacuum fluctuations or the quantum zero-point fluctuations? (c) How does DE or vacuum energy describe the evolution of spacetime? In other words, is the hybrid Einstein equation with the expectation value of the energy–momentum tensor $G_{\mu\nu} = G\langle T_{\mu\nu}\rangle$ meaningful? (d) What is the relation between DE and DM? (e) How does the dark matter-like object acquire spatial structure ? Finally, (f) they are all related to quantum mechanics. Then, how are they related to the wave function of the Universe, not including space-time itself?

This wide range of cosmological and physical problems are often studied separately developing sophisticated ideas, such as supergravity, string theory, modified gravity and extra-dimensions. On the other hand, we would like to develop much fundamental study considering most of the above problems together. Among the above set of problems, the cosmological inflation [3–10] is the central problem that is deeply related to the others.

In particular, the viewpoint from quantum mechanics, phase transition, and spontaneous symmetry breaking (SSB) [11] would be relevant with respect to the above problems [7].

Therefore, we start our study regarding inflation as a phase transition dynamics that is described in full quantum mechanics, except the spacetime itself. In the standard theory of inflation, the potential energy of the slowly-rolling inflaton field drives the inflation, and its quantum fluctuations yield the seeds of density fluctuations.

If we start from this simple view, the following problems should be addressed:

(a) What is the inflaton field $\varphi(x)$? Is this a classical scalar field from the beginning? On the other hand, its inhomogeneous part $\delta\varphi_k$ is treated quantum mechanically to derive the primordial density fluctuations. Consistency is needed for the quantum and classical treatment. We would like to identify this field $\varphi(x)$ as a coherent state, which defines the order parameter of the inflationary phase transition.

(b) How does the field $\varphi(x)$ evolve from $\varphi(x) = 0$ in the case of small-field inflation models? We study this trigger from quantum fluctuations. This is a spontaneous symmetry breaking [12] of U(1) symmetry for the case of the complex scalar field.

(c) What is the fluctuation mode $\delta\varphi_k$? Was it originally a quantum fluctuation field but suddenly becomes a C-number field after crossing the horizon associated with the de Sitter space during inflation?

(d) How this mode $\delta\varphi_k$ evolves from the initial $\delta\varphi_k = 0$ in the fully uniform spacetime? We study this trigger from quantum fluctuations as well. This is SSB of the spatial translational invariance.

(e) What is vacuum energy? How do the vacuum fluctuations appear? We would like to distinguish the vacuum energy and vacuum fluctuations.

The above problems indicate that the inflationary dynamics may be a grand phase transition and SSB process, which should be fundamentally described in quantum theory. In the SSB process, macroscopic C-number degrees of freedom naturally appear and this becomes the order parameter that characterizes the specific branch among many equivalent branches. In order to answer the above questions, this SSB process should be described within full quantum theory.

In order to solve the above problems, decoherence [13,14] is often discussed as the key process. It is often the case that the decoherence is induced by the coarse-graining of the environment [15]. However, this process is not at all spontaneous but is a highly operational process from the observer side. Further, even if the decoherence has been naturally achieved, this is still not enough. What is needed is the sharp indicator of order, the order parameter, which characterizes the phase transition. This order parameter cannot be diffuse but should be coherent.

The best quantum state possessing the property of order parameter would be the coherent state [16,17]. It is the minimum uncertainty state and the vacuum expectation value of the Heisenberg field does not vanish. Therefore, if it develops, the coherent state can act as the order parameter of the phase transition. However, for a coherent state to develop, a linearly coupled C-number field to the system operator is needed. There seems to be no such origin in the ordinary quantum theories for inflation.

In our case, we derive the Langevin equation, a stochastic differential equation, with the C-number random field in it. This C-number field linearly couples to the system field and develops the coherent state. In order to derive the Langevin equation, we need a special form of action. The Langevin equation can be derived from the highly squeezed state which has strong quantum fluctuations. This stochastic equation appears and is consistently represented by the imaginary part of the action in the closed-time-path method as the stochastic kernel.

However, this random field never works in the free theory. This is because the activation of the random field or noise inevitably injects energy into the system, while there is no outflow of energy for the interaction-free system. Therefore, the random field cannot work as noise despite its appearance. We name this inactive random field dry noise. This situation will be explained afterward. If the interaction is introduced, the energy can flow from mode to mode and cascade in the phase space. Thus, in the above, random field dry noise begins to work.

If this flow is steady in the small system segmented as a part of the whole system, then the energy flow-in by fluctuation and the flow-out by dissipation can balance with each other and establish the standard fluctuation–dissipation relation. The energy flow and cascade are impossible in the isolated free systems and probably integrable systems since the energy is confined in a single mode and cannot cascade in the phase space. Summarizing the process for the order parameter to appear, within our model we discuss in this paper, the following stages are needed: (a) instability to make the system squeezed, (b) separation of the stochastic kernel from the dynamics to yield random fluctuation (or dry noise), (c) a coherent state developed by interaction and energy flow and (d) further development of the coherent state.

The construction of this paper is as follows. In Section 2, we develop our basic formalism describing the dynamical phase transition based on quantum theory. In particular, we start to show that the tachyonic instability in an adiabatic ground state develops squeezed state (Section 2.1). We further show in Section 2.2 that the divergence implies the separation of stochastic kernel and random fluctuations from the deterministic evolution. Then, we show in Section 2.3 that the interaction yields the Langevin equation and the development of a coherent state.

In Section 3.4, we show that further instability yields macroscopic order parameter and the completion of the phase transition. This formalism is applied in Section 3 to the inflation and the inflaton field. In particular, we begin to show that the de Sitter instability for massless scalar field develops squeezed state (Section 3.1). We further show in Section 3.2 that the infrared divergence implies the separation of stochastic kernel and dry noise from the deterministic evolution.

We show in Section 3.3 that the self-interaction of the field yields the Langevin equation and the development of a coherent state. Then, in Section 3.4, we show that further instability yields the zero-mode coherent state rolling down the potential, i.e., SSB of U(1), as well as the k-mode coherent state evolution corresponding to the density fluctuations, i.e., SSB of the translational invariance. In Section 4, we summarize our study and describe possible future work. In Appendix A, we itemize some specific discussions, which are closely related to the text.

2. Basic Formalism

In this section, we describe our basic formalism, which will be applied to cosmic inflation in the next section. This is schematically displayed at the end of this paragraph. This formalism starts from an unstable free quantum system that evolves into a squeezed state, in which the zero-point fluctuation grows. This squeezing often happens in cosmology [7,8] and is called particle production [18]. However, this state is a typical quantum state and is quite different from a statistical ensemble of particles [19,20].

For example, in the squeezed state, a particle pair in the mode, say wave number mode $\boldsymbol{k}$ and $-\boldsymbol{k}$, is entangled with each other and never yields any spatial inhomogeneity. Further, although it is possible to derive a Langevin Equation [19] with strong classical fluctuations in the Schwinger–Keldysh [21] or closed time path Green's function method [22–24], this fluctuation never appears to yield any randomness in the system. This is because the inevitable energy input from the fluctuations cannot flow or cascade in the free system.

This classical fluctuation simply represents enhanced zero-point fluctuations in the free state. On the other hand, if the interaction enters and the energy can flow, this c-number fluctuation works as a real source of the noise. Although this field is random, since it is a c-number field, it sources a coherent state.

These coherent state dynamics are best described by the effective action method. If some further instability, such as unstable potential or strong fluctuation exists in the system, then the coherent state further develops macroscopically and provides the order parameter of the phase transition. This order parameter is a single realization among many other possible equivalent values, and its dynamics represent the process of spontaneously symmetry breaking (SSB).

- *Quantum system*
  $\Downarrow \leftarrow$*2-1. tachyonic instability ($m \leq 0$ mode)*
- *Squeezed state*
  $\Downarrow \leftarrow$*2-2. divergence*
- *Separation of the stochastic kernel and Dry noise [SK formalism]*
  $\Downarrow \leftarrow$*2-3. interaction/non-linearity and energy flow*
- *Langevin equation and Coherent state [effective action]*
  $\Downarrow \leftarrow$*2-4. further instability*
- *Order parameter and Phase transition with SSB*

We now explain each process below.

2.1. Squeezed State from Tachyonic Instability

Let us consider a typical unstable system, an inverted harmonic oscillator (IHO). A pictorial example would be the ice-pick set exactly vertical on a flat surface [25]. In classical mechanics, the Newtonian equation is

$$\ddot{x} = (g/l)x, \tag{1}$$

where g is the gravitational acceleration and l is the length of the ice-pick. This can be solved immediately,

$$x(t) = c_1 e^{-\sqrt{g/l}t} + c_2 e^{\sqrt{g/l}t}. \tag{2}$$

Although this solution is unstable toward both future and past, it stays forever if $x(0) = \dot{x}(0) = 0$. On the other hand in quantum mechanics, the uncertainty relation

$$\Delta x \Delta p \geq \hbar/2, \tag{3}$$

which immediately comes from the Schrödinger equation, may indicate that this state is unstable and the ice-pick eventually falls within a time scale $\sqrt{l/g}$.

In general, time-dependent classical source of type $f(t)\hat{q}^2$ yields the squeezed state. We first consider a simple model, which yields the squeezed state, the inverted harmonic oscillator (IHO) [8]. The Hamiltonian is given by

$$\hat{H} = \frac{1}{2m}\hat{p}^2 - \frac{m\omega^2}{2}\hat{x}^2 = i\frac{\omega\hbar}{2}\left(\hat{a}^2 e^{-2i\phi} - h.c.\right), \tag{4}$$

where

$$\begin{array}{rl} \hat{a}^\dagger = & (\frac{m\omega}{2\hbar})^{1/2}(\hat{x} - i\frac{\hat{p}}{m\omega}) \\ \hat{a} = & (\frac{m\omega}{2\hbar})^{1/2}(\hat{x} + i\frac{\hat{p}}{m\omega}) \end{array}, \tag{5}$$

and $\phi = -\pi/4$. Ordinary cross term $\hat{a}^\dagger\hat{a}$ disappears because the second term in the middle of Equation (4) is negative. Then, the wave function at time t becomes

$$\begin{aligned} |\Psi(t)\rangle &= \exp\left[\frac{\omega t}{2}\left(\hat{a}^2 e^{-2i\phi} - h.c.\right)\right]|0\rangle \\ &\equiv S(t)\,|0\rangle \\ &= \exp\left[-\frac{\omega t}{2}\left(\hat{a}^\dagger\right)^2 e^{2i\phi} - \left(\frac{\omega t}{2}\right)^2\right]|0\rangle. \end{aligned} \tag{6}$$

Therefore, the particle pairs are 'condensed' in this state $|\Psi(t)\rangle$. The unitary operator $S(t)$ defines the Bogolubov transformation from the canonical pair $a, a^\dagger$ to the new pair:

$$\begin{cases} b = S^\dagger a S = \hat{a}\cosh\omega t - \hat{a}^\dagger e^{2i\phi}\sinh\omega t, \\ b^\dagger = S^\dagger a^\dagger S = \hat{a}^\dagger\cosh\omega t - \hat{a}e^{-2i\phi}\sinh\omega t, \end{cases} \tag{7}$$

and $SS^\dagger = S^\dagger S = 1$. This state $|\Psi(t)\rangle$ is unlimitedly squeezed in time toward the direction $\phi = -\pi/4$ in phase space as,

$$\langle\Psi(t)|\,(\hat{p}\cos\phi \pm \hat{x}\sin\phi)\,|\Psi(t)\rangle = \begin{cases} 4e^{-2t} \\ 4e^{2t} \end{cases}. \tag{8}$$

This state is often regarded as particle-creating state simply because the number operator average is non-vanishing:

$$N \equiv \langle\Psi(t)|\,a^\dagger a\,|\Psi(t)\rangle = \langle 0|\,b^\dagger b\,|0\rangle = (\sinh\omega t)^2. \tag{9}$$

However, $|\Psi(t)\rangle$ is a genuine quantum mechanical state and the particles, claimed to be created, are clearly not the classical object before any measurement process. In fact, the state is reversible to the original state $|0\rangle$ by the unitary operation $S^\dagger$. This means that this inverted harmonic oscillator, if prepared in the symmetric state initially, never falls toward a particular direction. The state is always in the symmetric neutral position $\langle\Psi(t)|\,\hat{x}\,|\Psi(t)\rangle = 0$ even if the quantum fluctuations develop infinitely.

Similarly in quantum field theory, the Hamiltonian Equation (4) is the infinite collection of the harmonic oscillator labeled by the three momentum $\boldsymbol{k}$. Since the momentum is conserved, the pair in the squeezed state must have exactly the opposite momentum. Thus, the Bogolubov transformation Equation (7) should now be the form,

$$\begin{cases} \hat{b}_{\boldsymbol{k}} = \alpha^*_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} - \beta_{\boldsymbol{k}}\hat{a}^\dagger_{-\boldsymbol{k}}, \\ \hat{b}^\dagger_{-\boldsymbol{k}} = \alpha_{\boldsymbol{k}}\hat{a}^\dagger_{-\boldsymbol{k}} - \beta^*_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} \end{cases}. \tag{10}$$

Therefore, the particle pair of the momentum $k, -k$ is entangled with each other. Many cases of the particle production, Unruh effect, accelerated mirror, Hawking radiation from the black hole [18], etc., correspond to this type of Bogolubov transformation, and the states are generalized squeezed states. These are simply the quantum states represented by the wave functions.

These states are also similar to the entangled spin pair of Silver atoms in the Stern–Gerlach experiments [26]. All the above quantum system, even under the external force, is free and therefore keeps quantum coherence and is even reversible before it is measured by an apparatus or disturbed by the interactions. This squeezing appears everywhere in the Universe when a time dependent C-number force $f(t)$ couples to the system in a quadratic form $f(t)\hat{\phi}^2$.

Incidentally, we summarize another typical quantum state, the coherent state [16], generated when a time dependent external force $\xi(t)$ couples to the system in a linear form $\xi(t)\hat{x}$. This coherent state plays a central role in the subsequent discussions. The coherent state is defined to be the eigenstate of the annihilation operator $\hat{a}$,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{11}$$

which has an explicit form,

$$\begin{aligned} |\alpha\rangle &= e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle \\ &\equiv C(\alpha)|0\rangle \\ &= e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^\dagger}|0\rangle. \end{aligned} \tag{12}$$

The unitary operator $C(\alpha)$ shifts the creation and annihilation operators by C-number

$$\begin{cases} \hat{b} = C^\dagger\hat{a}C = \hat{a} + \alpha, \\ \hat{b}^\dagger = C^\dagger\hat{a}^\dagger C = \hat{a}^\dagger + \alpha^*. \end{cases} \tag{13}$$

The particle number expectation value is finite,

$$N = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2, \tag{14}$$

and the two developed coherent states $\alpha\rangle$ and $\beta\rangle$, with large $|\alpha|, |\beta|$, only have exponentially small superposition

$$|\langle\alpha\,|\,\beta\rangle|^2 = e^{-|\alpha-\beta|^2}. \tag{15}$$

In particular, for the infinite system, the volume factor V enhances this isolation.

$$\propto e^{-V|\alpha-\beta|^2}. \tag{16}$$

If the quantum system $\hat{x}$ has a linear coupling $\xi(t)\hat{x}$ to a c-number source $\xi(t)$ and if the equation of motion is given by

$$\ddot{\hat{x}}(t) = -\gamma\dot{\hat{x}}(t) + \omega^2\hat{x}(t) + \xi(t), \tag{17}$$

in the linear form, the dominant solution is

$$\hat{x}(t) = \frac{1}{2}e^{\frac{1}{2}\left(\sqrt{\gamma^2+4\omega^2}-\gamma\right)t}\hat{x}(0) + \int_0^t dt' \frac{e^{\frac{1}{2}(\sqrt{\gamma^2+4\omega^2}-\gamma)(t-t')}}{\sqrt{\gamma^2+4\omega^2}}\xi(t'). \tag{18}$$

If the second term on RHS further dominates in Equation (18), either due to the accumulation of the c-number force $\xi(t)$ or any instability of the potential, then the variable $\hat{x}(t)$ approaches toward a c-number. We discuss these dynamics later.

2.2. Divergence Indicates the Separation of Fluctuation Kernel

The transient process of the phase transition and the development of the order parameter often accompanies SSB. Since the original symmetry is violated and some particular trajectory is selected among many other equivalent possibilities. In this process, some probabilistic and stochastic processes should be involved. This process is best described by the Langevin equation, a differential equation with a random force term. Therefore, before deriving this Langevin equation from a more basic theory, we describe this Langevin equation first.

This gives us a natural motivation for the closed-time-path formalism [27–29]. Starting from the classical Langevin equation, we derive the effective partition function. The quantization of this system almost deduces the desired formalism in quantum mechanics. The simplest Langevin equation for a particle moving in the environment, with the potential force $-V'$, random force ξ, and the friction γ, is given by

$$\ddot{x}(t) = -\gamma\dot{x}(t) - V'(x(t)) + \xi(t). \tag{19}$$

The statistical average

$$\langle \ldots \rangle_\xi = \int D[\xi] \ldots P[\xi] \tag{20}$$

is defined by the Gaussian weight functional $P[\xi]$,

$$P[\xi] = e^{-\int \xi(t)^2/(2\sigma^2)}. \tag{21}$$

We would like to know the action that drives this Langevin equation. We first construct the partition function of the system by summing all the possible trajectories in the whole phase space.

$$\begin{aligned} Z[J] &\equiv \left\langle \delta[\ddot{x}(t) + \gamma\dot{x}(t) + V'(x(t)) - \xi(t)] \right\rangle_\xi \\ &= \int D[\xi]P[\xi]\delta[\ddot{x}(t) + \gamma\dot{x}(t) + V'(x(t)) - \xi(t)] \exp\left[i\int dtJ(t)x(t)\right], \end{aligned} \tag{22}$$

where the source term $i\int dtJ(t)x(t)$ is inserted for convenience. Further, the integral form of the delta functional is utilized introducing a fictitious variable $x'(t)$,

$$\begin{aligned} Z[J] = & \int D[\xi]D[x']P[\xi]e^{i\int dtx'(t)\{\ddot{x}(t)+\gamma\dot{x}(t)+V'(x(t))-\xi(t)\}}\exp\left[i\int dtJ(t)x(t)\right] \\ = & \int D[\xi]D[x']P[\xi]e^{i\int dt\{-\dot{x}'(t)\dot{x}(t)+\gamma x'(t)\dot{x}(t)+x'(t)V'(x(t))-x'(t)\xi(t)\}+J(t)x(t)} \\ = & \int D[x']e^{i\tilde{S}[x,x']+i\int dtJ(t)x(t)}, \end{aligned} \tag{23}$$

where the boundary term is removed, and the final action $\tilde{S}[x,x']$ becomes

$$\tilde{S}[x,x'] \equiv \int dt\{-\dot{x}'(t)\dot{x}(t)+\gamma x'(t)\dot{x}(t)+x'(t)V'(x(t))+\frac{i}{2}\sigma^2 x'(t)^2+J(t)x(t)\}. \tag{24}$$

Note that the second last term in RHS, which represents classical statistical fluctuations, is purely imaginary. The rest of the terms are real and describe the deterministic dynamics even including the friction term. We now quantize this system by path integration on the variable $x(t)$,

$$Z[J] = \int D[x]D[x']e^{i\tilde{S}[x,x']+i\int dtJ(t)x(t)}. \tag{25}$$

It is possible to reverse the above logic: starting from the quantum action $\tilde{S}[x_+,x_-]$ to obtain the Langevin equation. In order to do so, we use the action in which the imaginary part is removed from the action as a statistical weight $P[\xi]$,

$$Z[J] \quad = \quad \int D[x]D[x']D[\xi]P[\xi]e^{i\tilde{S}_{eff}[x,x']+i\int dtJ(t)x(t)} \tag{26}$$

and this action is real,

$$\tilde{S}_{eff}[x,x'] \equiv \int dt\{-\dot{x}'(t)\dot{x}(t)+\gamma x'(t)\dot{x}(t)+x'(t)V'(x(t))-x'(t)\xi(t)\}. \tag{27}$$

By applying the least action principle,

$$\frac{\delta \tilde{S_{eff}}[x,x']}{\delta x'}|_{x'=0} = 0, \tag{28}$$

we have the quantum Langevin equation with the classical random force $\xi(t)$,

$$\ddot{\hat{x}}(t) = -\gamma\dot{\hat{x}}(t) - V'(\hat{x}(t)) + \xi(t). \tag{29}$$

The separated real action Equation (27) was essential to derive the Langevin equation. If we used the full complex action in the original form Equation (24), the least action principle yields a fully deterministic equation missing random field. This corresponds to the average evolution equation over statistical fluctuations, and this approach never yields SSB. On the other hand, we are interested in the individual evolution equation, which yields SSB.

The above complex action is crucial to obtain the classical stochastic force in the quantum Langevin equation. This is based on the fact that the real part of the action represents the deterministic dynamics of quantum mechanics, including possible friction, while the imaginary of the action represents the classical fluctuations. In fact, the action $e^{iS/\hbar}$ yields the deterministic dynamics as

$$\begin{aligned} &\psi(t+\varepsilon,x) = \int d\gamma\exp\left[\tfrac{i}{\hbar}\tfrac{m\gamma^2}{2\varepsilon}\right]\exp\left[-\tfrac{i}{\hbar}\varepsilon V\left(t+\tfrac{\varepsilon}{2},x+\tfrac{\gamma}{2}\right)\right]\psi(t,x+\gamma) \\ &\Rightarrow \varepsilon\tfrac{\partial\psi(t,x)}{\partial t} = -\tfrac{i}{\hbar}\varepsilon V(t,x)\psi(t,x) - \tfrac{\hbar\varepsilon}{2mi}\tfrac{\partial^2\psi(t,x)}{\partial x^2}, \end{aligned} \tag{30}$$

which is the Schrödinger equation. On the other hand, the action e^{-S} yields stochastic fluctuations as

$$\begin{aligned} &\psi(t+\varepsilon,x) = \int d\gamma \exp\left[\tfrac{m\gamma^2}{2\varepsilon}\right] \exp\left[\varepsilon V\left(t+\tfrac{\varepsilon}{2}, x+\tfrac{\gamma}{2}\right)\right]\psi(t,x+\gamma) \\ &\Rightarrow \varepsilon\tfrac{\partial\psi(t,x)}{\partial t} = \varepsilon V(t,x)\psi(t,x) + \tfrac{\varepsilon}{2m}\tfrac{\partial^2\psi(t,x)}{\partial x^2}, \end{aligned} \tag{31}$$

which is the Fokker–Plank equation. This distinction between quantum and classical is similar to the popular procedure of Wick rotation in appearance: $\beta \equiv (k_B T)^{-1} \leftrightarrow it/\hbar$. However, we are dealing with the intrinsic complex action and not the artificial operation.

If we further rewrite the variables as

$$x_\pm = x \pm \frac{1}{2}x', \tag{32}$$

then the description becomes the Schwinger–Keldysh closed time formalism [21–24], which is widely used in the non-equilibrium quantum field theory. Then, Equation (25) becomes

$$Z[J] = \int D[x_+]D[x_-]e^{i\tilde{S}[x_+,x_-]}, \tag{33}$$

where

$$\tilde{S}[x_+,x_-] = \int dt \left\{ \begin{array}{c} \left((\ddot{x}_+(t))^2 - V(x_+(t))\right) - \left((\ddot{x}_-(t))^2 - V(x_-(t))\right) \\ +\frac{\gamma}{2}(x_+(t)\dot{x}_-(t) - \dot{x}_+(t)x_-(t)) + \frac{i}{2}\sigma^2(x_+(t) - x_-(t))^2 \end{array} \right\}. \tag{34}$$

The first line on RHS of the above action represents the deterministic unitary dynamics for the variables $x_\pm(t)$ separately, and the second line represents dissipation γ and fluctuation $i\sigma^2$ terms where the variables $x_\pm(t)$ mix. This form of action often appears in many literature [27].

It is a natural extension of this expression to introduce a closed time-path C for the domain of the dynamical variables. The path C runs from $-\infty$ to ∞ (+ branch) and then comes back from ∞ to $-\infty$ ($-$ branch). We suppose the supports of the variables $x_\pm(t)$ are, respectively, the $+$ and $-$ branches. We denote the variable $\tilde{x}(t)$ on the contour C unifying the variables $x_\pm(t)$:

$$\tilde{x}(t) = \begin{cases} x_+(t) & t \in (+\text{branch}) \\ x_-(t) & t \in (-\text{branch}). \end{cases} \tag{35}$$

We use a tilde to show that the variable is defined on this closed time domain.

In this theory, the partition function for the system with the free action $S[\tilde{x}]$ is given by

$$\begin{aligned} \tilde{Z}[\tilde{J}] &= \int_C D\tilde{x} \exp[iS[\tilde{x}] + i\int dt\tilde{J}(t)\tilde{x}(t)] \equiv \exp i\tilde{W} \\ &= \int D\tilde{x} \exp[iS[x_+] - iS[x_-] + i\int dt\tilde{J}(t)\tilde{x}(t)]. \end{aligned}$$

This reduces to

$$\tilde{Z}[\tilde{J}] = \exp[-\frac{1}{2}\int dt\tilde{J}(t)\tilde{G}_0(t,t')\tilde{J}(t')]. \tag{36}$$

where the Green's functions are

$$\tilde{G}_0(t,t') = \begin{pmatrix} G_F(t,t') & G_+(t,t') \\ G_-(t,t') & G_{\overline{F}}(t,t') \end{pmatrix} \equiv \begin{pmatrix} \mathrm{Tr}[Tx(t)x(t')\rho] & \mathrm{Tr}[x(t')x(t)\rho] \\ \mathrm{Tr}[x(t)x(t')\rho] & \mathrm{Tr}[\overline{T}x(t)x(t')\rho] \end{pmatrix}, \tag{37}$$

where T and $\overline{T}$ denote, respectively, the time-ordering and reversed time-ordering operators, and ρ represents the initial density matrix. If we change the representation of the matrix by

$$J_\pm(t) = J_c \pm \frac{1}{2}J_\Delta \tag{38}$$

or

$$\tilde{J} = \begin{pmatrix} J_\Delta \\ J_C \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} J_+ \\ J_- \end{pmatrix}; \tag{39}$$

then, we have

$$\tilde{G}_0(t,t') = \begin{pmatrix} 0 & G_R(t,t') \\ G_A(t,t') & G_C(t,t') \end{pmatrix} \equiv \begin{pmatrix} 0 & \theta(t-t')\mathrm{Tr}[x(t'),x(t)]\rho \\ \theta(t'-t)\mathrm{Tr}[x(t'),x(t)]\rho & \mathrm{Tr}\{x(t),x(t')\}\rho \end{pmatrix}. \tag{40}$$

In our case of IHO, the relevant Green's functions are

$$\begin{aligned} G_R(t,t') &= \theta(t-t')\langle[x(t),x(t')]\rangle \\ &= \frac{i}{2}\left(\frac{\omega}{2\hbar}\right)^{-1}\sin(2\phi)\theta(t-t')\sinh(\omega(t-t')) \\ &= \frac{i}{2}\left(\frac{\omega}{2\hbar}\right)^{-1}\theta(t-t')\sinh(\omega(t-t')), \end{aligned} \tag{41}$$

$$\begin{aligned} G_C(t,t') &= \langle\{x(t),x(t')\}\rangle \\ &= \frac{1}{2}\left(\frac{\omega}{2\hbar}\right)^{-1}(\cosh(\omega(t+t')) - \cos(2\phi)\sinh(\omega(t+t'))) \\ &= \frac{1}{2}\left(\frac{\omega}{2\hbar}\right)^{-1}\cosh(\omega(t+t')), \end{aligned} \tag{42}$$

where $\phi = -\pi/4$. The Green's function $G_C(t,t')$ yields the imaginary part of the action.

The above two Green's functions show different time behavior:

$$\begin{aligned} G_R(t,t') &\propto i\omega^{-1}\sinh(\omega(t-t')), \\ G_C(t,t') &\propto \omega^{-1}\cosh(\omega(t+t')). \end{aligned} \tag{43}$$

For a large time $t,t' \to \infty$, $G_C(t,t')$ exponentially diverges, while $G_R(t,t')$ can have milder behavior. Further, for low frequency limit $\omega \to 0$, $G_C(t,t') \propto 1/\omega$, and therefore $G_C(t,t')$ has a stronger singularity than $G_R(t,t')$. If we want to obtain a regular evolution of the system avoiding the divergence in $G_C(t,t')$, we may separate $G_C(t,t')$ from the evolution equation and place it in the form of the weight functional $P[\xi]$ for the random field ξ.

This mathematical separation is always possible; however, the appearance of the divergence may be a strong motivation to do so. It is non-trivial and interesting that this divergent term corresponds to the imaginary part of the action. Therefore, the separated diverging contribution exactly yields the classical stochastic contribution to the evolution equation.

Now, we separate the diverging term from the proper dynamics. Since the diverging term is pure imaginary in the action, it is consistent to place it in the form of the stochastic weight $P(\xi)$ in the path integral form. Comparing Equations (41) and (42), the symmetric term in Equation (36)

$$-\frac{1}{2}\int dt J_\Delta(t)G_C(t,t')J_\Delta(t') \tag{44}$$

is real and positive. Therefore, we can factor out this part as c-number statistical fluctuations introducing an auxiliary classical field $\xi(t)$,

$$\tilde{Z}[\tilde{J}] = \int D\xi P(\xi)\exp[-\frac{1}{2}\int dt\tilde{J}(t)\tilde{G}_0'(t,t')\tilde{J}(t') + i\int dt J_\Delta(t)\xi(t)], \tag{45}$$

where

$$P(\xi) = \exp[-\frac{1}{2}\int dt\xi(t)G_C(t,t')^{-1}\xi(t')], \tag{46}$$

and $\tilde{G}_0'(t,t')$ is thus a separated Green's function. This separation procedure is simply the reverse of the previous section, where Equation (34) yields Equation (21) [30].

The above operation is consistent with the traditional Schwinger–Keldysh formalism giving the relation between the classical fluctuation and the imaginary part of the action. The stochastic/fluctuation kernel in the traditional Schwinger–Keldysh formalism includes the term arising from the zero-point fluctuations as well as the term arising from the thermal environment.

In our case, the latter term does not exist, and the former term is anomalously enhanced by the dynamical instability of the system. As in the above, we identified the classical statistical fluctuation in Equations (45) and (46). However, this classical fluctuation never results as it is in the present free theory including our IHO case. This random field cannot work because it conflicts with energy conservation.

A classical random field inevitably injects energy into the system; however, the energy is conserved, and there is no open channel to allow energy to flow from mode to mode. Therefore, within a free field, this random field never takes effect despite its appearance. Nothing happens in a free state before any interaction or measurement process. We name this failed random field dry noise.

The classical fluctuation (dry noise) in our case is different from the ordinary noise in the Schwinger–Keldysh formalism despite the similar appearance. The former is derived within a pure closed system, while the latter is often derived from any coarse-graining of an environment in an open system. The former represents the vacuum fluctuations, while the latter directly represents the classical statistical fluctuations.

The former is simply a mathematical rephrasing of the complex action, while the latter is the result of the operation of coarse-graining. The former does not allow the energy outflow from the system, while the latter allows the energy outflow from the system to the environment. Therefore, the former does not allow stochasticity, while the latter does.

2.3. Interaction Promotes a Coherent State

The above property of the classical fluctuations will drastically change if we introduce interactions or non-linearities. We now introduce the effective action $\tilde{\Gamma}[\tilde{X}]$, which is adequate to describe the coherent state evolution. This is because the argument $\tilde{X}$ is the vacuum expectation value of the operator $\hat{\tilde{x}}(t)$, $\tilde{X} \equiv \langle \hat{\tilde{x}}(t) \rangle$. In the previous argument, the effective action $\tilde{\Gamma}[\tilde{X}]$ is the Legendre transformation of the partition function $\tilde{Z}[\tilde{J}] = \exp i[\tilde{W}[\tilde{J}]]$,

$$\exp[i\tilde{\Gamma}[\tilde{X}]] = \exp i[\tilde{W}[\tilde{J}] - \int dt \tilde{J}(t)\tilde{X}(t)]. \tag{47}$$

This becomes

$$\begin{aligned}
\exp[i\tilde{\Gamma}[\tilde{X}]] &= \exp i[\tilde{W}[\tilde{J}] - \int dt \tilde{J}(t)\tilde{X}(t)] \\
&= \int_C D\tilde{x} \exp i[\tilde{S}[x] + \int dt \tilde{J}(t)(\tilde{x}(t) - \tilde{X}(t))] \\
&= \int_C D\tilde{x} \exp i[\tilde{S}[\tilde{X} + \tilde{x}] + \int dt \tilde{J}(t)\tilde{X}(t)],
\end{aligned} \tag{48}$$

where we shifted the path-integration variable. Expanding $\tilde{S}[\tilde{X} + \tilde{x}]$ in the series of $\tilde{x}$, we have

$$\tilde{S}[\tilde{X} + \tilde{x}] = \tilde{S}[\tilde{X}] + \tilde{S}'[\tilde{X}]\tilde{x} + \frac{1}{2}\tilde{S}''[\tilde{X}]\tilde{x}^2 + \frac{1}{3!}\tilde{S}'''[\tilde{X}]\tilde{x}^3 + \dots . \tag{49}$$

The first term on RHS represents the classical action for the field $\tilde{X}(t)$, and the remaining terms represent the interaction with the quantum variable.

If the interaction term is quartic, $(\lambda/4!)x(t)^4$, and the background vanishes $\tilde{X} = 0$ initially, then the term $\frac{1}{3!}\tilde{S}'''[\tilde{X}]\tilde{x}^3$ gives the dominant interaction term $\lambda\tilde{X}(t)\tilde{x}(t)^3$. The lowest contribution of this term yields the two-loop quantum effect for $\tilde{X}(t)$

$$\begin{aligned}
&\int dtdt'\lambda^2\tilde{X}(t)\mathrm{Tr}\Big[T_C\rho\tilde{x}(t)^3\tilde{x}(t')^3\Big]\tilde{X}(t') \\
=&\int dtdt'\lambda^2\tilde{X}(t)\mathrm{Tr}\big[T_C\rho\tilde{x}(t)\tilde{x}(t')\big]^3\tilde{X}(t') \\
=&\int dtdt'\lambda^2\tilde{X}(t)G_C(t,t')^3\tilde{X}(t') \\
=&\int dtdt'\lambda^2(X_C,X_\Delta)_t\begin{pmatrix} 0 & G_C^2G_A \\ G_RG_C{}^2 & G_C{}^3 \end{pmatrix}_{t,t'}\begin{pmatrix} X_C \\ X_\Delta \end{pmatrix}_{t'}.
\end{aligned} \tag{50}$$

Therefore, the only imaginary term $X_\Delta(t)G_C(t,t')^3X_\Delta(t)$ contributes to the fluctuation as before and can be separated from the real part of the action by the introduction of an auxiliary field $\xi(t)$. In particular, this separation is useful and is even inevitable when this imaginary part yields a singular contribution to the dynamics and the perturbation theory breaks down.

Then, in the lowest order in perturbation, the effective action becomes

$$\exp[i\Gamma[X]] = \int D\xi P(\xi)\exp[iS_{\mathrm{eff}}[X] + i\int dt\tilde{J}(t)\tilde{X}(t)]], \tag{51}$$

where the real action is given by

$$\begin{aligned}
S_{\mathrm{eff}}[X] = \quad & S[X] + \textstyle\iint dtdt' X_C(t)\Big(1+\lambda^2 G_C(t,t')^2\Big)G_A(t,t')X_\Delta(t') \\
&+ \textstyle\iint dtdt' X_\Delta(t)G_R(t,t')\Big(1+\lambda^2 G_C(t,t')^2\Big)X_C(t') + \int dt\xi(t)X_\Delta(t)],
\end{aligned} \tag{52}$$

and the fluctuation weight is given by

$$P(\xi) = \exp[-\frac{\lambda^2}{2}\iint dtdt'\xi(t)G_C(t,t')^{-3}\xi(t')]. \tag{53}$$

This weight and the corresponding random field dominate at the initial stage $\tilde{X}\approx 0$, while the other types of random fields gradually dominate when $\tilde{X}$ develops.

Note that the advanced term $\iint dtdt' X_C(t)\Big(1+\lambda^2 G_C(t,t')^2\Big)G_A(t,t')X_\Delta(t')$ becomes exactly the same as the retarded term $\iint dtdt' X_\Delta(t)G_R(t,t')\Big(1+\lambda^2 G_C(t,t')^2\Big)X_C(t')$ if we exchange the time variables $t\leftrightarrow t'$.

Now, the application of the least action principle for $\Gamma_{\mathrm{eff}}[X]$

$$\frac{\delta\Gamma_{\mathrm{eff}}[X]}{X_\Delta(t)}|_{X_\Delta=0} = 0, \tag{54}$$

yields the classical Langevin equation as

$$\ddot{X}_C(t) - \omega^2 X_C(t) + \frac{\lambda}{3!}X_C(t)^3 + 2\int dt' G_R(t,t')\Big(1+\lambda^2 G_C(t,t')^2\Big)X_C(t') + \xi(t) = 0. \tag{55}$$

This equation can describe the (c-number part of) coherent state evolution $\tilde{X}$, triggered by the classical noise $\xi(t)$ [30,31]. This equation describes a rapid evolution of $X_C(t)$ under (a) the strong fluctuation from $\xi(t)$ described by Equations (53) and (42) and (b) the original classical instability $-\omega^2 X_C(t)$ that promotes the exponential development of the system. The retarded term proportional to $G_R(t,t')$ sometimes shows the non-Markovian dissipative effects.

This Langevin equation (55) reflects the evolution of the coherent state sourced by the c-number random field $\xi(t)$ in Equation (29); the vacuum expectation of Equation (29) roughly corresponds to the present Equation (55). The above Equation (55) can describe the SSB and the phase transition dynamics. In fact, the classical variable $X_C(t)$ can develop exactly from zero (symmetric state) to some finite value (asymmetric state). Thus, $X_C(t)$ becomes the order parameter of the phase transition. After the development of $X_C(t)$, other interaction terms in Equation (49) gradually contribute to the dynamics. New types of noise may become active. Individual imaginary terms in the action yield each stochastic noise:

$$\exp[-\iint dtdt' X_\Delta X_C^2 G_C X_C^2 X_\Delta] = \int D\xi_1 \exp[-\iint dtdt' \xi_1 G_C^{-1} \xi_1 + i\int dt \xi_1 X_C^2 X_\Delta], \quad (56)$$

and

$$\exp[-\iint dtdt' X_\Delta X_C G_C^2 X_C X_\Delta] = \int D\xi_2 \exp[-\iint dtdt' \xi_2 G_C^{-2} \xi_2 + i\int dt \xi_2 X_C X_\Delta]. \quad (57)$$

However, it is formally possible to represent all the imaginary terms by a single noise ξ, if we allow the variable dependent kernels:

$$\begin{aligned} &\exp[-\lambda^2 \textstyle\iint dtdt' X_\Delta (X_C^2 G_C X_C^2 + X_C G_C^2 X_C + G_C^3) X_\Delta] \\ &= \textstyle\int D\xi \exp[-\iint dtdt' \xi [\lambda^2 (X_C^2 G_C X_C^2 + X_C G_C^2 X_C + G_C^3)]^{-1} \xi + i \int dt \xi X_\Delta] \end{aligned} \quad (58)$$

Since the fluctuation kernel is unique, the random field $\xi(t)$ is also unique at this stage. However, many higher order contributions to the action may yield a variety of random noises.

2.4. Development of Phase Transition

The whole phase transition proceeds along with the order parameter development described by Equation (55), in which we are considering the potential of the wine-bottle shape. The variable $X_C(t)$ can be a complex scalar field. Starting from the symmetric state $X_C(t) = 0$, random fields $\xi(t)$ act to develop $X_C(t)$ provided $|X_C(t)| < \omega/\sqrt{2\lambda}$.

However, further for $|X_C(t)| > \omega/\sqrt{2\lambda}$, the effective mass becomes positive, and the random field disappears. Then, $X_C(t)$ evolves deterministically along the potential until it reaches the stable position $|X_C(t)| = \omega/\sqrt{2\lambda/3}$. This rough evolution process would be modified by the rest of the terms in Equation (55), and the possible thermal noise enhances the fluctuation to promote the phase transition. This completes the SSB process.

However, the full description of the transition dynamics requires further study regarding the details of the following issues: (a) Exponentially time-dependent noise terms should be properly evaluated for numerical calculations. (b) A strongly non-Markovian dissipative term that makes the order parameter settle at the true stable vacuum should be evaluated. (c) Our description is now limited to the transient process on an unstable vacuum, while the ordinary quantum field theory always remains in the stability of the vacuum. We need to develop any necessary perturbation technique to be viable in the unstable case. These issues will be clarified, to some extent, in our next report, which includes the systematic numerical calculations of the stochastic evolution of the order parameter.

On the other hand, a standard method to describe SSB needs an infinitesimal explicit violation of the symmetry with the delicate order of the two limiting operations. For example, in the case of ferromagnetic materials, the order parameter magnetization is given by

$$M_\pm \equiv \lim_{B\to 0\pm} \lim_{V\to\infty} m_V(B), \quad (59)$$

where $m_V(B)$ is the local average of the magnetization [32], and B, V are, respectively, the magnetic field and the volume.

In the case of Bose–Einstein condensation (BEC), the argument starts from the assumption that the boson field can be separated as [33]

$$\hat{\phi} = \varphi + \delta\hat{\phi}, \quad (60)$$

where the classical parameter φ represents the 0-momentum condensation. This separation would correspond to Equation (49) in our case. Further, formally introducing the creation and annihilation operators $\hat{a}_0^\dagger, \hat{a}_0$ for φ, the condensation is characterized by

$$\langle \hat{a}_0^* \hat{a}_0 \rangle / V > 0, \tag{61}$$

as well as the condition [32]

$$|\langle \hat{a}_0 \rangle|^2 / V > 0. \tag{62}$$

This condition guarantees the existence of the off-diagonal long-range order or the fact that the BEC as a phase transition accompanying the spontaneous symmetry breaking. In BEC, the Gross–Pitaevskii equation is generally given as Equation (55) with random force. Our formalism extends these traditional methods, including the spatial dependence of the order parameter.

3. Inflation and Inflaton Fields

Inflation theory forms the firm base of modern cosmology. Inflation provides us with the basic mechanism for creating seeds of the large-scale structure in the Universe from almost nothing. Therefore, hundreds of inflationary models have been proposed thus far to best fit the observational data. However, plenty of problems remain, and the fundamental mechanism has not yet been established.

Applying the basic formalism, presented in the previous section, we attempt to fill the ditch toward a more basic formalism of inflation. Each subsection here corresponds to the individual subsection in the previous section.

3.1. Tachyonic Instability

During the inflation, the background de Sitter spacetime is characterized by the metric as

$$ds^2 = dt^2 - a(t)^2 dx^2 = a(t)^2 \left(d\eta^2 - dx^2 \right) \tag{63}$$

where the scale factor $a(t)$ and the time η are given by

$$a(t) = e^{Ht} = -(H\eta)^{-1} \tag{64}$$

The action of a scalar field $u(x)$, with the potential $V(u)$, is

$$S[u] = \int \sqrt{-g} d^4x \left(\partial_\mu u \partial^\mu u - V(u) \right), \tag{65}$$

and the standard coordinate transformation

$$\phi = zR, \; z = a\frac{\dot{u}}{H} \tag{66}$$

yields, for the relevant case of the massless minimal-coupling field,

$$S[\phi] = \frac{1}{2} \int d\eta d^3x \left(\phi'^2 - (\nabla\phi)^2 + \frac{z''}{z}\phi^2 \right), \tag{67}$$

where ϕ is the gauge-invariant scalar perturbation [9], and the prime is the derivative with respect to η. In this action, the effective mass becomes

$$m_{eff}^2(\eta) = \frac{z''}{z} = -\frac{H}{a\dot{u}} \partial_\eta^2 \left(\frac{a\dot{u}}{H} \right) < 0. \tag{68}$$

This negative time-dependent mass yields the squeezed state as before in the case of IHO.

Expanding the field as

$$\hat{\phi}(\eta, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left(\hat{a}_{\boldsymbol{k}} v_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\boldsymbol{x}} + h.c. \right), \tag{69}$$

we have the equation of motion,

$$v''_{\boldsymbol{k}}(\eta) + \omega_{\boldsymbol{k}}(\eta)^2 v_{\boldsymbol{k}}(\eta) = 0 \tag{70}$$

where $\omega_{\boldsymbol{k}}^2 = \boldsymbol{k}^2 - \frac{z''}{z} \approx \boldsymbol{k}^2 - \frac{2}{\eta^2}$, and

$$v_{\boldsymbol{k}}(\eta) = c_1 \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta} \right) + c_2 \frac{e^{ik\eta}}{\sqrt{2k}} \left(1 + \frac{i}{k\eta} \right). \tag{71}$$

The solution $v_{\boldsymbol{k}}(\eta)$ is often chosen so that it approaches the Minkowski vacuum in the past, $c_1 = 0$. The factor $-\frac{i}{k\eta} = i\frac{H}{k}e^{Ht}$yields a strong squeezed state, as already argued in many places [8].

3.2. Infrared Divergence

Let us consider the quantum field theory generalizing the previous section. The generating functional of many point functions is defined as

$$\begin{aligned} \tilde{Z}[\tilde{J}] \quad &\equiv \quad \mathrm{Tr}\left[\tilde{T}\left[\exp[i \int \tilde{J}\tilde{\phi}]\rho \right] \right] \\ &\equiv \quad \exp[i\tilde{W}[\tilde{J}]], \end{aligned} \tag{72}$$

where the tildes indicate that the associated quantities are defined on the closed time-contour: from $-\infty$ to $+\infty$ and then back to $-\infty$ again. $\tilde{T}$ is the time ordering operation on this contour, $\tilde{J}$ is an external source, and ρ is the initial density matrix for the field $\tilde{\phi}$.

The trace operation is over the functions on the closed time-contour. In the two by two matrix representation, $\tilde{\phi}(x) = (\phi_+(x), \phi_-(x))$, $\tilde{J}[x] = (J_+(x), J_-(x))$, and $\int \tilde{J}\tilde{\phi} = \int dx J_+(x)\phi_+(x) - \int dx J_-(x)\phi_-(x)$. Note the extra minus sign in the above comes from the reversed time contour part that has a negative measure. A pair of variables $\phi_\Delta \equiv \phi_+(x) - \phi_-(x)$ and $\phi_C \equiv (\phi_+(x) + \phi_-(x))/2$ are also often used. In the interaction picture: $\mathcal{L}[\phi] = \mathcal{L}_0[\phi] - V[\phi]$, we have

$$\tilde{Z}[\tilde{J}] = \exp\left[-i \int V\left[\frac{1}{i}\frac{\delta}{\delta \tilde{J}} \right] \right] \exp[-\frac{i}{2} \int \int \tilde{J}(x)\tilde{G}_0(x,y)\tilde{J}(y)]\mathrm{Tr}(: \exp(i \int \tilde{J}\tilde{\phi}) : \rho), \tag{73}$$

where ϕ is in the interaction picture, $\tilde{G}_0$ is a free propagator, and : . . . : represents the normal ordering [22–24]. We can develop perturbative calculations based on this expression. The C-number field $\tilde{\varphi}(x)$ is defined by

$$\tilde{\varphi}(x) \equiv \frac{\delta \tilde{W}}{\delta \tilde{J}(x)}. \tag{74}$$

Then, the effective action $\tilde{\Gamma}$ is defined as the Legendre transformation of $\tilde{W}$:

$$\tilde{\Gamma}[\tilde{\varphi}] \equiv \tilde{W}[\tilde{J}] - \int \tilde{J}\tilde{\varphi}. \tag{75}$$

The propagator part in the above $\tilde{J}(x)\tilde{G}_0(x,y)\tilde{J}(y)$ becomes

$$J_\Delta(x)G_R(x,y)J_C(y) + J_C(x)G_A(x,y)J_\Delta(y) - iJ_\Delta(x)G_C(x,y)J_\Delta(y) \tag{76}$$

where

$$
\begin{aligned}
G_R(x,y) &= i\theta\left(x^0 - y^0\right)\langle[\phi(x),\, \phi(y)]\rangle, &(77)\\
G_A(x,y) &= -i\theta\left(y^0 - x^0\right)\langle[\phi(x),\, \phi(y)]\rangle, &(78)\\
G_C(x,y) &= \langle\{\phi(x),\, \phi(y)\}\rangle.
\end{aligned}
$$

The last term $G_C(x,y)$ in Equation (76) contributes as imaginary part of the action. It comes from the symmetric part of the propagator, while the rest is from the anti-symmetric part of the propagator.

The Green's functions have the following form, in our case of massless minimal scalar in de Sitter spacetime,

$$
\begin{aligned}
G_C(\overrightarrow{k}) &= \frac{H^2}{k^3}\Big((1+k^2\eta\eta')\cos(k\Delta\eta) + k\Delta\eta\sin(k\Delta\eta)\Big) \propto \frac{H^2}{k^3}, &(79)\\
G_R(\overrightarrow{k}) &= -i\,\frac{H^2}{k^3}\Big(-k\Delta\eta\cos(k\Delta\eta) + (1+k^2\eta\eta')\sin(k\Delta\eta)\Big) \propto \frac{iH^2\Delta\eta}{k^2}. &(80)
\end{aligned}
$$

It is apparent that $G_C(x,y)$ is infrared (IR) divergent, while the others are finite. This IR divergence yields further divergence in higher loop corrections and ruins the perturbative calculations. We accept this divergence and adopt it to use positively to motivate the separation of the fluctuation kernel from the proper finite dynamics. This separated kernel yields the stochastic weight for the c-number random force, which potentially triggers the coherent state and eventually yields SSB.

However, within the free theory even in the curved spacetime, no energy flow can be allowed in the phase space. Therefore, this random force, which inevitably injects energy into the system, conflicts the energy conservation, and it cannot be active in the dynamics. The property of this classical noise is the same as in the previous section. The interaction would allow the energy flow/cascade from one mode to the others and makes this dry noise active to trigger the coherent state.

3.3. Interaction and Energy Flow

The above property of the classical fluctuations drastically changes if we introduce interactions or non-linearities in the same way as the previous section. We now introduce the effective action $\widetilde{\Gamma}[\tilde{\varphi}]$, which is adequate to describe the coherent state evolution,

$$
\begin{aligned}
\exp[i\widetilde{\Gamma}[\tilde{\varphi}]] &= \exp i[\widetilde{W}[\widetilde{J}] - \int d^4x \widetilde{J}(x)\widetilde{\varphi}(x)] &(81)\\
&= \int \mathcal{D}\widetilde{\phi} \exp i[\widetilde{S}[\widetilde{\phi}] + \int d^4x \widetilde{J}(x)(\widetilde{\phi}(x) - \widetilde{\varphi}(x))],\\
&= \int \mathcal{D}\widetilde{\phi} \exp i[\widetilde{S}[\widetilde{\varphi} + \widetilde{\phi}] + \int d^4x \widetilde{J}(x)\widetilde{\phi}(x)],
\end{aligned}
$$

where the integration field is shifted by φ. Then, expanding the action around φ, we further have

$$
\exp[i\widetilde{\Gamma}[\tilde{\varphi}]] = \int \mathcal{D}\widetilde{\phi} \exp i[\widetilde{S}_{int}[\widetilde{\phi};\widetilde{\varphi}] + \frac{1}{2}\int d^4x \widetilde{\phi}(x) G_0^{-1}(x,y)\widetilde{\phi}(y) - \int d^4x \widetilde{J}(x)\widetilde{\varphi}(x)], \quad (82)
$$

where $\widetilde{S}_{int}[\phi;\varphi]$ is the Taylor expansion of ϕ around φ.

The first-order term does not vanish because we do not assume that the field φ solves the free equation of motion from $\widetilde{S}_0$ as in the ordinary stationary approach. The second-order term is absorbed into the propagator $G_0(x-y)$. The third-order or higher terms are genuine interactions that yield the one-particle irreducible graphs as usual. The first term of the interaction yields the factor in the effective action

$$\exp[i\widetilde{S}_0[\tilde{\varphi}]]\int\mathcal{D}\widetilde{\phi}\exp i[\lambda\varphi^3\phi]+\frac{1}{2}\int d^4x\phi(x)G_0^{-1}(x,y)\phi(y)] \tag{83}$$
$$= \exp[i\widetilde{S}_0[\tilde{\varphi}]]\int\mathcal{D}\widetilde{\phi}\exp i[(\lambda\varphi(x)^3)_\Delta G_R(x,y)(\lambda\varphi(y)^3)_C+(\lambda\varphi(x)^3)_C G_A(x,y)(\lambda\varphi(y)^3)_\Delta$$
$$+i(\lambda\varphi(x)^3)_\Delta G_C(x,y)(\lambda\varphi(y)^3)_\Delta]$$

where $G_R(x,y)$, $G_C(x,y)$ have the form of Equation (80), and only $G_C(x,y)$ has IR divergence as well as pure imaginary part of the action. This IR-divergent term $G_C(x,y)$ becomes Gaussian and can be separated as in the previous way to yield statistical fluctuations. The finite terms $G_R(x,y)$, $G_A(x,y)$ may yield dissipative and non-Markovian contributions. However, the macroscopic friction term $-3H\dot{\varphi}$ directly associated with the cosmic expansion in the equation of motion dominates this friction.

Thus, the full effective action is found to be

$$\exp[i\widetilde{\Gamma}[\tilde{\varphi}]]=\int\mathcal{D}\xi P(\xi)\exp[i\Gamma[\tilde{\varphi;\xi}]], \tag{84}$$

where

$$\exp[i\Gamma[\tilde{\varphi;\xi}]] = \exp[i\widetilde{S}_0[\tilde{\varphi}]]\exp i[S'_{int}[\frac{\delta}{i\delta\widetilde{J}};\varphi]]\exp i[\frac{1}{2}\int d^4x\widetilde{J}(x)G'_0(x,y)\widetilde{J}(y) \tag{85}$$
$$+\int d^4x\xi(x)(x)(\lambda\varphi(x)^3)_\Delta].$$

where $S'_{int}$is the interaction term with the linear term removed, and $G'_0(x,y)$ is the propagator with the IR divergence separated.

This allows the ordinary perturbation calculations to be IR safe for higher-order quantum corrections. The IR-diverging term is fully separated in the fluctuation kernel $P(\xi)$, and, for example, one loop logarithmic divergence can be avoided. On the other hand, C-number statistical fluctuations represented by ξ act on the local quantum dynamics intermittently. However, the effect is mostly limited in the long-range IR region.

We can obtain the equation of motion for the order parameter $\tilde{\varphi}$ as

$$\frac{\delta\tilde{\Gamma}}{\delta\tilde{\varphi}(x)}=-\tilde{J}(x). \tag{86}$$

This becomes, in the lowest order of ξ_k and in the strong damping regime or in the slow-rolling stage,

$$3H\dot{\varphi}_k+(\lambda/2)\varphi_0^2\varphi_k=(\lambda/2)\varphi_0^2\xi_k. \tag{87}$$

This equation yields the same power spectrum for φ_k, as in the standard theory [9] but with a different amplitude,

$$\langle\varphi_k\varphi_k\rangle_\xi\approx\lambda^2\varphi_0^4\frac{H^2}{k^3}. \tag{88}$$

Thus, the inflaton fluctuation power spectrum becomes, at the horizon crossing time,

$$P_{\delta\phi}=\frac{4\pi k^3}{(2\pi)^3}|\phi_k|^2\underset{k/(aH)=1}{\rightarrow}\lambda^2(\Delta t\,\varphi_0)^4\left(\frac{H}{2\pi}\right)^2, \tag{89}$$

where $\Delta t\approx\phi_0/\dot{\phi}_0$ represents the time scale of the change of the field φ_0. The factor $\lambda^2(\Delta t\,\varphi_0)^4$ modifies the amplitude of the fluctuation. Thus, the density fluctuations shows the Zeldovich scale-free spectrum; however, the amplitude depends on the model of inflation.

3.4. Instability: Inflation and Inflaton Fields

As in the previous section, the above coherent state would evolve to become macroscopic if the potential is unstable. This is the case of the inflationary model in which the inflaton field evolves from zero to larger values, i.e., from the symmetric phase toward SSB phase. The evolution is governed by a similar equation to Equation (55) with the general random force Equation (58),

$$\Box\varphi_0(x) = -V'(\varphi_0(x)) + \int_{-\infty}^{t_x} dy G_{\rm ret}(x,y)\varphi_0(y) + \frac{1}{2}\lambda\varphi_0^2\xi_1(x) + \lambda\xi_2(x) + \dots. \quad (90)$$

However, the inflaton field is somewhat different from a simple potential: The inflaton potential is always supposed to guarantee the slow-rolling phase. In this phase, the mass of the field almost vanishes and the state is still a squeezed state. Therefore, the random field is still active. This is good for the generation of density perturbations at finite modes $\boldsymbol{k}$; however, the (almost) zero wave number inflaton field may be affected by the random force since the random field spectrum is IR divergent and since the long-range random field is infinitely strong. This may alter the standard picture of slow-rolling. The inflaton field is not only steadily rolling but also strongly fluctuating. Then, the ordinary model parameters, $\epsilon \equiv \frac{1}{2}(V'/V)^2, \eta_V \equiv V''/V$, may not be a good indicator to connect to observations. The details require further study beyond the scope of this paper.

4. Summary and Prospects

After proposing the basic formalism to describe a phase transition and spontaneous symmetry breaking (SSB) process in quantum mechanics, we applied it to the cosmic inflation process to obtain the systematic dynamics. The formalism is general. (a) Quantum system develops into a strong squeezed state by a tachyonic instability of the adiabatic ground state. (b) Possible divergence motivates the separation of the imaginary part of the action, which is enhanced by the squeezing.

The Schwinger–Keldysh formalism is used to obtain classical fluctuations in a free system. (c) The interaction allows the energy flow and makes this classical fluctuation active in the Langevin equation, which promotes the coherent state. The effective action method systematically describes these dynamics. (d) Instability and strong classical fluctuations further promote the development of the order parameter until the phase transition and SSB are completed.

We emphasize several crucial points in our study compared with the standard arguments.

1. The appearance of the squeezed state (adiabatic ground state) is essential in our formalism. A highly squeezed state is often possible in the inverted harmonic oscillator (IHO) or the de Sitter spacetime during inflation. These states are genuine quantum states and have nothing to do with classical statistical states [20]. However, this squeezing is often interpreted as particle production and structure formation. If one allows the coarse-graining operation, then the state may tern into the classical statistical ensemble. However, this artificial operation does not appear in nature; it is a special projection from the observer's side.
2. Energy flow or cascade in the phase space is essential for the appearance of active classical fluctuations. If the interaction exists, the energy can flow in the phase space in general. However, a detailed analysis is needed based on more rigorous discussions. For example, the concept of the resonance singularity [34,35] may help our discussion.
3. The effective action method that we utilized is not a coarse-graining method. This method is useful to describe the dynamics of a coherent state. On the other hand, coarse-graining is useful if a specific measurement process is fixed, specifically in laboratory experiments.
4. Decoherence is often discussed in the literature for deriving classical statistical properties in quantum systems [36]. However, for the dynamics of a phase transition, a single state must be autonomously selected, as an order parameter, among many other

candidates. This order parameter state is needed for describing the SSB process and the structure formation process. Thus, decoherence is not enough.

This state selection is also needed in the process of quantum measurement. In this measurement process, a single state is probabilistically selected among many other possible states [37]. This is briefly discussed in Section A.4.

5. The infrared (IR) divergence of inflaton field in de Sitter spacetime is often discussed [38–41]: How to remove this harmful divergence for the proper perturbation calculations. IR divergence is not special and appears everywhere in field theories, including QED and QCD. The standard approach would be the cancellation of the divergence or an appropriate choice of the asymptotic states. If these prescriptions work, perturbation and the deterministic dynamics are guaranteed. On the other hand in our formalism, we separate this IR divergence as the statistical weight. This is possible since all the IR divergence only appears in the imaginary part of the Schwinger–Keldysh action.

   Then, the IR divergence is separated from the deterministic dynamics. The IR divergence is simply transferred to the singular C-number fluctuations. This type of classical fluctuation or noise is quite common and appears as $1/f$ noise or pink noise in various fields of physics. The merit of our prescription is, on top of the separation of IR divergence, to derive the C-number order parameter, which triggers SSB and phase transitions. Further, we clarified one possible origin of IR singular noise.

We summarize several issues that we could not fully address in this paper. However, they are crucial for the further development of our formalism and useful applications.

1. We separated the IR divergence at the propagator level. Then, a simple one loop IR logarithmic divergence could be removed. However, there may appear further IR divergence in higher loops, and the problem of the mixture with the UV divergence may arise [39]. We need to analyze the IR divergence in all orders of perturbations.
2. We treated the energy flow in the phase space due to the interaction intuitively in this paper. However, the problem of energy flow and cascade would be a deeper problem related to quantum chaos, the eigenstate thermalization hypothesis and resonance singularity [34,35].
3. The nature and the prescription of the IR divergence in quantum electrodynamics (QED) are well-established subjects [42]. However, it would be interesting to describe the IR divergence in QED from the present point of view. The coherent state bases, which remedied the IR divergence in QED, may be dynamically created according to our formalism.
4. We did not directly study the DE and DM problems in this paper. In this dark sector, DE and DM may be the same field but in a different phase. For example, an ordinary gas phase of some complex scalar field may behave as DM, while the Bose–Einstein condensation of the same field may form DE [43]. Then, the same set of dark sectors may be the inflaton field in the early Universe. These two dark sectors, at the inflation and the present, maybe continuously connected [44].

The author is planning to study these problems and hopes to report on them soon.

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Appendix A

Appendix A.1. Comparison with the Stochastic Method

The standard method to derive the primordial density fluctuations is the stochastic method [45]. In this formalism, a separation of the inflaton field mode $\phi = \phi_< + \phi_>$ is introduced at around the scale of the de Sitter horizon: $\phi_> \equiv \int dk\theta(k-H)\phi$. Thus, the two modes appear to have a bi-linear interaction of the form $\dot{\phi}_>(x)\dot{\phi}_<(x)$. Then, the effective dynamics for the large scale mode $\phi_<$ is given by integrating out the mode $\phi_>$,

$$\begin{aligned}\widetilde{Z}[\widetilde{J}] &= \int \mathcal{D}\widetilde{\phi}_< \mathcal{D}\widetilde{\phi}_> \exp[i\widetilde{S}[\widetilde{\phi}] + i\int d^4x \widetilde{J}(x)\widetilde{\phi}(x)], \\ &= \int \mathcal{D}\widetilde{\phi}_< \exp[i\int d^4x \widetilde{\phi}_<(x) G_0(x,y)\widetilde{\phi}_<(y) + i\int d^4x \widetilde{J}(x)\widetilde{\phi}(x)], \\ &= \int \mathcal{D}\widetilde{\phi}_< \exp[-\frac{1}{4}\int d^3k \phi_{<\Delta}(\overrightarrow{k}) G_C(\overrightarrow{k})\phi_{<\Delta}(\overrightarrow{k}) + \\ &\quad + \int d^3k \phi_{<\Delta}(\overrightarrow{k})\theta(\Delta\eta) G_R(\overrightarrow{k})\phi_{<C}(\overrightarrow{k}) + i\int d^3k J(\overrightarrow{k})\phi(\overrightarrow{k})],\end{aligned} \tag{A1}$$

where

$$G_0 = -\int \frac{d^3k}{(2\pi)^3}\frac{H^2}{2k^3} e^{-ik(\eta-\eta')+i\overrightarrow{k}\cdot(\overrightarrow{x}-\overrightarrow{x'})}(i-k\eta)(i+k\eta'). \tag{A2}$$

Then, in the last equation, the statistical and deterministic parts are separated as

$$\begin{aligned}Z[J] &= \int \mathcal{D}\xi P(\xi)\int \mathcal{D}\phi_< \exp[\int d^3k \phi_{<\Delta}(\overrightarrow{k})\theta(\Delta\eta) G_R(\overrightarrow{k})\phi_{<C}(\overrightarrow{k}) \\ &\quad + i\int d^3k J(\overrightarrow{k})\phi_<(\overrightarrow{k}) + i\int d^3k \xi(\overrightarrow{k})\phi_<(\overrightarrow{k})],\end{aligned} \tag{A3}$$

where the statistical weight becomes the Gaussian form,

$$P(\xi) = \exp[-\frac{1}{4}\int d^3k \xi(\overrightarrow{k}) G_C(\overrightarrow{k})^{-1}\xi(\overrightarrow{k})]. \tag{A4}$$

The Langevin equation is derived by the least action principle for $\phi_<(\overrightarrow{k})$ and becomes, in the slow-rolling limit,

$$3H\frac{d\phi(\overrightarrow{k})}{d\eta} = \xi. \tag{A5}$$

The two-point correlation function of $\phi_<(\overrightarrow{k})$ becomes

$$\left\langle \phi_<(\overrightarrow{k})\phi_<(\overrightarrow{k})\right\rangle_\xi \approx \frac{H^2}{k^3} \tag{A6}$$

at the horizon crossing $\eta = -k^{-1}$. This is claimed to legitimate the derivation of the standard correlation function [9,46].

However, artificial separation of a free field $\phi = \phi_< + \phi_>$ at the horizon does not resolve the quantum-classical transition problem at the fundamental level. However, if we adopt such separation, Equation (A5) shows the quantum field $\phi_<$ evolution sourced by the classical random field ξ. Therefore, the coherent state directly evolves as in our case.

A seminal work of Caldeira-Legget [27] describes a similar theory. A bi-linear interaction between the system and the environment is set in their model as in the stochastic method. They coarse-grain the environment and derive the master equation for the system density matrix. The environment makes the energy flow possible, and the random force is always active, including the tiny zero-point fluctuations of the environment.

Appendix A.2. Case of Massive or Curvature Coupling

Although we only considered the massless inflaton field, massive and curvature coupling model can also be considered. Setting the parameter $\nu = (\frac{9}{4} - \frac{m^2\phi^2 + \kappa R\phi}{H^2})^{1/2}$, $(0 \leqq \nu \leqq 3/2)$, we have

$$\begin{aligned} G_C\left(\vec{k}\right) &= J_\nu(k\eta)J_\nu(k\eta') + Y_\nu(k\eta)Y_\nu(k\eta') \propto H^{2\nu}k^{-2\nu} \\ G_R\left(\vec{k}\right) &= i(J_\nu(k\eta)Y_\nu(k\eta') - J_\nu(k\eta')Y_\nu(k\eta)) \propto k^0. \end{aligned} \tag{A7}$$

where J_ν, Y_ν are the Hankel functions: $H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$ and $H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$. Thus, the infrared divergence, if any, is concentrated in the $G_C(\vec{k})$ term and $G_R(\vec{k})$ is always finite. In particular, negative κ, within positive curvature R, may yield strong IR divergence even for massive models.

Appendix A.3. How to Choose the Vacuum

In most of the cases, a Banch–Davies vacuum is chosen, with the mode $v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}}H_\nu^{(2)}(k|\eta|)$, in the standard cosmology since it smoothly connects to the initial empty vacuum. In fact, the vacuum choice is arbitrary. The general vacuum, with the mode

$$u_k^*(\eta) = \alpha_k v_k^*(\eta) + \beta_k v_k(\eta), (|\alpha|^2 - |\beta|^2 = 1), \tag{A8}$$

yields the correlation function

$$\langle 0|\varphi(x_1, \eta_1)\varphi(x_2, \eta_2)|0\rangle = \int d^3k H^2 |\eta_1\eta_2|^{3/2} e^{ik(x_1 - x_2)} u_k^*(\eta_1) u_k^*(\eta_2), \tag{A9}$$

and the IR diverging contribution is proportional to the $Y_\nu Y_\nu$ term. Then, the power spectrum of the fluctuation results as the same as the standard form; however, the amplitude is multiplied by $|\alpha - \beta|^2$. Therefore, IR divergent property is not changed for any choice of vacuum, and it never disappears: $\alpha \neq \beta$.

Appendix A.4. Transient Dynamics of the Quantum Measurement

The appearance of the C-number order parameter in our formalism is related to the quantum measurement process in which a particular state is probabilistically selected among multiple possibilities by a measurement. The situation is similar to the SSB process, although there must be a back-reaction to the quantum system from the emerged order parameter. This back-reaction guarantees the firm correlation between the quantum state of the system and the value of the order parameter.

A prototype of the quantum measurement model was analyzed in this line of thought introducing the external thermal bath in [12]. This model describes the transient dynamics of the detector field $\hat{\phi}$, which measures the spin $\hat{\mathbf{S}}$ in the fixed magnetic field $\boldsymbol{B}$. The Lagrangian is given by

$$L = \frac{1}{2}(\nabla\hat{\phi})^2 - \frac{1}{2}m^2\hat{\phi}^2 - \frac{1}{4!}\lambda\hat{\phi}^4 + \mu\hat{\phi}\hat{\mathbf{S}}\cdot\boldsymbol{B} + (bath). \tag{A10}$$

The detector order parameter $\varphi \equiv \langle\hat{\phi}\rangle$ corresponds to our variable $X_C(t)$ in Equation (55). On top of these dynamics of φ, the back reaction of it to the spin $\hat{\mathbf{S}}$ as well as an initial trigger of $\hat{\phi}$ are expressed in Equation (A10). The thermal bath degrees of freedom may not be essential and are replaced by the classical fluctuations associated with the initial squeezed state triggered by the unstable potential ($m^2 < 0, \lambda > 0$ and $\mu > 0$). Spontaneous appearance of the order parameter and SSB, as well as the establishment of a correlation between $\hat{\mathbf{S}}$ and φ, are the essence of the quantum detector. Thus, the quantum measurement process may also be a phase transition [11].

Incidentally, since we do not introduce any extra invisible degrees of freedom, the above model of quantum measurement has nothing to do with the hidden-variable theory [47]. Further, it may be interesting to compare our approach to the measurement theory based on decoherence and the einselection [37]. In our case, an essential role is played by the coherent state, which may correspond to the einselected pointer bases in [37]. However, in our formalism, the development of the coherent state varies rapidly triggered by the unstable potential and the strong noise. This is a sharp difference from the einselection by the slow thermodynamical process approach to the equilibrium [48].

We would like to emphasize the importance of the energy flow and cascade in the phase space for the state selection. In the case of the spontaneous emission of a photon, the random selection of the photon emission direction is associated with the quantum vacuum fluctuation as well as energy cascade through the interaction [49]. In the problem of an infinite series of the von Neumann measurement chain, the electron avalanche and the energy cascade in the Geiger–Müller tube is essential to achieving the macroscopic order parameter.

Appendix A.5. Transient Dynamics Which Shows Macroscopic Irreversibility

As seen above, our present study may be related to the quantum measurement dynamics, phase transition, and SSB. These are typical irreversible processes. A classical degree of freedom as a developed coherent state has appeared after the time evolution by the Langevin equation and a single specific state has been spontaneously chosen among possible many states. This is the nondeterministic irreversibility [50] and is associated with $G_C(x, y)$ in the last line of Equation (78).

On the other hand, careful argument on the appearance of the arrow of time in quantum mechanics [51] may indicate another type of irreversibility. They argue the difference between the complex conjugate pole pairs associated with the resonance in some quantum systems. This is the deterministic irreversibility and is associated with $G_A(x, y), G_R(x, y)$ in the first two lines of Equation (78).

The above two types of irreversibilities are deeply related to each other. In the system with nondeterministic irreversibility, it inevitably gains energy since $G_C(x, y)$ is always associated with the random force. If the system should be in a dynamically steady state, an energy loss process should also be associated with it to make balance the energy. This is the deterministic irreversibility represented by the term $G_R(x, y)$ in Equation (55). $G_R(x, y)$ is always selected and not $G_A(x, y)$ in the formalism because we are setting a causal description preparing the initial conditions.

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