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Standard model gauge fields localized on non-Abelian vortices in six dimensions

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A brane-world $SU(5)$ grand unified theory model with global non-Abelian vortices is constructed in six-dimensional spacetime. We find a solution with a vortex associated to $SU(3)$ separated from another vortex associated to $SU(2)$. This 3–2 split configuration achieves a geometric Higgs mechanism for $SU(5) \to SU(3) \times SU(2) \times U(1)$ symmetry breaking. A simple deformation potential induces a domain wall between non-Abelian vortices, leading to a linear confining potential. The confinement stabilizes the vortex separation moduli, and ensures the vorticities of the $SU(3)$ and $SU(2)$ groups are identical. This dictates the equality of the numbers of fermion zero modes in the fundamental representation of $SU(3)$ (quarks) and $SU(2)$ (leptons), leading to quark/lepton generations. The standard model massless gauge fields are localized on the non-Abelian vortices thanks to a field-dependent gauge kinetic function. We perform fluctuation analysis with an appropriate gauge fixing and obtain a four-dimensional effective Lagrangian of unbroken and broken gauge fields at quadratic order. We find that $SU(3) \times SU(2) \times U(1)$ gauge fields are localized on the vortices and exactly massless. Complications in analyzing the spectra of gauge fields with the nontrivial gauge kinetic function are neatly worked out by a vector-analysis-like method.

Subject Index B33, B35

1. Introduction

One of the most interesting paradigms of unified theories beyond the standard model is the brane-world scenario in which we live on a brane in higher-dimensional spacetime [1–3]. As candidates for the brane, topological solitons have a number of attractive features in contrast to thin delta-function-like branes, which may be regarded as somewhat artificial idealizations. Traditional reasons repeatedly emphasized in the literature are:

- The topological solitons are dynamically generated as a consequence of spontaneously broken symmetries. This phenomenon is referred to as dynamical compactification [4].
- They trap chiral fermions on their world volumes which are topological states and therefore inevitably appear irrespective of any details [5,6].

The models using topological solitons are often called fat brane-world models, and enjoy an additional merit, which is common to brane-world scenarios. The hierarchy problem of fermion masses can be naturally explained as a consequence of overlapping of wave functions of the localized fermions and the Higgs field [7].

In order to realize the standard model (SM) on a brane, a serious obstacle has been localization of massless gauge fields on the brane. Suppose the gauge symmetry H is broken in the bulk (Higgs phase) and is restored only in the vicinity of the topological soliton. One might expect that massless H gauge bosons will appear inside the soliton. However, this is not true, and in general they get masses of the order of the inverse width of the topological solitons [8]. This is because the bulk (not including the soliton core) is a sort of non-Abelian superconductor. Therefore, though the gauge symmetry is restored in the soliton, electric fluxes from a probe electric charge put inside the soliton are immediately absorbed into the superconducting bulk. Hence, the gauge fields can only propagate for a distance of about the width of the soliton, namely they are massive [2,8]. This argument is quite reasonable, and therefore localizing massless gauge bosons on the soliton generally seems to be quite difficult.

The key idea for getting over this difficulty was first proposed in Ref. [8]. It employs a sort of electromagnetic duality. If we put a probe magnetic charge inside the soliton in the superconducting bulk, the magnetic flux is entirely repelled from the bulk by the Meissner effect, and because of the conservation of flux, the field lines extend to infinity along the soliton [2,8]. In Ref. [8], a dual picture of this, namely replacing the Higgs phase by a confining phase under the assumption that magnetic charges are condensed in the bulk, was proposed. A massless H gauge field is localized inside the soliton if the H gauge group is unbroken inside the soliton and is enhanced to a large non-Abelian group G in the confining bulk. This mechanism, the so-called Dvali–Shifman (DS) mechanism, was rigorously proven in the super-Yang–Mills theory in four dimensions in Ref. [8], but a higher-dimensional version has not yet successfully been proven because we do not know how confinement works in higher dimensions. Then, a lot of papers followed [8] investigating fat brane scenarios, but most of them needed to assume the DS mechanism to work.

One of the earliest such works, which proposes a simple phenomenological model for the DS mechanism, was Ref. [2]. There, the Abrikosov–Nilsen–Olsen vortex is considered as a fat 3-brane in six dimensions. In addition to the SM fields, the model includes an extra scalar field T which condenses only inside the vortex core. Furthermore, the model has a phenomenological dress factor to the gauge kinetic term of the form $\Lambda^{-2}\,\mathrm{Tr}\,T^2 G_{\mu\nu}G^{\mu\nu}$, where Λ is a cut-off scale. Then, it was assumed that the model meets the following three conditions:

(i) Outside the vortex the SM gauge group H is extended into a larger non-Abelian gauge group G (note that H is not broken everywhere, and it is included as part of the unbroken large group G in the bulk).
(ii) There is no light matter in the bulk.
(iii) The tree-level gauge coupling (corresponding to the factor T^2 in the above example) becomes large away from the vortex core.

It was proposed in Ref. [2] that when these conditions are satisfied the localization of massless H gauge fields on the vortex takes place.

We should note that, although the seminal work of Ref. [8] gave the basic idea for the localization of massless gauge fields on fat branes, detailed analysis of how to get the physical mass spectrum on vortices in six dimensions has not yet appeared in the literature. The purpose of this paper is to provide concrete phenomenological models for the fat brane-world scenario using topological vortices in six dimensions along the lines of Ref. [2], and to give a systematic analysis of the physical mass spectra, including both massless and massive modes.

A classical realization of a confining vacuum in the bulk can be given in terms of a generic nonlinear kinetic term, namely a field-dependent gauge kinetic function [9–16] of the form

$$-\frac{\mathcal{B}^2}{4}F_{MN}F^{MN}, \tag{1}$$

where M, N are spacetime indices. If $\mathcal{B}$ is a constant, then this is the usual minimal gauge kinetic term with a coupling constant $1/\mathcal{B}$, but we allow $\mathcal{B}$ to be a function of scalar fields, such as $\mathcal{B}(T) = T$ in the above example. The scalar fields are not necessarily constant in the extra dimensions, but can be non-trivial functions of the coordinates of the extra dimensions as a consequence of the dynamics of the system under consideration. If this is the case, the inverse effective gauge coupling $\mathcal{B}$ is no longer constant and depends on the extra-dimensional coordinates nontrivially.

In a series of previous works [10–16] we have established a general criterion for obtaining massless gauge fields on topological solitons: if $\mathcal{B}^2$ is square-integrable with respect to the integration over the whole extra dimensions, the massless four-dimensional gauge fields are localized on the topological soliton. This corresponds to criterion (iii) of Ref. [2] mentioned above. Since the square-integrability does not depend on the details of the model, this localization mechanism is robust. We verified this statement in various concrete models in five-dimensional spacetime [10–14], and also gave a formal proof in generic spacetime dimensions [15,16]. With a nontrivial kinetic function, we also found an interesting localization mechanism of massless scalar fields [17] on domain walls similar to Eq. (1) for gauge fields. The mechanism is found to have a topological nature similar to the Jackiw–Rebbi topological mechanism [5] for localization of massless fermions on domain walls. The fat-brane scenario has been much discussed, mostly in other contexts, and has produced various different models with their own advantages/disadvantages [18–40].

Since models in five-dimensional spacetime are the simplest to analyze, there have been many concrete brane-world models using domain walls to obtain the symmetry breaking of grand unified theories (GUTs) down to the SM, such as in Refs. [41–47], without an explicit mechanism to localize the gauge fields. With our mechanism for localization of gauge fields, we have constructed a concrete model for the $SU(5)$ GUT on domain walls [12]. The GUT symmetry breaking is determined by the positions of the domain walls. This geometric Higgs mechanism is a characteristic feature of the brane-world model with topological solitons. By introducing a moduli stabilization potential we obtained the $SU(5) \to SU(3) \times SU(2) \times U(1)$ leading to the SM. As an alternative possibility, we have also obtained a five-dimensional model for the SM, where the condensation of the SM Higgs field Φ is driven by the formation of domain walls which localizes the SM. It predicts a new contribution to the $\Phi \to \gamma\gamma$ decay and the possible experimental accessibility of heavy monopoles [14].

To explain quark/lepton generations naturally, however, we need an additional idea in brane-world models with topological solitons. One such mechanism is to use vortices in the extra dimensions. If there is a vortex with k vorticity (k coincident vortices), the index theorem gives k zero modes for fermions. This mechanism has previously been proposed to explain fermion generations in the brane-world scenario [48,49], although the model was without the localization mechanism for gauge fields.

The purpose of this paper is to propose a class of models in six-dimensional spacetime which is a non-Abelian generalization of the simple model considered in Ref. [2]. Our model is based on a GUT-inspired $G = SU(5)$ gauge theory with a field-dependent gauge kinetic function in six spacetime dimensions. It contains complex scalar fields in a singlet and an adjoint representation of $SU(5)$, which are combined into a 5×5 matrix-valued field T. The model admits topologically stable non-Abelian global vortices. We find that the massless gauge fields of $H = SU(3) \times SU(2) \times U(1)$ are localized on the four-dimensional world volume of the non-Abelian global vortices with multiple winding numbers.[1]

By a simple potential with the 5×5 matrix field T we can obtain vortex equations for each diagonal component $T = \mathrm{diag}(T_1, T_2, T_3, T_4, T_5)$ independent of each other. Then we obtain five different species of vortices $I = 1, \ldots, 5$ corresponding to vortices in the Ith diagonal component T_I. We find two important phenomena due to the vortices:

- Geometric Higgs mechanism: The gauge symmetry $G = SU(5)$ is partially broken when topological vortices are present. Interestingly, the breaking pattern of the $SU(5)$ gauge symmetry depends on the positions of the vortices. Namely, the dynamics of the non-Abelian vortices determines which subgroup of $SU(5)$ remains unbroken. We are primarily interested in the configuration where vortices in the three diagonal components, say T_1, T_2, T_3, of the matrix field T are coincident at a point in the extra-dimensional plane, whereas vortices in the remaining two diagonal components T_4, T_5 are coincident at another point. This provides a symmetry breaking pattern $SU(5) \to SU(3) \times SU(2) \times U(1)$. It is important to notice that the origin of the symmetry breaking resides only locally near the non-Abelian vortices. This is the reason why the SM gauge fields are localized around the vortices. We call this vortex solution the 3–2 splitting configuration. Note that criterion (i) of Ref. [2] mentioned above is naturally satisfied by the generation of the non-Abelian vortices.
- Confinement of non-Abelian global vortices: The separation between the position of one group of vortices in T_1, T_2, T_3 and the position of another group in T_4, T_5 is a moduli that depends on the details of the scalar potential, namely on a particular ratio of potentials for singlet and adjoint in parts of T. If we perturb this ratio, we find that separation is no longer a moduli. Namely, a potential energy is induced and becomes constant for asymptotically large separations. Thus, a confining force emerges between non-Abelian global vortices. The confining force can only end at another non-Abelian global vortex. This process continues until they finally form an $SU(5)$ singlet combination of non-Abelian vortices. In other words, only when each diagonal component has the same number of vortices does the configuration become stable. Any vortices which cannot form a singlet combination are dynamically removed to spatial infinity by the confining energy density (a domain wall) extending to infinity. This is important once fermions couple to the vortices, because the

[1] An application of non-Abelian *local* vortices to brane-world physics without localization of the gauge fields was considered in Ref. [50].

number of fermion zero modes is identical to the winding number. Hence, the confinement of non-Abelian global vortices ensures that the number of fermion zero modes is common for different representations of the SM; namely, leptons and quarks should come in generations.

We also obtain a low-energy effective theory on the background of the 3–2 splitting configuration with a particular focus on the problem of localization of the unbroken $SU(3) \times SU(2) \times U(1)$ gauge fields. Deriving the effective Lagrangian can be quite complicated even at the quadratic order of small fluctuations, for a number of reasons.

- The background solution is non-trivial, namely non-Abelian vortices.
- The gauge kinetic Lagrangian in Eq. (1) is not in the canonical form.
- Apart from $SU(3) \times SU(2) \times U(1)$ four-dimensional vector fields, we have gauge fields corresponding to the broken generators of $SU(5)$. The extra-dimensional components of gauge fields are also present, and mix among themselves and with the matrix-valued complex scalar fields.
- We have to take care of these issues by choosing an appropriate gauge fixing.

In order to organize the calculations, we develop an effective and a compact formula generalizing the usual three-dimensional vector analysis. It turns out that this approach is useful to clean up complicated calculations and make things transparent. Furthermore, our method allows us to treat both unbroken and broken parts in a very similar manner. Armed with our vector-analysis-like method, we show most importantly that the 3–2 splitting configuration of non-Abelian global vortices localizes the massless degrees of freedom corresponding to the $SU(3) \times SU(2) \times U(1)$ SM gauge fields on the four-dimensional world-volume of the vortices. We also show that other fields, except for a mixing part of the extra-dimensional component of the broken gauge fields and the scalar fields, are either massive or unphysical in the sense that they are absorbed by massive Kaluza–Klein (KK) towers of the gauge fields. Hence, our 3–2 splitting background configuration of the non-Abelian global vortices in six dimensions is a promising platform for a fat brane-world scenario with GUT.

This paper is organized as follows. In Sect. 2.1 we present our model admitting non-Abelian global vortices in six dimensions, and study multiple vortices, especially the 3–2 splitting configuration that leads to the desired symmetry breaking $SU(5) \to SU(3) \times SU(2) \times U(1)$. In Sect. 2.2, the confinement of non-Abelian global vortices by domain walls are described. Section 3 is devoted to deriving a four-dimensional low-energy effective Lagrangian of the gauge fields. We develop the useful vector-analysis-like technique and examine which fields provide massless/massive, physical/unphysical, and normalizable/non-normalizable modes under the 3–2 splitting background. We give proofs of some theorems resembling Helmholtz's theorem for our vector-analysis-like method in Appendix A.

2. A brane-world model with non-Abelian vortices

2.1 *Non-Abelian global vortices for $SU(5) \to SU(3) \times SU(2) \times U(1)$*

Let us consider an $SU(5)$ non-Abelian gauge theory in six dimensions,

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{\mathcal{B}(T)\mathcal{B}^\dagger(T)}{2}\mathcal{F}_{MN}\mathcal{F}^{MN} + D_M T (D^M T)^\dagger\right] - V, \tag{2}$$

where $M = 0, 1, \ldots, 5$ denotes spacetime indices, and $\mathcal{F}_{MN} = \partial_M \mathcal{A}_N - \partial_N \mathcal{A}_M + i[\mathcal{A}_M, \mathcal{A}_N]$ is the field strength of the $SU(5)$ gauge fields $\mathcal{A}_M$. A 5×5 matrix-valued complex scalar field T contains a singlet T_0 and an adjoint $\hat{T}$ representation of $SU(5)$,

$$T_0 = \mathrm{Tr}\, T, \qquad \hat{T} = T - \frac{T_0}{5}\mathbf{1}_5, \tag{3}$$

with $\mathrm{Tr}\, \hat{T} = 0$. Then, a covariant derivative is defined by

$$D_M T = \partial_M T + i[\mathcal{A}_M, T]. \tag{4}$$

The Lagrangian in Eq. (2) has a peculiar factor $\mathcal{B}\mathcal{B}^\dagger$ in front of the gauge kinetic term $\mathcal{F}_{MN}^2$. If we take a constant $\mathcal{B} = 1/g$, the Lagrangian is just the standard one. Instead, we allow $\mathcal{B}$ to be a generic function of T, and call it a gauge kinetic function. We require $\mathcal{B}$ to be at least a Hermitian matrix and invariant under the $SU(5)$ gauge transformation. Otherwise, we leave it arbitrary for now, since it does not play any role in the rest of this section.[2] It will play an important role in the next section, and we will clarify concrete conditions on $\mathcal{B}$ for a physically meaningful brane-world GUT model, namely the conditions for having massless gauge bosons localized on non-Abelian vortices.[3] In the next section we will conclude that $\mathcal{B}\mathcal{B}^\dagger$ has to asymptotically go to zero far away from the solitons, which implies that the effective gauge coupling $g \sim 1/\sqrt{\mathcal{B}\mathcal{B}^\dagger}$ diverges. Therefore, the Lagrangian in Eq. (2) is not suitable for describing physics in a homogeneous vacuum. Instead, we interpret it as a phenomenological model which is suitable to describe non-trivial confinement physics (the DS mechanism) under the presence of topological solitons in higher dimensions, as proposed in Ref. [2].

There also seems to be no a priori condition for the scalar potential V of T except for the gauge invariance and reality condition. Hence, it can be an arbitrary function of gauge-invariant quantities such as $|T_0|^2$, $\mathrm{Tr}\big[\hat{T}\hat{T}^\dagger\big]$, $\mathrm{Tr}\big[(\hat{T}\hat{T}^\dagger)^2\big]$, $T_0^* \,\mathrm{Tr}\big[\hat{T}^2\hat{T}^\dagger\big] + \text{h.c.}$, $T_0^{*2}\,\mathrm{Tr}\big[\hat{T}^2\big] + \text{h.c.}$, and so on. Instead of surveying such vast possibilities, we concentrate on a simple potential and its deformations to obtain a platform for the brane-world and GUT:

$$V = \frac{\lambda^2}{2}\mathrm{Tr}\left[\left(TT^\dagger - v^2\mathbf{1}_5\right)^2\right]. \tag{5}$$

The vacuum configuration of V up to symmetry transformation is obviously

$$T = v\mathbf{1}_5. \tag{6}$$

This is the $SU(5)$-preserving vacuum, but it is important to realize that the $U(1)$ global symmetry $T \to e^{i\alpha}T$ is spontaneously broken. Hence, the vacuum manifold is isomorphic to S^1, which is not simply connected space, and the fundamental homotopy group is nontrivial as $\pi_1(S^1) = \mathbb{Z}$. This gives rise to global vortices (three-branes in six-dimensional spacetime) which are topologically stable. The Euler–Lagrange equations can be solved by consistently setting all the off-diagonal components of T and the gauge fields to vanish. This leads to five decoupled Euler–Lagrange equations for the diagonal elements of $T = \mathrm{diag}(T_1, T_2, T_3, T_4, T_5)$. Assuming T_I depends only on extra-dimensional coordinates x^a, $a = 4, 5$, we obtain

$$\partial_a^2 T_I - \lambda^2\left(|T_I|^2 - v^2\right)T_I = 0, \qquad I = 1, \ldots, 5. \tag{7}$$

Each equation is identical to the familiar equation for a global vortex, which can be derived from a one-scalar model,

$$\mathcal{L}' = |\partial_M \phi|^2 - \frac{\lambda^2}{2}\left(|\phi|^2 - v^2\right)^2, \tag{8}$$

[2] One can jump to Eq. (109) to see a concrete example for $\mathcal{B}$.

[3] The conditions for $\mathcal{B}$ are given in Eqs. (24) and (75).

if we identify ϕ with T_I.

To obtain the $k_I \in \mathbb{Z}$ coincident vortices at the origin for the Ith diagonal field T_I, we can solve the angular part of the equation by expressing T_I as

$$T_I = v f_I(r) e^{ik_I\theta} \tag{9}$$

in terms of the polar coordinates $x_4 + ix_5 = re^{i\theta}$. The remaining radial equation is given as

$$f_I'' + \frac{f_I'}{r} - \frac{k_I^2}{r^2} f_I - \lambda^2 v^2 (f_I^2 - 1) f_I = 0. \tag{10}$$

This should be solved under the boundary conditions

$$f_I(0) = 0, \qquad f_I(\infty) = 1. \tag{11}$$

Although these vortex solutions are mere five copies of the standard global vortex solutions embedded into the diagonal components of the matrix field T, they actually have a characteristic property of non-Abelian vortices. For example, take the simplest case with $k_1 = 1$ and all the others zero ($k_{2,3,4,5} = 0$). In terms of the matrix field T, the vortex solution is

$$T = v\,\mathrm{diag}(f_1 e^{i\theta}, 1, 1, 1, 1) = e^{i\frac{\theta}{5}} v\,\mathrm{diag}\left(f_1 e^{i\frac{4}{5}\theta}, e^{-i\frac{1}{5}\theta}, e^{-i\frac{1}{5}\theta}, e^{-i\frac{1}{5}\theta}, e^{-i\frac{1}{5}\theta}\right). \tag{12}$$

At spatial infinity ($f_1 \to 1$) we have

$$T\Big|_{r\to\infty} = v e^{i\frac{\theta}{5}} e^{i\frac{4\theta}{5}\lambda_{14}}, \tag{13}$$

with an $SU(5)$ generator $\lambda_{14} = \mathrm{diag}\left(1, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$. This decomposition shows that the winding number in the overall $U(1)$ group is $\frac{1}{5}$ when we go around the vortex once. In this sense, the vortex is called a $\frac{1}{5}$ fractionally quantized global vortex. Let us turn our eyes to the opposite limit $r = 0$. We have

$$T\Big|_{r\to 0} = v\,\mathrm{diag}(0, 1, 1, 1, 1). \tag{14}$$

Namely, symmetry breaking $SU(5) \to SU(4) \times U(1)$ occurs at the very center of the vortex. This implies that the vortices transform nontrivially under the non-Abelian global $SU(5)$ transformations, and such vortices are called non-Abelian vortices. In summary, the vortex solution in this model is the $\frac{1}{5}$ fractionally quantized non-Abelian global vortex. Note that the $SU(5)$ gauge fields do not play any role at all, and therefore the above vortex solutions are called non-Abelian global vortices as found in models without gauge fields [51–55].

Since the equations in Eq. (7) for five diagonal fields T_I are decoupled, we can freely choose positions of the vortices in each diagonal component T_I. Namely, they are moduli of the solutions. Let us consider a solution where the first three diagonal components, say T_1, T_2, T_3, have a common number of vortices at a single position, whereas the remaining two diagonal components (T_4, T_5) have the same number of vortices at another position,

$$T = v \begin{pmatrix} f_3(r_3) e^{ik_3\theta_3} \mathbf{1}_3 & 0 \\ 0 & f_2(r_2) e^{ik_2\theta_2} \mathbf{1}_2 \end{pmatrix}, \tag{15}$$

where (r_3, θ_3) and (r_2, θ_2) are the polar coordinates whose origins are at the respective vortex centers. We have $T \to v\,\mathrm{diag}(0, 0, 0, e^{i\theta_2}, e^{i\theta_2})$ at $r_3 \to 0$, and $T \to v\,\mathrm{diag}(e^{i\theta_3}, e^{i\theta_3}, e^{i\theta_3}, 0, 0)$ at $r_2 \to 0$. Since the vortex positions of T_1, T_2, T_3 and of T_4, T_5 are distinct, the solution breaks $SU(5)$ down to $SU(3) \times SU(2) \times U(1)$. We can identify this gauge symmetry breaking as the breaking of GUT to the SM. We should emphasize that this symmetry breaking occurs only locally in the vicinity of the vortex centers. This fact is important for having massless gauge fields *localized* on the vortices, as we will see later.

We note that the tensions of the non-Abelian global vortices in our model are logarithmically divergent, similar to usual Abelian global vortices. However, this is not harmful for the construction of the effective theory of four-dimensional fields, as we explain in the subsequent sections.

2.2 *Moduli stabilization through confinement of vortices*

Though we are satisfied with the vortex solution in Eq. (15), of course it is not a perfect solution. Firstly, the vortex positions are moduli. Therefore, they in general scatter around, and then the breaking pattern of the gauge symmetry becomes $SU(5) \to U(1)^4$ which is not acceptable with respect to the brane-world model. Secondly, the vortex numbers k_a are arbitrary. Indeed, the vortex number is related to the number of generations of fermions coupled to the T field. Therefore, we would like to have $k_1 = k_2 = \cdots = k_5$ at least, and, if possible, we further want $k_a = 3$ for all a. Unfortunately we cannot solve for all these conditions at once, but we can at least solve them partially.

Let us begin by extracting quadratic terms from the potential in Eq. (5),

$$V \supset -\lambda^2 v^2 \mathrm{Tr}\big[TT^\dagger\big] = -\lambda^2 v^2 \left(\frac{|T_0|^2}{5} + \mathrm{Tr}\big[\hat{T}\hat{T}^\dagger\big]\right). \tag{16}$$

The particular ratio $\frac{1}{5}$ between the coefficients of $|T_0|^2$ and $\mathrm{Tr}\big[\hat{T}\hat{T}^\dagger\big]$ is important for having the five decoupled equations in Eq. (7). However, there is no a priori reason to choose this specific ratio on symmetry grounds. We can single out, for example, the traceless part and modify the potential, adding

$$\delta V = \alpha^2\, \mathrm{Tr}\big[\hat{T}\hat{T}^\dagger\big]. \tag{17}$$

To see what happens because of this additional term, let us plug the minimal vortex solution in Eq. (12) with $k_1 = 1$ and $k_{2,\,3,\,4,\,5} = 0$ into the additional term. It reads

$$\delta V = \frac{22 v^2 \alpha^2}{25}\left(1 + f_1^2 - 2 f_1 \cos\theta\right) \to \frac{44 v^2\alpha^2}{25}\,(1 - \cos\theta) \tag{18}$$

as $r \to \infty$. This depends on the angular coordinate θ via $\cos\theta$. Therefore, when we traverse around the vortex, we inevitably cross the potential barrier once. Namely, the minimal vortex is attached by the domain wall, as familiar axion cosmic strings. Because the domain wall pulls the vortex towards the spatial infinity, we cannot retain the vortex. However, the domain wall can end on a different vortex, say with $k_2 = 1$ and $k_{1,\,3,\,4,\,5} = 0$, at some other point. When we are sufficiently far from both of the vortices we have $f_1 \sim f_2 \to 1$ and $\theta_1 \sim \theta_2 \equiv \theta$. Therefore, the additional potential asymptotically reduces to

$$\delta V \to \frac{12 v^2 \alpha^2}{5}(1 - \cos\theta). \tag{19}$$

There is still the domain wall. To eliminate the domain wall completely, we need the same number of vortices in all the diagonal components. For example, when $k_{1,\,2,\,\ldots,\,5} = 1$, we have $\hat{T} \to 0$ and

$$\delta V \to 0. \tag{20}$$

Now, the θ dependence asymptotically disappears. This can be understood as follows. The domain walls may exist but they are completely terminated by the five vortices. In other words, the vortices are confined to form the singlet (a set of five different vortices) by the linear force of the domain wall.[4] Note that the domain wall appears by adding $|T_0|^2$ instead of $\mathrm{Tr}\big[\hat{T}\hat{T}^\dagger\big]$ in Eq. (17). Only when they appear together with a particular ratio so that they are unified as $\mathrm{Tr}[TT^\dagger]$ does no domain wall appear even in the single vortex with $k_1 = 1$ with $k_{2,3,4,5} = 0$. Namely, the vortices are deconfined when the scalar potential can be described by T only (without T_0 and/or $\hat{T}$) as Eq. (5).

The vortex confinement is definitely an important piece of solving the problems described at the beginning of this subsection. Firstly, it provides the attractive confining force among the asymptotically separated vortices. This lifts the position moduli. Secondly, the domain walls discard unconfined constituent vortices to spatial infinity. Namely, the vortex winding numbers $k_1, k_2, \ldots, k_5$ are automatically adjusted to the same integer. Hence, this ensures unification of generations of localized fermions.

All is good news so far. However, there are still unclear points. First, we cannot explain the reason why the unified generation becomes 3, and second, we cannot predict very well what kind of singlet configuration remains under the presence of the confining force. If there is only an attractive force from the domain wall, all the vortices would completely coincide. This is not good for the brane-world scenario since $T \propto \mathbf{1}_5$ and $SU(5)$ is never broken. We can show this via a concrete numerical simulation.

In Fig. 1 we show typical configurations including five non-Abelian vortices. We numerically constructed these by making use of a standard relaxation method [add a dissipation to the equations of motion for the potential V in Eq. (5) with δV in Eq. (17)]. We first prepare an appropriate initial configuration with five vortices separately placed at the vertices of a pentagon. Then, we run the relaxation simulation. We show two snapshots at early, Fig. 1(a) and (b), and late, Fig. 1(c) and (d), stages. Panel (b) shows the amplitude $|\det T|$ for five well-separated non-Abelian vortices.Panel (a) shows the additional potential density δV, which is nothing but the domain wall that is in charge of the confinement. Panels (c) and (d) show the same information as (a) and (b) but at a much later stage. The pentagon is completely squashed, and we are left with an integer quantized Abelian vortex. This occurs because there is only a confining force among the non-Abelian vortices.

We should introduce a repulsive force that can compensate for the domain wall force. A candidate is

$$\tilde{V} = \frac{\tilde{\lambda}^2}{2}\left(\mathrm{Tr}\big[TT^\dagger - v^2\mathbf{1}_5\big]\right)^2 + \alpha^2\,\mathrm{Tr}\big[\hat{T}\hat{T}^\dagger\big]. \tag{21}$$

When $\alpha^2 > 0$, the vacuum is again given by $T = v\mathbf{1}_5$. Note that the first term looks very similar to Eq. (5), but is different. We can decompose this:

$$\tilde{V} = \frac{\tilde{\lambda}^2}{2}\left(\frac{1}{5}|T_0|^2 - 5v^2\right)^2 + \frac{\tilde{\lambda}^2}{2}\left(\mathrm{Tr}\big[\hat{T}\hat{T}^\dagger\big]\right)^2 + \tilde{\lambda}^2\left(\frac{1}{5}|T_0|^2 - 5v^2 + \frac{\alpha^2}{\tilde{\lambda}^2}\right)\mathrm{Tr}\big[\hat{T}\hat{T}^\dagger\big]. \tag{22}$$

We should pay attention to the coefficient of $\mathrm{Tr}\big[\hat{T}\hat{T}^\dagger\big]$ in the third term. It is positive ($\alpha^2 > 0$) in the bulk, but becomes negative in the vicinity of the vortex center since some of the diagonal components of T vanish and $|T_0|^2$ is smaller than $25v^2$. When the sign of the coefficient flips to negative, then $\hat{T}$ locally condenses around the vortices. This leads to a short-range repulsive force.

Figure 2 shows a numerical solution where the five vortices split into triply ($T_{1,2,3}$) and doubly

[4] This singlet is a sort of baryonic type with $k_I > 0$ or ($k_I < 0$) for all I. We could have a mesonic singlet of $k_1 = 1$ and $k_1 = -1$ with $k_{2,3,4,5} = 0$, but this is topologically trivial and would annihilate.

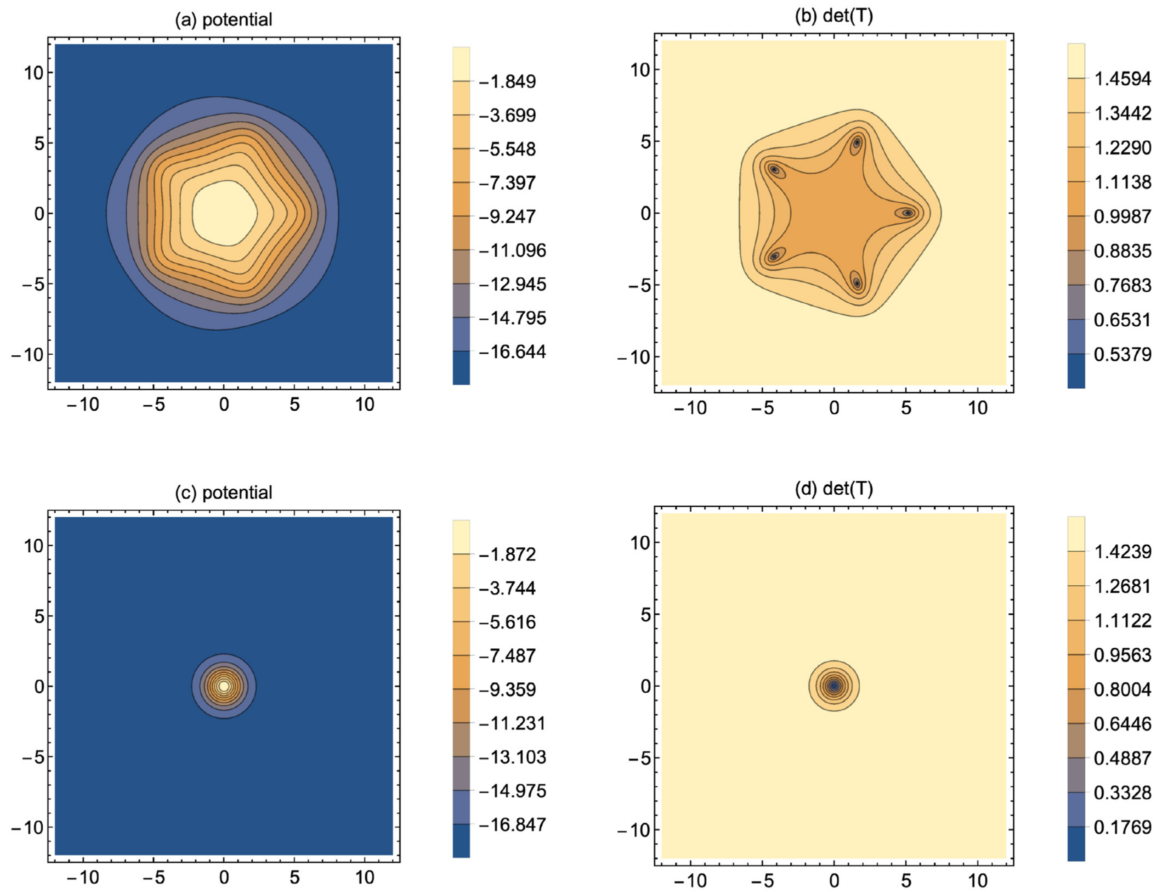

Fig. 1. The two snapshots of the relaxation process starting with the five separate vortices. We take $\lambda_0 = 1$, $v = 1$, and $m^2 = 0.3$. The two upper panels are at an early stage and the two lower panels are at a late stage.

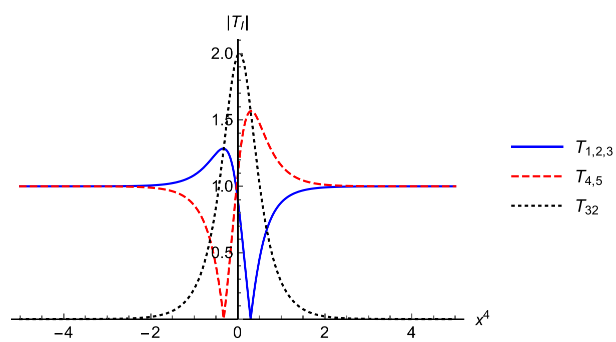

Fig. 2. A 3–2 splitting of the five vortices. The vortex centers are on the x^4 axes and the cross-sections of the absolute values $|T_I|$ are shown. The parameters are taken as $\tilde{\lambda} = 15$, $\alpha = \frac{3}{2}\sqrt{\frac{5}{2}}$, and $v = 1$. The blue curve corresponds to $|T_{1,2,3}|$ and the red dashed line to $|T_{4,5}|$. The blue dotted curve shows $T_{32} = \mathrm{Tr}[T\lambda_{32}]$.

($T_{4,5}$) coincident parts at a finite distance. As can be clearly seen from the figure, the condensations of $T_{4,5}$ increase at the center of the triply coincident vortices, whereas those of $T_{1,2,3}$ also increase near the doubly coincident vortices. This is a direct consequence of the condensation of $T_{32} \equiv \mathrm{Tr}[T\lambda_{32}]$ with $\lambda_{32} = \mathrm{diag}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{2}, -\frac{1}{2}\right)$, which is also depicted by the black dotted line in Fig. 2. The blue (red) curve must touch zero at the center of the $T_{1,2,3}$ ($T_{4,5}$) vortex but at the same time it tends to increase near the adjacent vortex cores $T_{4,5}$ ($T_{1,2,3}$). This opposite tendency conflicts with each other, and generates the desired short-range inter-vortex repulsion. A similar mechanism for producing vortex molecules plays an important role in very different contexts, such as in coherently coupled multicomponent Bose–Einstein condensates of cold atoms [56–67], and also in dense quantum chromodynamics [68].

A remark: though we found one particular numerical solution with the desired 3–2 splitting structure, we have not performed any systematic survey on the parameter space. To figure out how common this configuration is, is important future work. Furthermore, we have not established the stability of our numerical solution. We also leave this as future work. For this paper, we satisfy ourselves with the fact that we succeeded in obtaining the 3–2 splitting solutions in the models with the potentials in Eqs. (5) and (21).

In the next section we present a formal analysis of the localization of the $SU(5)$ gauge fields. For that purpose, concrete solutions are not needed, so we will assume that the 3–2 splitting vortex configuration is obtained by wisely setting a model up somehow.

3. Localization of the gauge fields on vortices

3.1 *Quadratic Lagrangian for gauge fields and gauge fixing*

In this section we study small fluctuations of the $SU(5)$ gauge fields $\mathcal{A}_M$ in the presence of the non-Abelian global vortices, in order to clarify the massless and massive modes. As mentioned above, we are primarily interested in the 3–2 splitting background in which the gauge symmetry is spontaneously broken as $SU(5) \to SU(3) \times SU(2) \times U(1)$. Our starting point is to divide the small fluctuations as

$$\mathcal{A}_M = \begin{pmatrix} \mathcal{G}_M & \mathcal{X}_M/\sqrt{2} \\ \mathcal{X}_M^\dagger/\sqrt{2} & \mathcal{W}_M \end{pmatrix} + \mathcal{Y}_M \frac{1}{\sqrt{60}} \begin{pmatrix} 2I_3 & \\ & -3I_2 \end{pmatrix}, \tag{23}$$

where $\mathcal{G}_M$ is an $SU(3)$ gauge field, $\mathcal{W}_M$ is an $SU(2)$ gauge field, and $\mathcal{Y}_M$ is a $U(1)$ gauge field. The off-diagonal gauge field $\mathcal{X}_M$ is a 2×3 rectangular complex matrix. Finding out the physical spectra of these gauge fields $\mathcal{G}_M$, $\mathcal{W}_M$, $\mathcal{Y}_M$, and $\mathcal{X}_M$ involves complicated calculation due to the following factors:

- Our background is a nontrivial configuration of vortices.
- We need a separate treatment for the gauge fields with the four-dimensional indices $\mathcal{A}_\mu$ and with the extra-dimensional indices $\mathcal{A}_a$.
- We also have to distinguish the unbroken gauge fields $\{\mathcal{G}_M, \mathcal{W}_M, \mathcal{Y}_M\}$ and the broken gauge field $\mathcal{X}_M$, which absorb the fluctuations from the scalar field T.
- We have to clarify the distinction between the physical modes and the unphysical modes by taking into account gauge invariance.

Despite these obstacles, we provide a reasonably simple scheme with which we can handle complicated fluctuation analysis. We can also apply our scheme to a wide class of models, since it does not depend on details of the background.

Our main goal in this section is to examine whether or not massless gauge fields corresponding to the SM gauge group $SU(3) \times SU(2) \times U(1)$ are localized on the 3–2 splitting vortex background. The field-dependent gauge kinetic function $\mathcal{B}(T)$ in Eq. (2), which did not play any role in the previous section, will play a crucial role here. We assume $\mathcal{B}$ is a function of T and $T^\dagger$ (more generically, a function of T_0, T_0^*, $\hat{T}$, and $\hat{T}^\dagger$), although we do not fix a concrete

$\mathcal{B}$ for now.[5] When the background solution T is a diagonal matrix, we obtain a diagonal 5×5 matrix $\mathcal{B}$:

$$T\big|_{\rm bg} = \begin{pmatrix} \tau_3(x^a)\mathbf{1}_3 & \\ & \tau_2(x^a)\mathbf{1}_2 \end{pmatrix}, \qquad \mathcal{B}\big|_{\rm bg} = \begin{pmatrix} \beta_3(\tau_3)\mathbf{1}_3 & \\ & \beta_2(\tau_2)\mathbf{1}_2 \end{pmatrix}. \tag{24}$$

For later convenience, let us define

$$\beta_1 \equiv \sqrt{\frac{3|\beta_2|^2 + 2|\beta_3|^2}{5}}, \qquad \beta_X \equiv \sqrt{\frac{|\beta_2|^2 + |\beta_3|^2}{2}}, \qquad \beta_\phi \equiv \tau_3 - \tau_2. \tag{25}$$

Note that β_3, β_2, and β_ϕ are in general complex, but β_1 and β_X are real by definition.

In addition to the fluctuations of the gauge fields in Eq. (23), we now introduce small fluctuations of the scalar field T. Since T is a 5×5 complex matrix, there are 50 real fluctuations. We can separate them into three parts (Γ, Ψ, Φ) as

$$T = e^{i\Phi}\left(\tilde{T} + \Gamma + \left[\Psi, \tilde{T}\right]\right) e^{-i\Phi} = \tilde{T} + \Gamma + \left[\Psi + i\Phi, \tilde{T}\right] + \cdots, \tag{26}$$

with

$$\Gamma = \begin{pmatrix} \gamma_3 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \qquad \Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \psi \\ \psi^\dagger & 0 \end{pmatrix}, \qquad \Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \varphi \\ \varphi^\dagger & 0 \end{pmatrix}. \tag{27}$$

Here, γ_3 is a 3×3 complex matrix and γ_2 is a 2×2 complex matrix, whereas ψ and φ are 3×2 rectangular complex matrices. The real degrees of freedom included in γ_3, γ_2, ψ, and φ are 18, 8, 12, and 12, respectively. Summing them all, we have the correct real degrees of freedom, namely 50. By construction, $e^{i\Phi}$ can be regarded as a gauge transformation with the generator broken by an amount φ. Hence, the fluctuation field φ contains the Nambu–Goldstone modes corresponding to the broken generators. In the following, we keep φ and ψ only, and ignore $\gamma_{2,3}$ since they are decoupled from the gauge sector or they have masses of the order of the GUT scale and they do not appear in the low-energy effective Lagrangian of the gauge fields at the quadratic order.

Let us next substitute $\mathcal{A}_M$ given in Eq. (23) into the first term in the square bracket in Eq. (2), and pick up only terms of the quadratic order. Those for the unbroken gauge fields can be expressed as

$$\begin{aligned} \mathcal{L}_\alpha = \mathrm{Tr}\Big[& \mathcal{A}^\alpha_\mu\big\{|\beta_\alpha|^2\left(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu\right) - \eta^{\mu\nu}\partial_a|\beta_\alpha|^2\partial_a\big\}\mathcal{A}^\alpha_\nu \\ & - 2(\partial^\mu\mathcal{A}^\alpha_\mu)\partial_a(|\beta_\alpha|^2\mathcal{A}^\alpha_a) \\ & - \mathcal{A}^\alpha_a\left(|\beta_\alpha|^2\delta_{ab}\Box - \delta_{ab}\partial_c|\beta_\alpha|^2\partial_c + \partial_b|\beta_\alpha|^2\partial_a\right)\mathcal{A}^\alpha_b\Big], \end{aligned} \tag{28}$$

which is valid regardless of the details of the gauge kinetic function $\mathcal{B}$, provided Eq. (24) holds. Here μ, ν are the four-dimensional spacetime indices, a, b, c are used for the extra-dimensional space indices, and $\alpha = 1, 2, 3$ are introduced to distinguish $SU(3)$, $SU(2)$, and $U(1)$ as $\mathcal{A}^{\alpha=3}_M = \mathcal{G}_M$, $\mathcal{A}^{\alpha=2}_M = \mathcal{W}_M$, and $\mathcal{A}^{\alpha=1}_M = \mathcal{Y}_M$, respectively. Note that the trace in Eq. (28) for $\mathcal{A}^{\alpha=1}_M = \mathcal{Y}_M$ should be understood as replacing Tr by $\frac{1}{2}$.

[5]The constraint on $\mathcal{B}$ for having localized massless non-Abelian gauge fields inside the vortices is given in Eq. (75), and a concrete example of $\mathcal{B}$ is given in Eq. (109).

The quadratic Lagrangian for the broken gauge field $\mathcal{X}_M$ can also be obtained in the same way:

$$\mathcal{L}_X = \mathrm{Tr}\Big[\mathcal{X}_\mu^\dagger\{\beta_X^2\,(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu) - \eta^{\mu\nu}\partial_a\beta_X^2\partial_a\}\mathcal{X}_\nu \\ - 2(\partial^\mu\mathcal{X}_\mu^\dagger)\partial_a\left(\beta_X^2\mathcal{X}_a\right) \\ - \mathcal{X}_a^\dagger\left(\beta_X^2\delta_{ab}\Box - \delta_{ab}\partial_c\beta_X^2\partial_c + \partial_b\beta_X^2\partial_a\right)\mathcal{X}_b\Big]. \tag{29}$$

Lastly, we write down the scalar Lagrangian which can be obtained by substituting T given in Eq. (26) into $K_\phi = \mathrm{Tr}[D_M T(D^M T)^\dagger]$ in Eq. (2) as

$$K_\phi = \mathrm{Tr}\Big\{\big[\mathcal{A}_M + \partial_M\Phi, \tilde{T}\big]\big[\mathcal{A}^M + \partial^M\Phi, \tilde{T}\big]^\dagger \\ - \big[\Psi, \tilde{T}\big]\left(\Box - \partial_a^2\right)\big[\Psi, \tilde{T}\big]^\dagger + \left(2i\big[\mathcal{A}_M + \partial_M\Phi, \tilde{T}\big]\big[\partial^M\tilde{T}, \Psi\big] + \mathrm{h.c.}\right)\Big\} \\ = \mathrm{Tr}\Big[-\varphi^\dagger\left(|\beta_\phi|^2\Box - \partial_a|\beta_\phi|^2\partial_a\right)\varphi + \varphi^\dagger\partial_a\left(|\beta_\phi|^2\mathcal{X}_a\right) - |\beta_\phi|^2\mathcal{X}_a^\dagger\partial_a\varphi \\ - 2(\partial^\mu\mathcal{X}_\mu^\dagger)|\beta_\phi|^2\varphi + \mathcal{X}_\mu^\dagger\eta^{\mu\nu}|\beta_\phi|^2\mathcal{X}_\nu - \mathcal{X}_a^\dagger\delta_{ab}|\beta_\phi|^2\mathcal{X}_b \\ - \psi^\dagger\left(|\beta_\phi|^2\Box - \beta_\phi\partial_a^2\beta_\phi^*\right)\psi + \left(J_a(\mathcal{X}_a^\dagger + (\partial_a\phi^\dagger))\psi + \mathrm{h.c.}\right)\Big], \tag{30}$$

where we defined

$$J_a = i\left\{\beta_\phi^*(\partial_a\beta_\phi) - \beta_\phi(\partial_a\beta_\phi^*)\right\}. \tag{31}$$

We retained $\mathcal{X}_\mu$, $\mathcal{X}_a$, φ, and ψ, since they are mixed with gauge fields. However, we omitted Γ, since it decouples at the quadratic order. We also need to denote one more quadratic term from the potential,

$$V_\phi = U\,\mathrm{Tr}\big[\psi^\dagger\psi\big], \tag{32}$$

where U is a certain function of the background solution $\tilde{T}$. Here, we also omitted Γ, since it decouples with ψ. Hence, the quadratic Lagrangian is

$$\mathcal{L}_\phi = K_\phi - V_\phi. \tag{33}$$

Finally, we introduce a gauge-fixing Lagrangian. For the unbroken generators, we introduce the following gauge-fixing Lagrangian:

$$\mathcal{L}_\alpha^{(\mathrm{gf})} = -\frac{|\beta_\alpha|^2}{\xi}\mathrm{Tr}\left[\left(\partial^\mu\mathcal{A}_\mu^\alpha - \frac{\xi}{|\beta_\alpha|^2}\partial_a\left(\beta_\alpha^2\mathcal{A}_a^\alpha\right)\right)^2\right]. \tag{34}$$

As before, Tr is understood to be replaced by $\frac{1}{2}$ for $\alpha = 1$. For the broken generators, we introduce another gauge-fixing term:

$$\mathcal{L}_{X\phi}^{(\mathrm{gf})} = -\frac{\beta_X^2}{\xi}\mathrm{Tr}\left[\left(\partial^\mu\mathcal{X}_\mu - \frac{\xi}{\beta_X^2}\left[\partial_a\left(\beta_X^2\mathcal{X}_a\right) + 2|\beta_\phi|^2\varphi\right]\right)(\mathrm{h.c.})\right]. \tag{35}$$

Here, ξ is an arbitrary gauge-fixing constant similar to the R_ξ gauge-fixing condition.

3.2 *Compact formulae for the unbroken gauge fields*

3.2.1 *Canonically normalized gauge fields.* The above quadratic Lagrangian is complicated and far from the standard expression due to the extra β^2 factor. In order to bring it into a more familiar[6] form, let us define

$$A_M^\alpha \equiv |\beta_\alpha| \mathcal{A}_M^\alpha \qquad (\alpha = 1, 2, 3). \tag{36}$$

Below we will also use the expressions $A_M^{\alpha=3} = G_M$, $A_M^{\alpha=2} = W_M$, and $A_M^{\alpha=1} = Y_M$. In the following we need to deal with the extra-dimensional components of the gauge field A_a^α differently from the four-dimensional fields A_μ^α due to the fact that $\beta_\alpha(x^a)$ depends not on x^μ but on x^a. The following vector notation turns out to be convenient for describing the low-energy effective Lagrangian:

$$\vec{A}^\alpha \equiv \begin{pmatrix} A_4^\alpha \\ A_5^\alpha \end{pmatrix} \qquad (\alpha = 1, 2, 3)\,. \tag{37}$$

3.2.2 *Vector-analysis-like method.* We now introduce differential operators useful for performing a vector-analysis-like method for analyzing mass spectra of gauge fields:

$$\vec{D}^\alpha \equiv \begin{pmatrix} D_4^\alpha \\ D_5^\alpha \end{pmatrix} \qquad (\alpha = 1, 2, 3)\,, \tag{38}$$

$$D_a^\alpha \equiv -|\beta_\alpha| \partial_a \frac{1}{|\beta_\alpha|} = -\partial_a + \left(|\beta_\alpha|^{-1} \partial_a |\beta_\alpha|\right) \qquad (\alpha = 1, 2, 3)\,, \tag{39}$$

where no sum is taken for the index α in the middle and the right-most equations. An adjoint operator of the above differential operator is defined by

$$\vec{D}^{\alpha\dagger} = \left(D_4^{\alpha\dagger},\ D_5^{\alpha\dagger}\right), \tag{40}$$

$$D_a^{\alpha\dagger} = |\beta_\alpha|^{-1} \partial_a |\beta_\alpha| = \partial_a + \left(|\beta_\alpha|^{-1} \partial_a |\beta_\alpha|\right) \qquad (\alpha = 1, 2, 3)\,, \tag{41}$$

where we do not sum in α.

To develop an analogue of the usual vector analysis in three spatial dimensions, we introduce analogues of gradient, divergence, and rotation in the following way. In order to avoid inessential complications, we will suppress the index α in the following.

- Gradient:

$$\mathrm{grad}\, f(x^a) \equiv \vec{D} \circ f(x^a) = \begin{pmatrix} D_4 f(x^a) \\ D_5 f(x^a) \end{pmatrix}. \tag{42}$$

- Divergence:

$$\mathrm{div}\, \vec{f}(x^a) \equiv \vec{D}^\dagger \cdot \vec{f}(x^a) = D_4^\dagger f_4(x^a) + D_5^\dagger f_5(x^a). \tag{43}$$

- Vector rotation:

$$\mathrm{rot_v} \vec{f}(x^a) \equiv \vec{D} \times \vec{f}(x^a) = D_5 f_4(x^a) - D_4 f_5(x^a). \tag{44}$$

- Scalar rotation:

$$\mathrm{rot_s} f(x^a) \equiv \vec{D}^\dagger \otimes f(x^a) = \begin{pmatrix} D_5^\dagger f(x^a) \\ -D_4^\dagger f(x^a) \end{pmatrix}. \tag{45}$$

[6]The analysis in this section is a generalization of that in Refs. [10–14] for the fat brane-world scenario with the domain wall in five dimensions, and is a refinement of that in Refs. [15,16], which also studied similar problems in higher dimensions.

- Laplacian:

$$\triangle f \equiv \mathrm{div}\,\mathrm{grad}\, f = \vec{D}^\dagger \cdot \vec{D} \circ f = \sum_{a=4,5} D_a^\dagger D_a f. \tag{46}$$

- Dual scalar Laplacian:

$$\tilde{\Delta}_\mathrm{s} f \equiv \mathrm{rot}_\mathrm{v}\,\mathrm{rot}_\mathrm{s}\, f = \vec{D} \times \vec{D}^\dagger \otimes f = \sum_{a=4,5} D_a D_a^\dagger f. \tag{47}$$

- Dual vector Laplacian:

$$\tilde{\Delta}_\mathrm{v} \vec{f} \equiv \mathrm{rot}_\mathrm{s}\,\mathrm{rot}_\mathrm{v}\, \vec{f} = \vec{D}^\dagger \otimes \vec{D} \times \vec{f} = \begin{pmatrix} D_5^\dagger D_5 & -D_5^\dagger D_4 \\ -D_4^\dagger D_5 & D_4^\dagger D_4 \end{pmatrix} \vec{f}. \tag{48}$$

Since $[D_4, D_5] = 0$ implies $\mathrm{rot}_\mathrm{v}\,\mathrm{grad}\, f = \vec{D} \times \vec{D} \circ f = D_5 D_4 f - D_4 D_5 f = 0$, we find

$$\mathrm{rot}_\mathrm{v}\,\mathrm{grad}\, f = 0. \tag{49}$$

Similarly, $\mathrm{div}\,\mathrm{rot}_\mathrm{s}\, f = \vec{D}^\dagger \cdot \vec{D}^\dagger \otimes f = D_4^\dagger D_5^\dagger f - D_5^\dagger D_4^\dagger f = 0$ gives

$$\mathrm{div}\,\mathrm{rot}_\mathrm{s}\, f = 0. \tag{50}$$

We can define the adjoint of div acting on a vector function $\vec{f}(x^a)$ in terms of the inner product between the scalar function $h(x^a)$ as

$$\begin{aligned} \int dx^4 dx^5\, h^*(x^a)\, \mathrm{div} \vec{f}(x^a) &= \int dx^4 dx^5 \left(h^* D_4^\dagger f_4 + h^* D_5^\dagger f_5\right) \\ &= \int dx^4 dx^5\ (D_4 h^*)\, f_4 + (D_5 h^*)\, f_5 \\ &= \int dx^4 dx^5\ (\mathrm{grad}\, h(x^a))^\dagger \cdot \vec{f}(x^a), \end{aligned} \tag{51}$$

where we assumed $\beta(x^a) h(x^a)$ goes to zero as $x^a \to \infty$, and the dagger † in the last line means the standard Hermite conjugation of a complex vector. Thus, we observe that the adjoint of div is given by grad and vice versa:

$$\mathrm{grad}^\dagger = \mathrm{div} \quad \text{or} \quad \left(\vec{D} \circ\right)^\dagger = \vec{D}^\dagger \cdot\,, \tag{52}$$

$$\mathrm{div}^\dagger = \mathrm{grad} \quad \text{or} \quad \left(\vec{D}^\dagger \cdot\right)^\dagger = \vec{D} \circ\ . \tag{53}$$

Similarly, the adjoint of rot_s acting on a scalar function $f(x^a)$ can be defined in terms of an inner product with a vector function $\vec{g}(x^a)$ as

$$\begin{aligned} \int dx^4 dx^5\, \vec{g}^\dagger \cdot \mathrm{rot}_\mathrm{s} f &= \int dx^4 dx^5 \left(g_4^* D_5^\dagger f - g_5^* D_4^\dagger f\right) \\ &= \int dx^4 dx^5 \left(D_5 g_4^* - D_4 g_5^*\right) f \\ &= \int dx^4 dx^5\ (\mathrm{rot}_\mathrm{v}\, \vec{g})^\dagger\, f, \end{aligned} \tag{54}$$

where the † in the last line means the standard Hermite conjugation of a complex vector. Thus, we find

$$\mathrm{rot_v}^\dagger = \mathrm{rot}_\mathrm{s} \quad \text{or} \quad \left(\vec{D} \times\right)^\dagger = \vec{D}^\dagger \otimes\,, \tag{55}$$

$$\mathrm{rot_s}^\dagger = \mathrm{rot}_\mathrm{v} \quad \text{or} \quad \left(\vec{D}^\dagger \otimes\right)^\dagger = \vec{D} \times\ . \tag{56}$$

We find also that rot_v grad and div rot_s are adjoints of each other:

$$(\text{rot}_\text{v}\,\text{grad})^\dagger = \text{div}\,\text{rot}_\text{s} \quad \text{or} \quad \left(\vec{D}\times\vec{D}\circ\right)^\dagger = \vec{D}^\dagger\cdot\vec{D}^\dagger\otimes . \tag{57}$$

The final piece of our vector-analysis-like method is a decomposition formula for a (two-extra-dimensional) vector field. Let $\vec{A}$ be an arbitrary two-component vector field. There exist scalar fields B and C with which $\vec{A}$ is decomposed as

$$\vec{A} = \text{grad}\,B + \text{rot}_\text{s}\,C. \tag{58}$$

With the identities in Eqs. (49) and (50), this theorem implies that a vector field can be decomposed into a rotation-free part and a divergence-free part. The theorem is proved in Appendix A, and it is an analogue of Helmholtz's theorem for a three-dimensional vector field.[7] Taking the divergence and rotation leads to Poisson-like equations:

$$\triangle B = \text{div}\,\vec{A}, \qquad \tilde{\triangle}_\text{s} C = \text{rot}_\text{v}\,\vec{A}. \tag{59}$$

By solving these equations under appropriate boundary conditions, we can determine B and C for a given $\vec{A}$. Note, however, that B and C are not uniquely determined. We can determine B and C only up to solutions of the Laplace-like equations

$$\triangle B_0 = 0, \qquad \tilde{\triangle}_\text{s} C_0 = 0. \tag{60}$$

Solutions to the above Laplace-like equations are given by the kernels of grad $=\vec{D}\circ$ and $\text{rot}_\text{s} = \vec{D}^\dagger\otimes$, respectively. From Eqs. (39) and (41), we find explicitly that the kernels B_0, C_0 are given by

$$\text{grad}\,B_0 = 0 \Rightarrow \quad B_0 \propto |\beta|, \tag{61}$$

$$\text{rot}_\text{s}\,C_0 = 0 \Rightarrow \quad C_0 \propto |\beta|^{-1}. \tag{62}$$

(Remember that we have suppressed the index $\alpha = 1, 2, 3$.) Hence, when B_0 is normalizable, C_0 is non-normalizable, and vice versa. On the other hand, our Laplacian and the dual Laplacian are positive semi-definite, as given in Eqs. (46) and (47).

3.2.3 *Quadratic Lagrangian for unbroken gauge fields.* We can now rewrite the effective Lagrangians in Eq. (28) in terms of the differential operators of the vector-analysis-like method,

$$\mathcal{L} = \text{Tr}\Big[A_\mu\left(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu + \eta^{\mu\nu}\triangle\right)A_\nu - 2\left(\partial^\mu A_\mu\right)\left(\text{div}\,\vec{A}\right) - \vec{A}^\dagger(\Box + \tilde{\triangle}_\text{v})\vec{A}\Big], \tag{63}$$

after performing a partial integration. Next, we rewrite the gauge-fixing Lagrangians as

$$\mathcal{L}^{(\text{gf})} = -\frac{1}{\xi}\text{Tr}\left[\left(\partial^\mu A_\mu - \xi\,\text{div}\,\vec{A}\right)^2\right]. \tag{64}$$

Summing these two, we get

$$\begin{aligned}\mathcal{L} + \mathcal{L}^{(\text{gf})} = \text{Tr}\Bigg[&A_\mu\left(\eta^{\mu\nu}\Box - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu + \eta^{\mu\nu}\triangle\right)A_\nu \\ &- \vec{A}^\dagger(\Box + \tilde{\triangle}_\text{v} + \xi\,\text{grad}\,\text{div})\vec{A}\Bigg],\end{aligned} \tag{65}$$

where we used Eq. (53) to obtain

$$\left(\text{div}\vec{A}\right)^\dagger\text{div}\vec{A} = \vec{A}^\dagger\,\text{grad}\,\text{div}\vec{A}. \tag{66}$$

[7] The theorem states that a three-vector field $\vec{A}$ can be decomposed into rotation-free and divergence-free components as $\vec{A} = \vec{\nabla}B + \vec{\nabla}\times\vec{C}$.

We decompose $\vec{A} = \mathrm{grad}\, B + \mathrm{rot_s}\, C$ using the generalized Helmholtz theorem, Eq. (58). Note that we can freely exclude the zero mode $B_0 \propto |\beta| (\mathrm{grad}\, B_0 = 0)$ from B and the zero mode $C_0 \propto |\beta|^{-1}$ $(\mathrm{rot_s}\, C_0 = 0)$ from C without the loss of any information contained in $\vec{A}$, because they never change $\vec{A}$. Plugging the decomposition into Eq. (65), we arrive at the following simple formula:

$$\mathcal{L} + \mathcal{L}^{(\mathrm{gf})} = \mathrm{Tr}\Bigg[A_\mu \left(\eta^{\mu\nu} \Box - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \triangle \right) A_\nu \\ - C^\dagger \tilde{\triangle}_\mathrm{s} \big(\Box + \tilde{\triangle}_\mathrm{s}\big) C - B^\dagger \triangle \left(\Box + \xi \triangle\right) B \Bigg]. \tag{67}$$

Since $\triangle$ and $\tilde{\triangle}_\mathrm{s}$ are semi-positive definite and zero modes are excluded from B and C, we can redefine the divergence-free and rotation-free parts by

$$b \equiv \sqrt{\triangle}\, B, \qquad c \equiv \sqrt{\tilde{\triangle}_\mathrm{s}}\, C. \tag{68}$$

Substituting these into the last expression, we obtain the final formula with the omitted index $\alpha = 1, 2, 3$ retained:

$$\left(\mathcal{L} + \mathcal{L}^{(\mathrm{gf})}\right)_{\alpha=1,2,3} = \mathrm{Tr}\Bigg[A^\alpha_\mu \left(\eta^{\mu\nu} \Box - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \triangle^\alpha \right) A^\alpha_\nu \\ - c^{\alpha\dagger} \big(\Box + \tilde{\triangle}^\alpha_\mathrm{s}\big) c^\alpha - b^{\alpha\dagger} \left(\Box + \xi \triangle^\alpha\right) b^\alpha \Bigg]. \tag{69}$$

Note that b^α and c^α are Hermitian matrix fields of the same size as A^α_μ, since they arise as rotation-free and divergence-free parts of $A^\alpha_{a=4,5}$. We do not take the sum over α on the right-hand side.

In order to make the KK expansion more explicit, let us define eigenvalues m_n $(\tilde{m}_n)$ and eigenfunctions $B_n(x^a)$ $(C_n(x^a))$ of $\triangle$ $(\tilde{\triangle}_\mathrm{s})$ as[8]

$$\triangle B_n = m_n^2 B_n, \qquad \tilde{\triangle}_\mathrm{s} C_n = \tilde{m}_n^2 C_n. \tag{70}$$

As mentioned before, $\triangle$ and $\tilde{\triangle}$ are positive semi-definite. We denote the possible zero eigenvalue as $m_{n=0} = \tilde{m}_{n=0} = 0$, and positive eigenvalues as $n > 0$. We set the normalizations of the mode functions as usual:

$$\int dx^4 dx^5\, B_m B_n = \delta_{mn}, \qquad \int dx^4 dx^5\, C_m C_n = \delta_{mn}. \tag{71}$$

Of course, these are meaningful only for normalizable modes.[9] If $|\beta|$ $(|\beta|^{-1})$ is square integrable, B_0 (C_0) is normalizable but C_0 (B_0) is not normalizable.

Now, we decompose B (C) as

$$B(x^M) = \sum_n f_n(x^\mu) B_n(x^a), \qquad C(x^M) = \sum_n g_n(x^\mu) C_n(x^a), \tag{72}$$

where the expansion coefficients f_n and g_n are effective fields in four dimensions, and the sum is taken only for the normalizable modes. Equation (68) implies that expansions for b and c do not have zero modes,

$$b(x^M) = \sum_{n>0} b^{(n)}(x^\mu) B_n(x^a), \qquad c(x^M) = \sum_{n>0} c^{(n)}(x^\mu) C_n(x^a), \tag{73}$$

with $b^{(n)} = f_n m_n$ and $c^{(n)} = g_n \tilde{m}_n$, since zero modes are excluded in defining B and C from $\vec{A}$.

[8] Again, we omit the subscript $\alpha = 1, 2, 3$ from now on.

[9] More precisely, mode functions should be bounded, so that they can be delta-function normalizable for continuum spectra.

In contrast, the four-dimensional components do have the zero mode of $\triangle$ because there is no reason to eliminate it:

$$A_\mu(x^M) = \sum_{n\geq 0} A_\mu^{(n)}(x^\mu) B_n(x^a). \tag{74}$$

Since we wish to have the massless gauge fields localized on the vortices, we should impose the square-integrable condition on β (this is one of the most important results of this work) as

$$\frac{1}{g^2} \equiv \int dx^4 dx^5 \, |\beta|^2 < \infty. \tag{75}$$

This gives the normalization of the mode function for $n = 0$ as

$$B_0(x^a) = g|\beta(x^a)|. \tag{76}$$

Substituting the expansions in Eqs. (73) and (74) into Eq. (69), and integrating it over the x^4–x^5 plane, we obtain the effective Lagrangians for the KK towers.

The massless mode ($n = 0$) only appears in the sector of the four-dimensional gauge fields,

$$\mathcal{L}_\alpha^{(0)} = \mathrm{Tr}\left[A_\mu^{\alpha(0)} \left(\eta^{\mu\nu}\Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right) A_\nu^{\alpha(0)}\right]. \tag{77}$$

This is nothing but the quadratic Lagrangian of massless vector fields. In order to recover self-interaction terms of non-Abelian gauge fields, we return to Eq. (36). We have $A_\mu^\alpha(x^M) = B_0^\alpha(x^a) A_\mu^{\alpha(0)}(x^\mu) + \cdots$, and therefore

$$\mathcal{A}_\mu^\alpha(x^M) = \beta_\alpha^{-1}(x^a) A_\mu^\alpha(x^M) = g_\alpha A_\mu^{\alpha(0)}(x^\mu) + \cdots . \tag{78}$$

It is remarkable that the mode function of the zero mode with respect to the original field $\mathcal{A}_\mu$ is constant, namely g_α. This ensures the universality of the gauge-coupling constant. Nevertheless, the zero mode effective Lagrangian is well defined because the extra factor $\mathcal{B}(T)\mathcal{B}(T)^\dagger$ in Eq. (2) gives the necessary suppression factor $|\beta(x^a)|^2$ in the extra dimensions. Keeping only the zero mode in Eq. (78), the field strength reads

$$\mathcal{F}_{\mu\nu}^\alpha = g_\alpha F_{\mu\nu}^{\alpha(0)}, \tag{79}$$

$$F_{\mu\nu}^{\alpha(0)} = \partial_\mu A_\nu^{\alpha(0)} - \partial_\nu A_\mu^{\alpha(0)} + i g_\alpha \left[A_\mu^{\alpha(0)}, A_\nu^{\alpha(0)}\right]. \tag{80}$$

This should be compared with the original field strength $\mathcal{F}_{MN} = \partial_M \mathcal{A}_N - \partial_N \mathcal{A}_M + i[\mathcal{A}_M, \mathcal{A}_N]$ in which the gauge-coupling constant is absorbed in the gauge field $\mathcal{A}_M$. The four-dimensional effective gauge coupling appears in Eq. (80) because of the normalization condition in Eq. (75). One can also understand this result as a consequence of unbroken four-dimensional local gauge invariance [9]. Now, including the self interactions of the non-Abelian gauge fields, the zero mode effective Lagrangian is found to be the standard one,

$$\int dx^4 dx^5 \, \mathrm{Tr}\left[-\frac{\beta_\alpha^2}{2} g_\alpha^2 F_{\mu\nu}^{\alpha(0)} F^{\alpha(0)\mu\nu}\right] = -\frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}^{\alpha(0)} F^{\alpha(0)\mu\nu}\right]. \tag{81}$$

We would like to emphasize an important new feature of our model compared to many previous works. The gauge kinetic function $\mathcal{B}\mathcal{B}^\dagger$ in Eq. (2) is not a scalar but a matrix as a nontrivial representation of the gauge group. A similar mechanism of localizing massless gauge fields on topological solitons has been studied, which utilizes a conformal factor. However, it is usually a singlet of the gauge group, since it usually originates from a spacetime metric or dilaton. Such a singlet conformal factor cannot distinguish various components of $SU(5)$ gauge fields, unlike our model.

As for the higher KK modes with $n > 0$, we have two separated parts. One is for massive vector fields $A_\mu^{\alpha(n)}$,

$$\mathcal{L}_{1,\alpha}^{(n>0)} = \mathrm{Tr}\Bigg[A_\mu^{\alpha(n)} \left(\eta^{\mu\nu}\Box - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu + \eta^{\mu\nu} (m_n^\alpha)^2 \right) A_\nu^{\alpha(n)} \\ - b^{\alpha(n)\dagger} \left(\Box + \xi (m_n^\alpha)^2 \right) b^{\alpha(n)} \Bigg]. \tag{82}$$

Note that the vector field $A_\mu^{\alpha(n)}$ has two components, with the mass squared $(m_n^\alpha)^2$ and $\xi(m_n^\alpha)^2$. On the other hand, the scalar b originates from the rotation-free part of $\vec{A}^\alpha$, and has mass squared $\xi(m_n^\alpha)^2$. It should be combined with the above components of the vector field $A_\mu^{\alpha(n)}$ with the same mass squared together with the ghost and anti-ghost fields to become unphysical, as can be recognized from their gauge-dependent mass. This situation is analogous to the usual situation in the R_ξ gauge. Thus, we see that the rotation-free part plays the role of an (unphysical) Nambu–Goldstone field to give mass to the KK tower of vector fields. The other part includes the divergence-free part c:

$$\mathcal{L}_{2,\alpha}^{(n>0)} = -\mathrm{Tr}\left[c^{\alpha(n)\dagger} \left(\Box + (\tilde{m}_n^\alpha)^2 \right) c^{\alpha(n)} \right]. \tag{83}$$

This should be a physical scalar field with the mass squared $(m_n^\alpha)^2$.

A summary of the unbroken sector $SU(3) \times SU(2) \times U(1)$ is the following: The six-dimensional gauge fields $\mathcal{A}_M^\alpha$ provide one massless gauge field $A_\mu^{\alpha(0)}$, a KK tower of massive vector fields $A_\mu^{\alpha(n>0)}$, and a KK tower of massive scalar fields $c^{\alpha(n>0)}$ to the four-dimensional effective theory.

3.3 *Compact formulae for the broken gauge fields*

We move to the broken sector with $\mathcal{X}_M$ and φ, which is more complicated than the unbroken sector in the previous subsection. We will make the effective Lagrangians in Eqs. (29), (33), and (35) as simple as possible.

Firstly, we define canonically normalized fields by

$$X_M \equiv \beta_X \mathcal{X}_M, \qquad X_6 \equiv |\beta_\phi| \varphi, \qquad X_7 \equiv |\beta_\phi| \psi. \tag{84}$$

We will also use the notation $X_6 = \phi$ later. We can treat the NG bosons φ and ψ as if they are the sixth and seventh components of a massive gauge field, by defining a four-component vector as

$$\boldsymbol{X} \equiv \begin{pmatrix} X_4 \\ X_5 \\ X_6 \\ X_7 \end{pmatrix}, \tag{85}$$

which will bring the benefit of simplification in many expressions below. Assigning the NG boson $X_6 = \phi$ to be the sixth gauge field is somehow natural because would-be eaten NG bosons can be regarded as part of the massive vector field in general. As before, let us introduce the differential operators by

$$\vec{D}^X \equiv -\beta_X \vec{\partial} \frac{1}{\beta_X} = -\vec{\partial} + \left(\beta_X^{-1} \vec{\partial} \beta_X \right), \tag{86}$$

$$\vec{D}^\phi \equiv -|\beta_\phi| \vec{\partial} \frac{1}{|\beta_\phi|} = -\vec{\partial} + \left(|\beta_\phi|^{-1} \vec{\partial} |\beta_\phi| \right). \tag{87}$$

Note that $\vec{D}^X$ can be expressed by $\vec{D}^\phi$ as

$$\vec{D}^X \circ = \mathcal{M}^{-1}\vec{D}^\phi \circ \mathcal{M}, \qquad \vec{D}^\phi \circ = \mathcal{M}\vec{D}^X \circ \mathcal{M}^{-1}, \qquad \mathcal{M} \equiv \frac{|\beta_\phi|}{\beta_X}. \tag{88}$$

Then, we define a four-component (effectively three-component) operator by

$$\boldsymbol{D} \equiv \begin{pmatrix} \vec{D}^X \\ \mathcal{M} \\ 0 \end{pmatrix}. \tag{89}$$

From this, we can define:

- Gradient:

$$\mathrm{Grad}\, f \equiv \boldsymbol{D} \circ f = \begin{pmatrix} \vec{D}^X \\ \mathcal{M} \\ 0 \end{pmatrix} \circ f = \begin{pmatrix} \vec{D}^X \circ f \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \vec{0} \\ \mathcal{M} f \\ 0 \end{pmatrix}. \tag{90}$$

- Divergence:

$$\mathrm{Div}\, \boldsymbol{f} \equiv \boldsymbol{D}^\dagger \cdot \boldsymbol{f} = \left(\vec{D}^{X\dagger},\ \mathcal{M},\ 0\right) \cdot \begin{pmatrix} \vec{f} \\ f_6 \\ f_7 \end{pmatrix} = \vec{D}^{X\dagger} \cdot \vec{f} + \mathcal{M} f_6. \tag{91}$$

- Laplacian:

$$\boldsymbol{\Delta} f \equiv \mathrm{Div}\,\mathrm{Grad}\, f = \boldsymbol{D}^\dagger \cdot \boldsymbol{D} \circ f = \triangle^X f + \mathcal{M}^2 f, \tag{92}$$

  with $\triangle^X = \mathrm{div}^X\, \mathrm{grad}^X = \vec{D}^{X\dagger} \cdot \vec{D}^X \circ$. Note that $\boldsymbol{\Delta}$ is positive definite since $\triangle^X$ is non-negative definite and $\mathcal{M}^2$ is positive definite.
- Dual vector Laplacian:

$$\tilde{\boldsymbol{\Delta}}_{\mathrm{v}} \equiv \begin{pmatrix} \tilde{\triangle}^X_{\mathrm{v}} + \mathcal{M}^2 \mathbf{1}_2 & -\mathcal{M}\vec{D}_\phi \circ & -\frac{1}{|\beta_\phi|^2}\mathcal{M}\vec{J} \\ -\vec{D}^\dagger_\phi \cdot \mathcal{M} & \triangle^\phi & \vec{D}^{\phi\dagger} \cdot \vec{J}\frac{1}{|\beta_\phi|^2} \\ -\frac{1}{|\beta_\phi|^2}\mathcal{M}\vec{J}^\dagger & \frac{1}{|\beta_\phi|^2}\vec{J}^\dagger \vec{D}^\phi \circ & \beta_\phi \vec{D}^{\phi\dagger} \cdot \frac{1}{|\beta_\phi|^2}\vec{D}^\phi \circ \beta^*_\phi + U \end{pmatrix}, \tag{93}$$

  with the 2×2 Laplacian $\tilde{\triangle}^X_{\mathrm{v}} = \vec{D}^{X\dagger} \otimes \vec{D}^X \times$ and the 1×1 Laplacian $\triangle^\phi = \vec{D}^{\phi\dagger} \cdot \vec{D}^\phi \circ$. The elements of the two-component vector $\vec{J}$ are given by Eq. (31). Note that $\tilde{\boldsymbol{\Delta}}_{\mathrm{v}}$ itself is 4×4.

It would be nice if we could factorize $\tilde{\boldsymbol{\Delta}}_{\mathrm{v}}$ as $\tilde{\boldsymbol{\Delta}}_{\mathrm{v}} = \mathrm{Rot}_{\mathrm{s}}\,\mathrm{Rot}_{\mathrm{v}}$ with appropriate rotation operators $\mathrm{Rot}_{\mathrm{s}}$ and $\mathrm{Rot}_{\mathrm{v}}$. Unfortunately, this does not seem very easy, so we abandon it. Nevertheless, we still have identities corresponding to $\mathrm{Rot}_{\mathrm{v}}\,\mathrm{Grad} = \mathbf{0}$ and $\mathrm{Div}\,\mathrm{Rot}_{\mathrm{s}} = 0$:

$$\tilde{\boldsymbol{\Delta}}_{\mathrm{v}}\,\mathrm{Grad} = \mathbf{0}, \tag{94}$$

$$\mathrm{Div}\,\tilde{\boldsymbol{\Delta}}_{\mathrm{v}} = 0. \tag{95}$$

The former is verified as

$$\tilde{\boldsymbol{\Delta}}_{\mathrm{v}}\,\mathrm{Grad}\, f = \tilde{\boldsymbol{\Delta}}_{\mathrm{v}} \begin{pmatrix} \vec{D}^X \circ f \\ \mathcal{M} f \\ 0 \end{pmatrix} = \begin{pmatrix} (\tilde{\triangle}^X_{\mathrm{v}} + \mathcal{M}^2)\vec{D}^X \circ f - \mathcal{M}\vec{D}^\phi \circ \mathcal{M} f \\ -\vec{D}^{\phi\dagger} \cdot \mathcal{M}\vec{D}^X \circ f + \vec{D}^{\phi\dagger} \cdot \vec{D}^\phi \circ \mathcal{M} f \\ -\frac{1}{|\beta_\phi|^2}\mathcal{M}\vec{J}^\dagger \vec{D}^X \circ f + \frac{1}{|\beta_\phi|^2}\vec{J}^\dagger \vec{D}^\phi \circ \mathcal{M} f \end{pmatrix} = \mathbf{0}, \tag{96}$$

where we used the identity $\tilde{\triangle}^X_{\rm v}\vec{D}^X\circ = \vec{D}^{X\dagger}\otimes(\vec{D}^X\times\vec{D}^X\circ) = 0$, which is similar to Eq. (49) for $\vec{D}^X$ and Eq. (88). The latter can also be verified as

$$\begin{aligned}
\mathrm{Div}\,\tilde{\boldsymbol{\Delta}}_{\rm v}\boldsymbol{f} &= \mathrm{Div}\begin{pmatrix} (\tilde{\triangle}^X_{\rm v} + \mathcal{M}^2)\vec{f} - \mathcal{M}\vec{D}_\phi\circ f_6 - \frac{1}{|\beta_\phi|^2}\mathcal{M}\vec{J}f_7 \\ \vec{D}^\dagger_\phi\cdot\mathcal{M}\vec{f} + \triangle^\phi f_6 + \vec{D}^{\phi\dagger}\cdot\vec{J}\frac{1}{|\beta_\phi|^2}f_7 \\ \frac{1}{|\beta_\phi|^2}\mathcal{M}\vec{J}^\dagger\vec{f} + \frac{1}{|\beta_\phi|^2}\vec{J}^\dagger\vec{D}^\phi\circ\ f_6 + (\beta_\phi\vec{D}^{\phi\dagger}\cdot\frac{1}{|\beta_\phi|^2}\vec{D}^\phi\circ\beta^*_\phi + U)f_7\end{pmatrix} \\
&= \vec{D}^{X\dagger}\cdot\left((\tilde{\triangle}^X_{\rm v} + \mathcal{M}^2)\vec{f} - \mathcal{M}\vec{D}_\phi\circ f_6 - \frac{1}{|\beta_\phi|^2}\mathcal{M}\vec{J}f_7\right) \\
&\quad + \mathcal{M}\left(-\vec{D}^\dagger_\phi\cdot\mathcal{M}\vec{f} + \triangle^\phi f_6 + \vec{D}^{\phi\dagger}\cdot\vec{J}\frac{1}{|\beta_\phi|^2}f_7\right) \\
&= 0,
\end{aligned} \tag{97}$$

where we used the identity $\vec{D}^{X\dagger}\cdot\tilde{\triangle}^X_{\rm v} = (\vec{D}^{X\dagger}\cdot\vec{D}^{X\dagger}\otimes)\vec{D}^X\times = 0$, similar to Eq. (50) for $\vec{D}^X$ and Eq. (88).

Since we did not succeed in defining appropriate rotation operators in the case of broken generators, we cannot introduce a Helmholtz-like decomposition for the four-vector $\boldsymbol{X}$. Instead, we introduce a projection operator

$$\boldsymbol{P} = \mathrm{Grad}\,\boldsymbol{\Delta}^{-1}\,\mathrm{Div}. \tag{98}$$

This satisfies $\boldsymbol{P}^2 = \boldsymbol{P}$, and is well defined because $\boldsymbol{\Delta}$ is positive definite. We decompose $\boldsymbol{X}$ as

$$\boldsymbol{X} = \boldsymbol{P}\boldsymbol{X} + (1-\boldsymbol{P})\boldsymbol{X}. \tag{99}$$

The first term is "rotation-free" because

$$\boldsymbol{P}\boldsymbol{X} = \mathrm{Grad}\,Y, \qquad \boldsymbol{\Delta}Y = \mathrm{Div}\,\boldsymbol{X}. \tag{100}$$

The second term is divergence-free because

$$\mathrm{Div}\,(1-\boldsymbol{P})\boldsymbol{X} = \mathrm{Div}\,\boldsymbol{X} - \mathrm{Div}\,\big(\mathrm{Grad}\,\boldsymbol{\Delta}^{-1}\,\mathrm{Div}\big)\,\boldsymbol{X} = 0. \tag{101}$$

The Hermitian conjugate of this is $(1-\boldsymbol{P})\,\mathrm{Grad} = 0$.

Now, we are ready to rewrite the Lagrangians in Eqs. (29), (33), and (35) in more compact forms. Let us first rewrite Eqs. (29) and (33):

$$\begin{aligned}
\mathcal{L}_X + \mathcal{L}_\phi = \mathrm{Tr}\Big[&X^\dagger_\mu\,(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu + \eta^{\mu\nu}\boldsymbol{\Delta})\,X_\nu \\
&- 2\big(\partial^\mu X^\dagger_\mu\big)\,\mathrm{Div}\,\boldsymbol{X} - \boldsymbol{X}^\dagger\big(\Box + \tilde{\boldsymbol{\Delta}}_{\rm v}\big)\boldsymbol{X}\Big].
\end{aligned} \tag{102}$$

Remarkably, the unification of $\vec{X}$ originated from the extra-dimensional component of $\mathcal{X}_M$, the NG boson $X_6 = \phi$, and the additional scalar X_7 into the four-component vector $\boldsymbol{X}$ is essential to get the quadratic Lagrangian in a compact form. Furthermore, Eq. (102) for X_μ and $\boldsymbol{X}$ is formally identical to Eq. (63) for the unbroken gauge field A_μ and $\vec{A}$.

Similarly, the gauge-fixing Lagrangian in Eq. (35) can be expressed as

$$\mathcal{L}^{(\rm gf)}_{X\phi} = -\frac{1}{\xi}\mathrm{Tr}\left[\,(\partial^\nu X_\nu - \xi\,\mathrm{Div}\,\boldsymbol{X})^\dagger\,\big(\partial^\mu X_\mu - \xi\,\mathrm{Div}\,\boldsymbol{X}\big)\,\right]. \tag{103}$$

This is a counterpart of Eq. (64).

Adding Eqs. (102) and (103), we have the quadratic Lagrangian

$$\begin{aligned}
\mathcal{L}_X + \mathcal{L}_\phi + \mathcal{L}^{(\rm gf)}_{X\phi} = \mathrm{Tr}\Big[&X^\dagger_\mu\left(\eta^{\mu\nu}\Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu + \eta^{\mu\nu}\boldsymbol{\Delta}\right)X_\nu \\
&- \boldsymbol{X}^\dagger\big(\Box + \tilde{\boldsymbol{\Delta}}_{\rm v} + \xi\,\mathrm{Grad\,Div}\big)\boldsymbol{X}\Big],
\end{aligned} \tag{104}$$

where we used

$$(\mathrm{Div}\,\boldsymbol{X})^\dagger\,(\mathrm{Div}\,\boldsymbol{X}) = \boldsymbol{X}^\dagger\,\mathrm{Grad}\,\mathrm{Div}\,\boldsymbol{X}. \tag{105}$$

This is clearly a counterpart of Eq. (65).

Thus, we have accomplished obtaining a much simpler formula compared to the initial one. But the most important point is that Eq. (104) for the broken gauge field with the NG fields is expressed in a form which is formally identical to the fashion found in Eqs. (65) for the unbroken gauge fields. Since we have developed a way to extract physical spectra for the latter, we just formally but partially repeat the same procedures.

Therefore, what we have to do next is to decompose $\boldsymbol{X}$ as $\boldsymbol{X} = \mathrm{Grad}\,Y + (1-\boldsymbol{P})\boldsymbol{X}$. Substituting this into Eq. (104) and using the identities in Eqs. (94) and (95), we get

$$\begin{aligned}\mathcal{L}_X + \mathcal{L}_\phi + \mathcal{L}^{(\mathrm{gf})}_{X\phi} = \mathrm{Tr}\Bigg[& X^\dagger_\mu\left(\eta^{\mu\nu}\Box - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu + \eta^{\mu\nu}\boldsymbol{\Delta}\right)X_\nu \\ & - ((1-\boldsymbol{P})\boldsymbol{X})^\dagger\,(\Box + \tilde{\boldsymbol{\Delta}}_{\mathrm{v}})(1-\boldsymbol{P})\boldsymbol{X} - Y\boldsymbol{\Delta}\,(\Box + \xi\,\boldsymbol{\Delta})\,Y\Bigg].\end{aligned} \tag{106}$$

The last treatment is making this canonical by redefining Y and $(1-\boldsymbol{P})\boldsymbol{X}$ by

$$y = \sqrt{\boldsymbol{\Delta}}\,Y, \qquad \boldsymbol{x} = (1-\boldsymbol{P})\boldsymbol{X}. \tag{107}$$

Substituting this into the above expression, we get the final form of the quadratic Lagrangian of the broken sector:

$$\begin{aligned}\mathcal{L}_X + \mathcal{L}_\phi + \mathcal{L}^{(\mathrm{gf})}_{X\phi} = \mathrm{Tr}\Bigg[& X^\dagger_\mu\left(\eta^{\mu\nu}\Box - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu + \eta^{\mu\nu}\boldsymbol{\Delta}\right)X_\nu \\ & - \boldsymbol{x}^\dagger(\Box + \tilde{\boldsymbol{\Delta}}_{\mathrm{v}})\boldsymbol{x} - y\,(\Box + \xi\,\boldsymbol{\Delta})\,y\Bigg].\end{aligned} \tag{108}$$

We have seen remarkable similarities between the unbroken and broken sectors. From now on, we will shed light on the differences between them. Firstly, the Laplacian $\boldsymbol{\Delta}$ given in Eq. (92) is positive definite. (This should be compared with $\triangle^\alpha$ in Eq. (69), which is only non-negative.) This leads to the important physical consequence that there are no massless modes in the four-dimensional component of the broken gauge field X_μ. This was, of course, expected from the beginning because X_μ corresponds to the gauge fields associated with the broken generators. What is not totally trivial here is to clarify which modes are eaten by X_μ. The answer is y. We are led to this conclusion by the fact that X_μ and y share the same mass operator $\boldsymbol{\Delta}$ except for the extra constant factor ξ for y. Again, we come across a generalized form of the R_ξ gauge (this should be compared to the relation between $\vec{A}_\mu$ and b in Eq. (69)). The appearance of ξ implies that y plays the role of an infinite tower of the NG modes which are absorbed by the infinite tower of X_μ including the bottom of the tower. Remembering the definition given in Eq. (100), the source for y is $\mathrm{Div}\,\boldsymbol{X} = \vec{D}^{X\dagger}\cdot\vec{X} + \mathcal{M}\phi$. Therefore, the broken gauge field eats not pure NG modes ϕ but a mixture of the rotation-free part $\vec{D}^{X\dagger}\cdot\vec{X}$ and the NG modes ϕ. Specifying the lowest mass of $\boldsymbol{\Delta}$ is in general difficult except for the very special case where $\mathcal{M}$ is a constant. If $\mathcal{M}$ is a constant, $\boldsymbol{\Delta} = \triangle^X + \mathcal{M}^2$ and $\triangle^X$ differ by just a constant, $\mathcal{M}^2$. We know that the zero mode of the non-negative-definite operator $\triangle^X$ is given by β_X. Therefore, the lowest mass is the constant $\mathcal{M}$ itself. However, in general $\mathcal{M}$ is not constant, so we are only sure that there is a finite mass gap but the explicit spectrum of $\boldsymbol{\Delta}$ depends on $\mathcal{B}$.

Our final comment is on the divergence-free component $\boldsymbol{x}$. As we mentioned above, we were not able to factorize the Laplacian $\tilde{\boldsymbol{\Delta}}_{\mathrm{v}}$ as a product of rotation operators. Remembering the unbroken part where we decomposed $\vec{A}$ with a Helmholtz-like decomposition, and we succeeded in splitting the two-component vector into two scalars b and c by replacing $\tilde{\triangle}_{\mathrm{v}} = \mathrm{rot}_{\mathrm{s}}\,\mathrm{rot}_{\mathrm{v}}$ by $\tilde{\triangle}_{\mathrm{s}} = \mathrm{rot}_{\mathrm{v}}\,\mathrm{rot}_{\mathrm{s}}$ as in Eq. (69). Furthermore, this rewriting allows us to conclude that the divergence-free component c does not have the zero mode of the non-negative operator $\tilde{\triangle}_{\mathrm{s}}$. At present, unfortunately, we cannot make a similar statement for $\tilde{\boldsymbol{\Delta}}_{\mathrm{v}}$. We are sure that there are no negative eigenvalues, but we cannot exclude a zero eigenvalue. This is an important open problem. But if we could do, it would not be so useful in the following sense. Recall that $\tilde{\boldsymbol{\Delta}}_{\mathrm{v}}$ acts on the four-component vector $(1-\boldsymbol{P})\boldsymbol{X}$. Precisely speaking, since the divergence part is projected out, the degrees of freedom of the four-component vector $(1-\boldsymbol{P})\boldsymbol{X}$ is not four but three. So, if we succeeded in dualizing $\tilde{\boldsymbol{\Delta}}_{\mathrm{v}}$, we are still left with the three-component vector. It is a complicated problem to find the spectrum of the 4×4 matrix operator $\tilde{\boldsymbol{\Delta}}_{\mathrm{v}}$ given in Eq. (93), since it highly depends on the model details. We leave the spectrum of this operator as a future problem.

3.4 *A typical example*

Let us illustrate our models more explicitly by a concrete choice of $\mathcal{B}$ as an example. One of the simplest choices of $\mathcal{B}$ is

$$\mathcal{B}\mathcal{B}^\dagger = \left(v^2\mathbf{1}_5 - T^\dagger T\right)^2. \tag{109}$$

Note that this is just a single possibility. One can consider, say, $\mathcal{B}\mathcal{B}^\dagger = \left(v^2\mathbf{1}_5 - T^\dagger T\right)^s$ with $s \geq 2$. The crucial thing is whether the condition in Eq. (75) is satisfied or not. We will verify that for the above simplest choice it is. T is from the 3–2 splitting vortex background solution, see also Eq. (15),

$$T = \begin{pmatrix} \tau_3\mathbf{1}_3 & \\ & \tau_2\mathbf{1}_2 \end{pmatrix}, \qquad \tau_2 = vf_2(r_2)e^{i\theta_2}, \qquad \tau_3 = vf_3(r_3)e^{i\theta_3}. \tag{110}$$

For simplicity, we consider the minimal winding number, namely each diagonal component has a single winding number. Here, (r_2, θ_2) and (r_3, θ_3) are two-dimensional polar coordinates defined as $x^4 - a_2 + i(x^5 - b_2) = r_2e^{i\theta_2}$ and $x^4 - a_3 + i(x^5 - b_3) = r_3e^{i\theta_3}$, where (a_3, b_3) is the position of the vortex associated to $SU(3)$ and (a_2, b_2) is that of the vortex associated to $SU(2)$. We require the profile functions to satisfy the boundary conditions $f_{2,3}(0) = 0$ and $f_{2,3}(\infty) = 1$. Solutions of the vortex equation have only been obtained numerically, but we do not need precise solutions for f_2 and f_3 in order to understand the qualitative aspects of gauge field localization. Hence, we use the approximation

$$f_\alpha(r) = \frac{r_\alpha}{\sqrt{1+r_\alpha^2}} \qquad (\alpha = 2, 3). \tag{111}$$

Note that these satisfy not only the correct boundary condition but also exhibit good asymptotic behavior, $f_{2,3} \to r_{2,3}$ $(r_{2,3} \to 0)$ and $f_{2,3} \to 1 - 1/(2r_{2,3}^2)$ $(r_{2,3} \to \infty)$, as a global vortex. Then, we have

$$|\beta_\alpha(r)| = v^2\left(1 - f_\alpha(r_\alpha)^2\right) = \frac{v^2}{1+r_\alpha^2} \qquad (\alpha = 2, 3), \tag{112}$$

which satisfy the square-integrability condition in Eq. (75) because their asymptotic behaviors are $|\beta_{2,3}| \to v^2/r^2$ as $r_{2,3} \sim r \to \infty$.[10] Now the corresponding $\triangle^\alpha$ for $\alpha = 2, 3$ are given by

[10] One might naively expect that the unbroken $H = SU(3)\times SU(2)\times U(1)$ gauge fields remain massless everywhere, for the scalar field profile of the global vortex only reduces as a power law. However, this is not correct. Localization of the H gauge fields does not depend on the scalar profile itself but the gauge

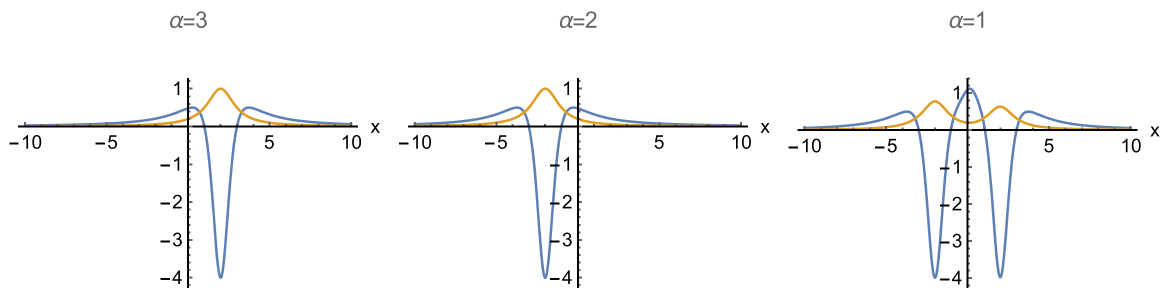

Fig. 3. $\mathcal{V}_\alpha$ and the zero mode β_α for $\alpha = 3, 2, 1$. We set $(a_3, b_3) = (2, 0)$ and $(a_2, b_2) = (-2, 0)$. The figures show cross sections on the x^4 axis ($x^5 = 0$). The left-most panel shows the gluon G_μ, the middle panel the weak boson W_μ, and the right-most panel the hyper $U(1)$ Y_μ.

$$\Delta^\alpha = \sum_{a=4,5}\left(-\partial_a^2 + \frac{\partial_a^2|\beta_\alpha|}{|\beta_\alpha|}\right) = -\nabla^2 + \mathcal{V}_\alpha(r), \quad \mathcal{V}_\alpha(r) = \frac{4(r_\alpha^2 - 1)}{(1 + r_\alpha^2)^2}, \tag{113}$$

$$\tilde{\Delta}_s^\alpha = \sum_{a=4,5}\left(-\partial_a^2 + \frac{\partial_a^2|\beta_\alpha|^{-1}}{|\beta_\alpha|^{-1}}\right) = -\nabla^2 + \tilde{\mathcal{V}}_\alpha(r), \quad \tilde{\mathcal{V}}_\alpha(r) = \frac{4}{1 + r_\alpha^2}. \tag{114}$$

The eigenvalue problems for these operators are mere two-dimensional Schrödinger equations with the axially symmetric potentials $\mathcal{V}_\alpha$ and $\tilde{\mathcal{V}}_\alpha$. The normalizable zero mode of $\mathcal{V}_\alpha$ is $\beta_\alpha(r)$ itself and it is the only discrete spectrum[11] of Δ^α. On the other hand, there is no discrete spectrum such as normalizable zero modes for $\tilde{\Delta}^\alpha$ since $\tilde{\mathcal{V}}_\alpha$ is a positive convex function. This means that massless four-dimensional gauge fields G_μ and W_μ exist only in the $SU(3)$ and $SU(2)$ gauge groups. For this particular choice of $\mathcal{B}$, massive localized modes also exist in the four-dimensional vector component in the adjoint representation of the $SU(3)$ and $SU(2)$ gauge groups. However, together with the continuum spectra in both four-dimensional and extra-dimensional components, all these massive modes have an energy gap of the order of the GUT scale.

The analysis for the $U(1)$ part runs almost parallel to those above. Firstly, we have, from Eq. (25),

$$\beta_1(r) = v^2\sqrt{\frac{2}{5(1 + r_3^2)^4} + \frac{3}{5(1 + r_2^2)^4}}. \tag{115}$$

This again leads to Schrödinger problems with the potentials

$$\mathcal{V}_{\alpha=1} = \frac{\partial_a^2\beta_1}{\beta_1}, \qquad \tilde{\mathcal{V}}_{\alpha=1} = \frac{\partial_a^2\beta_1^{-1}}{\beta_1^{-1}}. \tag{116}$$

Instead of showing the form, we just display $\mathcal{V}_\alpha$ and its zero mode β_α in Fig. 3.

Reflecting the fact that β_1 is a weighted average of $|\beta_2|$ and $|\beta_3|$, the corresponding potential $\mathcal{V}_1$ has two attractive valleys around the vortices at $(x^4, x^5) = (\pm 2, 0)$. Hence the zero mode wave function of Y_μ is concentrated around both the vortices associated to $SU(3)$ and $SU(2)$. This results in an interesting difference between the wave functions of Y_μ compared to G_μ and W_μ.

kinetic function β ($\mathcal{B}$). Even though our background solution is the global vortex, it can localize the massless non-Abelian gauge fields by the square-integrable β. Of course, if one considers a local vortex instead of the global one, satisfying the square-integrability condition becomes much easier, so that range of possibilites for $\mathcal{B}$ would become larger. We will investigate the case of a local vortex elsewhere.

[11] Of course, this is merely a property of the specific choice of $\mathcal{B}^2$ in Eq. (109), and not a generic property for other possible choices.

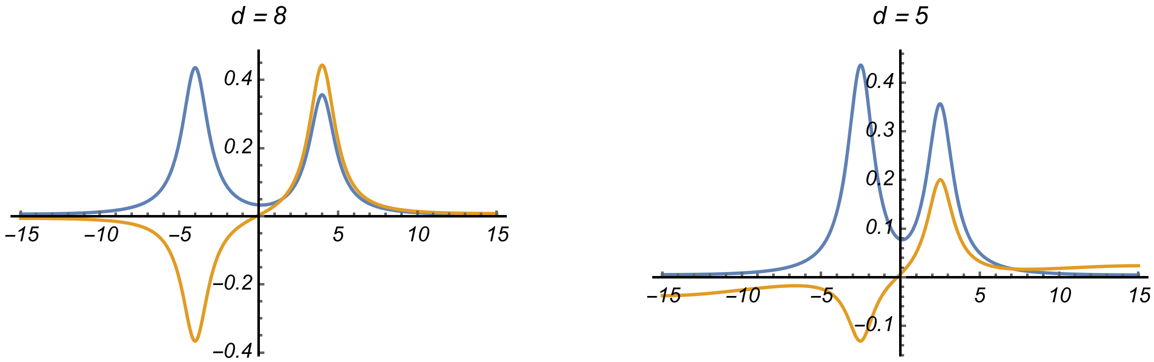

Fig. 4. The persistent zero mode β_1 (blue) and the lifted zero mode (orange). The positions of the 3- and 2-vortices are set to be $(a_3, b_3) = (d/2, 0)$ and $(a_2, b_2) = (-d/2, 0)$; the left figure is with $d = 8$ and the right one is with $d = 5$. The lifted zero mode in the left graph is normalizable while the one in the right graph is non-normalizable.

The spectra of the gluons and W bosons do not depend on the separation between vortices. The gluon (W boson) has a single massless mode localized around the vortex associated to $SU(3)$ ($SU(2)$). On the other hand, the spectrum of Y_μ depends on the vortex separations, though the single zero mode always exists. Suppose that the vortices associated to $SU(3)$ and $SU(2)$ are infinitely separated; then, two potential wells are also infinitely separated. Each isolated well has a localized zero mode. Hence, the whole spectrum should include two massless modes for the $U(1)$ gauge field. If, however, we bring the vortices together again, the degenerate zero modes now split (level repulsion) by a quantum tunneling effect between the two wells. The lowest one remains massless, while the other one is lifted by an exponentially suppressed nonperturbative tunneling effect. Note also that if the vortices associated to $SU(3)$ and $SU(2)$ are completely coincident, the $SU(5)$ is unbroken and β_1 becomes identical to β_3 and β_2. As we saw above, β_3 (β_2) admits only a single discrete spectrum which is massless, so is β_1 at the coincident limit. Therefore, the lifted zero mode that is normalizable for a well-separated 3–2 splitting background should enter into continuum spectrum at a particular value of the separation. Concrete examples are given in Fig. 4. Hence, an extra massive mode can exist in Y_μ, unlike in G_μ and W_μ. This extra massive vector particle

could be evidence for the underlying background of vortices. It may be observable if the 3–2 split configuration with a large separation of the vortices is realized as the stabilized configuration.

To see the mass spectra of the broken sector, we next need the other βs from Eq. (25),

$$\beta_X = v^2\sqrt{\frac{1}{2(1+r_3^2)^4} + \frac{1}{2(1+r_2^2)^4}}, \qquad \beta_\phi = v\left(\frac{r_3 e^{i\theta_3}}{\sqrt{1+r_3^2}} - \frac{r_2 e^{i\theta_2}}{\sqrt{1+r_2^2}}\right). \tag{117}$$

We can compute the Laplacian $\mathbf{\Delta}$ for the four-dimensional component X_μ,

$$\mathbf{\Delta} = \triangle^X + \mathcal{M}^2 = \sum_{a=4,5}\left(-\partial_a^2 + \frac{\partial_a^2\beta_X}{\beta_X}\right) + \mathcal{M}^2, \qquad \mathcal{M}^2 = \frac{|\beta_\phi|^2}{\beta_X^2}. \tag{118}$$

This is also a bit complicated, so we do not show the explicit expression. If we omit $\mathcal{M}$, β_X is quite similar to β_1 except for the weights for taking the average. Then, there exists a single normalizable zero mode which is proportional to β_X. However, this zero mode is completely swept out by the additional term $\mathcal{M}^2$ which dominates the potential. Figure 5 shows a concrete

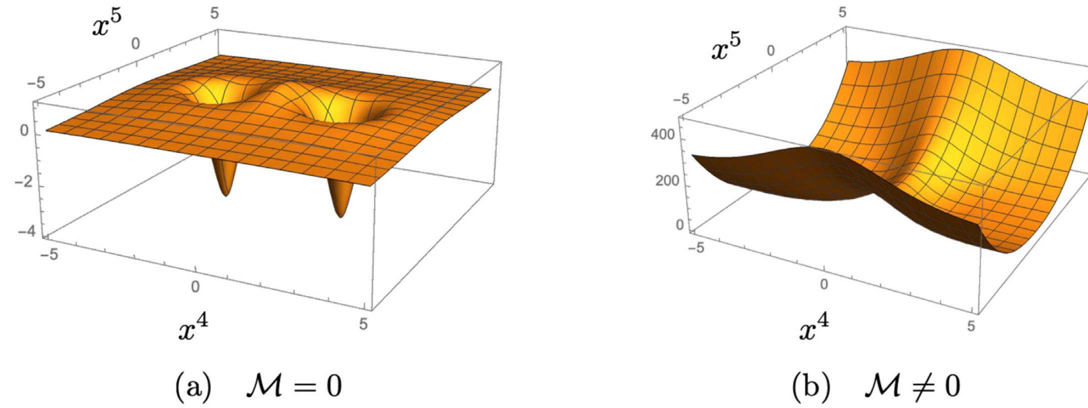

Fig. 5. The potential $V_X = \frac{\partial_a^2 \beta_X}{\beta_X} + \mathcal{M}^2$. Panel (a) shows V_X without $\mathcal{M}^2$, and panel (b) is with $\mathcal{M}^2$.

example of the potential with and without $\mathcal{M}^2$. The vortex potential wells are clearly visible for $\mathcal{M}^2 = 0$, but it is almost washed out for $\mathcal{M}^2 \neq 0$. Although we do not find eigenvalues and eigenfunctions exactly, we are sure that there are no massless modes and that a mass gap of the order of the GUT scale exists when $\mathcal{M}^2$ is of the order of the GUT scale. Showing the absence of the additional massless vector field rigorously is an important problem from the phenomenological viewpoint and is left as future work.

4. Conclusions and discussion

In this work we have examined issues associated with the SM gauge fields, gluons, weak bosons, and hypercharge $U(1)$ gauge fields, specifically their localization on the non-Abelian vortices as 3-branes in six-dimensional spacetime. Our six-dimensional Lagrangian has $SU(5)$ gauge symmetry, so the model has an aspect of $SU(5)$ GUT in addition to a fat brane-world scenario. The model also has additional global $U(1)$ symmetry, which is broken spontaneously in the bulk and gives rise to topologically stable vortices of the non-Abelian kind. On the other hand, the $SU(5)$ gauge symmetry is unbroken in the bulk, but breaks only *locally* near the cores of vortices. Hence, to which subgroup the $SU(5)$ gauge symmetry breaks depends on how many vortices are generated and where they are located.

In Sect. 2 we studied two specific models which exhibit the desired symmetry breaking $SU(5) \to SU(3) \times SU(2) \times U(1)$ by the non-Abelian vortices. The first model, in Sect. 2.1, admits the embedding of the usual $U(1)$ global vortices into diagonal elements of the 5×5 matrix field T. The embedded vortex is a $\frac{1}{5}$ fractionally quantized non-Abelian vortex. There are five different species of the fractional non-Abelian vortex associated with each diagonal element of T. The desired configuration can be constructed by putting a vortex with a common vorticity k_3 in the first three entries and another vortex with a common vorticity k_2 in the remaining two entries of the diagonal elements of T given as $(k_3, k_3, k_3, k_2, k_2)$. However, the positions (moduli) of vortices can be freely changed in this first model, leading to unwanted patterns of symmetry breaking such as $SU(5) \to U(1)^4$. We need to remove the zero modes of such a deformation. This moduli stabilization problem was solved in the second model by adding simple deformation potentials in Sect. 2.2. The key point is to deal with the traceless part $\hat{T}$ (adjoint representation) of T separately from the trace part T_0, since the vortex separation corresponds to nonvanishing adjoint fields. Our simple potential induces a domain wall attached to the fractional non-Abelian vortex. The domain walls glue the fractional non-Abelian vortices, and we found that the fractional non-Abelian vortices are confined to form an $SU(5)$ singlet. This dynamical process aligns the vorticity, so that a stable configuration has to have identical vorticities in all diagonal entries of T. Thus, we found that the domain walls tend to fully confine the fractional non-Abelian vortices, but this is not our desired solution since the $SU(5)$ gauge symmetry is unbroken. This point is resolved by a short-range repulsive interaction between the vortices. Competition between the long-range attraction by the domain wall and the short-range repulsion results in a singlet combination clustering into a 3–2 splitting molecule. Namely, the five vortices with identical vorticity split into a molecule configuration with three vortices at a point (associated to $SU(3)$) and two vortices (associated to $SU(2)$) at another point separated by a small distance. This is phenomenologically important once fermions are introduced, though we did not explicitly deal with them in this paper. The fact that there are identical vorticities associated to $SU(3)$ and to $SU(2)$ ensures the same number of fermion zero modes in the fundamental representation of $SU(3)$ (quark) and of $SU(2)$ (lepton) are localized.

In Sect. 3 we turned to clarifying the localization of the SM gauge fields on a 3–2 splitting background. We performed a standard fluctuation study by introducing small fluctuations of all the fields and expanded the Lagrangian to the quadratic order of fluctuations. We can then extract physical information such as the mass spectra of fluctuations. The procedure has a number of complications:

- The background configuration is an inhomogeneous vortex background, which is not axially symmetric because of the 3–2 splitting molecule.
- The $SU(5)$ gauge symmetry is locally broken to $SU(3) \times SU(2) \times U(1)$, and we need to treat the gauge fields in the unbroken and broken sectors differently.
- We need a gauge fixing suitable for the vortex molecule background.
- We need to treat four-dimensional and extra-dimensional components of the gauge fields differently.

We succeeded in showing that the massless gauge fields corresponding to the SM gauge group are localized on the non-Abelian vortex molecule, thanks to the field-dependent gauge kinetic function $\mathcal{B}\mathcal{B}^\dagger$ in Eq. (2). In order to perform the fluctuation analysis efficiently, we developed a vector-analysis-like method using derivative operators $\vec{D}$ of the two extra dimensions (x^4, x^5) in Eq. (38). The method helps not only to make the expressions compact, but also to distinguish physical and unphysical modes. The Helmholtz-like theorem turned out to be important for decomposing the extra-dimensional gauge field $\vec{A}$ of the unbroken sector into rotation-free and divergence-free parts. We found that the rotation-free part of $\vec{A}$ (b in Eq. (69)) is unphysical, playing the role of NG bosons for the KK towers (except for the bottom) of the four-dimensional gauge fields A_μ (the SM gauge fields: gluons, W bosons, and hypercharge gauge fields). On the other hand, the divergence-free part of $\vec{A}$ (c in Eq. (69)) is physical but massless modes are absent. In short, the vortex effective theory contains the SM gauge fields as the only massless modes, and infinite towers of massive KK modes. The extra-dimensional components (A_4 and A_5) provide only one (five-dimensional) scalar field degree of freedom whose spectrum does not have massless modes. We applied the same kind of technique to the unbroken sector, although the analysis becomes more complicated because of the mixing with the T field. Although the analogy with three-dimensional vector analysis is not as complete as in the unbroken sector, we still found the vector-analysis-like method useful. Similarly to the unbroken sector, the "rotation-free" component of $\boldsymbol{X}$ is found to be unphysical because it is absorbed by X_μ. The important difference from the unbroken sector is that *all* the KK modes of X_μ become massive, so that there are no massless vector modes in the unbroken sector. The remaining three degrees of freedom, the divergence-free components, of $\boldsymbol{X}$ were only partially understood. We gave a formal expression for the 4×4 Laplacian $\tilde{\boldsymbol{\Delta}}_{\rm v}$ in Eq. (93), whose eigenvalues depend on the model details. We also demonstrated the low-lying mass spectrum in a concrete model and pointed out the possibility that an exotic heavy (but not too heavy) vector field in the KK tower of the $U(1)$ hypercharge vector field may be observed.

Thus, we provided a class of new models suitable for the gauge sector of the fat brane-world scenario with GUT in six spacetime dimensions by non-Abelian vortices. As future work, we first need to include fermions to complete our mission. As usual in $SU(5)$ models, it is natural to have **5** and **10** representations of $SU(5)$ as fermions. The most important advantage of using the vortices as the host 3-branes is that the six-dimensional theory gives the same number of fermion zero modes for the fundamental representation of $SU(3)$ (quarks) and $SU(2)$ (leptons) forming generations. Namely, the number of fermion generations is identical to the vorticity (the topological number) of the background solution. Second, we also need to include a Higgs field which breaks the electroweak symmetry. There is a longstanding issue known as the doublet–triplet splitting problem in $SU(5)$ GUT models. Whether the 3–2 split non-Abelian vortices configuration can account for the doublet–triplet problem without fine tunings is a challenging future problem. Other phenomenologically challenging issues include the hierarchy in quark/lepton masses and mixing matrices, possible right-handed neutrinos, and proton decay.

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A. Generalized Helmholtz decomposition

Here we give a proof that a vector field $\vec{A}$ can always be decomposed into rotation- and divergence-free components as in Eq. (58). Firstly, we define a projection operator which projects out $\vec{A}$ onto the rotation-free component,

$$P = \vec{D} \circ \left(\vec{D}^\dagger \cdot \vec{D} \circ\right)^{-1} \vec{D}^\dagger \cdot = \text{ grad} \,\triangle^{-1} \,\text{div}. \tag{A1}$$

This is a projection operator since it satisfies $P^2 = P$:

$$P^2 = \left(\vec{D} \circ \left(\vec{D}^\dagger \cdot \vec{D} \circ\right)^{-1} \vec{D}^\dagger \cdot\right)^2 = P. \tag{A2}$$

Therefore, we can always decompose the vector $\vec{A} = P\vec{A} + (1 - P)\vec{A}$. However, P is well defined only when the inverse Laplacian $\triangle^{-1}$ is well defined. A dangerous case occurs if $\triangle^{-1}$ acts on its zero mode $\triangle B_0 = 0$ (grad $B_0 = 0$) as is given in Eq. (70). However, this is not the case here because $\triangle^{-1}$ acts right after the divergence div. Let $\vec{E}$ be an arbitrary two-component vector, and take an inner product as

$$\left(B_0, \operatorname{div} \vec{E}\right) = \left(\operatorname{grad} B_0, \vec{E}\right) = 0, \tag{A3}$$

where we defined the inner product by $(A, B) = \int dx^4 dx^5\, A^* B$ and $\left(\vec{A}, \vec{B}\right) = \int dx^4 dx^5\, \vec{A}^\dagger \vec{B}$. Hence, we conclude that $\operatorname{div} \vec{E}$ does not include B_0, and therefore $\triangle^{-1} \operatorname{div}$ is always well defined, and so is P.

Next, we show that $P\vec{A}$ is rotation free. This is trivial by definition because

$$P\vec{A} = \operatorname{grad}\left(\triangle^{-1} \operatorname{div} \vec{A}\right), \tag{A4}$$

and from Eq. (49) we always have $\mathrm{rot_v}\, P\vec{A} = 0$.

Let us next consider $(1-P)\vec{A}$. We expand it by eigenfunctions of the non-negative-definite Hermitian operator $\tilde{\triangle}_\mathrm{v} = \mathrm{rot_s}\, \mathrm{rot_v}$:

$$\tilde{\triangle}_\mathrm{v} \vec{C}_n = \mathrm{rot_s}\, \mathrm{rot_v}\, \vec{C}_n = \tilde{m}_n^2 \vec{C}_n, \tag{A5}$$

$$(1-P)\vec{A} = \sum_{n \geq 0} d_n \vec{C}_n. \tag{A6}$$

Note that the index n of the eigenvalues starts at $n = 0$, which corresponds to the lowest eigenvalue. Indeed, the lowest eigenvalue is $\tilde{m}_0 = 0$ and the corresponding eigenfunction is

$$\mathrm{rot_v}\, \vec{C}_0 = 0 \quad \to \quad \vec{C}_0 \propto \operatorname{grad} \xi, \tag{A7}$$

with ξ being an arbitrary function. We now show that d_0 is zero. This is because

$$\begin{aligned} \left((1-P)\vec{A}, \vec{C}_0\right) &\propto \left((1-P)\vec{A},\ \operatorname{grad} \xi\right) \\ &= \left(\operatorname{div}(1-P)\vec{A}, \xi\right) \\ &= \left(\operatorname{div} \vec{A} - \operatorname{div} \operatorname{grad} \triangle^{-1} \operatorname{div} \vec{A}, \xi\right) = 0. \end{aligned} \tag{A8}$$

Therefore, the above expansion is modified as

$$(1-P)\vec{A} = \sum_{n > 0} d_n \vec{C}_n. \tag{A9}$$

Note also that all the positive eigenvalues $\tilde{m}_n^2$ $(n > 0)$ of $\tilde{\triangle}_\mathrm{v}$ correspond one-to-one to those of the $\tilde{\triangle}_\mathrm{s}$ defined in Eq. (70). The actual correspondence is given by

$$\vec{C}_n \propto \mathrm{rot_s}\, C_n, \qquad C_n \propto \mathrm{rot_v}\, \vec{C}_n. \tag{A10}$$

This can be verified as follows for $n > 0$:

$$\tilde{\triangle}_\mathrm{v} (\mathrm{rot_s}\, C_n) = (\mathrm{rot_s}\, \mathrm{rot_v})\, \mathrm{rot_s}\, C_n = \mathrm{rot_s}\, \tilde{\triangle}_\mathrm{s} C_n = \tilde{m}_n^2 (\mathrm{rot_s}\, C_n). \tag{A11}$$

It is straightforward to show the other equation. Hence, we now arrive at the desired expression,

$$(1-P)\vec{A} = \mathrm{rot_s}\left(\sum_{n>0} d_n' \vec{C}_n\right), \tag{A12}$$

which implies that $(1-P)A$ is divergence free.

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