Noncommutative geometry, $D = 11$ supermembranes and the Seiberg-Witten map over the Weyl algebra bundle*

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We discuss an extension of the recent proposed noncommutative action dor $D = 11$ Supermembranes [1]. The new action depends only on the class of admissible symplectic connections introduced in Ref. 1 following Fedosov approach. The extended action is invariant under a new gauge symmetry relating symplectic connections on the Weyl algebra bundle.

Keywords: Noncommutative geometry; supermembranes; Weyl algebra.

Discutimos una extensión de la acción no conmutativa para la Supermembrana en D=11 recientemente propuesta en la Ref. 1. La nueva acción depende solamente de la clase admisible de conexiones simplécticas introducidas en la Ref. 1 siguiendo el procedimiento de Fedsov. La acción extendida es invariante respecto a una invariancia de calibre que relaciona conexiones simplécticas en el fibrado algebraico de Weyl.

Descriptores: Geometría no commutativa; supermembranas; algebra de Weyl.

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1. Introduction

The formulation of superstrings in 10 dimensions , supermembranes and super5 branes in 11 dimensions in terms of noncommutative geometry has been recently analized from different points of view, for a recent review see Ref. 2. Most of the work has been performed in the context of the interaction of open strings with constant antisymmetric background fields which are assumed to be much larger that the background metric. The precise limits in which the noncommutative geometry arises were discussed in Ref. 3.

The approach we followed in Ref. 1 was to emphasize the residual gauge symmetry of the $D = 11$ supermembrane in the light cone gauge, the area preserving diffeomorphisms. In two dimensions they are the same as the symplectomorphisms on the world volume . The idea was then to formulate a new action for the supermembrane in terms of a noncommutative symplectic geometry. The symplectic two-form arises naturally from the existence of a non trivial central charge on the supersymmetric algebra. In order to formulate the action in terms of a noncommutative geometry the fields describing the supermembrane action were lifted to Yang-Mills connections and associated gauge fields over the Weyl algebra bundle, where a general framework for the symplectic formulation of noncommutative geometry was introduced by Fedosov in Ref. 4. One of the results in Ref. 1 was the construction of a geometrical action over the Weyl bundle which reduces to the well known action of the $D = 11$ supermembrane in the light cone gauge when the geometrical objects are projected down from the Weyl algebra bundle to fields over the world volume and a formal parameter in the expansion of the elements of the Weyl algebra tends to zero.

In this paper we first briefly review Fedosov's general framework and the introduction of Yang Mills connections over the Weyl Bundle. We then propose an extension of the action in Ref. 1 which becomes independent of the symplectic connections introduced by Fedosov.

To do so we introduce a map similar to the one used by Seiberg and Witten in Ref. 3 but on the Weyl algebra bundle. This new map has the advantage that it is globally defined over the world volume, it is not a local map as the Seiberg-Witten one. This is an important advantage to the construction we followed in Ref. 1 with the usual formulations. The current approach considers constant symplectic structure, constructed from the constant background antisymmetric field, and it is then valid only over a Darboux chart. The price paid in order to have a global cobnstruction is the introduction of a symplectic connection over the world volume that, in general, is not unique. There is a class of admissible symplectic connections which may be introduced. One would like then to have an action which depends only on the class of symplectic connections. This is the problem we address in this paper.

2. Yang Mills connections over the Weyl bundle

In Ref. 4 a formal Weyl algebra W_x corresponding to a symplectic space $T_x\Sigma$, where Σ is a symplectic manifold (Σ, ω) of dimension $2n$ and $T_x\Sigma$ is its tangent space at x, is an algebra whose elements are given by a formal series

$$a(y, h) = \sum_{k,p\geq 0} h^k a_{k,\mu_1\ldots\mu_p} y^{\mu_1} \ldots y^{\mu_p}, \tag{1}$$

where h is a formal parameter, $\mu_0 = 0$ and μ_p runs from 1 to $2n$ when $p \neq 0$. To order terms in the summation, we give the following degrees to variables: deg $y^\mu = 1$, deg $h = 2$

and we order by increasing degrees $2k+p$. The Weyl product of elements $a, b \in W_x$ is defined as

$$a \circ b = \sum_{k=0}^{\infty} (-\frac{ih}{2})^k \frac{1}{k!} \omega^{\mu_1 \nu_1} \dots \omega^{\mu_k \nu_k} \times \frac{\partial^k a}{\partial y^{\mu_1} \dots \partial y^{\mu_k}} \frac{\partial^k b}{\partial y^{\nu_1} \dots \partial y^{\nu_k}}. \quad (2)$$

This product is associative and independent of the basis in $T_x\Sigma$.The union of W_x defines the Weyl algebra bundle W. We will consider q-forms on Σ with values in W. A torsion free symplectic connection preserving the symplectic structure is introduced on the tangent bundle of Σ. This connection is then lifted to the Weyl bundle W by considering its action directly on the coefficients of the expansion 1, we denote it $\mathcal{D}_S$ as

$$\mathcal{D}_S a = dx^\rho \wedge D_\rho a, \quad (3)$$

where D_ρ denotes the symplectic connection on Σ. General covariant derivatives $\mathcal{D}$ on the bundle may be considered with one-form connections γ globally defined on Σ and with values in W,

$$\mathcal{D}a = \mathcal{D}_S a + \frac{i}{h}[\gamma, a] \quad (4)$$

The two-form

$$\Omega = R + \mathcal{D}_S \gamma + \frac{i}{2h}[\gamma, \gamma] \quad (5)$$

is the Weyl curvature of the connection $\mathcal{D}$. R is the curvature of the connection $\mathcal{D}_S$. The Weyl curvature satisfies the Bianchi identity

$$\mathcal{D}\Omega = \mathcal{D}_S \Omega + \frac{i}{h}[\gamma, \Omega] = 0; \quad (6)$$

moreover, for any section $a \in W \otimes \Lambda$,

$$\mathcal{D}^2 a = \frac{i}{h}[\Omega, a]. \quad (7)$$

In general, transitions on the bundle $T\Sigma$ will induce transitions on the algebra W.The infinitesimal gauge transformations on elements of the algebra are expressed as automorphisms given by

$$a \to a + [a, \lambda] \quad (8)$$

with ‘infinitesimal’ $\lambda \in W$. The corresponding gauge transformations for the connections $\mathcal{D}$ are

$$\mathcal{D} \to \mathcal{D} + \mathcal{D}\lambda, \quad (9)$$

implying

$$\mathcal{D}a \to \mathcal{D}a + [\mathcal{D}a, \lambda]. \quad (10)$$

Abelian connections $\mathcal{D}_A$ are connections $\mathcal{D}$ with Weyl curvature Ω being a central form of the algebra. Let us denote it Ω_A. It then satisfies

$$[\Omega_A, a] = 0 \quad (11)$$

for any section $a \in W$. Associated with $\mathcal{D}_A$ there is a subalgebra of W, denoted W_A, defined by

$$W_A = \left\{ a \in W : \mathcal{D}_A a = 0 \right\}. \quad (12)$$

There is a one to one correspondence between the C^∞ functions $a_0(x)$ over Σ and the elements of W_A. In fact, given $a \in W_A$, the projection is defined as

$$\sigma a := a(x, y = 0, h) = a_0(x), \quad (13)$$

and given $a_0(x)$ there is a unique element $a \in W_A$ with such projection. If a and $b \in W_A$, its Weyl product is projected to the globally defined $\star$-product

$$\sigma(a \circ b) = a_0 \star b_0. \quad (14)$$

In the particular case when $\omega_{\mu\nu}$ is constant and the symplectic connection is zero, the formula agrees with the Moyal product. In [4] we constructed the Yang Mills connection over the Weyl algebra bundle. Let Σ be a symplectic manifold with a symplectic two-form $\omega_{\mu\nu} dx^\mu \wedge dx^\nu$. In this section, we assume ω to be an arbitrary non-degenerate closed two-form over Σ. A set of multi-beins is defined by

$$\omega_{\mu\nu} = \varepsilon^i_\mu \varepsilon^j_\nu \epsilon_{ij}, \quad (15)$$

where ϵ_{ij} is the canonical symplectic tensor. Because of Darboux theorem, locally we always have

$$\varepsilon^i_\mu = \partial_\mu g^i. \quad (16)$$

We may consider an atlas where on each chart we have (16) . The transitions on g^i between different charts preserve the symplectic structure (15). The multi-bein ε^i_μ will then have transitions over Σ, otherwise, one would have a set of $2n$ non-singular vector fields globally defined over Σ, but this is not true in general.

Let us discuss the transitions on intersection of charts in more detail. Consider two open sets U and $\hat{U}$, $U \bigcap \hat{U} = \emptyset$ in which

$$\varepsilon^i_\mu = \partial_\mu g^i, \text{ and } \hat{\varepsilon}^i_\mu = \partial_\mu \hat{g}^i, \quad (17)$$

respectively. In $U \cap \hat{U}$ we then have

$$\omega_{\mu\nu} = \varepsilon^i_\mu \varepsilon^j_\nu \epsilon_{ij} = \hat{\varepsilon}^i_\mu \hat{\varepsilon}^j_\nu \epsilon_{ij}, \quad (18)$$

from which we obtain

$$\hat{\varepsilon}^i_\mu = S^i_j \varepsilon^j_\mu \text{ where } S^i_j = \epsilon^{ik} \hat{\varepsilon}^\mu_k \varepsilon^l_\mu \epsilon_{lj}; \quad (19)$$

we define the inverse of ϵ by $\epsilon^{ij}\epsilon_{jk} = \delta^i_k$. One may verify that S preserves the canonical symplectic tensor and hence $S \in Sp(2n)$. Consequently in order to have a global construction over Σ, one must begin by introducing a symplectic

$Sp(2n)$ connection on the tangent bundle. We first consider the following symplectic connection over Σ:

$$\Theta^{\lambda}_{\mu\nu} \equiv \zeta_{\mu\nu\rho}\omega^{\rho\lambda} + \frac{1}{3}\frac{\partial\omega_{\nu\rho}}{\partial x^{\mu}}\omega^{\rho\lambda} + \frac{1}{3}\frac{\partial\omega_{\mu\rho}}{\partial x^{\nu}}\omega^{\rho\lambda}, \qquad (20)$$

where $\zeta_{\mu\nu\rho}$ is a totally symmetric tensor. This is the most general expression for a connection satisfying

$$(\frac{\partial}{\partial x^{\mu}} + \Theta_{\mu})\omega_{\rho\lambda} = 0. \qquad (21)$$

Here $\Theta^{\lambda}_{\mu\nu}$ is expressed in terms of $\omega_{\mu\nu}$ and its derivatives. It is invariant under the $Sp(2n)$ transition of the multi-bein. We now consider the following torsion free connection on the tangent space:

$$\Gamma^{j}_{\mu i} = \varepsilon^{\nu}_{i}(\frac{\partial\varepsilon^{j}_{\nu}}{\partial x^{\mu}} - \Theta^{\lambda}_{\mu\nu}\varepsilon^{j}_{\lambda}); \qquad (22)$$

it transforms as a $Sp(2n)$ connection under $Sp(2n)$ transformations on the tangent space. In fact,

$$\hat{\Gamma}^{j}_{\mu i} = (S^{-1})^{l}_{i}\Gamma^{k}_{\mu l}S^{j}_{k} - (S^{-1})^{k}_{i}\frac{\partial S^{j}_{k}}{\partial x^{\mu}}. \qquad (23)$$

This connection is symplectic on the tangent space. We may construct from it the most general symplectic connection on the tangent space in the following way. Let us denote

$$\breve{D}_{\mu} \equiv \partial_{\mu} + \Gamma_{\mu} \qquad (24)$$

a symplectic connection must satisfy

$$(\breve{D}_{\mu} + \Delta\Gamma_{\mu})\epsilon_{ij} = 0. \qquad (25)$$

This equation has the general solution

$$\Delta\Gamma^{j}_{\mu i} = \frac{1}{3}(\breve{D}_{\mu}\epsilon_{il})\epsilon^{lj} + \frac{1}{3}\varepsilon^{k}_{\mu}\varepsilon^{\nu}_{i}(\breve{D}_{\nu}\epsilon_{kl})\epsilon^{lj} + \varepsilon^{k}_{\mu}\tilde{\zeta}_{(ilk)}\epsilon^{lj}; \qquad (26)$$

$\Delta\Gamma^{j}_{\mu i}$ is a covariant vector on the world volume and a tensor under $Sp(2n)$ transformations. Since the connection (24) is symplectic the first two terms of the right hand side member in (26) are zero. We may finally construct our symplectic connection D , when acting on mixed indices vectors V^{i}_{ν} it yields

$$D_{\mu}V^{i}_{\nu} = \frac{\partial V^{i}_{\nu}}{\partial x^{\mu}} + (\Gamma_{\mu} + \Delta\Gamma_{\mu})^{i}_{l}V^{l}_{\nu} - \Theta^{\lambda}_{\mu\nu}V^{i}_{\lambda}, \qquad (27)$$

it satisfies

$$D_{\mu}\omega_{\rho\lambda} = 0, \quad D_{\mu}\epsilon_{ij} = 0, \qquad (28)$$

and it has the right transformation law on the world volume and in the tangent space.

The final form of the Yang Mills connection is

$$\mathcal{D} = \frac{i}{h}[G_i e^i, \bullet] + \frac{i}{h}[\gamma, \bullet], \qquad (29)$$

where G_i obeys the following equations:

$$\mathcal{D}_A G_i = 0, \ \sigma G_i = \epsilon_{ij}g^{j}(x), \qquad (30)$$

where $g^{j}(x)$ is defined in (16) and γ_i also obeys

$$\mathcal{D}_A\gamma_i = 0. \qquad (31)$$

The curvature of the connection $\mathcal{D}$ is then given by

$$\Omega = \frac{i}{2h}[G,G] + \frac{i}{h}[G,\gamma] + \frac{i}{2h}[\gamma,\gamma], \qquad (32)$$

it satisfies the Bianchi identity:

$$\mathcal{D}\Omega = 0. \qquad (33)$$

this property follows from the Jacobi identity for the bracket. The first term in (32) reduces in the flat limit to

$$\frac{i}{2h}[G,G] = -\frac{1}{2}e^{i}\wedge e^{j}\epsilon_{ij} = -\omega \qquad (34)$$

The projection of Ω has in general the expression

$$\begin{aligned}\sigma\Omega = -\omega + \mathcal{F} - \frac{h^2}{96}\left(R_{jkli}(D_{\hat{j}}D_{\hat{k}}D_{\hat{l}}\mathcal{A}_m)\right) \\ - \left(\frac{1}{4}R_{\hat{j}\hat{k}\hat{l}p}\epsilon^{pq}D_q\mathcal{A}_m\right)\epsilon^{j\hat{j}}\epsilon^{k\hat{k}}\epsilon^{l\hat{l}}e^{i}\wedge e^{m} \\ - \frac{h^2}{96\cdot 8}R_{jkli}R_{\hat{j}\hat{k}\hat{l}m}\epsilon^{j\hat{j}}\epsilon^{k\hat{k}}\epsilon^{l\hat{l}}e^{i}\wedge e^{m} \\ + O(h^3)\ldots,\end{aligned} \qquad (35)$$

where the curvature is constructed from the $Sp(2n)$ symplectic connection (27), the remaining terms are higher order in h and depend also on the derivatives of the curvature. The curvature $\mathcal{F}$ is the Yang Mills field strength

$$\mathcal{F} = \frac{1}{2}e^{i}\wedge e^{j}\left(D_i\mathcal{A}_j - D_j\mathcal{A}_i + \frac{i}{h}\{\mathcal{A}_i,\mathcal{A}_j\}_{\text{star}}\right) \qquad (36)$$

constructed now with the $Sp(2n)$ covariant symplectic derivative introduced in (27) , notice that the $star$ bracket in (36) is the global generalization of the Moyal bracket over the whole symplectic manifold obtained in Ref. 5. We notice that, because of (16) and (19), the first covariant symplectic derivative of g^i is a simple derivative. We will assume the same transformation law under $Sp(2n)$ for $\mathcal{A}$.

3. Supermembrane action and Seiberg-Witten map

The starting point in the construction of the noncommutative action for the supermembrane in Ref. 1 was to consider a nontrivial central charge of the supersymmetric algebra. This condition defines in a natural way a nondegenerate closed two-form ω over the spatial world volume, which may be taken to be a Riemann surface. This closed two-form is invariant under the area preserving diffeomorphisms which is the residual gauge symmetry of the supermembrane in the light cone gauge. We may then write the following Hamiltonian density on the Weyl algebra, (see Ref. 4 for more details):

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}(\acute{P}^M \circ *\acute{P}^M) + \frac{1}{2}(\acute{\Pi} \circ *\acute{\Pi}) \\ &- \frac{1}{2h^2}([G_i e^i + \gamma, X^M] \circ * \circ *[G_i e^i + \gamma, X^M]) \\ &- \frac{1}{4h^2}([X^M, X^N] \circ *[X^M, X^N]) \\ &+ \frac{1}{2}(\Omega \circ *\Omega), \end{aligned} \tag{37}$$

where the Hodge $*$ is constructed using an induced Riemannian metric.

We may inmediately project out the center components of this Hamiltonian yielding

$$\begin{aligned}\sigma\mathcal{H} &= \frac{1}{2}(\acute{P}^M)^2\omega + \frac{1}{2}(\acute{\Pi} \wedge *\acute{\Pi}) - (D_{\mathcal{A}}X^M \wedge *D_{\mathcal{A}}X^M) \\ &- \frac{1}{4h^2}\{X^M, X^N\}^2_{\text{star}}\omega + \frac{1}{2}(\mathcal{F} - \omega)(*\mathcal{F} - *\omega) \\ &+ \text{more terms}\ , \end{aligned} \tag{38}$$

where $D_{\mathcal{A}} = D_S X^M + \{\mathcal{A}, X^M\}_{\text{star}}$ corresponds to the first terms in a projection of the gauge covariant derivative,

$$\sigma\mathcal{D}_i X^M = \frac{i}{h}\{G_i, X^M\}_{\text{star}} + \frac{i}{h}\{\mathcal{A}_i, X^M\}_{\text{star}} \tag{39}$$

$$= D_{S\,i}X^M + \frac{i}{h}\{\mathcal{A}_i, X^M\}_{\text{star}} + \text{curvature terms}\ldots \tag{40}$$

If we consider the terms of $\sigma\mathcal{H}$ independent on the formal parameter h we obtain exactly the $D = 11$ supermembrane Hamiltonian in the light cone gauge. Let us know consider a generalization of the Seiberg-Witten map to the Weyl bundle, we will then use this map in order to write an action for the supermembrane depending only on the class of symplectic connections introduced on the previous section. the construction will be based on the Hamiltonian (37).

The Seiberg-Witten map [2] is a map between gauge equivalent classes of noncommutative gauge fields and commutative ones. That is,

$$\mathcal{A}(x) \overset{S-Wmap}{\Leftrightarrow} A(x), \tag{41}$$

such that

$$\Delta_g\mathcal{A} = d\lambda + \{\mathcal{A}, \lambda\} \Leftrightarrow \Delta_g A(x) = d\tilde{\lambda}, \tag{42}$$

where λ and $\tilde{\lambda}$ are infinitesimal gauge parameters. This map was explicitly constructed for a constant background antisymmetric field. It may be generalized to the Weyl algebra bundle for arbitrary symplectic structures in the following way. We may associate, in a unique way, to each Yang Mills connection over the Weyl algebra bundle an abelian connection $\mathcal{D}_A$ whose curvature is of the form

$$\Omega_A = F + h\Omega_1 + h^2\Omega_2 + \ldots, \tag{43}$$

where F $\Omega_1, \Omega_2, \ldots$ are closed two-forms which are gauge invariant. Moreover, this map associates to each non-commutative gauge equivalent class, corresponding to $\mathcal{D}$, the gauge equivalent class of D_A:

$$\mathcal{D}\lambda = \partial\lambda + [\gamma, \lambda] \rightarrow d(\lambda + s),$$

where $s = s(\gamma, \lambda)$. If we define the projection

$$\sigma\gamma = \mathcal{A}(x, h), \tag{44}$$

then

$$\mathcal{A}(x, h) = A(x) + hA_1 + h^2 A_2 + \ldots \tag{45}$$

It turns out that

$$F = F(A) \tag{46}$$

$$\Omega_1 = \Omega_1(A, A_1) \tag{47}$$

$$\Omega_2 = \Omega_2(A, A_1, A_2). \tag{48}$$

and so on. If we impose now the conditions

$$\Omega_i = 0, i = 1, \ldots, \infty \tag{49}$$

we exactly recover the Seiberg-Witten map over any Darboux chart. The procedure provides then a global extension of the Seiberg=Witten map.

There is also a geometrical construction based on a Poisson bracket, instead of the Weyl bracket which is relevant in our discussion. We consider the vector bundle constructed with the same geometrical objects defined in Sec. 2 but instead of constructing a Weyl bracket from the Weyl product, we introduce a Poisson bracket.

All the analysis of Sec. 2 in terms of connection may be developed in the same way. Moreover the Seiberg Witten map may also be considered for a Yang-Mills connections over this vector bundle (with a Poisson structure) which we will denote P. It is then possible to map the gauge equivalent classes of Yang-Mills connections over the Weyl algebra bundle to the gauge equivalent classes of Yang-Mills connections over P.

It was shown in Ref. 3 and Ref. 4 that the geometrical hamiltonian (35) constructed over P is exactly the hamiltonian of the D=11 Supermembrane over a compactified target space. It depends only on the class of simplectic connections which differ by a totally symmetric symbol. It is then possible to rewrite this hamiltonian using the Seiberg-Witten map in terms of non-commutative Yang-Mills connection and associated gauge fields over the Weyl algebra bundle. Since the original hamiltonian depends only on the class of symplectic connection, the same is true for the non-commutative one. The expression (35) corresponds to the first relevant terms of the extension.

We thus conclude that it is possible to extend (35) in a way which depends only on the class of symplectic connections differing by a totally symmetric symbol.

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