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Detecting horizons of symmetric black holes using relative differential invariants

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Abstract

Let $\mathfrak{k}$ be a nontrivial finite-dimensional Lie algebra of vector fields on a manifold M, and consider the family of Lorentzian metrics on M whose Killing algebra contains $\mathfrak{k}$. We show that scalar relative differential invariants of such metrics, with respect to a Lie algebra of vector fields on M preserving $\mathfrak{k}$, can be used to detect the horizons of several well-known black holes. In particular, using the Lie algebra structure of $\mathfrak{k}$, we construct a general relative differential invariant of order 0 that always vanishes on $\mathfrak{k}$-invariant Killing horizons. While the current work is meant to demonstrate the relevance of jet bundles and relative differential invariants in physical applications, we also provide a computationally simple approach for finding a relative differential invariant that detects Killing horizons. The computation and use of this relative differential invariant is comparable in difficulty to other horizon detection methods when there is an obvious Killing vector field that generates the Killing horizon, and often simpler when the preferred Killing vector field is not obvious.

Keywords: Killing vector fields, Killing horizons, differential invariants, jet bundles, Lorentzian geometry

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1. Introduction

Black holes are solutions to Einstein's field equations describing the result of the gravitational collapse of stellar objects. The study of these solutions gives insight into higher curvature regimes in the Universe and by studying the boundaries of these surfaces, it is possible to model the possible gravitational waves arising from perturbed solutions [1, 2]. However, to do this, it is necessary to have a firm definition of a black hole horizon. In the case of common stationary black holes appearing in astrophysics, such as the Kerr solution [3], the event horizon coincides with a null hypersurface that can be determined locally. For more general black holes that do not admit a time-like symmetry, determining a hypersurface that acts as a boundary for the black hole and evolves in time with the black hole solution is a difficult problem.

The event horizon is not an ideal candidate as a boundary for a black hole. It is generally not quasi-local, and in black hole solutions admitting a positive cosmological constant it may not exist at all [4]. In practice, quasi-local surfaces, known as future trapped outer horizons or more briefly as apparent horizons are used to determine boundaries of dynamical black holes [1]. However, such surfaces are often not coordinate independent as they depend on the foliation of spacetime and therefore cannot be considered fully physical. The observation that the horizon can be detected as a quasi-local invariant surface in the case of stationary and weakly isolated black hole solutions [5] has motivated the Geometric Horizon conjectures which posit that an appropriate quasi-local invariant hypersurface that bounds the black hole must arise as the zero-set of some scalar curvature invariant [6].

The classification of black hole solutions has been investigated using scalar polynomial curvature invariants and the horizon has been shown to be detectable in terms of these curvature invariants [7–9]. More generally, the classification of black hole solutions can be accomplished using Cartan invariants [10, chapter 9] and the Killing horizons were shown to be zero-sets of certain Cartan invariants [11].

Initially the investigation of horizon detecting curvature invariants presupposed knowledge of the horizon's location for a given spacetime and a curvature invariant was found that vanishes on the horizon. Using symmetry arguments, [9] described the construction of scalar polynomial curvature invariants that would detect the Killing horizon. However, it might be difficult to find the necessary number of functionally independent scalar polynomial curvature invariants. Later, by employing the framework of weakly isolated horizons [5] a first order curvature invariant was found that would always detect the Killing horizon, without knowledge of its location [12].

There are other approaches to classification of spacetimes, and in particular black hole solutions, beyond curvature invariants. In principle, an IDEAL classification of spherically symmetric black hole solutions is possible [13]. However, a general approach to the IDEAL classification has yet to be determined for general spacetimes. Lastly, one can obtain differential invariants by directly investigating an appropriate Lie pseudogroup action on jet bundles. For instance, this approach has been used to classify Kundt spacetimes [14] and also spacetimes admitting two commuting Killing vectors [15, 16]. Kerr-like solutions can be treated within the framework of [16], but the differential invariants and analysis presented there are not adapted to horizon detection.

In this paper we will focus on spacetimes admitting Killing vectors, with the goal of determining relative differential invariants that vanish on physically important hypersurfaces, such as black hole horizons and Killing horizons. Let $\mathfrak{k} \subset \mathfrak{X}(M)$ be a given Lie algebra of vector fields on a manifold M and consider the family of Lorentzian metrics on M with Killing algebras containing $\mathfrak{k}$. We will assume that $\mathfrak{k} \subset \mathfrak{X}(M)$ is a fixed, but essentially arbitrary, Lie algebra of globally defined vector fields on M that is the Killing algebra for at least one Lorentzian metric. By investigating the Lie pseudogroup of diffeomorphisms that preserve $\mathfrak{k}$, and by computing the corresponding scalar differential invariants, we aim to find conditions that determine important hypersurfaces in M and, in particular, black hole horizons.

Our approach is as follows. We consider a family of metrics with Killing algebra containing $\mathfrak{k}$. They are sections of the bundle $\pi : S^2_{\mathrm{Lor}}T^*M \to M$ of symmetric 2-forms with Lorentzian signature that satisfy a PDE (system), which we will in general denote by $\mathcal{E}$. In the extreme case, $\mathcal{E}$ is defined by the system

$$L_X g = 0, \qquad \forall X \in \mathfrak{k}, \tag{1}$$

where g denotes a general section of π, and L_X the Lie derivative along the vector field X. In this case the space $\mathrm{Sol}(\mathcal{E})$ of solutions consists of all metrics for which every element in $\mathfrak{k}$ is a Killing vector field. In general, we will also allow for $\mathcal{E}$ to be further constrained by additional equations. We treat the PDE $\mathcal{E}$ geometrically, namely as a family of submanifolds in the spaces of jets of sections of the bundle π, i.e. $\mathcal{E}^i \subset J^i\pi$ for $i \in \{0,1,2,\dots\}$.

Let $G \subset \mathrm{Diff}_{\mathrm{loc}}(M)$ be the Lie pseudogroup of (local) symmetries of $\mathcal{E}$ that also preserve the Lie algebra $\mathfrak{k}$ of vector fields on M. It encodes the remaining coordinate freedom, after fixing $\mathfrak{k}$ and $\mathcal{E}$. The Lie algebra (sheaf) $\mathfrak{g}$ of vector fields corresponding to the Lie pseudogroup G contains $\mathfrak{k}$ as an ideal, and it is always a Lie subalgebra of the Lie algebra of all locally defined vector fields preserving $\mathfrak{k}$,

$$\{X \in \mathfrak{X}(M) \mid [X,K] \in \mathfrak{k}, \forall K \in \mathfrak{k}\}. \tag{2}$$

In this paper, we will in general not impose the Einstein equation on our Lorentzian manifolds, but simply note that it is possible to do that within our framework. Due to covariance of the Einstein equations, the sub-PDE of (1) obtained by imposing the Einstein equation has the full Lie algebra (2) as symmetries.

The Lie algebra $\mathfrak{g}$ prolongs to a Lie algebra $\mathfrak{g}^{(i)}$ of vector fields on $\mathcal{E}^i$ for $i \in \{0,1,2,\dots\}$. Our goal is to investigate to which extent scalar relative $\mathfrak{g}^{(i)}$-invariants on $\mathcal{E}^i$ (or relative differential invariants of order i) can be used to detect black hole horizons of metrics admitting Killing vectors. It is natural to conjecture that they would do that, at least in some cases, as they determine hypersurfaces in a coordinate-independent way. We outline the structure and main results of the paper.

Main results

In section 3.1 we show how an ideal of $\mathfrak{g}$ can be used to construct one relative differential invariant of order 0, and also one of order 1, leading to the following theorem.

Theorem 1. *Let* $\mathfrak{i} = \langle K_1, \dots, K_r\rangle$ *be an ideal of a Lie algebra* $\mathfrak{g}$ *of symmetries of the PDE* $\mathcal{E}$*, and assume that* $\dim \mathfrak{i} = r \geqslant 1$. *Then*

$$L_X (K_1 \wedge \cdots \wedge K_r) = \lambda_X (K_1 \wedge \cdots \wedge K_r), \qquad \forall X \in \mathfrak{g}$$

where $\lambda \in \mathfrak{g}^*$. *Consequently, the function*

$$R^{\mathfrak{i}} = \|K_1^{(0)} \wedge \cdots \wedge K_r^{(0)}\|_h^2$$

on $\mathcal{E}^0 \subset J^0\pi$ *is a relative differential invariant with weight* 2λ. *Moreover, the function* $S^{\mathfrak{i}} = \|\bar{d}R^{\mathfrak{i}}\|_h^2$ *on* $\mathcal{E}^1 \subset J^1\pi$ *is a relative differential invariant with weight* 4λ.

Here h is the canonical horizontal symmetric 2-form on $S^2_{\mathrm{Lor}}T^*M$, which is defined in section 2.1. It can be thought of as a placeholder for any particular metric satisfying the PDE

$\mathcal{E}$. We note that for some ideals $R^{\mathfrak{i}}$ may be an absolute differential invariant (with $\lambda = 0$), and it is even possible that $R^{\mathfrak{i}}$ is simply a constant function.

We will focus on $\mathfrak{g}$-invariant ideals of $\mathfrak{k}$ (which are also ideals of $\mathfrak{g}$). Depending on the initial Lie algebra $\mathfrak{k}$ there may be several ways of choosing the ideal $\mathfrak{i}$, and different choices may lead to different relative differential invariants. Due to the importance of $\mathfrak{i}$ being an ideal of $\mathfrak{g}$, special attention is paid to characteristic ideals of $\mathfrak{k}$. In particular, the radical ideal $\mathfrak{r}$ of $\mathfrak{k}$, and the elements of its derived sequence $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_k$ are characteristic ideals. In section 3.2 we use the terms of the derived sequence of $\mathfrak{r}$ to construct a characteristic abelian ideal $\mathfrak{a}(\mathfrak{r})$, and show that it intersects nontrivially with any other ideal of $\mathfrak{r}$.

In section 3.3 we introduce the concept of $\mathfrak{k}$-invariant Killing horizon, and show that the existence of such implies that $\mathfrak{k}$ contains a nontrivial solvable ideal, spanned by the generators of the Killing horizon. Thus, the radical $\mathfrak{r}$ in the Levi decomposition $\mathfrak{k} = \mathfrak{s} \ltimes \mathfrak{r}$ is nontrivial, which in turn guarantees that $\mathfrak{a}(\mathfrak{r})$ is nontrivial. We show that the relative invariant constructed from $\mathfrak{a}(\mathfrak{r})$ by theorem 1 always vanishes on $\mathfrak{k}$-invariant Killing horizons.

Theorem 2. *Let (M, g) be a Lorentzian manifold with Killing algebra $\mathfrak{k} = \mathfrak{s} \ltimes \mathfrak{r}$, let $\mathfrak{g}$ be a Lie algebra of vector fields on M containing $\mathfrak{k}$ as an ideal, and let $\mathfrak{a}(\mathfrak{r}) = \langle K_1, \ldots, K_r \rangle$ be the abelian ideal defined by (6). If $\mathcal{H}$ is a $\mathfrak{k}$-invariant Killing horizon, then the function*

$$R_g^{\mathfrak{a}(\mathfrak{r})} = \|K_1 \wedge \cdots \wedge K_r\|_g^2$$

vanishes on $\mathcal{H}$. The function $R_g^{\mathfrak{a}(\mathfrak{r})}$ is the restriction of the relative invariant of theorem 1 to g: $R_g^{\mathfrak{a}(\mathfrak{r})} = R^{\mathfrak{a}(\mathfrak{r})} \circ g$.

If a Killing vector K is a generator of the Killing horizon $\mathcal{H}$, then the function $\|K\|_g^2$ will also vanish on $\mathcal{H}$, but if $\dim \mathfrak{k} > 1$ finding the appropriate Killing vector K may be nontrivial.

For example, if g is the metric for for the Kerr black hole ([17, section 5.3]), we have $\mathfrak{a}(\mathfrak{r}) = \mathfrak{k} = \langle \partial_t, \partial_\varphi \rangle$ and

$$R_g^{\mathfrak{a}(\mathfrak{r})} = -\sin^2(\theta)\left(r^2 - 2Mr + a^2\right).$$

We recognize the condition $r^2 - 2Mr + a^2 = 0$ as the defining equation for the event horizon $r = r_+$ and for the inner apparent horizon $r = r_-$, where r_+ and r_- are (respectively) the largest and smallest of the two solutions of the quadratic equation. It is well-known that these horizons are Killing horizons, with generators

$$\left(r_+^2 + a^2\right)\partial_t + a\partial_\phi, \qquad \left(r_-^2 + a^2\right)\partial_t + a\partial_\phi,$$

respectively, but the difficulty of finding the correct linear combination of Killing vectors seems to be comparable to finding the horizons themselves, even when all the Killing vectors are known [17]. Computing $R_g^{\mathfrak{a}(\mathfrak{r})}$, on the other hand, is often straightforward once a basis for the Killing vectors is known.

In comparison, the calculation of the required horizon detecting curvature invariants require the computation of the curvature tensor and its covariant derivatives. These differential invariants are in general functions on $\mathcal{E}^k$ for some $k \geqslant 2$, while $R^{\mathfrak{a}(\mathfrak{r})}$ is a function on $\mathcal{E}^0$. Furthermore, in the case of scalar polynomial curvature invariants [8, 9], there is no guarantee that the necessary functionally independent invariants can be found, and the resulting invariant may vanish on surfaces other than the Killing horizon. The horizon detecting invariant based on the Cartan algorithm will work in higher dimensions and will only detect the horizon [11]. However, this requires further calculation in order to determine the geometrically preferred frame, which can be a difficult task in dimension greater than 3.

We note that the vanishing of $R_g^{\mathfrak{a}(\mathfrak{r})}$ is a necessary condition for detecting Killing horizons, but it is not a sufficient condition. One must still check that the hypersurfaces given by $R_g^{\mathfrak{a}(\mathfrak{r})} = 0$ are null hypersurfaces, and that one of the Killing vector fields in $\mathfrak{a}(\mathfrak{r})$ is proportional to the gradient of the defining function of the hypersurface. In the case one is interested in Killing horizons, we note that this is a necessary step also when the horizon is determined from the norm of a Killing vector field or by a scalar polynomial curvature invariant.

In section 4 we focus on several specific examples of Killing algebras. We apply the results of section 3 together with a more detailed orbit analysis on $\mathcal{E}^0$ to get a complete picture of the relative differential invariants of order 0, and in some cases we also make a similar analysis on $\mathcal{E}^1$. When $\mathfrak{k}$ is the Killing algebras of the Schwarzschild metric, the Kerr metric, or the generic Near Horizon geometries we see that the well-known horizons can be deduced from the orbit structure on $\mathcal{E}^0$. Furthermore, since these horizons are $\mathfrak{k}$-invariant Killing horizons, they are also detected by the relative differential invariant of theorem 2.

Even in some cases where the relative differential invariant of theorem 2 is not defined (for example if $\mathfrak{k}$ is simple), the orbit structure on $\mathcal{E}^i$ can be used to detect horizons for $i > 0$. In section 4.4 we consider the family of spherically symmetric spacetimes, for which $\mathfrak{k}$ contains no nontrivial ideals. In this case, $\mathcal{E}^0$ is 8-dimensional and all orbits on $\mathcal{E}^0$ are 7-dimensional. In particular, there are no proper relative differential invariants on $\mathcal{E}^0$ (only an absolute differential invariant). However, there are relative differential invariants on $\mathcal{E}^1$, and we show that one of these determines the unique spherically symmetric apparent horizon of imploding spherically symmetric metrics.

To provide background for these results, we begin in section 2 with an introduction to jets and differential invariants, after which we outline the general set-up of our approach in more detail.

2. Jets and differential invariants

In this section we give a brief introduction to some language and notations related to jet spaces and differential invariants. For more information about these subjects, we refer to [18–20].

2.1. Jet spaces

Consider the vector bundle $S^2T^*M \to M$ of symmetric covariant 2-tensors. We will refer to sections of this bundle as symmetric 2-forms on M, and denote by $\pi : S^2_{\mathrm{Lor}}T^*M \to M$ the subbundle of tensors of Lorentzian signature. A Lorentzian metric on a manifold M is a section of the bundle π.

A metric $g \in \Gamma(\pi)$ defined on a domain $U \subset M$ determines a submanifold $g(U) \subset S^2_{\mathrm{Lor}}T^*M$. We say that two metrics $g, \tilde{g}$ are k-equivalent at $a \in M$ in a neighborhood U of a if the submanifolds $g(U), \tilde{g}(U) \subset S^2_{\mathrm{Lor}}T^*M$ have kth order tangency at a, for $k \in \{0, 1, 2, \dots\}$. The notion of k-equivalence can be formulated in coordinates in the following way: If $x^1, \dots, x^n$ are coordinates on U, then $g = \sum_{i \leqslant j} g_{ij}(x)\mathrm{d}x^i\,\mathrm{d}x^j$ and $\tilde{g} = \sum_{i \leqslant j} \tilde{g}_{ij}(x)\mathrm{d}x^i\,\mathrm{d}x^j$ are k-equivalent at $a \in U$ if and only if $\frac{\partial^{|\sigma|} g_{ij}(x)}{\partial x^\sigma}\big|_{x=a} = \frac{\partial^{|\sigma|} \tilde{g}_{ij}(x)}{\partial x^\sigma}\big|_{x=a}$ for each multi-index σ with $|\sigma| \leqslant k$, and each pair i,j with $i \leqslant j$.

We call the equivalence class of g the k-jet of g at a, and denote it by $[g]^k_a$. It can be thought of as the collection of kth-degree Taylor polynomials of the components of g at the point a. We denote by $J^k_a\pi$ the space of k-jets of metrics at the point a, and define the jet space $J^k\pi = \bigsqcup_{a \in M} J^k_a\pi$. Note that $J^0\pi = S^2_{\mathrm{Lor}}T^*M$. We have the natural fiber bundle projections $\pi_{k,l} : J^k\pi \to J^l\pi$ for $0 \leqslant l < k$ and $\pi_k : J^k\pi \to M$.

A local diffeomorphism φ, defined on $U \subset M$, prolongs to a local diffeomorphism defined on $\pi_k^{-1}(U) \subset J^k\pi$ in the following way. Consider a general point $\theta = [g]^k_a \in \pi_k^{-1}(U) \subset J^k\pi$, where g is a section of π defined in a neighborhood of $a \in U$. Then $\varphi^{(k)}(\theta) = [(\varphi^{-1})^* g]^k_{\varphi(a)}$. In a similar way, a vector field X on $U \subset M$ can be uniquely prolonged to a vector field $X^{(k)}$ on $\pi_k^{-1}(U) \subset J^k\pi$. The prolongation formula for vector fields is given in (3).

The total space $S^2_{\mathrm{Lor}}T^*M$ is naturally equipped with a horizontal symmetric 2-form h, defined by $h_\theta = \theta$ for any point $\theta \in S^2_{\mathrm{Lor}}T^*_aM$. In particular, for $g \in \Gamma(\pi)$ we have $g^*h = g$. The symmetric 2-form h is invariant under the prolongation of any local diffeomorphism φ and any vector field X on M, meaning

$$\left(\varphi^{(0)}\right)^* h = h, \qquad L_{X^{(0)}} h = 0.$$

Often we use these properties for computing formulas for $\varphi^{(0)}$ and $X^{(0)}$ from the formulas for φ and X, respectively. We refer to [21, section 3] for more details on this type of universal constructions.

For example, if $x^1, \ldots, x^n$ are local coordinates on M, they can be extended to a coordinate system on $S^2_{\mathrm{Lor}}T^*M$ by adding coordinates u_{ij}, $1 \leqslant i \leqslant j \leqslant n$, in such a way that h takes the form

$$h = \sum_{i \leqslant j} u_{ij} \mathrm{d}x^i \, \mathrm{d}x^j.$$

Furthermore, they can be extended canonically to coordinates $(u_{ij})_\sigma$ on $J^k\pi$, where σ is a multi-index with $|\sigma| \leqslant k$.

If $g = \sum_{i \leqslant j} g_{ij}(x) \mathrm{d}x^i \, \mathrm{d}x^j$ is a (local) section of π, we write $u_{ij}|_g = u_{ij} \circ g = g_{ij}(x)$, and easily verify that $h \circ g = g$. The lift of a general vector field $X = a^i(x)\partial_{x^i}$ takes the form

$$X^{(0)} = \sum_i a^i(x) \, \partial_{x^i} + \sum_{i \leqslant j} b_{ij}(x,u) \, \partial_{u_{ij}},$$

where the functions $b_{ij}(x,u)$ are uniquely determined from the linear algebraic system $L_{X^{(0)}} h = 0$. Higher prolongations $X^{(k)}$ are then computed by the standard jet-prolongation formulas (see for example [19, section 1.5]):

$$X^{(k)} = \sum_l a^l(x) \, D^{(k+1)}_{x^l} + \sum_{i \leqslant j, |\sigma| \leqslant k} D_{x^\sigma}(\varphi_{ij}) \, \partial_{(u_{ij})_\sigma}. \tag{3}$$

Here $\varphi_{ij} = b_{ij}(x,u) - \sum_l (u_{ij})_l a^l(x)$. The total derivative operator D_{x^l} satisfies $D_{x^l}(f) \circ j^{k+1}g = \partial_{x^l}(f \circ j^k g)$ for a function f on $J^k\pi$, and in coordinates it takes the form

$$D_{x^l} = \partial_{x^l} + \sum_{i \leqslant j} \sum_\sigma (u_{ij})_{\sigma + 1_l} \partial_{(u_{ij})_\sigma}.$$

In (3) D_{x^σ} denotes the iterated total derivative according to the multi-index σ, and $D^{(k+1)}_{x^l}$ is the truncated total derivative:

$$D^{(k+1)}_{x^l} = \partial_{x^l} + \sum_{i \leqslant j} \sum_{\sigma \leqslant k} (u_{ij})_{\sigma + 1_l} \partial_{(u_{ij})_\sigma}.$$

A metric g uniquely defines a section $j^k g$ of π_k, called the kth prolongation of g and defined by $(j^k g)(a) = [g]^k_a$. A function f on $J^k\pi$ can be composed with the kth prolongation of a section g of π to give a function $f_g := f \circ j^k g$ on the base M. The horizontal exterior derivative, denoted by $\bar{d}$, satisfies $(\bar{d}f)\circ j^{k+1}g = \mathrm{d}(f\circ j^k g)$ for any $g\in\Gamma(\pi)$, where d is the exterior derivative on M. In local coordinates it can be written as $\bar{d}f = \sum_l D_{x^l}(f)\mathrm{d}x^l$.

Remark 1. The horizontal symmetric 2-form h on $J^0\pi = S^2_{\mathrm{Lor}}T^*M$ can be pulled back to a horizontal symmetric 2-form $\pi^*_{k,0}h$ on $J^k\pi$ for any integer $k>0$. It is given by exactly the same coordinate formula as h, and it is thus clear that $L_{X^{(k)}}(\pi^*_{k,0}h) = 0$ for every positive integer k. We also define the universal horizontal tensor field $h^{-1} = \sum_{i,j}([h]^{-1})^{ij}D_{x^i}\otimes D_{x^j}$, where $[h]^{-1}$ denotes the inverse of the symmetric matrix $[h]$ which defines $h = \sum_{i,j}[h]_{ij}\mathrm{d}x^i\otimes \mathrm{d}x^j$. It satisfies $h^{-1}\circ g = g^{-1}$ and $L_{X^{(\infty)}}h^{-1} = 0$, where we prolong X to $J^\infty\pi$ because the total derivatives depend on the ∞-jet.

In our examples involving jet computations, we have been using the DifferentialGeometry and JetCalculus packages in Maple™. They have built-in tools for doing these types of computations, such as computing the prolongation of vector fields.

2.2. PDE coming from Killing vectors

In this paper, $\mathfrak{k}$ will be a fixed finite-dimensional Lie algebra of vector fields on M. The elements of $\mathfrak{k}$ are Killing vectors of a section $g\in\Gamma(\pi)$ if and only if

$$L_K g = 0, \qquad \forall K\in\mathfrak{k}. \tag{4}$$

This gives an (in general overdetermined) system of PDEs on the components of g. The framework of jet spaces gives us a geometric interpretation of differential equations. By replacing all occurences of $\partial_{x^l}(g_{ij})$ with $(u_{ij})_l$, we obtain from (4) a submanifold $\mathcal{E}^1\subset J^1\pi$. Furthermore, by differentiating (4) the appropriate number of times we obtain in a similar way, for each $k\geqslant 1$, a submanifold $\mathcal{E}^k\subset J^k\pi$. Geometrically, the fact that g is a solution to (4) implies that $[g]^k_a\in\mathcal{E}^k$ for each k and each a.

In this set-up it may happen that some algebraic combination of the defining equations for $\mathcal{E}^k$ is of order strictly less than k, but still does not vanish on $\mathcal{E}^{k-1}$. This means that the map $\pi_{k,k-1}|_{\mathcal{E}^k}\colon \mathcal{E}^k\to\mathcal{E}^{k-1}$ is not surjective. In such cases we may add this new equation to the defining equations of $\mathcal{E}^{k-1}$. We may continue this process for various k until all the maps $\pi_{i,i-1}|_{\mathcal{E}^i}\colon \mathcal{E}^i\to\mathcal{E}^{i-1}$ are submersions. In this case the collection $\{\mathcal{E}^0,\mathcal{E}^1,\mathcal{E}^2,\dots\}$ is called formally integrable. We will denote this collection by $\mathcal{E}$, and refer to it as a PDE (system). It is important to note that any solution g of (4) will still satisfy $[g]^k_a\in\mathcal{E}^k$ after this procedure, and we will therefore in general assume that our PDE $\mathcal{E}$ is formally integrable. For more details on formal integrability we refer to [19].

It is possible that the inclusion $\mathcal{E}^0\subset J^0\pi = S^2_{\mathrm{Lor}}T^*M$ is strict. Let $\tau^i:\mathcal{E}^i\to J^i\pi$ be the inclusion map. Then $(\tau^0)^*h$ is a horizontal symmetric 2-form on $\mathcal{E}^0$, and we will denote also this by h.

We denote the space of (smooth) solutions of $\mathcal{E}$ by $\mathrm{Sol}(\mathcal{E})$:

$$\mathrm{Sol}(\mathcal{E}) = \{g\in\Gamma(\pi)\mid L_K g = 0, \forall K\in\mathfrak{k}\}.$$

We will throughout this paper assume that $\mathfrak{k}$ is the Lie algebra of Killing vectors for at least one Lorentzian metric on M, implying that $\mathrm{Sol}(\mathcal{E})$ is nonempty. If $\mathfrak{k}$ is the Lie algebra of Killing vectors of a section $g\in\Gamma(\pi)$, we call it the Killing algebra of g.

In some cases it is desirable to consider a sub-PDE $\tilde{\mathcal{E}}$ given by submanifolds $\tilde{\mathcal{E}}^i\subset\mathcal{E}^i$, $i\in\{0,1,2,\dots\}$. For example, we may add to (4) the constraints of the Einstein equation.

2.3. The Lie algebra preserving $\mathfrak{k}$

There exists a Lie pseudogroup G consisting of the local diffeomorphisms on M preserving $\mathfrak{k}$:

$$G = \{\varphi \in \mathrm{Diff}_{\mathrm{loc}}(M) \mid \varphi_* \mathfrak{k} = \mathfrak{k}\}.$$

We denote the corresponding Lie algebra (sheaf) of locally defined vector fields by $\mathfrak{g}$:

$$\mathfrak{g} = \{X \in \mathfrak{X}(M) \mid [X, K] \in \mathfrak{k}, \forall K \in \mathfrak{k}\}.$$

Both G and $\mathfrak{g}$ are often infinite-dimensional, and it is clear that $\mathfrak{k}$ is an ideal of $\mathfrak{g}$.

Any diffeomorphism in G lifts to a diffeomorphism of $J^k\pi$ for any integer $k \geqslant 0$, and it is a symmetry of $\mathcal{E}$, i.e. it preserves the set $\mathcal{E}^k$ for each k, and also the space of solutions $\mathrm{Sol}(\mathcal{E})$. In most cases we will work with $\mathfrak{g}$ rather than G. The Lie algebra $\mathfrak{g}$ consists of (infinitesimal) symmetries of $\mathcal{E}$:

$$X_\theta^{(k)} \in T_\theta \mathcal{E}^k, \qquad \forall \theta \in \mathcal{E}^k, \forall X \in \mathfrak{g}.$$

We will use the notation

$$\mathfrak{g}^{(i)} = \left\{X^{(i)} \mid X \in \mathfrak{g}\right\}$$

for the Lie algebra of prolonged vector fields.

As mentioned in the previous section, it is possible to restrict to a sub-PDE $\tilde{\mathcal{E}}$. If $\tilde{\mathcal{E}}$ is not $\mathfrak{g}$-invariant, one should then also restrict to a subalgebra $\tilde{\mathfrak{g}} \subset \mathfrak{g}$ of symmetries of $\tilde{\mathcal{E}}$.

2.4. Differential invariants

Let $\mathcal{E}$ be a PDE, defined as a collection of submanifolds $\mathcal{E}^i \subset J^i\pi$, $i \in \{0, 1, 2, \dots\}$, and let $\mathfrak{g}$ be a Lie algebra of symmetries of $\mathcal{E}$. We define absolute and relative differential invariants in this context.

Definition 1. A function I on $\mathcal{E}^k$ is called an absolute differential invariant of order k if

$$X^{(k)}(I) = 0, \qquad \forall X \in \mathfrak{g}.$$

A function R on $\mathcal{E}^k$ is called a relative differential invariant of order k if

$$X^{(k)}(R) = \lambda_X R, \qquad \forall X \in \mathfrak{g},$$

for some linear map $\lambda \in \mathrm{Hom}(\mathfrak{g}, C^\infty(\mathcal{E}^k))$. We will call λ the weight of the relative differential invariant.

Both equations in the definitions are required to hold only on $\mathcal{E}^k$ (in general, differential invariants on $\mathcal{E}^k$ do not extend to differential invariants on $J^k\pi$). In particular, all scalar polynomial curvature invariants are examples of absolute differential invariants, but the special symmetric spacetimes we consider may admit additional absolute (and relative) differential invariants which can not be extended to generic spacetimes. Invariants of this type are sometimes referred to as 'conditional invariants' (see [22]). All the differential invariants that we consider will turn out to be rational. We will not give a detailed explanation of this fact, but simply note that it is related to transitivity of $\mathfrak{g}$ and algebraicity of the corresponding (transitive) Lie pseudogroup (see [20] for more details). Moreover, for polynomial relative invariants we actually have $\lambda \in \mathrm{Hom}(\mathfrak{g}, C^\infty(M))$, as explained in [22].

Absolute differential invariants of order k are constant on $\mathfrak{g}^{(k)}$-orbits, meaning that their level sets are $\mathfrak{g}^{(k)}$-invariant. For relative invariants, only the zero sets are invariant in general.

Any nonvanishing function on $\mathcal{E}^k$ technically satisfies the conditions of definition 1, but in general we will not pay attention to these. (The space of relative differential invariants forms a group under multiplication. This group can be endowed with an equivalence relation that identifies relative invariants that differ by a nonvanishing factor, under which any nonvanishing function is equivalent to a constant function. See [22] and references therein for more details.)

Example 1. The function $\det(h)$ on $\mathcal{E}^0 \subset J^0\pi$ satisfies the condition

$$X^{(0)}\left(\det(h)\right) = \lambda_X \det(h), \qquad \forall X \in \mathfrak{g},$$

but $\det(h)$ never vanishes on $\mathcal{E}^0 \subset J^0\pi = S^2_{\mathrm{Lor}}T^*M$. We therefore do not consider it to be a proper relative differential invariant.

A differential invariant is in general not a function on M, but on $\mathcal{E}^k \subset J^k\pi$. Let I be an absolute differential invariant of order k. Then, for any $g \in \mathrm{Sol}(\mathcal{E})$ the function $I|_g = I \circ j^k g$ is a function on M. Intuitively, its invariance means the following: Its definition in terms of the components of g and their partial derivatives up to order k does not depend on the coordinates in which g was expressed, with the coordinate freedom being given by the (Zariski connected component of the) Lie pseudogroup G corresponding to $\mathfrak{g}$ [23, section 1.1]. Assume $I|_g$ is non-constant in every neighborhood of $a \in M$. Then we can, in a neighborhood of a define a hypersurface $I|_g = C$ in a $\mathfrak{g}$-invariant way. Here C can be any constant in the image of $I|_g$, and by varying C one easily realizes that M is foliated by such hypersurfaces in a neighborhood of a. For a relative invariant R of order k, we define the restriction $R|_g = R \circ j^k g$ in the same way. If we assume that $R|_g$ does not vanish entirely on M, and that it does vanish on some points, we will often get a unique invariantly defined hypersurface $\{R|_g = 0\} \subset M$.

Since relative differential invariants single out particular hypersurfaces, rather than providing a foliation of such, they will be our main object of study. However, note that for a hypersurface to be 'special' it is strictly speaking not sufficient that it is given in terms of a relative differential invariant, as the following argument shows. Let $I = R/Q$ be a rational absolute differential invariant, where R and Q are polynomial relative differential invariants of the same weight, and assume that the level sets of $I|_g$ foliates the spacetime. Any leaf of this foliation is given by $I|_g = C$ for some constant C. Away from singular points, this equation is equivalent to $R|_g - CQ|_g = 0$, where $R - CQ$ is a relative invariant, but in general, is not an absolute invariant.

Given a Lie algebra $\mathfrak{g}$ of vector fields on M, and its prolongation $\mathfrak{g}^{(k)}$ to $\mathcal{E}^k$, the important relative invariants of order k are often found by locating the points in $\mathcal{E}^k$ where the orbit dimension drops. In particular, if $\mathfrak{g}^{(k)}$ is transitive on a Zariski dense subset of $\mathcal{E}^k$, then all proper relative invariants are found in this way.

If generic $\mathfrak{g}^{(k)}$-orbits have smaller dimension than $\mathcal{E}^k$, it is possible (but not necessary) that the orbit dimension drops on the level set of some absolute invariant. To illustrate this, let us consider the following two very simple examples of Lie algebras of vector fields on $\mathbb{R}^2$:

$$\mathfrak{g}_1 = \langle y\partial_x \rangle, \qquad \mathfrak{g}_2 = \langle x\partial_x \rangle.$$

In both cases $I = y$ is an absolute invariant. In the first case $\{I = 0\}$ is exactly the hypersurface consisting of points where the orbit dimension drops. In the second case $R = x$ is an additional relative invariant, and $\{R = 0\}$ is the hypersurface of points where the orbit dimension drops. These examples illustrate general phenomena that can also occur for differential invariants on $\mathcal{E}^k$.

3. Differential invariants, and ideals of $\mathfrak{k}$

In this section we use the structure of the Killing algebra $\mathfrak{k}$ to construct relative differential invariants with respect to a Lie algebra $\mathfrak{g}$ of vector fields on M that contains $\mathfrak{k}$ as an ideal. In section 3.1 we show how to construct a general relative differential invariant of order 0 (and one of order 1) from an ideal of $\mathfrak{g}$. We discuss candidates for such ideals in section 3.2. In particular, we construct an abelian characteristic ideal $\mathfrak{a}(\mathfrak{r})$ of $\mathfrak{k}$ from the derived sequence of the radical ideal $\mathfrak{r}$ of $\mathfrak{k}$ and show that it intersects nontrivially with all ideals of $\mathfrak{r}$. In section 3.3 we show that the (solvable) Lie algebra $\mathfrak{k}_{\mathcal{H}}$ of generators of a Killing horizon $\mathcal{H}$ is an ideal in $\mathfrak{k}$ if $\mathcal{H}$ is $\mathfrak{k}$-invariant. The fact that $\mathfrak{a}(\mathfrak{r}) \cap \mathfrak{k}_{\mathcal{H}} \neq \{0\}$ implies that the relative differential invariant given by the ideal $\mathfrak{a}(\mathfrak{r})$ vanishes on $\mathcal{H}$.

3.1. *Ideals of $\mathfrak{g}$ giving rise to relative invariants*

As in section 2, let $\mathfrak{k}$ be a finite-dimensional Lie algebra, $\mathfrak{g}$ a larger Lie algebra (sheaf) that contains $\mathfrak{k}$ as an ideal, and let $\mathcal{E}$ be $\mathfrak{g}$-invariant PDE system on the space of $\mathfrak{k}$-invariant Lorentzian metrics, meaning that the vector fields of $\mathfrak{g}^{(i)}$ are tangent to $\mathcal{E}^i \subset J^i\pi$.

Our first theorem does not involve $\mathfrak{k}$ explicitly, but lets us construct relative differential invariants from an ideal of $\mathfrak{g}$. Before we prove it, we need to introduce the concept of vertical 1-forms. We denote the Cartan forms on $J^\infty\pi$ by ω^{ij}_σ for $i \leqslant j$ and σ a multi-index. They can be defined in a coordinate-free fasion, but for simplicity we will here just give their coordinate formulas: $\omega^{ij}_\sigma = \mathrm{d}(u_{ij})_\sigma - (u_{ij})_{\sigma+1_l}\mathrm{d}x^l$. We call a 1-form on $J^\infty\pi$ vertical if it is a $C^\infty(J^\infty\pi)$-linear combination of Cartan forms.

Lemma 1. *Let f be a function on $J^\infty\pi$, and X a vector field on M. Then $L_{X^{(\infty)}}(\bar{d}f) - \bar{d}(L_{X^{(\infty)}}f)$ is a vertical 1-form.*

Note that a function on $J^\infty\pi$ is by definition the pullback of a function on $J^k\pi$ for some k. This lemma can be found for example in [24, chapter 0.9]. It is a consequence of a the fact that $df - \bar{d}f$ is vertical, and that the Lie derivative commutes with the exterior derivative. We recall two important properties of the vertical 1-forms. First, if $|\sigma| = k$ we have

$$L_{X^{(\infty)}}\omega^{j}_\sigma = \sum_{i\leqslant j, |\rho|\leqslant k} B^{\rho}_{ij}\omega^{ij}_\rho$$

for some collection of functions B^ρ_{ij} on $J^k\pi$, implying that the Lie derivative of a prolonged vector field preserves the space of vertical 1-forms. Second, the contraction of any vertical 1-form with total derivatives D_{x^i} gives 0.

Theorem 1. *Let $\mathfrak{i} = \langle K_1, \ldots, K_r\rangle$ be an ideal of a Lie algebra $\mathfrak{g}$ of symmetries of the PDE $\mathcal{E}$, and assume that* $\dim\mathfrak{i} = r \geqslant 1$. *Then*

$$L_X(K_1 \wedge \cdots \wedge K_r) = \lambda_X(K_1 \wedge \cdots \wedge K_r), \qquad \forall X \in \mathfrak{g}$$

where $\lambda \in \mathfrak{g}^*$. Consequently, the function

$$R^{\mathfrak{i}} = \|K_1^{(0)} \wedge \cdots \wedge K_r^{(0)}\|_h^2$$

on $\mathcal{E}^0 \subset J^0\pi$ is a relative differential invariant with weight 2λ. Moreover, the function $S^{\mathfrak{i}} = \|\bar{d}R^{\mathfrak{i}}\|_h^2$ on $\mathcal{E}^1 \subset J^1\pi$ is a relative differential invariant with weight 4λ.

Proof. Let $X \in \mathfrak{g}$ be a general element, and $X^{(0)}$ its prolongation to $\mathcal{E}^0$. The expression $L_X(K_1 \wedge \cdots \wedge K_r)$ is by the Leibniz rule a sum of exterior products that have factors $K_1, \ldots, K_{i-1}, L_XK_i, K_{i+1}, \ldots, K_r$ where i runs from 1 to r. Since $\mathfrak{i}$ is an ideal of $\mathfrak{g}$, we have

$L_XK_i = [X,K_i] = C_i^jK_j$ for some set of constants C_i^j. Thus the only terms that remain in this sum are constant multiples of $K_1\wedge\cdots\wedge K_r$, which proves the first part of the proposition.

Using the notation

$$||K_1^{(0)}\wedge\cdots\wedge K_r^{(0)}||_h^2 = h\left(K_1^{(0)}\wedge\cdots\wedge K_r^{(0)},K_1^{(0)}\wedge\cdots\wedge K_r^{(0)}\right),$$

we have that

$$\begin{aligned} L_{X^{(0)}}R^{\mathfrak{i}} &= L_{X^{(0)}}\left(h\left(K_1^{(0)}\wedge\cdots\wedge K_r^{(0)},K_1^{(0)}\wedge\cdots\wedge K_r^{(0)}\right)\right)\\ &= 2h\left(L_{X^{(0)}}\left(K_1^{(0)}\wedge\cdots\wedge K_r^{(0)}\right),K_1^{(0)}\wedge\cdots\wedge K_r^{(0)}\right)\\ &= 2\lambda_X h\left(K_1^{(0)}\wedge\cdots\wedge K_r^{(0)},K_1^{(0)}\wedge\cdots\wedge K_r^{(0)}\right)\\ &= 2\lambda_X R^{\mathfrak{i}}, \qquad \forall X\in\mathfrak{g}. \end{aligned}$$

In the second equality, we used that $L_{X^{(0)}}h = 0$ for every $X\in\mathfrak{g}$, which implies that $L_{X^{(0)}}\left(h^{\otimes r}\right) = 0$. In the third equality, we used the argument from the first paragraph of the proof and the fact that $X\mapsto X^{(0)}$ is a Lie algebra homomorphism. Next, using lemma 1 together with the facts that $R^{\mathfrak{i}}$ is a function on $\mathcal{E}^0$ and that λ_X is constant, we see that

$$\begin{aligned} L_{X^{(1)}}\left(\bar{d}R^{\mathfrak{i}}\right) &= \bar{d}\left(L_{X^{(0)}}R^{\mathfrak{i}}\right)+\nu_X\\ &= \bar{d}\left(2\lambda_XR^{\mathfrak{i}}\right)+\nu_X\\ &= 2\bar{d}\left(\lambda_X\right)R^{\mathfrak{i}}+2\lambda_X\bar{d}R^{\mathfrak{i}}+\nu_X\\ &= 2\lambda_X\bar{d}R^{\mathfrak{i}}+\nu_X \end{aligned}$$

where ν_X is a vertical 1-form. We have written $X^{(1)}$ and $X^{(0)}$ instead of $X^{(\infty)}$ since these vector fields act in the same way as $X^{(\infty)}$ on $\bar{d}R^{\mathfrak{i}}$ and $R^{\mathfrak{i}}$, respectively.

It follows that

$$\begin{aligned} L_{X^{(1)}}\left(h^{-1}\left(\bar{d}R^{\mathfrak{i}},\bar{d}R^{\mathfrak{i}}\right)\right) &= L_{X^{(\infty)}}\left(h^{-1}\left(\bar{d}R^{\mathfrak{i}},\bar{d}R^{\mathfrak{i}}\right)\right)\\ &= 2h^{-1}\left(L_{X^{(\infty)}}\bar{d}R^{\mathfrak{i}},\bar{d}R^{\mathfrak{i}}\right)\\ &= 2h^{-1}\left(2\lambda_X\bar{d}R^{\mathfrak{i}}+\nu_X,\bar{d}R^{\mathfrak{i}}\right)\\ &= 4\lambda_Xh^{-1}\left(\bar{d}R^{\mathfrak{i}},\bar{d}R^{\mathfrak{i}}\right). \end{aligned}$$

Note that the first equality holds because $(h^{-1}(\bar{d}R^{\mathfrak{i}},\bar{d}R^{\mathfrak{i}}))$ involves only first-order variables. The second equality holds because $L_{X^{(\infty)}}(h^{-1}) = 0$. Remark 1 explains the necessity of prolonging $X^{(0)}$ to $\mathcal{E}^\infty$. The fourth equality is due to the fact that any contraction of h^{-1} with vertical 1-forms gives 0. ☐

Notice that the choice of basis of $\mathfrak{i}$ above influences the relative invariants R and S only by a constant scalar factor. Although the statement is technically true in general, it is possible that $R^{\mathfrak{i}}$ is constant or even constantly zero. In particular, when $\dim\mathfrak{i} > \dim M$, the function $R^{\mathfrak{i}}$ vanishes everywhere due to the fact that $\Lambda^rT_aM = \{0\}$ when $r\geqslant\dim M$.

Given a metric $g\in\mathrm{Sol}(\mathcal{E})$ on M, we can compute the restriction of $R^{\mathfrak{i}}$ and $S^{\mathfrak{i}}$ (or any other function on $\mathcal{E}^i$) to g:

$$R_g^{\mathfrak{i}} = R^{\mathfrak{i}}\circ g, \qquad S_g^{\mathfrak{i}} = S^{\mathfrak{i}}\circ j^1g.$$

The functions $R_g^{\mathfrak{i}}$ and $S_g^{\mathfrak{i}}$ are functions on M.

Our main application of theorem 1 will be to $\mathfrak{g}$-invariant ideals of $\mathfrak{k}$. If $\mathfrak{i}$ is an ideal of $\mathfrak{k}$, then the vector fields of $\mathfrak{i}$ span a (not necessarily regular) distribution in TM which is invariant with respect to the vector fields of $\mathfrak{k}$. If in addition $\mathfrak{i}$ is an ideal in $\mathfrak{g}$, this distribution is $\mathfrak{g}$-invariant. This happens in particular when $\mathfrak{i}$ is a characteristic ideal of $\mathfrak{k}$.

Definition 2. An ideal $\mathfrak{i}$ in $\mathfrak{k}$ is called a characteristic ideal if it satisfies $D(\mathfrak{i}) \subset \mathfrak{i}$ for any derivation $D\colon \mathfrak{k} \to \mathfrak{k}$.

Recall that a derivation of the Lie algebra $\mathfrak{k}$ is a linear map $D\colon \mathfrak{k} \to \mathfrak{k}$ satisfying $D[X,Y] = [DX,Y] + [X,DY]$ for every pair $X, Y \in \mathfrak{k}$. It follows that if $\mathfrak{g}$ is a (possibly infinite-dimensional) Lie algebra that contains $\mathfrak{k}$ as an ideal, and $\mathfrak{i}$ is a characteristic ideal of $\mathfrak{k}$, then $\mathfrak{i}$ is an ideal of $\mathfrak{g}$. This is due to the fact that for any $Z \in \mathfrak{g}$, the operation $\mathrm{ad}_Z = [Z,\cdot]$ is a derivation of $\mathfrak{k}$. Thus any characteristic ideal of $\mathfrak{k}$ can play the role of $\mathfrak{i}$ in theorem 1.

3.2. *An abelian ideal of $\mathfrak{k}$*

In this section we will state some general results involving the Lie algebras $\mathfrak{k}$ and $\mathfrak{g}$. In fact, none of these results rely on $\mathfrak{k}$ being the Killing algebra of any metric, but we prefer to keep this notation as it emphasizes the applications we have in mind.

For any finite-dimensional Lie algebra $\mathfrak{k}$, over a field of characteristic zero, the Levi decomposition lets us write [25]:

$$\mathfrak{k} = \mathfrak{s} \ltimes \mathfrak{r},$$

where $\mathfrak{s}$ is a semisimple Lie algebra and $\mathfrak{r}$ the radical (largest solvable ideal) of $\mathfrak{k}$, and we will be using this decomposition throughout the paper, so that $\mathfrak{s}$ and $\mathfrak{r}$ will always denote the components of the Levi decomposition of $\mathfrak{k}$.

The derived sequence of the Lie algebra $\mathfrak{r}$ is defined as follows:

$$\mathfrak{r}_0 = \mathfrak{r}, \qquad \mathfrak{r}_i = [\mathfrak{r}_{i-1}, \mathfrak{r}_{i-1}]. \tag{5}$$

As $\mathfrak{r}$ is solvable, there exists some $k \in \mathbb{N}$ such that $\mathfrak{r}_k \neq \{0\}$ and $\mathfrak{r}_{k+1} = \{0\}$. In this case $\mathfrak{r}_k$ is abelian. The radical $\mathfrak{r} = \mathfrak{r}_0$ is a characteristic ideal of $\mathfrak{k}$ by theorem 2.5.13 of [25]. Furthermore, if $\mathfrak{r}_i$ is a characteristic ideal, then $\mathfrak{r}_{i+1}$ is a characteristic ideal, since

$$D\mathfrak{r}_{i+1} = D[\mathfrak{r}_i, \mathfrak{r}_i] \subset [D\mathfrak{r}_i, \mathfrak{r}_i] + [\mathfrak{r}_i, D\mathfrak{r}_i] \subset [\mathfrak{r}_i, \mathfrak{r}_i] = \mathfrak{r}_{i+1}.$$

It follows by induction that all the Lie algebras $\mathfrak{r}_0, \ldots, \mathfrak{r}_k$ are characteristic ideals of $\mathfrak{k}$. If $\mathfrak{k}$ is an ideal of a larger Lie algebra, $\mathfrak{g}$, the characteristic ideals of $\mathfrak{k}$ are ideals of $\mathfrak{g}$, since $ad_Z\colon \mathfrak{k} \to \mathfrak{k}$ is a derivation for each $Z \in \mathfrak{g}$, this implies that $ad_Z(\mathfrak{r}_i) \subset \mathfrak{r}_i$. This observation leads to the following statement:

Lemma 2. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{k} = \mathfrak{s} \ltimes \mathfrak{r}$ an ideal of $\mathfrak{g}$. If $\mathfrak{r}_0, \mathfrak{r}_1, \ldots$ is the derived sequence of $\mathfrak{r}$, then $\mathfrak{r}_i$ is an ideal of $\mathfrak{g}$ for each $i = 0, 1, \ldots$.*

For any Lie algebra $\mathfrak{h}$, we let $\mathfrak{z}(\mathfrak{h})$ denote the center of $\mathfrak{h}$:

$$\mathfrak{z}(\mathfrak{h}) = \{X \in \mathfrak{h} \mid [X,Y] = 0, \forall Y \in \mathfrak{h}\}.$$

Since the last nonzero term $\mathfrak{r}_k$ of the derived sequence is abelian, we have $\mathfrak{z}(\mathfrak{r}_k) = \mathfrak{r}_k$. The Lie algebra $\mathfrak{z}(\mathfrak{r}_i)$ is a characteristic ideal of $\mathfrak{k}$, for each i, because $\mathfrak{r}_i$ is a characteristic ideal:

$$[D\mathfrak{z}(\mathfrak{r}_i), \mathfrak{r}_i] \subset D[\mathfrak{z}(\mathfrak{r}_i), \mathfrak{r}_i] + [\mathfrak{z}(\mathfrak{r}_i), D\mathfrak{r}_i] \subset [\mathfrak{z}(\mathfrak{r}_i), \mathfrak{r}_i] = \{0\}.$$

This argument was adapted from [26]. It is obvious that the sum of characteristic ideals is a characteristic ideal, resulting in the following statement:

Proposition 1. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{k} = \mathfrak{s} \ltimes \mathfrak{r}$ *an ideal of* $\mathfrak{g}$*. If* $\mathfrak{r}_0, \mathfrak{r}_1, \ldots, \mathfrak{r}_k$ *is the derived sequence of* $\mathfrak{r}$ *and* $\mathfrak{r}_{k+1} = \{0\}$*, then the abelian Lie algebra* $\mathfrak{a}(\mathfrak{r}) \subset \mathfrak{r}$ *defined by*

$$\mathfrak{a}(\mathfrak{r}) = \mathfrak{z}(\mathfrak{r}_0) + \mathfrak{z}(\mathfrak{r}_1) + \ldots + \mathfrak{z}(\mathfrak{r}_k). \tag{6}$$

is a characteristic ideal of $\mathfrak{k}$*. In particular,* $\mathfrak{a}(\mathfrak{r})$ *is an ideal of* $\mathfrak{g}$*.*

It turns out that the abelian ideal $\mathfrak{a}(\mathfrak{r})$ intersects nontrivially with each ideal of $\mathfrak{r}$. This useful property will be utilized in the next section.

Proposition 2. *Let* $\mathfrak{i} \neq \{0\}$ *be an ideal of* $\mathfrak{r}$*. Then* $\mathfrak{i} \cap \mathfrak{a}(\mathfrak{r}) \neq \{0\}$*.*

Proof. Let $\mathfrak{r}_k$ be the last nontrivial element of the derived sequence of $\mathfrak{r}$. If $\mathfrak{i} \cap \mathfrak{r}_k \neq \{0\}$, then it is obvious that $\mathfrak{i} \cap \mathfrak{a}(\mathfrak{r}) \neq \{0\}$. If $\mathfrak{i} \cap \mathfrak{r}_k = \{0\}$, then there exists an integer $j < k$ such that $\mathfrak{i} \cap \mathfrak{r}_j \neq \{0\}$ and $\mathfrak{i} \cap \mathfrak{r}_{j+1} = \{0\}$. We have

$$[\mathfrak{i} \cap \mathfrak{r}_j, \mathfrak{r}_j] \subset \mathfrak{i} \cap \mathfrak{r}_{j+1} = \{0\},$$

which implies that $\mathfrak{i} \cap \mathfrak{r}_j \subset \mathfrak{z}(\mathfrak{r}_j) \subset \mathfrak{a}(\mathfrak{r})$. □

3.3. Killing horizons

Theorem 1 explains how to construct a relative differential invariant from an ideal of $\mathfrak{g}$, and proposition 1 gives us one particular such ideal, $\mathfrak{a}(\mathfrak{r})$. In this section we will show that $\mathfrak{a}(\mathfrak{r})$ can be used to construct a relative differential invariant that always vanishes on $\mathfrak{k}$-invariant Killing horizons.

We use the following definition, adapted from [27]:

Definition 3. Let (M, g) be a Lorentzian manifold with Killing algebra $\mathfrak{k}$. A Killing horizon on (M, g) is a null hypersurface $\mathcal{H} \subset M$ for which there exists a Killing vector $K \in \mathfrak{k}$ which satisfies $g(K, K)|_{\mathcal{H}} = 0$ and $K_a \in T_a\mathcal{H} \setminus \{0\}$ for each $a \in \mathcal{H}$.

The Killing vector K in the definition is called a generator of the Killing horizon, and it satisfies $K_a^{\perp} = T_a\mathcal{H}$ for each $a \in \mathcal{H}$. More generally, we will call K a generator of the Killing horizon $\mathcal{H}$ if $g(K, K)|_{\mathcal{H}} = 0$ and $K_a \in T_a\mathcal{H} \setminus \{0\}$ for each a in an open dense subset of $\mathcal{H}$. Let $\mathfrak{k}_{\mathcal{H}}$ denote the union of all generators of $\mathcal{H}$ and the trivial Killing vector $0 \in \mathfrak{k}$. By theorem 2 of [27], the set $\mathfrak{k}_{\mathcal{H}}$ is a Lie subalgebra of $\mathfrak{k}$. If $\dim(\mathfrak{k}_{\mathcal{H}}) > 1$, then $\mathcal{H}$ is called a multiple Killing horizon. Such horizons were treated in [27].

Remark 2. We will not worry about topological properties of $\mathcal{H}$, for example whether it is an embedded or injectively immersed hypersurface.

In order to focus in on Killing horizons that are geometrically special, we introduce the following definition:

Definition 4. Let (M, g) be a Lorentzian manifold, and $\mathfrak{k}$ the Lie algebra of its Killing vector fields. A Killing horizon $\mathcal{H} \subset M$ is $\mathfrak{k}$-invariant if each vector field in $\mathfrak{k}$ is tangent to $\mathcal{H}$. That is, $X_a \in T_a\mathcal{H} \subset T_aM$ for each $X \in \mathfrak{k}$ and $a \in \mathcal{H}$.

To motivate our focus on $\mathfrak{k}$-invariant Killing horizons, notice that if $\mathcal{H}$ is not $\mathfrak{k}$-invariant, then there exists a vector field $X \in \mathfrak{k}$ satisfying $X_a \notin T_a\mathcal{H}$ for some $a \in \mathcal{H}$. In this case, the flow φ_s of X gives a continuous family $\mathcal{H}_s = \varphi_s(\mathcal{H})$ of distinct Killing horizons in M, at least in a neighborhood of a. Since φ_s is an isometry on (M, g) for each s, there is no way to pick out a single, special Killing horizon using local arguments, as they are all geometrically (locally) equivalent. There are several well-known spacetimes that are foliated by Killing horizons in this way, the simplest example being the 2-dimensional Minkowski space $(\mathbb{R}^2, -\mathrm{d}t^2 + \mathrm{d}x^2)$, with Killing algebra $\langle \partial_t, \partial_x, x\partial_t + t\partial_x \rangle$, and Killing horizons generated by $(x - x_0)\partial_t + (t - t_0)\partial_x$ for each $(t_0, x_0) \in \mathbb{R}^2$.

The following statement follows straight from the definitions, but involves a useful property that we will use several times.

Lemma 3. *Let (M, g) be a Lorentzian manifold, and* $\mathfrak{k}$ *the Lie algebra of its Killing vector fields. Assume that there exists a Killing horizon* $\mathcal{H} \subset M$ *with generator K. Then* $\mathcal{H}$ *is* $\mathfrak{k}$*-invariant if and only if* $g(X,K)|_{\mathcal{H}} \equiv 0$ *for every* $X \in \mathfrak{k}$.

Proof. The Killing horizon $\mathcal{H}$ is $\mathfrak{k}$-invariant if and only if $X_a \in T_a\mathcal{H}$ for every $X \in \mathfrak{k}$ and every $a \in \mathcal{H}$. Since K is a generator of $\mathcal{H}$, we have $K_a \in (T_a\mathcal{H})^{\perp}$ for each $a \in \mathcal{H}$, implying that $g(X,K)|_{\mathcal{H}} \equiv 0$ for every $X \in \mathfrak{k}$. Conversely, if $g(X,K)|_{\mathcal{H}} \equiv 0$, then $X_a \in K_a^{\perp}$ for each $a \in \mathcal{H}$, implying that $X_a \in T_a\mathcal{H}$ for each a in an open dense subset of $\mathcal{H}$. By continuity of the vector field X, $\mathcal{H}$ is $\mathfrak{k}$-invariant. ☐

The following proposition is a consequence of lemma 3.

Proposition 3. *Let (M, g) be a Lorentzian manifold,* $\mathfrak{k}$ *its Lie algebra of Killing vectors, and assume that there exists a* $\mathfrak{k}$*-invariant Killing horizon* $\mathcal{H} \subset M$*. Let* $\mathfrak{i} = \langle K_1, \ldots, K_r \rangle$ *be a* $\mathfrak{g}$*-invariant ideal of* $\mathfrak{k}$ *satisfying* $\mathfrak{i} \cap \mathfrak{k}_{\mathcal{H}} \neq \emptyset$*. Then the function*

$$R_g^{\mathfrak{i}} = \|K_1 \wedge \cdots \wedge K_r\|_g^2$$

vanishes on $\mathcal{H}$*. Furthermore,* $R_g^{\mathfrak{i}}$ *is simply the restriction of the relative differential invariant of theorem 1 to g:* $R_g^{\mathfrak{i}} = R^{\mathfrak{i}} \circ g$.

Proof. We assume without loss of generality that $K_1 \in \mathfrak{k}_{\mathcal{H}}$. The quantity $g(K_1 \wedge \cdots \wedge K_r, K_1 \wedge \cdots \wedge K_r)$ can be expressed in terms of a sum of products of $g(K_i, K_j)$ where each term in the sum contains a factor of the form $g(K_1, K_j)$. Since each $X \in \mathfrak{k}$ is tangent to $\mathcal{H}$ (by $\mathfrak{k}$-invariance), and since K_1 is a generator of the null-hypersurface $\mathcal{H}$, it follows from lemma 3 that $g(K_1, X)|_{\mathcal{H}} \equiv 0$ for every $X \in \mathfrak{k}$. Thus

$$\left(R_g^{\mathfrak{i}}\right)|_{\mathcal{H}} = g\left(K_1 \wedge \cdots \wedge K_r, K_1 \wedge \cdots \wedge K_r\right)|_{\mathcal{H}} \equiv 0.$$

The last sentence of the proposition follows directly from theorem 1 (and this is the only place we use that $\mathfrak{i}$ is $\mathfrak{g}$-invariant). ☐

Remark 3. Our focus is on $\mathfrak{k}$-invariant Killing horizons for the reasons explained above. Still, it is worth pointing out that it is sufficient that $\mathcal{H}$ is $\mathfrak{i}$-invariant (not necessarily $\mathfrak{k}$-invariant) for the conclusion of proposition 3 to hold.

When $\mathcal{H}$ is a $\mathfrak{k}$-invariant Killing horizon, the Lie algebra $\mathfrak{k}_{\mathcal{H}}$ is not only a Lie subalgebra, but also an ideal in $\mathfrak{k}$. Before we prove that, we recall the following lemma from appendix A of [27].

Lemma 4 ([27]). *If* $X \neq 0$ *is a Killing vector of a Lorentzian manifold (M, g), then the subset on which X vanishes has codimension at least 2.*

This lemma is very useful. For example, if $\mathcal{H} \subset M$ is a $\mathfrak{k}$-invariant hypersurface, then there exists a Lie algebra homomorphism

$$\mathfrak{k} \to \mathfrak{k}|_{\mathcal{H}},$$

defined by restricting the vector fields of $\mathfrak{k}$ to $\mathcal{H}$ ($\mathfrak{k}|_{\mathcal{H}}$ should not be confused with $\mathfrak{k}_{\mathcal{H}}$). From lemma 4 it follows that this is a Lie algebra isomorphism.

Proposition 4. *Let (M, g) be a Lorentzian manifold, and $\mathfrak{k}$ the Lie algebra of its Killing vector fields. For any $\mathfrak{k}$-invariant Killing horizon $\mathcal{H} \subset M$, the Lie algebra $\mathfrak{k}_{\mathcal{H}}$ of generators of $\mathcal{H}$ forms an ideal in $\mathfrak{k}$.*

Proof. For $X \in \mathfrak{k}$ we have $L_X g = 0$, implying that

$$L_X\left(g\left(Y,Z\right)\right) = g\left(\left[X,Y\right],Z\right) + g\left(Y,\left[X,Z\right]\right), \qquad X,Y,Z \in \mathfrak{k}.$$

If $K \in \mathfrak{k}_{\mathcal{H}}$, then $g(K,Z)|_{\mathcal{H}} \equiv 0$ for every $Z \in \mathfrak{k}$, by lemma 3. By setting $Y = K$ in the above equation, we see that

$$g\left(\left[X,K\right],Z\right)|_{\mathcal{H}} = L_X\left(g\left(K,Z\right)\right)|_{\mathcal{H}} \equiv 0, \qquad X,Z \in \mathfrak{k}.$$

The last equality holds since $g(K,Z)|_{\mathcal{H}} \equiv 0$ and X is tangent to $\mathcal{H}$. Thus $[X,K]$ is either a generator for $\mathcal{H}$ or it vanishes on $\mathcal{H}$, in which case it vanishes everywhere by lemma 4. ☐

If $\mathfrak{k}_{\mathcal{H}}$ is one-dimensional, it is trivially solvable. Theorem 3 of [27] says that if $m = \dim \mathfrak{k}_{\mathcal{H}} \geqslant 2$, then there exists an abelian ideal $\mathfrak{k}_{\mathcal{H}}^{\mathrm{deg}} \subset \mathfrak{k}_{\mathcal{H}}$ of dimension at least $m-1$. This implies that $\mathfrak{k}_{\mathcal{H}}$ is solvable. Therefore, if $\mathcal{H}$ is $\mathfrak{k}$-invariant, then $\mathfrak{k}_{\mathcal{H}}$ is a solvable ideal of $\mathfrak{k}$. It follows that $\mathfrak{k}_{\mathcal{H}}$ is an ideal of the radical $\mathfrak{r}$ of $\mathfrak{k} = \mathfrak{s} \ltimes \mathfrak{r}$. Thus proposition 4 leads to the following corollary:

Corollary 1. *Let (M, g) be a Lorentzian manifold with Killing algebra $\mathfrak{k}$, and let $\mathfrak{s} \ltimes \mathfrak{r}$ be the Levi decomposition of $\mathfrak{k}$. If $\mathcal{H} \subset M$ is a $\mathfrak{k}$-invariant Killing horizon, then the ideal $\mathfrak{k}_{\mathcal{H}}$ of $\mathfrak{k}$ is contained in $\mathfrak{r}$. In particular, $\mathfrak{k}$ has a nontrivial radical ideal.*

Let us take the ideal $\mathfrak{a}(\mathfrak{r})$ from proposition 1. Since $\mathfrak{k}_{\mathcal{H}}$ is an ideal of $\mathfrak{r}$, it follows from proposition 2 that $\mathfrak{k}_{\mathcal{H}} \cap \mathfrak{a}(\mathfrak{r}) \neq \{0\}$. Proposition 3 then gives us a relative invariant that vanishes on $\mathcal{H}$. We summarize this in a theorem:

Theorem 2. *Let (M, g) be a Lorentzian manifold with Killing algebra $\mathfrak{k} = \mathfrak{s} \ltimes \mathfrak{r}$, let $\mathfrak{g}$ be a Lie algebra of vector fields on M containing $\mathfrak{k}$ as an ideal, and let $\mathfrak{a}(\mathfrak{r}) = \langle K_1, \ldots, K_r\rangle$ be the abelian ideal defined by (6). If $\mathcal{H} \subset M$ is a $\mathfrak{k}$-invariant Killing horizon, then the function*

$$R_g^{\mathfrak{a}(\mathfrak{r})} = \|K_1 \wedge \cdots \wedge K_r\|_g^2$$

vanishes on $\mathcal{H}$. The function $R_g^{\mathfrak{a}(\mathfrak{r})}$ is the restriction of the relative invariant of theorem 1 to g: $R_g^{\mathfrak{a}(\mathfrak{r})} = R^{\mathfrak{a}(\mathfrak{r})} \circ g$.

We have thus shown that the relative differential invariant $R^{\mathfrak{a}(\mathfrak{r})}$ defined on $\mathcal{E}^0 \subset J^0\pi$ detects $\mathfrak{k}$-invariant Killing horizons, in the sense that the function $R_g^{\mathfrak{a}(\mathfrak{r})} = R \circ g \colon M \to \mathbb{R}$ vanishes on them.

Note that while $R_g^{\mathfrak{a}(\mathfrak{r})}$ always vanishes on $\mathfrak{k}$-invariant Killing horizons, it is possible that a hypersurface given by $R_g^{\mathfrak{a}(\mathfrak{r})} = 0$ is not a Killing horizon. An example of this can be seen in section 4.2.2. It is also possible in general that $\{R_g^{\mathfrak{a}(\mathfrak{r})} = 0\} \subset M$ is not a hypersurface at all. We end this section by looking at two examples where $R_g^{\mathfrak{a}(\mathfrak{r})}$ vanishes everywhere on M, for two different reasons.

Example 2. The pp-wave

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 + \left(2\mathrm{d}v + H\left(x,y,u\right)\mathrm{d}u\right)\mathrm{d}u$$

on $\mathbb{R}^4$ has Killing algebra $\mathfrak{k} = \langle\partial_v\rangle$ for generic H. In this case $\mathfrak{a}(\mathfrak{r}) = \mathfrak{k}$, and the function $R_g^{\mathfrak{a}(\mathfrak{r})} = \|\partial_v\|_g^2$ vanishes everywhere. This can be explained by the fact that $(\mathbb{R}^4, g)$ is foliated by $\mathfrak{k}$-invariant Killing horizons. They are the level sets of the coordinate function u, and they have generator ∂_v.

Example 3. The particular highly symmetric pp-wave (see [28, p 194])

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 + \left(2\mathrm{d}v + x^2 \mathrm{d}u\right)\mathrm{d}u$$

on $\mathbb{R}^4$ has the six-dimensional Killing algebra

$$\mathfrak{k} = \langle \partial_y, \partial_u, \partial_v, u\partial_y - y\partial_v, e^{-u}\left(\partial_x + x\partial_v\right), e^{u}\left(\partial_x - x\partial_v\right)\rangle .$$

The abelian ideal $\mathfrak{a}(\mathfrak{r}) = \langle \partial_y, \partial_v \rangle$ gives $R_g^{\mathfrak{a}(\mathfrak{r})} = \|\partial_y \wedge \partial_v\|_g^2$. Since $\mathfrak{k}$ is transitive, there are no $\mathfrak{k}$-invariant Killing horizons. Still, it is clear that $R_g^{\mathfrak{a}(\mathfrak{r})}$ vanishes everywhere. This is because ∂_y and ∂_v are both tangent to the Killing horizons given by $u = \text{const}$.

4. Analysis of orbits on $\mathcal{E}^0$ and $\mathcal{E}^1$

In this section we collect examples of computation. We look at cases where $\mathfrak{k}$ is isomorphic to the Lie algebras

$$\mathfrak{so}(3) \oplus \mathbb{R}, \quad \mathbb{R}^2, \quad \mathfrak{sol}(2), \quad \mathfrak{so}(3),$$

which is the case for several well-known metrics. For each of these cases, we will compute the Lie algebra $\mathfrak{g}$ of vector fields preserving the coordinate form of $\mathfrak{k}$, and a PDE $\mathcal{E}$ whose solutions have Killing algebras containing $\mathfrak{k}$. Then we will analyze the orbits on $\mathcal{E}^0$, and in some cases also on $\mathcal{E}^1$. From this analysis we find a relative invariant that vanishes on physically important hypersurfaces. In the first example we will be quite detailed in order to clearly explain how the computations are done, while in the later examples we will be more brief.

When $\mathfrak{k}$ has a nontrivial ideal, in all the examples we examine, we will see that the relative invariant $R^{\mathfrak{a}(\mathfrak{r})}$ vanishes on the $\mathfrak{g}^{(0)}$-invariant hypersurface in $\mathcal{E}^0 \subset J^0\pi$ containing orbits of submaximal dimension.

4.1. Static and spherically symmetric ($\mathfrak{k} \simeq \mathfrak{so}(3) \oplus \mathbb{R}$)

Consider the Lie algebra

$$\mathfrak{k} = \left\langle \partial_t, \partial_\varphi, \sin(\varphi)\partial_\theta + \frac{\cos(\varphi)}{\tan(\theta)}\partial_\varphi, \cos(\varphi)\partial_\theta - \frac{\sin(\varphi)}{\tan(\theta)}\partial_\varphi \right\rangle$$

of vector fields defined on $\mathbb{R} \times (0,\infty) \times (0,\pi) \times (0,2\pi)$, where t, r, θ, ϕ are the respective coordinates on each factor.

Proposition 5. *The Lie algebra $\mathfrak{g}$ preserving the Lie algebra $\mathfrak{k}$ of Killing vectors is infinite-dimensional, and spanned by the vector fields*

$$t\partial_t, \quad a(r)\partial_t, \quad b(r)\partial_r,$$

in addition to the last three generators of $\mathfrak{k}$:

$$\partial_\varphi, \quad \sin(\varphi)\partial_\theta + \frac{\cos(\varphi)}{\tan(\theta)}\partial_\varphi, \quad \cos(\varphi)\partial_\theta - \frac{\sin(\varphi)}{\tan(\theta)}\partial_\varphi .$$

The functions a and b are arbitrary locally defined C^∞ functions on the interval $(0,\infty) \subset \mathbb{R}$.

Notice that the vector field ∂_t is already included in $\mathfrak{g}$, just set $a(r) \equiv 1$. A metric has Killing algebra containing $\mathfrak{k}$ if and only if it takes the form

$$g = g_{11}(r)\,\mathrm{d}t^2 + g_{12}(r)\,\mathrm{d}t\mathrm{d}r + g_{22}(r)\,\mathrm{d}r^2 + g_{33}(r)\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\right),$$

where $g_{11}, g_{12}, g_{22}, g_{33}$ are functions on $(0,\infty)$. The PDE $\mathcal{E}$ determining g is given by

$$\mathcal{E}^0 = \left\{u_{13} = 0, u_{14} = 0, u_{23} = 0, u_{24} = 0, u_{34} = 0, u_{44} = \sin^2(\theta)\, u_{33}\right\}$$

on $J^0\pi$. The differential constraints of $\mathcal{E}^1 \subset J^1\pi$ are given by the total derivatives of the constraints of $\mathcal{E}^0$ and the additional constraints $(u_{ij})_t = 0, (u_{ij})_\theta = 0, (u_{ij})_\varphi = 0$ for $1 \le i \le j \le 2$ and $i = j = 3$. For each $k \geqslant 0$ we have $\dim \mathcal{E}^k = 8 + 4k$. The restriction of the horizontal symmetric 2-form h from section 2.1 to $\mathcal{E}^0$ is given by

$$h = u_{11}\mathrm{d}t^2 + u_{12}\mathrm{d}t\mathrm{d}r + u_{22}\mathrm{d}r^2 + u_{33}\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\right).$$

Each element $X \in \mathfrak{g}$ can be lifted to a vector field $X^{(0)}$ on $\mathcal{E}^0$, by the requirement that $L_{X^{(0)}} h = 0$. The lifts of the first three vector fields in proposition 5 are given by

$$t\partial_t - 2u_{11}\partial_{u_{11}} - u_{12}\partial_{u_{12}},\ a\partial_t - 2a'u_{11}\partial_{u_{12}} - a'u_{12}\partial_{u_{22}},\ b\partial_r - b'u_{12}\partial_{u_{12}} - 2b'u_{22}\partial_{u_{22}},$$

respectively, while the lifts of the elements in $\mathfrak{k}$ are trivial (they have no vertical components).

As $\mathfrak{a}(\mathfrak{r}) = \langle\partial_t\rangle$, the relative differential invariant of theorem 1 is given by

$$R^{\mathfrak{a}(\mathfrak{r})} = \|\partial_t\|_h^2 = u_{11}.$$

Proposition 6. *Generic $\mathfrak{g}^{(0)}$-orbits on $\mathcal{E}^0$ are 7-dimensional. The field of absolute invariants on $\mathcal{E}^0$ is generated by the absolute invariant u_{33}. The subset of $\mathcal{E}^0$ on which the orbit dimension is less than 7 is given exactly by $R^{\mathfrak{a}(\mathfrak{r})} = 0$. All orbits on this 7-dimensional subset are 6-dimensional.*

Proof. It is easy to verify that u_{33} is an absolute invariant. Since $\dim \mathcal{E}^0 = 8$ this implies that the dimension of $\mathfrak{g}^{(0)}$-orbits is at most 7. To verify that this is upper bound is attained, we look at the lift of the 8-dimensional Lie subalgebra of $\mathfrak{g}$ spanned by

$$\left\langle t\partial_t, \partial_t, r\partial_t, \partial_r, r\partial_r, \partial_\varphi, \sin(\varphi)\,\partial_\theta + \frac{\cos(\varphi)}{\tan(\theta)}\partial_\varphi, \cos(\varphi)\,\partial_\theta - \frac{\sin(\varphi)}{\tan(\theta)}\partial_\varphi \right\rangle.$$

Lining the generators up in an 8×8 matrix and computing the rank, shows that the generic rank is 7, implying that the upper bound is attained by generic orbits. The rank of this matrix drops when $u_{11}u_{22} - 4u_{12}^2 = 0$ or $u_{11} = 0$. (These computations are well suited for computer algebra systems. We have used Maple™ with its PolynomialIdeals package.) The first of these equations never holds since we assume that h is nondegenerate. The second equation is exactly $R^{\mathfrak{a}(\mathfrak{r})} = 0$. To see that the same result is true for the infinite-dimensional Lie algebra $\mathfrak{g}^{(0)}$, we look at the expressions for the lifted Lie algebras. When $u_{11} = 0$, the vector fields of $\mathfrak{g}^{(0)}$ have vanishing $\partial_{u_{11}}$-component and vanishing $\partial_{u_{33}}$-component, implying that the rank on the subset given by $u_{11} = 0$ is never greater than 6. It is also easy to check that the rank of the 8×8 matrix never drops below 6 as long as h is nondegenerate. □

Notice that the absolute invariant u_{33} can also be described in terms of the Lie algebra $\mathfrak{k}$:

$$\|\partial_\varphi\|_h^2 + \left\|\sin(\varphi)\,\partial_\theta + \frac{\cos(\varphi)}{\tan(\theta)}\partial_\varphi\right\|_h^2 + \left\|\cos(\varphi)\,\partial_\theta - \frac{\sin(\varphi)}{\tan(\theta)}\partial_\varphi\right\|_h^2 = 2u_{33}.$$

More generally, let Y_1, Y_2, Y_3 be a basis of the ideal $\mathfrak{so}(3)$ of $\mathfrak{k}$ which is orthonormal with respect to the Killing form on $\mathfrak{so}(3)$. Then $\|Y_1\|_h^2 + \|Y_2\|_h^2 + \|Y_3\|_h^2$ is proportional to u_{33}.

4.1.1. The Reissner–Nordström metric. As an example, for the Reissner–Nordström metric

$$g = -\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$$

we have

$$R_g^{\mathfrak{a}(\mathfrak{r})} = \|\partial_t\|_g^2 = -\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) c^2.$$

This function vanishes exactly on the event horizon, which is a $\mathfrak{k}$-invariant Killing horizon with generator ∂_t. For the Schwarzschild metric, we have $r_Q = 0$ and $R_g^{\mathfrak{a}(\mathfrak{r})} = -(1 - r_s/r)c^2$.

4.2. Stationary and axisymmetric ($\mathfrak{k} \simeq \mathbb{R}^2$)

Consider the 2-dimensional abelian Lie algebra

$$\mathfrak{k} = \langle \partial_{x^0}, \partial_{x^1} \rangle$$

on $M = \mathbb{R}^4(x^0, x^1, x^2, x^3)$.

Proposition 7. *The Lie algebra $\mathfrak{g}$ of vector fields preserving the Lie algebra $\mathfrak{k}$ of Killing vectors is infinite-dimensional and spanned by the vector fields*

$$x^0\partial_{x^0}, \quad x^0\partial_{x^1}, \quad x^1\partial_{x^0}, \quad x^1\partial_{x^1}, \quad \sum_{i=0}^{3} a^i\left(x^2, x^3\right)\partial_{x^i},$$

where the functions a^{i} are arbitrary locally defined C^∞ functions on the real plane $\mathbb{R}^2$.

The general metric admitting these Killing vectors is given by

$$g = \sum_{i \leqslant j} g_{ij}\left(x^2, x^3\right) \mathrm{d}x^i \mathrm{d}x^j.$$

It is the solution to the PDE system defined by

$$\mathcal{E}^1 = \left\{(u_{ij})_{x^0} = 0, (u_{ij})_{x^1} = 0 \mid 0 \leqslant i \leqslant j \leqslant 3\right\}.$$

In this case $\mathcal{E}^0 = S^2_{\mathrm{Lor}} T^* M$. The horizontal symmetric 2-form h is given by

$$h = \sum_{i \leqslant j} u_{ij} \mathrm{d}x^i \mathrm{d}x^j.$$

Since $\mathfrak{k}$ is abelian, there is *a priori* no special Killing vectors. In fact, the Lie pseudogroup corresponding to $\mathfrak{g}$ acts transitively on the 2-dimensional space of Killing vectors. We have $\mathfrak{a}(\mathfrak{r}) = \mathfrak{k}$, and the relative invariant $R^{\mathfrak{a}(\mathfrak{r})}$ of theorem 1 is given by

$$R^{\mathfrak{a}(\mathfrak{r})} = \|\partial_{x^0} \wedge \partial_{x^1}\|_h^2 = u_{00}u_{11} - u_{01}^2/4.$$

Proposition. *There is a (14-dimensional) open $\mathfrak{g}^{(0)}$-orbit on $\mathcal{E}^0$, and thus no absolute invariants on $\mathcal{E}^0$. The subset of $\mathcal{E}^0$ on which the orbit dimension is less than 14 is a reducible algebraic set, whose 13-dimensional component is given exactly by $R^{\mathfrak{a}(\mathfrak{r})} = 0$.*

Proof. To show that there is a 14-dimensional open orbit on $\mathcal{E}^0$, take for example the 16 independent vector fields in $\mathfrak{g}$ having polynomial coefficients of degree $\leqslant 1$, and lift them to $\mathcal{E}^0$. It is easily verified that the rank at a generic point is 14. Next, one can check that the rank of these 16 vector fields drops exactly on the algebraic subset $\mathcal{S}_{13} \cup \mathcal{S}_{11} \subset \mathcal{E}^0$ where $\mathcal{S}_{13} = \{R^{\mathfrak{a}(\mathfrak{r})} = 0\}$ has dimension 13 and $\mathcal{S}_{11}$ is a subvariety of dimension 11 (we remind that $\det(h)$, which appears in these computations, never vanishes on $S^2_{\mathrm{Lor}} T^*M$). In the end we verify that the set $\mathcal{S}_{13} \cup \mathcal{S}_{11}$ is $\mathfrak{g}^{(0)}$-invariant (since we so far used only a 16-dimensional Lie subalgebra of $\mathfrak{g}^{(0)}$). ☐

Thus, there is a unique $\mathfrak{g}^{(0)}$-invariant hypersurface in $\mathcal{E}^0$, and it is given by $R^{\mathfrak{a}(\mathfrak{r})} = 0$.

We note that the analysis provided in this section is similar to that in [16] where they provide a complete list of all absolute differential invariants for such spacetimes. However, our focus is on finding relative differential invariants of the lowest possible order that detect the Killing horizon.

4.2.1. The Kerr–Newman metric.

As an example, consider the Kerr–Newman metric in coordinates $x^0 = t, x^1 = \varphi, x^2 = r, x^3 = \theta$ whose nonzero components are given by [17]:

$$g_{00} = -\frac{\Delta - a^2 \sin^2(\theta)}{\Sigma}, \qquad g_{01} = -\frac{2a \sin^2(\theta)\left(r^2 + a^2 - \Delta\right)}{\Sigma}$$

$$g_{11} = \frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2(\theta)}{\Sigma} \sin^2(\theta), \qquad g_{22} = \frac{\Sigma}{\Delta}, \qquad g_{33} = \Sigma,$$

where $\Sigma = r^2 + a^2 \cos^2(\theta)$ and $\Delta = r^2 + a^2 + e^2 - 2Mr$. In this case, we have

$$R_g^{\mathfrak{a}(\mathfrak{r})} = -\sin^2(\theta)\left(r^2 - 2Mr + \left(a^2 + e^2\right)\right),$$

which vanishes exactly on the event horizon, which is well-known to be a Killing horizon. In this case, we can compute $R_g^{\mathfrak{a}(\mathfrak{r})}$ directly from the two Killing vector fields, which is simpler than computing the preferred Killing vector field which becomes null on the horizon or computing up to the first covariant derivative of the curvature tensor to calculate a horizon detecting invariant.

4.2.2. A metric of Eichhorn and Held.

The relative differential invariant $R^{\mathfrak{a}(\mathfrak{r})}$ may also detect hypersurfaces that are not Killing horizons. This is expected as the vanishing of $R^{\mathfrak{a}(\mathfrak{r})}$ is a necessary condition but not a sufficient condition for a ($\mathfrak{k}$-invariant) Killing horizon. In the previous examples, the Killing horizon was already known and no further work was needed to determine them. When the Killing horizon is not known *a priori*, one must check among other things that each of the hypersurfaces defined by $R^{\mathfrak{a}(\mathfrak{r})} = 0$ is null and that all Killing vector-fields in $\mathfrak{a}(\mathfrak{r})$ are non-zero when evaluated on each hypersurface. We note that these steps must also be taken when determining the Killing horizon from the norm of a Killing vector field [17], or when using scalar polynomial curvature invariants [8, 9]. Consider the stationary and axisymmetric spacetime given in [29]:

$$\begin{aligned} g = &-\frac{r^2 - 2\tilde{m}r + a^2\chi^2}{r^2 + a^2\chi^2} \mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r - \frac{4\tilde{m}ar}{r^2 + a^2\chi^2}\left(1 - \chi^2\right)\mathrm{d}u\mathrm{d}\phi \\ &- 2a\left(1 - \chi^2\right)\mathrm{d}r\mathrm{d}\phi + \frac{r^2 + a^2\chi^2}{1 - \chi^2}\mathrm{d}\chi^2 \\ &+ \frac{1 - \chi^2}{r^2 + a^2\chi^2}\left(\left(a^2 + r^2\right)^2 - a^2\left(r^2 - 2\tilde{m}r + a^2\right)\left(1 - \chi^2\right)\right)\mathrm{d}\phi^2. \end{aligned} \tag{7}$$

Here $u, r, \phi, \chi = \cos(\theta)$ are coordinates and $\tilde{m}$ is a function of r and χ. The Lie algebra of Killing vectors is 2-dimensional: $\mathfrak{k} = \langle \partial_u, \partial_\phi \rangle$. The invariant $R^{\mathfrak{a}(\mathfrak{k})}$ restricted to g takes the form

$$R_g^{\mathfrak{a}(\mathfrak{k})} = \|\partial_\varphi \wedge \partial_u\|_g^2 = \left(1-\chi^2\right)\left(2\tilde{m}\left(r,\chi\right)r - a^2 - r^2\right).$$

We have $\chi \in (-1,1)$ since $\theta \in (0,\pi)$. Therefore, the invariant vanishes if and only if $f := (2\tilde{m}(r,\chi)r - a^2 - r^2) = 0$, and we use the notation $\mathcal{H} = \{f = 0\}$. It is clear that none of the Killing vector fields vanish on $\mathcal{H}$. However, the surface $\mathcal{H}$ is not null, since the normal vector field

$$\begin{aligned} g^{-1}\left(\mathrm{d}f,\cdot\right)|_{\mathcal{H}} = {} & \frac{2\left(a^2+r^2\right)\left(r\partial_r\tilde{m}+\tilde{m}-r\right)}{a^2\chi^2+r^2}\partial_u \\ & - \frac{2\left(\chi^2-1\right)r\partial_\chi\tilde{m}}{a^2\chi^2+r^2}\partial_\chi + \frac{2a\left(r\partial_r\tilde{m}+\tilde{m}-r\right)}{a^2\chi^2+r^2}\partial_\phi, \end{aligned} \tag{8}$$

has, in general, non-vanishing norm:

$$\|\mathrm{d}f\|_g^2|_{\mathcal{H}} = -\frac{4\left(\chi^2-1\right)r^2\left(\partial_\chi\tilde{m}\right)^2}{a^2\chi^2+r^2}.$$

4.3. Two-dimensional solvable ($\mathfrak{k} \simeq \mathrm{sol}(2)$)

Consider the Lie algebra

$$\mathfrak{k} = \langle \partial_v, v\partial_v - r\partial_r \rangle$$

on $\mathbb{R}^4(x^1, x^2, r, v)$.

Proposition 9. *The Lie algebra $\mathfrak{g}$ of vector fields preserving the Lie algebra $\mathfrak{k}$ of Killing vectors are spanned by the vector fields*

$$\partial_v, \quad v\partial_v - r\partial_r, \quad a_1(x)\partial_{x^1} + a_2(x)\partial_{x^2} + a_3(x)r\partial_r + a_4(x)r^{-1}\partial_v.$$

The functions a^i, $i = 1,2,3,4$ are arbitrary locally defined C^∞functions on the real plane, $\mathbb{R}^2$.

The general invariant metric takes the form

$$\begin{aligned} g = {} & \frac{g_{33}(x)}{r^2}\mathrm{d}r^2 + \frac{g_{i3}(x)}{r}\mathrm{d}x^i\,\mathrm{d}r + \mathrm{d}v\left(g_{34}(x)\,\mathrm{d}r + rg_{i4}(x)\,\mathrm{d}x^i + r^2g_{44}(x)\,\mathrm{d}v\right) \\ & + g_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j, \end{aligned} \tag{9}$$

where Einstein notation is used and the indices satisfy $i \leqslant j$ in the second line. If we write the horizontal symmetric form as

$$h = \frac{u_{33}}{r^2}\mathrm{d}r^2 + \frac{u_{i3}}{r}\mathrm{d}x^i\,\mathrm{d}r + \mathrm{d}v\left(u_{34}\mathrm{d}r + ru_{i4}\mathrm{d}x^i + r^2u_{44}\mathrm{d}v\right) + u_{ij}\mathrm{d}x^i\,\mathrm{d}x^j,$$

then the PDE $\mathcal{E}$ is given by the following first-order system and its derivatives:

$$\mathcal{E}^1 = \left\{\left(u_{ij}\right)_r = 0, \left(u_{ij}\right)_v = 0\right\}.$$

Let us focus on a neighborhood of $r = 0$, and consider the sub-PDE $\mathcal{E}_0^1 \subset \mathcal{E}^1$ given by the additional constraints $u_{13} = u_{23} = u_{33} = 0$, and their derivatives (in particular $\mathcal{E}_0^0 = \{u_{13} = u_{23} = u_{33} = 0\}$). Its general solution has the form of g above, but with g_{13}, g_{23}, g_{33} identically equal to zero. This is the general expression of a Near Horizon geometry (see for example [27, section 4.5]).

As we have considered the subclass of regular spacetimes, we will consider vector fields that are regular around $r = 0$ as well. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be the Lie algebra of vector fields defined around $r = 0$, i.e. those with $a_4 \equiv 0$.

From the two invariant ideals, $\mathfrak{a}(\mathfrak{r}) = \langle \partial_v \rangle$ and $\mathfrak{k}$ itself, we get from theorem 1 two relative differential invariants (with respect to $\mathfrak{g}_0$) of order 0:

$$R^{\mathfrak{a}(\mathfrak{r})}|_{\mathcal{E}^0_0} = \|\partial_v\|_h^2 = r^2 u_{44}, \qquad R^{\mathfrak{k}}|_{\mathcal{E}^0_0} = \|\partial_v \wedge (v\partial_v - r\partial_r)\|_h^2 = -\frac{r^2 u_{34}^2}{4}.$$

Due to assumed nondegeneracy of h, u_{34} never vanishes. It is easily verified that the two relative invariants have the same weight, implying that their ratio u_{44}/u_{34}^2 is an absolute differential invariant.

Proposition 10. *Generic $\mathfrak{g}_0^{(0)}$-orbits on $\mathcal{E}^0_0$ are 10-dimensional. The field of rational absolute invariants on $\mathcal{E}^0$ is generated by the absolute invariant u_{44}/u_{34}^2. The orbit dimension drops exactly on the set $\{r = 0\} \subset \mathcal{E}^0_0$.*

4.3.1. Near Horizon geometries. For the Near Horizon geometries, the hypersurface $\mathcal{H} = \{r = 0\}$ is well-known to be a Killing horizon [30]. In fact, it is a multiple Killing horizon since $\dim(\mathfrak{k}_{\mathcal{H}}) = 2$ (see [27]):

$$\|(v - v_0)\partial_v - r\partial_r\|_g^2 = r(v - v_0)(g_{44}(x)\, r(v - v_0) - g_{34}(x)).$$

This computation also shows that the spacetime is foliated by Killing horizons, given by $v = v_0$. The latter ones are not $\mathfrak{k}$-invariant, because $\partial_v \in \mathfrak{k}$, which explains why they are not detected by $R^{\mathfrak{a}(\mathfrak{r})}|_{\mathcal{E}^0_0}$. On the other hand, the restriction of $R^{\mathfrak{a}(\mathfrak{r})}|_{\mathcal{E}^0_0}$ to a Near Horizon metric does vanish on $\mathcal{H}$, consistent with theorem 2.

4.4. Spherical symmetry ($\mathfrak{k} \simeq \mathfrak{so}(3)$)

Consider the Lie algebra

$$\mathfrak{k} = \left\langle \partial_\varphi, \sin(\varphi)\partial_\theta + \frac{\cos(\varphi)}{\tan(\theta)}\partial_\varphi, \cos(\varphi)\partial_\theta - \frac{\sin(\varphi)}{\tan(\theta)}\partial_\varphi \right\rangle$$

defined on the open chart $\mathbb{R} \times (0,\infty) \times (0,\pi) \times (0,2\pi)$ with coordinates t, r, θ, φ. This Lie algebra is abstractly $\mathfrak{so}(3)$. Since $\mathfrak{k}$ is simple, the only ideal that can be used in the context of theorem 1 is $\mathfrak{k}$ itself. However, since $\mathfrak{k}$ is 3-dimensional while the distribution it spans in $T\mathcal{E}^0$ is 2-dimensional, the function $R^{\mathfrak{k}}$ is just vanishing identically. However, we can still find relative invariants by analyzing orbits on $J^0\pi$ and $J^1\pi$.

Proposition 11. *The Lie algebra $\mathfrak{g}$ of vector fields preserving $\mathfrak{k}$ is spanned by the elements of $\mathfrak{k}$ and the vector fields of the form:*

$$a(t,r)\partial_t + b(t,r)\partial_r.$$

The functions a and b are arbitrary locally defined C^∞ functions on the direct product of $\mathbb{R}$ and the interval $(0,\infty) \subset \mathbb{R}$.

The general metric admitting these Killing vectors is given by

$$g = g_{11}(t,r)\,\mathrm{d}t^2 + g_{12}(t,r)\,\mathrm{d}t\mathrm{d}r + g_{22}(t,r)\,\mathrm{d}r^2 + g_{33}(t,r)\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\right).$$

Writing the horizontal symmetric form h as

$$h = u_{11}\mathrm{d}t^2 + u_{12}\mathrm{d}t\mathrm{d}r + u_{22}\mathrm{d}r^2 + u_{33}\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\right),$$

the PDE $\mathcal{E}$ determining g is given by

$$\mathcal{E}^0 = \left\{u_{13} = 0, u_{14} = 0, u_{23} = 0, u_{24} = 0, u_{34} = 0, u_{44} = \sin^2(\theta)\, u_{3,3}\right\}$$

on $J^0\pi$. The differential constraints of $\mathcal{E}^1 \subset J^1\pi$ are given by the total derivatives of the constraints of $\mathcal{E}^0$, in addition to the constraints $(u_{ij})_\theta = 0, (u_{ij})_\varphi = 0$ for $1 \leq i \leq j \leq 2$ and $i = j = 3$. In particular $\dim \mathcal{E}^0 = 8$ and $\dim \mathcal{E}^1 = 16$.

Proposition 12. *All $\mathfrak{g}^{(0)}$-orbits on $\mathcal{E}^0$ are 7-dimensional. Generic $\mathfrak{g}^{(1)}$-orbits on $\mathcal{E}^1$ are 14-dimensional, and the orbit dimension drops on the subset given by $(u_{33})_t = 0, (u_{33})_r = 0$. The field of absolute invariants on $\mathcal{E}^1$ is generated by the two invariants*

$$I = u_{33}, \qquad J = \|\bar{d}I\|_h^2 = \frac{4\left(u_{11}\,(u_{33})_r^2 - u_{12}\,(u_{33})_r\,(u_{33})_t + u_{22}\,(u_{33})_t^2\right)}{4u_{11}u_{22} - u_{12}^2}.$$

Notice that the absolute invariant I can be described in terms of the Lie algebra $\mathfrak{k}$ in the same way as was done in section 4.1.

Among the invariant hypersurfaces that can be singled out by a function of the form $f(I,J) = 0$, there is a special (irreducible) one, namely the one containing the $\mathfrak{g}^{(1)}$-invariant subset $\{(u_{33})_t = 0, (u_{33})_r = 0\} \subset \mathcal{E}^1$. It is given by $J = 0$ or, equivalently, by the vanishing of the relative invariant

$$Q = u_{11}\,(u_{33})_r^2 - u_{12}\,(u_{33})_r\,(u_{33})_t + u_{22}\,(u_{33})_t^2\,.$$

4.4.1. The imploding spherically symmetric metric.

Let us consider the imploding spherically symmetric metric in advanced coordinates (see [12]):

$$g = -2e^{\beta(t,r)}\left(1 - \frac{2m(t,r)}{r}\right)\mathrm{d}t^2 + 2e^{\beta(t,r)}\mathrm{d}t\mathrm{d}r + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\right).$$

We have $Q_g = -8e^{\beta(t,r)}(r - 2m(t,r))r$, which vanishes on the 'future outer trapping horizon' given by $r = 2m(t,r)$. In general this horizon will not be a null hypersurface, except when $m_{,t} = 0$ when it becomes an isolated horizon. Otherwise it will be a spacelike or timelike hypersurface [31].

5. Discussion

Invariantly defined hypersurfaces play an important role in general relativity. These hypersurfaces give physical insight into the nature of solutions to Einstein's field equations. An important example of such a hypersurface is the boundary of a black hole region. For most black hole spacetimes, determining this boundary can be difficult. However, for idealized black hole spacetimes satisfying Einstein's field equations, the existence of a timelike Killing vector field implies that the boundary of the black hole spacetime is a null hypersurface known as a Killing horizon. Such hypersurfaces can be characterized using the norm of a Killing vector field that acts as a generator of the hypersurface or using curvature invariants [9, 11].

However, finding the appropriate Killing vector field for a Killing horizon can be a non-trivial task when there are several Killing vector fields. This is best illustrated in the case of the Kerr spacetime [17]. Similarly, the use of curvature invariants to construct detectors requires the calculation of the curvature tensor and its covariant derivatives, and may only guarantee necessary conditions for detection of a Killing horizon. Necessary and sufficient conditions can be determined using Cartan invariants but this requires further calculation to determine a geometrically preferred frame for the spacetime.

While it is reasonable to suspect that for a given class of black hole solutions, the horizon can be detected by a relative differential invariant, such as with the stationary axisymmetric black hole solutions contained in [15, 16], it is less obvious that such invariants can be singled out in a systematic way without *a priori* knowledge of the location of the horizon. Our construction resolves this by explicitly providing relative differential invariants that always vanish on $\mathfrak{k}$-invariant Killing horizons. In this paper we focused on the Lie algebra $\mathfrak{g}$ preserving a fixed, but general, finite-dimensional Lie algebra $\mathfrak{k}$ of Killing vectors of a family of spacetimes, and computed relative invariants with respect to the prolongation of $\mathfrak{g}$ on appropriate jet bundles. We showed in theorem 1 that any finite-dimensional ideal of $\mathfrak{g}$, $\mathfrak{i} = \langle K_1, \ldots, K_r \rangle$, gives rise to a relative differential invariant of order 0:

$$R^{\mathfrak{i}} = \|K_1^{(0)} \wedge \cdots \wedge K_r^{(0)}\|_h^2.$$

To properly model black hole horizons, which are located in a specific region of a spacetime, we considered $\mathfrak{k}$-invariant Killing horizons, which are invariant under the group of isometries. We showed that there exists a particular $\mathfrak{g}$-invariant abelian ideal $\mathfrak{i} = \mathfrak{a}(\mathfrak{r})$ of $\mathfrak{k}$. Theorem 2 guarantees that the corresponding relative differential invariant $R^{\mathfrak{a}(\mathfrak{r})}$ always vanishes on $\mathfrak{k}$-invariant Killing horizons. In comparison to other horizon detection approaches, relative differential invariants are easier to use, as the calculation of the relative differential invariants is algorithmic and only involves an arbitrary basis of the Killing vector fields. Furthermore, once the Killing algebra is known, the approach presented here yields necessary conditions for the detection of a Killing horizon.

For several concrete examples of $\mathfrak{k}$, we also directly analyzed the $\mathfrak{g}^{(0)}$-orbits in $J^0\pi$ and $\mathfrak{g}^{(1)}$-orbits in $J^1\pi$ to successfully produce relative differential invariants of order 0 or 1 that vanish on horizons of several well-known black hole spacetimes. The obtained relative invariants are compared with $R^{\mathfrak{a}(\mathfrak{r})}$ in the cases where it makes sense. While this second approach is computationally more cumbersome than the first, it is also more general. For example, when the Killing algebra is $\mathfrak{so}(3)$ there are no $\mathfrak{k}$-invariant Killing horizons, but there is a relative differential invariant of order 1 that detects the unique spherically symmetric apparent horizon of imploding spherically symmetric metrics. We note that while the latter approach is more technical, it allows for all relative differential invariants to be determined up to a particular order. If one has no initial physical intuition about a spacetime, the study of relative differential invariants and their zero-sets can give insight into the spacetime.

This last example motivates the investigation of horizon detecting relative differential invariants for more general black hole solutions. The methods used in this paper could be extended to study conformal Killing horizons [32] by examining conformal Killing algebras. This could be applied to black hole solutions conformal to stationary black holes and give further insight into conformal Killing horizons as a valid black hole boundary for dynamical black holes [33, 34]. More generally, for explicit classes of dynamical black hole solutions, such as the Robinson-Trautmann class of solutions [35] or black hole spacetimes within the class of LRS spacetimes [36], this approach may be able to provide insight on the appropriate boundary.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary information files).

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