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# Induced Currents and Aharonov–Bohm Effect in Effective Fermion Models and in Spaces with a Compact Dimension

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**Abstract:** We consider fermion models in 3D- and 5D-space-time with an Aharonov–Bohm potential and a domain wall. Induced current is calculated, which is due to vacuum effects in the topologically nontrivial space-time. Violation of chiral symmetry and appearance of induced current is demonstrated in a simple example of quantum mechanical violation of symmetry in a model of a massless Dirac fermion moving in a background vector field and domain walls as barriers for the electron propagation. The effective Dirac equation for massless electrons modeling monolayer graphene is used. One of the solutions to the problem of describing domain walls in planar systems is reduced to finding exact analytic solutions. In this paper, we consider appearance of induced current in two-fermion model with a compact dimension as a result of vacuum polarization in the field of the external gauge field in the 4 + 1 and the 2 + 1 dimensional models with one type of fermions and with two types of fermions living in the brane and in the bulk. Two different approaches (Kaluza–Klein and Aharonov–Bohm) to the problem of induced current are used. Production of an induced current in a planar model with a thin solenoid is also studied.

**Keywords:** low-dimensional models; graphene; polyacetylene; induced current; chiral symmetry

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## 1. Introduction

Since the fundamental works of Kaluza and Klein [1,2], the idea of compactification of extra dimensions has been extensively developed: we mention here the idea of brane world [3,4]; the two-fermion model with additional dimension [5–7] was studied; and the idea of Yukawa unification, extended to the so-called "Hosotani mechanism" [8], explained Higgs particle appearance due to extra dimensional gauge field $A_5$ which plays the role of the Higgs field[9]. In this way, we also mention development of low-dimensional models in field theory and their realization in condensed-matter physics, such as a condensed-matter simulation of 3-dimensional anomaly [10], graphene [11–13] with simulation of anomalous Hall effect, where Dirac equation is effectively used to describe the behavior of the model.

The role of electromagnetic fields in low-dimensional models with nontrivial topology was discussed in the graphene model [14], fullerenes and carbon nanotubes [15–20], as well as the influence of an external magnetic field on the fermion mass generation was studied like in the Aharonov–Bohm problem in [21].

In these low-dimensional systems, studies of symmetry properties such as breaking of chiral symmetry [22], as well as of problems with fermion mass generation [23], were performed.

Nontrivial topological properties of space-time/gauge field lead to appearance of fermion zero modes [24]. In turn, this may cause induced vacuum fermion current [8,25–28]. Recently, in [25], the effect of vacuum polarization in the field of a solenoid at distances much larger than its radius was investigated. The induced current was calculated as finite periodical function of the magnetic flux. As an application of their result, they considered the graphene in the field of solenoid perpendicular to the plane of a sample.

Two-dimensional tight-binding quantum systems can be effectively described in the continuum approximation by the Dirac equation in (2 + 1)-dimensional space-time. This equation is obtained by linearizing the energy as a function of a momentum near Dirac points. The topological properties in this framework were accounted for in [27,29,30].

Clearly, more new important applications of low-dimensional structures can be realized when the transport problems in them are well understood. For instance, very interesting results were obtained in studying graphene under the influence of an Aharonov–Bohm flux and a constant magnetic field [31], where induced vacuum current, as well as an electric current were shown to arise (see also in [32], where induced current and transmission through the barrier in the four-fermion model in 2 + 1 dimensions were considered). We mention here yet another publication in this direction [33], where vacuum current induced by an axial-vector condensate and electron anomalous magnetic moment in a magnetic field was studied.

The aims of this paper is to study violation of chiral symmetry as a fundamental phenomenon (a simple example of quantum mechanical violation of symmetry is considered in a model of a massless Dirac fermion moving in a background vector field (Section 2)) and domain walls (Section 3) as barriers for the electron propagation [34] by using the effective Dirac equation for massless electrons in (monolayer) graphene [35–38]. In this way, the problem of describing domain walls in planar systems is reduced to finding exact analytic solutions in a simple way. Note, in particular, that the pseudopotentials (for definition of pseudopotential in graphene structures, see, e.g., in [39]) in the form of kink-type barriers with Pauli-matrix coefficients, mimicking the pseudospin and valley structure of the barrier, may be considered as limiting cases of induced gauge fields arising due to perturbations in the hopping parameters [38,40,41]. Therefore, we consider appearance of induced current in two-fermion model with a compact dimension (Section 4), and in Section 5, we also study the induced current originating as a result of vacuum polarization in the field of the gauge field component $A_5$ in the 4 + 1 and $A_3$ in the 2 + 1 dimensional models [7] with one type of fermions and with two types of fermions living in the brane and in the bulk. It should be mentioned that two different approaches (Kaluza–Klein and Aharonov–Bohm) to the problem of induced current appearance are used in these Sections. In Section 6, the induced current in a planar model with a thin solenoid is studied. Here, in particular, the effect of vanishing induced current at the half-integer values of the ratio of the magnetic flux and the elementary magnetic flux is demonstrated in accordance with the result of R.Jackiw et al. [25].

## 2. Violation of Chiral Symmetry

As was demonstrated in [26] for the appearance of the induced current in the chiral magnetic effect, two violations of symmetries are necessary, i.e., violation of axial current conservation due to topological properties of the background gauge field (the axial anomaly of QCD), and violation of symmetry in QED (electromagnetic anomaly). The QCD anomaly provides the chirality, the electromagnetic anomaly the current.

Let us consider as an example of quantum mechanical violation of chiral symmetry by a massless Dirac fermion moving in a vector field $A_\mu$. The problem derives from the second quantization procedure, as was demonstrated in [42]. Setting $A^0$ to zero, we can find the eigenmodes of the Hamiltonian in the vector field $\mathbf{A}$. We then fill the negative energy modes and leave the modes of positive energy empty. This is the way to define the model of the second quantized vacuum. Following the work in [42], we let $\mathbf{A}$ be constant and we suppose $\mathbf{A}$ to be adiabatically changing $\mathbf{A} \to \mathbf{A} + \delta\mathbf{A}$, which can arise due to nearest neighbor hopping.

According to our model of graphene we consider 2D Dirac equation for massless electrons in a zero energy mode. Assume that $\mathbf{A}$ is directed along $x-$axis ($A_x \equiv A$), then the divergence of the axial current is proportional to $\epsilon^{\mu\nu}F_{\mu\nu} \propto \partial_t A$. The eigenmodes of the zero mode satisfy a Dirac equation,

$$H\Psi = \sigma_1(-i\partial_x - A_x)\Psi = E\Psi,$$

where $A_x = A$ is constant. They are given by

$$\Psi_1(x) = \begin{pmatrix} e^{ip_x x} \\ 0 \end{pmatrix} \tag{1}$$

for $E = -p_x + A$, and

$$\Psi_2(x) = \begin{pmatrix} 0 \\ e^{ip_x x} \end{pmatrix} \tag{2}$$

for $E = p_x - A$.

The second quantized model corresponds to the filled Dirac energy sea, $E < 0$. With zero vector potential $A = 0$, branches with $p_x > 0$ and $p_x < 0$ correspond to different chirality and only the negative energy states are filled.With $A \to \delta A$ right-handed antiparticles and left-handed particles are produced. Thus, the sum of right and left charges is conserved, while they are not conserved separately. One should see that the change of $A = 0$ to $A = \delta A$ means gauge transformation, and this leads to particle production. One may conclude that the filled Dirac negative energy sea leads to non-conservation of chirality, while dynamics demonstrates chiral invariance.

## 3. Model with a Domain Wall

### *3.1. Dirac Equation in the Model with a Domain Wall*

Monolayer graphene can be modeled by Dirac equation in $D = 2 + 1$ spacetime. Various physical mechanisms that are due to defects in graphene can lead to interaction terms in the effective Dirac equation. (see, e.g., in [41,43] and the references therein).

Let us start with a field theoretical statement of the problem. The Euclidean action of a 3-d Gross–Neveu model has the form

$$S[\overline{\Psi}, \Psi] = \int d^3x \left[ \overline{\Psi}\gamma_\mu\partial_\mu\Psi + \overline{\Psi}\gamma_3\partial_3\Psi - \frac{G}{2N}\left(\overline{\Psi}\Psi\right)^2 \right], \tag{3}$$

where $\mu = 1,2$, $\gamma_1 = i\sigma_1, \gamma_2 = i\sigma_2, \gamma_5 = \gamma_3 = \sigma_3$, and an additional 3-d direction is a space direction. The action of the model in addition to $U(N)$ has also $\mathbf{Z}(2)$-symmetry

$$\psi_{R,L}(x) = \frac{1 \pm \gamma_3}{2}\psi(x),\ \overline{\psi}_{R,L}(x) = \overline{\psi}(x)\frac{1 \mp \gamma_3}{2}, \tag{4}$$

accompanied by space inversion in the 3d direction

$$\begin{aligned} \Psi_L(x_1, x_2, x_3)' &= \pm\Psi_L(x_1, x_2, -x_3),\ \overline{\Psi}_L(x_1, x_2, x_3)' = \pm\overline{\Psi}_L(x_1, x_2, -x_3), \\ \Psi_R(x_1, x_2, x_3)' &= \mp\Psi_R(x_1, x_2, -x_3),\ \overline{\Psi}_R(x_1, x_2, x_3)' = \mp\overline{\Psi}_R(x_1, x_2, -x_3). \end{aligned} \tag{5}$$

Consider therefore the Dirac equation for a $D = (2+1)$ system

$$H\Psi = i\partial_t \Psi, \tag{6}$$

where the Dirac Hamiltonian operator

$$H = \sigma_2(\sigma_1 \partial_x + \sigma_3 \partial_z - m(z)) = -i\sigma_3 \partial_x + i\sigma_1 \partial_z - \sigma_2 m(z) \tag{7}$$

includes a domain wall described by pseudopotential $\sigma_2 m(z)$ formed by the kink [44,45]

$$m(z) = m_0 \tanh(m_0 z). \tag{8}$$

This potential, as demonstrated in [45], can be self-consistently reproduced by complete set of solutions of Equations (6) and (7) summed over filled Dirac sea of negative energy states

$$\frac{G}{N}\overline{\Psi}(z)\Psi(z) = m(z).$$

The equation

$$[-i\sigma_3 \partial_x + i\sigma_1 \partial_z - \sigma_2 m(z)]\Psi(z) = E\Psi(z),$$

written explicitly

$$i(\partial_z + m(z))\Psi_2 = (E - k_x)\Psi_1,$$

$$(E + k_x)\Psi_2 = i(\partial_z - m(z)\Psi_1$$

has stationary solutions

$$\Psi = \begin{pmatrix} \Psi_1(z) \\ \Psi_2(z) \end{pmatrix} e^{ik_x x - iEt}. \tag{9}$$

There is a localized solution

$$\Psi_1 = 0,\ E^{(0)} = -k_x$$

$$\Psi = \sqrt{\frac{m_0}{2}} \begin{pmatrix} 0 \\ \cosh^{-1}(m_0 z) \end{pmatrix} e^{ik_x x - iEt}, \tag{10}$$

which is a normalized zero mode for left-moving electrons with $k_x < 0$ for $E^{(0)} > 0$. The zero mode for right-moving electrons with $k_x > 0$ for $E > 0$ is obtained for

$$\Psi_2 = 0,\ E = k_x,$$

which gives

$$\Psi = \sqrt{\frac{m_0}{2}} \begin{pmatrix} \cosh(m_0 z) \\ 0 \end{pmatrix} e^{ik_x x - iEt}, \tag{11}$$

and it is not localized on the wall and not normalizable.

### 3.2. *Vacuum Energy and Induced Current*

Consider the Dirac Hamiltonian with an additional term with a chiral chemical potential $\mu_5$. To this end, the term $\mu_5 \sigma_3$ is added to the Dirac Hamiltonian

$$H = -i\sigma_3 \partial_x + i\sigma_1 \partial_z - \sigma_2 m(z) + \mu_5 \sigma_3. \tag{12}$$

This equation describes an asymmetry between the number of right- and left-handed quarks due to the axial anomaly. The energy spectrum for the free massless Dirac electron with a chiral chemical potential in the zero mode for left-moving electrons with $k_x < 0$ looks like

$$\Psi_1 = 0, \; E^{(0)} = -k_x - \mu_5.$$

Therefore, according to the Atiyah–Singer theorem [24], we have only one normalizable fermion zero mode for the motion in the field with topological number equal unity. The vacuum energy for fermions is defined through equation

$$E_{\text{vac}} = -\sum E^{(0)}.$$

In our problem, we consider the contribution of zero modes

$$E_{\text{vac}} = -\frac{L}{2\pi}\int_{-\Lambda}^{+\Lambda} dk_x \left|E^{(0)}\right|$$

Let $k_x \to k_x - eA_x$, where after calculations the vector potential $A_x \to 0$. Then, as $\frac{\partial}{\partial A_x} = e\frac{d}{dk_x}$, the induced current

$$j_x = \left.\frac{\partial E_{\text{vac}}}{\partial A_x}\right|_{A_x \to 0}.$$

In this way, we obtain

$$j_x = \frac{eL}{2\pi}\int_{-\Lambda}^{\Lambda} dk_x \frac{d}{dk_x}|-k_x - \mu_5|,$$

where the cutoff $\Lambda \to \infty$. As a result, we have

$$j_x = \frac{eL}{2\pi}[\Lambda + \mu_5 - (\Lambda - \mu_5)] = \frac{eL}{\pi}\mu_5.$$

In the case of an antikink the result for the induced current changes sign. The coefficient 1 in front of $\frac{eL}{\pi}\mu_5$ is the topological number of the kink (8) or the index of a 2D Dirac Hamiltonian in the presence of a kink with a nontrivial topology (see similar discussion of this point in the problem of CME [26]). The corresponding transverse (with respect to motion through the wall) zero modes may be not degenerate. Let $N_\pm$ be the number of transverse zero modes with eigenvalues $\sigma_1$ equal to $\pm 1$. Then, the index of a two-dimensional Dirac Hamiltonian is equal to the difference $N_+ - N_-$ in the presence of a domain wall (for the case with a magnetic field, see in [26]).

## 4. Two-Fermion Model with a Compact Dimension

A 5-dimensional fermion model assumes an existence of 5D bulk fermions $\Psi$ in interaction with fermions $L$ on the 4D brane, which resides at a fixed point of the extra dimension, and a bulk gauge field $A_M$. The four-fermion effective interactions among these fermions may be provided by the exchange of the Kaluza–Klein excited modes of the bulk graviton [46]. We suppose that the charged bulk fermion interacts with the bulk abelian gauge field $A_M$, and the fermion $L$ is neutral. The dynamical mass generation arising from the four-fermion interaction [6,7] (The 3D fermion model with the same interaction of two types of fermions was considered in [7]. We shall not give the details of computations of our problem for this case, rather we shall give the final results parallel to the 5D conclusions.) is influenced by the constant potential $A_5$. Suppose that the vacuum averages $\langle A_\mu \rangle = 0$, while $\langle A_5 \rangle = A_5 = \text{const}$. With the $1/N_f$ expansion technique, fermions may be considered as multiplets of a flavor $O(N_f)$ group with $N_f$ flavor components. Accordingly, the model can be considered as an extension of the Lagrangian considered in [5] (The notations in [5] and those in [6]will be mainly adopted.):

$$\mathcal{L}^{(5)} = \bar{\psi} i \Gamma^M D_M \psi + [\bar{L} i \gamma^\mu \partial_\mu L - \frac{g^2}{N_f} (\bar{\psi} \Gamma^M L)(\bar{L} \Gamma_M \psi)] \delta(y), \tag{13}$$

where $M = \mu, 5$; $\mu = 0, 1, 2, 3$, $\Gamma^M = \{\gamma^\mu, i\gamma^5\}$ and $D_M = \partial_M - ieA_M$ is the covariant derivative with $A_M$ being a bulk abelian gauge field (Note that for 5 (odd) dimensions no standard chiral symmetry exists, as no $\gamma_5$-type matrix anticommuting with all other $\gamma$ matrices can be defined. An irreducible representation of a 5-d fermion field is realized by a 4-component field, and the fifth component of the $\gamma$ matrix for 5-d fermion field is just $i\gamma_5$ in 4 dimensions). The fifth coordinate $x^5 = y$ varies in the interval $[0, 2\pi R]$.

Clearly, in the absence of the gauge field $A_M$ the Lagrangian is invariant under the $Z_2$ discrete "chiral" symmetry: $y \to -y$, $\psi(y) \to \gamma^5 \psi(-y)$, $L(y) \to \gamma^5 L(-y)$. The discrete symmetry thus prevents a mass term of the form $\bar{\psi} L$. To preserve this symmetry, the field $A_M$ should transform as $A_\mu(y) \to A_\mu(-y)$ and $A_5(y) \to -A_5(-y)$. Therefore, the presence of the constant potential $A_5$ spontaneously breaks the chiral symmetry. In our previous paper, we considered the influence of a constant $A_5$ on the dynamical mass generation of fermions. In this paper, we study generation of induced current due to vacuum effects under the influence of the gauge field $A_5$. We shall briefly repeat some steps of our former calculations in [6]. Making Hubbard–Stratonovich transformation, we include an auxiliary field $\sigma_M$, and in the mean field approximation we put $< \sigma_\mu >= 0$, $\langle \sigma_5 \rangle = \sigma_5 = -\sigma$.

Compactify the third dimension in a circle with radius $R$ and set an additional parameter, the phase shift $\alpha$. Then, the bulk fermion field can be decomposed into the Kaluza–Klein modes

$$\Psi(x, y) = N \sum_{n=-\infty}^{\infty} \psi_n(x) e^{i \frac{y}{R}(n+\alpha)}, \tag{14}$$

where we chose $N = 1/\sqrt{2\pi R}$ to obtain the properly normalized kinetic term. The effective 4-d Lagrangian is obtained through integration over coordinate $y$ $\mathcal{L}^{(4)} = \int\limits_0^{2\pi R} dy \mathcal{L}^{(5)}$. After going over to matrix representation for fermion fields $\Psi$

$$(\Psi)^T = (L, \Psi_0, \Psi_1, \Psi_{-1}, \Psi_2, \Psi_{-2}, \ldots) \tag{15}$$

the Lagrangian takes the form

$$\mathcal{L}^{(4)}_{\text{eff}} = \overline{\Psi} i \not{\partial} \Psi + \overline{\Psi} M \Psi - |\sigma|^2. \tag{16}$$

where the mass matrix has the form

$$M = \begin{pmatrix} 0 & m^* & m^* & m^* & m^* & . & . \\ m & \frac{\alpha}{R} - a & 0 & 0 & 0 & . & . \\ m & 0 & \frac{\alpha+1}{R} - a & 0 & 0 & . & . \\ m & 0 & 0 & \frac{\alpha-1}{R} - a & 0 & . & . \\ m & 0 & 0 & 0 & \frac{\alpha+2}{R} - a & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}. \quad (17)$$

and $eA_5 \equiv a$, $m = Ng\sigma$.

We put $m = Ng < \sigma >$ with a non-vanishing vacuum expectation value of $\sigma$. The masses of 4-dimensional fermions are the eigenvalues of matrix $M$.

### *4.1. Effective Potential of the Model*

With the use of the functional integration formalism we define the generating functional of our system

$$Z = \int [\mathcal{D}\bar{\Psi}][\mathcal{D}\Psi][\mathcal{D}\sigma][\mathcal{D}\sigma^*] e^{i\int d^4x \mathcal{L}^{(4)}} = \int [\mathcal{D}\sigma][\mathcal{D}\sigma^*] e^{-i\int d^4x V_{\text{eff}}(\sigma)}. \quad (18)$$

The effective potential for $\sigma$ is found by integrating over the fermion fields $\Psi$ and is represented by integrating over the Euclidean momentum $k_E$. After the $1/N_f$ expansion, in the leading order we have

$$V_{\text{eff}} = |\sigma|^2 - \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{2} \ln \det(M^2 + Ik_E^2). \quad (19)$$

After some calculations (see in [6]) we obtain

$$\begin{aligned} V_{\text{eff}}(\sigma, a) \;&=\; |\sigma|^2 - \frac{1}{4\pi^2} \int_0^\Lambda dx\, x^3 \Big\{ \ln\Big( x^2[\cosh 2\pi Rx - \cos(2\pi Ra)] + 2\pi Rx|m|^2 \sinh 2\pi Rx \\ &+\; (\pi R|m|^2)^2 [\cosh 2\pi Rx + \cos(2\pi Ra)] \Big) - \ln\Big( x^2[\cosh 2\pi Rx - 1)] + 2\pi Rx|m|^2 \sinh 2\pi Rx \\ &+\; (\pi R|m|^2)^2 [\cosh 2\pi Rx + 1)] \Big) \Big\}. \end{aligned} \quad (20)$$

where $\Lambda$ is the cut-off parameter, and we subtracted the result for $A_5 = 0$ and put for simplicity $\alpha = 0$.

This integral at the critical point $m = 0$ can be transformed to the following expression,

$$V_{\text{eff}}(\sigma = 0, a) \;=\; -\frac{1}{4\pi^2} \int_0^\Lambda dx\, x^3 \ln \frac{\cosh 2\pi Rx - \cos(2\pi Ra)}{\cosh 2\pi Rx - 1}. \quad (21)$$

### *4.2. Induced Current*

Let us find the effect of vacuum polarization resulting in the possibility of appearance of induced current. The induced current $J_{\text{ind}} = \frac{\partial V_{\text{eff}}}{\partial A_5}$ can be found from Equation (20):

$$J_{\text{ind}} \;=\; \frac{1}{2\pi} eR \sin(2\pi\nu) \int_0^\infty x^3 dx \frac{(m^4\pi^2R^2 - x^2)}{[\cosh 2\pi Rx - \cos(2\pi\nu)]x^2 + 2\pi Rx|m|^2 \sinh 2\pi R + [\cosh 2\pi Rx + \cos(2\pi\nu)](\pi R|m|^2)^2}, \quad (22)$$

where $\nu = aR = eA_5R$. It should be noted that the integral converges and we may extend the upper limit to infinity. It is clear that the current is explicitly periodic in $\nu = eA_5R$ and vanishes for $\nu = n/2$, $n = 0, \pm 1, \pm 2, \ldots$, i.e., for $e\Phi/2\pi = n/2$, where $\Phi = 2\pi RA_5$ is the flux of the gauge field $A_3$.

The integral (26) at the critical point with $m = 0$ according to (35) simplifies to

$$J_{\text{ind}} = -\frac{1}{2\pi} eR\sin(2\pi\nu)\int\limits_0^\infty \frac{x^3 dx}{\cosh 2\pi Rx - \cos(2\pi\nu)}. \tag{23}$$

For spacetime with dimension D = 3, instead of (20) we have

$$\begin{aligned} V_{\text{eff}}^{(3)}(\sigma, a) &= |\sigma|^2 - \frac{1}{4\pi}\int_0^\Lambda dx\, x\Big\{ \ln\Big( x^2[\cosh 2\pi Rx - \cos(2\pi Ra)] + 2\pi Rx|m|^2 \sinh 2\pi Rx \\ &+ (\pi R|m|^2)^2[\cosh 2\pi Rx + \cos(2\pi Ra)]\Big) - \ln\Big( x^2[\cosh 2\pi Rx - 1)] + 2\pi Rx|m|^2 \sinh 2\pi Rx \\ &+ (\pi R|m|^2)^2[\cosh 2\pi Rx + 1)]\Big)\Big\}, \end{aligned} \tag{24}$$

and at the critical point $m = 0$

$$V_{\text{eff}}(\sigma = 0, a) = -\frac{1}{4\pi}\int_0^\Lambda dx\, x \ln\frac{\cosh 2\pi Rx - \cos(2\pi Ra)}{\cosh 2\pi Rx - 1}. \tag{25}$$

and for the induced current

$$J_{\text{ind}} = \int\limits_0^\infty \frac{dx\, x\, \frac{1}{2} eR\sin(2\pi\nu)\,(m^4\pi^2R^2 - x^2)}{[\cosh 2\pi Rx - \cos(2\pi\nu)]x^2 + 2\pi Rx|m|^2\sinh 2\pi Rx + [\cosh 2\pi Rx + \cos(2\pi\nu)](\pi R|m|^2)^2}. \tag{26}$$

For values of $\nu = eA_3R$ such that the quantity $\tilde{\nu}$ ( $\tilde{\nu} = \nu - n$ for $\nu > 0$ and $\tilde{\nu} = \nu + n$ for $\nu < 0$ with $n$ as a maximal integer number less than $|\nu|$) takes small values we have

$$V_{\text{eff}}^{(3)}(\sigma = 0, a) = -\frac{1}{4\pi}\int_0^\Lambda dx\, x \ln\frac{\cosh 2\pi Rx - \cos(2\pi Ra)}{\cosh 2\pi Rx - 1} = -\frac{\pi\tilde{\nu}^2}{4R^2}\ln(\pi\tilde{\nu}) + \mathrm{O}(\tilde{\nu}^4). \tag{27}$$

This result can be obtained in the following way,

$$V_{\text{eff}}^{(3)}(\sigma = 0, a) = -\frac{1}{4\pi}\int_0^\Lambda dx\, x \ln\left(1 + \frac{2\sin^2 \pi Ra}{\cosh 2\pi Rx - 1}\right) \approx -\frac{\tilde{\nu}^2}{4\pi R^2}\int_{\tilde{\nu}}^1 \frac{dx}{x} = \frac{\tilde{\nu}^2}{8\pi R^2}\ln\tilde{\nu}^2. \tag{28}$$

The above Formulas (45) and (46) are in agreement with the result of [47] $V_{\text{Diak}}$, obtained for gluons in the field of central vortices, with the evident modification

$$V_{\text{eff}}^{(3)} = -\frac{1}{2} \times 2\pi RV_{\text{Diak}},$$

where minus stands for the fermions and $2\pi R$ for the circumference length. The current in this case looks like

$$J_{\text{ind}} \approx \frac{e\tilde{\nu}}{4\pi R}\ln\tilde{\nu}^2. \tag{29}$$

## 5. Induced Current in a Model with a Compact Dimension

In this section, we shall study a fermion model in D = ($d$ + 1)-dimensional spacetime with one compactified spatial dimension (cylinder): $M = M^d \times S^1$, where $M^d$ is a D-dimensional Minkowski space and $S^1$, a space-like extra dimension, is a circle with a circumference $L = 2\pi R$. Our $d$ + 1 fermion model assumes an existence of a bulk gauge field $A_M$. Extra dimensions are real and physical, i.e., $d + 1 = 5$ and $d + 1 = 3$, though the results may be easily generalized for any dimensionality. Contrary to the approach of the previous section, where Kaluza–Klein method was used, we here used the Hosotani mechanism and the Aharonov–Bohm picture.

Let us further suppose that fermion $\Psi$ is charged and thus may interact with the abelian gauge field $A_M$. The presence of the constant potential $A_{d+1} = \text{const}$ may provide the dynamical mass generation due to Hosotani mechanism [8]).

### 5.1. Model in (4 + 1)-Dimensional Spacetime

Let us consider the model in $D = 5$ with the vacuum averages $\langle A_\mu \rangle = 0$, while $\langle A_5 \rangle = A_5 = \text{const}$. Moreover, for applying the $1/N_f$ expansion technique, fermions are assumed to be multiplets of a flavor $O(N_f)$ group and thus to have $N_f$ flavor components. The Lagrangian has the form

$$\mathcal{L}^{(5)} = \bar{\psi} i \Gamma^M D_M \psi. \tag{30}$$

where $M = \mu, 5;\ \mu = 0,1,2,3,\ \Gamma^M = \{\gamma^\mu, i\gamma^5\}$ and $D_M = \partial_M - ieA_M$ is the covariant derivative with $A_M$ being an abelian gauge field. The fifth coordinate $x^5 = y$ varies in the interval $[0, 2\pi R]$.

Clearly, in the absence of the gauge field $A_M$ the Lagrangian is invariant under the $Z_2$ discrete "chiral" symmetry: $y \to -y$, $\psi(y) \to \gamma^5 \psi(-y)$, $L(y) \to \gamma^5 L(-y)$. The discrete symmetry thus prevents a mass term of the form $\bar{\psi} L$. To preserve this symmetry, the field $A_M$ should transform as $A_\mu(y) \to A_\mu(-y)$ and $A_5(y) \to -A_5(-y)$. Therefore, the presence of the constant potential $A_5$ spontaneously breaks the chiral symmetry. In [6], we considered the influence of a constant $A_5$ on the dynamical mass generation of fermions. In this paper we study generation of induced current due to vacuum effects under the influence of the gauge field $A_5$.

Generating functional of our system is as follows,

$$Z = \int [\mathcal{D}\bar{\psi}][\mathcal{D}\psi] e^{i\int d^5x \mathcal{L}^{(5)}} = e^{-iN\int d^5x V_{\text{eff}}}, \tag{31}$$

where after integrating over the fermion fields $\psi$, $\bar{\psi}$ we introduced the effective potential in the form of an integral over the Euclidean momentum $p_E$, in the leading order of the $1/N$ expansion

$$V_{\text{eff}}^{(5)} = -i\,\text{Tr}_{xs} \ln(i\gamma^\mu \partial_\mu + i\gamma^5(i\partial_5 - eA_5)) = -\int \frac{d^5p}{(2\pi)^5} \text{tr}_s \ln(-\gamma^\mu p_\mu - i\gamma^5(p_5 - eA_5)), \tag{32}$$

where $x$ and $s$ indices of Tr operator correspond to integrating over spacetime and taking trace over spinor indices respectively. Compactify the fifth dimension in a circle with radius $R$ and set an additional parameter, the phase shift $\alpha$. Then, we obtain the decomposition of the fermion field into the Kaluza–Klein modes

$$\Psi(x, y) = N \sum_{n=-\infty}^{\infty} \psi_n(x) e^{i\frac{y}{R}(n+\alpha)}, \tag{33}$$

and Equation (32) takes the form

$$V_{\text{eff}}^{(5)} = -2\int \frac{d^4p}{(2\pi)^4} \frac{1}{L} \sum_{n=-\infty}^{\infty} \ln\Big[p_1^2 + p_2^2 + p_3^2 + p_4^2 + (\frac{2\pi}{L})^2(n + \alpha - \nu)^2\Big], \tag{34}$$

where $\nu = aR = eA_5R$.

This integral can be transformed to the following expression,

$$V_{\text{eff}}^{(5)}(\sigma = 0, a) = -\frac{1}{4\pi^2}\int_0^\Lambda dx\, x^3 \ln\frac{\cosh(2\pi Rx) - \cos(2\pi Ra + \alpha)}{\cosh(2\pi Rx) - 1}, \tag{35}$$

where we introduced subtraction at the point $2\pi Ra + \alpha = 0$ (Although the integral converges with subtraction, introduction of the cut-off $\Lambda$ is convenient for further numerical estimations of the results, while the final results prove to be independent of $\Lambda$). In what follows, we put for simplicity $\alpha = 0$, which corresponds for periodic condition for the $\Psi$ field. The induced current $J = \frac{\partial V_{\text{eff}}}{\partial A_5}$ can be found from Equation (35)

$$J_{\text{ind}}^{(5)} = -\frac{1}{2\pi} eR \sin(2\pi\nu) \int_0^\Lambda \frac{x^3 dx}{\cosh(2\pi Rx) - \cos(2\pi\nu)}. \tag{36}$$

It is clear that the current is explicitly periodic in $\nu = eA_5R$ and vanishes for $\nu = n/2$, $n = 0, \pm 1, \pm 2, \ldots$, i.e., for $e\Phi/2\pi = n/2$, where $\Phi = 2\pi RA_5$ is the flux of the gauge field $A_5$.

### *5.2. Model in (2 + 1)-Dimensional Spacetime*

In particular case of the 3D fermion model with $N$ fermion flavors, the action is

$$S = \int d^3x \overline{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi, \tag{37}$$

where $A_\mu$ is a vector potential. Summation over flavors will not be written explicitly and $\overline{\psi}\hat{O}\psi$ should be understood as $\sum_{k=1}^{N} \overline{\psi}_k \hat{O}\psi_k$.

We will use rank 4 (reducible) representation of the $\gamma$-matrices

$$\gamma^0 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}, \gamma^2 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{pmatrix}, \tag{38}$$

where $\sigma^i$ are the Pauli matrices.

The generating functional of our system is

$$Z = \int [\mathcal{D}\bar{\psi}][\mathcal{D}\psi] e^{i\int d^3x \mathcal{L}^{(3)}} = e^{-iN\int d^3x V_{\text{eff}}}, \tag{39}$$

where after integrating over the fermion fields $\psi$, $\bar{\psi}$ we introduced the effective potential in the form of an integral over the Euclidean momentum $p_E$, in the leading order of the $1/N$ expansion $V_{\text{eff}}$.

Let us now suppose that spacetime topology is given by $M = M^2 \times S^1$, where $M^2$ is a 2-dimensional Minkowski space and $S^1$, a space-like extra dimension, which is a circle with a circumference $L = 2\pi R$. As $S^1$ is not a simply-connected space, boundary conditions must be specified for the theory. Physical observables must be single-valued on $M^2 \times S^1$. Spatial compactification of one dimension in the circle $0 \leq x_2 \leq 2\pi R$, is realized through Fourier expansion due to periodicity condition for fermions $\psi|_{x_2=0} = \psi|_{x_2=2\pi R}$,

resulting in the following expression,

$$V_{\text{eff}}^{(3)} = \frac{\sigma^2}{2G} - 2\int \frac{d^2p}{(2\pi)^2} \frac{1}{L} \sum_{n=-\infty}^{\infty} \ln\left[p_1^2 + p_3^2 + (\frac{2\pi}{L}n)^2 + \sigma^2\right].$$

The Aharonov–Bohm phase $\nu = \frac{eL}{2\pi}A_\phi$ is introduced through the asimuthal component of the vector potential $A_\phi$. This can be done by assumption that at the axis of the cylinder there exists along it an infinitely thin solenoid with finite magnetic flux $\Phi = 2\pi RA_\phi$ and magnetic field

$$H_z = \frac{\Phi}{2\pi}\delta(r^2).$$

Then, the resulting effective potential takes the form

$$V_{\text{eff}}^{(3)} = \frac{\sigma^2}{2G} - 2\int \frac{d^2p}{(2\pi)^2} \frac{1}{L} \sum_{n=-\infty}^{\infty} \ln\left[p_1^2 + p_3^2 + (\frac{2\pi}{L})^2(n-\nu)^2\right]. \tag{40}$$

On the mass shell $p_1^2 + p_3^2 + (\frac{2\pi}{L})^2(n-\nu)^2 = 0$ the energy of a fermion $E = ip_3$ is evidently related to longitudinal momentum $p_1$ and transversal modes $p_2 = 2\pi(n-\nu)/L$ by the dispersion relation

$$E_{n,p_1}^2(\nu) = p_1^2 + (2\pi(n-\nu)/L)^2.$$

Note that for $\nu = n = 1, 2, \ldots$ the contribution of the asimuthal (Aharonov–Bohm) potential $A_\phi$ is absorbed by $n$ and we have for the energy the typical zero mode expression for the transversal motion

$$E_{n,p_1}^2(\nu = n) = p_1^2.$$

The appearance of zero modes for quantized flux $\nu = n = 1, 2, \ldots$ is the consequence of Aharonov–Bohm effect and is due to nontrivial topology of the model.

For the model with the four-fermion interaction we can transform the effective potential (40) to obtain the following form,

$$V^{(3)}_{\rm eff}(\sigma, a) \quad = \quad -\frac{1}{\pi L}\int_0^{\Lambda} dx\, x \ln \frac{\cosh Lx - \cos La}{\cosh Lx - 1}. \tag{41}$$

Now, we make use of the formula [48] for integration over Euclidean momenta:

$$\int \frac{d^{D-1}p_E}{(2\pi)^{D-1}} \frac{1}{2\pi R} \sum_{n=-\infty}^{\infty} \ln\left[p_E^2 + \left(\frac{n-\nu}{R}\right)^2\right] = -\frac{2\Gamma(\frac{1}{2}D)}{(2\pi R)^D \pi^{D/2}} f_D(2\nu) + (\nu\text{−independent terms}), \tag{42}$$

where

$$f_D(2\nu) = \sum_{m=1}^{\infty} \frac{\cos 2m\pi\nu}{m^D} = f_D(2\nu+2) = f_D(-2\nu)\ ,\ f_D(0) = \zeta_R(D)\ ,\ f_D(1) = -(1-2^{1-D})\,\zeta_R(D)\ . \tag{43}$$

Here, $\zeta_R(z)$ is the Riemann's zeta function. In our case, $D = 3$, $R = L/2\pi$. This formula gives the result

$$V^{(D)}_{\rm eff} = 2\frac{2\Gamma(\frac{1}{2}D)}{(2\pi R)^D \pi^{D/2}} f_D(2\nu) + (\nu\text{−independent terms}),$$

and for $D = 3$

$$V^{(3)}_{\rm eff} = 2\frac{2\Gamma(\frac{3}{2})}{(2\pi R)^3 \pi^{3/2}} \sum_{m=1}^{\infty} \frac{\cos 2m\pi\nu}{m^3} + (\nu\text{−independent terms}).$$

The induced current can be found by the formula

$$J^{(3)}_{\phi} = \frac{\partial V^{(3)}_{\rm eff}}{\partial A_\phi},$$

which with the help of Equation (42), (43) gives

$$J^{(3)}_{\phi} = -\frac{2e}{\pi L^2} \sum_{m=1}^{\infty} \frac{\sin(2\pi m\nu)}{m^2}.$$

The induced current is evidently periodic in $\nu \to \nu + n$ ($n = \pm 1, \pm 2, \pm 3, \ldots$), as $m \times n$ is also an integer. Here, it should be mentioned that Hosotani in his early paper [8], where he studied dynamical mass generation by compact extra dimensions in 4D Minkowski spacetime, also found a nonvanishing fermion condensate that may be considered as a current in an extra dimension

$$< \overline{\psi}\gamma_y\psi >= -\frac{4}{L^3\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^3}.$$

Next, we make use of the formula [49]

$$\sum_{m=1}^{\infty} \frac{\sin(2\pi m\nu)}{m^2} = -\int_0^{2\pi\nu} \ln(2\sin\frac{x}{2})dx$$

and obtain

$$J^{(3)}_{\phi} = \frac{2e}{\pi L^2} \int_0^{2\pi\nu} \ln(2\sin\frac{x}{2})dx. \tag{44}$$

As follows from our result (44), when $\nu = 1/2$, the current is equal to zero, as (see page 540 in [50])

$$\int_0^{\pi/2} \ln(\sin y))dy = -\frac{\pi}{2}\ln 2$$

and consequently

$$\int_0^{\pi} \ln(2\sin\frac{x}{2})dx = 2\int_0^{\pi/2} (\ln 2 + \ln(\sin y))dy = 2\Big[\frac{\pi}{2}\ln 2 + (-\frac{\pi}{2}\ln 2)\Big] = 0.$$

The induced current was also studied in [25] in the problem with a singular magnetic flux directed perpendicular to the plane with a moving electron. It is interesting to note that, similar to our situation, the current in [25] also vanishes at $\nu = 1/2$.

The induced current $J^{(3)}_{\text{ind}}$ as a function of the field $a = eA_3$ is shown in Figures 1 and 2 for various values of $R\Lambda$, where for convenience of numerical calculations we used dimensionless variables $J^{(3)}_{\text{ind}}/e \;\rightarrow\; J^{(3)}_{\text{ind}}(e\Lambda)^{-1}$, $a \rightarrow a\Lambda^{-1}$ and $R \rightarrow R\Lambda$ using the cuoff $\Lambda$ (see (35)). In this case, the quantity $\nu = aR$ remains dimensionless and independent of $\Lambda$.

The dependence of the induced current $J^{(3)}_{\text{ind}}$ on the mass $m$ is demonstrated in Figures 3 and 4, dimensionless variables are also used. In Figure 3, we plotted current $J^{(3)}_{\text{ind}}$ as a function of $m$ for different values of $\nu = aR$ for constant value of the compactification radius $R\Lambda = 1$. The values of field $a\Lambda^{-1} = 0.25$ and $a\Lambda^{-1} = 0.75$ are chosen on different sides from the extremum of the potential (35) $\nu = n/2$ (in this picture $\nu = 0.5$), so that the derivative of the potential has different signs, and consequently, current $J^{(3)}_{\text{ind}} = \frac{\partial V_{\text{eff}}}{\partial A_3}$ demonstrated various behavior. In Figure 4, the dependence of current $J^{(3)}_{\text{ind}}$ on mass $m$ is shown at different values of compactification radius $R\Lambda$, but at the same value of $\nu = aR$. For values of $\nu = eA_3R$ such that the quantity $\tilde{\nu}$ ( $\tilde{\nu} = \nu - n$ for $\nu > 0$ and $\tilde{\nu} = \nu + n$ for $\nu < 0$ with $n$ as a maximal integer number smaller than $|\nu|$) takes small values we have

$$V^{(3)}_{\text{eff}}(\sigma = 0, a) \;=\; -\frac{1}{4\pi}\int_0^{\Lambda} dx\, x \ln\frac{\cosh 2\pi Rx - \cos(2\pi Ra)}{\cosh 2\pi Rx - 1} = -\frac{\pi\tilde{\nu}^2}{4R^2}\ln(\pi\tilde{\nu}) + \mathrm{O}(\tilde{\nu}^4). \tag{45}$$

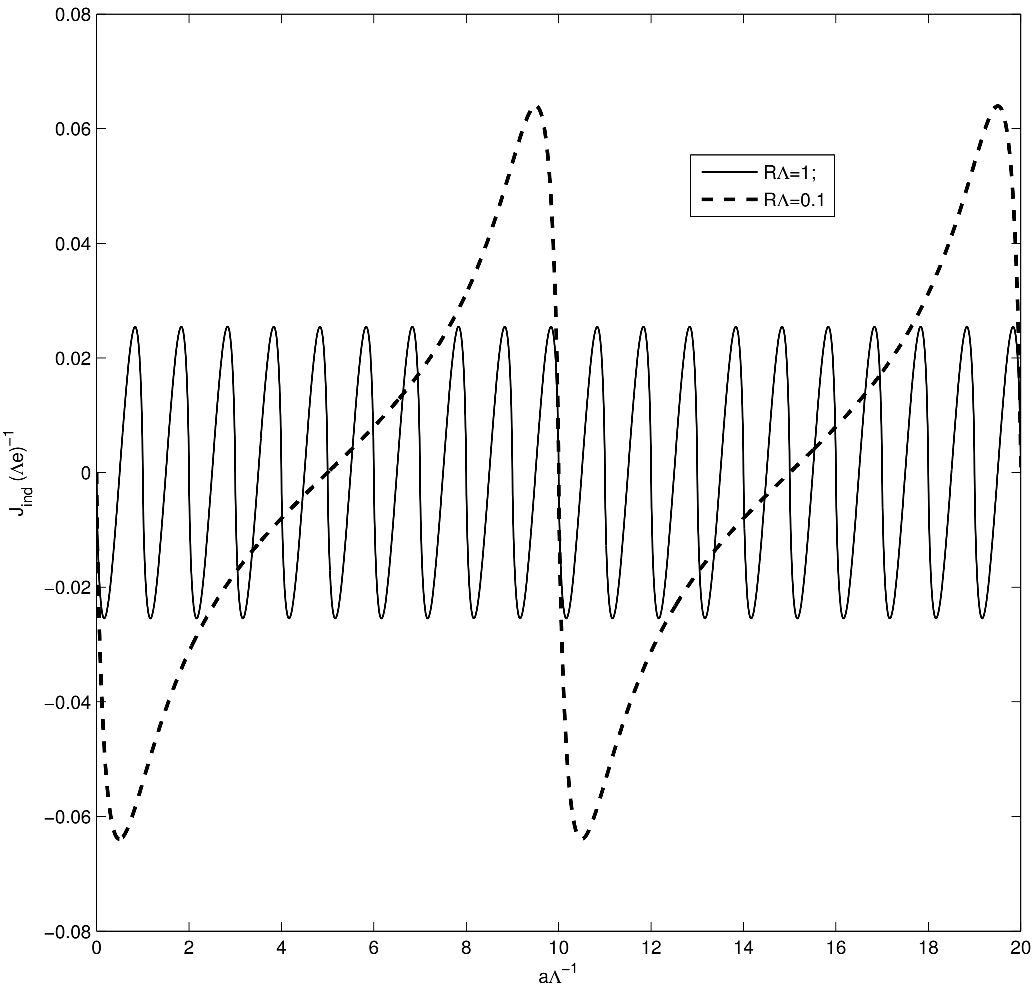


**Figure 1.** Induced current as a function of $a$ at the critical point $m = 0$ for various values of the compactification radius $R$. Induced current vanishes at $aR = n/2$.

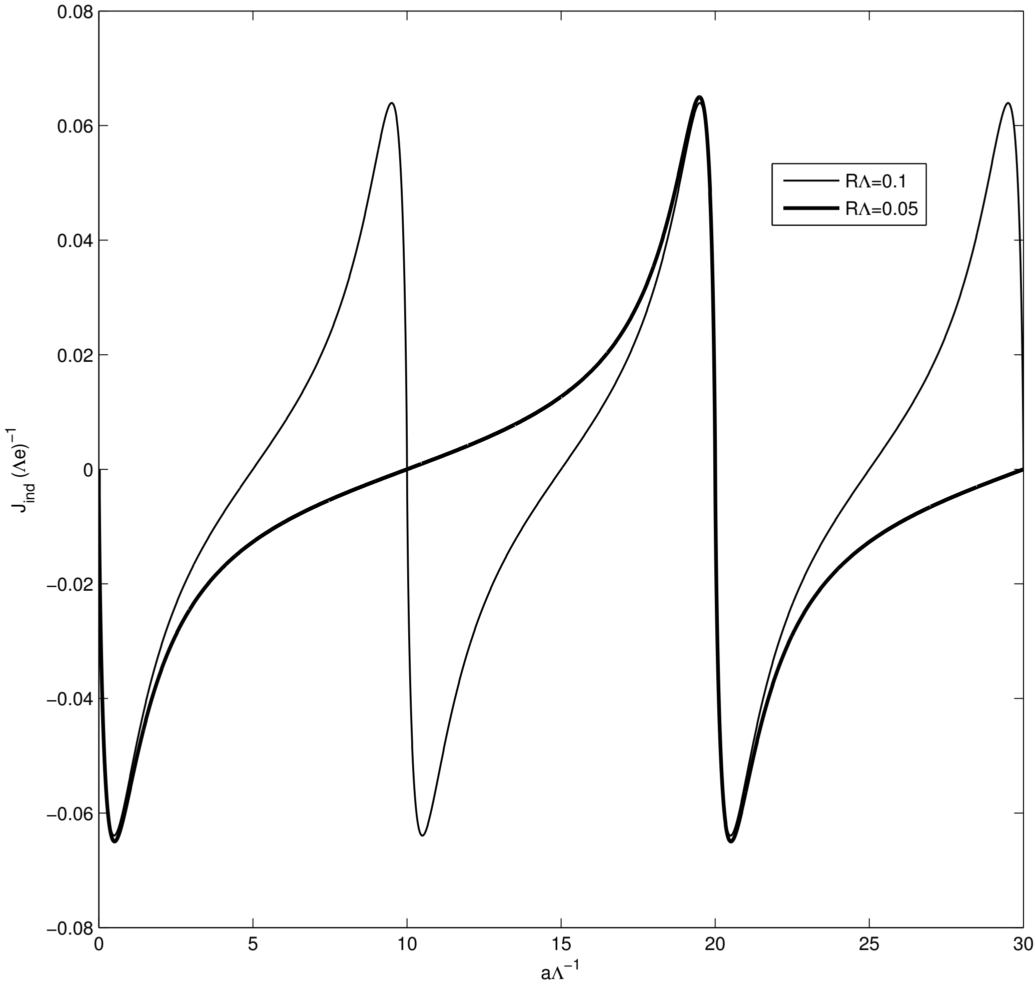


**Figure 2.** Induced current as a function of $a$ at the critical point $m = 0$ for various values of the compactification radius $R$. The induced current vanishes at $aR = n/2$.

This result can be obtained in the following way,

$$V^{(3)}_{\rm eff}(\sigma = 0, a) \; = \; -\frac{1}{4\pi}\int_0^\Lambda dx\, x \ln\left(1 + \frac{2\sin^2 \pi R a}{\cosh 2\pi R x - 1}\right) \; \approx \; -\frac{\tilde{\nu}^2}{4\pi R^2}\int_{\tilde{\nu}}^1 \frac{dx}{x} \; = \; \frac{\tilde{\nu}^2}{8\pi R^2}\ln \tilde{\nu}^2. \quad (46)$$

The current in this case looks like

$$J^{(3)}_{\rm ind} \; \approx \; \frac{e\tilde{\nu}}{4\pi R}\ln \tilde{\nu}^2. \quad (47)$$

For those values of $\nu = eA_3R$, at which the quantity $\tilde{\nu}$ is equal to $\nu - (2n+1)/2$ for $\nu > 0$, and $\tilde{\nu} = \nu + (2n+1)/2$ for $\nu < 0$, where $n$ is the maximal integer number smaller than $2|\nu|$), takes small values (these points correspond to the maximum of potential), the current is calculated as

$$J^{(3)}_{\rm ind} \; \approx \; \frac{e\tilde{\nu}}{2\pi R}\ln 2,$$

which corresponds to the behavior of the current demonstrated in Figure 1.

When values of $\nu = eA_\phi R$ are such that the quantity $\tilde{\nu}$ ( $\tilde{\nu} = \nu - n$ for $\nu > 0$ and $\tilde{\nu} = \nu + n$ for $\nu < 0$ with $n$ as a maximal integer number smaller than $|\nu|$) takes small values we have for the effective potential

$$V^{(3)}_{\rm eff} = \frac{4\pi\tilde{\nu}^2}{L^3}\ln \tilde{\nu}^2 \quad (48)$$

The above formula is in agreement with the result of [47] $V_{\rm Diak}$, obtained for gluons in the field of central vortices, with the evident modification

$$V^{(3)}_{\rm eff} = -4V_{\rm Diak},$$

where minus stands for the fermions and 4 for fermion spin and two signs of the energy.

For the same values of $\nu$, we have for the induced current

$$J^{(3)}_{\phi} = \frac{2e}{\pi L^2}\int_0^{2\pi\tilde{\nu}} \ln(2\sin\frac{x}{2})dx \approx \frac{2e\tilde{\nu}}{L^2}\ln\tilde{\nu}^2. \quad (49)$$

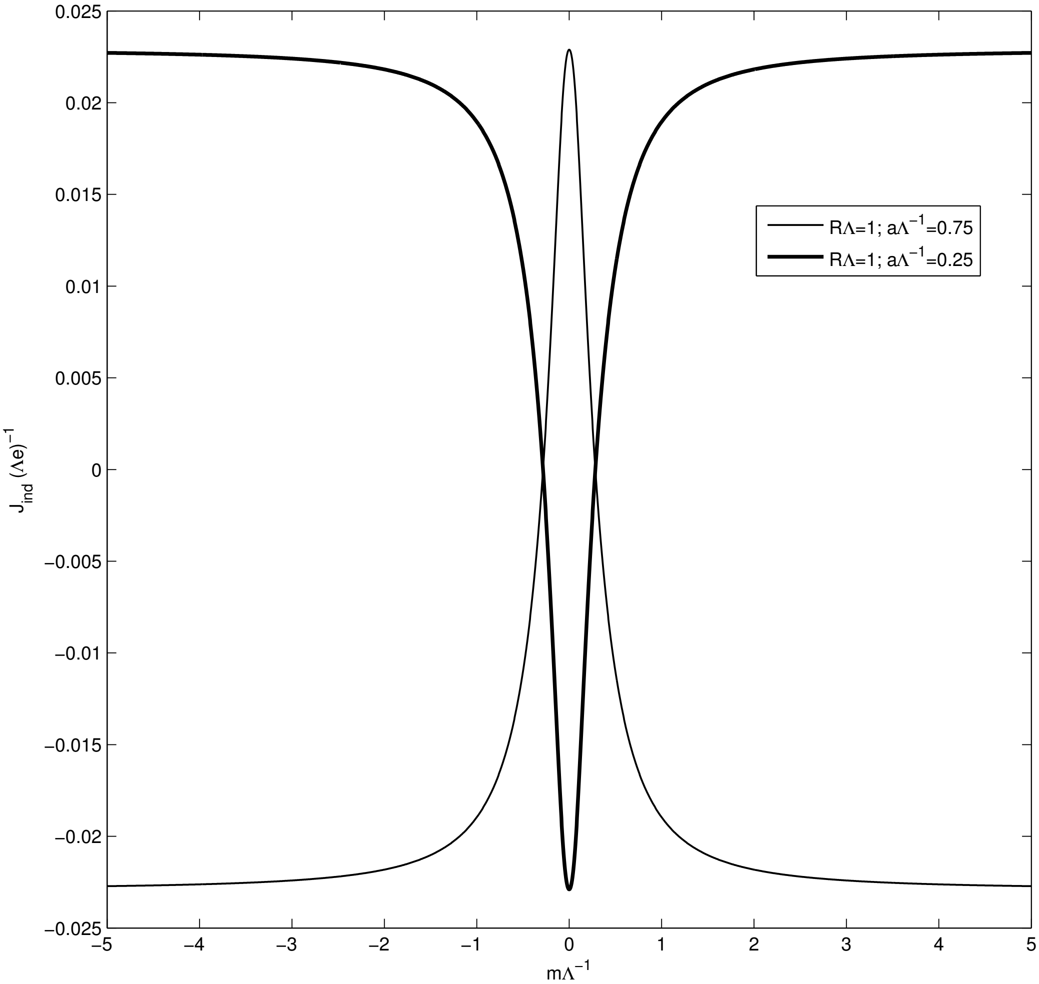


**Figure 3.** Induced current as a function of $m$.

For those values of $\nu = eA_\phi R$, at which the quantity $\tilde{\nu}$ is equal to $\nu - (2n+1)/2$ for $\nu > 0$, and $\tilde{\nu} = \nu + (2n+1)/2$ for $\nu < 0$, where $n$ is the maximal integer number smaller than $2|\nu|$), takes small values (these points correspond to the maximum of potential), the current is calculated to be

$$J^{(3)}_{\phi} \approx \frac{4e\tilde{\nu}}{L^2}\ln 2.$$

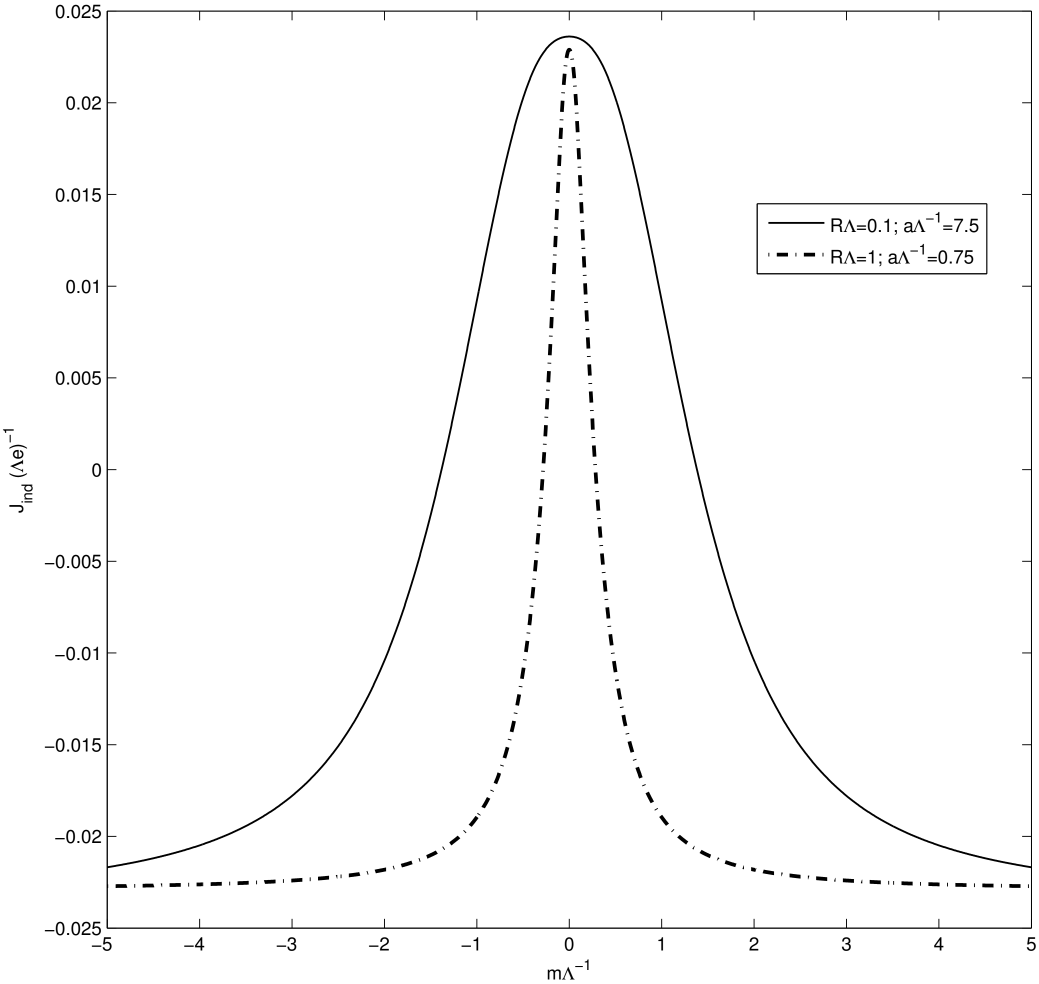


**Figure 4.** Induced current as a function of $m$ for various values of compactifications radius and fixed value of $\nu$.

## 6. Induced Current in a Planar Model with a Thin Solenoid

Diakonov [47] calculated the energy of a Yang–Mills vortex. The quantized $Z(2)$ vortices are dynamically preferred. The effective action was found to be expressed as

$$S_{eff}[\bar{A}] = \frac{1}{2}\ln\det(W_{\mu\nu}) - \ln\det(-D_\mu^2). \tag{50}$$

The Dirac determinant should be added to account for the presence of dynamical fermions in the fundamental representation:

$$-\ln\det(\nabla_\mu\gamma_\mu) = -\frac{1}{2}\ln\det\left(\nabla_\mu^2 - \frac{i}{2}[\gamma_\mu\gamma_\nu]F_{\mu\nu}^a t^a\right), \qquad \nabla_\mu = \partial_\mu - i\bar{A}_m u^a t^a. \tag{51}$$

As $\det(W_{\mu\nu}) = d\cdot\det(-\nabla_\mu^2)$ in $d$ full space dimension, $D^2$ in the cylindric coordinates can be found:

$$\nabla_\mu^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\left(\frac{\partial}{\partial\phi} + f^{acb}A_\phi^c\right)^2 + \partial_3^2 + \ldots + \partial_d^2. \tag{52}$$

The effective action (50) for slowly varying $\rho A_\phi$ is

$$S_{eff}[A_\phi] = -\left(\frac{d}{2} - 1\right)\int_{s_{min}}^{s_{max}}\frac{ds}{s}\ \mathrm{Tr}\ \exp(s\nabla_\mu^2) = \int d^{d-2}x_\parallel \int d^2x_\perp\ V^{(d)}(\mu), \qquad \mu = \rho\sqrt{A_\phi^a A_\phi^a}, \tag{53}$$

where $V^{(d)}(\mu)$ is the "vortex potential energy" in $d$ dimensions. Here, Tr denotes the functional and the color traces.

As it is convenient for our purpose, following the work in [47], we consider only the $SU(2)$ case. Three eigenvalues of the covariant Laplacian (52) in polar harmonics $\exp(im\phi)$, $m$ =integer, look as a $3\times 3$ color matrix:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}\cdot\begin{cases} m^2, \\ (m+\mu)^2, \\ (m-\mu)^2. \end{cases} \tag{54}$$

For the differential operator ((54), last line)we have the equation for eigenvalues

$$\hat{D}^2 J_{\pm(m-\mu)}(k\rho)=\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}(m-\mu)^2\right]J_{\pm(m-\mu)}(k\rho)=-k^2 J_{\pm(m-\mu)}(k\rho).$$

For a complete functional basis in the transverse plane $F_{m,k}(\phi,\rho)=\exp(im\phi)J_{|m-\mu|}(k\rho)$ we have

$$\frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\int_0^\infty dk\,k\,F_{m,k}(\phi,\rho)F^\star_{m,k}(\phi',\rho')=\frac{1}{\rho}\delta(\rho-\rho')\,\delta(\phi-\phi')|_{\text{mod }2\pi},\tag{55}$$

and also the ortho-normalization condition

$$\frac{1}{2\pi}\int_0^{2\pi}d\phi\int_0^\infty d\rho\,\rho\,F_{m,k}(\phi,\rho)F^\star_{m',k'}(\phi,\rho)=\frac{1}{k}\delta(k-k')\,\delta_{mm'}.\tag{56}$$

Equation (53) can be rewritten as

$$\mathrm{Tr}\ln(-\nabla^2_\mu)=\frac{1}{2\pi}\int d\phi\int\rho d\rho\frac{1}{2\pi}\sum_m\int\frac{dp_3}{2\pi}\int kdkF_{m,k}\ln(-\nabla^2_\mu)F^\star_{m,k}(\phi,\rho)$$

$$=\frac{1}{2\pi}\int d\phi\int\rho d\rho\frac{1}{2\pi}\sum_m\int\frac{dp_3}{2\pi}\int kdk\ln(k^2+p_3^2)\left[J^2_{|m-\mu|}(k\rho)-J^2_{|m|}(k\rho)\right]=\int d\phi\int\rho d\rho V^{(3)}(\mu),\tag{57}$$

where $V^{(3)}(\mu)$ is the $d=3$ potential energy

$$V^{(3)}(\mu)=\int\frac{dp_3}{2\pi}\int_{s_{min}}^{s_{max}}\frac{ds}{s}\int_0^\infty k\,dk\,\exp[-s(k^2+p_3^2)]$$

$$\times\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\left[J^2_{|m-\mu|}(k\rho)-J^2_{|m|}(k\rho)\right].\tag{58}$$

The trace operator in the above formulas

$$\mathrm{Tr}\,1=\int d^{d-2}x_\parallel\int d^2x_\perp\int\frac{p_3...p_d}{(2\pi)^{d-2}}\int\limits_0^\infty kdk\,\frac{1}{2\pi}\sum_{m=-\infty}^{m=+\infty}1$$

is applied to the exponent with the use of the eigen functions

$$\left(\mathrm{e}^{im\phi}\,J_{|m|}(k\rho)\,\mathrm{e}^{-i\vec{x}_\parallel\vec{p}_\parallel}\right)^\star\mathrm{e}^{s\hat{D}^2}\,\mathrm{e}^{im\phi}\,J_{|m|}(k\rho)e^{-i\vec{x}_\parallel\vec{p}_\parallel}+$$

$$\left(\mathrm{e}^{im\phi}\,J_{|m-\mu|}(k\rho)\,\mathrm{e}^{-i\vec{x}_\parallel\vec{p}_\parallel}\right)^\star\mathrm{e}^{s\hat{D}^2}\,\mathrm{e}^{im\phi}\,J_{|m-\mu|}(k\rho)\,\mathrm{e}^{-i\vec{x}_\parallel\vec{p}_\parallel}+$$

$$\left(\mathrm{e}^{im\phi}\,J_{|m+\mu|}(k\rho)\,\mathrm{e}^{-i\vec{x}_\parallel\vec{p}_\parallel}\right)^\star\mathrm{e}^{s\hat{D}^2}\,\mathrm{e}^{im\phi}\,J_{|m+\mu|}(k\rho)\,\mathrm{e}^{-i\vec{x}_\parallel\vec{p}_\parallel}\,.$$

In the sum over all $m$ the last two lines give identical contribution; one should also subtract the expression for the free theory

$$3\left(\mathrm{e}^{im\phi}\,J_{|m|}(k\rho)\,\mathrm{e}^{-i\vec{x}_\parallel\vec{p}_\parallel}\right)^\star\mathrm{e}^{s\hat{D}^2}\,\mathrm{e}^{im\phi}\,J_{|m|}(k\rho)\,\mathrm{e}^{-i\vec{x}_\parallel\vec{p}_\parallel}\,.$$

It should be taken into account that the operator acting on its eigenfunction is always replaced by its eigen-value. We see that only the indices of the Bessel functions include the dependence on the flux $\mu$. Moreover, the potential is explicitly periodic in $\mu$ with period 1. Integration over momenta can be easily performed using Equation 6.633.2 from in [50], yielding

$$V^{(d)}(\mu) = -(d-2)\int_{s_{min}}^{s_{max}} \frac{ds}{s}\left(\frac{1}{4\pi s}\right)^{\frac{d}{2}} \exp\left(-\frac{\rho^2}{2s}\right)\sum_{m=-\infty}^{\infty}\left[I_{|m-\mu|}\left(\frac{\rho^2}{2s}\right) - I_{|m|}\left(\frac{\rho^2}{2s}\right)\right]. \tag{59}$$

Here, we used formula ([50], p. 724)

$$\int_0^\infty e^{-sk^2} J_m^2(\rho k) k\, dk = \frac{1}{2s} e^{-\frac{\rho^2}{2s}} I_m\left(\frac{\rho^2}{2s}\right).$$

We next sum over $m$ using the integral representation for modified Bessel functions, Equation 8.431.5 of the work in [50]. Change of variables

$$t = \frac{\rho^2}{2s};\ t_{min} = \frac{\rho^2}{2s_{max}};\ t_{max} = \frac{\rho^2}{2s_{min}};\ \frac{dt}{t} = -\frac{ds}{s},$$

and a minus sign appears, so we interchange the integration limits (further we assume $\mu \in (0,1)$)

$$V^{(d)}(\mu) = -\frac{d-2}{(2\pi\rho^2)^{\frac{d}{2}}}\int_{t_{min}}^{t_{max}} dt\, t^{\frac{d}{2}-1}\,\mathrm{e}^{-t}\sum_{m=-\infty}^{m=+\infty}\Big(I_{|m-\mu|}(t) - I_{|m|}(t)\Big).$$

Integral representation of the Infeld functions

$$I_m(t) = \frac{1}{\pi}\int_0^\pi e^{t\cos\theta}\cos(m\theta)d\theta - \frac{\sin(m\pi)}{\pi}\int_0^\infty \mathrm{e}^{-t\,\mathrm{ch}\,x - mx}\,dx.$$

For the first integral in the representation of the Infeld functions, we can sum $\sum_{m=-\infty}^{m=+\infty}\Big(I_{|m-\mu|}(t) - I_{|m|}(t)\Big)$:

$$\sum_{m=-\infty}^{m=+\infty} -\frac{\sin(|m-\mu|\pi)}{\pi}\int_0^\infty \mathrm{e}^{-t\,\mathrm{ch}\,x-|m-\mu|x}\,dx + \frac{\sin(|m|\pi)}{\pi}\int_0^\infty \mathrm{e}^{-t\,\mathrm{ch}\,x-|m|x}\,dx.$$

The second term goes to zero for all integer $m$, the first term is transformed for $m > 1$, then for $m < -1$, and we consider separately the case $m = 0$

$$\sum_{m=1}^{m=+\infty} -\frac{\sin((m-\mu)\pi)}{\pi}\int_0^\infty \mathrm{e}^{-t\,\mathrm{ch}\,x-(m-\mu)x}\,dx$$

$$\sum_{m=-\infty}^{m=-1} -\frac{\sin((-m+\mu)\pi)}{\pi}\int_0^\infty \mathrm{e}^{-t\,\mathrm{ch}\,x-(-m+\mu)x}\,dx$$

$$-\frac{\sin(\mu\pi)}{\pi}\int_0^\infty \mathrm{e}^{-t\,\mathrm{ch}\,x-\mu x}\,dx$$

We make the change in the second term $m \to -m$, and as a result we obtain

$$\sum_{m=1}^{m=+\infty}(-1)^m\frac{\sin(\pi\mu)}{\pi}\int_0^\infty \mathrm{e}^{-t\,\mathrm{ch}\,x-mx}(\mathrm{e}^{\mu x} - \mathrm{e}^{-\mu x})dx +$$

$$\frac{\sin(\mu\pi)}{\pi}\int_0^\infty \mathrm{e}^{-t\,\mathrm{ch}\,x-\mu x}\,dx$$

Summing over $m$,

$$\sum_{m=1}^{m=+\infty} (-1)^m \, \mathrm{e}^{-mx} = -\frac{\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}}$$

we have

$$-\frac{\sin(\pi\mu)}{\pi} \int_0^\infty \mathrm{e}^{-t\,\mathrm{ch}\,x} \left[ \frac{\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}} (\mathrm{e}^{\mu x} - \mathrm{e}^{-\mu x}) + \mathrm{e}^{-\mu x} \right] dx =$$

$$-\frac{\sin(\pi\mu)}{\pi} \int_0^\infty \mathrm{e}^{-t\,\mathrm{ch}\,x} \frac{\mathrm{ch}(x(\mu-\frac{1}{2}))}{\mathrm{ch}(\frac{x}{2})} dx.$$

Thus without the first term of the integral representation of the Infeld function we have

$$V^{(d)}(\mu) = \frac{d-2}{(2\pi\rho^2)^{\frac{d}{2}}} \frac{\sin(\pi\mu)}{\pi} \int_0^\infty dx \frac{\mathrm{ch}((\frac{1}{2}-\mu)x)}{\mathrm{ch}(\frac{x}{2})} \int_{t_{min}}^{t_{max}} dt\, t^{\frac{d}{2}-1} \, \mathrm{e}^{-t(\mathrm{ch}\,x+1)}.$$

Now let us show that $\frac{1}{\pi}\int_0^\pi \mathrm{e}^{t\cos\theta} \cos(m\theta) d\theta$ gives a zero contribution

$$\sum_{m=-\infty}^{m=+\infty} \left[ \frac{1}{\pi} \int_0^\pi \mathrm{e}^{t\cos\theta} \cos(|m-\mu|\theta) d\theta - \frac{1}{\pi} \int_0^\pi \mathrm{e}^{t\cos\theta} \cos(|m|\theta) d\theta \right].$$

Then, after separate consideration of $m > 1$, $m < -1$, and $m = 0$:

$$\sum_{m=1}^{m=+\infty} \frac{1}{\pi} \int_0^\pi \mathrm{e}^{t\cos\theta} [\cos((m-\mu)\theta) + \cos((m+\mu)\theta) - \cos(m\theta) - \cos(m\theta)] d\theta +$$

$$\frac{1}{\pi} \int_0^\pi \mathrm{e}^{t\cos\theta} [\cos(\mu\theta) - 1] d\theta.$$

In the first integral, the first and the third terms are from $m > 1$, the second and the forth are from $m < 1$; the second integral is from $m = 0$.

$$\sum_{m=1}^{m=+\infty} \frac{1}{\pi} \int_0^\pi \mathrm{e}^{t\cos\theta} \, 2\cos(m\theta)(\cos(\mu\theta) - 1) d\theta + \frac{1}{\pi} \int_0^\pi \mathrm{e}^{t\cos\theta} [\cos(\mu\theta) - 1] d\theta.$$

With the use of the formula

$$\sum_{m=1}^{m=+\infty} \cos(m\theta) = \sum_{m=1}^{m=+\infty} \frac{\mathrm{e}^{im\theta}}{2} + \sum_{m=1}^{m=+\infty} \frac{\mathrm{e}^{-im\theta}}{2} = \frac{\frac{1}{2}\mathrm{e}^{\frac{i\theta}{2}}}{\mathrm{e}^{\frac{-i\theta}{2}} - \mathrm{e}^{\frac{i\theta}{2}}} + \frac{\frac{1}{2}\mathrm{e}^{-\frac{i\theta}{2}}}{\mathrm{e}^{\frac{i\theta}{2}} - \mathrm{e}^{\frac{-i\theta}{2}}} = -\frac{1}{2}$$

we proved the supposition.

The final expression for the effective potential

$$V^{(d)}(\mu) = \frac{d-2}{(2\pi\rho^2)^{\frac{d}{2}}} \frac{\sin(\pi\mu)}{\pi} \int_0^\infty dx \frac{\mathrm{ch}((\frac{1}{2}-\mu)x)}{\mathrm{ch}(\frac{x}{2})} \int_{t_{min}}^{t_{max}} dt\, t^{\frac{d}{2}-1} \, \mathrm{e}^{-t(\mathrm{ch}\,x+1)}$$

Calculate the integral for $t_{min} = 0$, $t_{max} = +\infty$, using the expressions for the gamma and beta functions

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1} \, \mathrm{e}^{-t}$$

$$B(x+y, x-y) = 4^{1-x} \int_0^\infty d\tau \frac{\mathrm{ch}(2y\tau)}{\mathrm{ch}^{2x}\tau}$$

make the change $t(\mathrm{ch}\,x+1) \to t$, then

$$\int_0^\infty dt\, t^{\frac{d}{2}-1}\, \mathrm{e}^{-t(chx+1)} \to \int_0^\infty dt\, t^{\frac{d}{2}-1}\, \mathrm{e}^{-t} \times \frac{1}{(\mathrm{ch}\,x+1)^{\frac{d}{2}}} = \frac{1}{(\mathrm{ch}\,x+1)^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}\right)$$

$$\mathrm{ch}\,x+1 = 2\,\mathrm{ch}^2\left(\frac{x}{2}\right).$$

Then, the effective potential

$$V^{(d)}(\mu) = \frac{d-2}{(2\pi\rho^2)^{\frac{d}{2}}}\frac{\sin(\pi\mu)}{\pi}\int_0^\infty dx \frac{\mathrm{ch}((\frac{1}{2}-\mu)x)}{2^{\frac{d}{2}}\,\mathrm{ch}^{d+1}(\frac{x}{2})}\Gamma\left(\frac{d}{2}\right) =$$

$$\frac{d-2}{(\pi\rho^2)^{\frac{d}{2}}}\frac{\sin(\pi\mu)}{\pi}B(\frac{d}{2}+1-\mu, \frac{d}{2}+\mu)\Gamma\left(\frac{d}{2}\right)$$

$$V^{(d)}(\mu) = \frac{d-2}{(\pi\rho^2)^{\frac{d}{2}}}\frac{\sin(\pi\mu)}{\pi}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+\mu\right)\Gamma\left(\frac{d}{2}+1-\mu\right)}{\Gamma(d+1)}$$

In the presence of fermions we should add the following term in the effective potential,

$$-\frac{1}{2}\ln\ \det\left((\partial_\mu - i\bar{A}^a_\mu t^a)^2\right)$$

then the eigenvalues correspond to the change $\mu \to \frac{\mu}{2}$, and existence of two types of fermion fields results in the additional multiplier $-2$. The result for the fermions

$$V^{(d)}(\mu) = -2\frac{d-2}{(\pi\rho^2)^{\frac{d}{2}}}\frac{\sin(\pi\frac{\mu}{2})}{\pi}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+\frac{\mu}{2}\right)\Gamma\left(\frac{d}{2}+1-\frac{\mu}{2}\right)}{\Gamma(d+1)}$$

According to Diakonov [47], we have for the effective potential for the vortex model in two cases with $d=3$ and $d=4$

$$V^{(d)}(\mu) = \frac{d-2}{(\pi\rho^2)^{\frac{d}{2}}}\cdot\frac{\sin(\pi\mu)}{\pi}\cdot\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+\mu\right)\Gamma\left(\frac{d}{2}+1-\mu\right)}{\Gamma(d+1)}$$

$$= \begin{cases} \frac{1}{\rho^4}\frac{1}{12\pi^2}\,\mu(1-\mu^2)(2-\mu)\Big|_{\mathrm{mod}\ 1} & \text{for}\ \ d=4, \\ \frac{1}{\rho^3}\frac{1}{96}\,\frac{\tan(\pi\mu)}{\pi}(1-4\mu^2)(3-2\mu)\Big|_{\mathrm{mod}\ 1} & \text{for}\ \ d=3. \end{cases} \quad (60)$$

The induced current in the planar world, $d=3$, is equal to

$$J = \frac{dV^{(3)}(\mu)}{d\mu} = \frac{1}{\rho^3}\frac{1}{16}\,\frac{\tan(\pi\mu)}{\pi}\left[(2\mu-1)^2 - \frac{4}{3}\right]$$

$$+\ \frac{1}{\rho^3}\frac{1}{96}\,\frac{1}{\cos^2(\pi\mu)}(1-4\mu^2)(3-2\mu)\Big|_{\mathrm{mod}\ 1}\ . \quad (61)$$

It can be easily verified that $J=0$ at $\mu=\frac{1}{2}$, in fact

$$6\frac{\tan(\pi\mu)}{\pi}(-\frac{4}{3}) + \frac{1}{\cos^2(\pi\mu)}2(1-4\mu^2)\Big|_{\mu\to\frac{1}{2}} = \frac{8}{\pi^2(\frac{1}{2}-\mu)} - \frac{8}{\pi^2}\frac{}{(\frac{1}{2}-\mu)} = 0, \quad (62)$$

which is in agreement with the result of Jackiw et al. [25].

## 7. Summary and Conclusions

In this paper, we have studied systems with extra dimensions, where induced current may be produced as a result of vacuum effects. Induced current of fermions appearing in this way is shown to be due to nontrivial topological structure of the models considered. In an example of quantum mechanical violation of chiral symmetry by a massless Dirac fermion moving in a vector field $A_\mu$, we demonstrated that the filled Dirac negative energy sea leads to non-conservation of chirality, while dynamics demonstrates chiral invariance. The models with a domain wall described by pseudopotential are studied with the use of the effective Dirac equation in $D = (2 + 1)$ and $D = (4 + 1)$ spaces. Vacuum energy and induced current are calculated. Induced current is found and is shown to be proportional to chiral chemical potential. This is in accordance with the Atiyah–Singer theorem, i.e., we have only one normalizable fermion zero mode for the motion in the field with topological number equal unity.

A two-fermion model with a compact dimension is considered. The model includes 5D bulk fermions $\Psi$ in interaction with fermions $L$ on the 4D brane, which resides at a fixed point of the extra dimension, and a bulk gauge field $A_M$. In the Kaluza–Klein formalism, we demonstrated that the four-fermion effective interaction among these fermions leads to appearance of vacuum energy and induced current related to "magnetic flux" of the gauge field $A_M$. In order to demonstrate another approach to the problem, we also used the Hosotani mechanism and the Aharonov–Bohm picture in derivation of the effective potential and the induced current. We finally examined the Diakonov model of a Yang–Mills vortex to solve the problem of induced current in a planar model with a thin solenoid. We re-derived his result for the effective potential for different dimensionalities ($d = 3$ and $d = 4$) and found a current induced by the nontrivial topological effect due to a Yang–Mills vortex modeling a thin solenoid.

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