

UNIVERSITY OF CANTERBURY

Department of Physics and Astronomy

CHRISTCHURCH NEW ZEALAND



Quantum Fields, Dark Matter, Elko Fields and
Non-Standard Wigner Classes

by

Adam Gillard

Doctor of Philosophy Thesis

2012

Supervisor: Associate Professor Ben Martin

Abstract

Quantum Fields, Dark Matter, Elko Fields and Non-Standard Wigner Classes

Adam Gillard

In this thesis we examine the Elko field dark matter candidate, its interactions, and possible theoretical origins. We discuss important areas in which Elko Field Theory is incomplete and propose what we consider to be the most natural ways of plugging the holes in the theory. The way we propose to plug these holes enables Elko fields to interact with Standard Model gauge quanta. Any possible Elko darkness may be then due to Elko non-locality. The possible existence of Elko gauge interactions constitutes a significant result in this thesis. We also explore how Elko quantum fields might arise on the state space. We show that the Elko field is not a quantum field in the sense of Weinberg and that the Elko field violates the symmetries of the Lorentz group; another significant result altering how we think about Elko Field Theory. We also show that subgroups of the Lorentz group do not give rise to Elko fields (or their VSR counterparts) on the state space.

We also examine the non-standard Wigner classes and show that in the context of our most natural ways of plugging the holes present in Elko Field Theory, Elko fields do not arise there either. We also show that in one of the non-standard Wigner classes, under certain conditions, there can exist a local massive spin-1/2 quantum field Majorana type dark matter candidate that is a well-defined quantum field in the sense of Weinberg. We give the dynamics of this new quantum field and also specify under what conditions this quantum field can exist. We finish the thesis by exploring Elko fields and their left and right-handed components in the context of the Electroweak Theory, in a more speculative way. We take the general concept of mass dimension transmutation introduced for Classical Spinor Theory by J.M. Hoff da Silva and R. da Rocha and apply it to the state space in the most natural way. We use this to derive a formula linking Dirac fields to the left-handed components of Elko fields and suggest the possibility of mass dimension transmutation being involved in electroweak interactions. Finally, we point out that although Elko fields cannot enter the Standard Model doublets, they can form their own doublets, the resulting symmetry currents of which can couple to the symmetry currents of the Standard Model.

Acknowledgements

I am indebted to many people who helped me complete this thesis, importantly including the following:

Associate Professor Ben Martin for investing a very significant amount of time into this project, for numerous enlightening discussions, for many useful insightful ideas, lots of excellent advice, and for his excellent advice on matters of exposition and miscellaneous help and advice in general across a whole range of issues.

Dr. Niels Gresnigt for interesting discussions on Clifford algebra, Electromagnetic Field Theory, and the Dirac equation that were indirectly beneficial to my thinking on matters directly related to the content of the thesis.

Prof. Phil Butler and Dr. William Joyce for investing a great amount of time into me in the early stages of my doctoral program, and helping to instill a positive can-do attitude when it comes to considering hard problems in physics at the foundational level.

Cheng-Yang Lee and Dimitri Schrittt for their contributions in group discussions and also private discussions on various technical matters relating to Elko.

I would like to again thank Ben Martin, Dimitri Schrittt and Cheng-Yang Lee for numerous discussions and ideas that came out of those discussions which are contained in this thesis.

Finally, I also thank both the Physics and Astronomy Department and the Mathematics and Statistics Department for their financial support.

Contents

Abstract	iii
Acknowledgements	iv
1 Introduction and Structure of the Thesis	1
1.1 Main Themes of the Thesis	1
1.2 Chapter Outline	1
1.3 Literature Review: Background and Motivation	2
1.4 Dark Matter	3
1.5 Elko Literature Review	4
1.6 Non-Standard Wigner Classes	7
1.7 Elements of the Thesis	8
2 Quantum Field Theory Review	15
2.1 Introduction	15
2.2 The Poincaré Group	16
2.3 Quantum Mechanical Axioms	18
2.4 Symmetry Operators	18
2.5 The Hilbert Space H_1 of One Particle Physical States	20
2.6 The Multiparticle Hilbert Space of Physical States	22
2.7 Creation and Annihilation Operators	24
2.8 Internal Symmetry Operators	25
2.9 Transformation Formulas for the Creation and Annihilation Operators under the Symmetry Operators	26
2.10 Scattering Theory and the S-Matrix	27
2.11 Symmetries of the S-Matrix	28
2.12 The Cluster Decomposition Principle	31
2.13 Matter Field Operators	31
2.14 Antiparticles	37
2.15 How to Work With Quantum Fields	38
2.15.1 The Gauge Principle	39
2.15.2 Dynamical Equations of Motion	41
2.16 Brief Overview of Some Important Parts of the Electroweak Theory	45
2.16.1 Standard Model Doublets	46

2.16.2	The Higgs field, Vacuum Screening Currents, and W^\pm , Z^0 and A^μ Gauge Fields	48
3	Elko Fields and Interactions	51
3.1	Introduction	51
3.2	Elko Field Review	51
3.2.1	Elko Spinors	53
3.2.2	Dual Elko Field and the Time Ordered Product	55
3.2.3	Elko Interactions	57
3.3	Elko $U(1)$ Gauge Transformation and Discussion	58
3.3.1	Elko $U(1)$ Gauge Transformation	58
3.3.2	Discussion of Elko Interactions	59
3.4	More Incompleteness in Elko Field Theory	61
3.5	General Elko Non-Abelian Gauge Symmetries	62
3.5.1	General Global Elko Non-Abelian Gauge Symmetries	62
3.5.2	General Local Elko Non-Abelian Gauge Symmetries	64
3.5.3	The Field Strength Tensor	66
3.6	Free Particle Hamiltonian	68
3.7	The Interaction Hamiltonian Density	69
3.8	Checking the Canonical Anti-Commutation Relations for the full Lagrangian	71
3.9	Elko and Dirac Symmetry Currents	73
3.10	Elko and Dirac Particle Scattering via Exchange of $U(1)$ Gauge Quanta . . .	74
3.10.1	The Cross-section	77
3.10.2	Problems With Elko at the Loop Correction Level	82
4	Elko Fields and the Weinberg Formalism	87
4.1	Introduction	87
4.2	Elko not a Quantum Field in the Sense of Weinberg	88
4.3	Subgroup of the Poincaré Group, and Elko	92
4.4	Locality	96
4.5	Elko and Very Special Relativity	102
5	Non-Standard Quantum Fields	105
5.1	Introduction	105
5.2	The Non-Standard Hilbert Spaces	106
5.3	Constructing Non-Standard Massive Spin-1/2 Quantum Fields	108
5.3.1	Case 2 Quantum Field Search	114
5.3.2	Case 3 Quantum Field Search	122
5.3.3	Case 4 Quantum Field Search	125
5.3.4	More General Non-Standard Representations of Parity and Time-Reversal	128

5.3.5	Cases 2 and 4 Revisited	131
5.3.6	Case 3 Revisited	133
5.4	Elko and Non-Standardness	136
5.5	Degeneracies, Internal Symmetries and the Theorem of Lee and Wick	137
5.6	The New Quantum Field	141
5.7	Darkness and the New Quantum Field	149
5.8	The New Quantum Field with $b^\dagger(\mathbf{p}, \sigma, \tau) \neq a^\dagger(\mathbf{p}, \sigma, \tau)$	151
5.9	Conclusion and Discussion	155
6	Elko Fields and the Electroweak Theory: an Elko-Dirac Connection	157
6.1	Introduction	157
6.2	Elko's Possible Involvement in Electroweak Interactions with Standard Model Matter	158
7	Conclusion	167
	References	173

1 Introduction and Structure of the Thesis

1.1 Main Themes of the Thesis

This thesis is about looking for “darkness” in Quantum Field Theory using Steven Weinberg’s approach to the development of Quantum Field Theory. Our search focuses on the Elko dark matter candidate, and extends to a search for viable dark matter candidates among the non-standard Wigner classes. We also take the Elko field and study its properties, independently of questions of Elko field origins, and critically analyze the existing Elko Field Theory, in particular, Elko particle interactions with Standard Model particles. One of the remaining large problems of physics is the problem of what dark matter is. This thesis helps fill a gap in this broad area of research by attempting to obtain the Elko dark matter candidate within the formalism of Quantum Field Theory as set out by Weinberg, and also by looking for possible dark matter candidates among the non-standard Wigner classes, also with the view of Quantum Field Theory adopted by Weinberg. Finally we consider Elko fields in the context of the Electroweak Theory. In this thesis, we subject the existing Elko Field Theory to critical analysis, which is a worthwhile activity as there are issues which require attention. Our thesis contributes to this process. A very important theme in this thesis is that we are taking a Quantum Field Theory approach in contrast to the usual spinorial approach.

1.2 Chapter Outline

We have divided this chapter into five main sections. We start the first section with a brief overview of the Standard Model and the main points proposed by Steven Weinberg as to why Quantum Field Theory is the way it is. We follow this by giving a brief overview of the motivation for believing in the existence of dark matter, and we give a short discussion on the main dark matter candidates. We then give a brief overview of the Elko dark matter candidate. We then have a short section introducing the idea of non-standard Wigner classes. In the final section of this chapter, we introduce the structure and content of the thesis. We specify what parts of the thesis constitute the author’s own work, and we also specify what parts are simply relevant review material which is well known and easily accessible in large portions of literature on the subject. Finally, we follow this with an account of what is contained in each section of the thesis together with why each section is there.

1.3 Literature Review: Background and Motivation

The Standard Model of particle physics is the most universally accepted theory of particle physics. It treats, in a unified way, interactions of all known visible matter by three of the four known fundamental interactions, the exception being the gravitational interaction.

The Standard Model of particle physics is built on the theory of quantum fields. Quantum Field Theory, in turn, started developing as a result of two main problems which were under investigation in the 1920's. The first main problem being looked at was how to have a quantum theory of the electromagnetic field. The second major problem under investigation was how to incorporate Special Relativity into Quantum Mechanics [1][2][3][4].

Special Relativity had been incorporated in Classical Field Theory [5][6] and historically, attempts to derive a quantum field were made by first examining the classical fields in Classical Field Theory. Plane wave solutions to the equations of Classical Field Theory were taken and the coefficients were “promoted” to become operators on a Hilbert space (see for example [7, p.126] and [8, p.24]), the creation and annihilation operators of which will be introduced in Chapter 2. Once a quantum field was written down, it would be put through the canonical formalism [7]. More will be said on this at the end of Chapter 2 and some core aspects of the canonical formalism will be illustrated in Chapter 3.

In 1964, Weinberg put together a coherent account of why quantum fields are the way they are, particularly by thinking about the second question concerning how to put Special Relativity and Quantum Mechanics together [9][10]. By considering physical states as rays in a Hilbert Space and considering the Poincaré group as part of a more general symmetry group including internal symmetries, which acts on the Hilbert space of physical states,* Weinberg put together a logically ordered account of why Quantum Field Theory is the way it is. The point of view expressed by Weinberg was that quantum fields (like the Dirac field for example) are the way they are in order that the interaction Hamiltonian gives rise to S-matrices which are Lorentz invariant and also local in the sense that sufficiently far away events do not affect each other. The locality condition is also needed to prove the Lorentz invariance of the S-matrix. We take the point of view in this thesis that Weinberg's approach to Quantum Field Theory is a good one, and this approach is the approach which we rely upon and take advantage of when looking at the Elko field and also when we look for dark matter candidates among the non-standard Wigner classes. We now turn our attention to Dark Matter and also to where the Elko dark matter candidate fits into the picture. After this, we present a brief review of the non-standard Wigner classes and state why we wanted to think about them in connection with dark matter.

*The possible representations of the Poincaré group on the Hilbert space of physical states were first methodically given by Wigner [11].

1.4 Dark Matter

In 1934, Fritz Zwicky [12] postulated that perhaps there exists dark matter. This is an unknown form of matter which is believed to not easily participate in the usual standard model interactions, with the only exception, being the gravitational interactions. He came to this conclusion while studying the orbital velocities of galaxies in clusters. There did not seem to be enough observed mass to generate the gravity whose effects were being observed. In particular, dark matter does not interact with light.

Similar conclusions have been reached from studies of the motions of individual galaxies [13, p.37]. All of these studies, concerning either clusters of galaxies or individual galaxy rotations, rely on the virial theorem, which was first properly formulated in 1870 by Clausius [14] and says roughly that the time averaged total kinetic energy should be half the total gravitational binding energy of the cluster of galaxies. It is based on the assumption that the system of massive objects is stable in some suitable way.

Not all dark matter tests have been limited to dependence on the virial theorem however. Measuring the mass density profiles of galaxies by gravitational lensing has yielded results compatible with the conclusions based on the application of the virial theorem. In 2003, a paper came out [15] adopting a further method for detecting the presence of dark matter using weak gravitational lensing and inferring the existence of dark matter from statistical arguments.

Other indirect detection attempts derive from analysis of the cosmic microwave background radiation [16][17].

There are lots of ideas about what might actually constitute dark matter. One such dark matter candidate is the axion [17][18][19]. The axion finds its origin by considering a possible term that can be added to the Lagrangian of Quantum Chromodynamics [20, p.93][21]:

$$\mathcal{L} = \frac{\theta g_s^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F_a^{\mu\nu} F_a^{\rho\sigma}. \quad (1.1)$$

with θ being some free parameter. There are terms in here which violate parity and time reversal symmetries but experiments show that, to a high degree of accuracy, both P and T are conserved which means that θ must be very small. The question of why θ should be so small, or why quantum chromodynamics preserves CP symmetry so well is known as the *strong CP problem*. A solution is to introduce the Peccei-Quinn symmetry [21] to the Standard Model and then spontaneously break the symmetry. If this is done, θ effectively becomes a quantum field and the corresponding θ particles are called axions. There have been several searches for the axion particles [22][23].

The most popular candidates for cold dark matter (dark matter that moves non-relativistically) are the weakly interacting massive particles (WIMPS) which could only have the gravitational or weak interactions. There are several experiments set up looking for WIMPS. One set up is an indirect detection method with the Super-Kamiokande in Japan. This idea works on

the principle that WIMPS would enter the sun and react, causing the sun to eject more neutrinos, which would in turn be detected by the Super-Kamiokande neutrino telescope [24]. Interpreting the data in this regard is difficult as the properties of the WIMPS themselves as well as the mass of the Higgs boson would ideally need to be known.

A direct detector effort is underway with the Cryogenic Dark Matter Search (CDMSII) detector at the Sudan Mine [25]. Researchers from there announced in December 2009 that two events had been observed, which may be WIMP related. It has also been suggested that this data is consistent with mirror matter models of dark matter [26]. A higher event rate has been observed by the CoGeNT detector which is smaller than CDMSII and designed to detect less massive WIMPS. This has given support to the idea that there exist dark $U(1)_D$ [†] gauge bosons which only interact with Standard Model particles by the kinetic mixing process [27]. Another direct detection effort is the Project in Canada to Search for Supersymmetric Objects (PICASSO). The PICASSO group have managed to put limits on the cross section for WIMP interactions on protons [28].

The main alternative to thinking about dark matter in order to account for Zwicky's observations is to consider modifications to gravity [29][30]. Another alternative idea is that there exist gravitational effects on the visible universe from multi-dimensional forces [31].

Another dark matter candidate, Elko, was proposed in 2005 [32]. It is the Elko quantum field that we look at extensively in Chapter 3, Chapter 4 and Chapter 6 which forms a significant portion of this thesis.

1.5 Elko Literature Review

In 2005 D.V. Ahluwalia and D. Grumiller announced the discovery of a new spin-1/2 matter field with mass dimension one, in contrast to the usual spin-1/2 fields which are Dirac fields of mass dimension-3/2 [32]. The spinors were eigenspinors of the finite-dimensional charge conjugation operator belonging to the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representations. This property of the spinors led to the German name *Eigenspinoren des Ladungskonjugationsoperators*, the acronym of which was settled to be *Elko* so the new quantum field was called the Elko field and the associated spinors were called the Elko spinors. This Elko field was non-local and satisfied $(CPT)^2 = -1$ for the representation space $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ so Elko became connected with the non-standard Wigner classes due to the CPT properties of the Elko spinors [32, p.18]. The non-standard Wigner classes are commented on further in Chapter 2 and then significantly further in Chapter 5. As will be explained in Chapter 5 when describing the possible irreducible representations of the full Poincaré group there are four isomorphism classes which differ from one another by their CPT properties. One of these isomorphism classes is the standard Wigner class in which the Standard Model fits, and the other three are known

[†]The group $U(1)_D$ is a $U(1)$ gauge group which is thought to be the gauge group for “dark photons,” the idea being that the dark sector is self referentially luminous.

as the non-standard Wigner classes. In the non-standard Wigner classes, there is a doubling of degrees of freedom compared to the standard Wigner class. Moreover, a theorem by Lee and Wick (see [33]) says roughly that if the symmetry group is enlarged to include internal symmetries, and if the *CPT* theorem is assumed to hold, then the non-standard classes can be made to look like two copies of the standard Wigner class.

The main candidates for Elko interactions with known matter were via the Higgs particle as well as through the gravitational interactions. This limited ability of Elko to interact with Standard Model matter, naturally led to Elko particles being put forward as a candidate for dark matter. It was proposed by D.V. Ahluwalia et al., that since Elko could not be made to fit into the Standard Model doublets due to its mass dimensionality mismatch with standard fermionic matter, Elko was a natural dark matter candidate [32][34]. In particular, Ahluwalia et al., claim there is no interaction with Standard Model photons.

Soon after this, R.da Rocha and W.A. Rodrigues Jr showed that the Elko spinors belonged to a wider class of flagpole spinors corresponding to class 5 of Lounesto's classification of spinors [35][36]. Lounesto classifies all spinors by the properties of their bilinear covariants. They also showed that all such spinors in the class of spinor which contain Elko spinors, must be built out of two-component spinors of opposite helicity.

In 2006, C.G. Bohmer applied the analysis of the Einstein-Cartan theory of gravitation to Elko spinors, leading to the idea that Elko spinors could act as sources of curvature and torsion [37]. Bohmer also argues that Elko spinors therefore help to solve the general problem of how Maxwell fields can be minimally coupled to Einstein-Cartan theory. Following this work, Bohmer then showed that the Elko spinors could be seen as a natural candidate as a driver of inflation, in addition to being a natural dark matter candidate [38].

In 2007, R. da Rocha and J.M. Hoff da Silva used the fact that Dirac and Elko spinors are classified in certain ways by Lounesto, in order to construct an algebraic way of relating Dirac spinors to Elko spinors in an effort to better understand the mathematical and physical properties of Elko spinor fields and how they relate to Dirac spinor fields [39]. They then showed in further work that Elko spinors cannot be used to describe instantons.

In 2009, da Rocha and da Silva also showed that the mapping from the Elko spinors to the Dirac spinors could be used in the process of deriving the Quadratic Spinor Lagrangian from the Einstein-Hilbert, Einstein-Palatini and Holst actions [40]. The Quadratic Spinor Lagrangian is popularly considered as a main candidate for a super gravity Lagrangian. Also, the Holst action is connected to Ashtekar's formulation of Quantum Gravity [41], so Elko spinor fields have also been linked to supersymmetric theories of gravity as well as Quantum Gravity.

In 2008, D.V. Ahluwalia, C.Y. Lee, D. Schrittt and T.F. Watson presented an Elko quantum field which is local in the sense that the fundamental anti-commutator conditions of the canonical formalism (see for example, [42, p.293]) were satisfied [34]. The spinors used were still eigenspinors of the charge conjugation operator in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ space of rep-

1 Introduction and Structure of the Thesis

representations and so are still Elko spinors. This new local Elko quantum field opened up the possibility of doing S-matrix calculations, if a suitable interaction Hamiltonian was given.

In 2011 M. Dias, F. de Campos and da Silva considered the Elko field operator when analyzing the prospects of detecting Elko particles at the LHC via interactions with the Higgs boson [43].

It has now become clear that Elko is worth studying. It has been proposed as a prime candidate for dark matter as well as a prime candidate for driving inflation. Elko has also been indirectly linked to Super Gravity and Quantum Gravity. The mapping between Dirac spinors and Elko spinors has become a candidate to use to provide a natural extension to the Standard Model to include dark matter. In Chapter 6 we look at the possibility of linking Elko in with the electroweak sector of the Standard Model by introducing the concept of mass dimension transmutation into Quantum Field Theory.

In 2009 and 2010, D. Gredat and S. Shankaranarayanan further analyzed the idea of an Elko spinor condensate driving inflation [44].

In 2010, L. Fabbri contributed to the general Elko research effort by showing that the problem of causal propagation for Elko fields is always solvable [45].

Also in 2010, H. Wei discussed the cosmological coincidence problem in the spinor dark energy model when using Elko spinors [46]. He also gave a method of reconstructing spinor dark energy from cosmological observations, which he claims “works fairly well.”

Much has been said about the Elko spinor fields but not much has been said in the literature about the use or the derivation of the associated Elko quantum field. This thesis helps to plug this hole in Elko research. We have not included our own contributions to Elko research in this list because we talk about this in great detail throughout the rest of this thesis.

We were first to show that there are problems at the level of the Hilbert space of physical states if we demand that Elko particles respect the full range of Poincaré spacetime symmetries [47][48]. Related results based on different arguments were given later in [49]. Specifically, the rotational symmetries are broken and a preferred direction results. We say more on these issues, as well as locality, during the course of this thesis, towards the end of Chapter 4. Our results have been taken into consideration by Dias, de Campos and da Silva [43, p.5]. Our results helped motivate more attention to the underlying symmetry groups that are compatible with Elko rest spinors in the sense described in Sec. (4.2). In 2010, Ahluwalia and S.P. Horvath presented a new quantum field based on spinors that transform according to the symmetry group of the more restricted Very Special Relativity symmetry group [50]. The rest spinors are Elko rest spinors but the boosted spinors are different from the Elko spinors so the resulting quantum field written down was referred to as an Elko cousin. We have more to say on this cousin field in Sec. (4.5). We finish this section by noting that [47][48] is joint work by the author of this thesis and it is based on material from this thesis.

Finally we note that more recently in 2011, Ahluwalia, Horvath, Lee and Schritt argue that an Elko theory which respects a subgroup of the Poincaré group along a preferred axis should

be regarded as a positive feature of the theory and they point to experimental evidence for the existence of a preferred axis [50][51].

1.6 Non-Standard Wigner Classes

In 1939, Wigner gave the irreducible unitary representations of the restricted Lorentz group [11]. In 1964, Wigner classified the various solutions concerning what the Hilbert space of physical states can look like under the action of the full Poincaré symmetry group, including the discrete space and time inversion symmetries [52]. Wigner showed that there are four solutions. All of the standard Quantum Field Theory is done with one of the solutions presented by Wigner. The solution universally used in the literature is known as the *Standard Wigner Class*. The other three solutions are known as the *Non-Standard Wigner Classes*. In these three classes, there is a doubling of the number of physical states. The existence of four distinct classes comes about when considering the projective representations of the full symmetry group. If the space and time inversion symmetries are ignored and assumed to not constitute physical symmetries, then the distinction is lost and the only solution is the standard Wigner class solution.

Two years later, in 1966, T.D. Lee and G.C. Wick argued that if the added assumptions were made to a field theory that it be local and also satisfy the *CPT* Theorem, then if there were internal symmetries, then either the non-standard classes did not occur, or else they could be reduced to the standard class by combining the state space representations of the space and time inversion symmetries with appropriate unitary representations of internal symmetry transformations [33]. We discuss these issues further in Sec. (5.5).

It has long been assumed that even if space or time reversal symmetries are broken, the *CPT* symmetry should still be universally valid. In 2002, O. Greenberg argued that a *CPT* violation implies a violation of Lorentz symmetries [53]. Evidence has come to light that the *CPT* symmetry might be violated since the antineutrinos seemingly have different masses to those of neutrinos. The results of these sorts of experimental searches are summarized in [54].

Weinberg only gives a proof of the *CPT* theorem in the context of the standard Wigner class. We embarked on a systematic study of the non-standard Wigner classes to see whether there could be any non-standard quantum fields which might serve as dark matter candidates and to see whether the Elko field could be derived in a non-standard state space. This is the subject of Chapter 5 of this thesis.

We now explain what is in the thesis and why various things are in the thesis.

1.7 Elements of the Thesis

This thesis has three main original components:

1. A critical analysis of Elko Field Theory, with particular emphasis on Elko interactions.
2. A look at the Elko dark matter candidate from the point of view of Quantum Field Theory, and in particular, the paradigm set out by Weinberg.
3. A search for possible dark matter candidates among the non-standard Wigner Classes from the point of view of Quantum Field Theory.

The first of these main components is found in Chapter 3, the second in Chapter 4 and the third in Chapter 5.

The first of these main original components of the thesis includes:

1. Establishing a necessary mathematical condition for Elko Lagrangians to be gauge invariant and critically analysing existing literature in Elko Field Theory in light of these results (Sec. (3.3)).
2. Describing general Elko gauge symmetries (Sec. (3.5)).
3. Deriving the Elko free particle Hamiltonian and the $U(1)$ interaction Hamiltonian density (Sec. (3.6) and Sec. (3.7)).
4. Checking that the anticommutators between the Elko fields and their new canonically conjugate field momenta have their usual form (Sec. (3.8)).
5. Deriving the Elko symmetry currents (Sec. (3.9)).
6. Investigating electromagnetic scattering between Elko and Dirac-type particles and bringing to light problems at the loop correction level of Elko quantum electrodynamic processes (Sec. (3.10)).

The second of these main original components of the thesis includes:

1. Showing that Elko fields are not quantum fields in the sense of Weinberg and that they break Lorentz symmetries (Sec. (4.2)).
2. Showing that restricting the number of Lorentz symmetries that Elko fields must satisfy, does not result in a formula for linking Elko rest spinors to boosted Elko spinors (Sec. (4.3)).
3. Observing that Elko fields do not obey an important causality condition (Sec. (4.4)).
4. Showing that Elko's VSR (Very Special Relativity) cousin fields do not arise in the sense of Weinberg from the VSR symmetry group, further suggesting that a move to non-commutative spacetime may be necessary (Sec. (4.5)).

The third of these main original components of the thesis includes:

1. Showing that there are no massive spin-1/2 quantum fields transforming according to any of the Case 2 and Case 4 non-standard representations of the Poincaré group (Sec. (5.3)).
2. Showing that there does exist a class of solutions giving rise to a quantum field transforming under the Case 3 non-standard representations of the Poincaré group. This involves a new finite-dimensional representation of the Lorentz group, which differs from the chiral representation by the forms of the discrete symmetry operators (Sec. (5.3)).
3. Discussing the absence of Elko fields among any of the Wigner classes (Sec. (5.4)).
4. Specifying the conditions that would have to exist in nature in order for the Case 3 non-standard representations of the Poincaré group to reduce to two copies of the standard Wigner class representations of the Poincaré group (Sec. (5.5)).
5. Elucidating the dynamics of the new massive spin-1/2 non-standard quantum field (Sec. (5.6)).
6. Giving the Majorana condition for the new quantum field to be a dark matter candidate (Sec. (5.7)).
7. Analyzing the non-standard quantum field in terms of its left and right-handed components, for the case where non-zero conserved quantum numbers are allowed. We suggest the identification of neutrinos with non-standard fields, and comment on a few strange aspects exhibited by the non-standard Lagrangian (Sec. (5.8)).

In addition to these three main original components, we also have a more speculative chapter where we look at mass dimension one fermionic fields in the context of the Electroweak Theory. In Chapter 6 this includes:

1. Devising and investigating a natural simplest first approach to incorporating the concept of mass dimension transmutation in the setting of the Electroweak Theory, and looking at a possible link between Elko and the electroweak sector of the Standard Model, resulting in a formula linking Standard Model Dirac fields to left-handed components of Elko fields. This, along with parity symmetry violation leads us to conjecture that perhaps the concept of mass dimension transmutation should be incorporated into Quantum Field Theory, and in particular, into the context of the Electroweak Theory (Sec. (6.2)).
2. Observing that in principle, Elko fields should be able to interact electroweakly with Standard Model particles. Any apparent darkness would not be due to a lack of ability for Elko to admit gauge interactions, but may come about as consequences of not possessing locality in all directions (Sec. (6.2)).

In the case of Elko, we initially assume the Elko quantum field to be just that, a quantum field, and we perform a series of calculations, using existing standard mathematical

1 Introduction and Structure of the Thesis

technology, motivated by physical considerations, to formally work out symmetries of the Lagrangian, determine what interactions are possible, by having gauge fields, both abelian and non-abelian, and illustrate with an example, the calculation of a cross-section of a hypothetical interaction between Elko particles and standard Dirac particles by exchange of a $U(1)$ gauge quantum. We also point out (Sec. (3.3.2)) that the Elko Quantum Field Theory research already existing in the literature, is incomplete, and we discuss the implications of this for Elko Field Theory, and in particular, Elko's ability to admit gauge interactions with Standard Model gauge quanta.

Some problems with the Elko field are pointed out as they come up. The existence of certain problems, such as that of the spin sums not being manifestly Lorentz covariant, provides a natural motivation to question whether the Elko quantum field respects the symmetries of the Poincaré group. In answering this question it becomes very important to understand how the axioms of Quantum Mechanics, together with the symmetries of Special Relativity, come together to result in quantum field operators acting on state kets in Hilbert space.

We show [47][48] (see also Sec. (4.2)) that the Elko quantum field does not respect the symmetries of the Lorentz group by examining the transformation properties of Elko fields under rotations. We do this by considering the interplay between the infinite-dimensional unitary representation of the Lorentz group on the state space and the finite-dimensional representations of the Lorentz group acting on the space of spinors. Elko fields are therefore not quantum fields in the sense of Weinberg. Later, in [49] Ahluwalia et al. explicitly stated that the rotational symmetries were violated and proposed an axis of locality for Elko fields. Our result had quickly caused a paradigm shift in the way Elko fields were understood, significantly altering the physics of Elko fields.

The question then arises whether a subgroup of the Poincaré group containing only one symmetry preserving rotation axis, and thereby automatically picking out a preferred direction, can give rise to the derivation of the Elko quantum field. The answer to this question quickly becomes apparent in the negative. See Sec. (4.5). The states are then labeled only by a single continuous label, giving the eigenvalues of the four-momentum operator. To have any hope of deriving the Elko quantum field, the states on the Hilbert space must have a two-valued discrete index in addition to the continuous index.

The next natural question which arises, is whether the Elko quantum field “feels” the whole Poincaré symmetry group but simply breaks the symmetry. We show in Sec. (4.3) that no field operator having spinors of the generic Elko form satisfies the locality conditions which must be satisfied in order to respect the Cluster Decomposition Principle.[‡]

Another question we ask, is whether the Elko quantum fields could fit in with one of the non-standard Wigner classes, which are the non-standard solutions to what the state space can look like given the unitary representations of the Poincaré spacetime symmetries. Af-

[‡]We say more on the Cluster Decomposition Principle in Sec. (2.12), but it is basically the principle that far away events should not interfere with each other.

ter all, the charge conjugation and parity operators, at the level of the finite-dimensional representations which act on the space of spinors, have the non-standard commutation/anti-commutation relations instead of the standard Dirac commutation/anti-commutation relations. We also asked, if one could find among the non-standard quantum fields, any dark matter candidates at all, whether they be Elko, or otherwise. We show in Chapter 5 that there is a possible massive spin-1/2 dark matter candidate which is both local and a quantum field in the sense of Weinberg. The Elko field does not arise from any of the non-standard Wigner classes despite the properties of the Elko spinors (see Sec. (5.4)).

All of these investigations we here undertake concerning Elko quantum fields and general massive spin-1/2 non-standard quantum fields are based on the fundamental tenets of Quantum Field Theory, some of the technical details of which become very important because Weinberg's formalism gives a clear way of answering the question of how to construct a well defined quantum field. Because of this, in the next chapter we review many of the core foundations on which Quantum Field Theory is built in the logical order of successive ideas as proposed by Weinberg. Many of the technical details in Chapter 2 are crucial to the analysis in Chapters 3 and 4. We have put Chapter 2 together in a logical order of theoretical development rather than of historical development. We see the crucial sequence of ideas which naturally follow on from one to the next as more enlightening and useful (and more relevant) to aid in developing a deeper understanding of the theory and for being able to more easily see with greater clarity how to handle the questions posed in Chapters 4, 5 and 6.

We start Chapter 2, entirely a review of existing material, by defining the Poincaré group (Sec. (2.2)). This is followed by a statement of the three fundamental axioms of Quantum Mechanics in Sec. (2.3). In Sec. (2.4) we introduce symmetry operators on the state space. We then show what the Hilbert space of physical states looks like in Sec. (2.5). After this, we give the formulas for how the symmetry operators corresponding to Poincaré spacetime symmetries act on the states, for the case of massive particles. Following this, we introduce the multiparticle Hilbert space in Sec. (2.6). We state how the symmetry operators act on these states for the case of massive particles. After this, we introduce the creation and annihilation operators in Sec. (2.7), followed by internal symmetry operators in Sec. (2.8), which correspond to internal symmetry transformations which do not necessarily depend on the Poincaré spacetime symmetries. After we have introduced these things, we move on to scattering theory and perturbation theory in Sec. (2.10) and Sec. (2.11), touching on it enough to present the essential ideas that are central to the logical flow of ideas needed to arrive at the quantum field operators. We also present the S -operator which we use when working out a hypothetical scattering cross-section between an Elko particle and a Dirac particle, via the exchange of a $U(1)$ gauge quantum in Sec. (3.10). In general, the S -operator links initial and final free particle states associated with some interaction. We explain this further in Chapter 2. Following this, we present the basic idea of the Cluster Decomposition Principle in Sec. (2.12). Finally, in Sec. (2.13), we introduce the quantum field operators and end that section

1 Introduction and Structure of the Thesis

with formulas for deriving the coefficient functions for massive quantum fields of arbitrary spin. In Sec. (2.14), we introduce the concept of antiparticles. In Sec. (2.15) we then outline how quantum fields are used, once they have been written down and introduce the concept of the gauge principle in Sec. (2.15.1) and illustrate how to get the free particle Lagrangian for the case of a scalar field in Sec. (2.15.2). We finish the chapter with Sec. (2.16) which introduces many core concepts in the Electroweak Theory which are relevant to the work in Chapter 6, where the concept of mass dimensionality transmutation is studied.

Chapter 3 starts with a short Elko review. We give the explicit form of the Elko spinors and also the form of the new dual. We then present the orthogonality and completeness relations for the Elko spinors and we point out that the Elko spinors do not obey the Dirac equation. The non-trivial connection between the spin sums and the wave operators for both Dirac and Elko spinors is given. The Elko propagator is then given, followed by the vacuum energy and the establishing of the fermionic statistics. We explain how the original version of the Elko quantum field was non-local, in a sense to be explained in Chapter 3, and how the modified Elko field is local in the sense to be explained at the end of Sec. (3.2.2) (but not local in the *causal* sense of Sec. (4.4)). We also explain how Elko might be viewed as a dark matter candidate.

Chapter 3 is about Elko fields and Elko interactions. We take the Elko field as a given, and ask *what can we do with it?* We first give a review of the Elko field. There are different variations of the Elko field so for consistency, we use the Elko field and take advantage of the results as found in [34]. The Elko Lagrangian is taken to be the Klein-Gordon Lagrangian and we argue that this Lagrangian is invariant under $U(1)$ gauge transformations in Sec. (3.3). In that section and in Sec. (3.4) we also describe some areas of incompleteness in Elko Field Theory. In Sec. (3.5) we proceed to illustrate general Elko non-abelian gauge symmetries of Elko Lagrangians of general Elko multiplets of Elko fields. We then introduce the associated field strength tensor in Sec. (3.5.3). Following this, in Sec. (3.6) we derive the form of the free particle Hamiltonian and then in Sec. (3.7) form the interaction Hamiltonian density that results from local $U(1)$ gauge invariance of the Elko Klein-Gordon Lagrangian density. In Sec. (3.8) we check the canonical anti-commutation relations with the altered canonically conjugate field momentum which is altered due to the $U(1)$ gauge transformations. In Sec. (3.9) we give the form of the Elko and Dirac symmetry currents and then use them in the following section on a hypothetical scattering of an Elko particle with a Dirac particle via the exchange of a $U(1)$ gauge quantum. We follow this with a cross-section calculation in Sec. (3.10.1) and then finish the chapter by looking at a contribution to the S-matrix at the loop correction level.

Chapter 4 concerns the question *where do Elko quantum fields fit into the general formalism of Quantum Field Theory?* Or, in other words, *how can we derive the Elko quantum field?* We start Chapter 4 by showing that the Elko field is not a quantum field in the sense of Weinberg. This (Sec. (4.2)) is one of the main results of the thesis. In Sec. (4.3) we consider

whether the Elko field can be derived by demanding the less restrictive condition that it only respects a subset of the Poincaré spacetime symmetries. We consider these results to also be of central importance in the thesis. In Sec. (4.4) we consider Elko field causality issues. We finish the chapter by looking at the Very Special Relativity (VSR) cousin of Elko fields. We argue based on the method of induced representations that if we follow the quantum field theoretic formalism of Weinberg in the most natural way for the VSR group, the cousin Elko fields do not arise, further hinting that Elko fields may find their natural theoretical setting in non-commutative spacetime.

We start Chapter 5 by showing what the non-standard state space looks like. After this, we take up the problem of how to go about constructing non-standard massive spin-1/2 quantum fields. After this, we undertake a systematic and somewhat lengthy search for massive spin-1/2 non-standard quantum fields. A candidate for a non-standard quantum field is finally found. We then discuss the absence of Elko fields among any of the Wigner classes despite the Elko spinors transforming in spinor space as members of the non-standard Wigner classes. We then focus on Case 3 among the non-standard Wigner classes and consider the Lee and Wick Theorem and the *CPT* theorem and the relationship between the non-standard Wigner class and the standard Wigner class. Following this, we examine the new quantum field and elucidate its basic dynamical properties, including giving the associated non-standard Lagrangian. We also show that the Majorana condition can be satisfied making the new quantum field a viable dark matter candidate when the field carries no conserved quantum numbers. We then move on to examining the new quantum field in the case where it carries one or more conserved quantum numbers. We conduct the analysis by looking at the left and right-handed components of the non-standard quantum field and its associated Lagrangian. We comment on its unusual aspects and suggest that perhaps neutrinos are non-standard quantum fields. Moreover, we also suggest, based on the unusual properties of the non-standard Lagrangian, that the right-handed neutrinos and left-handed antineutrinos may be prime dark matter candidates. We suggest also that their masses might be different from the left-handed neutrinos and right-handed antineutrinos.

Chapter 6 is more speculative in nature than the other chapters in the thesis. We consider Elko fields in the context of the Electroweak Theory. We get started by introducing a mass dimensionality transmutation operation taking the mass dimensionality of Standard Model fermionic fields from three halves to one. We split the fields up into their left and right-handed components and link them to the left-handed Elko field components. We then suggest what this might mean for electroweak interactions. We finish the chapter by once again taking Elko fields as a given, not worrying about their theoretical origin, and discuss their possible involvement in electroweak interactions. We argue that Elko darkness would not be due to any inability on Elko's part to participate in gauge interactions, but rather would be due to the general problem of how to detect Elko particles given their non-local aspects. We then finish the thesis in Chapter 7 with a conclusion.

1 Introduction and Structure of the Thesis

We finish Chapter 1 by remarking that we believe that in any physical theory, all of the allowed “wriggle room” in the mathematics should be fully explored so that the full extent of the physics may be elucidated and understood. Developing new theories has an important place in physics but we believe it to be *also* important to fully explore the existing physical theories. Checking the existence of existing theories and subjecting them to quality control may not be glamorous, but it is necessary.

Finally, we wish to state that we have a conference proceedings paper, [47] and another paper, [48], involving the contents of Sec. (4.2). We also have another paper, [55], which involves content taken from Chapter 3, Sec. (4.4) and Chapter 6, regarding Elko’s ability to interact with Standard Model particles via gauge interactions.

2 Quantum Field Theory Review

2.1 Introduction

In this thesis we take the point of view that Weinberg's way of understanding why quantum fields are the way they are, is a good way to understand Quantum Field Theory. Moreover, the insights that come with taking this approach are very important to our research into Elko quantum fields and also to our search for dark matter candidates among the non-standard Wigner classes.

Consequently, we here have a chapter which reviews those fundamental ideas of Weinberg's approach to Quantum Field Theory [9][10][56][42][21]. This produces the needed insights which we have used in our research on Elko quantum fields and non-standard quantum fields.

We here closely follow the main points of Weinberg's exposition of the fundamental tenets of Quantum Field Theory to the point of having explicit formulas for the construction of any massive quantum field of any spin for the case of the standard Wigner class. All of the material in this chapter is standard and is found throughout the literature on Quantum Field Theory but the sequence of ideas motivating and then defining quantum fields, is, to the best of our knowledge, unique to Weinberg. We then introduce key ideas in the canonical formalism, gauge theory and the Electroweak Theory that will be relevant for the work in this thesis.

The development of this chapter proceeds as follows:

We first define the Poincaré group. We then state the three fundamental axioms of Quantum Mechanics. We then define symmetry operators on the state space, which correspond to symmetry transformations belonging to underlying symmetry groups. We then say what the massive one particle Hilbert space of physical states can look like, according to Wigner's analysis [52]. Following this, we show how the operators corresponding to Poincaré spacetime symmetry transformations act on the basis kets in the one particle Hilbert space for the case of massive particles. We then give the multiparticle Hilbert space, according to the standard Wigner class and say how the Poincaré symmetry operators act on multiparticle states containing multiple massive particles.

After this we move on to defining the creation and annihilation operators on the multiparticle Hilbert space and we also introduce the internal symmetry operators corresponding to internal symmetry transformations, which do not explicitly depend on the Poincaré spacetime symmetries. Then we give the transformation formulas for the creation and annihilation

operators under the operators corresponding to the Poincaré spacetime transformations as well as under the operators corresponding to arbitrary internal symmetry transformations.

We then move on to giving a brief but broad outline of scattering theory and state some of the core parts of it which will allow us to make use of it when dealing with Elko field interactions in the following chapter. We also outline those essential aspects of perturbation theory which will give us an operational feel for how to use it.

We then briefly outline the remaining concept which is essential to setting the stage to introduce matter field operators, namely the Cluster Decomposition Principle.

Having done this, we introduce the field operators and use their transformation properties under the action of the symmetry operators corresponding to Poincaré spacetime transformations in order to derive explicit formulas enabling us to calculate the coefficient functions for the field operators. These formulas are the most essential part of the chapter as it is these which allow us to determine whether Elko fields respect Poincaré spacetime symmetries when the Hilbert space is defined by the standard Wigner class. Also, these same formulas turn out to be applicable to the construction of field operators on the state space which defines the non-standard Wigner classes. Following this, we introduce the notion of antiparticles and then summarize what to do with a quantum field once we have one, by saying a bit about the canonical formalism and the gauge principle. We finish the chapter by introducing some of the key ideas in the Electroweak Theory relevant to our work in Chapter 6 concerning mass dimensionality one fermions in the context of electroweak interactions.

2.2 The Poincaré Group

As will be discussed in the section on symmetry operators, a group of symmetry transformations has a representation on the state space given by a set of (anti)unitary operators on the state space. This in turn helps tell us what the Hilbert space of physical states must look like. Before discussing any of this, we need to understand what the underlying group of symmetry transformations on flat spacetime looks like. Quantum Field Theory results from combining Quantum Mechanics with Special Relativity, so the underlying spacetime is the Minkowski spacetime with the Minkowski metric being given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2 \quad (2.1)$$

where here $\eta_{\mu\nu}$ is the matrix

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (2.2)$$

Usually $\eta_{\mu\nu}$ is itself called the metric. The Poincaré group is the group of symmetries* of this metric. These symmetries are generated by three rotations parametrized by a total of three parameters, three boosts, also parametrized by a total of three parameters, four spacetime translations, parametrized by a total of four parameters, and two discrete transformations, space reversal and time reversal. The set of spacetime translation symmetry transformations is given by the four-vector a and the set of rotations and boosts is given by matrices denoted by Λ . The general form of a coordinate transformation under an arbitrary rotation, boost, and translation, is given by

$$x^{\mu'} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}. \quad (2.3)$$

If we now perform an arbitrary symmetry transformation of the same type (meaning no space or time reversal transformations involved), to this now transformed spacetime point, we get

$$x^{\mu''} = \Lambda^{\mu}_{\nu} x^{\nu'} + a^{\mu}, \quad (2.4)$$

which becomes

$$x^{\mu''} = \Lambda^{\mu}_{\nu} \Lambda^{\nu}_{\rho} x^{\rho} + \Lambda^{\mu}_{\nu} a^{\nu} + a^{\mu}. \quad (2.5)$$

We can now see that for a pair of group elements (Λ, a) , the group multiplication law is

$$(\bar{\Lambda}, \bar{a})(\Lambda, a) = (\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a}). \quad (2.6)$$

The space inversion discrete symmetry called *parity*, is given by the matrix \mathcal{P} , numerically equal to $\eta_{\mu\nu}$. The time reversal discrete symmetry, just called *time reversal*, is given by the matrix \mathcal{T} , numerically the negative of the matrix $\eta_{\mu\nu}$. The group composition law follows by inserting a \mathcal{P} or \mathcal{T} instead of a Λ in the above. The subgroup of the Poincaré group generated by the rotation, boost, and translation generators is called the *strict Poincaré group*. The subgroup of the Poincaré group generated by the rotations and boosts together with space and time inversions is called the *Lorentz group*. The subgroup of the Poincaré group generated by the generators of boosts and rotations is called the *strict Lorentz group*. The discrete transformations are very important, as it is the inclusion of these transformations which gives rise to the four Wigner classes which describe what the Hilbert space can look like, given the Poincaré group of spacetime symmetries. If there were no discrete symmetries, there would be no distinction between the Wigner classes and there would only be one way the (one particle) state space could be set up.

Furthermore, if we label the Poincaré group by \mathbb{P} , and the strict Poincaré group by \mathbb{P}^0 , the Poincaré group can be expressed as the union of four cosets:

$$\mathbb{P} = \mathbb{P}^0 \cup \mathcal{P}\mathbb{P}^0 \cup \mathcal{T}\mathbb{P}^0 \cup \mathcal{PT}\mathbb{P}^0. \quad (2.7)$$

If we know the representation $\rho|_{\mathbb{P}^0}$ of the strict Poincaré group, and if we also know $\rho(\mathcal{P})$, $\rho(\mathcal{T})$ and $\rho(\mathcal{PT})$, then we know ρ and can say what the multiplication law is for a given

*For a precise definition of what is meant by a symmetry, see, for example, [6, p.436].

2 Quantum Field Theory Review

representation of the entire Poincaré group \mathbb{P} , since, for example

$$\rho(\mathcal{P}\Lambda) = \rho(\mathcal{P})\rho(\Lambda). \quad (2.8)$$

With this essential knowledge, we now move on to giving the fundamental axioms of Quantum Mechanics, before moving on to symmetry operators, after which we will make use of some of the things which were pointed out here.

2.3 Quantum Mechanical Axioms

There are three central axioms which govern the mathematical setting in which Quantum Mechanics, and, therefore, Quantum Field Theory is done. The first axiom states that physical states are represented by rays in Hilbert space. A ray, \mathcal{R} , is a set of vectors whose elements differ from each other by a unimodular phase ξ . Usually there are more general definitions for a ray where the phases are not restricted to be unimodular (but still nonzero). The explanation for this restriction here is as follows. The norm $\langle\Psi|\Phi\rangle$ is positive definite. This, combined with the third axiom below (that the squares of inner products of state kets are interpreted as probabilities), and combined with our stated axiom here that physical states are represented by entire rays, rather than picking out special vectors within the ray, enforce that $\langle\Psi|\Psi\rangle = 1$ for every vector $|\Psi\rangle$ in the ray. If we now take another vector $|\Psi\rangle' = \xi|\Psi\rangle$ of the same ray and form the inner product of $|\Psi\rangle'$ with itself, we have

$$\langle\Psi|\Psi\rangle' = \langle\xi\Psi|\xi\Psi\rangle = \xi^*\xi\langle\Psi|\Psi\rangle = 1 \quad (2.9)$$

which forces $|\xi|$ to be 1.

The second axiom states that observables are represented by Hermitian operators on the Hilbert space. The third axiom states that given a system in a state represented by a ray \mathcal{R} , the probability of finding the state in one of a set of mutually orthogonal rays $\mathcal{R}_1, \mathcal{R}_2, \dots$ say the state \mathcal{R}_n , is given by

$$P(\mathcal{R} \rightarrow \mathcal{R}_n) = |\langle\alpha|\beta_n\rangle|^2 \quad (2.10)$$

where $|\alpha\rangle$ belongs to \mathcal{R} and $|\beta_n\rangle$ belongs to \mathcal{R}_n . For a complete set of states $|\alpha_n\rangle$, the probabilities must sum to unity:

$$\sum_n P(\mathcal{R} \rightarrow \mathcal{R}_n) = 1. \quad (2.11)$$

In the next section, we introduce the symmetry representation theorem, which allows us to set up operators on the Hilbert space of physical states, that are associated with symmetry transformations on Minkowski spacetime.

2.4 Symmetry Operators

In this section, we introduce the concept of symmetry operators on the Hilbert space.

The first crucial observation, is that a symmetry transformation should not affect the outcome of an experiment since a symmetry transformation just represents a change in our point of view. The results of an experiment should not be observer dependent. All observers of the same event should be able to compare their observations and describe the same physics. Given this, it follows that the probability of a state belonging to a ray \mathcal{R} going to a state belonging to a ray \mathcal{R}_n should be equal to the probability of going from a symmetry transformed ray \mathcal{R}' to the symmetry transformed state \mathcal{R}'_n , so we demand

$$P(\mathcal{R} \rightarrow \mathcal{R}_n) = P(\mathcal{R}' \rightarrow \mathcal{R}'_n). \quad (2.12)$$

The second key idea concerning symmetry operators is Wigner's Symmetry Representation Theorem [57], which says that for a symmetry transformation corresponding to a vector in a ray \mathcal{R} getting mapped to a vector in ray \mathcal{R}' , there exists an operator U on the Hilbert space which can be defined as either both unitary and linear, or else antiunitary and antilinear.

Since these unitary or antiunitary operators correspond to symmetry transformations, they also mimic the group composition law of the underlying symmetry group. However, a significant difference arises between the multiplication of the symmetry operators and the composition law for the symmetry transformations themselves, because the symmetry operators map an element of a ray to an element of a ray, rather than mapping a ray to a ray. The result is that, in general, the symmetry operators can form a projective representation, which is a representation up to a phase. So for example, if we have a group of symmetry transformations with group elements represented by $\rho(T_i)$, so that

$$\rho(T_1)\rho(T_2) = \rho(T_1T_2) \quad (2.13)$$

then the corresponding multiplication law for the symmetry operators is

$$U(T_1)U(T_2) = e^{i\phi(T_1, T_2)}U(T_1T_2) \quad (2.14)$$

where $\phi(T_1, T_2) \in \mathbb{R}$, and, in general $e^{i\phi(T_1, T_2)}$ cannot necessarily be made to equal unity.

We finish this section by focusing on Lie groups of transformations $T(\boldsymbol{\theta})$ which depend on a set of a continuous parameters θ^a . We continue following Weinberg [42, p.53-55]. In terms of these continuous parameters the group composition law takes the form

$$T(\bar{\boldsymbol{\theta}})T(\boldsymbol{\theta}) = T(f(\bar{\boldsymbol{\theta}}, \boldsymbol{\theta})). \quad (2.15)$$

For ordinary unitary representations of the Lie group on Hilbert space, the group composition law takes the form

$$U(T(\bar{\boldsymbol{\theta}}))U(T(\boldsymbol{\theta})) = U(T(f(\bar{\boldsymbol{\theta}}, \boldsymbol{\theta}))). \quad (2.16)$$

Since every element of the group is connected to the identity by a continuous path within the group, we can see what the representations look like in more detail by considering infinitesimal

2 Quantum Field Theory Review

translations in the neighborhood of the identity element. By expanding the functions $f^a(\bar{\theta}, \theta)$ and unitary operators $U(T(\theta))$ in power series we obtain, to second order:

$$f^a(\bar{\theta}, \theta) = \theta^a + \bar{\theta}^a + f_{bc}^a \bar{\theta}^b \theta^c + \dots \quad (2.17)$$

and

$$U(T(\theta)) = 1 + i\theta^a t_a + \frac{1}{2}\theta^b \theta^c t_{bc} + \dots \quad (2.18)$$

respectively. The f_{bc}^a are coefficients. The unitarity of $U(T(\theta))$ leads directly to the result that the t_a, t_{bc}, \dots are hermitian operators. By rewriting Eqn. (2.16) in its infinitesimal power series expanded form, and equating the $\bar{\theta}\theta$ terms on both sides of the equation, we obtain the condition

$$t_{bc} = -t_b t_c - i f_{bc}^a t_a. \quad (2.19)$$

The coefficients $\theta^b \theta^c$ of t_{bc} in the expansion Eqn. (2.16) imply that the operators t_{bc} are symmetric so that $t_{bc} = t_{cb}$ so if we note that

$$t_{cb} = -t_c t_b - i f_{cb}^a t_a \quad (2.20)$$

we can subtract Eqn. (2.19) from Eqn. (2.20) which yields

$$[t_b, t_c] = i(f_{bc}^a - f_{cb}^a)t_a. \quad (2.21)$$

Upon defining the structure constants $C_{bc}^a \equiv f_{bc}^a - f_{cb}^a$, we have a Lie algebra with Lie brackets given by

$$[t_b, t_c] = i C_{bc}^a t_a. \quad (2.22)$$

It should be noted here that the i appearing explicitly in the Lie algebra does not mean that the Lie algebra is complexified necessarily, but is there to make the generators end up being hermitian. The finite transformations represented in Hilbert space by $U(T(\theta))$ are obtained from the infinitesimal representations by exponentiation.

For symmetry operators $U(T(\theta))$ representing finite transformations T , of an abelian group which depend on a set of a continuous parameters θ , $U(T(\theta))$ is given by

$$U(T(\theta)) = \exp[i t_a \theta^a]. \quad (2.23)$$

For a general group, each $U(T(\theta))$ might consist of products of exponentials where the order of operation matters, and is often the case if the underlying symmetry group is non-Abelian.

Now that symmetry operators have been introduced, we can move on to saying what the Hilbert space of one particle physical states looks like.

2.5 The Hilbert Space H_1 of One Particle Physical States

If a physical state respects the group of Poincaré spacetime symmetries, the Hilbert space must carry a projective (anti)unitary representation of the Poincaré group. At this point, it

becomes a question of mathematics as to what the Hilbert space H_1 of one particle states can look like. Wigner showed that there are four solutions [52]. One of the solutions, the “standard Wigner class,” is the solution that we present here. We present the other three solutions, the “non-standard Wigner classes,” in Chapter 5. The one particle Hilbert space H_1 for a massive particle of mass m , is spanned by basis kets of the form $|\mathbf{p}, \sigma\rangle$.[†] These basis kets are simultaneous eigenkets of the four-momentum operator P^μ with eigenvalue p^μ , which has a three-momentum component of \mathbf{p} , and the spin- z angular momentum generator J_z with eigenvalue σ , so that

$$P^\mu |\mathbf{p}, \sigma\rangle = p^\mu |\mathbf{p}, \sigma\rangle, \quad J_z |\mathbf{0}, \sigma\rangle = \sigma |\mathbf{0}, \sigma\rangle. \quad (2.24)$$

There are infinitely many basis kets, each of the form $|\mathbf{p}, \sigma\rangle$ which span H_1 . The three-momentum \mathbf{p} runs over all possible values and there are infinitely many. The discrete index σ runs over a finite range of values, determined by the spin of the particle. For spin- j , σ has $2j + 1$ values. Any two kets $|\mathbf{p}, \sigma\rangle$ and $|\mathbf{p}', \sigma'\rangle$ are normalized so that the inner product is given by

$$(|\mathbf{p}', \sigma'\rangle, |\mathbf{p}, \sigma\rangle) = \langle \mathbf{p}', \sigma' | \mathbf{p}, \sigma \rangle = \delta^3(\mathbf{p}' - \mathbf{p}) \delta_{\sigma' \sigma}. \quad (2.25)$$

By the Symmetry Representation Theorem, for the pair of elements (Λ, a) there exists a corresponding unitary and linear operator $U(\Lambda, a)$ on H_1 . Similarly, for space inversion $(\mathcal{P}, 0)$ and time reversal $(\mathcal{T}, 0)$, there exists a corresponding unitary and linear parity operator $U(\mathcal{P}, 0)$ and antiunitary and antilinear operator $U(\mathcal{T}, 0)$ respectively, each also defined on H_1 .

Before we say what these operators $U(\Lambda, a)$, $U(\mathcal{P}, 0)$ and $U(\mathcal{T}, 0)$ do to the basis kets $|\mathbf{p}, \sigma\rangle$, we first observe that each representation with $p^0 > 0$, can be characterized by a standard four-momentum $k^\mu = (m, 0, 0, 0)$. Each momentum value can be given in terms of this standard four-momentum by applying a (matrix) boost operator $L(p)$ to k^μ so that

$$p^\mu = L^\mu{}_\nu(p) k^\nu. \quad (2.26)$$

There exists a subgroup of strict Lorentz transformations, $W(\Lambda, p)$, called the *little group*, which is defined as the group of spacetime transformations

$$W(\Lambda, p) = L^{-1}(\Lambda p) \Lambda L(p) \quad (2.27)$$

which leave the standard four-momentum invariant. By the symmetry representation theorem, there exist corresponding unitary and linear operators $U(W)$ on H_1 . These operators act on the kets $|k, \sigma\rangle$ by producing a linear combination of kets with a fixed index k so that

$$U(W) |k, \sigma\rangle = \sum_{\bar{\sigma}} D_{\bar{\sigma} \sigma}(W(\Lambda, p)) |k, \bar{\sigma}\rangle. \quad (2.28)$$

[†]The vectors spanning the Hilbert space H_1 , actually look like $\sum_{\sigma} \int d^4 p \psi(p, \sigma) |p, \sigma\rangle$ but it is sufficient to just work with the kets themselves, which is the standard practice, so we will speak of the basis kets as if they were the actual basis vectors spanning H_1 .

From this, and the group multiplication law that the $U(W)$ obey, it follows that these D 's satisfy

$$D_{\sigma'\sigma}(\bar{W}W) = \sum_{\sigma''} D_{\sigma'\sigma''}(\bar{W}) D_{\sigma''\sigma}(W). \quad (2.29)$$

For the case of positive energy massive particles, this little group is the group of rotations $SO(3)$ and the $D(W)$'s furnish an irreducible representation of $SO(3)$ for each fixed value of spin- j . With these D 's defined, the operation of acting on a basis ket $|\mathbf{p}, \sigma\rangle$ with $U(\Lambda, a)$ yields

$$U(\Lambda, a) |\mathbf{p}, \sigma\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} e^{-i(\Lambda p)^\mu a_\mu} \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p)) |\mathbf{p}_\Lambda, \sigma'\rangle. \quad (2.30)$$

The parity operator $U(\mathcal{P}, 0)$ on H_1 acts on $|\mathbf{p}, \sigma\rangle$ to give

$$U(\mathcal{P}, 0) |\mathbf{p}, \sigma\rangle = \eta |\mathbf{p}, \sigma\rangle \quad (2.31)$$

where η is the phase factor called the intrinsic parity of the particle. The time reversal operator $U(\mathcal{T}, 0)$ has the following action on $|\mathbf{p}, \sigma\rangle$:

$$U(\mathcal{T}, 0) |\mathbf{p}, \sigma\rangle = (-1)^{\frac{1}{2}-\sigma} |-\mathbf{p}, -\sigma\rangle. \quad (2.32)$$

We might like to put a phase factor on the right hand side here but since the time reversal operator is antilinear, the states can always be redefined so that the phase factor cancels with the phase factor coming from the redefined state. Explicitly, if we redefine a state

$$|\mathbf{p}, \sigma\rangle' = e^{i\beta} |\mathbf{p}, \sigma\rangle, \quad (2.33)$$

and redefine $U(\mathcal{T}, 0)$ so that

$$U(\mathcal{T}, 0) |\mathbf{p}, \sigma\rangle = e^{i\alpha} (-1)^{\frac{1}{2}-\sigma} |\mathbf{p}, \sigma\rangle, \quad (2.34)$$

we then have:

$$\begin{aligned} U(\mathcal{T}, 0) |\mathbf{p}, \sigma\rangle' &= U(\mathcal{T}, 0) e^{i\beta} |\mathbf{p}, \sigma\rangle \\ &= e^{-i\beta} U(\mathcal{T}, 0) |\mathbf{p}, \sigma\rangle \\ &= e^{i(\alpha-\beta)} (-1)^{\frac{1}{2}-\sigma} |-\mathbf{p}, -\sigma\rangle \\ &= e^{i(\alpha-\beta)} (-1)^{\frac{1}{2}-\sigma} e^{-i\beta} e^{i\beta} |-\mathbf{p}, -\sigma\rangle \\ &= e^{i(\alpha-2\beta)} (-1)^{\frac{1}{2}-\sigma} |-\mathbf{p}, -\sigma\rangle' \end{aligned} \quad (2.35)$$

which can be set equal to $(-1)^{\frac{1}{2}-\sigma} |-\mathbf{p}, -\sigma\rangle'$ by setting $\alpha = 2\beta$.

2.6 The Multiparticle Hilbert Space of Physical States

In this section, we introduce general multiparticle states in a multiparticle Hilbert space H . Firstly, the total Hilbert space $H^{(n)}$ of a particular species n , of particle, is given by either

2.6 The Multiparticle Hilbert Space of Physical States

the symmetrized or antisymmetrized tensor product of one particle Hilbert spaces $H_1^{(n)}$, depending on whether the particle is bosonic or fermionic[‡] respectively, so that

$$H^{(n)} = H_1^n \otimes H_1^n \otimes \cdots . \quad (2.36)$$

We abuse notation and simply write the symmetrized and antisymmetrized tensor products as \otimes .

The total Hilbert space H , of all numbers of particles of r species of particle, is given by

$$H = H^{(n_1)} \oplus H^{(n_2)} \oplus \cdots \oplus H^{(n_r)} = \bigoplus_{j=1}^r H^{(n_j)}. \quad (2.37)$$

Because we know what the one particle Hilbert space H_1 looks like, we automatically know what the total state space H looks like, and, furthermore, we also automatically know what the (anti)unitary symmetry operators corresponding to the Poincaré spacetime transformations must look like because the total state space H inherits a representation from the one particle state space H_1 . We denote the Hilbert space containing states with no particles as H_0 , and it is spanned by a single ket $|0\rangle$ called the *vacuum ket*, normalized so that

$$\langle 0|0\rangle = 1. \quad (2.38)$$

The vacuum state is defined to be a simultaneous eigenket of all symmetry operators with eigenvalue unity. As a matter of notation, we denote the basis kets of $H^{(n)}$ for a particle of species n by:

$$|\mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \cdots\rangle \equiv |\mathbf{p}_1, \sigma_1, n_1\rangle \otimes |\mathbf{p}_2, \sigma_2, n_2\rangle \otimes \cdots . \quad (2.39)$$

More generally, we represent a general base ket belonging to the total state space H , by

$$|\mathbf{p}_1, \sigma_1, n_1; \cdots; \mathbf{p}'_1, \sigma'_1, n_2; \cdots\rangle \equiv |\mathbf{p}_1, \sigma_1, n_1\rangle \otimes \cdots \otimes |\mathbf{p}'_1, \sigma'_1, n_2\rangle \otimes \cdots . \quad (2.40)$$

The formulas for the multiparticle symmetry operators are determined by the one particle case, so, for example, for the action of the multiparticle $U(\Lambda, a)$ on H (belonging to the standard Wigner class) we have[§]:

$$\begin{aligned} U(\Lambda, a) |\mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \cdots\rangle &= U(\Lambda, a) (|\mathbf{p}_1, \sigma_1, n_1\rangle \otimes |\mathbf{p}_2, \sigma_2, n_2\rangle \otimes \cdots) \\ &= (U_1(\Lambda, a) |\mathbf{p}_1, \sigma_1, n_1\rangle) \otimes (U_2(\Lambda, a) |\mathbf{p}_2, \sigma_2, n_2\rangle) \otimes \cdots = \sqrt{\frac{(\Lambda p_1)^0 (\Lambda p_2)^0 \cdots}{p_1^0 p_2^0 \cdots}} \times \\ &e^{-i\Lambda(p_1 + p_2 + \cdots)^\mu a_\mu} \sum_{\sigma'_1 \sigma'_2 \cdots} D_{\sigma'_1 \sigma_1}^{(j_1)}(W(\Lambda, p_1)) D_{\sigma'_2 \sigma_2}^{(j_2)}(W(\Lambda, p_2)) \cdots |\mathbf{p}_{1\Lambda}, \sigma'_1, n_1; \mathbf{p}_{2\Lambda}, \sigma'_2, n_2; \cdots\rangle. \end{aligned} \quad (2.41)$$

[‡]A bosonic particle is characterized by having integer spin and a fermionic particle is characterized by having half integer spin.

[§]Assuming that all of the particles in the state have positive mass, $m > 0$.

2 Quantum Field Theory Review

The action on H by the multiparticle parity operator $U(\mathcal{P}, 0)$ is given, for example, by

$$U(\mathcal{P}, 0) |\mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \dots\rangle = \eta_{n_1} \eta_{n_2} \dots |-\mathbf{p}_1, \sigma_1, n_1; -\mathbf{p}_2, \sigma_2, n_2; \dots\rangle. \quad (2.42)$$

where the intrinsic parities are species dependent, as indicated by the subscript n . We illustrate the action of the multiparticle time reversal operator $U(\mathcal{T}, 0)$ on H by:

$$U(\mathcal{T}, 0) |\mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \dots\rangle = (-1)^{j_1 - \sigma_1} (-1)^{j_2 - \sigma_2} \dots |-\mathbf{p}_1, -\sigma_1, n_1; -\mathbf{p}_2, -\sigma_2, n_2; \dots\rangle. \quad (2.43)$$

To illustrate a general action of an operator corresponding to a general Poincaré spacetime transformation, including rotations, boosts, translations, and either space reversal or time reversal, we have

$$\begin{aligned} U(\mathcal{P}, 0)U(\Lambda, a) |\mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \dots\rangle &= \sqrt{\frac{(\Lambda p_1)^0 (\Lambda p_2)^0 \dots}{p_1^0 p_2^0 \dots}} e^{-i(\Lambda p_1 + \Lambda p_2 + \dots)^\mu a_\mu} \quad (2.44) \\ &\times \sum_{\sigma'_1 \sigma'_2 \dots} D_{\sigma'_1 \sigma_1}^{(j_1)}(W(\Lambda, p_1)) D_{\sigma'_2 \sigma_2}^{(j_2)}(W(\Lambda, p_2)) \dots |-\mathbf{p}_{1\Lambda}, \sigma'_1, n_1; -\mathbf{p}_{2\Lambda}, \sigma'_2, n_2; \dots\rangle \end{aligned}$$

and

$$\begin{aligned} U(\mathcal{T}, 0)U(\Lambda, a) |\mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \dots\rangle &= (-1)^{j_1 - \sigma_1} (-1)^{j_2 - \sigma_2} \dots \sqrt{\frac{(\Lambda p_1)^0 (\Lambda p_2)^0 \dots}{p_1^0 p_2^0 \dots}} \quad (2.45) \\ &\times e^{-i(\Lambda p_1 + \Lambda p_2 + \dots)^\mu a_\mu} \sum_{\sigma'_1 \sigma'_2 \dots} D_{\sigma'_1, -\sigma_1}^{(j_1)*}(W(\Lambda, p_1)) D_{\sigma'_2, -\sigma_2}^{(j_2)*}(W(\Lambda, p_2)) \dots \times \\ &|-\mathbf{p}_{1\Lambda}, \sigma'_1, n_1; -\mathbf{p}_{2\Lambda}, \sigma'_2, n_2; \dots\rangle. \end{aligned}$$

In the above, the states are normalized so that the inner product is given by

$$\langle \mathbf{p}'_1, \sigma'_1, n'_1; \mathbf{p}'_2, \sigma'_2, n'_2; \dots | \mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \dots \rangle = \quad (2.46)$$

$$\delta^3(\mathbf{p}'_1 - \mathbf{p}_1) \delta_{\sigma'_1 \sigma_1} \delta_{n'_1 n_1} \delta^3(\mathbf{p}'_2 - \mathbf{p}_2) \delta_{\sigma'_2 \sigma_2} \delta_{n'_2 n_2} \dots \pm \text{permutations}.$$

2.7 Creation and Annihilation Operators

A creation operator $a^\dagger(\mathbf{p}, \sigma, n)$ on H is defined by its action on a state ket of adding a particle with an extra set of quantum labels to the start of the list of indices labeling the state ket. More explicitly:

$$a^\dagger(\mathbf{p}, \sigma, n) |\mathbf{p}_1, \sigma_1, n_1; \dots\rangle \equiv |\mathbf{p}, \sigma, n; \mathbf{p}_1, \sigma_1, n_1; \dots\rangle. \quad (2.47)$$

With this operator all state vectors can be related to the vacuum state by “pulling out” enough creation operators so that:

$$|\mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \dots\rangle = a^\dagger(\mathbf{p}_1, \sigma_1, n_1) a^\dagger(\mathbf{p}_2, \sigma_2, n_2) \dots |0\rangle. \quad (2.48)$$

The adjoint $a(\mathbf{p}, \sigma, n)$, of the creation operator $a^\dagger(\mathbf{p}, \sigma, n)$, is interpreted as being the operator which annihilates a particle of a given set of quantum numbers from the state vector.[¶] From the way creation and annihilation operators operate on multiparticle states, the following three operator relations are implied:

For two fermions, we have the anti-commutation relations:

$$\begin{aligned} \{a(\mathbf{p}_1, \sigma_1, n_1), a^\dagger(\mathbf{p}_2, \sigma_2, n_2)\} &= a(\mathbf{p}_1, \sigma_1, n_1)a^\dagger(\mathbf{p}_2, \sigma_2, n_2) + a^\dagger(\mathbf{p}_2, \sigma_2, n_2)a(\mathbf{p}_1, \sigma_1, n_1) \quad (2.49) \\ &= \delta^3(\mathbf{p}_1 - \mathbf{p}_2)\delta_{\sigma_1\sigma_2}\delta_{n_1n_2} \end{aligned}$$

$$\{a^\dagger(\mathbf{p}_1, \sigma_1, n_1), a^\dagger(\mathbf{p}_2, \sigma_2, n_2)\} = a^\dagger(\mathbf{p}_1, \sigma_1, n_1)a^\dagger(\mathbf{p}_2, \sigma_2, n_2) + a^\dagger(\mathbf{p}_2, \sigma_2, n_2)a^\dagger(\mathbf{p}_1, \sigma_1, n_1) = 0 \quad (2.50)$$

$$\{a(\mathbf{p}_1, \sigma_1, n_1), a(\mathbf{p}_2, \sigma_2, n_2)\} = a(\mathbf{p}_1, \sigma_1, n_1)a(\mathbf{p}_2, \sigma_2, n_2) + a(\mathbf{p}_2, \sigma_2, n_2)a(\mathbf{p}_1, \sigma_1, n_1) = 0, \quad (2.51)$$

and the following operator commutation relations are implied if one or both particles are bosons:

$$\begin{aligned} [a(\mathbf{p}_1, \sigma_1, n_1), a^\dagger(\mathbf{p}_2, \sigma_2, n_2)] &= a(\mathbf{p}_1, \sigma_1, n_1)a^\dagger(\mathbf{p}_2, \sigma_2, n_2) - a^\dagger(\mathbf{p}_2, \sigma_2, n_2)a(\mathbf{p}_1, \sigma_1, n_1) \quad (2.52) \\ &= \delta^3(\mathbf{p}_1 - \mathbf{p}_2)\delta_{\sigma_1\sigma_2}\delta_{n_1n_2} \end{aligned}$$

$$[a^\dagger(\mathbf{p}_1, \sigma_1, n_1), a^\dagger(\mathbf{p}_2, \sigma_2, n_2)] = a^\dagger(\mathbf{p}_1, \sigma_1, n_1)a^\dagger(\mathbf{p}_2, \sigma_2, n_2) - a^\dagger(\mathbf{p}_2, \sigma_2, n_2)a^\dagger(\mathbf{p}_1, \sigma_1, n_1) = 0 \quad (2.53)$$

$$[a(\mathbf{p}_1, \sigma_1, n_1), a(\mathbf{p}_2, \sigma_2, n_2)] = a(\mathbf{p}_1, \sigma_1, n_1)a(\mathbf{p}_2, \sigma_2, n_2) - a(\mathbf{p}_2, \sigma_2, n_2)a(\mathbf{p}_1, \sigma_1, n_1) = 0. \quad (2.54)$$

2.8 Internal Symmetry Operators

There are also other symmetry operators that act on the Hilbert space of physical states called internal symmetry operators $S_{\mathcal{I}}$. They correspond to other quantities which remain invariant in all inertial reference frames. The unitary internal symmetry operators $S_{\mathcal{I}}$, are defined on the state space as

$$\begin{aligned} S_{\mathcal{I}} |\mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \dots\rangle &= \sum_{n'_1 n'_2 \dots} \mathcal{D}_{n'_1 n_1}(S_{\mathcal{I}}) \mathcal{D}_{n'_2 n_2}(S_{\mathcal{I}}) \dots \quad (2.55) \\ &\times |\mathbf{p}_1, \sigma_1, n'_1; \mathbf{p}_2, \sigma_2, n'_2; \dots\rangle. \end{aligned}$$

for unitary matrices $\mathcal{D}_{n'_i n_i}(S_{\mathcal{I}})$. By direct calculation, we see that these internal symmetry operators commute with the operators $U(\Lambda, a)$ and $U(\mathcal{P}, 0)$ whereas in general, the $S_{\mathcal{I}}$ do not commute with $U(\mathcal{T}, 0)$ because of the antilinear nature of $U(\mathcal{T}, 0)$ and the fact that in general $D(S_{\mathcal{I}}) \neq D^*(S_{\mathcal{I}})$. Another important feature of internal symmetry operators is that by definition they commute with both the free particle and interaction parts of the Hamiltonian [42, p.122].

[¶]For more on how the annihilation operator works, see [42, p.173–174].

2.9 Transformation Formulas for the Creation and Annihilation Operators under the Symmetry Operators

Finally, in order for a multi-particle state

$$|\mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \dots\rangle = a^\dagger(\mathbf{p}_1, \sigma_1, n_1) a^\dagger(\mathbf{p}_2, \sigma_2, n_2) \dots |0\rangle \quad (2.56)$$

to transform correctly,^{||} the creation operators each transform as

$$U(\Lambda, a) a^\dagger(\mathbf{p}, \sigma, n) U(\Lambda, a)^{-1} = e^{-i(\Lambda p)^\mu a_\mu} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D_{\sigma'\sigma}^{(j_n)}(W(\Lambda, p)) a^\dagger(\mathbf{p}_\Lambda, \sigma', n) \quad (2.57)$$

$$U(\mathcal{P}, 0) a^\dagger(\mathbf{p}, \sigma, n) U(\mathcal{P}, 0)^{-1} = \eta_n a^\dagger(-\mathbf{p}, \sigma, n) \quad (2.58)$$

$$U(\mathcal{T}, 0) a^\dagger(\mathbf{p}, \sigma, n) U(\mathcal{T}, 0)^{-1} = (-1)^{j_n - \sigma} a^\dagger(-\mathbf{p}, -\sigma, n). \quad (2.59)$$

Also, we have

$$S_{\mathcal{I}} a^\dagger(\mathbf{p}, \sigma, n) S_{\mathcal{I}}^{-1} = \sum_{n'} \mathcal{D}_{n'n}(S_{\mathcal{I}}) a^\dagger(\mathbf{p}, \sigma, n'). \quad (2.60)$$

The annihilation operators transform as

$$U(\Lambda, a) a(\mathbf{p}, \sigma, n) U(\Lambda, a)^{-1} = e^{i(\Lambda p)^\mu a_\mu} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D_{\sigma'\sigma}^{(j_n)*}(W(\Lambda, p)) a(\mathbf{p}_\Lambda, \sigma', n) \quad (2.61)$$

$$U(\mathcal{P}, 0) a(\mathbf{p}, \sigma, n) U(\mathcal{P}, 0)^{-1} = \eta_n^* a(-\mathbf{p}, \sigma, n) \quad (2.62)$$

$$U(\mathcal{T}, 0) a(\mathbf{p}, \sigma, n) U(\mathcal{T}, 0)^{-1} = (-1)^{j_n - \sigma} a(-\mathbf{p}, -\sigma, n) \quad (2.63)$$

and

$$S_{\mathcal{I}} a(\mathbf{p}, \sigma, n) S_{\mathcal{I}}^{-1} = \sum_{n'} \mathcal{D}_{n'n}^*(S_{\mathcal{I}}) a(\mathbf{p}, \sigma, n'). \quad (2.64)$$

We motivate these formulas by deriving Eqn. (2.58). Consider the state

$$U(\mathcal{P}, 0) |\mathbf{p}, \sigma, n; \mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \dots\rangle. \quad (2.65)$$

If we rewrite this as

$$U(\mathcal{P}, 0) a^\dagger(\mathbf{p}, \sigma, n) |\mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \dots\rangle \quad (2.66)$$

and insert $1 = U(\mathcal{P}, 0)^{-1} U(\mathcal{P}, 0)$ to obtain

$$U(\mathcal{P}, 0) a^\dagger(\mathbf{p}, \sigma, n) U(\mathcal{P}, 0)^{-1} U(\mathcal{P}, 0) |\mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \dots\rangle, \quad (2.67)$$

then by recalling Eqn. (2.42), we get

$$U(\mathcal{P}, 0) a^\dagger(\mathbf{p}, \sigma, n) U(\mathcal{P}, 0)^{-1} \eta_{n_1} \eta_{n_2} \dots |-\mathbf{p}_1, \sigma_1, n_1; -\mathbf{p}_2, \sigma_2, n_2; \dots\rangle. \quad (2.68)$$

^{||}The vacuum state is Lorentz invariant so that $U(\Lambda, a) |0\rangle = |0\rangle$, $U(\mathcal{P}, 0) |0\rangle = |0\rangle$ and $U(\mathcal{T}, 0) |0\rangle = |0\rangle$.

But

$$U(\mathcal{P}, 0) |\mathbf{p}, \sigma, n; \mathbf{p}_1, \sigma_1, n_1; \mathbf{p}_2, \sigma_2, n_2; \dots\rangle = \eta_n \eta_{n_1} \eta_{n_2} \dots |-\mathbf{p}, \sigma, n; -\mathbf{p}_1, \sigma_1, n_1; -\mathbf{p}_2, \sigma_2, n_2; \dots\rangle \quad (2.69)$$

which can be rewritten as

$$\eta_n a^\dagger(-\mathbf{p}, \sigma, n) \eta_{n_1} \eta_{n_2} \dots |-\mathbf{p}_1, \sigma_1, n_1; -\mathbf{p}_2, \sigma_2, n_2; \dots\rangle \quad (2.70)$$

so we can compare Eqn. (2.68) with Eqn. (2.70) and write Eqn. (2.58).

2.10 Scattering Theory and the S-Matrix

In this section and the next, we review the central results and concepts used to think about particle interactions. In what follows, we have in mind situations such that free particles converge towards each other from large distances apart, and interact in a small region and then the resulting particles fly away and become free particles again at large distances from the region of interaction. In what has previously been described in this chapter until now, we have only been describing free particle states. The formulas given for how the states are transformed by symmetry operators hold for states that can be considered free particle states and it is not to be assumed that those formulas apply during the process of an interaction. We do not consider here the finer points of what will happen in the relatively small region where an interaction takes place.

In this section and the next, we first discuss in and out states, followed by defining the S -matrix. We then relate in and out states to free states and then use this to re-express the S -matrix in terms of free states, connected by the S -operator. The standard manifestly Lorentz invariant formula for the S -operator is then derived.

The physical states described so far are thought of as not evolving in time. The entire spacetime history of the state is encoded in the state ket. This is what is usually referred to as the Heisenberg picture of Quantum Mechanics. Because of this, Weinberg does not define free particle in and out states (“in” meaning “before the interaction” and “out” meaning “after” the interaction) as being the limits as $t \rightarrow \pm\infty$ of a time-dependent state vector because such a vector has not been defined here. The formalism of having time-dependent state vectors is usually referred to as the Schrodinger picture of Quantum Mechanics.

Weinberg defines in and out states, $|\mathbf{p}^+, \sigma^+, n^+\rangle$ and $|\mathbf{p}^-, \sigma^-, n^-\rangle$ to be states that would be accurately labeled by \mathbf{p} , σ and n at $t \rightarrow -\infty$ and $t \rightarrow \infty$ respectively. This definition depends on the inertial reference frame of the observer, and is not necessarily the same thing as a free state.

The S -matrix is the matrix whose entries are the probability amplitudes for starting with a certain state $|\text{in}\rangle$ and after an interaction, ending up with a certain out state $|\text{out}\rangle$ so the

elements $S_{\beta\alpha}^{**}$ of the S -matrix are given by

$$S_{\beta\alpha} = \langle \text{out} | \text{in} \rangle. \quad (2.71)$$

2.11 Symmetries of the S-Matrix

The following deals with the invariance of the S -matrix under various symmetries and with what the conditions on the Hamiltonian operator $H = P^0$ are that will ensure such invariance properties. The S -matrix is said to be invariant under strict Poincaré transformations, if

$$S_{\beta\alpha} = \langle \beta^- | \alpha^+ \rangle = \langle \beta^- | U^\dagger(\Lambda, a) U(\Lambda, a) | \alpha^+ \rangle \quad (2.72)$$

with the $U(\Lambda, a)$ acting the same way on both in and out states. More explicitly, we have

$$S_{p'_1 \sigma'_1 n'_1 \dots p_1 \sigma_1 n_1} = e^{ia_\mu \Lambda^\mu_\nu (p'_1{}^\nu + \dots - p_1{}^\nu - \dots)} \sqrt{\frac{(\Lambda p_1)^0 \dots (\Lambda p'_1)^0}{p_1^0 \dots p'_1{}^0 \dots}} \times \quad (2.73)$$

$$\sum_{\bar{\sigma}_1 \dots} D_{\bar{\sigma}_1 \sigma_1}^{j_{n_1}}(W(\Lambda, p_1)) \dots \sum_{\bar{\sigma}'_1 \dots} D_{\bar{\sigma}'_1 \sigma'_1}^{j_{n'_1}}(W(\Lambda, p'_1)) \dots S_{\Lambda p'_1 \bar{\sigma}'_1 n'_1 \dots \Lambda p_1 \bar{\sigma}_1 n_1 \dots}$$

From the Born approximation^{††}, we can write the part of the S -matrix that represents interactions among particular particles in the form

$$S_{\beta\alpha} - \delta(\beta - \alpha) = -2\pi i \mathcal{M}_{\beta\alpha} \delta^4(p_\beta - p_\alpha). \quad (2.74)$$

It is only for certain choices of Hamiltonian that there exists a unitary operator that acts on the definition as both “in” and “out” states. As will be explained in more detail in Sec. (2.15) on how to work with quantum fields, the Hamiltonian operator is the generator of time translations and consequently is fundamentally important to Quantum Field Theory. In the type of Quantum Field Theory we consider in this thesis, the effect of interactions is to add an interaction term V to the Hamiltonian (the full Hamiltonian operator for an interacting system is written by adding the free particle Hamiltonian with another operator that deals with the time evolution of an interaction, which is called the interaction Hamiltonian, or sometimes just called the interaction) while leaving the momentum and angular momentum unchanged^{‡‡}:

$$H = H_0 + V, \quad P = P_0, \quad J = J_0. \quad (2.75)$$

On the other hand, since

$$[K_0^i, P_0^j] = -iH_0 \delta^{ij}, \quad [K^i, P^j] = -iH \delta^{ij} \quad (2.76)$$

^{**}Usually the overall S -matrix itself is referred to by the same symbol as its elements and is also called $S_{\beta\alpha}$. The context is needed to work out which is which in the literature.

^{††}See, for example, [58, sec.7.2, p.386–390].

^{‡‡}Here, the subscripts “0” mean the operators previously defined on H by their action on the free particle states.

it is not possible to set the boost operator \mathbf{K} equal to its free particle counterpart K_0 , because then $H = H_0$ which is not the case in the presence of interactions. So when V is added to H_0 , a correction \mathbf{W} needs to be added to the boost generator:

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{W}. \quad (2.77)$$

Using this \mathbf{K} , it can be shown^{§§} that \mathbf{P} , \mathbf{J} and \mathbf{K} act the same way on both “in” and “out” states. ¶¶

The in and out states are related to free states carrying the same labels, by an operator given by***

$$\Omega(\mp\infty) \equiv \lim_{\tau \rightarrow \mp\infty} e^{iH\tau} e^{-iH_0\tau} \quad (2.78)$$

so that

$$|\text{in/out}\rangle = \Omega(\mp\infty) |\text{free}\rangle. \quad (2.79)$$

The S -matrix can now be related to free states by inserting this into the definition of what the S -matrix demands are:

$$\begin{aligned} S_{\beta\alpha} &= \langle \text{out} | \text{in} \rangle = \lim_{\tau \rightarrow \infty} \lim_{\tau_0 \rightarrow -\infty} \langle \text{free}_1 | e^{iH_0\tau} e^{-iH\tau} e^{iH\tau_0} e^{-iH_0\tau_0} | \text{free}_2 \rangle \\ &= \lim_{\tau \rightarrow \infty} \lim_{\tau_0 \rightarrow -\infty} \langle \text{free}_1 | e^{iH_0\tau} e^{-iH(\tau-\tau_0)} e^{-iH_0\tau_0} | \text{free}_2 \rangle. \end{aligned} \quad (2.80)$$

Following Weinberg, we refer to the operator above, without taking the indicated limits, as $U(\tau, \tau_0)$ so that

$$U(\tau, \tau_0) \equiv e^{iH_0\tau} e^{-iH(\tau-\tau_0)} e^{-iH_0\tau_0}, \quad (2.81)$$

and the S -operator is defined as being

$$S \equiv \lim_{\tau \rightarrow \infty} \lim_{\tau_0 \rightarrow -\infty} U(\tau, \tau_0). \quad (2.82)$$

We now proceed to derive a formula for the S -operator that is manifestly Lorentz invariant.

If we now differentiate $U(\tau, \tau_0)$ with respect to τ , we get

$$\frac{d}{d\tau} U(\tau, \tau_0) = iH_0 e^{iH_0\tau} e^{iH(\tau-\tau_0)} e^{-iH_0\tau} - iH e^{iH_0\tau} e^{iH(\tau-\tau_0)} e^{-iH_0\tau}. \quad (2.83)$$

Noting that $H = H_0 + V$ and substituting this in, and multiplying both sides by i gives

$$i \frac{d}{d\tau} U(\tau, \tau_0) = V(\tau) U(\tau, \tau_0) \quad (2.84)$$

where

$$V(\tau) \equiv e^{iH_0\tau} V e^{iH_0\tau}. \quad (2.85)$$

^{§§}See [42, p.120].

¶¶If the requirement is made that the matrix elements of \mathbf{W} should be smooth functions of energies and in particular, should not have singularities of the form $(E_\beta - E_\alpha)^{-1}$. See [42, p.119] for more on this.

***The minus sign in the limit is for relating in states to free states and the plus sign in the limit is for relating out states to the free states. This operator is only useful when acting on smooth superpositions of energy eigenstates [42, p.111].

2 Quantum Field Theory Review

Integrating both sides of the differential equation, using t as the dummy variable of integration yields

$$i \frac{d}{d\tau} \int_{t=\tau_0}^{t=\tau} U(t, \tau_0) dt = \int_{t=\tau_0}^{t=\tau} V(t) U(t, \tau_0) dt \quad (2.86)$$

which, by the fundamental theorem of calculus gives

$$i[U(\tau, \tau_0) - U(\tau_0, \tau_0)] = \int_{t=\tau_0}^{t=\tau} V(t) U(t, \tau_0) dt. \quad (2.87)$$

If we use the initial condition $U(\tau_0, \tau_0) = 1$ and rearrange, we get

$$U(\tau, \tau_0) = 1 - i \int_{\tau_0}^{\tau} dt V(t) U(t, \tau_0). \quad (2.88)$$

By iteration, we obtain a formula for $U(\tau, \tau_0)$ in terms of the interaction $V(t)$ to be

$$U(\tau, \tau_0) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{\tau_0}^{\tau} dt_1 \cdots \int_{\tau_0}^{t_{n-1}} dt_n V(t_1) \cdots V(t_n). \quad (2.89)$$

If we now have $\tau \rightarrow \infty$ and $\tau_0 \rightarrow -\infty$, we have the S operator which now takes the form

$$S = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n V(t_1) \cdots V(t_n). \quad (2.90)$$

There are two more steps to writing down an expression for the S -operator which is manifestly Lorentz invariant. The first of these steps is to define the time ordered product $T\{\cdots\}$ of operators $V(t_i)$. In the time ordered product the times to the right are always earlier times than the times to the left. For one operator, we have

$$T\{V(t)\} = V(t). \quad (2.91)$$

The time ordered product of two operators $V(t_1)$ and $V(t_2)$ is

$$T\{V(t_1)V(t_2)\} = \theta(t_1 - t_2)V(t_1)V(t_2) + \theta(t_2 - t_1)V(t_2)V(t_1) \quad (2.92)$$

where $\theta = +1$ for $t > 0$ and $\theta = 0$ for $t < 0$ (See [59, p.363–365] for more on the theta function). In general, the time ordered product of n operators $V(t_1) \cdots V(t_n)$ is

$$T\{V(t_1) \cdots V(t_n)\} = \sum_{\text{Permutations } i_1, i_2=1, i_1 \neq i_2} \prod_{i=1}^n \theta(t_{i_1} - t_{i_2}) V(t_{i_1}). \quad (2.93)$$

There are $n!$ permutations which need to be taken into account by dividing by $n!$ so when introducing the time ordered product, we also must introduce a compensating factor $\frac{1}{n!}$ to obtain

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n T\{V(t_1) \cdots V(t_n)\}. \quad (2.94)$$

Since, for free states $|i\rangle$ and $|f\rangle$ we have $S_{fi} = \langle f|S|i\rangle$, and we wish this to be Lorentz invariant, we want

$$U(\Lambda, a) S U(\Lambda, a)^{-1} = S. \quad (2.95)$$

This is accomplished if^{†††} the interaction can be written as the integral over space of a Lorentz scalar $\mathcal{H}(x)$ called the Hamiltonian density. By a Lorentz scalar, we mean that

$$U(\Lambda, a)\mathcal{H}(x)U(\Lambda, a)^{-1} = \mathcal{H}(\Lambda x + a). \quad (2.96)$$

Finally, the time ordering is Lorentz invariant between two timelike separated space time points[42, p.144], if^{†††} the Hamiltonian density commutes with itself at spacelike (or lightlike) separation so that^{§§§}

$$[\mathcal{H}(x), \mathcal{H}(x')] = 0 \quad \text{for} \quad (x - x')^2 \leq 0. \quad (2.97)$$

Expressed in terms of the Lorentz scalar $\mathcal{H}(x)$, the formula for the S -operator becomes

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d^4x_1 \cdots \int_{-\infty}^{\infty} d^4x_n T\{\mathcal{H}(x_1) \cdots \mathcal{H}(x_n)\}. \quad (2.98)$$

We now move on to introduce the concept of the cluster decomposition principle.

2.12 The Cluster Decomposition Principle

If we express the Hamiltonian as a sum of products of creation and annihilation operators, with suitable non-singular coefficients, the S -matrix will automatically satisfy a crucial requirement, the cluster decomposition principle, which says in effect that distant experiments yield uncorrelated results.

The cluster decomposition principle implies that an overall S -matrix for interactions taking place in mutually far away locations, factorizes. There is a theorem which says that the only known way of constructing a Hamiltonian which will give an S -matrix that satisfies the cluster decomposition principle, is to have a Hamiltonian that contains only one three-dimensional momentum-conservation delta function, and the Hamiltonian must also be expressed as a sum of products of creation and annihilation operators [42, p.182].

In the next section we introduce matter field operators and finish the section with some formulas that are crucial to showing explicitly in Chapter 4 that the Elko quantum field is not a quantum field in the sense of Weinberg. The formulas at the end of the next section will also come in useful when searching for potential dark matter candidates among the non-standard Wigner classes in Chapter 5.

2.13 Matter Field Operators

In this section, we use the Hamiltonian density Lorentz scalar to identify the quantum field operators (otherwise known as *matter fields*) and their form. Weinberg argues that the

^{†††}It is unclear whether it is also “only if.”

^{†††}But not “only if.”

^{§§§}Weinberg uses the metric $-dt^2 + dx^2 + dy^2 + dz^2$ in contrast to the one we here use so the spacelike interval here is expressed as $(x - x')^2 \leq 0$ rather than $(x - x')^2 \geq 0$ like it is expressed in [42, p.145] . We chose the given metric out of habit, with no special reason.

2 Quantum Field Theory Review

Hamiltonian density should be constructed out of creation and annihilation operators in order to make use of the Cluster Decomposition Principle [42, p.191]. The transformation properties of the creation and annihilation operators $a^\dagger(\mathbf{p}, \sigma, n)$ and $a(\mathbf{p}, \sigma, n)$ under Lorentz transformations involve matrices $D_{\sigma\sigma'}^{(j_n)}(W(\Lambda, p))$ which present a complication to the task of constructing a Hamiltonian density that is a Lorentz scalar. Weinberg argues that the solution to this construction problem is to use *field operators*. It is at this point that the concept of a field operator enters into the formalism with the approach taken by Weinberg.

We start this section by showing where the creation and annihilation fields $\psi_\ell^-(x)$ and $\psi_\ell^+(x)$ fit into the picture, whilst simultaneously giving their mathematical form. We then give the transformation properties these fields must obey in order for the Hamiltonian density composed from them to transform like a Lorentz scalar. These conditions are then used to derive formulas which allow us to construct the coefficient functions $u_\ell(x; \mathbf{p}, \sigma, n)$ and $v_\ell(x; \mathbf{p}, \sigma, n)$ of arbitrary massive quantum fields of any spin.

It is a fundamental theorem of Quantum Field Theory [42, p.171] that every operator on the Hilbert space of physical states can be written in the general form

$$\sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \int dq'_1 \dots dq'_N dq_1 \dots dq_M a^\dagger(q'_1) \dots a^\dagger(q'_N) a(q_M) \dots a(q_1) C_{NM}(q'_1 \dots q'_N q_1 \dots q_M), \quad (2.99)$$

for a set of coefficient functions $C_{NM}(q'_1 \dots q'_N q_1 \dots q_M)$. The Hamiltonian can then be written as

$$H = \int dp_1^{0'} \dots dp_N^{0'} dp_1^0 \dots dp_M^0 \mathcal{H} \quad (2.100)$$

where

$$\mathcal{H} = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \sum_{\sigma} \sum_n \int d^3\mathbf{p}'_1 \dots d^3\mathbf{p}'_N d^3\mathbf{p}_1 \dots d^3\mathbf{p}_M \times \quad (2.101)$$

$$C_{NM}(\mathbf{p}'_1, \sigma'_1, n'_1; \dots; \mathbf{p}'_N, \sigma'_N, n'_N; \mathbf{p}_1, \sigma_1, n_1; \dots; \mathbf{p}_M, \sigma_M, n_M) \times \\ a^\dagger(\mathbf{p}'_1, \sigma'_1, n'_1) \dots a^\dagger(\mathbf{p}'_N, \sigma'_N, n'_N) a(\mathbf{p}_M, \sigma_M, n_M) \dots a(\mathbf{p}_1, \sigma_1, n_1), \quad (2.102)$$

which, can be written in terms of constants $g_{\ell'_1 \dots \ell'_N, \ell_1 \dots \ell_M}$ and coefficient functions $u_\ell(x; \mathbf{p}, \sigma, n)$ and $v_\ell(x; \mathbf{p}, \sigma, n)$ as

$$\mathcal{H}(x) = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \sum_{\sigma} \sum_n \sum_{\ell'_1 \dots \ell'_N} \sum_{\ell_1 \dots \ell_M} g_{\ell'_1 \dots \ell'_N, \ell_1 \dots \ell_M} \int d^3\mathbf{p}'_1 \dots d^3\mathbf{p}'_N d^3\mathbf{p}_1 \dots d^3\mathbf{p}_M \times \quad (2.103)$$

$$u_{\ell'_1}(x; \mathbf{p}'_1, \sigma'_1, n'_1) \dots u_{\ell'_N}(x; \mathbf{p}'_N, \sigma'_N, n'_N) v_{\ell_1}(x; \mathbf{p}_1, \sigma_1, n_1) \dots v_{\ell_M}(x; \mathbf{p}_M, \sigma_M, n_M) \times \\ a^\dagger(\mathbf{p}'_1, \sigma'_1, n'_1) \dots a^\dagger(\mathbf{p}'_N, \sigma'_N, n'_N) a(\mathbf{p}_M, \sigma_M, n_M) \dots a(\mathbf{p}_1, \sigma_1, n_1).$$

Rearranging things slightly, we have:

$$\mathcal{H}(x) = \sum_{NM} \sum_{\sigma n} \sum_{\ell'_1 \dots \ell'_N} \sum_{\ell_1 \dots \ell_M} g_{\ell'_1 \dots \ell'_N, \ell_1 \dots \ell_M} \int d^3\mathbf{p}'_1 u_{\ell'_1}(x; \mathbf{p}'_1, \sigma'_1, n'_1) a^\dagger(\mathbf{p}'_1, \sigma'_1, n'_1) \times \dots \times \quad (2.104)$$

$$\int d^3\mathbf{p}'_N u_{\ell'_N}(x; \mathbf{p}'_N, \sigma'_N, n'_N) a^\dagger(\mathbf{p}'_N, \sigma'_N, n'_N) \times \int d^3\mathbf{p}_M v_{\ell_M}(x; \mathbf{p}_M, \sigma_M, n_M) a(\mathbf{p}_M, \sigma_M, n_M) \times \cdots \times \int d^3\mathbf{p}_1 v_{\ell_1}(x; \mathbf{p}_1, \sigma_1, n_1) a(\mathbf{p}_1, \sigma_1, n_1).$$

The Hamiltonian density can then be expressed as

$$\mathcal{H}(x) = \sum_{NM} \sum_{\ell'_1 \dots \ell'_N} \sum_{\ell_1 \dots \ell_M} g_{\ell'_1 \dots \ell'_N \ell_1 \dots \ell_M} \psi_{\ell'_1}^-(x) \dots \psi_{\ell'_N}^-(x) \psi_{\ell_1}^+(x) \dots \psi_{\ell_M}^+(x), \quad (2.105)$$

where $\psi_\ell^-(x)$ and $\psi_\ell^+(x)$ are operators having the form

$$\psi_\ell^+(x) = \sum_{\sigma, n} \int d^3\mathbf{p} u_\ell(x; \mathbf{p}, \sigma, n) a(\mathbf{p}, \sigma, n) \quad (2.106)$$

$$\psi_\ell^-(x) = \sum_{\sigma, n} \int d^3\mathbf{p} v_\ell(x; \mathbf{p}, \sigma, n) a^\dagger(\mathbf{p}, \sigma, n). \quad (2.107)$$

We then have, for a transformed coordinate $\Lambda x + a$:

$$\begin{aligned} \mathcal{H}(\Lambda x + a) &= \sum_{NM} \sum_{\bar{\ell}'_1 \dots \bar{\ell}'_N} \sum_{\bar{\ell}_1 \dots \bar{\ell}_M} g_{\bar{\ell}'_1 \dots \bar{\ell}'_N \bar{\ell}_1 \dots \bar{\ell}_M} \psi_{\bar{\ell}'_1}^-(\Lambda x + a) \dots \psi_{\bar{\ell}'_N}^-(\Lambda x + a) \\ &\times \psi_{\bar{\ell}_1}^+(\Lambda x + a) \dots \psi_{\bar{\ell}_M}^+(\Lambda x + a) = \sum_{NM} \sum_{\bar{\ell}'_1 \dots \bar{\ell}'_N} \sum_{\bar{\ell}_1 \dots \bar{\ell}_M} \sum_{\ell'_1 \dots \ell'_N} \sum_{\ell_1 \dots \ell_M} \times \\ &D_{\ell'_1 \dots \bar{\ell}'_1}(\Lambda^{-1}) \dots D_{\ell'_N \bar{\ell}'_N}(\Lambda^{-1}) D_{\ell_1 \bar{\ell}_1}(\Lambda^{-1}) \dots D_{\ell_M \bar{\ell}_M}(\Lambda^{-1}) \times \\ &g_{\ell'_1 \dots \ell'_N \ell_1 \dots \ell_M} \psi_{\ell'_1}^-(\Lambda x + a) \dots \psi_{\ell'_N}^-(\Lambda x + a) \psi_{\ell_1}^+(\Lambda x + a) \dots \psi_{\ell_M}^+(\Lambda x + a). \end{aligned} \quad (2.108)$$

If we compare the expressions for $\mathcal{H}(\Lambda x + a)$ and $\mathcal{H}(x)$, we see that in order for the Hamiltonian density to transform as a Lorentz scalar, it is sufficient that the field operators ψ^+ and ψ^- transform under the strict Poincaré group as

$$U(\Lambda, a) \psi_\ell^+(x) U(\Lambda, a)^{-1} = \sum_{\bar{\ell}} D_{\ell \bar{\ell}}(\Lambda^{-1}) \psi_{\bar{\ell}}^+(\Lambda x + a) \quad (2.109)$$

$$U(\Lambda, a) \psi_\ell^-(x) U(\Lambda, a)^{-1} = \sum_{\bar{\ell}} D_{\ell \bar{\ell}}(\Lambda^{-1}) \psi_{\bar{\ell}}^-(\Lambda x + a). \quad (2.110)$$

The matrices $D_{\ell \bar{\ell}}(\Lambda)$ furnish a finite-dimensional representation of the Lorentz group. These matrices are not the same as the $D_{\sigma \bar{\sigma}}$'s. The $D_{\sigma \bar{\sigma}}$'s come from the infinite-dimensional representation of the Poincaré group on the state space. The $D_{\ell \bar{\ell}}$'s however, come from a finite-dimensional representation and act on the set of u and v coefficient functions which, in the spin-1/2 case, form 4×1 -dimensional column matrices usually called spinors. In order for the fields to transform under the infinite-dimensional representation of the Poincaré group, and in order for the Hamiltonian density to transform as a Lorentz scalar, the u and v coefficient functions have to transform correctly under the action of a finite-dimensional representation of the Lorentz group.

2 Quantum Field Theory Review

In order to construct particular Lorentz scalar Hamiltonian densities, we need to calculate explicit expressions for the coefficient functions $u_\ell(x; \mathbf{p}, \sigma, n)$ and $v_\ell(x; \mathbf{p}, \sigma, n)$. In order for the S -matrix to be Lorentz invariant, for spacelike separation the Hamiltonian density must commute with itself

$$[\mathcal{H}(x_1), \mathcal{H}(x_2)] = 0, \quad (2.111)$$

which requires that the fields ψ making up $\mathcal{H}(x)$ satisfy

$$[\psi_\ell(x), \psi_{\ell'}(x')]_{\mp} = [\psi_\ell(x), \psi_{\ell'}^\dagger(x')]_{\mp} = 0 \quad (2.112)$$

for a spacelike interval $x - x'$. If we just take our fields to be $\psi_\ell^+(x)$ and $\psi_{\ell'}^-(y)$ we see that

$$\begin{aligned} [\psi_\ell^+(x), \psi_{\ell'}^-(y)]_{\mp} &= \sum_{\sigma, n} \sum_{\sigma', n'} \int d^3\mathbf{p} d^3\mathbf{p}' \times \\ &\quad \left\{ u(x; \mathbf{p}, \sigma, n) v(y; \mathbf{p}', \sigma', n') a(\mathbf{p}, \sigma, n) a^\dagger(\mathbf{p}', \sigma', n') \mp \right. \\ &\quad \left. u(x; \mathbf{p}, \sigma, n) v(y; \mathbf{p}', \sigma', n') a^\dagger(\mathbf{p}', \sigma', n') a(\mathbf{p}, \sigma, n) \right\} = \\ &= \sum_{\sigma n} \sum_{\sigma' n'} \int d^3\mathbf{p} \int d^3\mathbf{p}' u_\ell(x; \mathbf{p}, \sigma, n) v_{\ell'}(y; \mathbf{p}', \sigma', n') [a(\mathbf{p}, \sigma, n), a^\dagger(\mathbf{p}', \sigma', n')]_{\mp} \\ &= \sum_{\sigma n} \sum_{\sigma' n'} \int d^3\mathbf{p} \int d^3\mathbf{p}' u_\ell(x; \mathbf{p}, \sigma, n) v_{\ell'}(y; \mathbf{p}', \sigma', n') (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \\ &= (2\pi)^3 \sum_{\sigma n} \int d^3\mathbf{p} u_\ell(x; \mathbf{p}, \sigma, n) v_{\ell'}(y; \mathbf{p}, \sigma, n), \end{aligned} \quad (2.113)$$

which, in general, does not vanish. If, however, we choose our ψ 's such that

$$\psi_\ell(x) = \sum_{\sigma n} \int d^3\mathbf{p} [u_\ell(x; \mathbf{p}, \sigma, n) a(\mathbf{p}, \sigma, n) + v_\ell(x; \mathbf{p}, \sigma, n) a^\dagger(\mathbf{p}, \sigma, n)] \quad (2.114)$$

we see that $[\psi_\ell(x), \psi_{\ell'}(y)]_{\mp}$ becomes

$$(2\pi)^3 \int d^3\mathbf{p} [u_\ell(x; \mathbf{p}, \sigma, n) v_{\ell'}(y; \mathbf{p}, \sigma, n) \mp u_{\ell'}(y; \mathbf{p}, \sigma, n) v_\ell(x; \mathbf{p}, \sigma, n)] \quad (2.115)$$

which, for suitably chosen u 's and v 's can be made to vanish. These operators $\psi_\ell(x)$, are called quantum fields, or field operators, or sometimes matter fields. These are quantum fields where the particles are their own antiparticles. The slightly more general form of the quantum field, allowing particles and their antiparticles to be distinct particle species, is given in the next section. The resulting formulas hold independently of whether the particles are their own antiparticles.

In what follows we use Eqn. (2.109) and Eqn. (2.110) to derive formulas which can be used to help determine the u and v coefficient functions. After this any remaining freedom must satisfy Eqns. (2.112) to ensure locality. Given Eqn. (2.109) and Eqn. (2.110) and given that the creation and annihilation fields for each species of particle transform as

$$U(\Lambda, a) a^\dagger(\mathbf{p}, \sigma) U(\Lambda, a)^{-1} = e^{-i(\Lambda p)^\mu a_\mu} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D_{\sigma\sigma'}^{(j)*}(W^{-1}(\Lambda, p)) a^\dagger(\mathbf{p}_\Lambda, \sigma') \quad (2.116)$$

$$U(\Lambda, a)a(\mathbf{p}, \sigma)U(\Lambda, a)^{-1} = e^{i(\Lambda p)^\mu a_\mu} \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D_{\sigma\sigma'}^{(j)}(W^{-1}(\Lambda, p))a(\mathbf{p}_\Lambda, \sigma'), \quad (2.117)$$

taking

$$d^3\mathbf{p} \sqrt{\frac{(\Lambda p)^0}{p^0}} = \frac{d^3\mathbf{p}(\Lambda p)^0}{p^0} \frac{p^0}{(\Lambda p)^0} \sqrt{\frac{(\Lambda p)^0}{p^0}} = d^3(\Lambda p) \sqrt{\frac{p^0}{(\Lambda p)^0}} \quad (2.118)$$

gives

$$U(\Lambda, a)\psi_\ell^+(x)U(\Lambda, a)^{-1} = \sum_{\sigma n} \sum_{\sigma'} \int d^3(\Lambda p) u_\ell(x; \mathbf{p}, \sigma, n) e^{i(\Lambda p)^\mu a_\mu} \times \quad (2.119)$$

$$\sqrt{\frac{p^0}{(\Lambda p)^0}} D_{\sigma\sigma'}^{(j_n)}(W^{-1}(\Lambda, p))a(\mathbf{p}_\Lambda, \sigma', n)$$

and

$$U(\Lambda, a)\psi_\ell^-(x)U(\Lambda, a)^{-1} = \sum_{\sigma n} \sum_{\sigma'} \int d^3(\Lambda p) v_\ell(x; \mathbf{p}, \sigma, n) e^{-i(\Lambda p)^\mu a_\mu} \times \quad (2.120)$$

$$\sqrt{\frac{p^0}{(\Lambda p)^0}} D_{\sigma\sigma'}^{(j_n)*}(W^{-1}(\Lambda, p))a^\dagger(\mathbf{p}_\Lambda, \sigma', n).$$

It follows from Eqn. (2.109), Eqn. (2.110), Eqn. (2.119) and Eqn. (2.120) that

$$\sum_{\bar{\ell}} D_{\bar{\ell}\bar{\ell}}(\Lambda^{-1})u_{\bar{\ell}}(\Lambda x + a; \mathbf{p}_\Lambda, \sigma, n) = \sqrt{\frac{p^0}{(\Lambda p)^0}} D_{\sigma\sigma'}^{(j_n)}(W^{-1}(\Lambda, p))e^{i(\Lambda p)^\mu a_\mu} u_\ell(x; \mathbf{p}, \sigma', n) \quad (2.121)$$

$$\sum_{\bar{\ell}} D_{\bar{\ell}\bar{\ell}}(\Lambda^{-1})v_{\bar{\ell}}(\Lambda x + a; \mathbf{p}_\Lambda, \sigma, n) = \sqrt{\frac{p^0}{(\Lambda p)^0}} D_{\sigma\sigma'}^{(j_n)*}(W^{-1}(\Lambda, p))e^{-i(\Lambda p)^\mu a_\mu} v_\ell(x; \mathbf{p}, \sigma', n). \quad (2.122)$$

Rearranging Eqn. (2.121) and Eqn. (2.122) yields

$$\sum_{\sigma'} u_{\bar{\ell}}(\Lambda x + a; \mathbf{p}_\Lambda, \sigma', n) D_{\sigma'\sigma}^{(j_n)}(W(\Lambda, p)) = \sum_{\ell} D_{\bar{\ell}\ell}(\Lambda) e^{i(\Lambda p)^\mu a_\mu} \sqrt{\frac{p^0}{(\Lambda p)^0}} u_\ell(x; \mathbf{p}, \sigma, n) \quad (2.123)$$

$$\sum_{\sigma'} v_{\bar{\ell}}(\Lambda x + a; \mathbf{p}_\Lambda, \sigma', n) D_{\sigma'\sigma}^{(j_n)*}(W(\Lambda, p)) = \sum_{\ell} D_{\bar{\ell}\ell}(\Lambda) e^{-i(\Lambda p)^\mu a_\mu} \sqrt{\frac{p^0}{(\Lambda p)^0}} v_\ell(x; \mathbf{p}, \sigma, n) \quad (2.124)$$

which are the fundamental formulas needed to calculate the u and v coefficient functions in terms of a finite number of free parameters. When $\Lambda = 1$, these “fundamental requirements” reduce to^{¶¶¶}

$$u_\ell(x; \mathbf{p}, \sigma, n) = (2\pi)^{-3/2} e^{ip^\mu a_\mu} u_\ell(\mathbf{p}, \sigma, n) \quad (2.125)$$

$$v_\ell(x; \mathbf{p}, \sigma, n) = (2\pi)^{-3/2} e^{-ip^\mu a_\mu} v_\ell(\mathbf{p}, \sigma, n). \quad (2.126)$$

The fields $\psi_\ell^+(x)$ and $\psi_\ell^-(x)$ then become

$$\psi_\ell^+(x) = (2\pi)^{-3/2} \int d^3\mathbf{p} u_\ell(\mathbf{p}, \sigma, n) e^{ip \cdot x} a(\mathbf{p}, \sigma, n) \quad (2.127)$$

^{¶¶¶}The factor $(2\pi)^{-3/2}$ is inserted by convention following [42, p.195].

$$\psi_{\bar{\ell}}^{-}(x) = (2\pi)^{-3/2} \int d^3\mathbf{p} v_{\ell}(\mathbf{p}, \sigma, n) e^{-ip \cdot x} a^{\dagger}(\mathbf{p}, \sigma, n). \quad (2.128)$$

Comparing Eqn. (2.125) and Eqn. (2.126) to the fundamental requirements reveals that

$$\sum_{\sigma'} u_{\ell}(\mathbf{p}_{\Lambda}, \sigma', n) D_{\sigma'\sigma}^{(j_n)}(W(\Lambda, p)) = \sum_{\ell} D_{\bar{\ell}\ell}(\Lambda) \sqrt{\frac{p^0}{(\Lambda p)^0}} u_{\ell}(\mathbf{p}, \sigma, n) \quad (2.129)$$

$$\sum_{\sigma'} v_{\ell}(\mathbf{p}_{\Lambda}, \sigma', n) D_{\sigma'\sigma}^{(j_n)*}(W(\Lambda, p)) = \sum_{\ell} D_{\bar{\ell}\ell}(\Lambda) \sqrt{\frac{p^0}{(\Lambda p)^0}} v_{\ell}(\mathbf{p}, \sigma, n) \quad (2.130)$$

for arbitrary Lorentz transformations Λ . Looking at Eqn. (2.129) and Eqn. (2.130), if we take $\mathbf{p} = \mathbf{0}$ and $\Lambda = L(q)$, then $W = 1$ and the new fundamental requirements Eqn. (2.129) and Eqn. (2.130) then reduce to

$$u_{\bar{\ell}}(\mathbf{q}, \sigma, n) = \sqrt{\frac{m}{q^0}} \sum_{\ell} D_{\bar{\ell}\ell}(L(q)) u_{\ell}(\mathbf{0}, \sigma, n) \quad (2.131)$$

$$v_{\bar{\ell}}(\mathbf{q}, \sigma, n) = \sqrt{\frac{m}{q^0}} \sum_{\ell} D_{\bar{\ell}\ell}(L(q)) v_{\ell}(\mathbf{0}, \sigma, n). \quad (2.132)$$

Furthermore and finally, if we take $\mathbf{p} = \mathbf{0}$, $\mathbf{p}_{\Lambda} = \mathbf{0}$ and $W(\Lambda, p) = R$, where R is a rotation, then the fundamental requirements that the coefficient functions $u_{\ell}(\mathbf{0}, \sigma, n)$ and $v_{\ell}(\mathbf{0}, \sigma, n)$ must satisfy become:

$$\sum_{\sigma'} u_{\bar{\ell}}(\mathbf{0}, \sigma', n) D_{\sigma'\sigma}^{(j_n)}(R) = \sum_{\ell} D_{\bar{\ell}\ell}(R) u_{\ell}(\mathbf{0}, \sigma, n) \quad (2.133)$$

$$\sum_{\sigma'} v_{\bar{\ell}}(\mathbf{0}, \sigma', n) D_{\sigma'\sigma}^{(j_n)*}(R) = \sum_{\ell} D_{\bar{\ell}\ell}(R) v_{\ell}(\mathbf{0}, \sigma, n). \quad (2.134)$$

These equations will turn out to be very useful when considering whether an Elko quantum field is a quantum field in the sense of Weinberg, in Chapter 4 and they will also be of fundamental importance when hunting for non-standard quantum fields in Chapter 5.

We now close this section by summarizing Weinberg's definition of a quantum field. The ingredients we need are the following. We consider massive particles with positive energy and mass m . We take H and $U(\Lambda, a)$ to be as given in Sec. (2.5), for some choice of irreducible representation $D_{\sigma\sigma}^{(j)}(W(\Lambda, p))$ of $\text{SO}(3)$. Let $D = D_{\bar{\ell}\ell}(\Lambda)$ be a t -dimensional representation of \mathcal{L}^0 for some positive integer t . We define a *Weinberg quantum field based on the data* $(H, D_{\sigma\sigma}^{(j)}(W(\Lambda, p)), U(\Lambda, a), D_{\bar{\ell}\ell}(\Lambda))$ to be a collection of functions $\psi(x) = (\psi_{\ell}(x))_{1 \leq \ell \leq t}$ from \mathbb{R}^4 to $L(H)$ (where $L(H)$ is the set of continuous linear operators on H .) such that for all $(\Lambda, a) \in \mathbb{P}^0$, we have

$$U(\Lambda, a) \psi_{\ell}(x) U(\Lambda, a)^{-1} = \sum_{\bar{\ell}} D_{\bar{\ell}\ell}(\Lambda^{-1}) \psi_{\bar{\ell}}(\Lambda x + a). \quad (2.135)$$

The data $D_{\sigma\sigma}^{(j)}(W(\Lambda, p))$ and $U(\Lambda, a)$ are not independent — each determines the other — but we include them both for emphasis.

Before we turn our attention to what we do with quantum fields once we have them, we first make a few remarks about antiparticles.

2.14 Antiparticles

We here make a few remarks about antiparticles, which did not nicely fit into the exposition in the preceding section, but are important. Antiparticles come about in Quantum Field Theory because there are particles that are created or destroyed by quantum fields, that possess one or more conserved quantum numbers. These quantum numbers are the eigenvalues of internal symmetry operators.

Internal symmetry operators are required to commute with the Hamiltonian since the Hamiltonian is the generator of time translations. In order for this to be possible, the commutation relations between the quantum field operator and the symmetry generator need to be “simple” in some sense. For example, consider electric charge with a symmetry generator charge operator Q with eigenvalues $\pm q(n)$, which depend on the particle species n . The commutation relation between Q and $\psi_\ell(x)$ takes the form [42, p.199]

$$[Q, \psi_\ell(x)] = -q_\ell \psi_\ell(x). \quad (2.136)$$

$\mathcal{H}(x)$ can be made to commute with Q by constructing $\mathcal{H}(x)$ so that [42, p.199]

$$q_{\ell_1} + q_{\ell_2} + \cdots - q_{m_1} - q_{m_2} - \cdots = 0. \quad (2.137)$$

In order for Eqn. (2.137) to hold, for every particle species n , which carries a conserved quantum number $q(n) = q_\ell$, there must correspond a particle species \bar{n} , with a conserved quantum number equal in magnitude and opposite in sign to the quantum number carried by the particle species n , so that $q(\bar{n}) = -q_\ell$. The immediate consequence of this is that there is a doubling of particle species. These additional particles, which are identical to the original ones in every way apart from opposite values of conserved quantum numbers, are called *antiparticles*. By convention, negatively charged fermions are called particles whereas positively charged fermions are called antiparticles. The opposite convention has been adopted for bosons.

A quantum field is then written with a term that destroys particles and a term that creates antiparticles so, in general, a quantum field $\psi(x)$ is written as

$$\psi_\ell(x) = \sum_{\sigma} \int d^3\mathbf{p} [u_\ell(x; \mathbf{p}, \sigma) a(\mathbf{p}, \sigma, n) + v_\ell(x; \mathbf{p}, \sigma) a^\dagger(\mathbf{p}, \sigma, \bar{n})]. \quad (2.138)$$

The adjoint operator

$$\psi_\ell^\dagger(x) = \sum_{\sigma} \int d^3\mathbf{p} [u_\ell^*(x; \mathbf{p}, \sigma) a^\dagger(\mathbf{p}, \sigma, n) + v_\ell^*(x; \mathbf{p}, \sigma) a(\mathbf{p}, \sigma, \bar{n})] \quad (2.139)$$

creates particles and destroys antiparticles. Here the particle species is denoted by n and the associated antiparticle is denoted by \bar{n} .

We now turn our attention to outlining how quantum fields are used.

2.15 How to Work With Quantum Fields

We have seen the sequence of ideas that lead to the concept of a quantum field operator and we have also seen the formulas which provide a systematic way of determining the precise form of a quantum field of a given fixed spin. We here give a short summary concerning how quantum fields are put to use in physics.

Quantum fields are used to construct Lagrangians [42, p.297] which give rise to Lagrange's equations of motion [42, p.300] which describe the dynamics of a free particle. The Lagrangian is a functional of the field $\psi_\ell(x)$ and its time derivative $\dot{\psi}_\ell(x)$. There is an equivalent formulation of the dynamics based on the Hamiltonian formalism, where, instead of working directly with the Lagrangian, we work directly with the Hamiltonian H , which is the generator of time translations. The Hamiltonian is a functional of the quantum field $\psi_\ell(x)$ and its canonically conjugate momentum $\Pi_\ell(x)$, given by

$$\Pi_\ell(x) = \frac{\partial L}{\partial \dot{\psi}_\ell} \quad (2.140)$$

instead of the time derivative of the field. Transforming between the Lagrangian given in terms of the variables $\{\psi_\ell(x), \dot{\psi}_\ell(x)\}$ and the Hamiltonian, given in terms of the variables $\{\psi_\ell(x), \Pi_\ell(x)\}$, amounts to performing a Legendre transformation [42, sec.7.1][5, sec.8.1] which, in this case, takes the form:

$$L = \Pi \dot{\psi} - H. \quad (2.141)$$

Both the Lagrangian and Hamiltonian formulations of quantum dynamics are extensively used in Quantum Field Theory, particularly where particle interactions are involved. The interaction Hamiltonian (or, more precisely, the associated Hamiltonian density $\mathcal{H}_{\text{int}}(x)$ whose integral over all space gives the Hamiltonian) can be obtained when one knows the free particle Lagrangian density, and the interaction Lagrangian density, by a Legendre transformation. This will be illustrated for the case of Elko particles interacting with a massless $U(1)$ gauge field, which we explain in the next chapter.

Once the interaction Hamiltonian density has been obtained, S-matrix calculations can be done to obtain probability amplitudes for various particle interactions. Cross sections can then be calculated and phrased in terms of “laboratory variables.” Experiments can be done and the physical cross sections measured. These can then be compared with the results coming out of the Quantum Field Theory formalism, and in this way, the theory can be tested to give us a measure of how well the theory models reality.

The remaining link yet to be touched on in all this, is how to write down an expression for a Lagrangian density whether for the free particle case, or for the interacting cases. We now comment on these two issues. In Sec. (2.15.2) we will demonstrate how to construct a Lagrangian density for a free spin-zero massive particle. Before we do this, we will make a few comments about how the interaction Lagrangian density is obtained, once we know the Lagrangian density for the case of a free particle, or a system of free particles.

2.15.1 The Gauge Principle

Interaction Lagrangians are usually either written down based on dimensional arguments, only being restricted by the requirement that the term be a Lorentz scalar, or else, interaction terms are arrived at by the gauge principle. We shall here outline the concept of the gauge principle.

The gauge principle ties the dynamics of force-carrying particles to a symmetry principle. Usually the gauge principle is used to deduce that a particle with zero mass has unit spin. The approach taken by Weinberg [42], is consistent with the approach taken in modern string theories, which is that we observe massless particles of unit spin and conclude that there must be a resulting gauge invariance of the field theory of the massless particles of unit spin.

The gauge principle asserts that if the field operator $\psi(x)$ appearing in the dynamical equation of motion is changed by multiplying it by a space-time dependent phase transformation so that

$$\psi(x) \rightarrow \psi'(x) = e^{iq\chi(x)}\psi(x) \quad (2.142)$$

where $\chi(x)$ is a scalar field, the invariance of the field equation is not possible for a free particle theory, so an interacting theory is required. This interacting theory involves a spin-1 massless field A^μ which undergoes the transformation

$$A^\mu \rightarrow A^{\mu'} = A^\mu + \partial^\mu \chi. \quad (2.143)$$

This will dictate the form of the interaction and enters by replacing partial derivatives with appropriate covariant derivatives which contain A^μ in a way to be shown shortly. The field $A^\mu(x)$ has a more complicated transformation law under the Lorentz transformations [42, p.251] than fields which we refer to as being “quantum fields in the sense of Weinberg.” To illustrate how the gauge principle works, we here consider the Dirac Lagrangian density [7]

$$\mathcal{L} = \frac{i}{2}[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi, \quad (2.144)$$

for the case of a free Dirac Fermion particle. The γ^μ are 4×4 matrices satisfying the Clifford algebra $C\ell(1, 3)$ [36][60][61] and $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$. The precise details of what constitutes the Dirac quantum field can be found, for example, in [42, sec.5.5].

If we make the transformation $\psi(x) \rightarrow \psi'(x) = e^{iq\chi(x)}\psi(x)$, it follows by inspection that $\bar{\psi}'(x) = \bar{\psi}(x)e^{-iq\chi}$. In terms of the Lagrangian density, the gauge principle asserts that the Lagrangian density is invariant under this phase transformation so that $\mathcal{L}' = \mathcal{L}$. The problem with this Lagrangian density is that $\mathcal{L}' \neq \mathcal{L}$ because it involves partial derivatives of the fields, which will produce extra terms by acting on the space-time dependent phase factor. For the Lagrangian density to be invariant so that $\mathcal{L}' = \mathcal{L}$, we need to modify the Lagrangian density by adding extra terms which will cancel with the extra terms generated by the derivatives of the space-time dependent phase factor. The solution is to define a covariant derivative D_μ

2 Quantum Field Theory Review

which satisfies the crucial condition:

$$D'_\mu \psi'(x) = e^{iq\chi(x)} D_\mu \psi(x). \quad (2.145)$$

If we have such a covariant derivative and replace the free particle Lagrangian density with

$$\mathcal{L} = \frac{i}{2} [\bar{\psi} \gamma^\mu (D_\mu \psi) - (D_\mu \bar{\psi}) \gamma^\mu \psi] - m \bar{\psi} \psi, \quad (2.146)$$

and use this as our Lagrangian density, then $\mathcal{L}' = \mathcal{L}$ as required. In this particular case, the covariant derivative is defined to be

$$D_\mu = \partial_\mu + iqA_\mu \quad (2.147)$$

and

$$D'_\mu = \partial_\mu + iqA'_\mu \quad \text{where} \quad A'_\mu = A_\mu + \partial_\mu \chi. \quad (2.148)$$

The above Lagrangian density can be expanded out and written as a sum of the original free particle Lagrangian density $\mathcal{L}_{\text{Free}}$ and an extra part called the *interaction Lagrangian density* \mathcal{L}_{int} so that

$$\begin{aligned} \mathcal{L} &= \frac{i}{2} [\bar{\psi} \gamma^\mu (D_\mu \psi) - (D_\mu \bar{\psi}) \gamma^\mu \psi] - m \bar{\psi} \psi \\ &= \frac{i}{2} [\bar{\psi} \gamma^\mu ((\partial_\mu + iqA_\mu) \psi) - ((\partial_\mu - iqA_\mu) \bar{\psi}) \gamma^\mu \psi] - m \bar{\psi} \psi \\ &= \frac{i}{2} [\bar{\psi} \gamma^\mu (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma^\mu \psi] - m \bar{\psi} \psi - q \bar{\psi} \gamma^\mu \psi A_\mu \\ &= \mathcal{L}_{\text{Free}} + \mathcal{L}_{\text{int}}, \end{aligned} \quad (2.149)$$

where

$$\mathcal{L}_{\text{int}} = -q \bar{\psi} \gamma^\mu \psi A_\mu. \quad (2.150)$$

Exactly how to set up the covariant derivative for the generalized gauge principle involving multiplets of field operators, will be explained in the next chapter, in the context of Elko multiplets of Elko fields.

At this point, we wish to point out that the Dirac Lagrangian is invariant under $U(1)$ gauge transformations but the *Majorana Lagrangian density* is *not* $U(1)$ gauge invariant. The Majorana Lagrangian is made out of spin-1/2 quantum fields corresponding to particles that do not have any conserved quantum numbers. The Dirac field has a term that destroys a particle of species n and creates an antiparticle of species \bar{n} . A Majorana field takes the form:

$$\psi_\ell(x) = \int \frac{d^3 \mathbf{p}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2p^0}} \sum_\sigma [e^{ip \cdot x} u_\ell(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{-ip \cdot x} v_\ell(\mathbf{p}, \sigma) a^\dagger(\mathbf{p}, \sigma)] \quad (2.151)$$

where $a^\dagger(\mathbf{p}, \sigma)$ is the adjoint of $a(\mathbf{p}, \sigma)$. The free particle Lagrangian density made from these Majorana fields is [8, p.248–250]:

$$\mathcal{L} = \frac{1}{2} \psi^T C \gamma^\mu \partial_\mu \psi - \frac{1}{2} m \psi^T C \psi \quad (2.152)$$

where C is a constant matrix defined in [8, p.242]. The ψ^T on the left hand side instead of a field adjoint ψ^\dagger results in there being no e^{-iqx} term to cancel with the e^{+iqx} term so $\mathcal{L}' \neq \mathcal{L}$ under $U(1)$ gauge transformations when the Lagrangian density is made out of Majorana particles.

If we take multiplets of non-Majorana fields, Lagrangians can be found that are invariant under non-Abelian gauge groups. We illustrate this generalized gauge principle in Chapter 3. If a Majorana Lagrangian density is taken to have multiplets of Majorana fields then these Lagrangians will not be invariant under any group of gauge transformations. We show in Chapter 5 that there is a non-standard massive spin-1/2 quantum field which can satisfy the Majorana condition making the new quantum field a dark matter candidate (Sec. (5.7)).

We now turn to the final issue mentioned here which has not yet been elaborated on, which is the question of how to obtain the free particle Lagrangian density for a given field. We will illustrate this for the case of a real massive spin-zero scalar field.

2.15.2 Dynamical Equations of Motion

To finish off the review of the core elements of Quantum Field Theory, before introducing the Electroweak Theory, we here derive the dynamical equations of motion for the scalar field. Scalar fields transform according to the scalar representation of the Lorentz group so $D(\Lambda) = 1$. Eqn. (2.133) and Eqn. (2.134) only have solutions when $j = 0$. The rest coefficient functions are set by convention to be

$$u(\mathbf{0}, \sigma, n) = u(\mathbf{0}) = \frac{1}{\sqrt{2m}}, \quad v(\mathbf{0}, \sigma, n) = v(\mathbf{0}) = \frac{1}{\sqrt{2m}}. \quad (2.153)$$

The boosted coefficient functions are then given by

$$u(\mathbf{p}) = \sqrt{\frac{m}{p_0}} \frac{1}{\sqrt{2m}} = \frac{1}{\sqrt{2p_0}} = v(\mathbf{p}), \quad (2.154)$$

so, for a scalar field $\phi(x)$ with no conserved quantum numbers, $\phi(x)$ is given by

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} [e^{ip \cdot x} a(\mathbf{p}) + e^{-ip \cdot x} a^\dagger(\mathbf{p})]. \quad (2.155)$$

In order to derive the kinematical equations of motion for this scalar field, we take advantage of the canonical formalism. A local bosonic quantum field $q(\mathbf{x}, t)$ in the canonical formalism is local, if there can be found a canonically conjugate momentum $p(\mathbf{y}, t)$ such that

$$[q(\mathbf{x}, t), p(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}) \quad (2.156)$$

$$[q(\mathbf{x}, t), q(\mathbf{y}, t)] = 0 \quad (2.157)$$

$$[p(\mathbf{x}, t), p(\mathbf{y}, t)] = 0. \quad (2.158)$$

If we choose $q(\mathbf{x}, t)$ to be the scalar field so that $q(\mathbf{x}, t) = \phi(\mathbf{x}, t)$, then the field which satisfies the above conditions to be identified as the canonically conjugate field momentum is $\dot{\phi}$ so

2 Quantum Field Theory Review

here $p(\mathbf{x}, t) = \dot{\phi}(\mathbf{x}, t)$. To make progress, these fields are to be inverted in order to express the creation and annihilation operators in terms of the field and its canonically conjugate field momentum. These expressions will then be substituted into the free particle Hamiltonian which must always have the form [42, p.296–297]

$$H_0 = \sum_{n,\sigma} \int \frac{d^3\mathbf{p}}{(2\pi)^3} p_0 a^\dagger(\mathbf{p}, \sigma, n) a(\mathbf{p}, \sigma, n). \quad (2.159)$$

Having obtained the Hamiltonian in terms of the fields $q(x)$ and $p(x)$, the Lagrangian is then obtained by a Legendre transformation. The free field Lagrangian is given by [42, p.297]

$$L = \sum_n \int d^3\mathbf{x} p_n(\mathbf{x}, t) \dot{q}^n(\mathbf{x}, t) - H_0. \quad (2.160)$$

The kinematical equations of motion are then derived from the Lagrangian density (the Lagrangian density is the scalar function which, when integrated over all space, gives the Lagrangian functional: $L = \int d^3\mathbf{x} \mathcal{L}$) \mathcal{L} via Lagrange's equations of motion which are

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \quad (2.161)$$

In order to take the first step in this process of expressing the creation and annihilation operators in terms of the fields and field momenta we follow [8] (although [8] does not then go on to calculate the Lagrangian density or dynamical equations of motion with these expressions however), and calculate $\int d^3\mathbf{x} e^{-ip \cdot x} \phi(x)$ and $\int d^3\mathbf{x} e^{-ip \cdot x} p(x)$. We have, for the first expression,

$$\int d^3\mathbf{x} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{1}{\sqrt{2p'_0}} [e^{i(p'-p) \cdot x} a(\mathbf{p}') + e^{-i(p'+p) \cdot x} a^\dagger(\mathbf{p}')]. \quad (2.162)$$

Performing the integration over space yields

$$\int \frac{d^3\mathbf{p}'}{\sqrt{2p'_0}} [\delta^3(\mathbf{p}' - \mathbf{p}) a(\mathbf{p}') e^{i(p'-p_0)t} + e^{-i(p'_0+p_0)t} \delta^3(\mathbf{p}' + \mathbf{p}) a^\dagger(\mathbf{p}')]. \quad (2.163)$$

Performing the remaining integration yields

$$\frac{1}{\sqrt{2p_0}} [a(\mathbf{p}) + e^{-2ip_0 t} a^\dagger(-\mathbf{p})]. \quad (2.164)$$

Calculating $\int d^3\mathbf{x} e^{-ip \cdot x} p(x)$ now, gives

$$\int d^3\mathbf{x} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{(ip'_0)}{\sqrt{2p'_0}} [e^{i(p'-p) \cdot x} a(\mathbf{p}') - e^{-i(p'+p) \cdot x} a^\dagger(\mathbf{p}')]. \quad (2.165)$$

Performing the integration over space gives

$$\int d^3\mathbf{p}' i \sqrt{\frac{p'_0}{2}} [\delta^3(\mathbf{p}' - \mathbf{p}) a(\mathbf{p}') e^{i(p'-p_0)t} - e^{-i(p'_0+p_0)t} \delta^3(\mathbf{p}' - \mathbf{p}) a^\dagger(\mathbf{p}')]. \quad (2.166)$$

Performing the final integration now gives

$$i \sqrt{\frac{p_0}{2}} [a(\mathbf{p}) - e^{-2ip_0 t} a^\dagger(-\mathbf{p})]. \quad (2.167)$$

We now observe Eqn. (2.164) and Eqn. (2.167) and notice that

$$i\sqrt{2p_0}a(\mathbf{p}) = ip_0 \int d^3\mathbf{x} e^{-ip\cdot x} \phi(x) + \int d^3\mathbf{x} e^{-ip\cdot x} p(x). \quad (2.168)$$

Rearranging this gives

$$a(\mathbf{p}) = \frac{-i}{\sqrt{2p_0}} \int d^3\mathbf{x} e^{-ip\cdot x} [ip_0 q(x) + p(x)]. \quad (2.169)$$

A similar calculation evaluating $\int d^3\mathbf{x} e^{ip\cdot x} q(x)$ and $\int d^3\mathbf{x} e^{ip\cdot x} p(x)$ yields

$$a^\dagger(\mathbf{p}) = \frac{i}{2p_0} \int d^3\mathbf{x} e^{ip\cdot x} [-ip_0 q(x) + p(x)]. \quad (2.170)$$

Substituting the expressions for the creation and annihilation operators into the free particle Hamiltonian yields

$$H_0 = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{p_0}{2p_0} \int d^3\mathbf{x} \int d^3\mathbf{x}' e^{ip\cdot(x-x')} [-ip_0 q(x) + p(x)][ip_0 q(x') + p(x')]. \quad (2.171)$$

Performing the integration over \mathbf{p} yields

$$\frac{1}{2} \int d^3\mathbf{x} \int d^3\mathbf{x}' \delta^3(\mathbf{x} - \mathbf{x}') [-ip_0 q(x) + p(x)][ip_0 q(x') + p(x')]. \quad (2.172)$$

Performing the integration over \mathbf{x}' yields

$$\frac{1}{2} \int d^3\mathbf{x} [-ip_0 q(x) + p(x)][ip_0 q(x) + p(x)], \quad (2.173)$$

which, upon expanding the brackets gives

$$\frac{1}{2} \int d^3\mathbf{x} [p_0^2 q^2 + p^2 + ip_0(pq - qp)] \quad (2.174)$$

which reduces down to

$$\frac{1}{2} \int d^3\mathbf{x} [p_0^2 q^2 + p^2 - p_0 \delta^3(\mathbf{0})]. \quad (2.175)$$

where we have taken advantage of the fact that, for a local bosonic quantum field, $[q(\mathbf{x}, t), p(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y})$.

The last term only affects the zero of energy and only has physical significance in the presence of gravity [42, p.297] so in the absence of gravity this term can be ignored, so the free particle Hamiltonian becomes

$$H_0 = \frac{1}{2} \int d^3\mathbf{x} [p_0^2 q^2 + p^2]. \quad (2.176)$$

The Lagrangian is then

$$\begin{aligned} L &= \int d^3\mathbf{x} [p\dot{q} - \frac{1}{2}p_0^2 q^2 - \frac{1}{2}p^2] \\ &= \int d^3\mathbf{x} [p^2 - \frac{1}{2}p_0^2 q^2 - \frac{1}{2}p^2] \\ &= \int d^3\mathbf{x} [\frac{1}{2}p^2 - \frac{1}{2}p_0^2 q^2] \end{aligned} \quad (2.177)$$

2 Quantum Field Theory Review

but $p_0^2 = m^2 + \mathbf{p}^2$ so the Lagrangian then becomes

$$L = \int d^3\mathbf{x} [\frac{1}{2}p^2 - \frac{1}{2}\mathbf{p}^2 q^2 - \frac{1}{2}m^2 q^2]. \quad (2.178)$$

We now wish to write the Lagrangian in terms of the original field $\phi(x)$. Before doing that, it would be good to re-write the Lagrangian as

$$L = \int d^3\mathbf{x} [\frac{1}{2}p^2 - \frac{1}{2}(\nabla q)^2 - \frac{1}{2}m^2 q^2]. \quad (2.179)$$

At first sight it may not be at all obvious that this can be done. To see this, we observe that

$$(\nabla q)^2 = [-i\mathbf{p}(q^+ - q^-)]^2 \quad (2.180)$$

(where $q^+ = \phi^+$ and $q^- = \phi^-$) which, upon multiplying out the brackets gives

$$\mathbf{p}^2[q^{+2} + q^{-2} - q^+q^- - q^-q^+] \quad (2.181)$$

which is

$$\mathbf{p}^2[q^{+2} + q^{-2} + q^+q^- + q^-q^+] - 2\mathbf{p}^2(q^+q^- + q^-q^+) \quad (2.182)$$

which becomes

$$\mathbf{p}^2 q^2 - 2\mathbf{p}^2(q^+q^- + q^-q^+). \quad (2.183)$$

Inserting the mode expansions into the last term (and integrating over all space, since the term in the Lagrangian has this integral over all space with it) gives

$$\int d^3\mathbf{x} \int \int \frac{d^3\mathbf{p}' d^3\mathbf{p}}{(2\pi)^6 2p_0} e^{i(p' - p) \cdot x} \{a(\mathbf{p}'), a^\dagger(\mathbf{p})\}. \quad (2.184)$$

Evaluating this integral over all space gives

$$\int \int \frac{d^3\mathbf{p}' d^3\mathbf{p}}{(2\pi)^3 2p_0} \delta^3(\mathbf{p}' - \mathbf{p}) \{a(\mathbf{p}'), a^\dagger(\mathbf{p})\}, \quad (2.185)$$

which, upon performing the integral over \mathbf{p}' yields

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3 2p_0} \delta^3(\mathbf{0}). \quad (2.186)$$

Once again, this only affects the zero point of energy and is physically irrelevant in the absence of gravity, so assuming gravity to be absent, or at the very least ignorable, yields the Lagrangian

$$L = \int d^3\mathbf{x} [\frac{1}{2}p^2 - \frac{1}{2}(\nabla q)^2 - \frac{1}{2}m^2 q^2]. \quad (2.187)$$

This can be written in terms of ϕ as

$$L = \int d^3\mathbf{x} [\frac{1}{2}\partial_0\phi\partial^0\phi + \frac{1}{2}\partial_k\phi\partial^k\phi - \frac{1}{2}m^2\phi^2] \quad (2.188)$$

which becomes

$$L = \int d^3\mathbf{x} [\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2]. \quad (2.189)$$

2.16 Brief Overview of Some Important Parts of the Electroweak Theory

Thus the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \quad (2.190)$$

Before using Lagrange's equations, we re-write the Lagrangian density as

$$\mathcal{L} = \frac{1}{2}g_{\nu\alpha}\partial^\nu\phi\partial^\alpha\phi - \frac{1}{2}m^2\phi^2. \quad (2.191)$$

We now see that

$$-\frac{\partial\mathcal{L}}{\partial\phi} = m^2\phi \quad (2.192)$$

and that

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}g_{\nu\alpha} \left[\left(\frac{\partial}{\partial(\partial_\mu\phi)} \partial^\nu\phi \right) \partial^\alpha\phi + \partial^\nu\phi \left(\frac{\partial}{\partial(\partial_\mu\phi)} \partial^\alpha\phi \right) \right] \quad (2.193)$$

which becomes

$$\frac{1}{2}g_{\mu\alpha}\delta_\mu^\nu\partial^\alpha\phi + \frac{1}{2}g_{\nu\alpha}\delta_\mu^\alpha\partial^\nu\phi \quad (2.194)$$

which simplifies to

$$\frac{1}{2}[g_{\mu\alpha}\partial^\alpha\phi + g_{\mu\nu}\partial^\nu\phi] = \frac{1}{2}[\partial_\mu\phi + \partial_\mu\phi] = \partial_\mu\phi. \quad (2.195)$$

Thus

$$\partial^\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) = \partial^\mu\partial_\mu\phi = \partial_\mu\partial^\mu\phi \quad (2.196)$$

so the equations of motion are immediately seen to be

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = 0. \quad (2.197)$$

We now briefly introduce some of the fundamental concepts of the electroweak sector of the Standard Model, relevant to the contents of Chapter 6.

2.16 Brief Overview of Some Important Parts of the Electroweak Theory

In this section we introduce some of the main components of the Electroweak Theory that are relevant to be conscious of when thinking of mass dimension one fermion fields in the context of electroweak interactions in Chapter 6. In this section we mention such concepts as multiplets and symmetry currents. These concepts will be introduced properly in Chapter 3, when concentrating on general non-abelian gauge symmetries of the Elko Lagrangian.

In 1956, Lee and Yang predicted that parity is violated in weak interactions [62]. In 1957, Wu observed that in β decay, there was a dependence of the angular distribution of decaying electrons on the polarization of the decaying nucleus [63]. The observed decay angular distribution contained both scalar and pseudoscalar quantities. The current-current interactions describing weak processes were modified to incorporate the resulting parity violation by including axial vector terms like $\bar{u}\gamma^\mu\gamma_5 u$ to the currents which already had polar vector

terms like $\bar{u}\gamma^\mu u$. Hence, weak currents took on the well known ‘V-A’ structure. This ‘V-A’ structure required the conclusion that the weak field quanta must be *vector* particles. The short range of the weak force led to the conclusion that these vector particles must have mass.

That these massive vector particles should be part of a gauge theory, was corroborated by experiments showing that the weak charge g proved to be a universal coupling strength associated in all weak interactions involving both leptons and quarks. Another argument in favor of incorporating those vector particles into a gauge theory was the demand of renormalizability. Gauge theories are in general renormalizable. The non-zero mass of the vector particles seemed to imply that they could not constitute the gauge field quanta in a gauge theory, by virtue of breaking explicit gauge invariance, but this problem was overcome by incorporating the Higgs mechanism [64][65][66] which has the concept of a *vacuum screening current* [67, ch.13] which interacts with the weak gauge field quanta in such a way that the weak gauge field quanta behave as if they have mass. The vacuum screening currents are said to generate the mass of the weak gauge quanta. The field that the weak particles acquire their mass by interacting with, is the Higgs field. The quantum of the Higgs field is the Higgs boson. In 1971, ‘t Hooft [68] showed that these sorts of theories where massive vector particles acquire mass through vacuum screening currents, are renormalizable.

Another feature related to weak gauge interactions is that there are weak gauge quanta, the W^\pm particles, which have electromagnetic interactions. Thus, this whole vacuum screening process had to be understood in order to properly understand relevant electromagnetic phenomena too. The Standard Model treats electromagnetic currents as being closely connected to the neutral weak currents corresponding to the neutral massive vector Z^0 particle.

Another important feature of the Electroweak Theory is that only left-handed components of weak quark currents participate in weak interactions, and similarly with weak interactions for the generational lepton pairs (ν_e, e^-) , (ν_μ, μ^-) , (ν_τ, τ^-) . The gauge group $SU(2)_L \times U(1)$ describing this Electroweak Theory was first proposed by Glashow (1961). Weinberg and Salam treated the symmetry group as a hidden one (hidden in the sense of spontaneous symmetry breaking [69, p.290][20, p.193][21, p.163]). The resulting Electroweak Theory is in agreement with all known electroweak phenomena.

2.16.1 Standard Model Doublets

The (hidden, or spontaneously broken) $SU(2)_L \times U(1)$ *local* gauge transformations mediate interactions between members of an associated symmetry multiplet. General multiplets as well as the origin of symmetry currents, will be more properly introduced and described in some detail in Chapter 3. The multiplets of the Electroweak Theory are doublets respecting the *weak isospin* symmetry [69, sec.8.5], so the $t = \frac{1}{2}$ representation of $SU(2)_L$ consisting of 2×2 matrices $\frac{1}{2}\tau_i$ is used, where the τ_i are numerically identical to the Pauli matrices σ_i .

2.16 Brief Overview of Some Important Parts of the Electroweak Theory

The doublets have as their elements, Dirac fields. The quark doublets are of the form

$$q_L(u, d_c) = \begin{pmatrix} u \\ d_c \end{pmatrix}_L, \quad q_L(c, s_c) = \begin{pmatrix} c \\ s_c \end{pmatrix}_L, \quad q_L(b, t_c) = \begin{pmatrix} t \\ b_c \end{pmatrix}_L. \quad (2.198)$$

where d_c , s_c and b_c are related to the Dirac quark matter fields d , s and b by

$$\begin{pmatrix} d_c \\ s_c \\ b_c \end{pmatrix} = U_{\text{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \quad (2.199)$$

U_{CKM} is the Cabibbo-Kobayashi-Maskawa mixing matrix. The unitary Cabibbo-Kobayashi-Maskawa mixing matrix U_{CKM} contains three mixing angles and one CP -violating phase. The numerical values for these angles can be found in, for example, [70]. This flavor mixing was introduced into the Electroweak Theory because (u, c, t) quarks can be changed into any (d, s, b) quarks and vice versa by the absorbing or emitting of a W boson. Charge conservation holds because this is consistent with the charges of the top quark entries in the SM doublets differing from the charges of the bottom quark entries in the SM doublets by one unit of charge. The (u, c, t) quarks all have electric charge $\frac{2}{3}$ whereas the (d, s, b) quarks all have electric charge $-\frac{1}{3}$. The square of each entry in the matrix U_{CKM} gives the probabilities for various weakly induced quark flavor transformations to occur. The probabilities show that each quark has a strong tendency to change into the flavor that is contained in the same doublet. Quark flavor changing interactions are only known to be brought about by electroweak interactions.

The lepton Standard Model doublets are grouped together as follows:

$$l_e = \begin{pmatrix} \nu_{e,m} \\ e^- \end{pmatrix}_L, \quad l_\mu = \begin{pmatrix} \nu_{\mu,m} \\ \mu^- \end{pmatrix}_L, \quad l_\tau = \begin{pmatrix} \nu_{\tau,m} \\ \tau^- \end{pmatrix}_L \quad (2.200)$$

where

$$\begin{pmatrix} \nu_{e,m} \\ \nu_{\mu,m} \\ \nu_{\tau,m} \end{pmatrix} = U_{\text{MNS}} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \quad (2.201)$$

with U_{MNS} being the Pontecorvo-Maki-Nakagawa-Sakatu (MNS) matrix [71], the leptonic analogue of the CKM matrix U_{CKM} introduced into the Standard Model because of the observation that neutrinos undergo flavor oscillations over macroscopic distances.

The pairs of fields in these quark and lepton doublets are not mass degenerate but this is fine because the symmetry is hidden.

Because the weak force only interacts through left-handed fermion fields, the left-handed fermion fields which have non-zero isospin, are grouped in doublets whereas the right-handed fermion fields form singlets. As a result, there are two covariant derivatives. The left-handed covariant derivative, D_L^μ , for spin- $\frac{1}{2}$ particles is

$$D_L^\mu = \partial^\mu + ig \frac{1}{2} \boldsymbol{\tau} \cdot \mathbf{W}^\mu - ig' \frac{1}{2} B^\mu, \quad (2.202)$$

while the right-handed covariant derivative, D_R^μ , is

$$D_R^\mu = \partial^\mu - ig'B^\mu. \quad (2.203)$$

D_L^μ acts on left-handed leptons which have isospin- $\frac{1}{2}$ and hypercharge $y = -1$ corresponding to the new global $U(1)$ symmetry. The covariant derivative D_R^μ acts on singlet right-handed leptons that have an isospin of zero, and a hypercharge value of $y = -2$. There are two fundamental constants above, g and g' , called the weak charges. The weak charge g is from the $SU(2)_L$ part of the gauge group while g' is the weak charge corresponding to the $U(1)$ part of the gauge group. We here speak of hypercharge rather than charge because the field B^μ is not the Standard Model photon field A^μ , but rather related to it in a way to be explained soon. First however, we introduce the Higgs field ϕ , and then explain how to get the vacuum screening currents responsible for giving mass to the otherwise massless W^\pm and Z^0 gauge bosons.

2.16.2 The Higgs field, Vacuum Screening Currents, and W^\pm , Z^0 and A^μ Gauge Fields

The screening currents which provide terms that give mass to the gauge quanta of the weak force, come about by having an absolute vacuum away from where the vacuum expectation value of the scalar field ϕ is zero. The Higgs field has a non-zero vacuum expectation value at any of an infinite number of vacua that are all related to each other by a $U(1)$ transformation. The picking of a *particular* vacuum breaks the symmetry. We will not try to motivate the form of the Higgs doublet used here, but instead refer the reader to the literature, and carry on highlighting some core parts of the Electroweak Theory.

The Higgs field doublet

$$\phi(x) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}[f + \rho(x)] \end{pmatrix}, \quad (2.204)$$

with f a constant and $\rho(x)$ a scalar field with vanishing vacuum expectation value, is a spinless scalar field with a Lagrangian of the form

$$\mathcal{L}(\phi) = (D_\mu \phi)^\dagger (D^\mu \phi) + \frac{\mu^2}{2} \phi^\dagger \phi - \frac{1}{2} \frac{\mu^2}{f^2} (\phi^\dagger \phi)^2 + \text{interaction terms} \quad (2.205)$$

where the covariant derivatives are the ones such that $\mathcal{L}(\phi)$ is invariant under the hidden gauge group $SU(2)_L \times U(1)$. The *interaction terms* mentioned here are referring to the Yukawa couplings between the Higgs field and the otherwise massless fermionic fields [69, p.310][67, p.465]. The $\frac{f}{\sqrt{2}}$ is the value of the non-zero vacuum expectation value $\langle 0 | \phi | 0 \rangle$ of the Higgs field.

To find the vacuum screening currents, we consider the terms in this Lagrangian that come from the covariant derivatives and we ignore all of the interaction terms that arise from the

2.16 Brief Overview of Some Important Parts of the Electroweak Theory

ρ part of the Higgs field and just consider that part of the Higgs field concerning the pure vacuum with vacuum value $\frac{f}{\sqrt{2}}$. That is, we consider:

$$j^{a\mu}W_{a\mu} + j^\mu B_\mu \quad (2.206)$$

where

$$j^{a\mu} = \frac{ig}{2} \left[\phi^\dagger \tau^a (\partial^\mu \phi) - (\partial^\mu \phi)^\dagger \tau^a \phi \right] - \frac{g^2}{2} \phi^\dagger \phi W^{a\mu} - \frac{gg'}{2} \phi^\dagger \tau^a \phi B^\mu \quad (2.207)$$

and

$$j^\mu = \frac{ig'}{2} \left[\phi^\dagger (\partial^\mu \phi) - (\partial^\mu \phi)^\dagger \phi \right] - \frac{gg'}{2} \phi^\dagger \tau \phi \cdot \mathbf{W}^\mu - \frac{1}{2} g'^2 \phi^\dagger \phi B^\mu. \quad (2.208)$$

Given that we are ignoring the terms involving $\rho(x)$ in the Higgs doublet, it follows that for $a = 1, 2$:

$$j^{1\mu} = \frac{-g^2 f^2}{4} W^{1\mu} = -M_W^2 W^{1\mu} \quad \text{and} \quad j^{2\mu} = \frac{-g^2 f^2}{4} W^{2\mu} = -M_W^2 W^{2\mu} \quad (2.209)$$

where

$$M_W \equiv \frac{gf}{2}. \quad (2.210)$$

With $M' \equiv \frac{g'f}{2}$, the $a = 3$ current $j^{3\mu}$ and the current j^μ take the form:

$$j^{3\mu} = -M_W^2 W^{3\mu} + MM' B^\mu \quad \text{and} \quad j^\mu = M_W M' W^{3\mu} - M'^2 B^\mu. \quad (2.211)$$

These currents mix up the $W^{3\mu}$ and B^μ fields, each of which have indefinite mass leading to the conclusion that these are not the physical fields. If we define

$$g = \sqrt{g^2 + g'^2} \cos(\theta_W), \quad g' = \sqrt{g^2 + g'^2} \sin(\theta_W) \quad (2.212)$$

where θ_W is the Weinberg angle [69, p.312], and also define two fields Z^μ and A^μ by the linear combinations

$$W^{3\mu} = \cos(\theta_W) Z^\mu + \sin(\theta_W) A^\mu \quad \text{and} \quad B^\mu = -\sin(\theta_W) Z^\mu + \cos(\theta_W) A^\mu, \quad (2.213)$$

then a direct calculation reveals that $j^{3\mu} W_{3\mu} + j^\mu B_\mu$ becomes

$$-M_Z^2 Z^\mu Z_\mu \quad (2.214)$$

where $M_Z \equiv \frac{f}{2} \sqrt{g^2 + g'^2} = \frac{M_W}{\cos(\theta_W)}$. Thus, ignoring the self interaction terms arising from the $\rho(x)$ part of the Higgs field that would create a self interaction current $j^\mu(W)$ on the right hand side, the dynamical equations of motion for the gauge fields $W^{1\mu}, W^{2\mu}, Z^\mu$ and A^μ are

$$(\partial_\nu \partial^\nu + M_W^2) W^{1\mu} - \partial^\mu \partial_\nu W^{1\nu} = 0, \quad (\partial_\nu \partial^\nu + M_W^2) W^{2\mu} - \partial^\mu \partial_\nu W^{2\nu} = 0 \quad (2.215)$$

and

$$(\partial_\nu \partial^\nu + M_Z^2) Z^\mu - \partial^\mu \partial_\nu Z^\nu = 0, \quad \partial_\nu \partial^\nu A^\mu - \partial^\mu \partial_\nu A^\nu = 0. \quad (2.216)$$

2 Quantum Field Theory Review

By rewriting the left-handed covariant derivative D_L^μ in terms of the fields Z^μ and A^μ instead of $W^{3\mu}$ and B^μ , we can identify A^μ with the photon field by the relation

$$e = g \sin(\theta_W) \tag{2.217}$$

where e is the unit of electromagnetic charge. In Chapter 6 we consider the concept of transmuting the mass dimensionality of the Dirac-type spin-1/2 fermion fields in the Electroweak Theory from mass dimensionality three halves to mass dimensionality one, and relate their left and right-handed components to the left-handed Elko field components.

In the next chapter, we introduce the Elko field and then as closely and naturally as possible, mimic the usual procedures that people go through, once they have a quantum field to work with, and see what happens. In Chapter 3, we ignore the question of whether Elko is a quantum field in the sense of Weinberg. In Chapter 4, we wish to understand how the Elko field fits into the Quantum Field Theory formalism with the state space as defined here in Chapter 2. The question of where quantum fields come from and the question of what quantum fields should look like, is answered nicely in Weinberg's way of looking at things which we have presented here in Chapter 2. We use these insights in Weinberg's approach to try to understand how the Elko field could most naturally fit into the state space setting.

Finally we note that there are limitations to Weinberg's theory. For instance he does not address fields which are topologically twisted. He does however refer to them [42, p.119].

3 Elko Fields and Interactions

3.1 Introduction

In this chapter we first give a review of the Elko field, presenting many of the key results in [34]. For the remainder of this chapter, we then take the Elko field as given and apply the gauge principle to the Elko field and mimic the standard canonical formalism as closely as we can, to see what happens. Importantly, in Sec. (3.3.2) we discuss how our results concerning Elko's ability to interact with Standard Model particles interact with ideas and results in the Elko Field Theory literature. There we explain where our results agree and disagree with existing material. In the cases where our results disagree, we also explain why. No insights gleaned from Weinberg's approach are taken advantage of in this chapter. We find Weinberg's approach particularly useful when answering the question of "what can a quantum field look like?" In this chapter however, we simply start out with the Elko field and address some of the natural questions regarding "what can we do with it?" In particular, we look at some questions regarding the Elko field's $U(1)$ interactions. Our operating principle throughout this chapter is to try to fit in with the usual canonical formalism in the most obvious and natural way. We say this, because there are aspects of the Elko dual (the details of which we explain as they become relevant) which put certain quantities, like the vacuum expectation value of the time-ordered product of the Elko field with its dual for example, outside the usual mathematical apparatus of the standard canonical formalism.

We leave it to the following chapter to attempt to reconcile the Elko quantum field with Weinberg's formalism for defining quantum fields, so it is in the next chapter that we make use of some of Weinberg's key insights relevant to the question "what can a quantum field look like?"

3.2 Elko Field Review

In this Elko field review, we present the Elko field and the anti-commutation relations for the creation and annihilation operators. We then give the key properties of the Elko spinors. We then present the Elko spinor definitions, and define the Elko spinor dual and then give the Elko spinor orthonormality and completeness relations. After this, we present the Elko dual quantum field, followed by the vacuum expectation value of the time ordered product of the Elko field with its dual. We then present the Elko free particle Lagrangian along with the

3 Elko Fields and Interactions

Elko canonically conjugate field momenta. We then present the canonical anti-commutation relations between the Elko field and its canonically conjugate field momenta, and finish the section with a review of Elko interactions.

We start the review here, by introducing the Elko field presented as a local dark matter candidate in [34], and discuss its darkness with respect to the Standard Model.

The Elko field we will be using in this chapter is*

$$\Lambda(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2mE(\mathbf{p})}} \sum_{\sigma} [e^{ip \cdot x} \xi(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{-ip \cdot x} \zeta(\mathbf{p}, \sigma) b^{\dagger}(\mathbf{p}, \sigma)] \quad (3.1)$$

where here σ is a discrete two-valued index and $\xi(\mathbf{p}, \sigma)$ and $\zeta(\mathbf{p}, \sigma)$ are spinors defined by certain properties to be shortly discussed, $a(\mathbf{p}, \sigma)$ is the annihilation operator which destroys an Elko particle of three-momentum \mathbf{p} with a discrete index value σ , and $b^{\dagger}(\mathbf{p}, \sigma)$ creates an anti-Elko particle with three-momentum \mathbf{p} and discrete index value σ .

In [34, p.2] another Elko field is also presented:

$$\lambda(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2mE(\mathbf{p})}} \sum_{\sigma} [e^{ip \cdot x} \xi(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{-ip \cdot x} \zeta(\mathbf{p}, \sigma) a^{\dagger}(\mathbf{p}, \sigma)]. \quad (3.2)$$

We work with $\Lambda(x)$ rather than $\lambda(x)$ because $\Lambda(x)$ has antiparticles which are distinct from the Elko particles, whereas for $\lambda(x)$, there is no distinction between particle and antiparticle and we do not wish to complicate things with Majorana type conditions. In order to stay as close as possible to the usual canonical formalism, in order to be able to take advantage of the gauge principle, we want distinct particles and antiparticles. For more on this, see Sec. (2.15.1) and Sec. (5.7)

In [34] the label “ α ” is used to distinguish it from the usual “ σ ” which labels the eigenvalue of the angular momentum operator in the z -direction. The point of view taken in [34] is that in the low energy limit, dark matter must be described by irreducible representations of the full Poincaré group.

We take the view here that since the question of what the underlying state space can look like is a purely mathematical one, once given an underlying symmetry group, and since Wigner has shown that the only solution for what the Hilbert space can look like, is the standard Wigner class where the two-valued discrete index is σ , we will here use the label “ σ ” instead of “ α .”

The creation and annihilation operators were assumed in [34] to satisfy

$$\{a(\mathbf{p}, \sigma), a^{\dagger}(\mathbf{p}', \sigma')\} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \delta_{\sigma\sigma'} \quad (3.3)$$

$$\{a(\mathbf{p}, \sigma), a(\mathbf{p}', \sigma')\} = 0, \quad \{a^{\dagger}(\mathbf{p}, \sigma), a^{\dagger}(\mathbf{p}', \sigma')\} = 0 \quad (3.4)$$

$$\{b(\mathbf{p}, \sigma), b^{\dagger}(\mathbf{p}', \sigma')\} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \delta_{\sigma\sigma'} \quad (3.5)$$

$$\{b(\mathbf{p}, \sigma), b(\mathbf{p}', \sigma')\} = 0, \quad \{b^{\dagger}(\mathbf{p}, \sigma), b^{\dagger}(\mathbf{p}', \sigma')\} = 0. \quad (3.6)$$

*Here the signs in the exponents are opposite to the convention adopted by [34, p.4] to be consistent with the convention throughout this thesis.

3.2.1 Elko Spinors

In [34], four component spinors were introduced of the form

$$\chi(\mathbf{p}) = \begin{pmatrix} \eta\Theta\phi^*(\mathbf{p}) \\ \phi(\mathbf{p}) \end{pmatrix}. \quad (3.7)$$

Here, η is a phase, Θ is the Wigner 2×2 matrix time reversal operator defined by the property

$$\Theta \left[\frac{\boldsymbol{\sigma}}{2} \right] \Theta^{-1} = - \left[\frac{\boldsymbol{\sigma}}{2} \right]^*, \quad (3.8)$$

and numerically equal to the matrix $-i\sigma_2$ for the spin-1/2 case, and $\phi(\mathbf{p})$ is a spin-1/2 left-handed Weyl spinor. Weyl spinors are eigenspinors of the helicity operator $\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}$ so that

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}\phi_{\pm}(\mathbf{p}) = \pm\phi_{\pm}(\mathbf{p}), \quad (3.9)$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and the σ_i are the *Pauli matrices*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.10)$$

The top spinor, $\eta\Theta\phi^*(\mathbf{p})$, in $\chi(\mathbf{p})$ transforms as a right-handed Weyl spinor with

$$\eta\Theta\phi^*(\mathbf{p}) = \kappa^{(\frac{1}{2},0)}\eta\Theta\phi^*(\mathbf{0}) \quad (3.11)$$

where $\kappa^{(\frac{1}{2},0)}$ is the right-handed finite-dimensional boost operator

$$\frac{1}{\sqrt{2m(m+p_0)}}[(p_0+m)I_2 + \boldsymbol{\sigma} \cdot \mathbf{p}], \quad (3.12)$$

while for the left-handed Weyl spinor $\phi(\mathbf{p})$ we have

$$\phi(\mathbf{p}) = \kappa^{(0,\frac{1}{2})}\phi(\mathbf{0}), \quad (3.13)$$

where $\kappa^{(0,\frac{1}{2})}$ is the left-handed finite-dimensional boost operator

$$\frac{1}{\sqrt{2m(m+p_0)}}[(p_0+m)I_2 - \boldsymbol{\sigma} \cdot \mathbf{p}]. \quad (3.14)$$

I_2 is the 2×2 identity matrix. In the above,

$$\hat{\mathbf{p}} = (\sin(\theta)\cos(\Phi), \sin(\theta)\sin(\Phi), \cos(\theta)) \quad (3.15)$$

and

$$\phi_+(\mathbf{0}) = \sqrt{m} \begin{pmatrix} \cos(\frac{\theta}{2})e^{\frac{-i\Phi}{2}} \\ \sin(\frac{\theta}{2})e^{\frac{i\Phi}{2}} \end{pmatrix}, \quad (3.16)$$

$$\phi_-(\mathbf{0}) = \sqrt{m} \begin{pmatrix} -\sin(\frac{\theta}{2})e^{\frac{-i\Phi}{2}} \\ \cos(\frac{\theta}{2})e^{\frac{i\Phi}{2}} \end{pmatrix}. \quad (3.17)$$

3 Elko Fields and Interactions

Before continuing with the Elko spinor review, we first make a few comments about where the above boost operators $\kappa^{(\frac{1}{2},0)}$ and $\kappa^{(0,\frac{1}{2})}$ come from. The boost operators $\kappa^{(\frac{1}{2},0)}$ and $\kappa^{(0,\frac{1}{2})}$ come about by considering the Lorentz algebra of three angular momentum generators J_i , and three boost generators K_i , which is:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}K_k. \quad (3.18)$$

A complexified Lie algebra is formed from these generators (which forms a real Lie algebra, even though the conventional presence of the “ i ” obscures this) by defining two new generators \mathbf{A} and \mathbf{B} , such that:

$$\mathbf{A} = \mathbf{J} + i\mathbf{K}, \quad \mathbf{B} = \mathbf{J} - i\mathbf{K}. \quad (3.19)$$

The complexified Lie algebra with these generators is given by

$$[A_i, A_j] = iA_k, \quad [B_i, B_j] = iB_k, \quad [A_i, B_j] = 0. \quad (3.20)$$

The \mathbf{A} ’s and \mathbf{B} ’s are each usually thought of as generating a group $SU(2)$ which commute with each other [7, p.38]. As a result of this, spinors are given two angular momentum labels (j, j') to specify how they transform under the (complexified) Lorentz group. The usual convention for spin-1/2 spinors is to specify spinors that transform according to the $(\frac{1}{2}, 0)$ representation, and also specify spinors that transform according to the $(0, \frac{1}{2})$ representation. The matrix representation for $\mathbf{J}^{(\frac{1}{2})}$ is given by $\frac{1}{2}\boldsymbol{\sigma}$ for both representations. The $\mathbf{K}^{(\frac{1}{2})}$ takes the form $\frac{-i\boldsymbol{\sigma}}{2}$ for the $(\frac{1}{2}, 0)$ representation, and $\frac{i\boldsymbol{\sigma}}{2}$ for the $(0, \frac{1}{2})$ representation. The finite boost operators $\kappa^{(\frac{1}{2},0)}$ and $\kappa^{(0,\frac{1}{2})}$ then become $e^{\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\phi}}$ and $e^{\frac{-\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\phi}}$ respectively. It can be shown (see [7, p.41]) that these boost operators can be re-written in the form as presented in Eqn. (3.12) and Eqn. (3.14).

Another ingredient we need in order to define the Elko spinors presented in [34], is the matrix charge conjugation operator $S(C)$ belonging to the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representations, which, in the chiral representation [72, p.41], is given by

$$S(C) = \begin{pmatrix} 0_2 & i\Theta \\ -i\Theta & 0_2 \end{pmatrix} K \quad (3.21)$$

where K complex conjugates everything to the right of it. Finally, if we set $\eta = \pm i$, the spinors $\chi(\mathbf{p})$ become eigenspinors of the charge conjugation operator with eigenvalues ± 1 so that

$$S(C)\chi(\mathbf{p}) = \pm\chi(\mathbf{p}). \quad (3.22)$$

The Elko rest spinors were defined to be[†]

$$\xi\left(\mathbf{0}, \frac{1}{2}\right) \equiv \chi(\mathbf{0}) \Big|_{\phi(\mathbf{0}) \rightarrow \phi_+(\mathbf{0}), \eta = +i} \quad (3.23)$$

[†]In [34], the values of the discrete index were labeled by $\{-, +\}$ and $\{+, -\}$ referring to the dual helicity structure of the Elko spinors. Since we are here choosing to call the discrete index σ , in view of what the Hilbert space must look like under the full Poincaré group, we represent the labels $\{-, +\}$ and $\{+, -\}$ as $\frac{1}{2}$ and $\frac{-1}{2}$ respectively.

$$\xi\left(\mathbf{0}, \frac{-1}{2}\right) \equiv \chi(\mathbf{0}) \Big|_{\phi(\mathbf{0}) \rightarrow \phi_-(\mathbf{0}), \eta=+i} \quad (3.24)$$

$$\zeta\left(\mathbf{0}, \frac{1}{2}\right) \equiv \chi(\mathbf{0}) \Big|_{\phi(\mathbf{0}) \rightarrow \phi_-(\mathbf{0}), \eta=-i} \quad (3.25)$$

$$\zeta\left(\mathbf{0}, \frac{-1}{2}\right) \equiv -\chi(\mathbf{0}) \Big|_{\phi(\mathbf{0}) \rightarrow \phi_+(\mathbf{0}), \eta=-i}. \quad (3.26)$$

The boosted spinors $\xi(\mathbf{p}, \frac{1}{2})$, $\xi(\mathbf{p}, \frac{-1}{2})$, $\zeta(\mathbf{p}, \frac{1}{2})$ and $\zeta(\mathbf{p}, \frac{-1}{2})$ are all obtained by multiplying on the left by the matrix $\kappa^{(\frac{1}{2}, 0)} \oplus \kappa^{(0, \frac{1}{2})}$. To define a basis for the Elko spinor dual space, the dual spinors $\bar{\chi}(\mathbf{p}, \sigma)$ are defined as

$$\bar{\chi}(\mathbf{p}, \pm\sigma) \equiv \mp i \chi^\dagger(\mathbf{p}, \mp\sigma) \gamma^0. \quad (3.27)$$

The orthonormality relations are then

$$\bar{\xi}(\mathbf{p}, \sigma) \xi(\mathbf{p}, \sigma') = 2m \delta_{\sigma\sigma'}, \quad (3.28)$$

$$\bar{\zeta}(\mathbf{p}, \sigma) \zeta(\mathbf{p}, \sigma') = -2m \delta_{\sigma\sigma'}, \quad (3.29)$$

$$\bar{\xi}(\mathbf{p}, \sigma) \zeta(\mathbf{p}, \sigma') = 0, \quad (3.30)$$

$$\bar{\zeta}(\mathbf{p}, \sigma) \xi(\mathbf{p}, \sigma') = 0, \quad (3.31)$$

and the completeness relation is given by

$$\frac{1}{2m} \sum_{\sigma} [\xi(\mathbf{p}, \sigma) \bar{\xi}(\mathbf{p}, \sigma) - \zeta(\mathbf{p}, \sigma) \bar{\zeta}(\mathbf{p}, \sigma)] = I_4. \quad (3.32)$$

The Elko spin sums $\sum_{\sigma} \xi(\mathbf{p}, \sigma) \bar{\xi}(\mathbf{p}, \sigma)$ and $\sum_{\sigma} \zeta(\mathbf{p}, \sigma) \bar{\zeta}(\mathbf{p}, \sigma)$ are respectively given by [49, p.3]:

$$\sum_{\sigma} \xi(\mathbf{p}, \sigma) \bar{\xi}(\mathbf{p}, \sigma) = +m[1 + G(\mathbf{p})] \quad (3.33)$$

and

$$\sum_{\sigma} \zeta(\mathbf{p}, \sigma) \bar{\zeta}(\mathbf{p}, \sigma) = -m[1 - G(\mathbf{p})] \quad (3.34)$$

where $G(\mathbf{p})$ is a matrix of odd valued functions of \mathbf{p} so that

$$G(\mathbf{p}) = -G(-\mathbf{p}). \quad (3.35)$$

3.2.2 Dual Elko Field and the Time Ordered Product

A field analogous to the Dirac field $\bar{\psi}(x)$ was also defined in [34]. The Dirac field $\bar{\psi}$ is based on the Dirac dual of its spinors $\bar{u}(\mathbf{p}, \sigma) = u^\dagger(\mathbf{p}, \sigma) \gamma^0$, $\bar{v}(\mathbf{p}, \sigma) = v^\dagger(\mathbf{p}, \sigma) \gamma^0$, and so the Elko dual quantum field is also defined based on its dual spinors [34, p.4]:

$$\bar{\Lambda}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2mE(\mathbf{p})}} \sum_{\sigma} [e^{-ip \cdot x} \bar{\xi}(\mathbf{p}, \sigma) a^\dagger(\mathbf{p}, \sigma) + e^{ip \cdot x} \bar{\zeta}(\mathbf{p}, \sigma) b(\mathbf{p}, \sigma)]. \quad (3.36)$$

3 Elko Fields and Interactions

The Elko Lagrangian, being Klein-Gordon in form, is inferred from the quantity

$$\langle 0 | T \{ \Lambda(x') \bar{\Lambda}(x) \} | 0 \rangle \quad (3.37)$$

where T here is the fermionic time ordered product defined on two operators $A_1(x')$ and $A_2(x)$ by

$$T \{ A_1(x') A_2(x) \} = \theta(t' - t) A_1(x') A_2(x) - \theta(t - t') A_2(x) A_1(x') \quad (3.38)$$

with $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$. The quantity $\langle 0 | T \{ \Lambda(x') \bar{\Lambda}(x) \} | 0 \rangle$ is given in [34] to be

$$\langle 0 | T \{ \Lambda(x') \bar{\Lambda}(x) \} | 0 \rangle = \lim_{\epsilon \rightarrow 0^+} i \int \frac{d^4 p}{(2\pi)^4} e^{ip_\mu(x'^\mu - x^\mu)} \left[\frac{1 + G(\mathbf{p})}{p_\mu p^\mu - m^2 + i\epsilon} \right]. \quad (3.39)$$

It is then asserted in [34] that in the absence of a preferred direction[‡], the integral over $G(\mathbf{p})$ can be shown to vanish, in which case the quantity $\langle 0 | T \{ \Lambda(x') \bar{\Lambda}(x) \} | 0 \rangle$ becomes proportional to the Feynman-Dyson propagator

$$- \int \frac{d^4 p}{(2\pi)^4} e^{ip_\mu(x'^\mu - x^\mu)} \left[\frac{m^2 \mathbf{1}}{p_\mu p^\mu - m^2 + i\epsilon} \right], \quad (3.40)$$

where $\mathbf{1}$ is here the 4×4 identity matrix.

The propagator is interpreted physically, as the probability amplitude for the particle to propagate from spacetime point x to spacetime point x' . This result established the free particle Lagrangian for Elko fields to be Klein-Gordon in form [34, p.5]:

$$\mathcal{L}(x) = \partial_\mu \bar{\Lambda}(x) \partial^\mu \Lambda(x) - m^2 \bar{\Lambda}(x) \Lambda(x) \quad (3.41)$$

with the fields $\Lambda(x)$ and $\bar{\Lambda}(x)$ having mass dimension one since the Lagrangian density is of mass dimension four. In [34], the field momentum $\Pi(x)$ is given by

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\Lambda}} = \frac{\partial}{\partial t} \bar{\Lambda}(x) \quad (3.42)$$

and the canonical anti-commutators are given to be

$$\{ \Lambda(\mathbf{x}, t), \Pi(\mathbf{x}', t) \} = i \delta^3(\mathbf{x} - \mathbf{x}') \quad (3.43)$$

$$\{ \Lambda(\mathbf{x}, t), \Lambda(\mathbf{x}', t) \} = 0 \quad (3.44)$$

$$\{ \Pi(\mathbf{x}, t), \Pi(\mathbf{x}', t) \} = 0 \quad (3.45)$$

establishing the assertion in [34] that the Elko field $\Lambda(x)$ is local. We finish this section by remarking that Eqn. (3.43) only holds if there is no preferred axis. See Sec. (3.8).

[‡]In following sections, we will hold to this assumption but in the next chapter, we will not restrict ourselves to this assumption when we look at how the Elko field might be consistent with the Weinberg formalism.

3.2.3 Elko Interactions

In this section we review work that has been done on Elko quantum field interactions in order that later, in Sec. (3.3) we can clarify how our work relates to existing work on Elko field interactions, and make more explicit where we agree or disagree with various points and why. Elko quantum fields were introduced in [32] as a prime candidate for dark matter, having no Standard Model interactions other than with the Higgs particle. In this section we will also make clear the reasons given for considering Elko fields being put forward as dark matter candidates.

The interaction Lagrangian associated with Elko interactions with the Higgs particle was given in [32, p.44] for an Elko field $\eta(x)$ and a Higgs doublet $\phi(x)$ as:

$$\mathcal{L}_{\phi\eta}^{\text{int}}(x) = \lambda_E \phi^\dagger(x) \phi(x) \bar{\eta}(x) \eta(x) \quad (3.46)$$

with more interactions of this form possible if more scalar fields exist in nature. The symbol λ_E is a dimensionless coupling constant. An Elko quartic self interaction was also introduced [32, p.44] of the form

$$\mathcal{L}^{\text{self}}(x) = \alpha_E [\bar{\eta}(x) \eta(x)]^2 \quad (3.47)$$

with α_E another dimensionless coupling constant. In [32, p.44] Ahluwalia and Grumiller also state that the Elko field could couple to an abelian gauge field, via the field strength tensor $F_{\mu\nu}$ associated to the gauge field. Such an interaction has the form

$$\mathcal{L}_{\eta F}^{\text{int}} = \epsilon_E \bar{\eta}(x) [\gamma^\mu, \gamma^\nu] \eta(x) F_{\mu\nu}(x). \quad (3.48)$$

This interaction does not come from the gauge process. Rather, it comes from an attempt to write down a scalar constrained by considerations of mass dimensionality. Ahluwalia and Grumiller argued however, that the coupling constant ϵ_E would have to be vanishingly small because terms like these generate an effective mass for the photon but the possible mass of a photon has been experimentally severely constrained. Ahluwalia and Grumiller did not put the Elko field through the gauge process, but state [32, p.44] that “the Elko field is neutral with respect to local $U(1)$ gauge transformations.” They then conclude that the Elko-Higgs interaction of Eqn. (3.46) is the dominant interaction between Elko particles and the Standard Model particles.

In [32, p.61], Ahluwalia and Grumiller clarify that it is the mass dimensionality of the Elko field being one and not three halves that “forbids a large class of interactions with gauge and matter fields of the Standard Model while allowing for an interaction with the Higgs field.”

In [34], the Elko field $\Lambda_\ell(x)$ was introduced. This is the field we have introduced already in the present chapter. In [34, p.1] it was stated that Elko could not enter the fermionic doublets of the Standard Model due to the mismatch in mass dimensionality (this same idea was also strongly implied by [34]). On the same page, the position was stated that at least in the low energy limit, dark matter ‘must be described by the irreducible representations of

3 Elko Fields and Interactions

the full Poincaré group.” The Elko interactions given, are of the same general form as those given in [32].

A few pages later, [34, p.7] a stronger statement is made concerning Standard Model gauge transformations:

“ $\mathcal{L}^\Lambda(x)$ and $\mathcal{L}^\lambda(x)$ do not carry invariance under Standard Model gauge transformations.”

On the same page however, it was also stated that the Elko fields, although having Lagrangians that are not invariant under gauge transformations, are not forced to be self-referentially dark. The Elko spinors could undergo abelian gauge transformations of the form

$$\chi(\mathbf{p}) \rightarrow \exp[iM\alpha(x)]\chi(\mathbf{p}) \quad (3.49)$$

if and only if M is the 4×4 matrix γ_0 . Any non-abelian generalizations would have to retain this γ_0 .

The reason for claiming that the Elko *spinor* Lagrangian is not $U(1)$ gauge invariant, is that we have $S(C)\chi = \pm\chi$ but $S(C)\chi' \neq \pm\chi'$ where $\chi' = e^{i\alpha}\chi$ for some α .

Having reviewed the relevant parts of Elko Field Theory literature concerning interactions, we now examine standard $U(1)$ gauge invariance of the Elko *quantum field* Lagrangian, and then discuss the implications of our results in the context of the review of Elko interactions.

3.3 Elko $U(1)$ Gauge Transformation and Discussion

3.3.1 Elko $U(1)$ Gauge Transformation

In this section, we examine under what conditions $U(1)$ gauge invariance is possible for the Elko Lagrangian with the partial derivatives replaced by covariant derivatives. The main contents here concerning Elko gauge interactions is also present in our paper [55]. With the covariant derivative replacing the partial derivative, the Elko Lagrangian becomes

$$\mathcal{L} = (D_\mu^\top \Lambda)(D^\mu \Lambda) - m^2 \bar{\Lambda} \Lambda. \quad (3.50)$$

We wish to see if $\mathcal{L}' = \mathcal{L}$ where

$$\mathcal{L}' = (D_\mu^{\prime\top} \Lambda')(D^{\mu'} \Lambda') - m^2 \bar{\Lambda}' \Lambda' \quad (3.51)$$

with

$$D^{\mu'} = \partial^\mu + iqA^\mu + iq(\partial^\mu \chi) \quad (3.52)$$

and

$$\Lambda' = e^{-iq\chi} \Lambda. \quad (3.53)$$

It follows from how the covariant derivative is constructed that

$$D^{\mu'} \Lambda' = e^{-iq\chi} D^\mu \Lambda. \quad (3.54)$$

3.3 Elko $U(1)$ Gauge Transformation and Discussion

For $U(1)$ gauge invariance of the Elko Lagrangian, we would need

$$D'_\mu \bar{\Lambda}' = D_\mu \bar{\Lambda} e^{iq\chi} \quad (3.55)$$

and

$$\bar{\Lambda}' = \bar{\Lambda} e^{iq\chi} \quad (3.56)$$

to be true as well, so we would need the Elko dual operation to transform the product of an Elko field operator Λ with a non-Elko field operator A as:

$$(A\Lambda)^\dagger = \bar{\Lambda} A^\dagger. \quad (3.57)$$

This corresponds to the Elko dual of a non-Elko field operator simply being its adjoint. We believe this most natural assumption to be reasonable and moreover, are unaware of any reasons or hints that the Elko operation being applied to a standard field should produce anything different from its adjoint.

3.3.2 Discussion of Elko Interactions

Before moving on to examining general non-abelian Elko gauge interactions, we pause to make contact between the Elko interaction results in existing literature (reviewed in Sec. (3.2.3)) and what we have done in this thesis.

Our results are consistent with the claim that Elko fields can couple to the Higgs field, and the claim that Elko fields can have quartic self-interactions. Our results are compatible with these results. Our results do however contradict the assertion made in [34, p.7] that $\mathcal{L}^\Lambda(x)$ and $\mathcal{L}^\lambda(x)$ *do not carry invariance under Standard Model gauge transformations*. The justification for this claim made in the cited paper was made from considerations at the *spinor* level, considering Elko spinors as eigenspinors of an antilinear $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation space operator $S(C)$. We take the view that at the level of the *state space*, the spinor coefficients present in the quantum field operator do not determine whether a Lagrangian composed of *quantum field operators* is invariant under a gauge transformation. An argument of the nature described in [34, p.7] applies if we consider Lagrangians composed of Elko *spinors* where the Lagrangian is then defined in a classical spinor space and is not viewed as an operator in Hilbert space. We believe that this result does not automatically nullify the gauge invariance of Lagrangians that are operators on the Hilbert space of physical states, composed of Elko *quantum field operators*.

As pointed out in Sec. (3.2.3), another point raised in [34] was that Elko fields must be dark with respect to (non-Higgs) Standard Model particles on account of the Elko mass dimensionality being different from the mass dimensionality of the Standard Model fermions, thus forbidding Elko fields from being allowed to enter into the fermionic doublets of the Standard Model. We agree that Elko fields cannot enter the Standard Model doublets. However, as we point out in Chapter 6, we do *not* agree that this implies that Elko fields are

3 Elko Fields and Interactions

dark with respect to these Standard Model doublets. The reason for our divergent viewpoint is as follows:

As we have already pointed out in this section, it is possible at the level of the Hilbert space of physical states for Lagrangian operators composed of Elko quantum field operators to be gauge invariant. Once we see this, we can then take Elko fields through the gauge process (see Sec. (3.5)) and hence it is possible to set up Elko doublets and Elko Lagrangians that are invariant under $SU(2) \times U(1)$ gauge transformations. As we point out in Chapter 6, there is no reason why we cannot also choose to take the left-handed Elko components to form the doublets. We can then (even without demanding that we form the Elko doublets purely from the left-handed components of Elko fields) automatically write down symmetry currents in the usual way that arise from the invariance of the Lagrangian under $SU(2) \times U(1)$ gauge transformations. Once we have written down an Elko symmetry current consisting of doublets whose entries are solely Elko fields, we can couple this symmetry current four-vector operator with the Standard Model symmetry current four-vector operators and form *new* interaction terms in an expanded Hamiltonian density operator that now admits new additional interactions; specifically, interactions between the Elko fields and the Standard Model fields. This is at the level of the Electroweak Theory, involving the W^\pm and Z^0 vector bosons. By similar reasoning, simpler arguments apply in the simpler theory of Quantum Electrodynamics. There are also no obvious reasons why we could not also consider Elko triplets transforming under the natural representation of the $SU(3)$ gauge group and couple the corresponding Elko symmetry currents to the symmetry currents in Quantum Chromodynamics.

Although we have made the case for Elko particles in principle being able to interact with the usual Standard Model particles (not just the Higgs particle) we stop short of claiming that Elko is not a viable dark matter candidate. We say this for three main reasons.

The first reason that Elko particles might still be viable dark matter candidates is that in the particular sense described in Sec. (4.4), Elko fields have an element of non-locality, even along the axis of locality. If a particle is not local (in this sense, not causal), it is not clear to us exactly what this implies when it comes to the issue of how we can detect such a particle. The non-local nature of Elko fields may give Elko particles the *appearance* of being dark in the sense that we have trouble finding them, even though they admit gauge interactions.

The second reason that Elko particles might still be viable dark matter candidates is that Elko fields are not quantum fields in the sense of Weinberg (see Sec. (4.2)), and that, more specifically, they do not transform correctly under the Lorentz group (see Sec. (2.13)). They break rotational invariance in particular. This implies that if we were to construct Hamiltonian densities out of the Elko fields $\Lambda_\ell(x)$ and their complex conjugate transpose adjoint's $\Lambda_\ell^\dagger(x)$, we would not have a Hamiltonian density which is a Lorentz scalar under rotations. If we were then to couple a Standard Model symmetry current with an Elko symmetry current, the resulting object may *look* like a scalar when in reality it is not a scalar under rotations. This in turn may affect the detectability of Elko particles.

The third reason that Elko particles might still be viable dark matter candidates (which also relates to the first two reasons) resides in the fact that the Elko Field Theory is incomplete. There are areas of Elko Field Theory that are incomplete, leaving holes with no well-defined way of definitively stating whether certain things *must* be so. For example, Eqn. (3.56) is to us the most obvious and natural thing to write down but it cannot be proven because there are holes in Elko Field Theory. In this example the specific hole is that the Elko dual operation has not been defined for non-Elko operators or for products of Elko field operators with non-Elko field operators.

Another important element of Elko Field Theory that is incomplete is the definition of the two-valued discrete index. It has been given meaning at the classical spinor level, but it lacks a precise definition on the Hilbert space of physical states. This is another significant hole in the theory because there are no well-defined rules to use to determine something as simple as whether the two-valued discrete index for the Elko field should be interpreted as σ (the eigenvalues of the angular momentum generator J_z) or something different. We cannot therefore prove that the Elko two-valued index *must* be σ . Instead, we do the next best thing, and argue that the initial intended premises of the Elko field were built on the guiding principle that it respect the irreducible representations of the full Poincaré group [34, p.1], which, makes it natural for us to define a state space which carries such representations. This in turn, forces the two-valued discrete index to be interpreted as σ . This interpretation of σ as the conventional one also makes sense when remembering that Elko fields are introduced as being spin-1/2 fields.

3.4 More Incompleteness in Elko Field Theory

Another hole in the theory is the introduction of a new adjoint to the Hilbert space [34, p.4]. In this citation, a new operator is introduced, which is referred to as the adjoint of the Elko field operator. But if the underlying state space is still the usual Hilbert space of physical states, then the adjoint of an operator on the Hilbert space is unique, being specifically the complex conjugate transpose of the original operator. No further explanation was given on in what sense the new operator was the adjoint of the Elko field operator, giving rise to another ambiguity, or hole in Elko Field Theory. In the absence of well-defined new mathematical structures to accommodate a new adjoint, we are left to plug this hole as we want. We therefore choose what to us is the most natural assumption, namely that Elko was intended to operate on the usual Hilbert space of physical states, in which case, the adjoint is unique, and is fixed to be $\Lambda_\ell^\dagger(x)$.

In this thesis therefore, we take what we see as the most natural positions, given the stated ambiguities and holes in Elko Field Theory, and give the natural consequences of plugging the holes in Elko Field Theory in what we see as the most natural way of doing it. We take the view that our approach is valid and therefore that it constitutes a significant contribution

to our knowledge of the Elko Field Theory landscape.

3.5 General Elko Non-Abelian Gauge Symmetries

In this section we look at general symmetries of the Elko Lagrangian for general Elko multiplets. We first look at the global gauge symmetries then look at the local gauge symmetries. We finish the section by looking at the field strength tensor, and, how it transforms. The general concepts, which will be here spelt out in detail, can be found in a lot more sketchy form, in, for example, [21, p.1–5] or [73, p.373–376].

3.5.1 General Global Elko Non-Abelian Gauge Symmetries

Here we consider a multi-component Elko matter field (an Elko multiplet) $\mathbf{\Lambda} = \{\Lambda_N(x)\}$, which transforms according to some Lie group of internal symmetries. We start by varying the action. This will lead to the general Elko global symmetry currents, which, upon taking the time component and integrating over space, give rise to the general global Elko symmetry operators.

An infinitesimal transformation of the Elko multiplet $\mathbf{\Lambda}(x)$ then has the form

$$\mathbf{\Lambda}'(x) = \mathbf{\Lambda}(x) + \delta_0 \mathbf{\Lambda}(x) \quad (3.58)$$

where

$$\delta_0 \mathbf{\Lambda}(x) = \alpha^a T_a \mathbf{\Lambda}(x) \equiv \alpha \mathbf{\Lambda}(x), \quad (a = 1, \dots, n), \quad (3.59)$$

where the α^a are (temporarily) constant parameters, the T_a are the group generators satisfying the commutation relations

$$[T_a, T_b] = f_{ab}^c T_c, \quad (3.60)$$

and the structure constants f_{ab}^c satisfy the Jacobi identity. Since the α^a are here constant parameters, $\partial_\mu \mathbf{\Lambda}$ transforms like the field itself so we have

$$\delta_0(\partial_\mu \mathbf{\Lambda}(x)) = \partial_\mu \delta_0 \mathbf{\Lambda}(x) = \partial_\mu \alpha \mathbf{\Lambda}(x) = \alpha \partial_\mu \mathbf{\Lambda}(x) \quad (3.61)$$

because δ_0 and ∂_μ commute. Varying the action, we have

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Lambda} \delta \Lambda + \frac{\partial \mathcal{L}}{\partial \bar{\Lambda}} \delta \bar{\Lambda} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Lambda)} \delta (\partial_\mu \Lambda) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Lambda})} \delta (\partial_\mu \bar{\Lambda}) \right]. \quad (3.62)$$

Using integration by parts and taking[§]

$$u_1 = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Lambda)}, \quad u_2 = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Lambda})} \quad (3.63)$$

[§]Integrating by parts has the form $\int u dv = uv - \int v du$.

and

$$\delta v_1 = \delta(\partial_\mu \Lambda) d^4 x, \quad \delta v_2 = \delta(\partial_\mu \bar{\Lambda}) d^4 x \quad (3.64)$$

it follows directly that

$$\delta u_1 = \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Lambda)} \right) d^4 x, \quad \delta u_2 = \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Lambda})} \right) d^4 x \quad (3.65)$$

and

$$v_1 = \delta \Lambda, \quad v_2 = \delta \bar{\Lambda} \quad (3.66)$$

so the variation of the action becomes

$$\begin{aligned} \delta S = \int d^4 x \left[\frac{\partial \mathcal{L}}{\partial \Lambda} \delta \Lambda + \frac{\partial \mathcal{L}}{\partial \bar{\Lambda}} \delta \bar{\Lambda} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Lambda)} \right) \delta \Lambda - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Lambda})} \right) \delta \bar{\Lambda} \right] \\ + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Lambda)} \delta \Lambda \right]_{\Lambda_i}^{\Lambda_f} + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Lambda})} \delta \bar{\Lambda} \right]_{\bar{\Lambda}_i}^{\bar{\Lambda}_f}. \end{aligned} \quad (3.67)$$

The last two terms vanish because of the assumption that the fields disappear at sufficiently large distances so we get

$$\delta S = \int d^4 x \left[- \left(\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Lambda)} \right) - \frac{\partial \mathcal{L}}{\partial \Lambda} \right) \delta \Lambda - \left(\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Lambda})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\Lambda}} \right) \delta \bar{\Lambda} \right]. \quad (3.68)$$

For δS to vanish under arbitrary $\delta \Lambda$ and $\delta \bar{\Lambda}$, we must have Lagrange's equations:

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Lambda)} \right) = \frac{\partial \mathcal{L}}{\partial \Lambda}, \quad \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Lambda})} \right) = \frac{\partial \mathcal{L}}{\partial \bar{\Lambda}}. \quad (3.69)$$

Now turning our attention to the variation of the Lagrangian density $\delta_0 \mathcal{L}$, we have

$$\delta_0 \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \Lambda} \delta \Lambda + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Lambda)} \delta(\partial_\mu \Lambda) = \frac{\partial \mathcal{L}}{\partial \Lambda} \alpha \Lambda + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Lambda)} \alpha(\partial_\mu \Lambda) = 0 \quad (3.70)$$

which implies N identities:[¶]

$$\frac{\partial \mathcal{L}}{\partial \Lambda} T_a \Lambda + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Lambda)} T_a(\partial_\mu \Lambda) = 0. \quad (3.71)$$

By using this, and Lagrange's equations, we obtain:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Lambda)} \right) T_a \Lambda + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Lambda)} T_a(\partial_\mu \Lambda) = 0 \quad (3.72)$$

from which it is immediately seen from the product rule that

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Lambda)} T_a \Lambda \right] = 0. \quad (3.73)$$

[¶]The T_a appear by recalling that $\alpha = T_a \alpha^a$.

3 Elko Fields and Interactions

We define the global symmetry current $T_{\Lambda,a}^\mu$ to be

$$T_{\Lambda,a}^\mu \equiv \frac{-\partial\mathcal{L}}{\partial(\partial_\mu\Lambda)} T_a \Lambda \quad (3.74)$$

satisfying the equation

$$\partial_\mu T_{\Lambda,a}^\mu = 0. \quad (3.75)$$

From the Elko free particle Lagrangian we see that

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Lambda)} = \partial^\mu \bar{\Lambda} \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Lambda})} = \partial^\mu \Lambda \quad (3.76)$$

so we get

$$T_{\Lambda,a}^\mu = (\partial^\mu \bar{\Lambda}) T_a \Lambda + \bar{\Lambda} T_a (\partial^\mu \Lambda). \quad (3.77)$$

Expanding on the equation of continuity,

$$\partial_\mu T_{\Lambda,a}^\mu = \frac{\partial T_{\Lambda,a}^0}{\partial t} + \nabla \cdot \mathbf{T}_{\Lambda,a} = 0 \quad (3.78)$$

and integrating over all space yields

$$\frac{d}{dt} \int_{V \rightarrow \infty} T_{\Lambda,a}^0 d^3\mathbf{x} + \int_{V \rightarrow \infty} \nabla \cdot \mathbf{T}_{\Lambda,a} d^3\mathbf{x} = 0. \quad (3.79)$$

But by the divergence theorem

$$\int_{V \rightarrow \infty} \nabla \cdot \mathbf{T}_{\Lambda,a} d^3\mathbf{x} = \int_{S \rightarrow \infty} \mathbf{T}_{\Lambda,a} \cdot d\mathbf{S} \quad (3.80)$$

which vanishes, given that the fields die off sufficiently fast, so the operator

$$T_{\Lambda,a} = \int_{V \rightarrow \infty} T_{\Lambda,a}^0 d^3\mathbf{x} \quad (3.81)$$

is constant in time and is the general global Elko symmetry operator. A global symmetry operator gives rise to conservation laws like the conservation of electric charge for example.

3.5.2 General Local Elko Non-Abelian Gauge Symmetries

In this section, we introduce the covariant derivative and derive the general transformation law for the resulting gauge field. Let us consider transformations with the constant parameters replaced by arbitrary functions of spacetime position $\alpha^a = \alpha^a(x)$. The Lagrangian is no longer invariant because the transformation law of $\partial_\mu\Lambda$ is modified:

$$\partial_\mu(\delta_0\Lambda) = \partial_\mu(\alpha(x)\Lambda(x)) = (\partial_\mu\alpha)\Lambda + \alpha\partial_\mu\Lambda = \delta_0\partial_\mu\Lambda(x). \quad (3.82)$$

A direct calculation reveals that that the variation of the Lagrangian no longer vanishes:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Lambda}\delta\Lambda + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Lambda)}\delta(\partial_\mu\Lambda) = \frac{\partial\mathcal{L}}{\partial\Lambda}\alpha^a T_a \Lambda + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Lambda)}\alpha^a T_a \partial_\mu\Lambda + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Lambda)}(\partial_\mu\alpha)\Lambda, \quad (3.83)$$

so

$$\delta\mathcal{L} = -(\partial_\mu\alpha^a) \left[\frac{-\partial\mathcal{L}}{\partial(\partial_\mu\Lambda)} T_a\Lambda \right] = -(\partial_\mu\alpha^a) T_{\Lambda,a}^\mu \neq 0 \quad (3.84)$$

in general. We now introduce a covariant derivative D_μ . We start by introducing a new Lagrangian

$$\mathcal{L} = \mathcal{L}(\Lambda, D_\mu\Lambda) = D_\mu\Lambda^\dagger D^\mu\Lambda - m^2 \Lambda^\dagger \Lambda \quad (3.85)$$

where $D_\mu\Lambda$ transforms under local transformations in the same way as $\partial_\mu\Lambda$ does under the global ones:

$$\delta_0 D_\mu\Lambda = \alpha D_\mu\Lambda. \quad (3.86)$$

Now we have

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Lambda} \alpha\Lambda + \frac{\partial\mathcal{L}}{\partial(D_\mu\Lambda)} \alpha D_\mu\Lambda = 0. \quad (3.87)$$

We here construct the covariant derivative by introducing compensating gauge potentials (fields) A_μ , so that

$$D_\mu\Lambda = (\partial_\mu + A_\mu)\Lambda, \quad \text{where} \quad A_\mu \equiv T_a A_\mu^a. \quad (3.88)$$

We use the demand that $\delta_0 D_\mu\Lambda = \alpha D_\mu\Lambda$ to derive the transformation law for the gauge potentials as follows. Expanding out the left hand side of this demand gives

$$\begin{aligned} \delta_0 D_\mu\Lambda &= \delta_0 \partial_\mu\Lambda + \delta_0(A_\mu\Lambda) = \partial_\mu(\delta_0\Lambda) + (\delta_0 A_\mu)\Lambda + A_\mu(\delta_0\Lambda) \\ &= (\partial_\mu\alpha)\Lambda + \alpha\partial_\mu\Lambda + (\delta_0 A_\mu)\Lambda + A_\mu(\delta_0\Lambda), \end{aligned} \quad (3.89)$$

whereas the right hand side of this demand gives

$$\alpha\partial_\mu\Lambda + \alpha A_\mu\Lambda. \quad (3.90)$$

Setting the expanded left hand side equal to the expanded right hand side and rearranging yields

$$(\delta_0 A_\mu)\Lambda = -(\partial_\mu\alpha)\Lambda - [A_\mu, \alpha]\Lambda \quad (3.91)$$

from which it follows that

$$\delta_0 A_\mu^a = (-\partial_\mu\alpha - [A_\mu, \alpha])^a, \quad (3.92)$$

but

$$[A_\mu, \alpha]^a = (T_b A_\mu^b \alpha^c T_c - \alpha^c T_c T_b A_\mu^b)^a = ([T_b, T_c] \alpha^c A_\mu^b)^a = (f_{bc}^d T_d A_\mu^b \alpha^c)^a \quad (3.93)$$

and

$$f_{bc}^d \delta_0^a A_\mu^b \alpha^c = f_{bc}^a A_\mu^b \alpha^c,$$

so the gauge potential A transforms as

$$\delta A_\mu^a = -\partial_\mu\alpha^a - f_{bc}^a A_\mu^b \alpha^c. \quad (3.94)$$

The form of the covariant derivative of the Elko field $D_\mu\Lambda$ is determined by the transformation rule of Λ :

$$D_\mu\Lambda = \partial_\mu\Lambda + \delta_0\Lambda|_{\alpha \rightarrow A_\mu}. \quad (3.95)$$

3 Elko Fields and Interactions

We finish this section by commenting that Eqn. (3.86) is a non-abelian generalization of Eqn. (3.54). If we hold to the assumption stated in Sec.(3.3) that the Elko dual of a non-Elko field operator, say $\Omega(x)$, gives the corresponding adjoint $\Omega^\dagger(x)$, and if we take the Elko dual of both sides of Eqn. (3.86), we get the corresponding non-abelian generalization of Eqn. (3.55) so no extra assumptions are needed in order to ensure non-abelian gauge invariance. The Lagrangian, Eq. (3.85), is invariant under non-abelian gauge transformations because of the assumption we made in Sec. (3.3), namely, Eqn. (3.57).

3.5.3 The Field Strength Tensor

In order to write down Lagrangians for the gauge fields, we need to construct the field strength tensor and derive its transformation properties. The field strength tensor $F_{\mu\nu}$ is defined by the action of the commutator of covariant derivatives:

$$[D_\mu, D_\nu]\Lambda = (\partial_\mu + A_\mu)(\partial_\nu + A_\nu)\Lambda - (\partial_\nu + A_\nu)(\partial_\mu + A_\mu)\Lambda \quad (3.96)$$

$$\begin{aligned} & \partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda + (\partial_\mu A_\nu)\Lambda + A_\nu \partial_\mu \Lambda + A_\mu \partial_\nu \Lambda + A_\mu A_\nu \Lambda - (\partial_\nu A_\mu)\Lambda - A_\mu (\partial_\nu \Lambda) \\ & - A_\nu \partial_\mu \Lambda - A_\nu A_\mu \Lambda = (\partial_\mu A_\nu - \partial_\nu A_\mu)\Lambda + [A_\mu, A_\nu]\Lambda, \end{aligned}$$

so we have

$$[D_\mu, D_\nu]\Lambda = F_{\mu\nu}^a T_a \Lambda \equiv F_{\mu\nu} \Lambda. \quad (3.97)$$

Varying the field to see how it transforms yields

$$\begin{aligned} \delta_0 F_{\mu\nu}^a &= \delta_0(\partial_\mu A_\nu^a) - \delta_0(\partial_\nu A_\mu^a) + \delta_0 f_{bc}^a A_\mu^b A_\nu^c \\ &= \partial_\mu(\delta_0 A_\nu^a) - \partial_\nu(\delta_0 A_\mu^a) + \delta_0 f_{bc}^a A_\mu^b A_\nu^c \\ &= \partial_\mu[-\partial_\nu \alpha^a - f_{bc}^a A_\nu^b \alpha^c] - \partial_\nu[-\partial_\mu \alpha^a - f_{bc}^a A_\mu^b \alpha^c] + \delta_0 f_{bc}^a A_\mu^b A_\nu^c \\ &= -\partial_\mu f_{bc}^a A_\nu^b \alpha^c + \partial_\nu f_{bc}^a A_\mu^b \alpha^c + \delta_0 f_{bc}^a A_\mu^b A_\nu^c \\ &= -\partial_\mu f_{cb}^a A_\nu^c \alpha^b + \partial_\nu f_{cb}^a A_\mu^c \alpha^b + \delta_0 f_{bc}^a A_\mu^b A_\nu^c \\ &= f_{bc}^a [\partial_\mu A_\nu^c - \partial_\nu A_\mu^c] \alpha^b + f_{bc}^a A_\nu^c (\partial_\mu \alpha^b) - f_{bc}^a A_\mu^c (\partial_\nu \alpha^b) + \delta_0 f_{bc}^a A_\mu^b A_\nu^c. \end{aligned} \quad (3.98)$$

The last term gives:

$$\begin{aligned} \delta_0 f_{bc}^a A_\mu^b A_\nu^c &= f_{bc}^a (\delta_0 A_\mu^b) A_\nu^c + f_{bc}^a A_\mu^b (\delta_0 A_\nu^c) \\ &= f_{bc}^a [-\partial_\mu \alpha^b - f_{de}^b A_\mu^d \alpha^e] A_\nu^c + f_{bc}^a A_\mu^b [-\partial_\nu \alpha^c - f_{de}^c A_\nu^d \alpha^e] \\ &= -f_{bc}^a (\partial_\mu \alpha^b) A_\nu^c - f_{bc}^a (\partial_\nu \alpha^c) A_\mu^b - f_{bc}^a f_{de}^b A_\mu^d \alpha^e A_\nu^c - f_{bc}^a f_{de}^c A_\nu^d \alpha^e A_\mu^b \\ &= -f_{bc}^a (\partial_\mu \alpha^b) A_\nu^c - f_{bc}^a (\partial_\nu \alpha^b) A_\mu^c - f_{bc}^a f_{de}^b A_\mu^d \alpha^e A_\nu^c - f_{bc}^a f_{de}^c A_\mu^b \alpha^e A_\nu^d \\ &= -f_{bc}^a (\partial_\mu \alpha^b) A_\nu^c + f_{bc}^a (\partial_\nu \alpha^b) A_\mu^c - f_{bc}^a f_{de}^b A_\mu^d \alpha^e A_\nu^c - f_{bc}^a f_{de}^c A_\mu^b \alpha^e A_\nu^d \end{aligned} \quad (3.99)$$

so

$$\delta_0 F_{\mu\nu}^a = f_{bc}^a [\partial_\mu A_\nu^c - \partial_\nu A_\mu^c] \alpha^b + f_{bc}^a A_\nu^c (\partial_\mu \alpha^b) - f_{bc}^a A_\mu^c (\partial_\nu \alpha^b) - f_{bc}^a A_\mu^c (\partial_\nu \alpha^b) \quad (3.100)$$

3.5 General Elko Non-Abelian Gauge Symmetries

$$\begin{aligned}
& + f_{bc}^a (\partial_\nu \alpha^b) A_\mu^c - f_{bc}^a f_{de}^b A_\mu^d \alpha^e A_\nu^c - f_{bc}^a f_{de}^c A_\mu^b A_\nu^d \alpha^e \\
& = f_{bc}^a [\partial_\mu A_\nu^c - \partial_\nu A_\mu^c] \alpha^b - f_{bc}^a f_{de}^b A_\mu^d \alpha^e A_\nu^c - f_{bc}^a f_{de}^c A_\mu^b A_\nu^d \alpha^e.
\end{aligned}$$

Relabeling dummy indices in the last two terms to get α^b gives

$$f_{bc}^a [\partial_\mu A_\nu^c - \partial_\nu A_\mu^c] \alpha^b + [-f_{ec}^a f_{db}^e A_\mu^d A_\nu^c - f_{ec}^a f_{db}^c A_\mu^e A_\nu^d] \alpha^b. \quad (3.101)$$

Swapping the d and e dummy indices around on the last term now gives

$$f_{bc}^a [\partial_\mu A_\nu^c - \partial_\nu A_\mu^c] \alpha^b + [-f_{ec}^a f_{db}^e A_\mu^d A_\nu^c - f_{dc}^a f_{eb}^c A_\mu^d A_\nu^e] \alpha^b. \quad (3.102)$$

Also, from the Jacobi identity,

$$-f_{ce}^a f_{db}^c = f_{bc}^a f_{de}^c + f_{eb}^a f_{dc}^c \quad (3.103)$$

and

$$-f_{dc}^a f_{eb}^c = f_{bd}^a f_{ec}^c + f_{cb}^a f_{ed}^c \quad (3.104)$$

and given that in this adjoint representation, the f_{bc}^a are completely antisymmetric in all three indices, the terms with the repeated indices are zero so we now get

$$\begin{aligned}
& f_{bc}^a [\partial_\mu A_\nu^c - \partial_\nu A_\mu^c] \alpha^b + [f_{bc}^a f_{de}^c A_\mu^d A_\nu^e + f_{cb}^a f_{ed}^c A_\mu^d A_\nu^e] \alpha^b \\
& = f_{bc}^a [\partial_\mu A_\nu^c - \partial_\nu A_\mu^c] \alpha^b + f_{bc}^a [f_{de}^c A_\mu^d A_\nu^e + f_{de}^c A_\mu^d A_\nu^e] \alpha^b,
\end{aligned} \quad (3.105)$$

but the last two terms are just commutators of components of the gauge fields so we now have

$$f_{bc}^a \alpha^b [(\partial_\mu A_\nu - \partial_\nu A_\mu) + [A_\mu, A_\nu]]^c. \quad (3.106)$$

It then follows that the components of the field strength tensor transform as

$$\delta_0 F_{\mu\nu}^a = f_{bc}^a \alpha^b F_{\mu\nu}^c. \quad (3.107)$$

Contracting with T_a reveals that the field strength tensor transforms as

$$\delta_0 F_{\mu\nu} = [T_b, T_c] \alpha^b F_{\mu\nu}^c. \quad (3.108)$$

By expanding this out, we see that it can be written in a slightly more slick form:

$$\delta_0 F_{\mu\nu} = \alpha^b T_b T_c F_{\mu\nu}^c - T_c F_{\mu\nu}^c \alpha^b T_b = \alpha F_{\mu\nu} - F_{\mu\nu} \alpha = [\alpha, F_{\mu\nu}], \quad (3.109)$$

so, finally, we have

$$\delta_0 F_{\mu\nu}^a = [\alpha, F_{\mu\nu}]^a. \quad (3.110)$$

We may now construct a Lagrangian $\mathcal{L}_F(A, \partial_\mu A)$ for the field by varying the Lagrangian and demanding that $\delta \mathcal{L}_F = 0$. It can be shown, [73], that the solution is:

$$\mathcal{L}_F = -\frac{1}{4g^2} g_{ab} F_{\mu\nu}^a F^{b\mu\nu} \quad (3.111)$$

3 Elko Fields and Interactions

where g is a scaling constant, which would be iq in the electromagnetic case for example, and g_{ab} is the Cartan metric of the Lie algebra [74, p.46].

If we hold to what we see as being the most natural ways of filling the stated holes in Elko Field Theory, then the Elko Lagrangian is $U(1)$ gauge invariant. Non-abelian gauge fields can correspond to the particles mediating the weak and strong forces. In the next section, we take multiplets of Elko fields and look at general non-abelian gauge symmetries, which immediately implies that Elko Lagrangians can be invariant under non-abelian $SU(2)$ and $SU(3)$ gauge transformations in addition to the abelian $U(1)$ gauge transformations.

We now move on to deriving the free particle Hamiltonian density, followed by deriving the interaction Hamiltonian density that results from $U(1)$ gauge invariance of the Elko Lagrangian density.

3.6 Free Particle Hamiltonian

In this section the free particle Hamiltonian is written down in terms of the canonically conjugate momenta Π and $\bar{\Pi}$. This is done so that later on, after the interaction Lagrangian has been obtained from the $U(1)$ gauge invariance of the Elko Lagrangian, the total Hamiltonian can be written down, and the free particle Hamiltonian can be split off and the interaction Hamiltonian clearly identified. The free particle Hamiltonian is written as a functional in terms of Π_0 , $\bar{\Pi}_0$, Λ , $\bar{\Lambda}$, $\partial_k \bar{\Lambda}$ and $\partial^k \Lambda$. The free particle Lagrangian is given by

$$\mathcal{L}_0 = \partial_\mu \bar{\Lambda} \partial^\mu \Lambda - m^2 \bar{\Lambda} \Lambda. \quad (3.112)$$

Given that

$$\Pi_0 = \frac{\partial \mathcal{L}_0}{\partial \dot{\Lambda}} = \partial_0 \bar{\Lambda}, \quad \bar{\Pi}_0 = \frac{\partial \mathcal{L}_0}{\partial \dot{\bar{\Lambda}}} \quad (3.113)$$

the Lagrangian can be re-written so that

$$\mathcal{L} = \partial_0 \bar{\Lambda} \partial^0 \Lambda + \partial_k \bar{\Lambda} \partial^k \Lambda - m^2 \bar{\Lambda} \Lambda = \Pi_0 \bar{\Pi}_0 + \partial_k \bar{\Lambda} \partial^k \Lambda - m^2 \bar{\Lambda} \Lambda. \quad (3.114)$$

The free particle Hamiltonian density is then given by the Legendre transformation as

$$\begin{aligned} \mathcal{H}_0 &= \Pi_0 \dot{\Lambda} + \dot{\bar{\Lambda}} \bar{\Pi}_0 - \mathcal{L}_0 \\ &= \partial_0 \bar{\Lambda} \partial^0 \Lambda + \partial_0 \bar{\Lambda} \partial^0 \Lambda - \partial_\mu \bar{\Lambda} \partial^\mu \Lambda + m^2 \bar{\Lambda} \Lambda \\ &= 2\partial_0 \bar{\Lambda} \partial^0 \Lambda - \partial_0 \bar{\Lambda} \partial^0 \Lambda + \partial_k \bar{\Lambda} \partial_k \Lambda + m^2 \bar{\Lambda} \Lambda \\ &= \partial_0 \bar{\Lambda} \partial^0 \Lambda + \partial_k \bar{\Lambda} \partial_k \Lambda + m^2 \bar{\Lambda} \Lambda \\ &= \Pi_0 \bar{\Pi}_0 + \partial_k \bar{\Lambda} \partial_k \Lambda + m^2 \bar{\Lambda} \Lambda \end{aligned} \quad (3.115)$$

so the free particle Hamiltonian is therefore

$$H_0 = \int d^3 \mathbf{x} \left[\Pi_0 \bar{\Pi}_0 + \partial_k \bar{\Lambda} \partial_k \Lambda + m^2 \bar{\Lambda} \Lambda \right]. \quad (3.116)$$

We now turn our attention to the task of calculating the interaction Hamiltonian density.

3.7 The Interaction Hamiltonian Density

As was discussed in Chapter 2, the S -matrix is the matrix whose entries are the probability amplitudes of various particle interactions happening. These probability amplitudes are crucially dependent on the interaction Hamiltonian density. In a gauge theory, the interaction Hamiltonian density can be determined by the symmetries of the Lagrangian. In this section we use the standard procedure for the Klein-Gordon Lagrangians applied to the scalar fields, on Elko fields, in order to derive the interaction Hamiltonian density for the Elko fields when interacting with a $U(1)$ gauge quantum.

In this section we continue to assume that the Elko discrete index σ , and the operation of the Elko dual, are such that the Elko Lagrangian density is invariant under both global and local $U(1)$ gauge transformations. Firstly, the interaction Lagrangian density is obtained directly from the Elko Lagrangian partial derivatives being replaced with covariant derivatives. We then calculate the canonically conjugate momenta Π and $\bar{\Pi}$. We then perform a Legendre transformation and write the Hamiltonian as a functional of the Λ 's and Π 's. Once the free particle Hamiltonian is identified, we split it off from the total Hamiltonian, leaving the interaction Hamiltonian from which we obtain the interaction Hamiltonian density. We require the interaction Hamiltonian density for S -matrix calculations.

In order to make the form of the interaction Lagrangian that arises from $U(1)$ gauge invariance clear, we now write the Elko Lagrangian as

$$\mathcal{L} = D_\mu^\dagger \Lambda D^\mu \Lambda - m^2 \bar{\Lambda} \Lambda \quad (3.117)$$

$$= (\partial_\mu - iqA_\mu) \bar{\Lambda} (\partial^\mu + iqA^\mu) \Lambda - m^2 \bar{\Lambda} \Lambda. \quad (3.118)$$

The A_μ that appears here on the left, more properly should be written as A_μ^\dagger but we here work with gauge quanta that are massless and contain no conserved quantum numbers so that $A_\mu = A_\mu^\dagger$ (Since the u and v coefficient functions are complex conjugates of each other, [42, p.211].). Separating out the free particle Lagrangian, the total Lagrangian \mathcal{L} takes the form

$$\mathcal{L} = \mathcal{L}_{\text{KG}} + \mathcal{L}_{\text{int}} \quad (3.119)$$

where the interaction Lagrangian density \mathcal{L}_{int} is read off from above to give^{||}

$$\mathcal{L}_{\text{int}} = -iq[\bar{\Lambda} (\partial^\mu \Lambda) - (\partial^\mu \bar{\Lambda}) \Lambda] A_\mu + q^2 A^\mu A_\mu \bar{\Lambda} \Lambda. \quad (3.120)$$

We now calculate the canonically conjugate momenta Π and $\bar{\Pi}$ to be^{**}

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Lambda}} = \dot{\bar{\Lambda}} - iqA^0 \bar{\Lambda} \quad (3.121)$$

^{||} $\partial_\mu A^\mu = g_{\mu\nu} g^{\mu\sigma} \partial^\nu A_\sigma = \delta_\nu^\sigma \partial^\nu A_\sigma = \partial^\nu A_\nu = \partial^\mu A_\mu$.

^{**} $A^0 = A_0$.

3 Elko Fields and Interactions

and

$$\bar{\Pi} = \frac{\partial \mathcal{L}}{\partial \dot{\bar{\Lambda}}} = \dot{\Lambda} + iqA^0 \Lambda. \quad (3.122)$$

We are now in a position to calculate the form of the interaction Hamiltonian, which is needed in order to take advantage of the S -matrix formalism. Firstly, the Hamiltonian is given by a Legendre transformation to be

$$H(\Lambda, \Pi) = H_{\text{KG}} + H_{\text{int}} = \int d^3 \mathbf{x} [\Pi \dot{\Lambda} + \bar{\Pi} \dot{\bar{\Lambda}} - \mathcal{L}_{\text{KG}} - \mathcal{L}_{\text{int}}]. \quad (3.123)$$

By observing from the expressions for the canonically conjugate momenta that

$$\dot{\bar{\Lambda}} = \Pi + iqA^0 \bar{\Lambda} \quad (3.124)$$

and

$$\dot{\Lambda} = \bar{\Pi} - iqA^0 \Lambda, \quad (3.125)$$

the Hamiltonian then becomes

$$H = \int d^3 \mathbf{x} [\Pi(\bar{\Pi} - iqA^0 \Lambda) + (\Pi + iqA^0 \bar{\Lambda}) \bar{\Pi} - \partial_\mu \bar{\Lambda} \partial^\mu \Lambda + m^2 \bar{\Lambda} \Lambda + iq(\bar{\Lambda} (\partial^\mu \Lambda) - (\partial^\mu \bar{\Lambda}) \Lambda) A_\mu - q^2 A^\mu A_\mu \bar{\Lambda} \Lambda], \quad (3.126)$$

which, upon multiplying out, yields^{††}

$$H = \int d^3 \mathbf{x} [\Pi \bar{\Pi} + \Pi \bar{\Pi} - iqA^0 \Pi \Lambda + iqA^0 \bar{\Lambda} \bar{\Pi} - \dot{\bar{\Lambda}} \dot{\Lambda} + \partial_k \bar{\Lambda} \partial_k \Lambda + m^2 \bar{\Lambda} \Lambda + iq(\bar{\Lambda} \dot{\Lambda} - \dot{\bar{\Lambda}} \Lambda) A_0 - q^2 A^\mu A_\mu \bar{\Lambda} \Lambda + iq(\bar{\Lambda} (\partial^k \Lambda) - (\partial^k \bar{\Lambda}) \Lambda) A_k]. \quad (3.127)$$

If we now note that

$$\begin{aligned} \dot{\bar{\Lambda}} \dot{\Lambda} &= [\Pi + iqA_0 \bar{\Lambda}] [\bar{\Pi} - iqA_0 \Lambda] = \\ &\Pi \bar{\Pi} - iqA_0 \Pi \Lambda + iqA_0 \bar{\Pi} \bar{\Lambda} + q^2 A_0^2 \bar{\Lambda} \Lambda \end{aligned} \quad (3.128)$$

and also that

$$\begin{aligned} iq[\bar{\Lambda} \dot{\Lambda} - \dot{\bar{\Lambda}} \Lambda] A_0 &= iq[\bar{\Lambda} [\bar{\Pi} - iqA_0 \Lambda] - [\Pi + iqA_0 \bar{\Lambda}] \Lambda] A^0 \\ &= iq[\bar{\Lambda} \bar{\Pi} - \Pi \Lambda] A^0 + 2q^2 A_0^2 \bar{\Lambda} \Lambda, \end{aligned} \quad (3.129)$$

the Hamiltonian is seen to take the form

$$\begin{aligned} H &= \int d^3 \mathbf{x} [2\Pi \bar{\Pi} + iqA^0 [\bar{\Pi} \bar{\Lambda} - \Pi \Lambda] - \Pi \bar{\Pi} - iqA^0 [\bar{\Lambda} \bar{\Pi} - \Pi \Lambda] + \partial_k \bar{\Lambda} \partial_k \Lambda \\ &\quad + m^2 \bar{\Lambda} \Lambda + iq(\bar{\Lambda} \bar{\Pi} - \Pi \Lambda) A^0 + 2q^2 A_0^2 \bar{\Lambda} \Lambda - q^2 A_0^2 \bar{\Lambda} \Lambda + q^2 A_k A_k \bar{\Lambda} \Lambda \\ &\quad + iq[\bar{\Lambda} (\partial^k \Lambda) - (\partial^k \bar{\Lambda}) \Lambda] A_k - q^2 A_0^2 \bar{\Lambda} \Lambda]. \end{aligned} \quad (3.130)$$

^{††}The term with the ∂_k 's has a plus sign because $\partial_k = -\partial^k$.

3.8 Checking the Canonical Anti-Commutation Relations for the full Lagrangian

Simplifying this expression down a bit before extracting the part that corresponds to the interaction Hamiltonian, yields

$$H = \int d^3\mathbf{x} [\Pi \bar{\Pi} + iq(\bar{\Pi}\bar{\Lambda} - \Pi\Lambda)A^0 + (\partial_k \bar{\Lambda})(\partial_k \Lambda) + m^2 \bar{\Lambda} \Lambda + q^2 A_0^2 \bar{\Lambda} \Lambda + iq(\bar{\Lambda}(\partial_k \Lambda) - (\partial_k \bar{\Lambda})\Lambda)A_k - q^2 A_0^2 \bar{\Lambda} \Lambda + q^2 A_k A_k \bar{\Lambda} \Lambda]. \quad (3.131)$$

Finally, noting that the free particle Hamiltonian is

$$H_{\text{KG}} = \int d^3\mathbf{x} [\Pi \bar{\Pi} + \partial_k \bar{\Lambda} \partial_k \Lambda + m^2 \bar{\Lambda} \Lambda], \quad (3.132)$$

splitting this off from the full Hamiltonian reveals the interaction Hamiltonian to be

$$H_{\text{int}} = \int d^3\mathbf{x} [iq(\bar{\Lambda}\bar{\Pi} - \Pi\Lambda)A^0 + 2iq(\bar{\Lambda} \overleftrightarrow{\partial}^k \Lambda)A_k - q^2 A_k A^k \bar{\Lambda} \Lambda]. \quad (3.133)$$

Taking the interaction Hamiltonian density from this and rewriting it as $\mathcal{H}(\Lambda, \dot{\Lambda}, \bar{\Lambda}, \dot{\bar{\Lambda}})$ yields

$$\mathcal{H}_{\text{int}} = iq[\bar{\Lambda}(\dot{\Lambda} + iqA_0\Lambda) - (\dot{\bar{\Lambda}} - iqA_0\bar{\Lambda})\Lambda]A^0 + 2iq(\bar{\Lambda} \overleftrightarrow{\partial}^k \Lambda)A_k - q^2 A_k A^k \bar{\Lambda} \Lambda, \quad (3.134)$$

which becomes

$$\mathcal{H}_{\text{int}} = iq(\bar{\Lambda} \overleftrightarrow{\partial}^0 \Lambda)A_0 - q^2 A_0^2 \bar{\Lambda} \Lambda - q^2 A_0^2 \bar{\Lambda} \Lambda + iq(\bar{\Lambda} \overleftrightarrow{\partial}^k \Lambda)A_k - q^2 A_k A^k \bar{\Lambda} \Lambda. \quad (3.135)$$

Simplifying yields

$$\mathcal{H}_{\text{int}} = 2iq(\bar{\Lambda} \overleftrightarrow{\partial}^\mu \Lambda)A_\mu - q^2 A_\mu A^\mu \bar{\Lambda} \Lambda - q^2 A_0 A^0 \bar{\Lambda} \Lambda. \quad (3.136)$$

By inspection of this expression, and recalling the expression for the interaction Lagrangian density \mathcal{L}_{int} , we see that the interaction Hamiltonian density \mathcal{H}_{int} simply becomes

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}} - q^2 A_0^2 \bar{\Lambda} \Lambda. \quad (3.137)$$

This last term, although non-covariant, can be made to disappear [75, sec.6.14][42]. The next section is just a check to make sure that the canonical anti-commutation relations still hold with the new canonically conjugate field momentum.

3.8 Checking the Canonical Anti-Commutation Relations for the full Lagrangian

In this section we calculate the anticommutation relations $\{\Lambda_\alpha(\mathbf{x}, t), \Pi(\mathbf{y}, t)\}$, $\{\Lambda_\alpha(\mathbf{x}, t), \Lambda_\beta(\mathbf{y}, t)\}$ and $\{\Pi_\alpha(\mathbf{x}, t), \Pi_\beta(\mathbf{y}, t)\}$, where the canonically conjugate field momenta are as given in the previous section. We take the full Lagrangian $\mathcal{L} = \mathcal{L}_{\text{KG}} + \mathcal{L}_{\text{int}}$, to be

$$\mathcal{L} = (\partial_\mu \bar{\Lambda})(\partial^\mu \Lambda) - m^2 \bar{\Lambda} \Lambda - iq[\bar{\Lambda}(\partial^\mu \Lambda) - (\partial^\mu \bar{\Lambda})\Lambda]A_\mu + q^2 A^\mu A_\mu \bar{\Lambda} \Lambda. \quad (3.138)$$

3 Elko Fields and Interactions

Taking the canonically conjugate momenta from the full Lagrangian, the first anti-commutator becomes

$$\{\Lambda_\alpha(\mathbf{x}, t), \Pi_\beta(\mathbf{y}, t)\} = \{\Lambda_\alpha(\mathbf{x}, t), \dot{\bar{\Lambda}}_\beta(\mathbf{y}, y)\} - iqA^0\{\Lambda_\alpha(\mathbf{x}, t), \bar{\Lambda}_\beta(\mathbf{y}, t)\}. \quad (3.139)$$

The first of these two anticommutators is just the original anticommutation relation corresponding to the Elko Klein-Gordon Lagrangian, the result of which has already been stated. In order for the overall anticommutator to agree with this one, it is therefore necessary that the second anti-commutator above vanish. A direct calculation reveals that this is so. Substituting the mode expansions into the second anti-commutator yields:

$$\begin{aligned} & \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2mE(\mathbf{p})}} \sum_h [e^{-ip^0t + i\mathbf{p}\cdot\mathbf{x}} \xi_\alpha(\mathbf{p}, h) a(\mathbf{p}, h) + e^{ip^0t - i\mathbf{p}\cdot\mathbf{x}} \zeta_\alpha(\mathbf{p}, h) b^\dagger(\mathbf{p}, h)] \times \\ & \int \frac{d^3\mathbf{p}'}{(2\pi)^3 \sqrt{2mE(\mathbf{p}')}} \sum_{h'} [e^{ip'^0t - i\mathbf{p}'\cdot\mathbf{y}} \bar{\xi}_\beta(\mathbf{p}', h') a^\dagger(\mathbf{p}', h') + e^{-ip'^0t + i\mathbf{p}'\cdot\mathbf{y}} \bar{\zeta}_\beta(\mathbf{p}', h') b(\mathbf{p}', h')] + \\ & \int \frac{d^3\mathbf{p}'}{(2\pi)^3 \sqrt{2mE(\mathbf{p}')}} \sum_{h'} [e^{ip'^0t - i\mathbf{p}'\cdot\mathbf{y}} \bar{\xi}_\beta(\mathbf{p}', h') a^\dagger(\mathbf{p}', h') + e^{-ip'^0t + i\mathbf{p}'\cdot\mathbf{y}} \bar{\zeta}_\beta(\mathbf{p}', h') b(\mathbf{p}', h')] \times \\ & \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2mE(\mathbf{p})}} \sum_h [e^{-ip^0t + i\mathbf{p}\cdot\mathbf{x}} \xi_\alpha(\mathbf{p}, h) a(\mathbf{p}, h) + e^{ip^0t - i\mathbf{p}\cdot\mathbf{x}} \zeta_\alpha(\mathbf{p}, h) b^\dagger(\mathbf{p}, h)] \end{aligned} \quad (3.140)$$

which becomes^{††}

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{2mE(\mathbf{p})} \sum_h [\xi_\alpha(\mathbf{p}, h) \bar{\xi}_\beta(\mathbf{p}, h) + \zeta_\alpha(-\mathbf{p}, h) \bar{\zeta}_\beta(-\mathbf{p}, h)]. \quad (3.141)$$

Evaluating the spin sums yields

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{2mE(\mathbf{p})} [m(1 + G(\mathbf{p})_{\alpha\beta}) + (-m(1 - G(-\mathbf{p}))_{\alpha\beta})]. \quad (3.142)$$

Since $-G(-\mathbf{p}) = G(\mathbf{p})$, we see that the above becomes

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{2mE(\mathbf{p})} [m(1 + G(\mathbf{p})_{\alpha\beta}) + -m(1 + G(\mathbf{p}))_{\alpha\beta}] \quad (3.143)$$

which vanishes so we are left with the result that

$$\{\Lambda_\alpha(\mathbf{x}, t), \Pi_\beta(\mathbf{y}, t)\} = i\delta^3(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta} + i \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} G(\mathbf{p}). \quad (3.144)$$

The second term on the right hand side of Eqn. (3.144) in general does not vanish. There is a preferred frame along the z -direction in which this term vanishes. The anti-commutation

^{††}In getting to the previous step to this one, we have collected the terms and the anti-commutation relations between the creation and annihilation operators have been evaluated and the resulting delta functions have been used to evaluate the integral over \mathbf{p}' . Finally, we have made the change of variable $\mathbf{p} \rightarrow -\mathbf{p}$ in the second term and the now common exponential factor has been taken out.

relation $\{\Lambda_\alpha(\mathbf{x}, t), \Lambda_\beta(\mathbf{y}, t)\}$ is the same as before so does not need to be checked. Moving on to the anti-commutator $\{\Pi_\alpha(\mathbf{x}, t), \Pi_\beta(\mathbf{y}, t)\}$, we see that:

$$\begin{aligned} \{\Pi_\alpha(\mathbf{x}, t), \Pi_\beta(\mathbf{y}, t)\} &= \{\dot{\bar{\Lambda}}_\alpha(\mathbf{x}, t), \dot{\bar{\Lambda}}_\beta(\mathbf{y}, t)\} - iqA^0\{\dot{\bar{\Lambda}}_\alpha(\mathbf{x}, t), \bar{\Lambda}_\beta(\mathbf{y}, t)\} \\ &\quad - iqA^0\{\bar{\Lambda}_\alpha(\mathbf{x}, t), \dot{\bar{\Lambda}}_\beta(\mathbf{y}, t)\} - q^2A^{02}\{\bar{\Lambda}_\alpha(\mathbf{x}, t), \bar{\Lambda}_\beta(\mathbf{y}, t)\}. \end{aligned} \quad (3.145)$$

The first anti-commutator is already known to vanish and the fourth one clearly vanishes also, since $\{\Lambda_\alpha(\mathbf{x}, t), \Lambda_\beta(\mathbf{y}, t)\} = 0$. Taking the second of the four anti-commutators yields:

$$\begin{aligned} \int \int \frac{d^3\mathbf{p}d^3\mathbf{p}'}{(2\pi)^6\sqrt{2mEE'}} \sum_{h,h'} [e^{ip' \cdot t - i\mathbf{p}' \cdot \mathbf{y}} e^{ip \cdot t - i\mathbf{p} \cdot \mathbf{x}} \bar{\xi}_\alpha(\mathbf{p}, h) \bar{\xi}_\beta(\mathbf{p}', h') \{a^\dagger(\mathbf{p}, h), a^\dagger(\mathbf{p}', h')\} \\ + 3 \text{ similar terms with vanishing anti-commutators}] = 0 \end{aligned} \quad (3.146)$$

which completes the check.

3.9 Elko and Dirac Symmetry Currents

For a system of N Elko particles the free particle Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \bar{\Lambda}_1 \partial^\mu \Lambda_1 + \cdots + \frac{1}{2} \partial_\mu \bar{\Lambda}_N \partial^\mu \Lambda_N + \frac{1}{2} m^2 \bar{\Lambda}_1 \Lambda_1 + \cdots + \frac{1}{2} m^2 \bar{\Lambda}_N \Lambda_N. \quad (3.147)$$

If we define $\mathbf{\Lambda}$ to be the $N \times 1$ column matrix whose N entries are the N Elko fields, and we vary the Lagrangian, and demand that it be set equal to zero, we get

$$\mathbf{0} = \delta \mathcal{L} = \partial_\mu \left[\sum_{i=1}^N \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Lambda_i)} \delta \Lambda_i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Lambda}_i)} \delta \bar{\Lambda}_i \right) \right]. \quad (3.148)$$

From the Lagrangian, we see that

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Lambda_i)} = \partial^\mu \bar{\Lambda}_i, \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Lambda}_i)} = \partial^\mu \Lambda_i, \quad (3.149)$$

so $\delta \mathcal{L}$ becomes

$$\delta \mathcal{L} = \partial_\mu [(\partial^\mu \bar{\mathbf{\Lambda}}) \delta \mathbf{\Lambda} + \delta \bar{\mathbf{\Lambda}} (\partial^\mu \mathbf{\Lambda})] = 0 \quad (3.150)$$

We now look at a global $U(1)$ phase transformation, with variation in the fields given by

$$\delta \Lambda = -i\alpha \Lambda, \quad \delta \bar{\Lambda} = i\alpha \bar{\Lambda}. \quad (3.151)$$

Putting this into the variation of the Lagrangian density yields:

$$\delta \mathcal{L} = \alpha \partial_\mu [i(\bar{\mathbf{\Lambda}} (\partial^\mu \mathbf{\Lambda}) - (\partial^\mu \bar{\mathbf{\Lambda}}) \mathbf{\Lambda})] = 0. \quad (3.152)$$

We identify the part in the square brackets with the global $U(1)$ symmetry current so we have

$$j_{\text{Free}, \Lambda}^\mu = i(\bar{\mathbf{\Lambda}} (\partial^\mu \mathbf{\Lambda}) - (\partial^\mu \bar{\mathbf{\Lambda}}) \mathbf{\Lambda}). \quad (3.153)$$

3 Elko Fields and Interactions

Recalling the Elko interaction Lagrangian

$$\mathcal{L}_{\text{int}} = -iq[\bar{\Lambda} (\partial^\mu \Lambda) - (\partial^\mu \bar{\Lambda}) \Lambda] A_\mu + q^2 A^\mu A_\mu \bar{\Lambda} \Lambda, \quad (3.154)$$

the local $U(1)$ Elko current is

$$j_{\text{Elko}}^\mu = \frac{-\partial \mathcal{L}}{\partial A_\mu} \quad (3.155)$$

which gives^{§§}

$$j_{\text{Elko}}^\mu = iq[\bar{\Lambda} (\partial^\mu \Lambda) - (\partial^\mu \bar{\Lambda}) \Lambda] - 2q^2 A^\mu \bar{\Lambda} \Lambda. \quad (3.156)$$

The Dirac interaction which arises from local $U(1)$ gauge invariance is [59, p.191]

$$\mathcal{L}_{\text{int}} = -iq\bar{\psi}\gamma^\mu\psi A_\mu \quad (3.157)$$

from which the Dirac local $U(1)$ symmetry current

$$j_{\text{Dirac}}^\mu = \frac{-\partial \mathcal{L}}{\partial A_\mu} \quad (3.158)$$

is immediately seen to be

$$j_{\text{Dirac}}^\mu = iq\bar{\psi}\gamma^\mu\psi. \quad (3.159)$$

We now look at a hypothetical scattering scenario between Elko and Dirac particles via exchange of a common $U(1)$ gauge quantum.

3.10 Elko and Dirac Particle Scattering via Exchange of $U(1)$ Gauge Quanta

In this section we show that, to second order, local $U(1)$ gauge interactions with other particles give rise to invariant amplitudes but the invariance is lost at fourth order due to the Elko time ordered product. We illustrate this point here by considering the scattering of an electron with a hypothetical negatively charged Elko particle via the exchange of a $U(1)$ gauge quantum. When such language as “electromagnetic interactions” or “photon” is used here, the notion of the Standard Model photon is not necessarily meant. For the purposes of this section, we simply take a local $U(1)$ gauge symmetry and assume that both Elko and Dirac particles share this particular gauge quantum but leave it as an open question as to whether it corresponds to the $U(1)$ gauge quantum of the Standard Model or not. The S -matrix elements S_{fi} are given by

$$\langle f | \hat{S} | i \rangle = S_{fi} = \mathcal{A}_{fi}^{(0)} + \mathcal{A}_{fi}^{(1)} + \mathcal{A}_{fi}^{(2)} + \cdots = \sum_{n=0}^{\infty} \mathcal{A}_{fi}^{(n)} \quad (3.160)$$

^{§§}The “ $2A^\mu$ ” comes about as follows: $\frac{\partial}{\partial A_\mu}(A^\nu A_\nu) = g^{\alpha\nu} \frac{\partial}{\partial A_\mu}(A_\alpha A_\nu)$ which, upon using the product rule, gives $g^{\alpha\nu} \left(\frac{\partial A_\alpha}{\partial A_\mu} A_\nu + A_\alpha \frac{\partial A_\nu}{\partial A_\mu} \right)$, which, in turn becomes $g^{\alpha\nu} (\delta_\mu^\alpha A_\nu + \delta_\mu^\nu A_\alpha) = g^{\mu\nu} A_\nu + g^{\mu\alpha} A_\alpha = A^\mu + A^\mu = 2A^\mu$.

3.10 Elko and Dirac Particle Scattering via Exchange of U(1) Gauge Quanta

where

$$\mathcal{A}_{fi}^{(0)} = \langle f | 1 | i \rangle \quad (3.161)$$

$$\mathcal{A}_{fi}^{(1)} = -i d^4 x \langle f | \mathcal{H}_{\text{int}}(x) | i \rangle \quad (3.162)$$

$$\mathcal{A}_{fi}^{(2)} = \frac{-1}{2} \int \int d^4 x_1 d^4 x_2 \langle f | T \{ \mathcal{H}_{\text{int}}(x_1) \mathcal{H}_{\text{int}}(x_2) \} | i \rangle \quad (3.163)$$

etc., and

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int \cdots \int d^4 x_1 d^4 x_2 \cdots d^4 x_n T \{ \mathcal{H}_{\text{int}}(x_1) \mathcal{H}_{\text{int}}(x_2) \cdots \mathcal{H}_{\text{int}}(x_n) \}. \quad (3.164)$$

We here consider the scattering process (here the \mathbf{k} 's are not the same as the standard momentum $k^\mu = (k^0, \mathbf{k})$ given in Chapter 2.)

$$e^-(\mathbf{k}, \sigma) + \Lambda^-(\mathbf{p}, h) \rightarrow e^-(\mathbf{k}', \sigma') + \Lambda^-(\mathbf{p}', h')$$

and wish to calculate the second order scattering amplitude $\mathcal{A}_{fi}^{(2)}$, with the Elko and Dirac electromagnetic symmetry currents being

$$j_{\text{Elko}}^\mu = iq \left[\bar{\Lambda} (\partial^\mu \Lambda) - \left(\partial^\mu \bar{\Lambda} \right) \Lambda \right] - 2q^2 A^\mu \bar{\Lambda} \Lambda \quad \text{and} \quad (3.165)$$

$$j_{\text{Dirac}}^\mu = iq \bar{\psi} \gamma^\mu \psi. \quad (3.166)$$

We will however, neglect the $2q^2 A^\mu \bar{\Lambda} \Lambda$ term of the Elko symmetry current because we are considering the scattering process at order q^2 where this term will not contribute. The total interaction Hamiltonian density for this scattering process at second order is simply

$$\mathcal{H}_{\text{int}}(x) = [j_E^\mu + j_D^\mu] A_\mu = \left[-iq \left[\bar{\Lambda} (\partial^\mu \Lambda) - \left(\partial^\mu \bar{\Lambda} \right) \Lambda \right] - q \bar{\psi} \gamma^\mu \psi \right] A_\mu. \quad (3.167)$$

The fields that will be needed are the Elko fields as well as the Dirac fields

$$\psi(x) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3 \sqrt{2\omega}} \sum_{\sigma} \left[e^{ip \cdot x} u(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{-ip \cdot x} v(\mathbf{p}, \sigma) a^\dagger(\mathbf{p}, \sigma) \right] \quad (3.168)$$

$$\bar{\psi}(x) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3 \sqrt{2\omega}} \sum_{\sigma} \left[e^{-ip \cdot x} \bar{u}(\mathbf{p}, \sigma) a^\dagger(\mathbf{p}, \sigma) + e^{ip \cdot x} \bar{v}(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) \right], \quad (3.169)$$

and the fields for the U(1) gauge quanta,

$$A^\mu(x) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3 \sqrt{2\omega}} \sum_{\lambda} \left[e^{-ik \cdot x} \epsilon^\mu(\mathbf{k}, \lambda) \alpha(\mathbf{k}, \lambda) + e^{ik \cdot x} \epsilon^{*\mu}(\mathbf{k}, \lambda) \alpha^\dagger(\mathbf{k}, \lambda) \right], \quad (3.170)$$

where $\alpha(\mathbf{k}, \lambda)$ is the operator which destroys a photon of three-momentum \mathbf{k} and helicity λ , and $\alpha^\dagger(\mathbf{k}, \lambda)$ is the operator which creates a photon of three-momentum \mathbf{k} and helicity λ , and $\epsilon^\mu(\mathbf{k}, \lambda)$ and $\epsilon^{*\mu}(\mathbf{k}, \lambda)$ are the u and v coefficient functions respectively. The helicity label λ is a two-valued discrete index.

With the following normalization choice:

$$|\mathbf{p}_i, \sigma_i\rangle = \sqrt{2E_i} a_i(\mathbf{p}_i, \sigma_i) |0\rangle \quad (3.171)$$

3 Elko Fields and Interactions

for particle species i with $\sigma_i = \{\pm 1/2, \pm h\}$, the second order amplitude becomes

$$\mathcal{A}^{(2)} = \frac{(-i)^2}{2!} \int \int d^4 x_1 d^4 x_2 \langle 0 | a(\mathbf{p}_2, \sigma_2) c(\mathbf{p}_2, h_2) T \{ \mathcal{H}_{\text{int}}(x_1) \mathcal{H}_{\text{int}}(x_2) \} \times \\ c^\dagger(\mathbf{p}_1, h_1) a^\dagger(\mathbf{p}_1, \sigma_1) | 0 \rangle (16 E_{p_1}^D E_{p_2}^D E_{p_1}^\Lambda E_{p_1}^\Lambda)^{\frac{1}{2}}.$$

Putting the expression for the interaction Hamiltonian density into this and only keeping the surviving terms yields

$$\frac{(-i)^2}{2!} \int \int d^4 x_1 d^4 x_2 \{ \langle \mathbf{p}_2^\Lambda, h_2^\Lambda | j_\Lambda^\mu(x_1) | \mathbf{p}_1^\Lambda, h_1^\Lambda \rangle \langle 0 | T \{ A_\mu(x_1) A_\nu(x_2) \} | 0 \rangle \times \\ \langle \mathbf{p}_2^D, \sigma_2^D | j_D^\nu(x_2) | \mathbf{p}_1^D, \sigma_1^D \rangle + (x_1 \leftrightarrow x_2) \}.$$

Calculating the matrix elements separately gives

$$\langle \mathbf{p}_2^D, \sigma_2^D | j_D^\nu(x_2) | \mathbf{p}_1^D, \sigma_1^D \rangle = -q \bar{u}(\mathbf{p}_2^D, \sigma_2^D) \gamma^\nu u(\mathbf{p}_1^D, \sigma_1^D) e^{-i(p_1^D - p_2^D) \cdot x_2} \quad (3.172)$$

$$\langle \mathbf{p}_2^\Lambda, h_2^\Lambda | j_\Lambda^\mu(x_1) | \mathbf{p}_1^\Lambda, h_1^\Lambda \rangle = -q(p_1^{\Lambda\mu} + p_2^{\Lambda\mu}) e^{-i(p_1^\Lambda - p_2^\Lambda) \cdot x_1} \quad (3.173)$$

$$\langle 0 | T \{ A_\mu(x_1) A_\nu(x_2) \} | 0 \rangle = \left[\frac{i [g_{\mu\nu} + (1 - \xi) q_\mu q_\nu / q^2]}{q^2} \right], \quad (3.174)$$

where the first and third of these matrix elements are just the usual ones for the Dirac symmetry current and photon propagator respectively. The q^2 factors here are momentum factors and are not to be confused with the coupling constant q . The Elko symmetry current of Eqn. (3.156) has exactly the same form as that of the scalar field case. Taking this, and noting that the $(x_1 \leftrightarrow x_2)$ term in the second order amplitude is the same except for the x_1 and x_2 being in different places, performing the x_1 and x_2 integrals and taking advantage of the symmetry of the delta function yields the second order amplitude

$$\mathcal{A}^{(2)} = i(2\pi)^4 \delta^4(p_1^\Lambda + p_2^D - p_1^\Lambda - p_2^D) \mathcal{M}_{e^-, \Lambda^-} \quad (3.175)$$

where

$$i\mathcal{M}_{e^-, \Lambda^-} = (-i)^2 q(p_1^{\Lambda\mu} + p_2^{\Lambda\mu}) \left[\frac{i [g_{\mu\nu} + (1 - \xi) q_\mu q_\nu / q^2]}{q^2} \right] q \bar{u}(\mathbf{p}_2^D, \sigma_2^D) \gamma^\nu u(\mathbf{p}_1^D, \sigma_1^D) \quad (3.176) \\ = (-i)^2 j_{\Lambda^-}^\mu(p_1^\Lambda, p_2^\Lambda) \left[\frac{i [g_{\mu\nu} + (1 - \xi) q_\mu q_\nu / q^2]}{q^2} \right] j_{e^-}^\nu(p_1^D, p_2^D).$$

The second term in the photon propagator is contracted onto the symmetry currents and therefore disappears leaving the invariant amplitude looking more obviously invariant:

$$i\mathcal{M}_{e^-, \Lambda^-} = (-i)^2 j_{\Lambda^-}^\mu(p_1^\Lambda, p_2^\Lambda) \frac{i g_{\mu\nu}}{q^2} j_{e^-}^\nu(p_1^D, p_2^D). \quad (3.177)$$

3.10.1 The Cross-section

In this section, we perform a cross section calculation for a beam of electrons scattering off a hypothetical target made of Elko particles. The cross section is a measure of the interacting cross sectional area as seen by the incoming particles. The unit of a cross section is the unit of area. To illustrate the idea a bit more clearly, the differential cross section element $d\sigma$ means [58, p.385][76]:

$$d\sigma = \frac{\text{scattered electron flux}}{\text{incoming electron flux}} \times \text{cross sectional area of interaction.} \quad (3.178)$$

If we use the *box normalization* procedure, imagining that space has a finite volume V , and also that the interaction is turned on for a finite period of time T , then $(2\pi)^4\delta^4(0)$ is effectively VT [42, sec.3.4][59, sec.6.3]. The transition rate per unit volume is then

$$\dot{P}_{fi} = \frac{|\mathcal{A}_{e^-, \Lambda^-}^{(2)}|^2}{VT} = (2\pi)^4\delta^4(p_1^\Lambda + p_1^D - p_2^\Lambda - p_2^D)|\mathcal{M}_{e^-, \Lambda^-}|^2, \quad (3.179)$$

where

$$\mathcal{A}_{e^-, \Lambda^-}^{(2)} = (2\pi)^4\delta^4(p_1^\Lambda + p_1^D - p_2^\Lambda - p_2^D)i\mathcal{M}_{e^-, \Lambda^-}. \quad (3.180)$$

Dividing through by the number of scattering particles per unit volume will give a normalization independent quantity. Also, if we wish to have a quantity which can be compared from experiment to experiment, the dependence of the transition rate on the incident flux of particles and on the number of target particles per unit volume needs to be removed. Towards this end, we integrate over the number density of final states, consistent with energy-momentum conservation. We choose the normalization so that there are $2E_i$ particles of species i per unit volume. If we choose the incident beam of particles to be the electrons with an incident flux of $2E_{e^-}|\mathbf{v}|$, where $|\mathbf{v}|$ is the velocity of the incoming electrons, and if we choose the number of target Elko particles per unit volume to be $2E_{\Lambda^-}$, then the number density of final states, which is the number of final states per particle in momentum space around the momenta \mathbf{p}_2^D and \mathbf{p}_2^Λ , is given by

$$d\mathcal{N}_f = \frac{d^3\mathbf{p}_2^D}{(2\pi)^3 2E_2^D} \frac{d^3\mathbf{p}_2^\Lambda}{(2\pi)^3 2E_2^\Lambda} \quad (3.181)$$

and the cross section $\sigma = \int d\sigma$ is then given by:

$$\frac{(2\pi)^4}{(2E_{\Lambda^-})(2E_{e^-})|\mathbf{v}|} \int \int \delta^4(p_1^\Lambda + p_1^D - p_2^\Lambda - p_2^D)|\mathcal{M}_{e^-, \Lambda^-}|^2 \frac{d^3\mathbf{p}_2^D}{(2\pi)^3 2E_2^D} \frac{d^3\mathbf{p}_2^\Lambda}{(2\pi)^3 2E_2^\Lambda}. \quad (3.182)$$

If we define the Lorentz invariant phase space by

$$d\text{Lips}(e^-, p_2^\Lambda, p_2^D) = \frac{1}{(4\pi)^2} \delta^4(p_1^\Lambda + p_1^D - p_2^\Lambda - p_2^D) \frac{d^3\mathbf{p}_2^D}{E_2^D} \frac{d^3\mathbf{p}_2^\Lambda}{E_2^\Lambda}, \quad (3.183)$$

the cross-section then becomes

$$\sigma = \frac{1}{4E_{\Lambda^-}E_{e^-}|\mathbf{v}|} \int |\mathcal{M}_{e^-, \Lambda^-}|^2 d\text{Lips}(E^-; P_2^\Lambda, p_2^D). \quad (3.184)$$

3 Elko Fields and Interactions

Next, the flux factor can be written in an invariant form. Since we can write

$$E_{e^-} = \frac{m_{e^-}}{\sqrt{1 - \mathbf{v}^2}}, \quad (3.185)$$

it follows by rearranging that

$$\mathbf{v}^2 = 1 - \frac{m_{e^-}^2}{E_{e^-}^2}, \quad (3.186)$$

so the flux factor becomes

$$E_{\Lambda^-}^2 E_{e^-}^2 |\mathbf{v}|^2 = E_{\Lambda^-}^2 E_{e^-}^2 - E_{\Lambda^-}^2 m_{e^-}^2. \quad (3.187)$$

If we take $E_{\Lambda^-} = m_{e^-}$ then $E_{\Lambda^-} E_{e^-} = p_{\Lambda^-} \cdot p_{e^-}$ so we get

$$E_{\Lambda^-} E_{e^-} |\mathbf{v}| = [(p_{\Lambda^-} \cdot p_{e^-})^2 - m_{\Lambda^-}^2 m_{e^-}^2]^{\frac{1}{2}}, \quad (3.188)$$

so the cross section becomes

$$\sigma = \frac{1}{4[(p_{\Lambda^-} \cdot p_{e^-})^2 - m_{\Lambda^-}^2 m_{e^-}^2]^{\frac{1}{2}}} \int |\mathcal{M}|^2 d\text{Lips}(e^-; p_2^\Lambda, p_2^D). \quad (3.189)$$

Moving to the CM frame, we have

$$\mathbf{p}_1^\Lambda + \mathbf{p}_1^D = \mathbf{0} = \mathbf{p}_2^\Lambda + \mathbf{p}_2^D \quad (3.190)$$

so the $d^3 \mathbf{p}_2^\Lambda$ integral becomes

$$\int \frac{d^3 \mathbf{p}_2^\Lambda}{E_2^\Lambda} \delta^4(p_1^\Lambda + p_1^D - p_2^\Lambda - p_2^D) = \frac{1}{E_2^\Lambda} \delta(E_1^\Lambda + E_1^D - E_2^\Lambda - E_2^D). \quad (3.191)$$

Our Lorentz invariant phase space now ends up being

$$\frac{1}{(4\pi)^2} \delta(E_1^\Lambda + E_1^D - E_2^\Lambda - E_2^D) \frac{d^3 \mathbf{p}_2^D}{E_2^D E_2^\Lambda}. \quad (3.192)$$

By observing that

$$E_2^D = (m_2^{D2} + \mathbf{p}_2^{D2})^{\frac{1}{2}} \quad (3.193)$$

from which it follows that

$$\frac{dE_2^D}{d|\mathbf{p}_2^D|} = \frac{|\mathbf{p}_2^D|}{E_2^D}, \quad (3.194)$$

rearranging yields

$$E_2^D dE_2^D = |\mathbf{p}_2^D| d|\mathbf{p}_2^D|. \quad (3.195)$$

Re-writing $d^3 \mathbf{p}_2^D$ in terms of angular variables as

$$d^3 \mathbf{p}_2^D = \mathbf{p}_2^{D2} d|\mathbf{p}_2^D| d\Omega \quad (3.196)$$

and substituting Eqn. (3.196) into Eqn. (3.192) gives

$$\frac{1}{(4\pi)^2} \delta(E_1^\Lambda + E_1^D - E_2^\Lambda - E_2^D) \frac{\mathbf{p}_2^{D2} d|\mathbf{p}_2^D| d\Omega}{E_2^D E_2^\Lambda}. \quad (3.197)$$

3.10 Elko and Dirac Particle Scattering via Exchange of $U(1)$ Gauge Quanta

Now we substitute Eqn. (3.195) into Eqn. (3.197) to obtain

$$d\text{Lips}(e^-; p_2^\Lambda, p_2^D) = \frac{|\mathbf{p}_2^D| dE_2^D}{(4\pi)^2 E_2^\Lambda} d\Omega \delta(E_1^\Lambda + E_1^D - E_2^\Lambda - E_2^D). \quad (3.198)$$

In order to further rewrite the Lorentz invariant phase space, we observe that in the CM frame

$$\mathbf{p}_1^D = \mathbf{p} = -\mathbf{p}_1^\Lambda, \quad \mathbf{p}_2^D = \mathbf{p}_2 = -\mathbf{p}_2^\Lambda \quad (3.199)$$

and

$$E_2^D dE_2^D = |\mathbf{p}_2| d|\mathbf{p}_2| = E_2^\Lambda dE_2^\Lambda \quad (3.200)$$

and define

$$W \equiv E_2^D + E_2^\Lambda \rightarrow dW = dE_2^D + dE_2^\Lambda, \quad (3.201)$$

from which it follows that

$$dW = \frac{|\mathbf{p}_2^D| d|\mathbf{p}_2^D|}{E_2^D} + \frac{|\mathbf{p}_2^\Lambda| d|\mathbf{p}_2^\Lambda|}{E_2^\Lambda} = \frac{W_2 |\mathbf{p}_2| d|\mathbf{p}_2|}{E_2^D E_2^\Lambda} = \frac{W_2 dE_2^D}{E_2^\Lambda} \quad (3.202)$$

so the Lorentz invariant phase space now becomes

$$d\text{Lips}(e^-; p_2^\Lambda, p_2^D) = \frac{|\mathbf{p}_2| dW_2}{(4\pi)^2 W_2} \delta(W_1 - W_2) d\Omega. \quad (3.203)$$

Performing the W_2 integral gives

$$d\text{Lips}(e^-; p_2^\Lambda, p_2^D) = \frac{|\mathbf{p}_1|}{(4\pi)^2 W_1} d\Omega \quad (3.204)$$

so now the cross section reads

$$\sigma = \int \frac{1}{4[(p_{\Lambda^-} \cdot p_{e^-})^2 - m_{\Lambda^-}^2 m_{e^-}^2]^{\frac{1}{2}}} \frac{|\mathbf{p}_1| |\mathcal{M}^2|}{(4\pi)^2 W_1} d\Omega. \quad (3.205)$$

Before writing the cross-section in its final form, we note that since

$$p_1^D \cdot p_1^\Lambda = E_1^D E_1^\Lambda + \mathbf{p}^2, \quad (3.206)$$

it follows that

$$\begin{aligned} (p_1^D \cdot p_1^\Lambda)^2 - m_1^{D2} m_1^{\Lambda2} &= (E_1^D E_1^\Lambda + \mathbf{p}^2)(E_1^D E_1^\Lambda + \mathbf{p}^2) - m_1^{D2} m_1^{\Lambda2} = \\ &= (E_1^D E_1^\Lambda)^2 + 2E_1^D E_1^\Lambda + \mathbf{p}^4 - [E_1^{D2} - \mathbf{p}^2][E_1^{\Lambda2} - \mathbf{p}^2] = \\ &= (E_1^D E_1^\Lambda)^2 + 2E_1^D E_1^\Lambda + \mathbf{p}^4 - (E_1^D E_1^\Lambda)^2 + E_1^{D2} \mathbf{p}^2 + E_1^{\Lambda2} \mathbf{p}^2 - \mathbf{p}^4 \\ &= (E_1^D + E_1^\Lambda)^2 \mathbf{p}^2 = W_1^2 \mathbf{p}^2. \end{aligned} \quad (3.207)$$

With this observation made, the cross-section now becomes

$$\sigma = \int \frac{1}{(8\pi W_1)^2} |\mathcal{M}|^2 d\Omega. \quad (3.208)$$

3 Elko Fields and Interactions

It follows directly, that the differential cross-section in the CM frame is given by

$$\left. \frac{d\sigma}{d\Omega} \right|_{CM} = \frac{1}{(8\pi W_1)^2} |\mathcal{M}|^2. \quad (3.209)$$

It just remains to evaluate the invariant amplitude

$$|\mathcal{M}|^2 \rightarrow \frac{1}{2} \sum_{\sigma_1, \sigma_2} |\mathcal{M}_{e^-, \Lambda^-}(\sigma_1, \sigma_2)|^2 \quad (3.210)$$

which is given by

$$\left(\frac{e^2}{q^2} \right)^2 \frac{1}{2} \sum_{\sigma_1, \sigma_2} [\bar{u}(\mathbf{p}_2^D, \sigma_2^D) \gamma^\mu u(\mathbf{p}_1^D \sigma_1^D)] [\bar{u}(\mathbf{p}_2^D, \sigma_2^D) \gamma^\nu u(\mathbf{p}_1^D \sigma_1^D)]^\dagger (p_1^\Lambda + p_2^\Lambda)_\mu (p_1^\Lambda + p_2^\Lambda)_\nu. \quad (3.211)$$

Firstly, the factor with the dagger can be re-written by observing that

$$[\bar{u}(\mathbf{p}_2^D, \sigma_2^D) \gamma^\nu u(\mathbf{p}_1^D \sigma_1^D)]^\dagger = u^\dagger(\mathbf{p}_1^D, \sigma_1^D) \gamma^{\nu\dagger} \gamma^0 u(\mathbf{p}_2^D, \sigma_2^D) \quad (3.212)$$

and inserting $1 = (\gamma^0)^2$ to obtain

$$u^\dagger(\mathbf{p}_1^D, \sigma_1^D) \gamma^0 \gamma^0 \gamma^{\nu\dagger} \gamma^0 u(\mathbf{p}_2^D, \sigma_2^D) = \bar{u}(\mathbf{p}_1^D, \sigma_1^D) \gamma^\nu u(\mathbf{p}_2^D, \sigma_2^D). \quad (3.213)$$

The invariant amplitude therefore becomes

$$\left(\frac{e^2}{q^2} \right)^2 \frac{1}{2} \sum_{\sigma_1, \sigma_2} [\bar{u}(\mathbf{p}_2^D, \sigma_2^D) \gamma^\mu u(\mathbf{p}_1^D \sigma_1^D) \bar{u}(\mathbf{p}_1^D, \sigma_1^D) \gamma^\nu u(\mathbf{p}_2^D, \sigma_2^D)] (p_1^\Lambda + p_2^\Lambda)_\mu (p_1^\Lambda + p_2^\Lambda)_\nu. \quad (3.214)$$

The sum over the factor in square brackets, in terms of matrix components becomes

$$\begin{aligned} & \sum_{\sigma_1^D, \sigma_2^D} \bar{u}_\alpha(\mathbf{p}_2^D, \sigma_2^D) (\gamma^\mu)_{\alpha\beta} (\gamma_\rho p_1^{\rho D} + m)_{\beta\gamma} (\gamma^\nu)_{\gamma\delta} u_\delta(\mathbf{p}_2^D, \sigma_2^D) \\ &= (\gamma_\rho p_2^{\rho D} + m)_{\delta\alpha} (\gamma^\mu)_{\alpha\beta} (\gamma_\rho p_1^{\rho D} + m)_{\beta\gamma} (\gamma^\nu)_{\gamma\delta} \\ &= \sum_\delta [(\gamma_\rho p_2^{\rho D} + m) \gamma^\mu (\gamma_\lambda p_1^{\lambda D} + m) \gamma^\nu]_{\delta\delta} \\ &= \text{Tr}[(\gamma_\rho p_2^{\rho D} + m) \gamma^\mu (\gamma_\lambda p_1^{\lambda D} + m) \gamma^\nu]. \end{aligned} \quad (3.215)$$

Using the following trace theorems [59, p.379–382]:

$$\text{Tr}1 = 4 \quad (3.216)$$

$$\text{Tr}(\text{odd number of } \gamma\text{'s}) = 0 \quad (3.217)$$

$$\text{Tr}(\gamma_\mu a^\mu \gamma_\nu b^\nu) = 4(a \cdot b) \quad (3.218)$$

$$\text{Tr}(\gamma_\mu a^\mu \gamma_\nu b^\nu \gamma_\rho c^\rho \gamma_\lambda d^\lambda) = 4[(a \cdot b)(c \cdot d) + (a \cdot d)(b \cdot c) - (a \cdot c)(b \cdot d)] \quad (3.219)$$

the second rank contravariant tensor becomes

$$\frac{1}{2} \left(\text{Tr}[\gamma_\alpha p_2^{\alpha D} \gamma^\mu \gamma_\beta p_1^{\beta D} \gamma^\nu] + \text{Tr}[\gamma_\alpha p_2^{\alpha D} \gamma^\mu m \gamma^\nu] \right) \quad (3.220)$$

3.10 Elko and Dirac Particle Scattering via Exchange of $U(1)$ Gauge Quanta

$$\begin{aligned}
& +\text{Tr}[m\gamma^\mu\gamma_\alpha p_1^{\alpha D}m\gamma^\nu] + \text{Tr}[m^2\gamma^\mu\gamma^\nu]) \\
& = \frac{1}{2}\left(\text{Tr}[\gamma_\alpha p_2^{\alpha D}\gamma^\mu\gamma_\beta p_1^{\beta D}\gamma^\nu] + m^2\text{Tr}[\gamma^\mu\gamma^\nu]\right) \\
& = 2[p_2^{D\mu}p_1^{D\mu} + p_2^{D\nu}p_1^{D\mu} - (p_2^D \cdot p_1^D)g^{\mu\nu}] + 2m^2g^{\mu\nu}.
\end{aligned}$$

By defining

$$q^2 = (p_1^D - p_2^D)^2 = 2m_{e^-}^2 - 2p_1^D \cdot p_2^D, \quad (3.221)$$

the tensor becomes

$$2\left[p_2^{D\mu}p_1^{D\nu} + p_2^{D\nu}p_1^{D\mu} + \left(\frac{q^2}{2}\right)g^{\mu\nu}\right]. \quad (3.222)$$

To evaluate the invariant amplitude, we now just have to contract this with the second rank covariant tensor given by

$$(p_1^\Lambda + p_2^\Lambda)_\mu(p_1^\Lambda + p_2^\Lambda)_\nu. \quad (3.223)$$

Doing this yields

$$\begin{aligned}
& 2[p_2^{D\mu}p_1^{D\mu} + p_2^{D\nu}p_1^{D\mu} + \left(\frac{q^2}{2}\right)g^{\mu\nu}](p_1^\Lambda + p_2^\Lambda)_\mu(p_1^\Lambda + p_2^\Lambda)_\nu \\
& = 2[p_2^{D\mu}p_2^{D\nu} + p_2^{D\nu}p_1^{D\mu}][p_{1\mu}^\Lambda p_{1\nu}^\Lambda + p_{1\mu}^\Lambda p_{2\nu}^\Lambda + p_{2\mu}^\Lambda p_{1\nu}^\Lambda + p_{2\mu}^\Lambda p_{2\nu}^\Lambda] \\
& \quad + q^2 g^{\mu\nu}[p_{1\mu}^\Lambda p_{1\nu}^\Lambda p_{1\mu}^\Lambda p_{2\nu}^\Lambda + p_{2\mu}^\Lambda p_{1\nu}^\Lambda + p_{2\mu}^\Lambda p_{2\nu}^\Lambda] = \\
& 2[(p_2^D \cdot p_1^\Lambda)(p_1^D \cdot p_1^\Lambda) + (p_2^D \cdot p_1^\Lambda)(p_1^D \cdot p_2^\Lambda) + (p_2^D \cdot p_2^\Lambda)(p_1^D \cdot p_1^\Lambda) + (p_2^D \cdot p_2^\Lambda)(p_1^D \cdot p_2^\Lambda) + \\
& (p_1^D \cdot p_1^\Lambda)(p_2^D \cdot p_1^\Lambda) + (p_1^D \cdot p_1^\Lambda)(p_2^D \cdot p_2^\Lambda) + (p_1^D \cdot p_2^\Lambda)(p_2^D \cdot p_1^\Lambda) + (p_1^D \cdot p_2^\Lambda)(p_2^D \cdot p_2^\Lambda) \\
& + q^2[p_{1\mu}^\Lambda p_1^{\Lambda\mu} + p_{1\mu}^\Lambda p_2^{\Lambda\mu} + p_{2\mu}^\Lambda p_1^{\Lambda\mu} p_{2\mu}^\Lambda p_2^{\Lambda\mu}]] \\
& = 2[8(p_1^\Lambda \cdot p_1^D)(p_1^\Lambda \cdot p_2^D)] + 4q^2 m_\Lambda^2 \\
& = 8[2(p_1^\Lambda \cdot p_1^D)(p_1^\Lambda \cdot p_2^D) + \left(\frac{q^2}{2}\right)m_\Lambda^2]
\end{aligned} \quad (3.224)$$

since $p_2^D \cdot p_2^\Lambda = p_1^D \cdot p_1^\Lambda$, $p_1^\Lambda \cdot p_2^D = p_1^D \cdot p_2^\Lambda$ and $p_1^{\Lambda 2} = p_2^{\Lambda 2} = m_\Lambda^2$. Finally, if we define

$$\alpha^2 = \frac{e^2}{4\pi} \quad (3.225)$$

the expression for the differential cross-section in the center of mass frame becomes

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{2\alpha^2}{W_1^2 q^4} \left[2(p_1^\Lambda \cdot p_1^D)(p_1^\Lambda \cdot p_2^D) + \frac{q^2}{2}m_\Lambda^2\right]. \quad (3.226)$$

The invariant amplitude at second order, is just that, invariant. We now look at an example of a fourth order correction which makes up part of a fourth order amplitude $\mathcal{A}_{fi}^{(4)}$.

3.10.2 Problems With Elko at the Loop Correction Level

Here, we consider just a single contribution to a Feynman tree diagram for Λ^+, Λ^- scattering via the exchange of a photon, with a 1 loop correction where one branch of the loop is a virtual Λ^+ Elko particle and the other branch of the loop, a virtual Λ^- Elko particle. The contribution here considered is

$$\begin{aligned}
 & (-ig)^4 \int \int \int \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i(p'_{\Lambda^-} - p_{\Lambda^+}) \cdot x_1} e^{i(p'_{\Lambda^+} - p_{\Lambda^-}) \cdot x_2} \times \\
 & \langle 0 | T \{ A(x_1) A(x_3) \} | 0 \rangle \langle 0 | T \{ A(x_2) A(x_4) \} | 0 \rangle \times \\
 & \langle 0 | T \{ \Lambda^+(x_3) \bar{\Lambda}^+(x_4) \} | 0 \rangle \langle 0 | T \{ \Lambda^-(x_3) \bar{\Lambda}^-(x_4) \} | 0 \rangle,
 \end{aligned} \tag{3.227}$$

where p'_{Λ^-} is the outgoing Λ^- particle momentum, p_{Λ^+} is the incoming Λ^+ momentum, p'_{Λ^+} is the outgoing Λ^+ momentum and p_{Λ^-} is the incoming Λ^- momentum. Each propagator is a function of coordinate differences so we here introduce coordinates $x = x_1 - x_3$, $y = x_2 - x_4$, $z = x_3 - x_4$ and the center of mass coordinate $X = \frac{1}{4}(x_1 + x_2 + x_3 + x_4)$. The Jacobian of the coordinate transformation is unity. If we introduce the usual shorthand notation for the propagators, namely $D_A(x)$ etc, we get

$$\begin{aligned}
 & (-ig)^4 \int \int \int \int d^4X d^4x d^4y d^4z e^{i(p'_{\Lambda^-} + p'_{\Lambda^+} - p_{\Lambda^-} - p_{\Lambda^+}) \cdot X} e^{i(p'_{\Lambda^-} - p_{\Lambda^+}) \cdot (3x - y + 2z)/4} \\
 & \times e^{i(p'_{\Lambda^+} - p_{\Lambda^-}) \cdot (-x + 3y - 2z)/4} D_A(x) D_A(y) D_{\Lambda^-}(z) D_{\Lambda^+}(z).
 \end{aligned} \tag{3.228}$$

Integrating over X and setting $q = p'_{\Lambda^-} - p_{\Lambda^+} = p_{\Lambda^-} - p'_{\Lambda^+}$ yields

$$\begin{aligned}
 & (-ig)^4 (2\pi)^4 \delta^4(p'_{\Lambda^-} + p'_{\Lambda^+} - p_{\Lambda^-} - p_{\Lambda^+}) \int \int \int d^4x d^4y d^4z e^{iq \cdot x} \\
 & \times D_A(x) e^{-iq \cdot y} D_A(y) D_{\Lambda^+}(z) D_{\Lambda^-}(z).
 \end{aligned} \tag{3.229}$$

The integrals over x and y give the Fourier transforms of the photon propagators. The remaining factor, which represents the loop, is given by

$$(-ig)^2 \int d^4z e^{iq \cdot z} D_{\Lambda^+}(z) D_{\Lambda^-}(z). \tag{3.230}$$

To make the non-covariance of this term more explicit, this becomes

$$\begin{aligned}
 & (-ig)^2 \int d^4z e^{iq \cdot z} \int \int e^{-i(p_1 + p_2) \cdot z} \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{1 + G(\mathbf{p}_1)}{p_1^2 - m_{\Lambda^+}^2 + i\epsilon} \frac{1 + G(\mathbf{p}_2)}{p_2^2 - m_{\Lambda^-}^2 + i\epsilon} \\
 & = (-ig)^2 (2\pi)^4 \delta^4(p_1 + p_2 - q) \int \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{1 + G(\mathbf{p}_1)}{p_1^2 - m_{\Lambda^+}^2 + i\epsilon} \frac{1 + G(\mathbf{p}_2)}{p_2^2 - m_{\Lambda^-}^2 + i\epsilon} \\
 & = (-ig)^2 \int \frac{d^4p}{(2\pi)^4} \frac{1 + G(\mathbf{p}_1)}{p^2 - m_{\Lambda^+}^2 + i\epsilon} \frac{1 + G(\mathbf{p}_2)}{(q - p)^2 - m_{\Lambda^-}^2 + i\epsilon},
 \end{aligned}$$

which is not manifestly Lorentz covariant.

Simply having non-covariant terms in the propagator is not enough to know whether the physical observables are Lorentz invariant. Before commenting further on the non-manifest

3.10 Elko and Dirac Particle Scattering via Exchange of $U(1)$ Gauge Quanta

Lorentz invariance of the Elko propagator, we pause to review a well known similar situation in the spin-1 massive vector field case. This review is based on [42, sec.6,2] but with the metric $\text{diag}(+1, -1, -1, -1)$ consistent with that used in the Elko papers cited in this thesis.

The propagator for Dirac type fermions, has the general form

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{\sigma} \theta(x-y) u_{\ell}(\mathbf{p}, \sigma, n) u_m^*(\mathbf{p}, \sigma, n) e^{ip \cdot (x-y)} - \theta(y-x) v_m^*(\mathbf{p}, \sigma, n) v_{\ell}(\mathbf{p}, \sigma, n) e^{-ip \cdot (x-y)}. \quad (3.231)$$

where $\theta(x-y)$ is the theta function [59, p.363]. Also, in general the spin sums have the form

$$\sum_{\sigma} u_{\ell}(\mathbf{p}, \sigma, n) u_m^*(\mathbf{p}, \sigma, n) = \frac{1}{2\sqrt{\mathbf{p}^2 + m_n^2}} P_{\ell m}(\mathbf{p}, \sqrt{\mathbf{p}^2 + m_n^2}) \quad (3.232)$$

$$\sum_{\sigma} v_{\ell}(\mathbf{p}, \sigma, n) v_m^*(\mathbf{p}, \sigma, n) = \frac{-1}{2\sqrt{\mathbf{p}^2 + m_n^2}} P_{\ell m}(-\mathbf{p}, -\sqrt{\mathbf{p}^2 + m_n^2}) \quad (3.233)$$

where $P_{\ell m}(\mathbf{p}, \sqrt{\mathbf{p}^2 + m_n^2})$ is a polynomial depending on the spin of the particle. For standard Dirac type massive spin-1/2 particles we have

$$P_{\ell m} = [(-i\gamma^{\mu} p_{\mu} + m)\beta]_{\ell m} \quad (3.234)$$

and for massive spin-1 bosons,

$$P_{\mu\nu}(p) = \eta_{\mu\nu} + \frac{1}{m^2} p_{\mu} p_{\nu}. \quad (3.235)$$

The polynomial $P_{\ell m}(\mathbf{p}, \sqrt{\mathbf{p}^2 + m_n^2})$ is only defined on the mass shell with $p^0 = \sqrt{\mathbf{p}^2 + m^2}$. In a propagator calculation we integrate over all four-momenta so a more general polynomial is needed. By observing that

$$(p^0)^{2\nu} = (\mathbf{p}^2 + m^2) \quad \text{and} \quad (p^0)^{2\nu+1} = p^0(\mathbf{p}^2 + m^2)^{\nu}, \quad (3.236)$$

Weinberg defines the more general polynomials $P^{(L)}(p)$, such that

$$P^{(L)}(p) = P(p) \quad \text{when} \quad p^0 = \sqrt{\mathbf{p}^2 + m^2} \quad (3.237)$$

and

$$P^{(L)}(q) = P^{(0)}(\mathbf{q}) + q^0 P^{(1)}(\mathbf{q}) \quad \text{otherwise.} \quad (3.238)$$

A little bit more detail concerning the different parts of the generalized polynomial $P^{(L)}(q)$ will be touched on after Eq. (3.240).

A standard calculation of the massive spin-1 propagator using the theta function $\theta(t)$ integral representation

$$\theta(t) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-ist}}{s + i\epsilon} ds \quad (3.239)$$

and inserting the change of variables $\mathbf{q} = \mathbf{p}$, $q^0 = p^0 + s$ into the first term in Eqn. (3.231) gives the propagator to be

$$\frac{1}{(2\pi)^4} \int \frac{P_{\ell m}^{(L)}(q) e^{iq \cdot (x-y)}}{q^2 - m^2 + i\epsilon} d^4 q. \quad (3.240)$$

3 Elko Fields and Interactions

When $p^2 = m^2$, the polynomial $P^{(L)}(q)$ is Lorentz invariant. The integration is over all q^μ , the result being that the expression only preserves its Lorentz invariance if the polynomial is linear in q^0 and also linear in each q^i . For the Dirac case $P_{\ell m}^{(L)}(q) = P_{\ell m}(q)$ but for the massive spin-1 vector field case we have

$$P_{\mu\nu}(q) = \eta_{\mu\nu} + \frac{1}{m^2} q_\mu q_\nu \quad (3.241)$$

which is quadratic in q_0 so we take the component $P_{00}(q)$. In order to add an extra term to $P_{\mu\nu}(q)$ to get an expression for $P_{\mu\nu}^{(L)}(q)$, the extra term must cancel the $(q_0)^2$ term in $P_{00}(q)$ and also vanish when q^μ is on the mass shell. Hence we have

$$P_{\mu\nu}^{(L)}(q) = \eta_{\mu\nu} + \frac{1}{m^2} q_\mu q_\nu - m^2 \delta_\mu^0 \delta_\nu^0 (q^2 - m^2) = P_{\mu\nu}(q) - m^2 (q^2 - m^2) \delta_\mu^0 \delta_\nu^0. \quad (3.242)$$

Thus, the massive spin-1 propagator becomes

$$\frac{1}{(2\pi)^4} \int \frac{P_{\mu\nu}(q) e^{iq \cdot (x-y)}}{q^2 - m^2 + i\epsilon} - \frac{m^2}{(2\pi)^4} \int \frac{(q^2 - m^2) \delta_\mu^0 \delta_\nu^0 e^{iq \cdot (x-y)}}{q^2 - m^2 + i\epsilon} d^4 q \quad (3.243)$$

which simplifies to

$$\frac{1}{(2\pi)^4} \int \frac{P_{\mu\nu}(q) e^{iq \cdot (x-y)}}{q^2 - m^2 + i\epsilon} - m^2 \delta^4(x-y) \delta_\mu^0 \delta_\nu^0. \quad (3.244)$$

The second term in the propagator here is non-covariant. The essential observation here however, is that it is *local*, containing a $\delta^4(x-y)$ factor.

Weinberg explains in [42, p.278], that in such cases, the non-covariant term can be canceled by adding a non-covariant local term to the Hamiltonian density. Weinberg points out that if the interaction between a vector field $V_\mu(x)$ and other fields happens via a coupling with a symmetry current $J^\mu(x)$, then the non-covariant term in the propagator yields an interaction term

$$-i\mathcal{H}_{\text{eff}}(x) = \frac{1}{2} [-iJ^\mu(x)] [-iJ^\nu(x)] \left[\frac{-i}{m^2} \delta_\mu^0 \delta_\nu^0 \right]. \quad (3.245)$$

The Hamiltonian density in its entirety now has the general form

$$\mathcal{H}(x) = \dots + \mathcal{H}_{\text{eff}}. \quad (3.246)$$

We are free to add a non-covariant local term \mathcal{H}_{NC} to the Hamiltonian to get

$$\mathcal{H}(x) = \dots + \mathcal{H}_{\text{eff}} + \mathcal{H}_{\text{NC}}. \quad (3.247)$$

If we choose

$$\mathcal{H}_{\text{NC}} = -\mathcal{H}_{\text{eff}} = \frac{1}{2m^2} [J^0(x)]^2 \quad (3.248)$$

the result is a Lorentz invariant S -matrix [42, sec.7.5].

The Elko propagator is proportional to

$$\int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot (x-y)} \left[\frac{1}{p_\mu p^\mu - m^2 + i\epsilon} \right] + \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot (x-y)} \left[\frac{G(\mathbf{p})}{p_\mu p^\mu - m^2 + i\epsilon} \right]. \quad (3.249)$$

3.10 Elko and Dirac Particle Scattering via Exchange of $U(1)$ Gauge Quanta

In [49, sec.2.6] the spacelike unit four-vector $g^\mu = (0, \mathbf{g})$ is introduced where $\mathbf{g} = (\sin(\phi), -\cos(\phi), 0)$ so that $g_\mu p^\mu = 0$. The matrix $G(\mathbf{p})$ is related to the four-vector g^μ by

$$G(\mathbf{p}) = \gamma^5 \gamma_\mu g^\mu. \quad (3.250)$$

The appearance of g^μ makes clear the presence of a preferred axis. In order for the second term in the Elko propagator to disappear, the vector $\mathbf{x} - \mathbf{y}$ must be aligned along the z -axis, and it need not be demanded that $x - y = 0$ everywhere except at a point. This is why [49] speaks of an axis of locality. In general, the second term in the Elko propagator is non-vanishing and non-local. We cannot add a non-local term to the Hamiltonian density to compensate so the problem of a lack of manifest Lorentz invariance at the loop correction level which contains the Elko propagator remains.

The presence of these $G(\mathbf{p})$ functions in the Elko “propagator” also puts question marks over what the free particle Elko Lagrangian density should be in the first place. It becomes a natural priority to try to answer the question *can the Elko quantum field be derived from considerations of the state space?* We examine this question now, in Chapter 4.

4 Elko Fields and the Weinberg Formalism

4.1 Introduction

At this point we wish to comment on two different perspectives in Quantum Field Theory which are related to each other. Historically, the search for quantum fields tended to be undertaken by starting off with a classical equation of motion, looking for classical solutions to the classical field equation, and thinking about how to quantize the classical solution. Plane wave solutions to the classical equations would be found and then the coefficients were “promoted” to being operators, otherwise known as the creation and annihilation operators.

This approach to setting up quantum fields is radically different to the approach given by Weinberg, who, by contrast, focuses directly on the Hilbert space of physical states from the start. As shown in Chapter 2, the quantum fields naturally arise from considering what the Hilbert space must look like under the Poincaré group and considering what the locality of physical events implies for the Hilbert space of physical states.

When taking Weinberg’s perspective on setting up quantum fields, we believe that it is clear what is needed to have a quantum field, namely, a Hilbert space, a (anti) unitary representation of the Poincaré group on the Hilbert space, and a finite-dimensional representation of the strict Poincaré group. There is an interplay between the two representations, the details of which are explicitly manifest in Weinberg’s formalism.

This interplay between the two representations which need to work together in an appropriate way to define quantum fields would be more obscure if one were to take the usual approach of starting with a classical field equation with classical plane wave solutions and then quantizing the classical fields. That approach makes it difficult to think about the transformation properties of the field operators on the state space.

The Elko field was postulated based on the transformation properties of the Elko spinors under the finite-dimensional strict Poincaré group. The transformation properties of the overall Elko quantum field itself were obscure for a long time.

In this chapter, we turn our attention to the transformation properties of the Elko quantum field under the Poincaré spacetime symmetries and we think about how the Elko quantum field can fit into the general structure as set up by Weinberg. This is where the advantages of Weinberg’s formalism become clearer since the Weinberg formalism makes explicit the interplay between the two representations which is needed to both satisfy the Cluster Decomposition Principle and preserve the Lorentz invariance of the S-matrix.

We finish the chapter by considering Elko causality issues and also the Elko cousin field which was brought about by considering the symmetries of Very Special Relativity. We argue based on the method of induced representations that such a field does not arise, at least in the usual way one approaches Quantum Field Theory.

4.2 Elko not a Quantum Field in the Sense of Weinberg

In this section we investigate whether the Elko field can be interpreted as a quantum field in the sense of Weinberg. We explain how Eqn. (2.133) and Eqn. (2.134) given near the end of Sec. (2.13), are used to give the Dirac rest spinors $u(\mathbf{0}, \sigma)$ and $v(\mathbf{0}, \sigma)$. We then show that these equations are incompatible with Elko rest spinors. This section is a slightly more expanded form of [47][48].

We assume, as in [32], that the finite-dimensional representation $D_{\ell\bar{\ell}}(\Lambda)$ is the chiral representation of the restricted Lorentz group $D_{\ell\bar{\ell}}^{\text{ch}}(\Lambda)$ [72, p.41]. We have already given the matrices $D_{\ell\bar{\ell}}(\Lambda)$ for the case where Λ is a boost, being $\kappa^{(\frac{1}{2}, 0)} \oplus \kappa^{(0, \frac{1}{2})}$. The rotation matrices are $e^{i\frac{\sigma}{2} \cdot \boldsymbol{\theta}} \oplus e^{i\frac{\sigma}{2} \cdot \boldsymbol{\theta}}$, which may be directly inferred from [7, p.38]. The state space H and the unitary representation $U(\Lambda, a)$ of \mathbb{P}^0 are not given explicitly in [32], so we need to specify them. Under the assumption that $U(\Lambda, a)$ is irreducible, H and $U(\Lambda, a)$ must be of the form given in Sec. (2.5) for some spin- j and some irreducible representation $D_{\bar{\sigma}\sigma}^{(j)}(W(\Lambda, p))$ of $\text{SO}(3)$. The derivation below shows that j must be one half: this need not be assumed *a priori*.

Weinberg shows that the Dirac field is a Weinberg quantum field which is well defined given the data $(H, D_{\bar{\sigma}\sigma}^{(\frac{1}{2})}(W(\Lambda, p)), U(\Lambda, a), D^{\text{ch}}(\Lambda))$, where $D_{\bar{\sigma}\sigma}^{(\frac{1}{2})}(W(\Lambda, p))$ is the standard spin-1/2 representation of $\text{SO}(3)$. Below we recall the relevant parts of his derivation; a slight extension of the argument shows that the Elko field is not a Weinberg quantum field based on the data $(H, D_{\bar{\sigma}\sigma}(W(\Lambda, p)), U(\Lambda, a), D^{\text{ch}}(\Lambda))$ for any choice of $D_{\bar{\sigma}\sigma}(\Lambda)$. This argument is valid regardless of whether the particle is its own antiparticle. Before moving on, we pause to clarify the notation. H stands for the Hilbert space of physical states. $U(\Lambda, a)$ are the strict Poincaré operators which act on kets in the state space. The result is that states will have their σ values mixed up and momentum values changed in a way governed by the matrices $D_{\bar{\sigma}\sigma}(\Lambda)$. In contrast, the $D_{\ell\bar{\ell}}(\Lambda)$ are matrices that act on the spinors. A particular representation which acts on the spinors is the chiral representation $D_{\ell\bar{\ell}}^{\text{ch}}(\Lambda)$.

Let \mathbf{J} be the generators of angular momentum corresponding to the representation $D_{\bar{\sigma}\sigma}^{(\frac{1}{2})}(W(\Lambda, p))$ of $\text{SO}(3)$. Each of the three components of \mathbf{J} is a $(2j+1) \times (2j+1)$ matrix. We relabel the rest spinors $u_{\ell}(\mathbf{0}, \sigma)$ as $u_{m\pm}(\mathbf{0}, \sigma)$, where m takes the values \pm and $--, -+, +-, ++$ correspond to $\ell = 1, 2, 3, 4$ respectively. We similarly relabel $v_{\ell}(\mathbf{0}, \sigma)$ as $v_{m\pm}(\mathbf{0}, \sigma)$. We now define $(2j+1) \times (2j+1)$ matrices U_{\pm}, V_{\pm} by

$$(U_{\pm})_{m\sigma} = u_{m\pm}(\mathbf{0}, \sigma) \quad \text{and} \quad (V_{\pm})_{m\sigma} = v_{m\pm}(\mathbf{0}, \sigma). \quad (4.1)$$

It follows from Eqn. (2.135) in Sec. (2.13) that the matrices U_{\pm} , V_{\pm} satisfy the equations

$$U_+ \mathbf{J} = \frac{1}{2} \boldsymbol{\sigma} U_+, \quad U_- \mathbf{J} = \frac{1}{2} \boldsymbol{\sigma} U_- \quad (4.2)$$

$$-V_+ \mathbf{J}^* = \frac{1}{2} \boldsymbol{\sigma} V_+, \quad -V_- \mathbf{J}^* = \frac{1}{2} \boldsymbol{\sigma} V_-; \quad (4.3)$$

see [42, p.220]. Here $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ where the σ_i are the Pauli matrices, which in the chiral representation take the form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.4)$$

By Schur's lemma [77, p.31][78, p.21], we must have $j = 1/2$ and the \mathbf{J} must be the same as $\frac{1}{2} \boldsymbol{\sigma}$ up to a similarity transformation; moreover, U_{\pm} and V_{\pm} are determined up to scalar multiple. Suppose we choose \mathbf{J} to be equal to $\frac{1}{2} \boldsymbol{\sigma}$; this amounts to choosing $D_{\bar{\sigma}\sigma}(W(\Lambda, p))$ to be $D_{\bar{\sigma}\sigma}^{(\frac{1}{2})}(W(\Lambda, p))$. It then follows that the U_{\pm} matrices must be proportional to the identity and the V_{\pm} matrices must be proportional to σ_2 , because if $\mathbf{J} = \frac{1}{2} \boldsymbol{\sigma}$ then $-\mathbf{J}^* = \frac{1}{2} \sigma_2 \boldsymbol{\sigma} \sigma_2$, so, multiplying on the right by σ_2 and taking advantage of the relation $\sigma_2^2 = I_2$, yields

$$(V_{\pm} \sigma_2) \boldsymbol{\sigma} = \boldsymbol{\sigma} (V_{\pm} \sigma_2) \quad (4.5)$$

which implies that $V_{\pm} \sigma_2$ must be proportional to the identity, which in turn implies that V_{\pm} must therefore be proportional to σ_2 . We then get, explicitly:

$$U_+ = \begin{pmatrix} u_1(\mathbf{0}, 1/2) & u_1(\mathbf{0}, -1/2) \\ u_2(\mathbf{0}, 1/2) & u_2(\mathbf{0}, -1/2) \end{pmatrix} = \begin{pmatrix} c_+ & 0 \\ 0 & c_+ \end{pmatrix} \quad (4.6)$$

$$U_- = \begin{pmatrix} u_3(\mathbf{0}, 1/2) & u_3(\mathbf{0}, -1/2) \\ u_4(\mathbf{0}, 1/2) & u_4(\mathbf{0}, -1/2) \end{pmatrix} = \begin{pmatrix} c_- & 0 \\ 0 & c_- \end{pmatrix} \quad (4.7)$$

$$V_+ = \begin{pmatrix} v_1(\mathbf{0}, 1/2) & v_1(\mathbf{0}, -1/2) \\ v_2(\mathbf{0}, 1/2) & v_2(\mathbf{0}, -1/2) \end{pmatrix} = \begin{pmatrix} 0 & -d_+ \\ d_+ & 0 \end{pmatrix} \quad (4.8)$$

$$V_- = \begin{pmatrix} v_3(\mathbf{0}, 1/2) & v_3(\mathbf{0}, -1/2) \\ v_4(\mathbf{0}, 1/2) & v_4(\mathbf{0}, -1/2) \end{pmatrix} = \begin{pmatrix} 0 & -d_- \\ d_- & 0 \end{pmatrix}. \quad (4.9)$$

We then easily read off the form of the Dirac rest spinors to be

$$u\left(\mathbf{0}, \frac{1}{2}\right) = \begin{pmatrix} c_+ \\ 0 \\ c_- \\ 0 \end{pmatrix}, \quad u\left(\mathbf{0}, \frac{-1}{2}\right) = \begin{pmatrix} 0 \\ c_+ \\ 0 \\ c_- \end{pmatrix} \quad (4.10)$$

$$v\left(\mathbf{0}, \frac{1}{2}\right) = \begin{pmatrix} 0 \\ d_+ \\ 0 \\ d_- \end{pmatrix}, \quad v\left(\mathbf{0}, \frac{-1}{2}\right) = \begin{pmatrix} -d_+ \\ 0 \\ -d_- \\ 0 \end{pmatrix}. \quad (4.11)$$

4 Elko Fields and the Weinberg Formalism

A further analysis allows one to determine the value of the constants c_{\pm} , d_{\pm} . The extra conditions

$$U(\mathcal{P}, 0)\psi_{\ell}(x)U(\mathcal{P}, 0)^{-1} = \sum_{\bar{\ell}} D_{\ell\bar{\ell}}(\mathcal{P}^{-1})\psi_{\bar{\ell}}(\mathcal{P}x) \quad (4.12)$$

together with the demand from locality considerations that

$$\{\psi_{\ell}(x), \psi_{\bar{\ell}}^{\dagger}(y)\} = 0 \quad (4.13)$$

for spacelike separated x and y , fixes the constants uniquely. One finds that the resulting rest spinors are then precisely the Dirac rest spinors [42, p.224]. Hence the Dirac field is the only Weinberg quantum field based on the data $(H_1, U(\Lambda, a), U(\mathcal{P}), D_{\ell\bar{\ell}}(\Lambda), D_{\ell\bar{\ell}}(\mathcal{P}))$. The Majorana field is also a quantum field in the sense of Weinberg.

If we want to obtain a different Weinberg quantum field, the only freedom we have is to replace the representation $D_{\bar{\sigma}\sigma}^{(\frac{1}{2})}(W(\Lambda, p))$ with another representation $D_{\bar{\sigma}\sigma}^{(\frac{1}{2})}(W(\Lambda, p))'$ in the same isomorphism class. The corresponding angular momentum \mathbf{J} is not the same as $\frac{1}{2}\boldsymbol{\sigma}$ but is related to it by a similarity transform. For good measure, let us also allow the representation $D_{\ell\bar{\ell}}(\Lambda)$ to be not the chiral representation $D_{\ell\bar{\ell}}^{\text{ch}}(\Lambda)$, but another representation in the same isomorphism class. The corresponding angular momentum \mathbf{M} is also related to $\frac{1}{2}\boldsymbol{\sigma}$ by a similarity transform. Eqns. (4.2) become

$$U_+\mathbf{J} = \mathbf{M}U_+, \quad U_-\mathbf{J} = \mathbf{M}U_- \quad (4.14)$$

$$V_+\mathbf{J}^* = \mathbf{M}V_+, \quad V_-\mathbf{J}^* = \mathbf{M}V_- \quad (4.15)$$

It follows again from Schur's Lemma that U_+ and U_- are proportional and that V_+ and V_- are proportional. Here, we take a general Elko field with four Elko rest spinors of the form given by Eqn. (3.7) in Sec. (3.2.1):

$$u\left(\mathbf{0}, \frac{1}{2}\right) = \begin{pmatrix} -\eta b_1^* \\ \eta a_1^* \\ a_1 \\ b_1 \end{pmatrix}, \quad u\left(\mathbf{0}, \frac{-1}{2}\right) = \begin{pmatrix} -\eta b_2^* \\ \eta a_2^* \\ a_2 \\ b_2 \end{pmatrix} \quad (4.16)$$

$$v\left(\mathbf{0}, \frac{1}{2}\right) = \begin{pmatrix} -\eta d_1^* \\ \eta c_1^* \\ c_1 \\ d_1 \end{pmatrix}, \quad v\left(\mathbf{0}, \frac{-1}{2}\right) = \begin{pmatrix} -\eta d_2^* \\ \eta c_2^* \\ c_2 \\ d_2 \end{pmatrix}, \quad (4.17)$$

4.2 Elko not a Quantum Field in the Sense of Weinberg

where the a 's, b 's, c 's and d 's are arbitrary numbers. The matrices U_{\pm} and V_{\pm} then become:

$$U_+ = \begin{pmatrix} u_1(\mathbf{0}, \frac{1}{2}) & u_1(\mathbf{0}, \frac{-1}{2}) \\ u_2(\mathbf{0}, \frac{1}{2}) & u_2(\mathbf{0}, \frac{-1}{2}) \end{pmatrix} = \begin{pmatrix} -\eta b_1^* & -\eta b_2^* \\ \eta a_1^* & \eta a_2^* \end{pmatrix} \quad (4.18)$$

$$U_- = \begin{pmatrix} u_3(\mathbf{0}, \frac{1}{2}) & u_3(\mathbf{0}, \frac{-1}{2}) \\ u_4(\mathbf{0}, \frac{1}{2}) & u_4(\mathbf{0}, \frac{-1}{2}) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \quad (4.19)$$

$$V_+ = \begin{pmatrix} v_1(\mathbf{0}, \frac{1}{2}) & v_1(\mathbf{0}, \frac{-1}{2}) \\ v_2(\mathbf{0}, \frac{1}{2}) & v_2(\mathbf{0}, \frac{-1}{2}) \end{pmatrix} = \begin{pmatrix} -\eta d_1^* & -\eta d_2^* \\ \eta c_1^* & \eta c_2^* \end{pmatrix} \quad (4.20)$$

$$V_- = \begin{pmatrix} v_3(\mathbf{0}, \frac{1}{2}) & v_3(\mathbf{0}, \frac{-1}{2}) \\ v_4(\mathbf{0}, \frac{1}{2}) & v_4(\mathbf{0}, \frac{-1}{2}) \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix}. \quad (4.21)$$

If we set $U_+ = AU_-$ for some proportionality constant A , then by inspection, we see that we would have to have

$$Aa_1 = -\eta b_1^*, \quad \text{and} \quad \eta a_1^* = Ab_1. \quad (4.22)$$

Rearranging both of these relations for a_1 gives

$$a_1 = -A^{-1}(\eta b_1^*) \quad \text{and} \quad a_1 = A^*(\eta b_1^*), \quad (4.23)$$

from which it immediately follows that we require

$$-A^{-1} = A^* \quad (4.24)$$

to hold. To look at this more closely consider A to be a general complex number $A = re^{i\theta}$ where $r, \theta \in \mathbb{R}$. We immediately get

$$re^{-i\theta} = \frac{-1}{r}e^{-i\theta} \quad (4.25)$$

which implies that

$$r^2 = -1, \quad (4.26)$$

but $r \in \mathbb{R}$ so there exists no solution. Elko fields are therefore not quantum fields in the sense of Weinberg. This is one of the main results of this thesis.

The form of the spinors $u(\mathbf{p}, \sigma)$ and $v(\mathbf{p}, \sigma)$ in the Dirac field is completely determined by Eqns. (2.135), (4.12) and the requirement of locality. We finish the section by recalling briefly the rest of the argument, which involves the transformation properties under the full Lorentz and Poincaré groups. The representation $U(\Lambda, a)$ of \mathbb{P}^0 on H_1 can be extended to give a representation of \mathbb{P} : for instance, the operator $U(\mathcal{P}, 0)$ is multiplication by a scalar $\eta = \pm 1$. We obtain a representation of \mathbb{P}^0 on H . In order to pin down the constants c_{\pm} and d_{\pm} in Eqns. (4.6), Weinberg assumes that the overall Hamiltonian density $\mathcal{H}(x)$ is parity-invariant. Since the Dirac field $\psi(x)$ appears in the Hamiltonian density, its parity transform $U(\mathcal{P}, 0)\psi(x)U(\mathcal{P}, 0)^{-1}$ also does. Locality requires that $\psi(x)$ and $U(\mathcal{P}, 0)\psi(x)U(\mathcal{P}, 0)^{-1}$ commute [42, Ch.5], and this — together with the requirement that $\psi(x)$ is local — determines the values of c_{\pm} and d_{\pm} .

Our work shows that the most direct attempt to interpret the Elko field as a field in the sense of Weinberg fails. The only Weinberg quantum field based on the standard spin-1/2 representation of the full Poincaré group and the Dirac spinor representation of the full Lorentz group is the Dirac field.

The Elko field must break the rotational symmetries because we cannot write

$$U(R, 0)\Lambda_\ell(x)U(R, 0)^{-1} = \sum_{\bar{\ell}} D_{\ell\bar{\ell}}(R^{-1})\Lambda_{\bar{\ell}}(Rx) \quad (4.27)$$

for any combination of $U(R, 0)$ and $D_{\ell\bar{\ell}}(R^{-1})$. Two more questions arise from the observation that the rotational symmetries are violated, which give a couple more avenues in which to see if Elko fields can emerge in a rotational symmetry violating setting. This will be the issue to be examined in the following section.

4.3 Subgroup of the Poincaré Group, and Elko

In this section we address two more ideas which might have given rise to the Elko field. The first idea was that Elko fields might be confined to $2 + 1$ dimensions, and thus, belong to a subgroup of the Poincaré group containing two boosts and one rotation. It was suggested (C.Y. Lee) that perhaps these $2 + 1$ -dimensional hyperplanes were free to move around in the larger $3 + 1$ - dimensional spacetime. If we consider the symmetry group generated by the four spacetime translations, K_x , K_y and J_z , and we ask the question *do the unitary representations of this symmetry group on the space of physical states give rise to Elko fields?* We can use the method of induced representations to help tell us the answer [77, ch.9,10]. The states can be labeled by the eigenvalues of the translation generators and also the eigenvalues of the maximal set of commuting generators from the non-translation generators, that leave the chosen characteristic four-momentum vector unchanged. The set of such generators generate the little group. Here, the little group is $SO(2)$, being generated solely by J_z . The unitary representations of $SO(2)$ are one-dimensional, defined by a number $\alpha = 0, \pm 1, \pm 2, \dots$. These states are labeled by a continuous momentum index and a single number for each $SO(2)$ representation, so for any particular representation there is no two-valued discrete index so this approach cannot give rise to Elko.

The second suggestion was that maybe Elko fields are free to propagate freely in $1 + 3$ -dimensional spacetime but they might simply break the rotational symmetry so that only one rotational symmetry is respected, so that

$$U(R_y)\Lambda_\ell(x)U(R_y)^{-1} = \sum_{\bar{\ell}} D(R_y^{-1})_{\ell\bar{\ell}}\Lambda_{\bar{\ell}}(R_yx) \quad (4.28)$$

would be required to hold but not

$$U(R_x)\Lambda_\ell(x)U(R_x)^{-1} = \sum_{\bar{\ell}} D(R_x^{-1})_{\ell\bar{\ell}}\Lambda_{\bar{\ell}}(R_xx) \quad (4.29)$$

or

$$U(R_z)\Lambda_\ell(x)U(R_z)^{-1} = \sum_{\bar{\ell}} D(R_z^{-1})_{\ell\bar{\ell}}\Lambda_{\bar{\ell}}(R_z x). \quad (4.30)$$

In this way, the two valued discrete index σ may be preserved and the formulas for the rest spinors used in the previous section may be used here except instead of having $\sigma/2$ on the right hand side, we relax this condition so that this is replaced by just $\sigma_2/2$.

Dropping the n index and looking at the one-particle state space, we can again replace the index ℓ with a pair of 2-valued indices m and \pm such that

$$\sum_{\bar{\sigma}} u_{\bar{m}\pm}(\mathbf{0}, \bar{\sigma})[J_2]_{\bar{\sigma}\sigma} = \sum_m \frac{1}{2}[\sigma_2]_{\bar{m}m} u_{m\pm}(\mathbf{0}, \sigma) \quad (4.31)$$

$$-\sum_{\bar{\sigma}} v_{\bar{m}\pm}(\mathbf{0}, \bar{\sigma})[J_2]_{\bar{\sigma}\sigma}^* = \sum_m \frac{1}{2}[\sigma_2]_{\bar{m}m} v_{m\pm}(\mathbf{0}, \sigma). \quad (4.32)$$

If $u_{m\pm}(\mathbf{0}, \sigma)$ and $v_{m\pm}(\mathbf{0}, \sigma)$ are the m, σ elements of matrices U_{\pm} and V_{\pm} then we have

$$U_{\pm}J_2 = \frac{1}{2}\sigma_2 U_{\pm} \quad (4.33)$$

$$-V_{\pm}J_2^* = \frac{1}{2}\sigma_2 V_{\pm}. \quad (4.34)$$

Choosing $J_2 = \frac{1}{2}\sigma_2$ and noting that $-\sigma_2^* = \sigma_2$ yields:

$$U_+\sigma_2 = \sigma_2 U_+ \quad (4.35)$$

$$U_-\sigma_2 = \sigma_2 U_- \quad (4.36)$$

$$V_+\sigma_2 = \sigma_2 V_+ \quad (4.37)$$

$$V_-\sigma_2 = \sigma_2 V_- \quad (4.38)$$

or, upon multiplying on the right by σ_2 and noting that $\sigma_2^2 = 1$:

$$U_+ = \sigma_2 U_+ \sigma_2 \quad (4.39)$$

$$U_- = \sigma_2 U_- \sigma_2 \quad (4.40)$$

$$V_+ = \sigma_2 V_+ \sigma_2 \quad (4.41)$$

$$V_- = \sigma_2 V_- \sigma_2. \quad (4.42)$$

Explicitly, the matrices are

$$U_+ = \begin{pmatrix} u_1(\mathbf{0}, \frac{1}{2}) & u_1(\mathbf{0}, \frac{-1}{2}) \\ u_2(\mathbf{0}, \frac{1}{2}) & u_2(\mathbf{0}, \frac{-1}{2}) \end{pmatrix} \quad (4.43)$$

$$U_- = \begin{pmatrix} u_3(\mathbf{0}, \frac{1}{2}) & u_3(\mathbf{0}, \frac{-1}{2}) \\ u_4(\mathbf{0}, \frac{1}{2}) & u_4(\mathbf{0}, \frac{-1}{2}) \end{pmatrix} \quad (4.44)$$

$$V_+ = \begin{pmatrix} v_1(\mathbf{0}, \frac{1}{2}) & v_1(\mathbf{0}, \frac{-1}{2}) \\ v_2(\mathbf{0}, \frac{1}{2}) & v_2(\mathbf{0}, \frac{-1}{2}) \end{pmatrix} \quad (4.45)$$

$$V_- = \begin{pmatrix} v_3(\mathbf{0}, \frac{1}{2}) & v_3(\mathbf{0}, \frac{-1}{2}) \\ v_4(\mathbf{0}, \frac{1}{2}) & v_4(\mathbf{0}, \frac{-1}{2}) \end{pmatrix}. \quad (4.46)$$

Putting everything together, explicitly we have

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} u_1(\mathbf{0}, \frac{1}{2}) & u_1(\mathbf{0}, \frac{-1}{2}) \\ u_2(\mathbf{0}, \frac{1}{2}) & u_2(\mathbf{0}, \frac{-1}{2}) \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \quad (4.47)$$

$$\begin{pmatrix} -iu_2(\mathbf{0}, \frac{1}{2}) & -iu_2(\mathbf{0}, \frac{-1}{2}) \\ iu_1(\mathbf{0}, \frac{1}{2}) & iu_1(\mathbf{0}, \frac{-1}{2}) \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} =$$

$$\begin{pmatrix} u_2(\mathbf{0}, \frac{-1}{2}) & -u_2(\mathbf{0}, \frac{1}{2}) \\ -u_1(\mathbf{0}, \frac{-1}{2}) & u_1(\mathbf{0}, \frac{1}{2}) \end{pmatrix} = \begin{pmatrix} u_1(\mathbf{0}, \frac{1}{2}) & u_1(\mathbf{0}, \frac{-1}{2}) \\ u_2(\mathbf{0}, \frac{1}{2}) & u_2(\mathbf{0}, \frac{-1}{2}) \end{pmatrix}.$$

The other three matrix equations are very similar in the obvious way. By equating each matrix element on the left hand side of these matrix equations with the corresponding matrix elements on the right hand side of these matrix equations we get

$$u_1\left(\mathbf{0}, \frac{1}{2}\right) = u_2\left(\mathbf{0}, \frac{-1}{2}\right) \quad (4.48)$$

$$u_2\left(\mathbf{0}, \frac{1}{2}\right) = -u_1\left(\mathbf{0}, \frac{-1}{2}\right) \quad (4.49)$$

$$u_3\left(\mathbf{0}, \frac{1}{2}\right) = u_4\left(\mathbf{0}, \frac{-1}{2}\right) \quad (4.50)$$

$$u_4\left(\mathbf{0}, \frac{1}{2}\right) = -u_3\left(\mathbf{0}, \frac{-1}{2}\right) \quad (4.51)$$

$$v_1\left(\mathbf{0}, \frac{1}{2}\right) = v_2\left(\mathbf{0}, \frac{-1}{2}\right) \quad (4.52)$$

$$v_2\left(\mathbf{0}, \frac{1}{2}\right) = -v_1\left(\mathbf{0}, \frac{-1}{2}\right) \quad (4.53)$$

$$v_3\left(\mathbf{0}, \frac{1}{2}\right) = v_4\left(\mathbf{0}, \frac{-1}{2}\right) \quad (4.54)$$

$$v_4\left(\mathbf{0}, \frac{1}{2}\right) = -v_3\left(\mathbf{0}, \frac{-1}{2}\right). \quad (4.55)$$

Setting

$$u_1\left(\mathbf{0}, \frac{1}{2}\right) = c_+^1, \quad u_2\left(\mathbf{0}, \frac{1}{2}\right) = c_+^2 \quad (4.56)$$

$$u_3\left(\mathbf{0}, \frac{1}{2}\right) = c_-^1, \quad u_4\left(\mathbf{0}, \frac{1}{2}\right) = c_-^2 \quad (4.57)$$

4.3 Subgroup of the Poincaré Group, and Elko

$$v_1\left(\mathbf{0}, \frac{1}{2}\right) = d_+^1, \quad v_2\left(\mathbf{0}, \frac{1}{2}\right) = d_+^2 \quad (4.58)$$

$$v_3\left(\mathbf{0}, \frac{1}{2}\right) = d_-^1, \quad v_4\left(\mathbf{0}, \frac{1}{2}\right) = d_-^2 \quad (4.59)$$

yields

$$u\left(\mathbf{0}, \frac{1}{2}\right) = \begin{pmatrix} c_+^1 \\ c_+^2 \\ c_-^1 \\ c_-^2 \end{pmatrix}, \quad u\left(\mathbf{0}, \frac{-1}{2}\right) = \begin{pmatrix} -c_+^2 \\ c_+^1 \\ -c_-^2 \\ c_-^1 \end{pmatrix} \quad (4.60)$$

$$v\left(\mathbf{0}, \frac{1}{2}\right) = \begin{pmatrix} d_+^1 \\ d_+^2 \\ d_-^1 \\ d_-^2 \end{pmatrix}, \quad v\left(\mathbf{0}, \frac{-1}{2}\right) = \begin{pmatrix} -d_+^2 \\ d_+^1 \\ -d_-^2 \\ d_-^1 \end{pmatrix}. \quad (4.61)$$

If we identify $u(\mathbf{0}, \frac{1}{2})$ with

$$\begin{pmatrix} -\eta b^* \\ \eta a^* \\ a \\ b \end{pmatrix} \quad (4.62)$$

then it follows from Eqns. (4.60) that

$$u\left(\mathbf{0}, \frac{-1}{2}\right) = \begin{pmatrix} -\eta a^* \\ -\eta b^* \\ -b \\ a \end{pmatrix}. \quad (4.63)$$

If we replace the a 's with b 's and b 's with $-a$'s in this expression for $u(\mathbf{0}, \frac{-1}{2})$, we see that $u(\mathbf{0}, \frac{-1}{2})$ is also an Elko spinor. The same reasoning establishes by inspection that $v(\mathbf{0}, \frac{-1}{2})$ is also an Elko spinor so the spinors in Eqns. (4.60) and Eqns. (4.61) are compatible with Elko rest spinors.

We now return to Eqn. (2.129) and Eqn. (2.130) which we reproduce here for convenience:

$$\sum_{\sigma'} u_\ell(\mathbf{p}_\Lambda, \sigma', n) D_{\sigma'\sigma}^{(j_n)}(W(\Lambda, p)) = \sum_{\ell} D_{\bar{\ell}\ell}(\Lambda) \sqrt{\frac{p^0}{(\Lambda p)^0}} u_\ell(\mathbf{p}, \sigma, n) \quad (4.64)$$

$$\sum_{\sigma'} v_\ell(\mathbf{p}_\Lambda, \sigma', n) D_{\sigma'\sigma}^{(j_n)*}(W(\Lambda, p)) = \sum_{\ell} D_{\bar{\ell}\ell}(\Lambda) \sqrt{\frac{p^0}{(\Lambda p)^0}} v_\ell(\mathbf{p}, \sigma, n). \quad (4.65)$$

Consider boosts. Since we cannot boost in the y -direction, we will take $\mathbf{p} = (0, p_y, 0)$ and let Λ be the standard boost that takes a spin-1/2 particle of mass m from four-momentum $(m, 0, p_y, 0)$ to (p_0, p_x, p_y, p_z) . Then

$$W(\Lambda, p) = L^{-1}(\Lambda p) \Lambda L(p) = L^{-1}(p) L(p) = 1. \quad (4.66)$$

In this case, Eqn. (2.129) and Eqn. (2.130) becomes

$$u_{\bar{\ell}}(\mathbf{p}, \sigma) = \sqrt{\frac{m}{p^0}} \sum_{\ell} D_{\bar{\ell}\ell}(L(p)) u_{\ell}((0, p_y, 0), \sigma) \quad (4.67)$$

$$v_{\bar{\ell}}(\mathbf{p}, \sigma) = \sqrt{\frac{m}{p^0}} \sum_{\ell} D_{\bar{\ell}\ell}(L(p)) v_{\ell}((0, p_y, 0), \sigma). \quad (4.68)$$

A problem has emerged. Although the new relaxed conditions give Elko rest spinors, we do not have a way of obtaining the boosted spinors. The best we can do, without additional information, would be to apply a boost in the x and z directions and obtain the spinors $u((p_x, 0, p_z), \sigma)$ and $v((p_x, 0, p_z), \sigma)$. Furthermore, in [34] and [49], a boost in the y -direction is allowed. We cannot derive a formula for the boosted spinors from the rest spinors. It remains an open question whether the Elko field transforms correctly under this subgroup.

4.4 Locality

In the Standard Model, to every matter field operator $\psi_{\ell}(x)$ acting on the Hilbert space of physical states, there corresponds a unique field operator, the adjoint $\psi_{\ell}^{\dagger}(x)$, which is also a matter field operator, with clear physical interpretation. In order for both matter field operators to be *causal*, they have to anticommute (or commute in the case of bosons) with themselves and each other at spacelike separated x and x' [42, p.198]:

$$\{\psi_{\ell}(x), \psi_{\ell'}(x')\} = \{\psi_{\ell}^{\dagger}(x), \psi_{\ell'}^{\dagger}(x')\} = \{\psi_{\ell}(x), \psi_{\ell'}^{\dagger}(x')\} = 0. \quad (4.69)$$

It is clear from the Elko literature (see [34, p.6] for example) that the Elko fields anticommute with themselves at spacelike separation but not clear as to whether they anticommute with their *adjoints* at spacelike separation. In this section we examine the anticommutator of the Elko field with its adjoint at spacelike separation.* Since we think of the Elko field operator as an operator on Hilbert space that destroys Elko particles and create antiparticles, it follows that its unique adjoint has the physical interpretation of creating Elko particles and destroying Elko antiparticles. In this section we show that the Elko field in general does not anticommute with its adjoint at spacelike separation, even along the axis of locality. This means that Elko fields have an additional element of non-locality (in the sense of acausality) that needs to be considered in any attempts to detect Elko particles.

*We are calculating whether Elko fields have an element of acausality. Since this has implications for where in time an Elko event happened, relative to other events, we also view this calculation as having implications for locality. It is in this sense that we speak of locality in this section.

Here, we take a general Elko field with four Elko rest spinors of the form:

$$u\left(\mathbf{0}, \frac{1}{2}\right) = \begin{pmatrix} -\eta b_1^* \\ \eta a_1^* \\ a_1 \\ b_1 \end{pmatrix}, \quad u\left(\mathbf{0}, \frac{-1}{2}\right) = \begin{pmatrix} -\eta b_2^* \\ \eta a_2^* \\ a_2 \\ b_2 \end{pmatrix} \quad (4.70)$$

$$v\left(\mathbf{0}, \frac{1}{2}\right) = \begin{pmatrix} -\eta d_1^* \\ \eta c_1^* \\ c_1 \\ d_1 \end{pmatrix}, \quad v\left(\mathbf{0}, \frac{-1}{2}\right) = \begin{pmatrix} -\eta d_2^* \\ \eta c_2^* \\ c_2 \\ d_2 \end{pmatrix} \quad (4.71)$$

for general complex numbers $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$, and look at the anticommutator of the field $\psi_\ell(x)$ at spacetime point x , with its adjoint $\psi_\ell^\dagger(y)$ at a spacetime point y which is at spacelike separation from the point x . The vanishing of this anticommutator is important because it is necessary in order that the Cluster Decomposition Principle hold.

This section once again assumes Elko fields can somehow be obtained and that the rest spinors are connected to the boosted spinors by the matrix $\sqrt{\frac{m}{p_0}}D(L(p))$ as stated in [49][†]. This section therefore is not linked to the previous section. The previous section was exploring a possible way of obtaining an Elko field where as in this section, we pose the question: *what choice of Elko rest spinors can yield causal Elko fields if we assume that we have an Elko field with boosted spinors connected to the rest spinors by multiplying the rest spinors by the matrix $\sqrt{\frac{m}{p_0}}D(L(p))$?*

For convenience of presentation, we will temporarily use $m_+ \equiv m+p_0+p_z$, $m_- \equiv m+p_0-p_z$, $p_- \equiv p_x - ip_y$, and $p_+ \equiv p_x + ip_y$, so the boost matrix $\sqrt{\frac{m}{p_0}}D(L(p))$ takes the form:

$$\frac{1}{\sqrt{2p_0(m+p_0)}} \begin{pmatrix} m_+ & p_- & 0 & 0 \\ p_+ & m_- & 0 & 0 \\ 0 & 0 & m_- & -p_- \\ 0 & 0 & -p_+ & m_+ \end{pmatrix}. \quad (4.72)$$

The boosted spinors $u(\mathbf{p}, \sigma)$ and $v(\mathbf{p}, \sigma)$ now take the form:

$$u\left(\mathbf{p}, \frac{1}{2}\right) = \frac{1}{\sqrt{2p_0(m+p_0)}} \begin{pmatrix} -\eta m_+ b_1^* + \eta p_- a_1^* \\ -\eta p_+ b_1^* + \eta m_- a_1^* \\ m_- a_1 - p_- b_1 \\ -p_+ a_1 + m_+ b_1 \end{pmatrix} \quad (4.73)$$

[†]The cited paper does not have the factor $\sqrt{\frac{m}{p_0}}$ though.

$$u\left(\mathbf{p}, \frac{-1}{2}\right) = \frac{1}{\sqrt{2p_0(m+p_0)}} \begin{pmatrix} -\eta m_+ b_2^* + \eta p_- a_2^* \\ -\eta p_+ b_2^* + \eta m_- a_2^* \\ m_- a_2 - p_- b_2 \\ -p_+ a_2 + m_+ b_2 \end{pmatrix} \quad (4.74)$$

$$v\left(\mathbf{p}, \frac{1}{2}\right) = \frac{1}{\sqrt{2p_0(m+p_0)}} \begin{pmatrix} -\eta m_+ d_1^* + \eta p_- c_1^* \\ -\eta p_+ d_1^* + \eta m_- c_1^* \\ m_- c_1 - p_- d_1 \\ -p_+ c_1 + m_+ d_1 \end{pmatrix} \quad (4.75)$$

$$v\left(\mathbf{p}, \frac{-1}{2}\right) = \frac{1}{\sqrt{2p_0(m+p_0)}} \begin{pmatrix} -\eta m_+ d_2^* + \eta p_- c_2^* \\ -\eta p_+ d_2^* + \eta m_- c_2^* \\ m_- c_2 - p_- d_2 \\ -p_+ c_2 + m_+ d_2 \end{pmatrix}. \quad (4.76)$$

The adjoint's of these spinors are

$$u^\dagger\left(\mathbf{p}, \frac{1}{2}\right) = \frac{1}{\sqrt{2p_0(m+p_0)}} (-\eta^* m_+ b_1 + \eta^* p_-^* a_1, -\eta^* p_+^* b_1 + \eta^* m_- a_1, m_- a_1^* - p_-^* b_1^*, -p_+^* a_1^* + m_+ b_1^*) \quad (4.77)$$

$$u^\dagger\left(\mathbf{p}, \frac{-1}{2}\right) = \frac{1}{\sqrt{2p_0(m+p_0)}} (-\eta^* m_+ b_2 + \eta^* p_-^* a_2, -\eta^* p_+^* b_2 + \eta^* m_- a_2, m_- a_2^* - p_-^* b_2^*, -p_+^* a_2^* + m_+ b_2^*) \quad (4.78)$$

$$v^\dagger\left(\mathbf{p}, \frac{1}{2}\right) = \frac{1}{\sqrt{2p_0(m+p_0)}} (-\eta^* m_+ d_1 + \eta^* p_-^* c_1, -\eta^* p_+^* d_1 + \eta^* m_- c_1, m_- c_1^* - p_-^* d_1^*, -p_+^* c_1^* + m_+ d_1^*) \quad (4.79)$$

$$v^\dagger\left(\mathbf{p}, \frac{-1}{2}\right) = \frac{1}{\sqrt{2p_0(m+p_0)}} (-\eta^* m_+ d_2 + \eta^* p_-^* c_2, -\eta^* p_+^* d_2 + \eta^* m_- c_2, m_- c_2^* - p_-^* d_2^*, -p_+^* c_2^* + m_+ d_2^*). \quad (4.80)$$

We take our field to be

$$\psi_\ell(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{\sigma} [e^{ip \cdot x} u_\ell(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{-ip \cdot x} v_\ell(\mathbf{p}, \sigma) b^\dagger(\mathbf{p}, \sigma)]. \quad (4.81)$$

The anticommutator of the field $\psi_\ell(x)$ with $\psi_\ell^\dagger(y)$ with $x - y$ spacelike, gives:

$$\{\psi_\ell(x), \psi_\ell^\dagger(y)\} = \psi_\ell(x) \psi_\ell^\dagger(y) + \psi_\ell^\dagger(y) \psi_\ell(x) \quad (4.82)$$

$$\begin{aligned} &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \sum_{\sigma, \sigma'} \{ [e^{ip \cdot x} u_\ell(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{-ip \cdot x} v_\ell(\mathbf{p}, \sigma) b^\dagger(\mathbf{p}, \sigma)] \times \\ &[e^{-ip' \cdot y} u_\ell^\dagger(\mathbf{p}', \sigma') a^\dagger(\mathbf{p}', \sigma') + e^{ip' \cdot y} v_\ell^\dagger(\mathbf{p}', \sigma') b(\mathbf{p}', \sigma')] + [e^{-ip' \cdot y} u_\ell^\dagger(\mathbf{p}', \sigma') a^\dagger(\mathbf{p}', \sigma') + e^{ip' \cdot y} v_\ell^\dagger(\mathbf{p}', \sigma') b(\mathbf{p}', \sigma')] \\ &\times [e^{ip \cdot x} u_\ell(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{-ip \cdot x} v_\ell(\mathbf{p}, \sigma) b^\dagger(\mathbf{p}, \sigma)] \} \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \sum_{\sigma, \sigma'} \{ e^{ip \cdot x} e^{-ip' \cdot y} u_\ell(\mathbf{p}, \sigma) u_\ell^\dagger(\mathbf{p}', \sigma') \{ a(\mathbf{p}, \sigma), a^\dagger(\mathbf{p}', \sigma') \} + \\ &e^{-ip \cdot x} e^{ip' \cdot y} v_\ell(\mathbf{p}, \sigma) v_\ell^\dagger(\mathbf{p}', \sigma') \{ b^\dagger(\mathbf{p}, \sigma), b(\mathbf{p}', \sigma') \} \} \end{aligned} \quad (4.83)$$

$$\begin{aligned}
& e^{-ip \cdot x} e^{ip' \cdot y} v_\ell(\mathbf{p}, \sigma) v_\ell^\dagger(\mathbf{p}', \sigma') \{b^\dagger(\mathbf{p}, \sigma), b(\mathbf{p}', \sigma')\} + e^{ip \cdot x} e^{ip' \cdot y} u_\ell(\mathbf{p}, \sigma) v_\ell^\dagger(\mathbf{p}', \sigma') \{a(\mathbf{p}, \sigma), b(\mathbf{p}', \sigma')\} \\
& + e^{-ip \cdot x} e^{-ip' \cdot y} v_\ell(\mathbf{p}, \sigma) u_\ell^\dagger(\mathbf{p}', \sigma') \{b^\dagger(\mathbf{p}, \sigma), a^\dagger(\mathbf{p}', \sigma')\} \} = \\
& \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \int \frac{d^3 \mathbf{p}'}{(2\pi)^3} \sum_{\sigma, \sigma'} [e^{ip \cdot x} e^{-ip' \cdot y} u_\ell(\mathbf{p}, \sigma) u_\ell^\dagger(\mathbf{p}', \sigma') \delta^3(\mathbf{p} - \mathbf{p}') \delta_{\sigma\sigma'} + e^{-ip \cdot x} e^{ip' \cdot y} v_\ell(\mathbf{p}, \sigma) v_\ell^\dagger(\mathbf{p}', \sigma') \delta^3(\mathbf{p} - \mathbf{p}') \delta_{\sigma\sigma'}] \\
& = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \sum_{\sigma} [u_\ell(\mathbf{p}, \sigma) u_\ell^\dagger(\mathbf{p}, \sigma) e^{ip \cdot (x-y)} + v_\ell(\mathbf{p}, \sigma) v_\ell^\dagger(\mathbf{p}, \sigma) e^{-ip \cdot (x-y)}].
\end{aligned}$$

If we temporarily adopt the shorthand notation:

$$A_i^+ = -\eta m_+ b_1^* + \eta p_- a_1^*, \quad A_i^- = -\eta p_+ b_i^* + \eta m_- a_i^* \quad (4.84)$$

$$B_i^+ = m_- a_i - p_- b_i, \quad B_i^- = -p_+ a_i + m_+ b_i \quad (4.85)$$

$$C_i^+ = -\eta m_+ d_1^* + \eta p_- c_1^*, \quad C_i^- = -\eta p_+ d_i^* + \eta m_- c_i^* \quad (4.86)$$

$$D_i^+ = m_- c_i - p_- d_i, \quad D_i^- = -p_+ c_i + m_+ d_i, \quad (4.87)$$

$$E = \frac{1}{2p_0(m + p_0)} \quad (4.88)$$

where $i = 1, 2$, then we calculate the spin sums to be

$$u\left(\mathbf{p}, \pm \frac{1}{2}\right) u^\dagger\left(\mathbf{p}, \pm \frac{1}{2}\right) = E \times \quad (4.89)$$

$$\begin{pmatrix}
A_{1,2}^+(A_{1,2}^+)^* & A_{1,2}^+(A_{1,2}^-)^* & A_{1,2}^+(B_{1,2}^+)^* & A_{1,2}^+(B_{1,2}^-)^* \\
A_{1,2}^-(A_{1,2}^+)^* & A_{1,2}^-(A_{1,2}^-)^* & A_{1,2}^-(B_{1,2}^+)^* & A_{1,2}^-(B_{1,2}^-)^* \\
B_{1,2}^+(A_{1,2}^+)^* & B_{1,2}^+(A_{1,2}^-)^* & B_{1,2}^+(B_{1,2}^+)^* & B_{1,2}^+(B_{1,2}^-)^* \\
B_{1,2}^-(A_{1,2}^+)^* & B_{1,2}^-(A_{1,2}^-)^* & B_{1,2}^-(B_{1,2}^+)^* & B_{1,2}^-(B_{1,2}^-)^*
\end{pmatrix}$$

$$v\left(\mathbf{p}, \pm \frac{1}{2}\right) v^\dagger\left(\mathbf{p}, \pm \frac{1}{2}\right) = E \times \quad (4.90)$$

$$\begin{pmatrix}
C_{1,2}^+(C_{1,2}^+)^* & C_{1,2}^+(C_{1,2}^-)^* & C_{1,2}^+(D_{1,2}^+)^* & C_{1,2}^+(D_{1,2}^-)^* \\
C_{1,2}^-(C_{1,2}^+)^* & C_{1,2}^-(C_{1,2}^-)^* & C_{1,2}^-(D_{1,2}^+)^* & C_{1,2}^-(D_{1,2}^-)^* \\
D_{1,2}^+(C_{1,2}^+)^* & D_{1,2}^+(C_{1,2}^-)^* & D_{1,2}^+(D_{1,2}^+)^* & D_{1,2}^+(D_{1,2}^-)^* \\
D_{1,2}^-(C_{1,2}^+)^* & D_{1,2}^-(C_{1,2}^-)^* & D_{1,2}^-(D_{1,2}^+)^* & D_{1,2}^-(D_{1,2}^-)^*
\end{pmatrix}.$$

We now turn our attention to the matrix elements $[u(\mathbf{p}, \sigma) u^\dagger(\mathbf{p}, \sigma)]_{11}$ and $[v(\mathbf{p}, \sigma) v^\dagger(\mathbf{p}, \sigma)]_{11}$.

Writing these elements in terms of the original arbitrary numbers a_1, \dots etc, we have:

$$[u(\mathbf{p}, \sigma) u^\dagger(\mathbf{p}, \sigma)]_{11} = E(A_1^+(A_1^+)^* + A_2^+(A_2^+)^*) = \quad (4.91)$$

$$\begin{aligned}
& E\eta\eta^* \{(-m_+ b_1^* + p_- a_1^*)(-m_+ b_1 + p_-^* a_1) + (-m_+ b_2^* + p_- a_2^*)(-m_+ b_2 + p_-^* a_2)\} = \\
& E(m_+^2(b_1^* b_1 + b_2^* b_2) + p_- p_-^*(a_1^* a_1 + a_2^* a_2) - m_+ p_-^*(b_1^* a_1 + b_2^* a_2) - m_+ p_-(a_1^* b_1 + a_2^* b_2))
\end{aligned}$$

and

$$[v(\mathbf{p}, \sigma) v^\dagger(\mathbf{p}, \sigma)]_{11} = \quad (4.92)$$

$$E(m_+^2(d_1^*d_1 + d_2^*d_2) + p_-p_-^*(c_1^*c_1 + c_2^*c_2) - m_+p_-^*(d_1^*c_1 + d_2^*c_2) - m_+p_-(c_1^*d_1 + c_2^*d_2)).$$

We now consider $\{\psi_1(x), \psi_1^\dagger(y)\}$ which takes the form:

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3 2p_0(m+p_0)} [u(\mathbf{p}, \sigma) u^\dagger(\mathbf{p}, \sigma)]_{11} e^{ip \cdot (x-y)} + \int \frac{d^3\mathbf{p}}{(2\pi)^3 2p_0(m+p_0)} [v(\mathbf{p}, \sigma) v^\dagger(\mathbf{p}, \sigma)]_{11} e^{-ip \cdot (x-y)}, \quad (4.93)$$

which, upon substituting in the spin sum expansions yields:

$$\begin{aligned} & \int \frac{d^3\mathbf{p}}{(2\pi)^3 2p_0(m+p_0)} [-(m+p_0+p_z)(p_x+ip_y)(b_1^*a_1+b_2^*a_2) - (m+p_0+p_z)(p_x-ip_y)(a_1^*b_1+a_2^*b_2) + \\ & \quad (m+p_0+p_z)(m+p_0+p_z)(b_1^*b_1+b_2^*b_2) + (p_x^2+p_y^2)(a_1^*a_1+a_2^*a_2)] e^{ip \cdot (x-y)} + \\ & \int \frac{d^3\mathbf{p}}{(2\pi)^3 2p_0(m+p_0)} [-(m+p_0+p_z)(p_x+ip_y)(d_1^*c_1+d_2^*c_2) - (m+p_0+p_z)(p_x-ip_y)(c_1^*d_1+c_2^*d_2) + \\ & \quad (m+p_0+p_z)(m+p_0+p_z)(d_1^*d_1+d_2^*d_2) + (p_x^2+p_y^2)(c_1^*c_1+c_2^*c_2)] e^{-ip \cdot (x-y)}. \end{aligned} \quad (4.94)$$

Before moving further, we here pause to look at how the Dirac field turns out to be local in the sense of being causal. This will make a few issues clear as to what has to happen with the above spin sums in order for the anti-commutator of the field with its adjoint to vanish at spacelike separation. For the following, the Dirac rest spinors are taken to be [42, p.224]:[‡]

$$u\left(\mathbf{p}, \frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u\left(\mathbf{p}, \frac{-1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad (4.95)$$

$$v\left(\mathbf{p}, \frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v\left(\mathbf{p}, \frac{-1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad (4.96)$$

The anticommutator $\{\psi(x), \psi^\dagger(y)\}$ between the Dirac field and its adjoint with x and y at spacelike separation is

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3 2p_0} \sum_{\sigma} [u(\mathbf{p}, \sigma) u^\dagger(\mathbf{p}, \sigma) e^{ip \cdot (x-y)} + v(\mathbf{p}, \sigma) v^\dagger(\mathbf{p}, \sigma) e^{-ip \cdot (x-y)}]. \quad (4.97)$$

Explicit calculation reveals that

$$\sum_{\sigma} u(\mathbf{p}, \sigma) u^\dagger(\mathbf{p}, \sigma) = \left(\frac{\gamma_\mu p^\mu + mI_4}{2m} \right) \gamma_0 \quad (4.98)$$

[‡]Here we drop the superscript “ D ” because the whole calculation only involves Dirac spinors so no confusion should result.

and

$$\sum_{\sigma} v(\mathbf{p}, \sigma) v^{\dagger}(\mathbf{p}, \sigma) = \left(\frac{\gamma_{\mu} p^{\mu} - m I_4}{2m} \right) \gamma_0 \quad (4.99)$$

where here,

$$\gamma_0 = \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix} \quad \text{and} \quad \gamma_i = \begin{pmatrix} 0_2 & \sigma_i \\ -\sigma_i & 0_2 \end{pmatrix}, \quad (4.100)$$

so the anticommutator becomes

$$\int \frac{d^3 \mathbf{p}}{(2\pi)^3 4mp_0} [(\gamma_{\mu} p^{\mu} + m I_4) e^{ip \cdot (x-y)} + (\gamma_{\mu} p^{\mu} - m I_4) e^{-ip \cdot (x-y)}] \gamma_0. \quad (4.101)$$

Next, we observe that

$$\partial^{\mu} e^{ip \cdot x} = ip^{\mu} e^{ip \cdot x} \quad \text{and} \quad \partial^{\mu} e^{-ip \cdot x} = -ip^{\mu} e^{-ip \cdot x} \quad (4.102)$$

so the anticommutator can now be re-written as

$$\int \frac{d^3 \mathbf{p}}{(2\pi)^3 4mp_0} [(-i\gamma_{\mu} \partial^{\mu} + m I_4) e^{ip \cdot (x-y)} + (i\gamma_{\mu} \partial^{\mu} - m I_4) e^{-ip \cdot (x-y)}] \gamma_0. \quad (4.103)$$

We now take advantage of a result [42, sec.5.2] used by Weinberg, that the function

$$\int \frac{d^3 \mathbf{p}}{(2\pi)^3 2p_0} e^{ip \cdot (x-y)} \quad (4.104)$$

is an even function of $x - y$ when $x - y$ is spacelike. This allows us to pull out a common factor of the exponentials and write the anticommutator as

$$\int \frac{d^3 \mathbf{p}}{(2\pi)^3 4mp_0} [-i\gamma_{\mu} \partial^{\mu} + m I_4 + i\gamma_{\mu} \partial^{\mu} - m I_4] e^{ip \cdot (x-y)} \gamma_0 \quad (4.105)$$

which identically vanishes. The key observations which will be of direct use are:

1. The matrices formed from the u and v spin sums are both of the same form.
2. The matrix elements in the spin sums with no p 's in them were of opposite sign so that they could cancel once the exponential factor was taken out.
3. The other matrix elements had the same signs. This, coupled with there being one “ p ” term meant that when expressed as partial derivatives, there was a sign change in those matrix elements belonging to the v spin sums which then enabled cancellation with the corresponding matrix elements belonging to the u spin sums. The critical sign change happened because the *single* derivative of the even function of $x - y$ was an odd function so that when the “ $x - y$ ” was turned into a “ $y - x$ ”, a minus sign appeared.

Returning now to our problem, by replacing p_{μ} with $-i\partial_{\mu}$ in Eqn. (4.94), we get:

$$\begin{aligned} & [-(m - i\partial_0 - i\partial_z)(-i\partial_x + i(-i\partial_y))(b_1^* a_1 + b_2^* a_2) - (m - i\partial_0 - i\partial_z)(-i\partial_x - i(-i\partial_y)) \times \\ & (a_1^* b_1 + a_2^* b_2) + (m - i\partial_0 - i\partial_z)(m - i\partial_0 - i\partial_z)(b_1^* b_1 + b_2^* b_2) - (\partial_x^2 + \partial_y^2)(a_1^* a_1 + a_2^* a_2)] H(x - y) + \\ & [-(m - i\partial_0 - i\partial_z)(-i\partial_x + i(-i\partial_y))(d_1^* c_1 + d_2^* c_2) - (m - i\partial_0 - i\partial_z)(-i\partial_x - i(-i\partial_y)) \times \end{aligned} \quad (4.106)$$

4 Elko Fields and the Weinberg Formalism

$$(c_1^* d_1 + c_2^* d_2) + (m - i\partial_0 - i\partial_z)(m - i\partial_0 - i\partial_z)(d_1^* d_1 + d_2^* d_2) - (\partial_x^2 + \partial_y^2)(c_1^* c_1 + c_2^* c_2)]H(y - x)$$

where $H(x - y)$ is an even function of $x - y$ for spacelike intervals:

$$H(x - y) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3 2p_0(m + p_0)} e^{-ip \cdot (x - y)}. \quad (4.107)$$

By inspection of the coefficients in Eqn. (4.106) we deduce that if $\{\Lambda_\ell(x), \Lambda_\ell^\dagger(y)\} = 0$ then

$$a_1^* a_1 + a_2^* a_2 = -c_1^* c_1 - c_2^* c_2 \quad (4.108)$$

and

$$b_1^* b_1 + b_2^* b_2 = -d_1^* d_1 - d_2^* d_2. \quad (4.109)$$

The only solution to this however, is the trivial solution:

$$a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = 0. \quad (4.110)$$

so in general

$$\{\Lambda_\ell(x), \Lambda_\ell^\dagger(y)\} \neq 0 \quad (4.111)$$

for a spacelike interval $(x - y)^2 < 0$.

Elko fields thus have an element of acausality to them *even along* the axis of locality. This makes it less clear how Elko particles and the Standard Model particles they interact with, manifest themselves during Elko particle interactions.

As mentioned in Sec. (3.10.2), there can be elements of the S-matrix which are not manifestly Lorentz covariant which can be made to disappear by adding an appropriate non-covariant term to the Hamiltonian density so long as the original non-covariant term is local in the sense we described. This applies to gauge theories too [42, sec. 5.9, sec. 8.1]. However in the case of Elko this cannot be done because the required type of locality described in Sec. (3.10.2) is not applicable to the Elko case.

We now turn our attention to a possibility for incorporating Elko fields in a quantum field theoretic framework whose systematic study would take us well outside the intended scope of this thesis. We include the following section because a new insight is added which further strengthens the suspicions mentioned in [50, sec.5.2] that the search for Elko type fields should be undertaken in a non-commutative spacetime setting.

4.5 Elko and Very Special Relativity

In [50] a spinor representation is given which transforms under the SIM(2) group together with the group of spacetime translations \mathcal{T}_4 , and thus transforms under the Very Special Relativity (VSR) group, instead of the whole Lorentz group. The rest spinors were Elko in form but the boost operator was different, while still taking a momentum vector of momentum $(m, 0, 0, 0)$ to a general momentum (p_0, p_1, p_2, p_3) . It is discussed in [50, sec.5.2] that the

locality structure of the VSR Elko quantum field dark matter candidate might imply that the underlying momentum space should be non-commutative instead of commutative. In this section, we arrive at this conclusion in a different way, thus hopefully giving added impetus and motivation for Elko to be studied in the context of non-commutative spacetime.

Here there is a well defined symmetry group. Moreover, this symmetry group has an abelian subgroup (the group of spacetime translations \mathcal{T}_4) so we can use the method of induced representations (see [77, ch.9,10]) to tell us what the state space looks like on which this symmetry group acts. This in turn, will tell us what kinds of quantum fields are allowed that respect these symmetries, and we will be able to see if a quantum field is allowed that has coefficient functions making spinors of the form given in [50]. We will keep all of the assumptions from Weinberg about such things as the S-matrices and the cluster decomposition principle.

We begin here by giving the Lie algebra of the VSR group adjoined with spacetime translations. This group is generated by $P^0, P^1, P^2, P^3, J_z, K_z$ together with two new generators $T_1 \equiv K_x + J_y$ and $T_2 \equiv K_y - J_x$. The Lie algebra satisfied by these generators is

$$[P^\mu, P^\nu] = 0 \quad [T_1, T_2] = 0, \quad [T_1, K_z] = iT_1, \quad [T_2, K_z] = iT_2, \quad [T_1, J_z] = -iT_2 \quad (4.112)$$

$$[T_2, J_z] = iT_1 \quad [J_z, K_z] = 0, \quad [J_z, P^0] = 0, \quad [J_z, P^1] = iP^2, \quad [J_z, P^2] = -iP^1$$

$$[J_z, P^3] = 0, \quad [K_z, P^0] = -iP^3, \quad [K_z, P^1] = 0, \quad [K_z, P^2] = 0$$

$$[K_z, P^3] = -ip^0, \quad [T_1, P^0] = -iP^1, \quad [T_1, P^1] = -i(P^0 + P^3) \quad [T_1, P^3] = 0 \quad [T_1, P^2] = iP^1$$

$$[T_2, P^0] = -iP^2, \quad [T_2, P^1] = 0, \quad [T_2, P^2] = -i(P^0 + P^3), \quad [T_2, P^3] = iP^2.$$

The Lie algebra satisfied by the generators T_1, T_2, J_z and K_z is called $\text{sim}(2)$. The group of spacetime translations generated by P^0, P^1, P^2, P^3 is commonly called \mathcal{T}_4 . We will call the group that corresponds to the total Lie algebra expressed above as G .

We can use the method of induced representations if there exists an abelian subgroup of G . \mathcal{T}_4 is an abelian subgroup of G so we may proceed. We start by choosing a characteristic vector. We will choose $k^\mu = (M, 0, 0, 0)$, and label a subset of the state kets by k^μ . These states are eigenstates of the operator P^0 with eigenvalue M . We now turn to the factor group G/\mathcal{T}_4 and search this group for generators that commute with P^0 and also commute amongst themselves. The set of generators that commute with P^0 generates the little group associated with the given choice of characteristic vector. By inspection of the Lie algebra we see that J_z is the only generator which commutes with P^0 . The angular momentum generator J_z generates the group $SO(2)$, which is the little group associated with the characteristic vector. The state vectors are labeled by the eigenvalues of the generators of the abelian subgroup \mathcal{T}_4 and also the eigenvalues which label the representations of the little group, which in this case, is $SO(2)$. The irreducible representations of $SO(2)$ are all one-dimensional and are labeled by the eigenvalue σ of J_z , which can take on the values $\sigma = 0, \pm 1, \pm 2, \dots$. Each vector will be labeled by (M, σ) where M and σ take on fixed values for each irreducible representation of G .

4 Elko Fields and the Weinberg Formalism

We take these labels as implicit and label the state vectors corresponding to the characteristic vector by the three momentum part of k^μ which is $\mathbf{0}$ so we have $|\mathbf{0}\rangle$. At this point we see that there is no two-valued discrete index labeling the state kets so if we keep as many of Weinberg's assumptions and formalism as possible and only make the simplest departure by changing the actual symmetry group which acts on the space of physical states then the VSR Elko fields do not fit into this formalism.[§] The VSR Elko spinors in themselves are mathematically well-defined. It seems natural to us to assume that the associated VSR Elko quantum field must have a well-defined mathematical structure too. The above argument convinces us that most likely, the quantum field would be compatible with spacetime symmetries where the momentum generators are non-commutative. From discussions with Ahluwalia, P. Butler, B. Martin, P. Renaud and N. Gresnigt we feel that the Stabilized Poincaré Heisenberg algebra [79] is a natural first place to start looking, one reason being that it is intimately related directly to the Clifford algebra $Cl(1, 3)$ which has a natural direct geometric interpretation [61]. We will not pursue this avenue of investigation as it takes us well beyond the intended scope of this thesis. We now move on to a systematic study of the Wigner classes, which was initially motivated by the non-standard behavior of Elko spinors in the spin-1/2 spinor representation space under discrete symmetry transformations.

[§]For completeness, the other eigenstates of the four-momentum operator P^μ are obtained by considering the group elements of G that are in G/\mathcal{T}_4 but are not in the little group. The representations of G can then be written down, and the states can be normalized.

5 Non-Standard Quantum Fields

5.1 Introduction

Wigner described the possible irreducible unitary representations of the strict Poincaré group in 1939 [11]: they are the representations $U(\Lambda, a)$ of \mathbb{P}^0 on H_1 given in Sec. (2.5). Later he extended this work to give a classification of the irreducible unitary representations $U(\Lambda, a)$ of the full Poincaré group (including the discrete symmetries of space inversion \mathcal{P} and time reversal \mathcal{T}) [52]. There are four isomorphism classes: one so-called standard Wigner class and three non-standard Wigner classes. The standard Wigner class is the representation of \mathbb{P} on H_1 discussed at the end of Sec. (2.5).

Our argument that the Elko field is not a Weinberg quantum field involves only Eqn. (2.135) applied to elements of \mathbb{P}^0 and \mathcal{L}^0 : the discrete symmetries do not appear to play any part. Nevertheless, to make further progress, we need to consider representations of the full Poincaré and Lorentz groups. The explanation for this seeming paradox is as follows. We assume the one-particle state space carries an irreducible unitary representation of \mathbb{P} . The restriction of the representation to \mathbb{P}^0 is isomorphic to a direct sum of irreducible representations of \mathbb{P}^0 . If the representation of \mathbb{P} we started with is in the standard Wigner class, then this restriction is irreducible: there is only one irreducible summand, namely H_1 endowed with the representation $U(\Lambda, a)$ of \mathbb{P}^0 . This is the case considered in the previous chapter. If the representation of \mathbb{P} is in one of the non-standard Wigner classes then the restriction to \mathbb{P}^0 is the sum of two or more irreducible representations. These turn out to be isomorphic to each other; one may choose basis kets of the form $|p, \sigma, \tau\rangle$, where p and σ are as before and τ is a degeneracy index which distinguishes between the irreducible components. These cases are new; for non-standard Wigner classes, there is an extra degree of freedom to explore.

In the next section, we extract explicit formulas from [52]. We then use these formulas to search for massive spin-1/2 local quantum fields.

We show that there are no non-trivial massive local spin-1/2 quantum fields given the data

$$\left(H_1^{\text{NS}}, U(\Lambda, a), U(\mathcal{P}, 0), U(\mathcal{T}, 0), D_{\sigma\sigma}^{(\frac{1}{2})}(W(\Lambda, p)), D_{\ell\ell}^{\text{ch}}(\Lambda), D_{\ell\ell}^{\text{ch}}(\mathcal{P}), D_{\ell\ell}^{\text{ch}}(\mathcal{T}) \right).$$

where $U(\Lambda, a)$ is one of the non-standard representations. We then search for more general non-standard finite-dimensional representations of the discrete space and time inversion symmetries and then search again for non-trivial massive spin-1/2 quantum fields, this time, exploiting the available freedom to choose an appropriate finite-dimensional representation

of the space and time inversions as seen to be appropriate. We show that in two of the three non-standard cases, there exist no non-trivial local massive spin-1/2 quantum fields for any extension of the chiral representation. Finally, we show that for the remaining non-standard Wigner class, there do exist non-standard representations of the full Lorentz group $(D_{\ell\bar{\ell}}^{\text{ch}}(\Lambda), D_{\ell\bar{\ell}}(\mathcal{P}), D_{\ell\bar{\ell}}(\mathcal{T}))$ which give rise to a new massive spin-1/2 local quantum field. In the final sections of this chapter, we make observations on the new quantum field. We also give the mathematical conditions that would all have to be realised in nature in order for the non-standard class to be able to be transformed to look like two copies of the standard Wigner class. We did not find the Elko field among any of the non-standard Wigner classes, a point on which we expand in Sec. (5.4).

5.2 The Non-Standard Hilbert Spaces

We here describe what the massive one-particle non-standard Hilbert spaces H_1^{NS} look like and then give the representation of the full Poincaré group on H_1^{NS} . The form these spaces take was determined by Wigner in [52]. In all three cases, the state space is spanned by kets of the form $|\mathbf{p}, \sigma, \tau\rangle$ where \mathbf{p} and σ have their usual definitions from H_1 as discussed in Chapter 2, and τ is a new, additional, two-valued discrete index $\tau = \pm 1$. The inner product is given by

$$\langle \mathbf{p}', \sigma', \tau' | \mathbf{p}, \sigma, \tau \rangle = \delta^3(\mathbf{p}' - \mathbf{p}) \delta_{\sigma'\sigma} \delta_{\tau'\tau}. \quad (5.1)$$

The representations $U(\Lambda, a)$ of the strict Poincaré group act on states as specified in Chapter 2, apart from the inclusion of the additional index τ :

$$U(\Lambda, a) |\mathbf{p}, \sigma, \tau\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} e^{-i(\Lambda p)^\mu a_\mu} \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p)) |\mathbf{p}_\Lambda, \sigma', \tau\rangle. \quad (5.2)$$

The index τ is unaffected by the action of $U(\Lambda, a)$ on the state kets. It is the space and time inversion symmetries that mix up the τ values. Before showing what the parity and time reversal operators do in each of the three non-standard cases, we now choose to adopt a convenient notation which is not so cumbersome to work with. Following Weinberg, we define:

$$\mathbf{P} \equiv U(\mathcal{P}, 0), \quad \mathbf{T} \equiv U(\mathcal{T}, 0). \quad (5.3)$$

From a careful inspection of Table 3 in [52, p.72], we can read off from Table 3 that the discrete symmetries \mathbf{P} and \mathbf{T} act on the kets as follows:

Case 1:

$$\mathbf{P} |\mathbf{p}, \sigma\rangle = \eta |-\mathbf{p}, \sigma\rangle \quad (5.4)$$

$$\mathbf{A} |\mathbf{p}, \sigma\rangle = \eta (-1)^{j+\sigma} |\mathbf{p}, -\sigma\rangle \quad (5.5)$$

$$\mathbf{T} |\mathbf{p}, \sigma\rangle = (-1)^{j+\sigma} |-\mathbf{p}, -\sigma\rangle \quad (5.6)$$

Case 2:

$$\mathbf{P} |\mathbf{p}, \sigma, \tau\rangle = \eta \tau |-\mathbf{p}, \sigma, \tau\rangle \quad (5.7)$$

$$\mathbf{A} |\mathbf{p}, \sigma, \tau\rangle = \eta (-1)^{j+\sigma} |\mathbf{p}, -\sigma, -\tau\rangle \quad (5.8)$$

$$\mathbf{T} |\mathbf{p}, \sigma, \tau\rangle = \tau (-1)^{j+\sigma} |-\mathbf{p}, -\sigma, -\tau\rangle \quad (5.9)$$

Case 3:

$$\mathbf{P} |\mathbf{p}, \sigma, \tau\rangle = \eta \tau |-\mathbf{p}, \sigma, \tau\rangle \quad (5.10)$$

$$\mathbf{A} |\mathbf{p}, \sigma, \tau\rangle = \eta \tau (-1)^{j+\sigma} |\mathbf{p}, -\sigma, -\tau\rangle \quad (5.11)$$

$$\mathbf{T} |\mathbf{p}, \sigma, \tau\rangle = (-1)^{j+\sigma} |-\mathbf{p}, -\sigma, -\tau\rangle \quad (5.12)$$

Case 4:

$$\mathbf{P} |\mathbf{p}, \sigma, \tau\rangle = \eta |-\mathbf{p}, \sigma, \tau\rangle \quad (5.13)$$

$$\mathbf{A} |\mathbf{p}, \sigma, \tau\rangle = \eta \tau (-1)^{j+\sigma} |\mathbf{p}, -\sigma, -\tau\rangle \quad (5.14)$$

$$\mathbf{T} |\mathbf{p}, \sigma, \tau\rangle = \tau (-1)^{j+\sigma} |-\mathbf{p}, -\sigma, -\tau\rangle. \quad (5.15)$$

where \mathbf{A} is the antiunitary operator \mathbf{PT} , and η is the intrinsic parity of a particle which is set equal to 1 in [52, p.63–64] but in general, if the state space contains more than one type of particle, then the relative values of the intrinsic parities matter, so we include η .

The multiparticle Hilbert space is spanned by the basis kets

$$|\mathbf{p}_1, \sigma_1, \tau_1, n_1; \mathbf{p}_2, \sigma_2, \tau_2, n_2; \dots\rangle = |\mathbf{p}_1, \sigma_1, \tau_1, n_1\rangle \otimes |\mathbf{p}_2, \sigma_2, \tau_2, n_2\rangle \otimes \dots \quad (5.16)$$

suitably symmetrised or antisymmetrised, and normalized so that the inner product is given by

$$\langle \mathbf{p}'_1, \sigma'_1, \tau'_1, n'_1; \mathbf{p}'_2, \sigma'_2, \tau'_2, n'_2; \dots | \mathbf{p}_1, \sigma_1, \tau_1, n_1; \mathbf{p}_2, \sigma_2, \tau_2, n_2; \dots \rangle = \quad (5.17)$$

$$\delta^3(\mathbf{p}'_1 - \mathbf{p}_1) \delta_{\sigma'_1 \sigma_1} \delta_{\tau'_1 \tau_1} \delta_{n'_1 n_1} \delta^3(\mathbf{p}'_2 - \mathbf{p}_2) \delta_{\sigma'_2 \sigma_2} \delta_{\tau'_2 \tau_2} \delta_{n'_2 n_2} \dots \pm \text{permutations}.$$

The operators $U(\Lambda, a)$ of the strict Poincaré group have the following action on the multiparticle state space:

$$U(\Lambda, a) |\mathbf{p}_1, \sigma_1, \tau_1, n_1; \dots\rangle = \sqrt{\frac{(\Lambda p)^0 \dots}{p_1^0 \dots}} e^{-i(\Lambda p_1 + \dots)^\mu a_\mu} \times \quad (5.18)$$

$$\sum_{\sigma'_1 \dots} D_{\sigma'_1 \sigma_1}^{(j_1)}(W(\Lambda, p_1)) \dots |\mathbf{p}_{1\Lambda}, \sigma'_1 \tau_1, n_1; \dots\rangle.$$

For Case 2 the parity and time reversal operators act on the multiparticle states as

$$\mathbf{P} |\mathbf{p}_1, \sigma_1, \tau_1, n_1; \dots\rangle = \eta_{n_1} \dots \tau_{n_1} \dots |-\mathbf{p}_1, \sigma_1, \tau_1, n_1; \dots\rangle \quad (5.19)$$

$$\mathbf{T} |\mathbf{p}_1, \sigma_1, \tau_1, n_1; \dots\rangle = \zeta_{n_1} \dots \tau_{n_1} \dots (-1)^{j_1 + \sigma_1} \dots |-\mathbf{p}_1, -\sigma_1, -\tau_1, n_1; \dots\rangle. \quad (5.20)$$

5 Non-Standard Quantum Fields

Here $\zeta_{n_1} \dots$ are the relative time reversal intrinsic parities that are retained here for the same reasons as the space inversion intrinsic parities η .

The parity and time reversal operators for Case 3 act on the multiparticle states as

$$\mathbf{P} |\mathbf{p}_1, \sigma_1, \tau_1, n_1; \dots\rangle = \eta_{n_1} \dots \tau_{n_1} \dots |-\mathbf{p}_1, \sigma_1, \tau_1, n_1; \dots\rangle \quad (5.21)$$

$$\mathbf{T} |\mathbf{p}_1, \sigma_1, \tau_1, n_1; \dots\rangle = \zeta_{n_1} \dots (-1)^{j_1 + \sigma_1} \dots |-\mathbf{p}_1, -\sigma_1, -\tau_1, n_1; \dots\rangle \quad (5.22)$$

and the parity and time reversal operators act on the Case 4 multiparticle states as

$$\mathbf{P} |\mathbf{p}_1, \sigma_1, \tau_1, n_1; \dots\rangle = \eta_{n_1} \dots |-\mathbf{p}_1, \sigma_1, \tau_1, n_1; \dots\rangle \quad (5.23)$$

$$\mathbf{T} |\mathbf{p}_1, \sigma_1, \tau_1, n_1; \dots\rangle = \zeta_{n_1} \dots \tau_{n_1} \dots (-1)^{j_1 + \sigma_1} \dots |-\mathbf{p}_1, -\sigma_1, -\tau_1, n_1; \dots\rangle. \quad (5.24)$$

The creation operators are defined in a way analogous to Eqn. (2.48):

$$|\mathbf{p}_1, \sigma_1, \tau_1, n_1; \mathbf{p}_2, \sigma_2, \tau_2, n_2; \dots\rangle = a^\dagger(\mathbf{p}_1, \sigma_1, \tau_1, n_1) a^\dagger(\mathbf{p}_2, \sigma_2, \tau_2, n_2) \dots |0\rangle. \quad (5.25)$$

The annihilation operators $a(\mathbf{p}, \sigma, \tau, n)$ are the adjoints of the creation operators $a^\dagger(\mathbf{p}, \sigma, \tau, n)$.

Now that we have presented what the non-standard state spaces look like, we turn our attention to the task of constructing non-standard massive spin-1/2 quantum fields.

5.3 Constructing Non-Standard Massive Spin-1/2 Quantum Fields

Any non-standard quantum field, like the standard fields, can be grouped into two categories, one in which the quantum field carries no conserved quantum numbers in which case $b^\dagger = a^\dagger$ and one in which the quantum field carries one or more conserved quantum numbers in which case the particle is distinct from its associated antiparticle with $b^\dagger \neq a^\dagger$. In this chapter, we will be concerned with both categories of fields.

We have seen in the Quantum Field Theory review in Chapter 2 that it is required that

$$U(\Lambda, a) \psi_\ell(x) U(\Lambda, a)^{-1} = \sum_{\bar{\ell}} D_{\ell\bar{\ell}}(\Lambda^{-1}) \psi_{\bar{\ell}}(\Lambda x + a). \quad (5.26)$$

The view we take here is that if the symmetry group of operators which act on the Hilbert space of physical states includes space and time inversions, then, if we allow for projective representations, we should also demand for some numbers A and B that

$$\mathbf{P} \psi_\ell(x) \mathbf{P}^{-1} = A \sum_{\bar{\ell}} D_{\ell\bar{\ell}}(\mathcal{P}^{-1}) \psi_{\bar{\ell}}(\mathcal{P}x) \quad (5.27)$$

and

$$\mathbf{T} \psi_\ell(x) \mathbf{T}^{-1} = B \sum_{\bar{\ell}} D_{\ell\bar{\ell}}(\mathcal{T}^{-1}) \psi_{\bar{\ell}}(-\mathcal{P}x) \quad (5.28)$$

with $|A| = |B| = 1$ in order for the free particle Hamiltonian to transform like a Lorentz scalar under parity and time reversal transformations.

5.3 Constructing Non-Standard Massive Spin-1/2 Quantum Fields

Before we impose these demands on each of the three non-standard Wigner classes, we here list the rest spinors that are solutions of the constraints imposed by the connected part of the Lorentz group. Because there is here an additional two-valued discrete index, if we allow for the possibility of having distinct particle and antiparticle species, the general form of the quantum field will look like

$$\psi_\ell(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3\sqrt{2p_0}} \sum_{\sigma,\tau} \left[e^{ip \cdot x} u_\ell(\mathbf{p}, \sigma, \tau) a(\mathbf{p}, \sigma, \tau) + e^{-ip \cdot x} v_\ell(\mathbf{p}, \sigma, \tau) b^\dagger(\mathbf{p}, \sigma, \tau) \right], \quad (5.29)$$

so there will be eight spinors instead of four like there is in the standard Wigner class. The rest spinors can be constructed following the same argument in Sec. (4.2) leading to Eqns. (4.10). Explicitly, these rest spinors are

$$\begin{aligned} u\left(\mathbf{0}, \frac{1}{2}, 1\right) &= \sqrt{\frac{m}{2}} \begin{pmatrix} c_+^1 \\ 0 \\ c_-^1 \\ 0 \end{pmatrix}, & u\left(\mathbf{0}, \frac{1}{2}, -1\right) &= \sqrt{\frac{m}{2}} \begin{pmatrix} c_+^{-1} \\ 0 \\ c_-^{-1} \\ 0 \end{pmatrix} \\ u\left(\mathbf{0}, \frac{-1}{2}, 1\right) &= \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ c_+^1 \\ 0 \\ c_-^1 \end{pmatrix}, & u\left(\mathbf{0}, \frac{-1}{2}, -1\right) &= \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ c_+^{-1} \\ 0 \\ c_-^{-1} \end{pmatrix}, \\ v\left(\mathbf{0}, \frac{1}{2}, 1\right) &= \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ d_+^1 \\ 0 \\ d_-^1 \end{pmatrix}, & v\left(\mathbf{0}, \frac{1}{2}, -1\right) &= \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ d_+^{-1} \\ 0 \\ d_-^{-1} \end{pmatrix}, \\ v\left(\mathbf{0}, \frac{-1}{2}, 1\right) &= \sqrt{\frac{m}{2}} \begin{pmatrix} -d_+^1 \\ 0 \\ -d_-^1 \\ 0 \end{pmatrix}, & v\left(\mathbf{0}, \frac{-1}{2}, -1\right) &= \sqrt{\frac{m}{2}} \begin{pmatrix} -d_+^{-1} \\ 0 \\ -d_-^{-1} \\ 0 \end{pmatrix}. \end{aligned}$$

The finite-dimensional boost operator $D(L(\mathbf{p}))$ [7, p.41–42] in the chiral representation, is, explicitly:

$$D(L(\mathbf{p})) = \kappa^{(\frac{1}{2}, 0)} \oplus \kappa^{(0, \frac{1}{2})} \quad (5.30)$$

where

$$\kappa^{(\frac{1}{2}, 0)} = \frac{1}{\sqrt{2m(m+p_0)}} \begin{pmatrix} m+p_0+p_z & p_x - ip_y \\ p_x + ip_y & m+p_0-p_z \end{pmatrix} \quad (5.31)$$

and

$$\kappa^{(0, \frac{1}{2})} = \frac{1}{\sqrt{2m(m+p_0)}} \begin{pmatrix} m+p_0-p_z & -(p_x - ip_y) \\ -(p_x + ip_y) & m+p_0+p_z \end{pmatrix}. \quad (5.32)$$

By slightly modifying Eqns. (2.129) – (2.130) with the inclusion of the τ index, the boosted spinors are given by $u(\mathbf{p}, \sigma, \tau) = D(L(\mathbf{p}))u(\mathbf{0}, \sigma, \tau)$ and $v(\mathbf{p}, \sigma, \tau) = D(L(\mathbf{p}))v(\mathbf{0}, \sigma, \tau)$. Explicitly these are:

$$u\left(\mathbf{p}, \frac{1}{2}, 1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} c_+^1(m+p_0+p_z) \\ c_+^1(p_x+ip_y) \\ c_-^1(m+p_0-p_z) \\ -c_-^1(p_x+ip_y) \end{pmatrix} \quad (5.33)$$

$$u\left(\mathbf{p}, \frac{1}{2}, -1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} c_+^{-1}(m+p_0+p_z) \\ c_+^{-1}(p_x+ip_y) \\ c_-^{-1}(m+p_0-p_z) \\ -c_-^{-1}(p_x+ip_y) \end{pmatrix} \quad (5.34)$$

$$u\left(\mathbf{p}, \frac{-1}{2}, 1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} c_+^1(p_x-ip_y) \\ c_+^1(m+p_0-p_z) \\ -c_-^1(p_x-ip_y) \\ c_-^1(m+p_0+p_z) \end{pmatrix} \quad (5.35)$$

$$u\left(\mathbf{p}, \frac{-1}{2}, -1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} c_+^{-1}(p_x-ip_y) \\ c_+^{-1}(m+p_0-p_z) \\ -c_-^{-1}(p_x-ip_y) \\ c_-^{-1}(m+p_0+p_z) \end{pmatrix} \quad (5.36)$$

$$v\left(\mathbf{p}, \frac{1}{2}, 1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} d_+^1(p_x-ip_y) \\ d_+^1(m+p_0-p_z) \\ -d_-^1(p_x-ip_y) \\ d_-^1(m+p_0+p_z) \end{pmatrix} \quad (5.37)$$

$$v\left(\mathbf{p}, \frac{1}{2}, -1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} d_+^{-1}(p_x-ip_y) \\ d_+^{-1}(m+p_0-p_z) \\ -d_-^{-1}(p_x-ip_y) \\ d_-^{-1}(m+p_0+p_z) \end{pmatrix} \quad (5.38)$$

$$v\left(\mathbf{p}, \frac{-1}{2}, 1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} -d_+^1(m+p_0+p_z) \\ -d_+^1(p_x+ip_y) \\ -d_-^1(m+p_0-p_z) \\ d_-^1(p_x+ip_y) \end{pmatrix} \quad (5.39)$$

$$v\left(\mathbf{p}, \frac{-1}{2}, -1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} -d_+^{-1}(m+p_0+p_z) \\ -d_+^{-1}(p_x+ip_y) \\ -d_-^{-1}(m+p_0-p_z) \\ d_-^{-1}(p_x+ip_y) \end{pmatrix}. \quad (5.40)$$

By inspection, we see that

$$\begin{aligned}
 u^\dagger\left(\mathbf{p}, \frac{1}{2}, 1\right) &= \left(\frac{c_+^{1*}(m+p_0+p_z)}{\sqrt{2(m+p_0)}}, \frac{c_+^{1*}(p_x-ip_y)}{\sqrt{2(m+p_0)}}, \frac{c_-^{1*}(m+p_0-p_z)}{\sqrt{2(m+p_0)}}, \frac{-c_-^{1*}(p_x-ip_y)}{\sqrt{2(m+p_0)}} \right) \\
 u^\dagger\left(\mathbf{p}, \frac{1}{2}, -1\right) &= \left(\frac{c_+^{-1*}(m+p_0+p_z)}{\sqrt{2(m+p_0)}}, \frac{c_+^{-1*}(p_x-ip_y)}{\sqrt{2(m+p_0)}}, \frac{c_-^{-1*}(m+p_0-p_z)}{\sqrt{2(m+p_0)}}, \frac{-c_-^{-1*}(p_x-ip_y)}{\sqrt{2(m+p_0)}} \right) \\
 u^\dagger\left(\mathbf{p}, \frac{-1}{2}, 1\right) &= \left(\frac{c_+^{1*}(p_x+ip_y)}{\sqrt{2(m+p_0)}}, \frac{c_+^{1*}(m+p_0-p_z)}{\sqrt{2(m+p_0)}}, \frac{-c_-^{1*}(p_x+ip_y)}{\sqrt{2(m+p_0)}}, \frac{c_-^{1*}(m+p_0+p_z)}{\sqrt{2(m+p_0)}} \right) \\
 u^\dagger\left(\mathbf{p}, \frac{-1}{2}, -1\right) &= \left(\frac{c_+^{-1*}(p_x+ip_y)}{\sqrt{2(m+p_0)}}, \frac{c_+^{-1*}(m+p_0-p_z)}{\sqrt{2(m+p_0)}}, \frac{-c_-^{-1*}(p_x+ip_y)}{\sqrt{2(m+p_0)}}, \frac{c_-^{-1*}(m+p_0+p_z)}{\sqrt{2(m+p_0)}} \right) \\
 v^\dagger\left(\mathbf{p}, \frac{1}{2}, 1\right) &= \left(\frac{d_+^{1*}(p_x+ip_y)}{\sqrt{2(m+p_0)}}, \frac{d_+^{1*}(m+p_0-p_z)}{\sqrt{2(m+p_0)}}, \frac{-d_-^{1*}(p_x+ip_y)}{\sqrt{2(m+p_0)}}, \frac{d_-^{1*}(m+p_0+p_z)}{\sqrt{2(m+p_0)}} \right) \\
 v^\dagger\left(\mathbf{p}, \frac{1}{2}, -1\right) &= \left(\frac{d_+^{-1*}(p_x+ip_y)}{\sqrt{2(m+p_0)}}, \frac{d_+^{-1*}(m+p_0-p_z)}{\sqrt{2(m+p_0)}}, \frac{-d_-^{-1*}(p_x+ip_y)}{\sqrt{2(m+p_0)}}, \frac{d_-^{-1*}(m+p_0+p_z)}{\sqrt{2(m+p_0)}} \right) \\
 v^\dagger\left(\mathbf{p}, \frac{-1}{2}, 1\right) &= \left(\frac{-d_+^{1*}(m+p_0+p_z)}{\sqrt{2(m+p_0)}}, \frac{-d_+^{1*}(p_x-ip_y)}{\sqrt{2(m+p_0)}}, \frac{-d_-^{1*}(m+p_0-p_z)}{\sqrt{2(m+p_0)}}, \frac{d_-^{1*}(p_x-ip_y)}{\sqrt{2(m+p_0)}} \right) \\
 v^\dagger\left(\mathbf{p}, \frac{-1}{2}, -1\right) &= \left(\frac{-d_+^{-1*}(m+p_0+p_z)}{\sqrt{2(m+p_0)}}, \frac{-d_+^{-1*}(p_x-ip_y)}{\sqrt{2(m+p_0)}}, \frac{-d_-^{-1*}(m+p_0-p_z)}{\sqrt{2(m+p_0)}}, \frac{d_-^{-1*}(p_x-ip_y)}{\sqrt{2(m+p_0)}} \right).
 \end{aligned}$$

In what follows we calculate the anticommutator $\{\psi_\alpha(x), \psi_\beta^\dagger(y)\}$ for spacelike separated spacetime points x and y . For any field $\psi_\alpha(x)$ to be local, this anticommutator must vanish. Evaluating this will give us conditions for the $c_{+/-}^{\pm 1}$ and $d_{+/-}^{\pm 1}$ to satisfy. Explicitly, we have

$$\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = \psi_\alpha(x)\psi_\beta^\dagger(y) + \psi_\beta^\dagger(y)\psi_\alpha(x) \quad (5.41)$$

which, upon inserting the mode expansions, becomes

$$\begin{aligned}
 & \int \frac{d^3 \mathbf{p}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma, \tau} [e^{ip \cdot x} u_\alpha(\mathbf{p}, \sigma, \tau) a(\mathbf{p}, \sigma, \tau) + e^{-ip \cdot x} v_\alpha(\mathbf{p}, \sigma, \tau) b^\dagger(\mathbf{p}, \sigma, \tau)] \times \\
 & \int \frac{d^3 \mathbf{p}'}{(2\pi)^3 \sqrt{2p'_0}} \sum_{\sigma', \tau'} [e^{-ip' \cdot y} u_\beta^\dagger(\mathbf{p}', \sigma', \tau') a^\dagger(\mathbf{p}', \sigma', \tau') + e^{ip' \cdot y} v_\beta^\dagger(\mathbf{p}', \sigma', \tau') b(\mathbf{p}', \sigma', \tau')] \\
 & + \int \frac{d^3 \mathbf{p}'}{(2\pi)^3 \sqrt{2p'_0}} \sum_{\sigma', \tau'} [e^{-ip' \cdot y} u_\beta^\dagger(\mathbf{p}', \sigma', \tau') a^\dagger(\mathbf{p}', \sigma', \tau') + e^{ip' \cdot y} v_\beta^\dagger(\mathbf{p}', \sigma', \tau') b(\mathbf{p}', \sigma', \tau')] \\
 & \times \int \frac{d^3 \mathbf{p}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma, \tau} [e^{ip \cdot x} u_\alpha(\mathbf{p}, \sigma, \tau) a(\mathbf{p}, \sigma, \tau) + e^{-ip \cdot x} v_\alpha(\mathbf{p}, \sigma, \tau) b^\dagger(\mathbf{p}, \sigma, \tau)].
 \end{aligned}$$

Collecting terms so as to give anticommutators for the creation and annihilation operators yields

$$\begin{aligned}
 & \int \int \frac{d^3 \mathbf{p} d^3 \mathbf{p}'}{(2\pi)^6 \sqrt{2p_0} \sqrt{2p'_0}} \sum_{\sigma, \tau, \sigma', \tau'} [e^{i(p \cdot x - p' \cdot y)} u_\alpha(\mathbf{p}, \sigma, \tau) u_\beta^\dagger(\mathbf{p}', \sigma', \tau') \{a(\mathbf{p}, \sigma, \tau), a^\dagger(\mathbf{p}', \sigma', \tau')\} \\
 & + e^{-i(p \cdot x - p' \cdot y)} v_\alpha(\mathbf{p}, \sigma, \tau) v_\beta^\dagger(\mathbf{p}', \sigma', \tau') \{b^\dagger(\mathbf{p}, \sigma, \tau), b(\mathbf{p}', \sigma', \tau')\} \\
 & + e^{i(p \cdot x + p' \cdot y)} u_\alpha(\mathbf{p}, \sigma, \tau) v_\beta^\dagger(\mathbf{p}', \sigma', \tau') \{a(\mathbf{p}, \sigma, \tau), b(\mathbf{p}', \sigma', \tau')\} \\
 & + e^{-i(p \cdot x + p' \cdot y)} v_\alpha(\mathbf{p}, \sigma, \tau) u_\beta^\dagger(\mathbf{p}', \sigma', \tau') \{b^\dagger(\mathbf{p}, \sigma, \tau), a^\dagger(\mathbf{p}', \sigma', \tau')\}].
 \end{aligned}$$

Evaluating the anticommutators then gives*

$$\begin{aligned}
 & \int \int \frac{d^3 \mathbf{p} d^3 \mathbf{p}'}{(2\pi)^6 \sqrt{2p_0} \sqrt{2p'_0}} \sum_{\sigma, \tau, \sigma', \tau'} [(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \delta_{\sigma\sigma'} \delta_{\tau\tau'} e^{i(p \cdot x - p' \cdot y)} u_\alpha(\mathbf{p}, \sigma, \tau) u_\beta^\dagger(\mathbf{p}', \sigma', \tau') \\
 & + (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \delta_{\sigma\sigma'} \delta_{\tau\tau'} e^{-i(p \cdot x - p' \cdot y)} v_\alpha(\mathbf{p}, \sigma, \tau) v_\beta^\dagger(\mathbf{p}', \sigma', \tau')]
 \end{aligned}$$

which, upon evaluating the $d^3 \mathbf{p}'$ integral, becomes

$$\int \frac{d^3 \mathbf{p}}{(2\pi)^3 2p_0} \sum_{\sigma, \tau} [e^{ip \cdot (x-y)} u_\alpha(\mathbf{p}, \sigma, \tau) u_\beta^\dagger(\mathbf{p}, \sigma, \tau) + e^{ip \cdot (y-x)} v_\alpha(\mathbf{p}, \sigma, \tau) v_\beta^\dagger(\mathbf{p}, \sigma, \tau)],$$

so in order for this to vanish we need to know the spin sums. They are calculated to be

$$\sum_{\sigma} u(\mathbf{p}, \sigma, 1) u^\dagger(\mathbf{p}, \sigma, 1) = \tag{5.42}$$

$$\begin{pmatrix} c_+^{1*} c_+^1 (p_0 + p_z) & c_+^{1*} c_+^1 (p_x - ip_y) & mc_+^{1*} c_-^{1*} & 0 \\ c_+^{1*} c_+^1 (p_x + ip_y) & c_+^{1*} c_+^1 (p_0 - p_z) & 0 & mc_+^{1*} c_-^{1*} \\ mc_+^{1*} c_-^1 & 0 & c_-^1 c_-^{1*} (p_0 - p_z) & -c_-^1 c_-^{1*} (p_x - ip_y) \\ 0 & mc_+^{1*} c_-^1 & -c_-^1 c_-^{1*} (p_x + ip_y) & c_-^1 c_-^{1*} (p_0 + p_z) \end{pmatrix},$$

$$\sum_{\sigma} u(\mathbf{p}, \sigma, -1) u^\dagger(\mathbf{p}, \sigma, -1) = \tag{5.43}$$

*The results are unchanged if $b^\dagger(\mathbf{p}, \sigma, \tau) = a^\dagger(\mathbf{p}, \sigma, \tau)$.

5.3 Constructing Non-Standard Massive Spin-1/2 Quantum Fields

$$\begin{pmatrix} c_+^{-1*} c_+^{-1}(p_0 + p_z) & c_+^{-1*} c_+^{-1}(p_x - ip_y) & mc_+^{-1} c_-^{-1*} & 0 \\ c_+^{-1*} c_+^{-1}(p_x + ip_y) & c_+^{-1*} c_+^{-1}(p_0 - p_z) & 0 & mc_+^{-1} c_-^{-1*} \\ mc_+^{-1*} c_-^{-1} & 0 & c_-^{-1} c_-^{-1*}(p_0 - p_z) & -c_-^{-1} c_-^{-1*}(p_x - ip_y) \\ 0 & mc_+^{-1*} c_-^{-1} & -c_-^{-1} c_-^{-1*}(p_x + ip_y) & c_-^{-1} c_-^{-1*}(p_0 + p_z) \end{pmatrix}$$

$$\sum_{\sigma} v(\mathbf{p}, \sigma, 1) v^{\dagger}(\mathbf{p}, \sigma, 1) = \quad (5.44)$$

$$\begin{pmatrix} d_+^{1*} d_+^1(p_0 + p_z) & d_+^{1*} d_+^1(p_x - ip_y) & md_+^{1*} d_-^{1*} & 0 \\ d_+^{1*} d_+^1(p_x + ip_y) & d_+^{1*} d_+^1(p_0 - p_z) & 0 & md_+^{1*} d_-^{1*} \\ md_+^{1*} d_-^1 & 0 & d_-^1 d_-^{1*}(p_0 - p_z) & -d_-^1 d_-^{1*}(p_x - ip_y) \\ 0 & md_+^{1*} d_-^1 & -d_-^1 d_-^{1*}(p_x + ip_y) & d_-^1 d_-^{1*}(p_0 + p_z) \end{pmatrix},$$

$$\sum_{\sigma} v(\mathbf{p}, \sigma, -1) v^{\dagger}(\mathbf{p}, \sigma, -1) = \quad (5.45)$$

$$\begin{pmatrix} d_+^{-1*} d_+^{-1}(p_0 + p_z) & d_+^{-1*} d_+^{-1}(p_x - ip_y) & md_+^{-1*} d_-^{-1*} & 0 \\ d_+^{-1*} d_+^{-1}(p_x + ip_y) & d_+^{-1*} d_+^{-1}(p_0 - p_z) & 0 & md_+^{-1*} d_-^{-1*} \\ md_+^{-1*} d_-^{-1} & 0 & d_-^{-1} d_-^{-1*}(p_0 - p_z) & -d_-^{-1} d_-^{-1*}(p_x - ip_y) \\ 0 & md_+^{-1*} d_-^{-1} & -d_-^{-1} d_-^{-1*}(p_x + ip_y) & d_-^{-1} d_-^{-1*}(p_0 + p_z) \end{pmatrix}.$$

If we make the replacement $p_{\mu} \rightarrow -i\partial_{\mu}$, then the set of anticommutators for different combinations of (α, β) all take the general form

$$(\text{stuff}) \Delta_+(x - y) + (\text{stuff}) \Delta_+(y - x). \quad (5.46)$$

If we look at the upper left quadrant of the spin sums, we see that the anticommutators for those four cases take the general form:

$$\left(\sum c's \right) (\partial_1) \Delta_+(x - y) + \left(\sum d's \right) (\partial_2) \Delta_+(y - x) \quad (5.47)$$

where the symbols ∂_1 and ∂_2 here stand for linear combinations of derivative operators. Also, for each pair (α, β) we have $\partial_1 = \partial_2$. Since $\Delta_+(y - x)$ is an even function of $y - x$, its derivatives are odd functions of $y - x$ so we get

$$\left(\sum c's - \sum d's \right) (\partial_1) \Delta_+(x - y). \quad (5.48)$$

The factors $(\partial_1) \Delta_+(x - y)$ do not vanish in general, so for the anticommutators corresponding to $\alpha = 1, 2$ and $\beta = 1, 2$ to vanish, we require that the factors $(\sum c's - \sum d's)$ must vanish. Similarly for the anticommutators corresponding to the bottom right quadrant of the spin sums. For the anticommutators corresponding to the upper right and lower left quadrants, we have, (remembering the evenness of the function $\Delta_+(y - x)$):

$$\left(\sum c's + \sum d's \right) m \Delta_+(x - y) \quad (5.49)$$

5 Non-Standard Quantum Fields

which only vanishes in general if the factor $(\sum c's + \sum d's)$ vanishes. Explicitly, we therefore require that:

$$c_+^{1*}c_+^1 + c_+^{-1*}c_+^{-1} = d_+^{1*}d_+^1 + d_+^{-1*}d_+^{-1} \quad (5.50)$$

$$c_-^{1*}c_-^1 + c_-^{-1*}c_-^{-1} = d_-^{1*}d_-^1 + d_-^{-1*}d_-^{-1} \quad (5.51)$$

$$c_+^{1*}c_-^1 + c_+^{-1*}c_-^{-1} = -d_+^{1*}d_-^1 - d_+^{-1*}d_-^{-1}. \quad (5.52)$$

For any non-standard local massive spin-1/2 quantum field, it is desired that these equations be satisfied in order for the quantum field to be local. The other anticommutator that needs to vanish is between the field and itself at spacelike separation. We do this calculation in Sec. (5.6). In the following section, we begin our search for non-trivial spin-1/2 quantum fields.

5.3.1 Case 2 Quantum Field Search

In this section we start the search for non-trivial local massive non-standard spin-1/2 quantum fields by looking at Case 2. The final ingredient needed to begin the search is to know how the creation and annihilation operators transform under parity and time reversal. Creation and annihilation operators are defined on the multiparticle state space. The space is spanned by the kets

$$\bigotimes_{i=1}^N |\mathbf{p}_i, \sigma_i, \tau_i, n_i\rangle. \quad (5.53)$$

Looking at parity, following similar reasoning as that found in Sec. (2.9) leading up to Eqn. (2.58) we write:

$$\mathbf{P}a^\dagger(\mathbf{p}, \sigma, \tau)\mathbf{P}^{-1} = \eta\tau a^\dagger(-\mathbf{p}, \sigma, \tau). \quad (5.54)$$

Taking the adjoint of both sides and taking advantage of the fact that $\mathbf{P}^\dagger = \mathbf{P}^{-1}$ gives the transformation rule for the annihilation operators under parity to be

$$\mathbf{P}a(\mathbf{p}, \sigma, \tau)\mathbf{P}^{-1} = \eta^*\tau a(-\mathbf{p}, \sigma, \tau). \quad (5.55)$$

We now calculate the parity transformed arbitrary spin-1/2 field operator $\mathbf{P}\psi_\ell(x)\mathbf{P}^{-1}$ where the spinors have not been fixed yet. Taking advantage of the fact that \mathbf{P} is a linear operator, it can be freely moved through the exponential factors as well as the u and v coefficients so that we get

$$\begin{aligned} \mathbf{P}\psi_\ell(x)\mathbf{P}^{-1} &= \int \frac{d^3\mathbf{p}}{(2\pi)^3\sqrt{2p_0}} \sum_{\sigma,\tau} [e^{ip\cdot x} u_\ell(\mathbf{p}, \sigma, \tau) \mathbf{P}a(\mathbf{p}, \sigma, \tau)\mathbf{P}^{-1} + \\ &\quad e^{-ip\cdot x} v_\ell(\mathbf{p}, \sigma, \tau) \mathbf{P}a^\dagger(\mathbf{p}, \sigma, \tau)\mathbf{P}^{-1}], \end{aligned} \quad (5.56)$$

which becomes

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3\sqrt{2p_0}} \sum_{\sigma,\tau} [e^{ip\cdot x} \eta^*\tau u_\ell(\mathbf{p}, \sigma, \tau) a(-\mathbf{p}, \sigma, \tau) + e^{-ip\cdot x} \eta\tau v_\ell(\mathbf{p}, \sigma, \tau) a^\dagger(-\mathbf{p}, \sigma, \tau)]. \quad (5.57)$$

5.3 Constructing Non-Standard Massive Spin-1/2 Quantum Fields

Changing the variable of integration from \mathbf{p} to $-\mathbf{p}$ gives

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3\sqrt{2p_0}} \sum_{\sigma,\tau} [e^{i\mathcal{P}p \cdot x} \eta^* \tau u_\ell(-\mathbf{p}, \sigma, \tau) a(\mathbf{p}, \sigma, \tau) + e^{-i\mathcal{P}p \cdot x} \eta \tau v_\ell(-\mathbf{p}, \sigma, \tau) a^\dagger(\mathbf{p}, \sigma, \tau)]. \quad (5.58)$$

In order for this to be equal to the right hand side of Eqn. (5.27), it is necessary and sufficient that

$$\eta^* \tau u(-\mathbf{p}, \sigma, \tau) = AD(\mathcal{P}^{-1})u(\mathbf{p}, \sigma, \tau) \quad (5.59)$$

and

$$\eta \tau v(-\mathbf{p}, \sigma, \tau) = AD(\mathcal{P}^{-1})v(\mathbf{p}, \sigma, \tau) \quad (5.60)$$

for some A . To break this down a bit more, we observe that[†]

$$u(-\mathbf{p}, \sigma, \tau) = D(L(-\mathbf{p}))u(\mathbf{0}, \sigma, \tau) = \beta D(L(\mathbf{p}))\beta u(\mathbf{0}, \sigma, \tau) \quad (5.61)$$

and

$$v(-\mathbf{p}, \sigma, \tau) = D(L(-\mathbf{p}))v(\mathbf{0}, \sigma, \tau) = \beta D(L(\mathbf{p}))\beta v(\mathbf{0}, \sigma, \tau) \quad (5.62)$$

where $\beta = D(\mathcal{P}^{-1})$ is the constant matrix

$$\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (5.63)$$

so we would like

$$\eta^* \tau \beta u(\mathbf{0}, \sigma, \tau) = Au(\mathbf{0}, \sigma, \tau) \quad (5.64)$$

and

$$\eta \tau \beta v(\mathbf{0}, \sigma, \tau) = Av(\mathbf{0}, \sigma, \tau) \quad (5.65)$$

[†]We have written $D(L(-\mathbf{p})) = \beta D(L(\mathbf{p}))\beta$ rather than $\beta D(L(\mathbf{p}))\beta^{-1}$ because for the standard finite-dimensional chiral representation, $\beta = \beta^{-1}$.

or, more explicitly,

$$\eta^* \beta u \left(\mathbf{0}, \frac{1}{2}, 1 \right) = Au \left(\mathbf{0}, \frac{1}{2}, 1 \right) \quad (5.66)$$

$$\eta^* \beta u \left(\mathbf{0}, \frac{1}{2}, -1 \right) = -Au \left(\mathbf{0}, \frac{1}{2}, -1 \right) \quad (5.67)$$

$$\eta^* \beta u \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = Au \left(\mathbf{0}, \frac{-1}{2}, 1 \right) \quad (5.68)$$

$$\eta^* \beta u \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = -Au \left(\mathbf{0}, \frac{-1}{2}, -1 \right) \quad (5.69)$$

$$\eta \beta v \left(\mathbf{0}, \frac{1}{2}, 1 \right) = Av \left(\mathbf{0}, \frac{1}{2}, 1 \right) \quad (5.70)$$

$$\eta \beta v \left(\mathbf{0}, \frac{1}{2}, -1 \right) = -Av \left(\mathbf{0}, \frac{1}{2}, -1 \right) \quad (5.71)$$

$$\eta \beta v \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = Av \left(\mathbf{0}, \frac{-1}{2}, 1 \right) \quad (5.72)$$

$$\eta \beta v \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = -Av \left(\mathbf{0}, \frac{-1}{2}, -1 \right). \quad (5.73)$$

By writing these equations out in terms of the explicit spinors, a set of equations can be obtained for the c and d numbers which are present in the rest spinors. The matrix equations are

$$\eta^* \beta u \left(\mathbf{0}, \frac{1}{2}, 1 \right) = \quad (5.74)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta^* c_+^1 \\ 0 \\ \eta^* c_-^1 \\ 0 \end{pmatrix} = \begin{pmatrix} \eta^* c_-^1 \\ 0 \\ \eta^* c_+^1 \\ 0 \end{pmatrix} = \begin{pmatrix} Ac_+^1 \\ 0 \\ Ac_-^1 \\ 0 \end{pmatrix}$$

$$\eta^* \beta u \left(\mathbf{0}, \frac{1}{2}, -1 \right) = \quad (5.75)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta^* c_+^{-1} \\ 0 \\ \eta^* c_-^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} \eta^* c_-^{-1} \\ 0 \\ \eta^* c_+^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} -Ac_+^{-1} \\ 0 \\ -Ac_-^{-1} \\ 0 \end{pmatrix}$$

$$\eta^* \beta u \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = \quad (5.76)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \eta^* c_+^1 \\ 0 \\ \eta^* c_-^1 \end{pmatrix} = \begin{pmatrix} 0 \\ \eta^* c_-^1 \\ 0 \\ \eta^* c_+^1 \end{pmatrix} = \begin{pmatrix} 0 \\ Ac_+^1 \\ 0 \\ Ac_-^1 \end{pmatrix}$$

$$\eta^* \beta u \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = \quad (5.77)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \eta^* c_+^{-1} \\ 0 \\ \eta^* c_-^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \eta^* c_-^{-1} \\ 0 \\ \eta^* c_+^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ -Ac_+^{-1} \\ 0 \\ -Ac_-^{-1} \end{pmatrix}$$

$$\eta \beta v \left(\mathbf{0}, \frac{1}{2}, 1 \right) = \quad (5.78)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \eta d_+^1 \\ 0 \\ \eta d_-^1 \end{pmatrix} = \begin{pmatrix} 0 \\ \eta d_-^1 \\ 0 \\ \eta d_+^1 \end{pmatrix} = \begin{pmatrix} 0 \\ Ad_+^1 \\ 0 \\ Ad_-^1 \end{pmatrix}$$

$$\eta \beta v \left(\mathbf{0}, \frac{1}{2}, -1 \right) = \quad (5.79)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \eta d_+^{-1} \\ 0 \\ \eta d_-^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \eta d_-^{-1} \\ 0 \\ \eta d_+^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ -Ad_+^{-1} \\ 0 \\ -Ad_-^{-1} \end{pmatrix}$$

$$\eta \beta v \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = \quad (5.80)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\eta d_+^1 \\ 0 \\ -\eta d_-^1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\eta d_-^1 \\ 0 \\ -\eta d_+^1 \\ 0 \end{pmatrix} = \begin{pmatrix} -Ad_+^1 \\ 0 \\ -Ad_-^1 \\ 0 \end{pmatrix}$$

$$\eta \beta v \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = \quad (5.81)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\eta d_+^{-1} \\ 0 \\ -\eta d_-^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} -\eta d_-^{-1} \\ 0 \\ -\eta d_+^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} Ad_+^{-1} \\ 0 \\ Ad_-^{-1} \\ 0 \end{pmatrix}.$$

Before gathering up the above conditions on the c 's and d 's, the conditions on the c 's and d 's from time reversal will now be identified. To see what the transformation laws for creation and annihilation operators under time reversal are, we observe that for Case 2,

$$\mathbf{T} |\mathbf{p}, \sigma, \tau\rangle = \tau(-1)^{\frac{1}{2}+\sigma} a^\dagger(-\mathbf{p}, -\sigma, -\tau) |0\rangle \quad (5.82)$$

and also that[‡]

$$\mathbf{T} |\mathbf{p}, \sigma, \tau\rangle = \mathbf{T} a^\dagger(\mathbf{p}, \sigma, \tau) \mathbf{T}^{-1} |0\rangle. \quad (5.83)$$

Following the reasoning contained in Sec. (2.9) we see that the creation and annihilation operator time reversal transformation laws are

$$\mathbf{T} a^\dagger(\mathbf{p}, \sigma, \tau) \mathbf{T}^{-1} = \tau(-1)^{\frac{1}{2}+\sigma} a^\dagger(-\mathbf{p}, -\sigma, -\tau) \quad (5.84)$$

and

$$\mathbf{T} a(\mathbf{p}, \sigma, \tau) \mathbf{T}^{-1} = \tau(-1)^{\frac{1}{2}+\sigma} a(-\mathbf{p}, -\sigma, -\tau) \quad (5.85)$$

respectively. Since the anti-unitary time reversal operator is anti-linear, the field operator transforms as

$$\begin{aligned} \mathbf{T} \psi_\ell(x) \mathbf{T}^{-1} = \int \frac{d^3 \mathbf{p}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma, \tau} [e^{-ip \cdot x} u_\ell^*(\mathbf{p}, \sigma, \tau) \mathbf{T} a(\mathbf{p}, \sigma, \tau) \mathbf{T}^{-1} \\ + e^{ip \cdot x} v_\ell^*(\mathbf{p}, \sigma, \tau) \mathbf{T} a^\dagger(\mathbf{p}, \sigma, \tau) \mathbf{T}^{-1}]. \end{aligned} \quad (5.86)$$

which, after making the replacements $\mathbf{p} \rightarrow -\mathbf{p}$, $\sigma \rightarrow -\sigma$, and $\tau \rightarrow -\tau$, becomes

$$\int \frac{d^3 \mathbf{p}(-\tau)(-1)^{\frac{1}{2}-\sigma}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma, \tau} [e^{-i\mathcal{P}p \cdot x} u^*(-\mathbf{p}, -\sigma, -\tau) a(\mathbf{p}, \sigma, \tau) + e^{i\mathcal{P}p \cdot x} v^*(-\mathbf{p}, -\sigma, -\tau) a^\dagger(\mathbf{p}, \sigma, \tau)]. \quad (5.87)$$

In order for Eqn. (5.28) to hold, we require that

$$-\tau(-1)^{\frac{1}{2}-\sigma} u^*(-\mathbf{p}, -\sigma, -\tau) = BTu(\mathbf{p}, \sigma, \tau) \quad (5.88)$$

and

$$-\tau(-1)^{\frac{1}{2}-\sigma} v^*(-\mathbf{p}, -\sigma, -\tau) = BTv(\mathbf{p}, \sigma, \tau), \quad (5.89)$$

where $T = D(\mathcal{T}^{-1})$ is the constant matrix

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (5.90)$$

The u and v spinors on the left hand side can be re-written so that[§]

$$u^*(-\mathbf{p}, -\sigma, -\tau) = D^*(L(-\mathbf{p}))u^*(\mathbf{0}, -\sigma, -\tau) = \gamma_5 \mathcal{C} D(L(\mathbf{p})) \mathcal{C}^{-1} \gamma_5 u^*(\mathbf{0}, -\sigma, -\tau) \quad (5.91)$$

and

$$v^*(-\mathbf{p}, -\sigma, -\tau) = D^*(L(-\mathbf{p}))v^*(\mathbf{0}, -\sigma, -\tau) = \gamma_5 \mathcal{C} D(L(\mathbf{p})) \mathcal{C}^{-1} \gamma_5 v^*(\mathbf{0}, -\sigma, -\tau). \quad (5.92)$$

[‡]since $\mathbf{T} |0\rangle = |0\rangle$.

[§]The matrix \mathcal{C} appearing here is defined to be the matrix $\gamma_2 \beta$ in the chiral representation. The 4×4 matrix γ_5 is defined as $i\gamma_0 \gamma_1 \gamma_2 \gamma_3$.

5.3 Constructing Non-Standard Massive Spin-1/2 Quantum Fields

Identifying T with $\gamma_5 \mathcal{C}$, we require that

$$-(-1)^{\frac{1}{2}-\sigma} T u^*(\mathbf{0}, -\sigma, -\tau) = -\tau B u(\mathbf{0}, \sigma, \tau) \quad (5.93)$$

and

$$-(-1)^{\frac{1}{2}-\sigma} T v^*(\mathbf{0}, -\sigma, -\tau) = -\tau B v(\mathbf{0}, \sigma, \tau). \quad (5.94)$$

Explicitly, we have

$$T u^*\left(\mathbf{0}, \frac{1}{2}, 1\right) = B u\left(\mathbf{0}, \frac{-1}{2}, -1\right) \quad (5.95)$$

$$T u^*\left(\mathbf{0}, \frac{1}{2}, -1\right) = -B u\left(\mathbf{0}, \frac{-1}{2}, 1\right) \quad (5.96)$$

$$T u^*\left(\mathbf{0}, \frac{-1}{2}, 1\right) = -B u\left(\mathbf{0}, \frac{1}{2}, -1\right) \quad (5.97)$$

$$T u^*\left(\mathbf{0}, \frac{-1}{2}, -1\right) = B u\left(\mathbf{0}, \frac{1}{2}, 1\right) \quad (5.98)$$

$$T v^*\left(\mathbf{0}, \frac{1}{2}, 1\right) = B v\left(\mathbf{0}, \frac{-1}{2}, -1\right) \quad (5.99)$$

$$T v^*\left(\mathbf{0}, \frac{1}{2}, -1\right) = -B v\left(\mathbf{0}, \frac{-1}{2}, 1\right) \quad (5.100)$$

$$T v^*\left(\mathbf{0}, \frac{-1}{2}, 1\right) = -B v\left(\mathbf{0}, \frac{1}{2}, -1\right) \quad (5.101)$$

$$T v^*\left(\mathbf{0}, \frac{-1}{2}, -1\right) = B v\left(\mathbf{0}, \frac{1}{2}, 1\right). \quad (5.102)$$

The explicit matrix equations are

$$T u^*\left(\mathbf{0}, \frac{1}{2}, 1\right) = \quad (5.103)$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_+^{1*} \\ 0 \\ c_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ c_+^{1*} \\ 0 \\ c_-^{1*} \end{pmatrix} = \begin{pmatrix} 0 \\ B c_+^{-1} \\ 0 \\ B c_-^{-1} \end{pmatrix}$$

$$T u^*\left(\mathbf{0}, \frac{1}{2}, -1\right) = \quad (5.104)$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_+^{-1*} \\ 0 \\ c_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ c_+^{-1*} \\ 0 \\ c_-^{-1*} \end{pmatrix} = \begin{pmatrix} 0 \\ -B c_+^1 \\ 0 \\ -B c_-^1 \end{pmatrix}$$

$$Tu^* \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = \quad (5.105)$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ c_+^{1*} \\ 0 \\ c_-^{1*} \end{pmatrix} = \begin{pmatrix} -c_+^{1*} \\ 0 \\ -c_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} -Bc_+^{-1} \\ 0 \\ -Bc_-^{-1} \\ 0 \end{pmatrix}$$

$$Tu^* \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = \quad (5.106)$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ c_+^{-1*} \\ 0 \\ c_-^{-1*} \end{pmatrix} = \begin{pmatrix} -c_+^{-1*} \\ 0 \\ -c_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} Bc_+^1 \\ 0 \\ Bc_-^1 \\ 0 \end{pmatrix}$$

$$Tv^* \left(\mathbf{0}, \frac{1}{2}, 1 \right) = \quad (5.107)$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ d_+^{1*} \\ 0 \\ d_-^{1*} \end{pmatrix} = \begin{pmatrix} -d_+^{1*} \\ 0 \\ -d_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} -Bd_+^{-1} \\ 0 \\ -Bd_-^{-1} \\ 0 \end{pmatrix}$$

$$Tv^* \left(\mathbf{0}, \frac{1}{2}, -1 \right) = \quad (5.108)$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ d_+^{-1*} \\ 0 \\ d_-^{-1*} \end{pmatrix} = \begin{pmatrix} -d_+^{-1*} \\ 0 \\ -d_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} Bd_+^1 \\ 0 \\ Bd_-^1 \\ 0 \end{pmatrix}$$

$$Tv^* \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = \quad (5.109)$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -d_+^{1*} \\ 0 \\ -d_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -d_+^{1*} \\ 0 \\ -d_-^{1*} \end{pmatrix} = \begin{pmatrix} 0 \\ -Bd_+^{-1} \\ 0 \\ -Bd_-^{-1} \end{pmatrix}$$

$$Tv^* \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = \quad (5.110)$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -d_+^{-1*} \\ 0 \\ -d_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -d_+^{-1*} \\ 0 \\ -d_-^{-1*} \end{pmatrix} = \begin{pmatrix} 0 \\ Bd_+^1 \\ 0 \\ Bd_-^1 \end{pmatrix}.$$

5.3 Constructing Non-Standard Massive Spin-1/2 Quantum Fields

The constraints on the c 's and d 's imposed by Eqn. (5.27) and Eqn. (5.28) can be read off the above explicit matrix equations to be

$$\eta^* c_-^1 = A c_+^1 \quad (5.111)$$

$$\eta^* c_+^1 = A c_-^1 \quad (5.112)$$

$$\eta^* c_-^{-1} = -A c_+^{-1} \quad (5.113)$$

$$\eta^* c_+^{-1} = -A c_-^{-1} \quad (5.114)$$

$$\eta d_-^1 = A d_+^1 \quad (5.115)$$

$$\eta d_+^1 = A d_-^1 \quad (5.116)$$

$$\eta d_-^{-1} = -A d_+^{-1} \quad (5.117)$$

$$\eta d_+^{-1} = -A d_-^{-1} \quad (5.118)$$

and

$$c_+^{1*} = B c_+^{-1} \quad (5.119)$$

$$c_-^{1*} = B c_-^{-1} \quad (5.120)$$

$$c_+^{-1*} = -B c_+^1 \quad (5.121)$$

$$c_-^{-1*} = -B c_-^1 \quad (5.122)$$

$$d_+^{1*} = B d_+^{-1} \quad (5.123)$$

$$d_-^{1*} = B d_-^{-1} \quad (5.124)$$

$$d_+^{-1*} = -B d_+^1 \quad (5.125)$$

$$d_-^{-1*} = -B d_-^1. \quad (5.126)$$

We now demonstrate that there exist no non-trivial solutions to the above set of equations. Firstly, if we consider Eqn. (5.112) and Eqn. (5.113) we can take ratios and form the equation

$$\frac{c_+^1}{c_-^{-1}} = \frac{c_-^1}{-c_+^{-1}}. \quad (5.127)$$

Rearranging slightly and taking the complex conjugate of both sides yields

$$c_+^{1*} = \frac{c_-^{1*} c_-^{-1*}}{-c_+^{-1*}}. \quad (5.128)$$

Now if we take Eqn. (5.119) and Eqn. (5.120) and take ratios to combine the equations we obtain

$$\frac{c_+^{1*}}{c_-^{1*}} = \frac{c_+^{-1}}{c_-^{-1}}. \quad (5.129)$$

Rearranging this yields

$$c_+^{1*} = \frac{c_+^{-1} c_-^{1*}}{c_-^{-1}}. \quad (5.130)$$

Combining Eqn. (5.128) with Eqn. (5.130) yields

$$c_-^{-1*} c_-^{-1} = -c_+^{-1*} c_+^{-1} \quad (5.131)$$

which only has the trivial solution

$$c_+^1 = c_-^1 = c_+^{-1} = c_-^{-1} = 0. \quad (5.132)$$

Similarly, we can form the equations

$$d_+^{1*} = \frac{d_-^{1*} d_-^{-1*}}{-d_+^{-1*}} \quad (5.133)$$

and

$$d_+^{1*} = \frac{d_+^{-1} d_-^{1*}}{d_-^{-1}}. \quad (5.134)$$

and combine them to obtain

$$d_-^{-1*} d_-^{-1} = -d_+^{-1*} d_+^{-1} \quad (5.135)$$

for which the only solution is the trivial solution:

$$d_+^1 = d_-^1 = d_+^{-1} = d_-^{-1} = 0. \quad (5.136)$$

At this point we note that the η 's did not feature in this argument due to the taking of ratios. If we had have explicitly considered the case of distinct particles and antiparticles where η_n and $\eta_{\bar{n}}$ are not necessarily equal, the ratios would have removed these phases anyway and we would still have arrived at just the trivial solution. Thus the only massive spin-1/2 quantum field that transforms under parity and time reversal as in Case 2, is the trivial quantum field

$$\psi_\ell(x) = 0. \quad (5.137)$$

5.3.2 Case 3 Quantum Field Search

In this case (for $m > 0$, $j = 1/2$), we have $\mathbf{P} |\mathbf{p}, \sigma, \tau\rangle = \eta\tau |-\mathbf{p}, \sigma, \tau\rangle$ and also $\mathbf{T} |\mathbf{p}, \sigma, \tau\rangle = (-1)^{\frac{1}{2}+\sigma} |-\mathbf{p}, -\sigma, -\tau\rangle$. Demanding that Eqn. (5.27) and Eqn. (5.28) hold as with Case 2, and proceeding exactly as in Case 2, the parity transformation law is identical to Case 2, so yields the same set of conditions for the c 's and d 's that were obtained in Case 2. The time reversal transformation law differs from Case 2 only by a factor of τ . The absence of a τ in Case 3 results in a few sign changes in the results obtained for Case 2. We now demand that

$$(-1)^{\frac{1}{2}-\sigma} T u^*(\mathbf{0}, -\sigma, -\tau) = B u(\mathbf{0}, \sigma, \tau) \quad (5.138)$$

and

$$(-1)^{\frac{1}{2}-\sigma} T v^*(\mathbf{0}, -\sigma, -\tau) = B v(\mathbf{0}, \sigma, \tau), \quad (5.139)$$

5.3 Constructing Non-Standard Massive Spin-1/2 Quantum Fields

which gives rise to the following set of equations:

$$Tu\left(\mathbf{0}, \frac{1}{2}, 1\right) = -Bu\left(\mathbf{0}, -\frac{1}{2}, -1\right) \quad (5.140)$$

$$Tu\left(\mathbf{0}, \frac{1}{2}, -1\right) = -Bu\left(\mathbf{0}, \frac{1}{2}, 1\right) \quad (5.141)$$

$$Tu\left(\mathbf{0}, -\frac{1}{2}, 1\right) = Bu\left(\mathbf{0}, \frac{1}{2}, -1\right) \quad (5.142)$$

$$Tu\left(\mathbf{0}, -\frac{1}{2}, -1\right) = Bu\left(\mathbf{0}, \frac{1}{2}, 1\right) \quad (5.143)$$

$$Tv\left(\mathbf{0}, \frac{1}{2}, 1\right) = -Bv\left(\mathbf{0}, -\frac{1}{2}, -1\right) \quad (5.144)$$

$$Tv\left(\mathbf{0}, \frac{1}{2}, -1\right) = -Bv\left(\mathbf{0}, -\frac{1}{2}, 1\right) \quad (5.145)$$

$$Tv\left(\mathbf{0}, -\frac{1}{2}, 1\right) = Bv\left(\mathbf{0}, \frac{1}{2}, -1\right) \quad (5.146)$$

$$Tv\left(\mathbf{0}, -\frac{1}{2}, -1\right) = Bv\left(\mathbf{0}, \frac{1}{2}, 1\right). \quad (5.147)$$

This translates into the following set of matrix equations:

$$Tu^*\left(\mathbf{0}, \frac{1}{2}, 1\right) = \quad (5.148)$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_+^{1*} \\ 0 \\ c_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ c_+^{1*} \\ 0 \\ c_-^{1*} \end{pmatrix} = \begin{pmatrix} 0 \\ -Bc_+^{-1} \\ 0 \\ -Bc_-^{-1} \end{pmatrix}$$

$$Tu^*\left(\mathbf{0}, \frac{1}{2}, -1\right) = \quad (5.149)$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_+^{-1*} \\ 0 \\ c_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ c_+^{-1*} \\ 0 \\ c_-^{-1*} \end{pmatrix} = \begin{pmatrix} 0 \\ -Bc_+^1 \\ 0 \\ -Bc_-^1 \end{pmatrix}$$

$$Tu^*\left(\mathbf{0}, -\frac{1}{2}, 1\right) = \quad (5.150)$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ c_+^{1*} \\ 0 \\ c_-^{1*} \end{pmatrix} = \begin{pmatrix} -c_+^{1*} \\ 0 \\ -c_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} Bc_+^{-1} \\ 0 \\ Bc_-^{-1} \\ 0 \end{pmatrix}$$

$$\begin{aligned}
 Tu^* \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = & \quad (5.151) \\
 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ c_+^{-1*} \\ 0 \\ c_-^{-1*} \end{pmatrix} &= \begin{pmatrix} -c_+^{-1*} \\ 0 \\ -c_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} Bc_+^1 \\ 0 \\ Bc_-^1 \\ 0 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 Tv^* \left(\mathbf{0}, \frac{1}{2}, 1 \right) = & \quad (5.152) \\
 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ d_+^{1*} \\ 0 \\ d_-^{1*} \end{pmatrix} &= \begin{pmatrix} -d_+^{1*} \\ 0 \\ -d_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} Bd_+^{-1} \\ 0 \\ Bd_-^{-1} \\ 0 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 Tv^* \left(\mathbf{0}, \frac{1}{2}, -1 \right) = & \quad (5.153) \\
 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ d_+^{-1*} \\ 0 \\ d_-^{-1*} \end{pmatrix} &= \begin{pmatrix} -d_+^{-1*} \\ 0 \\ -d_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} Bd_+^1 \\ 0 \\ Bd_-^1 \\ 0 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 Tv^* \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = & \quad (5.154) \\
 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -d_+^{1*} \\ 0 \\ -d_-^{1*} \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ -d_+^{1*} \\ 0 \\ -d_-^{1*} \end{pmatrix} = \begin{pmatrix} 0 \\ Bd_+^{-1} \\ 0 \\ Bd_-^{-1} \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 Tv^* \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = & \quad (5.155) \\
 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -d_+^{-1*} \\ 0 \\ -d_-^{-1*} \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ -d_+^{-1*} \\ 0 \\ -d_-^{-1*} \end{pmatrix} = \begin{pmatrix} 0 \\ Bd_+^1 \\ 0 \\ Bd_-^1 \end{pmatrix}.
 \end{aligned}$$

5.3 Constructing Non-Standard Massive Spin-1/2 Quantum Fields

Reading off these matrix equations yields the equations which need to be satisfied, to be:

$$c_+^{1*} = -Bc_+^{-1} \quad (5.156)$$

$$c_-^{1*} = -Bc_-^{-1} \quad (5.157)$$

$$c_+^{-1*} = -Bc_+^1 \quad (5.158)$$

$$c_-^{-1*} = -Bc_-^1 \quad (5.159)$$

$$d_+^{1*} = -Bd_+^{-1} \quad (5.160)$$

$$d_-^{1*} = -Bd_-^{-1} \quad (5.161)$$

$$d_+^{-1*} = -Bd_+^1 \quad (5.162)$$

$$d_-^{-1*} = -Bd_-^1. \quad (5.163)$$

Combining Eqn. (5.156) and Eqn. (5.157) gives

$$c_+^{1*} = \frac{c_-^{1*}c_+^{-1}}{c_-^{-1}}. \quad (5.164)$$

Since Case 2 and Case 3 have the same formula for how the parity operator acts on kets, we can set Eqn. (5.164) equal to Eqn. (5.128) which leads to the same contradiction as pointed out in Case 2 unless the trivial solution is chosen. Similarly, by inspection

$$d_+^{1*} = \frac{d_-^{1*}d_+^{-1}}{d_-^{-1}}, \quad (5.165)$$

which when compared with Eqn. (5.133) leads to the same contradiction for non-zero d 's so we here only have the trivial solution

$$c_+^1 = c_-^1 = c_+^{-1} = c_-^{-1} = d_+^1 = d_-^1 = d_+^{-1} = d_-^{-1} = 0. \quad (5.166)$$

5.3.3 Case 4 Quantum Field Search

In this case the parity and time reversal operators are defined by

$$\mathbf{P} |\mathbf{p}, \sigma, \tau\rangle = \eta |-\mathbf{p}, \sigma, \tau\rangle \quad (5.167)$$

and

$$\mathbf{T} |\mathbf{p}, \sigma, \tau\rangle = \tau(-1)^{\frac{1}{2}-\sigma} |-\mathbf{p}, -\sigma, -\tau\rangle. \quad (5.168)$$

Here time reversal acts the same way as in Case 2, but parity is different so we will here set up the relevant equations which follow from this, in direct analogy to how things have been worked out in Cases 2 and 3. The equations are much the same as in the other two cases except for a sign change in a few places. The demanded conditions are

$$\eta^* \beta u(\mathbf{0}, \sigma, \tau) = A u(\mathbf{0}, \sigma, \tau) \quad (5.169)$$

and

$$\eta \beta v(\mathbf{0}, \sigma, \tau) = A v(\mathbf{0}, \sigma, \tau), \quad (5.170)$$

which translate to the following conditions:

$$\eta^* \beta u \left(\mathbf{0}, \frac{1}{2}, 1 \right) = Au \left(\mathbf{0}, \frac{1}{2}, 1 \right) \quad (5.171)$$

$$\eta^* \beta u \left(\mathbf{0}, \frac{1}{2}, -1 \right) = Au \left(\mathbf{0}, \frac{1}{2}, -1 \right) \quad (5.172)$$

$$\eta^* \beta u \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = Au \left(\mathbf{0}, \frac{-1}{2}, 1 \right) \quad (5.173)$$

$$\eta^* \beta u \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = Au \left(\mathbf{0}, \frac{-1}{2}, -1 \right) \quad (5.174)$$

$$\eta \beta v \left(\mathbf{0}, \frac{1}{2}, 1 \right) = Av \left(\mathbf{0}, \frac{1}{2}, 1 \right) \quad (5.175)$$

$$\eta \beta v \left(\mathbf{0}, \frac{1}{2}, -1 \right) = Av \left(\mathbf{0}, \frac{1}{2}, -1 \right) \quad (5.176)$$

$$\eta \beta v \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = Av \left(\mathbf{0}, \frac{-1}{2}, 1 \right) \quad (5.177)$$

$$\eta \beta v \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = Av \left(\mathbf{0}, \frac{-1}{2}, -1 \right) \quad (5.178)$$

which, more explicitly reads as

$$\eta^* \beta u \left(\mathbf{0}, \frac{1}{2}, 1 \right) = \quad (5.179)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta^* c_+^1 \\ 0 \\ \eta^* c_-^1 \\ 0 \end{pmatrix} = \begin{pmatrix} \eta^* c_-^1 \\ 0 \\ \eta^* c_+^1 \\ 0 \end{pmatrix} = \begin{pmatrix} Ac_+^1 \\ 0 \\ Ac_-^1 \\ 0 \end{pmatrix}$$

$$\eta^* \beta u \left(\mathbf{0}, \frac{1}{2}, -1 \right) = \quad (5.180)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta^* c_+^{-1} \\ 0 \\ \eta^* c_-^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} \eta^* c_-^{-1} \\ 0 \\ \eta^* c_+^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} Ac_+^{-1} \\ 0 \\ Ac_-^{-1} \\ 0 \end{pmatrix}$$

$$\eta^* \beta u \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = \quad (5.181)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \eta^* c_+^1 \\ 0 \\ \eta^* c_-^1 \end{pmatrix} = \begin{pmatrix} 0 \\ \eta^* c_-^1 \\ 0 \\ \eta^* c_+^1 \end{pmatrix} = \begin{pmatrix} 0 \\ Ac_+^1 \\ 0 \\ Ac_-^1 \end{pmatrix}$$

$$\eta^* \beta u \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = \quad (5.182)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \eta^* c_+^{-1} \\ 0 \\ \eta^* c_-^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \eta^* c_-^{-1} \\ 0 \\ \eta^* c_+^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ Ac_+^{-1} \\ 0 \\ Ac_-^{-1} \end{pmatrix}$$

$$\eta^* \beta v \left(\mathbf{0}, \frac{1}{2}, 1 \right) = \quad (5.183)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \eta d_+^1 \\ 0 \\ \eta d_-^1 \end{pmatrix} = \begin{pmatrix} 0 \\ \eta d_-^1 \\ 0 \\ \eta d_+^1 \end{pmatrix} = \begin{pmatrix} 0 \\ Ad_+^1 \\ 0 \\ Ad_-^1 \end{pmatrix}$$

$$\eta^* \beta v \left(\mathbf{0}, \frac{1}{2}, -1 \right) = \quad (5.184)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \eta d_+^{-1} \\ 0 \\ \eta d_-^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \eta d_-^{-1} \\ 0 \\ \eta d_+^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ Ad_+^{-1} \\ 0 \\ Ad_-^{-1} \end{pmatrix}$$

$$\eta^* \beta v \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = \quad (5.185)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\eta d_+^1 \\ 0 \\ -\eta d_-^1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\eta d_-^1 \\ 0 \\ -\eta d_+^1 \\ 0 \end{pmatrix} = \begin{pmatrix} -Ad_+^1 \\ 0 \\ -Ad_-^1 \\ 0 \end{pmatrix}$$

$$\eta^* \beta v \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = \quad (5.186)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\eta d_+^{-1} \\ 0 \\ -\eta d_-^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} -\eta d_-^{-1} \\ 0 \\ -\eta d_+^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} -Ad_+^{-1} \\ 0 \\ -Ad_-^{-1} \\ 0 \end{pmatrix}.$$

Reading off these equations, we see that

$$\eta^* c_-^1 = A c_+^1 \quad (5.187)$$

$$\eta^* c_+^1 = A c_-^1 \quad (5.188)$$

$$\eta^* c_-^{-1} = A c_+^{-1} \quad (5.189)$$

$$\eta^* c_+^{-1} = A c_-^{-1} \quad (5.190)$$

$$\eta d_-^1 = A d_+^1 \quad (5.191)$$

$$\eta d_+^1 = A d_-^1 \quad (5.192)$$

$$\eta d_-^{-1} = A d_+^{-1} \quad (5.193)$$

$$\eta d_+^{-1} = A d_-^{-1}. \quad (5.194)$$

Combining Eqn. (5.188) and Eqn. (5.189) and taking the complex conjugate gives

$$c_+^{1*} = \frac{c_-^{1*} c_-^{-1*}}{c_+^{-1*}}, \quad (5.195)$$

which does not contain a minus sign as is the case in Case 2. However if we observe Eqn. (5.121) and Eqn. (5.119) we see that both of them together imply that $c_+^{-1} = 0$, which, in turn, implies that $c_+^1 = 0$. Similarly, by looking at Eqn. (5.120) and Eqn. (5.122), we also see that $c_-^{-1} = 0$, and therefore we also have $c_-^1 = 0$. A similar thing can be done for the d 's. Since there exists no non-trivial solution, we conclude that there also cannot exist a non-trivial spin-1/2 quantum field for Case 4, given the chosen standard chiral representation for $D(\mathcal{P})$ and $D(\mathcal{T})$. In the following sections, the hunt for quantum fields extends to include more general $D(\mathcal{P})$ and $D(\mathcal{T})$. We will show that there are no massive spin-1/2 non-standard quantum fields belonging to Cases 2 and 4 but there is one for Case 3.

5.3.4 More General Non-Standard Representations of Parity and Time-Reversal

In this section we look for the most general forms that $D(\mathcal{P})$ and $D(\mathcal{T})$ can take, the idea being that the extra freedom might allow for the existence of non-trivial non-standard massive local spin-1/2 quantum fields. Previously we had chosen $D(\mathcal{P}) = \beta$. Now we look for a more general $D(\mathcal{P})$ such that

$$D(\mathcal{P})D(L(-\mathbf{p}))D(\mathcal{P})^{-1} = D(L(\mathbf{p})). \quad (5.196)$$

In doing this, we are not doing anything illegal as we are simply choosing a different finite-dimensional representation of the Lorentz group where the elements of the connected component are the same as before and only the space and time inversion matrices are being selected differently. It can be shown that

$$D(\mathcal{P}) = \begin{pmatrix} 0 & 0 & r_1 e^{i\alpha} & 0 \\ 0 & 0 & 0 & r_2 e^{i\beta} \\ r_3 e^{i\gamma} & 0 & 0 & 0 \\ 0 & r_4 e^{i\delta} & 0 & 0 \end{pmatrix}. \quad (5.197)$$

5.3 Constructing Non-Standard Massive Spin-1/2 Quantum Fields

We multiply this matrix on the left of

$$\begin{pmatrix} m + p_0 + p_z & p_x - ip_y & 0 & 0 \\ p_x + ip_y & m + p_0 - p_z & 0 & 0 \\ 0 & 0 & m + p_0 - p_z & -(p_x - ip_y) \\ 0 & 0 & -(p_x + ip_y) & m + p_0 + p_z \end{pmatrix}$$

to obtain

$$\begin{pmatrix} 0 & 0 & r_1 e^{i\alpha}(m + p_0 - p_z) & -r_1 e^{i\alpha}(p_x - ip_y) \\ 0 & 0 & -r_2 e^{i\beta}(p_x + ip_y) & r_2 e^{i\beta}(m + p_0 + p_z) \\ r_3 e^{i\gamma}(m + p_0 + p_z) & r_3 e^{i\gamma}(p_x - ip_y) & 0 & 0 \\ r_4 e^{i\delta}(p_x + ip_y) & r_4 e^{i\delta}(m + p_0 - p_z) & 0 & 0 \end{pmatrix}.$$

Finally, we multiply on the right of this by $D(\mathcal{P})^{-1}$ which gives

$$\begin{pmatrix} m + p_0 - p_z & -(p_x - ip_y) & 0 & 0 \\ -(p_x + ip_y) & m + p_0 + p_z & 0 & 0 \\ 0 & 0 & m + p_0 + p_z & p_x - ip_y \\ 0 & 0 & p_x + ip_y & m + p_0 - p_z \end{pmatrix} = D(L(-\mathbf{p}))$$

so long as

$$\frac{r_2}{r_1} e^{i(\beta-\alpha)} = \frac{r_1}{r_2} e^{i(\alpha-\beta)} = \frac{r_3}{r_4} e^{i(\gamma-\delta)} = \frac{r_4}{r_3} e^{i(\delta-\gamma)} = 1 \quad (5.198)$$

from which it follows that $r_1 e^{i\alpha} = r_2 e^{i\beta}$ and $r_3 e^{i\gamma} = r_4 e^{i\delta}$. For ease of typing, we redefine $a^{-1} = r_3 e^{i\gamma}$ and $b^{-1} = r_1 e^{i\alpha}$ so then the form of $D(\mathcal{P})$, and hence $D(\mathcal{P})^{-1}$ that we settle on, is

$$D(\mathcal{P}) = \begin{pmatrix} 0 & 0 & b^{-1} & 0 \\ 0 & 0 & 0 & b^{-1} \\ a^{-1} & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \end{pmatrix} \quad (5.199)$$

and

$$D(\mathcal{P})^{-1} = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix} \quad (5.200)$$

respectively. We now start hunting for a more general set of choices for $D(\mathcal{T})$. Firstly, we look for a general 2×2 matrix Θ which satisfies $\Theta \sigma \Theta^{-1} = -\sigma^*$. We here use

$$\Theta = \begin{pmatrix} 0 & g_1 e^{i\epsilon} \\ g_2 e^{i\iota} & 0 \end{pmatrix}, \quad (5.201)$$

5 Non-Standard Quantum Fields

for arbitrary real numbers g_1, g_2, ϵ and ι . Firstly, applying this to σ_1 gives

$$\begin{pmatrix} 0 & g_1 e^{i\epsilon} \\ g_2 e^{i\iota} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{g_2} e^{-i\iota} \\ \frac{1}{g_1} e^{-i\epsilon} & 0 \end{pmatrix} \quad (5.202)$$

which is calculated to be

$$\begin{pmatrix} 0 & \frac{g_1}{g_2} e^{i(\epsilon-\iota)} \\ \frac{g_2}{g_1} e^{-i(\epsilon-\iota)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \quad (5.203)$$

This last equality is from the above requirement that Θ must satisfy in order to be interpreted as a time reversal operator. Performing similar calculations for $\Theta\sigma_2\Theta^{-1}$ and $\Theta\sigma_3\Theta^{-1}$ we get

$$\begin{pmatrix} 0 & i\frac{g_1}{g_2} e^{i(\epsilon-\iota)} \\ -i\frac{g_2}{g_1} e^{-i(\epsilon-\iota)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (5.204)$$

and

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad (5.205)$$

respectively. By inspection, in order for $\Theta\sigma\Theta^{-1} = -\sigma^*$ to hold, we require that

$$\frac{g_1}{g_2} e^{i(\epsilon-\iota)} = \frac{g_2}{g_1} e^{-i(\epsilon-\iota)} = -1. \quad (5.206)$$

It follows immediately that

$$g_1 e^{i\epsilon} = -g_2 e^{i\iota}. \quad (5.207)$$

By labeling $g_1 e^{i\epsilon} = -g_2 e^{i\iota} = e^{-1}$, we settle with

$$\Theta = \begin{pmatrix} 0 & -e^{-1} \\ e^{-1} & 0 \end{pmatrix}. \quad (5.208)$$

It turns out that when we look for a general set of possibilities for $D(\mathcal{T})$ which satisfy

$$TD^*(L(-\mathbf{p}))T^{-1} = D(L(\mathbf{p})), \quad (5.209)$$

we arrive at the conclusion that the form of $D(\mathcal{T})$ must be

$$D(\mathcal{T}) = \begin{pmatrix} 0 & -e^{-1} \\ e^{-1} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -f^{-1} \\ f^{-1} & 0 \end{pmatrix}, \quad (5.210)$$

for some arbitrary numbers e and f . For later convenience, we write these possible choices for $D(\mathcal{T})$ and $D(\mathcal{T})^{-1}$ as

$$D(\mathcal{T}) = \begin{pmatrix} 0 & e^* & 0 & 0 \\ -e^* & 0 & 0 & 0 \\ 0 & 0 & 0 & f^* \\ 0 & 0 & -f^* & 0 \end{pmatrix} \quad (5.211)$$

and

$$D(\mathcal{T})^{-1} = \begin{pmatrix} 0 & -e & 0 & 0 \\ e & 0 & 0 & 0 \\ 0 & 0 & 0 & -f \\ 0 & 0 & f & 0 \end{pmatrix}. \quad (5.212)$$

We are now in a position to look for non-standard massive spin-1/2 quantum fields, this time by imposing the weaker demand that

$$\mathbf{P}\psi_\ell(x)\mathbf{P}^{-1} = A \sum_{\bar{\ell}} D(\mathcal{P}^{-1})_{\ell\bar{\ell}} \psi_{\bar{\ell}}(\mathcal{P}x) \quad (5.213)$$

and

$$\mathbf{T}\psi_\ell(x)\mathbf{T}^{-1} = B \sum_{\bar{\ell}} D(\mathcal{T}^{-1})_{\ell\bar{\ell}} \psi_{\bar{\ell}}(-\mathcal{P}x), \quad (5.214)$$

retaining the freedom to make an appropriate choice of $D(\mathcal{P})$ and $D(\mathcal{T})$ such that, if possible, a non-trivial quantum field can be obtained.

5.3.5 Cases 2 and 4 Revisited

In this section we show that there exist no non-trivial massive spin-1/2 quantum fields in Cases 2 or 4, for any representation of $D(\mathcal{P})$ and $D(\mathcal{T})$. Looking at parity in the same way as in the previous sections we arrive at the set of matrix equations:

$$\begin{pmatrix} \eta^* a c_-^1 \\ 0 \\ \eta^* b c_+^1 \\ 0 \end{pmatrix} = \begin{pmatrix} A c_+^1 \\ 0 \\ A c_-^1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \eta^* a c_-^{-1} \\ 0 \\ \eta^* b c_+^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} -A c_+^{-1} \\ 0 \\ -A c_-^{-1} \\ 0 \end{pmatrix} \quad (5.215)$$

$$\begin{pmatrix} 0 \\ \eta^* a c_-^1 \\ 0 \\ \eta^* b c_+^1 \end{pmatrix} = \begin{pmatrix} 0 \\ A c_+^1 \\ 0 \\ A c_-^1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \eta^* a c_-^{-1} \\ 0 \\ \eta^* b c_+^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ -A c_+^{-1} \\ 0 \\ -A c_-^{-1} \end{pmatrix} \quad (5.216)$$

$$\begin{pmatrix} 0 \\ \eta a d_-^1 \\ 0 \\ \eta b d_+^1 \end{pmatrix} = \begin{pmatrix} 0 \\ A d_+^1 \\ 0 \\ A d_-^1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \eta a d_-^{-1} \\ 0 \\ \eta b d_+^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ -A d_+^{-1} \\ 0 \\ -A d_-^{-1} \end{pmatrix} \quad (5.217)$$

$$\begin{pmatrix} -\eta a d_-^1 \\ 0 \\ -\eta b d_+^1 \\ 0 \end{pmatrix} = \begin{pmatrix} -A d_+^1 \\ 0 \\ -A d_-^1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -\eta a d_-^{-1} \\ 0 \\ -\eta b d_+^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} A d_+^{-1} \\ 0 \\ A d_-^{-1} \\ 0 \end{pmatrix}. \quad (5.218)$$

Similarly, the matrix equations for time reversal are

$$\begin{pmatrix} -ec_+^{1*} \\ 0 \\ -fc_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} Bc_+^{-1} \\ 0 \\ Bc_-^{-1} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} ec_+^{-1*} \\ 0 \\ fc_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} Bc_+^1 \\ 0 \\ Bc_-^1 \\ 0 \end{pmatrix} \quad (5.219)$$

$$\begin{pmatrix} 0 \\ -ec_+^{1*} \\ 0 \\ -fc_-^{1*} \end{pmatrix} = \begin{pmatrix} 0 \\ Bc_+^{-1} \\ 0 \\ Bc_-^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ ec_+^{-1*} \\ 0 \\ fc_-^{-1*} \end{pmatrix} = \begin{pmatrix} 0 \\ Bc_+^1 \\ 0 \\ Bc_-^1 \end{pmatrix} \quad (5.220)$$

$$\begin{pmatrix} 0 \\ -ed_+^{1*} \\ 0 \\ -fd_-^{1*} \end{pmatrix} = \begin{pmatrix} 0 \\ Bd_+^{-1} \\ 0 \\ Bd_-^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ ed_+^{-1*} \\ 0 \\ fd_-^{-1*} \end{pmatrix} = \begin{pmatrix} 0 \\ Bd_+^1 \\ 0 \\ Bd_-^1 \end{pmatrix} \quad (5.221)$$

$$\begin{pmatrix} ed_+^{1*} \\ 0 \\ fd_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} -Bd_+^{-1} \\ 0 \\ -Bd_-^{-1} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -ed_+^{-1*} \\ 0 \\ -fd_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} -Bd_+^1 \\ 0 \\ -Bd_-^1 \\ 0 \end{pmatrix}. \quad (5.222)$$

Reading off the conditions for the c 's and d 's gives the following sets of equations,

$$\eta^* ac_-^1 = Ac_+^1 \quad (5.223)$$

$$\eta^* bc_+^1 = Ac_-^1 \quad (5.224)$$

$$\eta^* ac_-^{-1} = -Ac_+^{-1} \quad (5.225)$$

$$\eta^* bc_+^{-1} = -Ac_-^{-1} \quad (5.226)$$

$$\eta ad_-^1 = Ad_+^1 \quad (5.227)$$

$$\eta bd_+^1 = Ad_-^1 \quad (5.228)$$

$$\eta ad_-^{-1} = -Ad_+^{-1} \quad (5.229)$$

$$\eta bd_+^{-1} = -Ad_-^{-1} \quad (5.230)$$

5.3 Constructing Non-Standard Massive Spin-1/2 Quantum Fields

from parity considerations, and

$$ec_+^{1*} = -Bc_+^{-1} \quad (5.231)$$

$$fc_-^{1*} = -Bc_-^{-1} \quad (5.232)$$

$$ec_+^{-1*} = Bc_+^1 \quad (5.233)$$

$$fc_-^{-1*} = Bc_-^1 \quad (5.234)$$

$$ed_+^{1*} = -Bd_+^{-1} \quad (5.235)$$

$$fd_-^{1*} = -Bd_-^{-1} \quad (5.236)$$

$$ed_+^{-1*} = Bd_+^1 \quad (5.237)$$

$$fd_-^{-1*} = Bd_-^1 \quad (5.238)$$

from time reversal considerations. By a quick inspection of these equations obtained from time reversal considerations, it is quickly apparent that the extra freedom brought in by the new phases fails to help. If we combine Eqn. (5.231) with Eqn. (5.233) and Eqn. (5.232) with Eqn. (5.234) we see that

$$c_+^1 c_+^{1*} = -c_+^{-1} c_+^{-1*} \quad (5.239)$$

and

$$c_-^{1*} c_-^1 = -c_-^{-1} c_-^{-1*}. \quad (5.240)$$

Also, by Eqn. (5.235) and Eqn. (5.237)

$$d_+^1 d_+^{1*} = -d_+^{-1} d_+^{-1*} \quad (5.241)$$

and by Eqn. (5.236) and Eqn. (5.238)

$$d_-^{1*} d_-^1 = -d_-^{-1} d_-^{-1*}. \quad (5.242)$$

Time reversal acts the same way in Case 4 so the same result applies there too, so in both cases only the trivial solution exists where

$$c_+^1 = c_-^1 = c_+^{-1} = c_-^{-1} = d_+^1 = d_-^1 = d_+^{-1} = d_-^{-1} = 0, \quad (5.243)$$

which implies that

$$\psi_\ell(x) = 0. \quad (5.244)$$

However, as we shall now see, there are non-trivial massive spin-1/2 quantum fields transforming according to Case 3, in the one particle state space H_1^{NS} .

5.3.6 Case 3 Revisited

The matrix equations for space inversion obtained in Case 3 are the same as in Case 2 and produce Eqns. (5.223) – (5.230). The matrix equations for time reversal are similar to those of Case 2 except for the fact that a few signs are different in just the right places to ensure

the existence of non-trivial solutions. The conditions on the rest spinors which lead directly to these matrix equations (compare with Eqn. (5.138) and Eqn. (5.139)), are the demands that

$$(-1)^{\frac{1}{2}-\sigma} D(\mathcal{T})^{-1} u^*(\mathbf{0}, -\sigma, -\tau) = Bu(\mathbf{0}, \sigma, \tau) \quad (5.245)$$

and

$$(-1)^{\frac{1}{2}-\sigma} D(\mathcal{T})^{-1} v^*(\mathbf{0}, -\sigma, -\tau) = Bv(\mathbf{0}, \sigma, \tau), \quad (5.246)$$

which, more explicitly, are written as:

$$D(\mathcal{T})^{-1} u^* \left(\mathbf{0}, \frac{1}{2}, 1 \right) = -Bu \left(\mathbf{0}, \frac{-1}{2}, -1 \right) \quad (5.247)$$

$$D(\mathcal{T})^{-1} u^* \left(\mathbf{0}, \frac{1}{2}, -1 \right) = -Bu \left(\mathbf{0}, \frac{-1}{2}, 1 \right) \quad (5.248)$$

$$D(\mathcal{T})^{-1} u^* \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = Bu \left(\mathbf{0}, \frac{1}{2}, -1 \right) \quad (5.249)$$

$$D(\mathcal{T})^{-1} u^* \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = Bu \left(\mathbf{0}, \frac{1}{2}, 1 \right) \quad (5.250)$$

$$D(\mathcal{T})^{-1} v^* \left(\mathbf{0}, \frac{1}{2}, 1 \right) = -Bv \left(\mathbf{0}, \frac{-1}{2}, -1 \right) \quad (5.251)$$

$$D(\mathcal{T})^{-1} v^* \left(\mathbf{0}, \frac{1}{2}, -1 \right) = -Bv \left(\mathbf{0}, \frac{-1}{2}, 1 \right) \quad (5.252)$$

$$D(\mathcal{T})^{-1} v^* \left(\mathbf{0}, \frac{-1}{2}, 1 \right) = Bv \left(\mathbf{0}, \frac{1}{2}, -1 \right) \quad (5.253)$$

$$D(\mathcal{T})^{-1} v^* \left(\mathbf{0}, \frac{-1}{2}, -1 \right) = Bv \left(\mathbf{0}, \frac{1}{2}, 1 \right). \quad (5.254)$$

These equations, explicitly, are

$$\begin{pmatrix} 0 & -e & 0 & 0 \\ e & 0 & 0 & 0 \\ 0 & 0 & 0 & -f \\ 0 & 0 & f & 0 \end{pmatrix} \begin{pmatrix} c_+^{1*} \\ 0 \\ c_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ ec_+^{1*} \\ 0 \\ fc_-^{1*} \end{pmatrix} = \begin{pmatrix} 0 \\ -Bc_+^{-1} \\ 0 \\ -Bc_-^{-1} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -e & 0 & 0 \\ e & 0 & 0 & 0 \\ 0 & 0 & 0 & -f \\ 0 & 0 & f & 0 \end{pmatrix} \begin{pmatrix} c_+^{-1*} \\ 0 \\ c_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ ec_+^{-1*} \\ 0 \\ fc_-^{-1*} \end{pmatrix} = \begin{pmatrix} 0 \\ -Bc_+^1 \\ 0 \\ -Bc_-^1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -e & 0 & 0 \\ e & 0 & 0 & 0 \\ 0 & 0 & 0 & -f \\ 0 & 0 & f & 0 \end{pmatrix} \begin{pmatrix} 0 \\ c_+^{1*} \\ 0 \\ c_-^{1*} \end{pmatrix} = \begin{pmatrix} -ec_+^{1*} \\ 0 \\ -fc_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} Bc_+^{-1} \\ 0 \\ Bc_-^{-1} \\ 0 \end{pmatrix}$$

5.3 Constructing Non-Standard Massive Spin-1/2 Quantum Fields

$$\begin{pmatrix} 0 & -e & 0 & 0 \\ e & 0 & 0 & 0 \\ 0 & 0 & 0 & -f \\ 0 & 0 & f & 0 \end{pmatrix} \begin{pmatrix} 0 \\ c_+^{-1*} \\ 0 \\ c_-^{-1*} \end{pmatrix} = \begin{pmatrix} -ec_+^{-1*} \\ 0 \\ -fc_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} Bc_+^1 \\ 0 \\ Bc_-^1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -e & 0 & 0 \\ e & 0 & 0 & 0 \\ 0 & 0 & 0 & -f \\ 0 & 0 & f & 0 \end{pmatrix} \begin{pmatrix} 0 \\ d_+^{1*} \\ 0 \\ d_-^{1*} \end{pmatrix} = \begin{pmatrix} -ed_+^{1*} \\ 0 \\ -fd_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} Bd_+^{-1} \\ 0 \\ Bd_-^{-1} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -e & 0 & 0 \\ e & 0 & 0 & 0 \\ 0 & 0 & 0 & -f \\ 0 & 0 & f & 0 \end{pmatrix} \begin{pmatrix} 0 \\ d_+^{-1*} \\ 0 \\ d_-^{-1*} \end{pmatrix} = \begin{pmatrix} -ed_+^{-1*} \\ 0 \\ -fd_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} Bd_+^1 \\ 0 \\ Bd_-^1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -e & 0 & 0 \\ e & 0 & 0 & 0 \\ 0 & 0 & 0 & -f \\ 0 & 0 & f & 0 \end{pmatrix} \begin{pmatrix} -d_+^{1*} \\ 0 \\ -d_-^{1*} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -ed_+^{1*} \\ 0 \\ -fd_-^{1*} \end{pmatrix} = \begin{pmatrix} 0 \\ Bd_+^{-1} \\ 0 \\ Bd_-^{-1} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -e & 0 & 0 \\ e & 0 & 0 & 0 \\ 0 & 0 & 0 & -f \\ 0 & 0 & f & 0 \end{pmatrix} \begin{pmatrix} -d_+^{-1*} \\ 0 \\ -d_-^{-1*} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -ed_+^{-1*} \\ 0 \\ -fd_-^{-1*} \end{pmatrix} = \begin{pmatrix} 0 \\ Bd_+^1 \\ 0 \\ Bd_-^1 \end{pmatrix}.$$

Reading off these matrices, we see that, together with Eqns. (5.223) – (5.230), the following needs to be satisfied:

$$-ec_+^{1*} = Bc_+^{-1} \quad (5.255)$$

$$-fc_-^{1*} = Bc_-^{-1} \quad (5.256)$$

$$-ec_+^{-1*} = Bc_+^1 \quad (5.257)$$

$$-fc_-^{-1*} = Bc_-^1 \quad (5.258)$$

$$-ed_+^{1*} = Bd_+^{-1} \quad (5.259)$$

$$-fd_-^{1*} = Bd_-^{-1} \quad (5.260)$$

$$-ed_+^{-1*} = Bd_+^1 \quad (5.261)$$

$$-fd_-^{-1*} = Bd_-^1. \quad (5.262)$$

It can be checked by inspection that the following constitute solutions to Eqns. (5.223) – (5.230) and Eqns. (5.255) – (5.262) if $a = b$, $e = -f$, $\eta = \pm i$:

$$c_+^1 = 1, \quad c_-^1 = \pm 1, \quad c_+^{-1} = 1, \quad c_-^{-1} = \mp 1 \quad (5.263)$$

$$d_+^1 = 1, \quad d_-^1 = \mp 1, \quad d_+^{-1} = 1, \quad d_-^{-1} = \pm 1. \quad (5.264)$$

Furthermore, these solutions are also consistent with Eqns. (5.50) – (5.52). If the quantum field also anticommutes with itself at spacelike separation, it will be local. This we show in Sec. (5.6).

5.4 Elko and Non-Standardness

At this point, we finish the section by making the observation that [32, p.17] refers to [52] and identifies the Elko spinors with the non-standard Wigner classes. In [32, p.18], the observation is made that, regarding the finite-dimensional representations,

$$\{D(\mathcal{P}), D(\mathcal{T})\} = 0, \quad (5.265)$$

as opposed to the standard Wigner class finite-dimensional result which says that

$$[D(\mathcal{P}), D(\mathcal{T})] = 0. \quad (5.266)$$

The solutions found in this section are consistent with this observation. The reason is that in the standard case, the matrix $D(\mathcal{T})$ has the form

$$\begin{pmatrix} A_2 & 0_2 \\ 0_2 & A_2 \end{pmatrix} \quad (5.267)$$

where A_2 is a constant 2×2 matrix. In the non-standard case here, the matrix $D(\mathcal{T})$ has the form

$$\begin{pmatrix} A_2 & 0_2 \\ 0_2 & -A_2 \end{pmatrix}. \quad (5.268)$$

This minus sign, resulting from the condition $e = -f$, is what results in Eqn. (5.265) holding rather than Eqn. (5.266).

In this chapter, our search for dark matter candidates and new possible quantum fields did not give rise to Elko fields. We can see this as follows. We have analyzed representations of the full Poincaré group on the state space, where the two-valued discrete index has the natural interpretation of labeling the eigenvalues of the spin-1/2 angular momentum generator J_z . In Sec. (3.3.2) we discussed the main senses in which the Elko Field Theory is incomplete. One of those senses was that the Elko two-valued discrete index was not well-defined on the state space. Since the initial premise was that Elko should respect all of the symmetries of the Poincaré group, it seemed most natural to examine Elko Field Theory under what we

5.5 Degeneracies, Internal Symmetries and the Theorem of Lee and Wick

consider to be the very reasonable assumption that the two-valued discrete index naturally arises from the representation of the Poincaré group on the state space. If we were to assume a very different interpretation for the Elko two-valued discrete index on the state space, and such Elko states were related to the usual states $|\mathbf{p}, \sigma, \tau\rangle$, perhaps Elko fields could be derived in some way. The most natural approach, interpreting the two-valued discrete index as σ , fails to give rise to Elko fields.

We might have been tempted to not bother looking for Elko fields among the non-standard Wigner classes with this interpretation of the discrete index by simply observing that Elko states have *one* two-valued discrete index whereas the non-standard states are labeled by *two* two-valued discrete indices. We believed however, that it was worthwhile to search. One reason once again comes down to incompleteness in Elko Field Theory. The Elko field was originally written down based on an analysis of how the Elko *spinors* transformed in spinor space. The transformation properties of the corresponding Elko *field* were not taken into consideration, or worked out. Elko might have been compatible with having an extra two-valued discrete index but still with only four visible spinors because the other four might have been identically null spinors.

By inspection of the spinors in Eqns. (5.297) – (5.304) there is no solution that will give spinors of the form given by Eqn. (3.7). We might consider using a representation other than the chiral representation and seeing whether we can find a transformation from the chiral representation to another equivalent representation that maps the rest spinors to spinors that look like Elko rest spinors. However, if we did this, the form of the finite-dimensional boost operator would be changed as well, so we would not get the Elko boosted spinors so we still would not have Elko fields. At best, we would have our new quantum field dark matter candidate in a disguised form, with associated spinors that looked like Elko rest spinors. We might consider changing the representation of the little group on the state space. If we do this we are forced to change the representation of the finite-dimensional Lorentz group in such a way that Schur's Lemma is satisfied, which results in the spinors looking like Dirac spinors. See Sec. (4.2) for details.

Before examining the new quantum field, we pause to reflect on the issue of whether the degenerate degree of freedom τ may be realised in nature as its existence is clearly crucial as to whether a new massive spin-1/2 non-standard quantum field could in principle exist.

5.5 Degeneracies, Internal Symmetries and the Theorem of Lee and Wick

Before examining the new quantum field, we first would like to say a few things that relate to the Lee and Wick theorem. In [33], Lee and Wick present a theorem which makes the case that any local Quantum Field Theory that also satisfies the *CPT* theorem can be made to look

like two independent copies of the standard Wigner class by enlarging the symmetry group by including an appropriate symmetry operator. They assert that such a symmetry operator can always be found. This amounts to using an appropriate internal symmetry operator S to combine with P to form a new operator $P' = SP$ which behaves like a standard operator.

If such an internal symmetry corresponds to an actual internal symmetry in nature, then there might be nothing new to be found in the non-standard Wigner classes, and the appearance of there being a new spin-1/2 massive quantum field may be artificial. So far, there have been no known observations of this two-fold degeneracy (which we have been representing by the degeneracy index τ) in nature.

We take the view that the lack of known discoveries of particles with this two-fold degeneracy is not proof of its non-existence, especially given the evidence which suggests that an abundance of dark matter may exist. Weinberg supports the possibility of there existing non-standard quantum fields [42, p.100–104].

We will finish this section by demonstrating the claims of the Lee and Wick theorem for Case 3, namely that by inclusion of an appropriately defined internal symmetry operator, the enlarged symmetry group acting on the state space makes the state space look like two distinct copies of the standard Wigner class. By enlarging the symmetry group like this, we will be moving from the one particles non-standard state space to a multiparticle non-standard state space.

We start by defining the mutually linearly independent states

$$|\mathbf{p}, \sigma, n\rangle_1 = \frac{1}{2} [|\mathbf{p}, \sigma, 1, n\rangle + |\mathbf{p}, \sigma, -1, n\rangle + |\mathbf{p}, \sigma, 1, \bar{n}\rangle + |\mathbf{p}, \sigma, -1, \bar{n}\rangle] \quad (5.269)$$

$$|\mathbf{p}, \sigma, \bar{n}\rangle_1 = \frac{1}{2} [-|\mathbf{p}, \sigma, 1, n\rangle + |\mathbf{p}, \sigma, -1, n\rangle - |\mathbf{p}, \sigma, 1, \bar{n}\rangle + |\mathbf{p}, \sigma, -1, \bar{n}\rangle] \quad (5.270)$$

$$|\mathbf{p}, \sigma, n\rangle_2 = \frac{1}{2} [|\mathbf{p}, \sigma, 1, n\rangle + |\mathbf{p}, \sigma, -1, n\rangle - |\mathbf{p}, \sigma, 1, \bar{n}\rangle - |\mathbf{p}, \sigma, -1, \bar{n}\rangle] \quad (5.271)$$

$$|\mathbf{p}, \sigma, \bar{n}\rangle_2 = \frac{1}{2} [|\mathbf{p}, \sigma, 1, n\rangle - |\mathbf{p}, \sigma, -1, n\rangle - |\mathbf{p}, \sigma, 1, \bar{n}\rangle + |\mathbf{p}, \sigma, -1, \bar{n}\rangle] \quad (5.272)$$

for the case where $n \neq \bar{n}$. For convenience, we display the action of the Case 3 parity and time reversal operators P^{NS} and T^{NS} on one state kets:

$$P^{\text{NS}} |\mathbf{p}, \sigma, \tau, n\rangle = \tau \eta'_n |-\mathbf{p}, \sigma, \tau, n\rangle, \quad P^{\text{NS}} |\mathbf{p}, \sigma, \tau, \bar{n}\rangle = \tau \eta'_{\bar{n}} |-\mathbf{p}, \sigma, \tau, \bar{n}\rangle \quad (5.273)$$

$$T^{\text{NS}} |\mathbf{p}, \sigma, \tau, n\rangle = \zeta'_n (-1)^{\frac{1}{2}+\sigma} |-\mathbf{p}, -\sigma, -\tau, n\rangle, \quad T^{\text{NS}} |\mathbf{p}, \sigma, \tau, \bar{n}\rangle = \zeta'_{\bar{n}} (-1)^{\frac{1}{2}+\sigma} |-\mathbf{p}, -\sigma, -\tau, \bar{n}\rangle. \quad (5.274)$$

We will now enlarge the symmetry group by defining an internal symmetry operator C^{NS} such that

$$C^{\text{NS}} |\mathbf{p}, \sigma, \tau, n\rangle = \tau \xi_n |\mathbf{p}, \sigma, \tau, \bar{n}\rangle, \quad C^{\text{NS}} |\mathbf{p}, \sigma, \tau, \bar{n}\rangle = \tau \xi_{\bar{n}} |\mathbf{p}, \sigma, \tau, n\rangle \quad (5.275)$$

for particle phase ξ_n and antiparticle phase $\xi_{\bar{n}}$. To show that C^{NS} is an internal symmetry operator we must check that it commutes with the Hamiltonian. Recalling Eqn. (2.159) from

[42, p.296–297], we see by obvious extension that the Hamiltonian must have the form:

$$H = \sum_{\sigma, \tau, n} \int d^3 \mathbf{p} p_0 a^\dagger(\mathbf{p}, \sigma, \tau, n) a(\mathbf{p}, \sigma, \tau, n). \quad (5.276)$$

Using the same logic that led to Eqn. (5.54) and Eqn. (5.55) we see that:

$$C^{\text{NS}} a^\dagger(\mathbf{p}, \sigma, \tau, n) (C^{\text{NS}})^{-1} = \xi_n a^\dagger(\mathbf{p}, \sigma, \tau, \bar{n}), \quad C^{\text{NS}} a(\mathbf{p}, \sigma, \tau, n) (C^{\text{NS}})^{-1} = \xi_n^* a(\mathbf{p}, \sigma, \tau, \bar{n}) \quad (5.277)$$

and

$$C^{\text{NS}} a^\dagger(\mathbf{p}, \sigma, \tau, \bar{n}) (C^{\text{NS}})^{-1} = \xi_{\bar{n}} a^\dagger(\mathbf{p}, \sigma, \tau, n), \quad C^{\text{NS}} a(\mathbf{p}, \sigma, \tau, \bar{n}) (C^{\text{NS}})^{-1} = \xi_{\bar{n}}^* a(\mathbf{p}, \sigma, \tau, n). \quad (5.278)$$

Calculating $C^{\text{NS}} H (C^{\text{NS}})^{-1}$ then yields

$$\begin{aligned} & \sum_{\sigma, \tau, n} \int d^3 \mathbf{p} p_0 C^{\text{NS}} a^\dagger(\mathbf{p}, \sigma, \tau, n) (C^{\text{NS}})^{-1} C^{\text{NS}} a(\mathbf{p}, \sigma, \tau, n) (C^{\text{NS}})^{-1} \\ &= \sum_{\sigma, \tau, \bar{n}} \int d^3 \mathbf{p} p_0 \xi_n \xi_n^* a^\dagger(\mathbf{p}, \sigma, \tau, \bar{n}) a(\mathbf{p}, \sigma, \tau, n). \end{aligned} \quad (5.279)$$

Since $\xi_n \xi_n^* = 1$ for each n , and since we are summing over the species index n , we can relabel the species summation index \bar{n} as n , we have

$$C^{\text{NS}} H (C^{\text{NS}})^{-1} = H \quad \rightarrow \quad [C^{\text{NS}}, H] = 0 \quad (5.280)$$

as required for C^{NS} to be an internal symmetry operator.

We will now combine this internal symmetry operator C^{NS} with the parity operator P^{NS} and consider their combined action on the states $|\cdots\rangle_1$. Defining

$$\eta_n \equiv \eta'_n \xi_n \quad \text{and} \quad \eta_{\bar{n}} \equiv \eta'_{\bar{n}} \xi_{\bar{n}} \quad (5.281)$$

then yields $C^{\text{NS}} P^{\text{NS}} |\mathbf{p}, \sigma, n\rangle_1$ to be

$$\frac{1}{\sqrt{2}} \eta_n [\eta_n^* \eta_{\bar{n}} |-\mathbf{p}, \sigma, 1, n\rangle + \eta_n^* \eta_{\bar{n}} |-\mathbf{p}, \sigma, -1, n\rangle + |-\mathbf{p}, \sigma, 1, \bar{n}\rangle + |-\mathbf{p}, \sigma, -1, \bar{n}\rangle] = \eta_n |-\mathbf{p}, \sigma, n\rangle_1 \quad (5.282)$$

if $\eta_n^* \eta_{\bar{n}} = 1$. By similar explicit calculations we see that

$$C^{\text{NS}} P^{\text{NS}} |\mathbf{p}, \sigma, \bar{n}\rangle_1 = \eta_{\bar{n}} |-\mathbf{p}, \sigma, \bar{n}\rangle_1 \quad (5.283)$$

if $\eta_{\bar{n}}^* \eta_n = 1$. Operating on these same states $|\cdots\rangle_1$ with the time reversal operator T^{NS} yields

$$T^{\text{NS}} |\mathbf{p}, \sigma, n\rangle_1 = \zeta_n (-1)^{\frac{1}{2} + \sigma} |-\mathbf{p}, -\sigma, n\rangle_1 \quad (5.284)$$

if $\zeta_n^* \zeta_{\bar{n}} = 1$ and

$$T^{\text{NS}} |\mathbf{p}, \sigma, \bar{n}\rangle_1 = \zeta_{\bar{n}} (-1)^{\frac{1}{2} + \sigma} |-\mathbf{p}, -\sigma, \bar{n}\rangle_1 \quad (5.285)$$

5 Non-Standard Quantum Fields

if $\zeta_{\bar{n}}^* \zeta_n = 1$. We now operate on the states $|\cdots\rangle_1$ with the internal symmetry operator C^{NS} which yields

$$C^{\text{NS}} |\mathbf{p}, \sigma, n\rangle_1 = -\xi_n |\mathbf{p}, \sigma, \bar{n}\rangle_1 \quad (5.286)$$

if $\xi_n^* \xi_{\bar{n}} = 1$ and

$$C^{\text{NS}} |\mathbf{p}, \sigma, \bar{n}\rangle_1 = -\xi_{\bar{n}} |\mathbf{p}, \sigma, n\rangle_1 \quad (5.287)$$

if $\xi_{\bar{n}}^* \xi_n = 1$. We now define

$$C^{\text{std}} \equiv C^{\text{NS}}, \quad P^{\text{std}} \equiv C^{\text{NS}} P^{\text{NS}}, \quad T^{\text{std}} \equiv T^{\text{NS}}. \quad (5.288)$$

Operating on the state $|\mathbf{p}, \sigma, n\rangle_1$ with the operator $[C^{\text{std}} P^{\text{std}} T^{\text{std}}]^2$ yields

$$\zeta_n^* \zeta_{\bar{n}} \eta_n^* \eta_{\bar{n}} \xi_n^* \xi_{\bar{n}} (-1)^{\frac{1}{2}-\sigma} (-1)^{\frac{1}{2}+\sigma} |\mathbf{p}, \sigma, n\rangle_1. \quad (5.289)$$

We immediately see that $(-1)^{\frac{1}{2}-\sigma} (-1)^{\frac{1}{2}+\sigma} = -1$ so that the expression simplifies to

$$-\zeta_n^* \zeta_{\bar{n}} \eta_n^* \eta_{\bar{n}} \xi_n^* \xi_{\bar{n}} |\mathbf{p}, \sigma, n\rangle_1. \quad (5.290)$$

In order for C^{std} to satisfy the hypothesis of the CPT theorem, we need $\zeta_n^* \zeta_{\bar{n}} \eta_n^* \eta_{\bar{n}} \xi_n^* \xi_{\bar{n}} = 1$. These are the conditions that are needed in order for the states to transform like Dirac states, and can be satisfied for a suitable choice of phases. Similarly, by explicit calculation, we find that

$$[C^{\text{std}} P^{\text{std}} T^{\text{std}}]^2 |\mathbf{p}, \sigma, \bar{n}\rangle_1 = -|\mathbf{p}, \sigma, \bar{n}\rangle_1 \quad (5.291)$$

so we have a copy of the standard Wigner class.

A similar analysis yields another copy of the standard Wigner class when looking at the states $|\mathbf{p}, \sigma, n\rangle_2$ and $|\mathbf{p}, \sigma, \bar{n}\rangle_2$. The above calculations demonstrate consistency with the Lee and Wick theorem.

We could also consider the case where the particle is its own antiparticle. If we do this, we can set

$$|\mathbf{p}, \sigma\rangle_1 = \frac{1}{\sqrt{2}} [|\mathbf{p}, \sigma, 1\rangle + |\mathbf{p}, \sigma, -1\rangle] \quad (5.292)$$

$$|\mathbf{p}, \sigma\rangle_2 = \frac{1}{\sqrt{2}} [|\mathbf{p}, \sigma, 1\rangle - |\mathbf{p}, \sigma, -1\rangle]. \quad (5.293)$$

If we define the internal symmetry operator S (which takes the place of C^{std} where we had $n = \bar{n}$) such that

$$S |\mathbf{p}, \sigma, \tau\rangle = \tau |\mathbf{p}, \sigma, \tau\rangle, \quad (5.294)$$

then we can get two copies of the standard Wigner class spanned by $|\mathbf{p}, \sigma\rangle_1$ and $|\mathbf{p}, \sigma\rangle_2$ respectively, if we set

$$P^{\text{std}} \equiv S P^{\text{NS}}, \quad T^{\text{std}} \equiv T^{\text{NS}} \quad (5.295)$$

and also have an operator C^{std} such that

$$C^{\text{std}} |\mathbf{p}, \sigma, \tau\rangle = \xi |\mathbf{p}, \sigma, \tau\rangle. \quad (5.296)$$

A straight forward check reveals that $(C^{\text{std}} P^{\text{std}} T^{\text{std}})^2 = -1$.

We end this section by emphasising that although one can define the states $|\cdots\rangle_{1,2}$ in the way we have done in this section, there is no reason why we *must* do this. It may be that there are degenerate particles that possess no such internal symmetry corresponding to C^{NS} . Furthermore, there may be superselection rules in nature that forbid the states $|\cdots\rangle_{1,2}$ from being physically realisable. We therefore proceed with examining the new quantum field, under the assumption that either there exists no such internal symmetry in nature, or else there exist superselection rules which rule out the states $|\cdots\rangle_{1,2}$ from being realised in nature for degenerate spin-1/2 massive particles. At this point, we also point out that even if we allowed the non-standard Wigner class to look like two copies of the standard Wigner class, the quantum field would still be different from the standard one because the finite-dimensional representation of the Lorentz group is different.

5.6 The New Quantum Field

Before applying the canonical formalism to this new quantum field, we here display the explicit solution. The rest spinors are taken to be

$$u\left(\mathbf{0}, \frac{1}{2}, 1\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (5.297)$$

$$u\left(\mathbf{0}, \frac{1}{2}, -1\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad (5.298)$$

$$u\left(\mathbf{0}, \frac{-1}{2}, 1\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad (5.299)$$

$$u\left(\mathbf{0}, \frac{-1}{2}, -1\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad (5.300)$$

$$v\left(\mathbf{0}, \frac{1}{2}, 1\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad (5.301)$$

$$v\left(\mathbf{0}, \frac{1}{2}, -1\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad (5.302)$$

$$v\left(\mathbf{0}, \frac{-1}{2}, 1\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (5.303)$$

$$v\left(\mathbf{0}, \frac{-1}{2}, -1\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \quad (5.304)$$

Applying the boost operator $\sqrt{\frac{m}{p_0}}D(L(\mathbf{p}))$ to these spinors yields the spinors $u(\mathbf{p}, \sigma, \tau)$ and $v(\mathbf{p}, \sigma, \tau)$ to be

$$u\left(\mathbf{p}, \frac{1}{2}, 1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} m+p_0+p_z \\ p_x+ip_y \\ m+p_0-p_z \\ -(p_x+ip_y) \end{pmatrix} \quad (5.305)$$

$$u\left(\mathbf{p}, \frac{1}{2}, -1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} m+p_0+p_z \\ p_x+ip_y \\ -(m+p_0-p_z) \\ p_x+ip_y \end{pmatrix} \quad (5.306)$$

$$u\left(\mathbf{p}, \frac{-1}{2}, 1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} p_x-ip_y \\ m+p_0-p_z \\ -(p_x-ip_y) \\ m+p_0+p_z \end{pmatrix} \quad (5.307)$$

$$u\left(\mathbf{p}, \frac{-1}{2}, -1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} p_x - ip_y \\ m + p_0 - p_z \\ p_x - ip_y \\ -(m + p_0 + p_z) \end{pmatrix} \quad (5.308)$$

$$v\left(\mathbf{p}, \frac{1}{2}, 1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} p_x - ip_y \\ m + p_0 - p_z \\ p_x - ip_y \\ -(m + p_0 + p_z) \end{pmatrix} \quad (5.309)$$

$$v\left(\mathbf{p}, \frac{1}{2}, -1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} p_x - ip_y \\ m + p_0 - p_z \\ -(p_x - ip_y) \\ m + p_0 + p_z \end{pmatrix} \quad (5.310)$$

$$v\left(\mathbf{p}, \frac{-1}{2}, 1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} -(m + p_0 + p_z) \\ -(p_x + ip_y) \\ m + p_0 - p_z \\ -(p_x + ip_y) \end{pmatrix} \quad (5.311)$$

$$v\left(\mathbf{p}, \frac{-1}{2}, -1\right) = \frac{1}{\sqrt{2(m+p_0)}} \begin{pmatrix} -(m + p_0 + p_z) \\ -(p_x + ip_y) \\ -(m + p_0 - p_z) \\ p_x + ip_y \end{pmatrix}. \quad (5.312)$$

With these spinors, the new quantum field simply looks like[¶]

$$\psi_\ell(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma, \tau} [e^{ip \cdot x} u_\ell(\mathbf{p}, \sigma, \tau) a(\mathbf{p}, \sigma, \tau) + e^{-ip \cdot x} v_\ell(\mathbf{p}, \sigma, \tau) b^\dagger(\mathbf{p}, \sigma, \tau)]. \quad (5.313)$$

This quantum field does not satisfy the Dirac equation. The Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\psi^D(x) = 0. \quad (5.314)$$

Taking the spacetime derivatives of the Dirac field

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma} [e^{ip \cdot x} u(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{-ip \cdot x} v(\mathbf{p}, \sigma) b^\dagger(\mathbf{p}, \sigma)] \quad (5.315)$$

[¶]Strictly speaking, the factor $\frac{1}{\sqrt{2p_0}}$ finds its origin from both the normalization of the rest spinors and from the chiral representation for the boost operator. This factor has been separated out and the remaining bits called “the spinors” to be in keeping with the convention of displaying explicitly the Lorentz invariance of the measure of integration.

5 Non-Standard Quantum Fields

yields the left hand side of Eqn. (5.314) to be:

$$\int \frac{d^3 \mathbf{p}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma} \left[-(\gamma^{\mu} p_{\mu} - m) u^D(\mathbf{p}, \sigma) e^{p \cdot x} a(\mathbf{p}, \sigma) + (\gamma^{\mu} p_{\mu} + m) v^D(\mathbf{p}, \sigma) e^{-ip \cdot x} b^{\dagger}(\mathbf{p}, \sigma) \right]. \quad (5.316)$$

Since

$$(\gamma^{\mu} p_{\mu} - m) u^D(\mathbf{p}, \sigma) = 0 \quad \text{and} \quad (\gamma^{\mu} p_{\mu} + m) v^D(\mathbf{p}, \sigma) = 0 \quad (5.317)$$

it follows that the Dirac field $\psi^D(x)$ satisfies the Dirac equation. However, this new quantum field $\psi(x)$ has spinors that relate to the Dirac spinors (see [42, p.224]). By inspection of the rest spinors we see that

$$u\left(\mathbf{p}, \frac{1}{2}, 1\right) = u^D\left(\mathbf{p}, \frac{1}{2}\right) \quad (5.318)$$

$$u\left(\mathbf{p}, \frac{1}{2}, -1\right) = -v^D\left(\mathbf{p}, \frac{-1}{2}\right) \quad (5.319)$$

$$u\left(\mathbf{p}, \frac{-1}{2}, 1\right) = u^D\left(\mathbf{p}, \frac{-1}{2}\right) \quad (5.320)$$

$$u\left(\mathbf{p}, \frac{-1}{2}, -1\right) = v^D\left(\mathbf{p}, \frac{1}{2}\right) \quad (5.321)$$

$$v\left(\mathbf{p}, \frac{1}{2}, 1\right) = v^D\left(\mathbf{p}, \frac{1}{2}\right) \quad (5.322)$$

$$v\left(\mathbf{p}, \frac{1}{2}, -1\right) = u^D\left(\mathbf{p}, \frac{-1}{2}\right) \quad (5.323)$$

$$v\left(\mathbf{p}, \frac{-1}{2}, 1\right) = v^D\left(\mathbf{p}, \frac{-1}{2}\right) \quad (5.324)$$

$$v\left(\mathbf{p}, \frac{-1}{2}, -1\right) = -u^D\left(\mathbf{p}, \frac{1}{2}\right). \quad (5.325)$$

Comparing these equations immediately above to Eqn. (5.317) reveals that

$$(\gamma^{\mu} p_{\mu} - m) u(\mathbf{p}, \sigma, 1) = 0 \quad \text{and} \quad (\gamma^{\mu} p_{\mu} + m) v(\mathbf{p}, \sigma, 1) = 0 \quad (5.326)$$

but the problem is that

$$(\gamma^{\mu} p_{\mu} - m) u(\mathbf{p}, \sigma, -1) \neq 0 \quad \text{and} \quad (\gamma^{\mu} p_{\mu} + m) v(\mathbf{p}, \sigma, -1) \neq 0 \quad (5.327)$$

so $\psi_{\ell}(x)$ does not vanish from the application of the Dirac operator $(i\gamma^{\mu} \partial_{\mu} - m)$.

Before moving on, we wish to clarify something about the non-standard spinors. In the case of the Dirac field, the sets of coefficient functions $u_{\ell}(\mathbf{p}, \sigma)$ and $v_{\ell}(\mathbf{p}, \sigma)$ form spinors $u(\mathbf{p}, \sigma)$ and $v(\mathbf{p}, \sigma)$ for fixed \mathbf{p} which are four-component spinors spanning a four-dimensional space. That is to say, the Dirac spinors are all linearly independent. When it comes to the spinors $u(\mathbf{p}, \sigma, \tau)$ and $v(\mathbf{p}, \sigma, \tau)$ formed from the coefficient functions $u_{\ell}(\mathbf{p}, \sigma, \tau)$ and $v_{\ell}(\mathbf{p}, \sigma, \tau)$ in the non-standard quantum field, there are eight of them. The associated spinor space is still only four-dimensional, so these spinors are linearly dependent. This might be of concern to us

if the spinors represented the physical states. The view we take is that it is the state kets $|\mathbf{p}, \sigma, \tau\rangle$ that span the space of physical states. For each subspace of fixed four-momentum, the states $|\mathbf{p}, \sigma, \tau\rangle$ span an eight-dimensional space, and the states, by definition, are linearly independent. The state kets of differing four-momentum are also linearly independent of one another. If we form states by adding basis kets of the form $|\mathbf{p}, \sigma, \tau\rangle$ together, the uniqueness of the physical states does not depend on whether each coefficient is different from one another. We take the view that the coefficient functions $u_\ell(\mathbf{p}, \sigma, \tau)$ and $v_\ell(\mathbf{p}, \sigma, \tau)$ are merely the coefficients and whether sets of coefficient functions form sets of linearly independent spinors spanning a spinor space does not strike us as being important in the context of Quantum Field Theory in Hilbert space.

In order to be able to use the canonical formalism, it is a fundamental requirement [42] that for a field $q(\mathbf{x}, t)$, there be a canonically conjugate field $p(\mathbf{y}, t)$ such that the following equations hold:^{||}

$$\{q(\mathbf{x}, t), p(\mathbf{y}, t)\} = i\delta^3(\mathbf{x} - \mathbf{y}) \quad (5.328)$$

$$\{q(\mathbf{x}, t), q(\mathbf{y}, t)\} = 0 \quad (5.329)$$

$$\{p(\mathbf{x}, t), p(\mathbf{y}, t)\} = 0. \quad (5.330)$$

If we take $q = \psi$ and $p = \frac{i}{2}\psi^\dagger$, Eqn. (5.328) is satisfied (see the calculation following Eqn. (5.41)). Checking Eqn. (5.329):**

$$\{q_\alpha(\mathbf{x}, t), q_\beta(\mathbf{y}, t)\} = \psi_\alpha(\mathbf{x}, t)^T \psi_\beta(\mathbf{y}, t) + \psi_\beta(\mathbf{y}, t)^T \psi_\alpha(\mathbf{x}, t) = \quad (5.331)$$

$$\begin{aligned} & \int \int \frac{d^3\mathbf{p} d^3\mathbf{p}'}{(2\pi)^6 \sqrt{2p_0} \sqrt{2p'_0}} \sum_{\sigma, \tau} \sum_{\sigma', \tau'} \times \left(\right. \\ & [e^{ip \cdot x} u_\alpha(\mathbf{p}, \sigma, \tau)^T a(\mathbf{p}, \sigma, \tau) + e^{-ip \cdot x} v_\alpha(\mathbf{p}, \sigma, \tau)^T b^\dagger(\mathbf{p}, \sigma, \tau)] \times \\ & [e^{ip' \cdot y} u_\beta(\mathbf{p}', \sigma', \tau') a(\mathbf{p}', \sigma', \tau') + e^{-ip' \cdot y} v_\beta(\mathbf{p}', \sigma', \tau') b^\dagger(\mathbf{p}', \sigma', \tau')] + \\ & [e^{ip' \cdot y} u_\beta(\mathbf{p}', \sigma', \tau') a(\mathbf{p}', \sigma', \tau') + e^{-ip' \cdot y} v_\beta(\mathbf{p}', \sigma', \tau') b^\dagger(\mathbf{p}', \sigma', \tau')] \times \\ & \left. [e^{ip \cdot x} u_\alpha(\mathbf{p}, \sigma, \tau)^T a(\mathbf{p}, \sigma, \tau) + e^{-ip \cdot x} v_\alpha(\mathbf{p}, \sigma, \tau)^T b^\dagger(\mathbf{p}, \sigma, \tau)] \right). \end{aligned}$$

Multiplying out and collecting terms to form anti-commutators with the creation and annihilation operators yields

$$\begin{aligned} & \int \int \frac{d^3\mathbf{p} d^3\mathbf{p}'}{(2\pi)^6 \sqrt{2p_0} \sqrt{2p'_0}} \sum_{\sigma, \tau} \sum_{\sigma', \tau'} \times \left(\right. \\ & e^{i(p \cdot x - p' \cdot y)} u_\alpha(\mathbf{p}, \sigma, \tau)^T v_\beta(\mathbf{p}', \sigma', \tau') \{a(\mathbf{p}, \sigma, \tau), b^\dagger(\mathbf{p}', \sigma', \tau')\} + \\ & e^{-i(p \cdot x - p' \cdot y)} v_\alpha(\mathbf{p}, \sigma, \tau)^T u_\beta(\mathbf{p}', \sigma', \tau') \{b^\dagger(\mathbf{p}, \sigma, \tau), a(\mathbf{p}', \sigma', \tau')\} + \\ & e^{i(p \cdot x + p' \cdot y)} u_\alpha(\mathbf{p}, \sigma, \tau)^T u_\beta(\mathbf{p}', \sigma', \tau') \{a(\mathbf{p}, \sigma, \tau), a(\mathbf{p}', \sigma', \tau')\} + \\ & \left. e^{-i(p \cdot x + p' \cdot y)} v_\alpha(\mathbf{p}, \sigma, \tau)^T v_\beta(\mathbf{p}', \sigma', \tau') \{b^\dagger(\mathbf{p}, \sigma, \tau), b^\dagger(\mathbf{p}', \sigma', \tau')\} \right). \end{aligned}$$

^{||}This is for fermions. If q corresponded to a bosonic field, the anticommutators would be replaced by commutators.

**The superscript “ T ” here stands for transpose.

5 Non-Standard Quantum Fields

$$e^{-i(p \cdot x + p' \cdot y)} v_\alpha(\mathbf{p}, \sigma, \tau)^T v_\beta(\mathbf{p}', \sigma', \tau') \{b^\dagger(\mathbf{p}, \sigma, \tau), b^\dagger(\mathbf{p}', \sigma', \tau')\}.$$

The anticommutators ensure that the expression vanishes. In the case where $b^\dagger(\mathbf{p}, \sigma, \tau) = a^\dagger(\mathbf{p}, \sigma, \tau)$, evaluating the anti-commutators and then performing the $d^3\mathbf{p}'$ integration like was done just previously yields

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3 2p_0} \sum_{\sigma, \tau} [e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} u_\alpha(\mathbf{p}, \sigma, \tau)^T v_\beta(\mathbf{p}, \sigma, \tau) + e^{-i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} v_\alpha(\mathbf{p}, \sigma, \tau)^T u_\beta(\mathbf{p}, \sigma, \tau)]. \quad (5.332)$$

The spinor relations vanish:

$$\sum_{\sigma, \tau} u(\mathbf{p}, \sigma, \tau)^T v(\mathbf{p}, \sigma, \tau) = \sum_{\sigma, \tau} v(\mathbf{p}, \sigma, \tau)^T u(\mathbf{p}, \sigma, \tau) = 0, \quad (5.333)$$

so the anti-commutator is zero, thus verifying that Eqn. (5.329) holds. Since $p = \frac{i}{2}\psi^\dagger$, it follows immediately that

$$\{p_\alpha(\mathbf{x}, t), p_\beta(\mathbf{y}, t)\} = \frac{-1}{8} [\psi_\beta(\mathbf{y}, t) \psi_\alpha(\mathbf{x}, t) + \psi_\alpha(\mathbf{x}, t) \psi_\beta(\mathbf{y}, t)]^\dagger = 0. \quad (5.334)$$

We now calculate the propagator, the vacuum expectation value of the time ordered product of the field with its adjoint times γ_0 , denoted by

$$\langle 0 | T \{ \psi_\alpha(y) (\psi^\dagger(x) \gamma_0)_\beta \} | 0 \rangle. \quad (5.335)$$

By the definition of fermionic time ordering this is

$$\langle 0 | \psi_\alpha(y) \bar{\psi}_\beta(x) | 0 \rangle \theta(y^0 - x^0) - \langle 0 | \bar{\psi}_\beta(x) \psi_\alpha(y) | 0 \rangle \theta(x^0 - y^0). \quad (5.336)$$

The first term becomes

$$\begin{aligned} & \theta(y^0 - x^0) \langle 0 | \int \int \frac{d^3\mathbf{p} d^3\mathbf{p}'}{(2\pi)^6 \sqrt{2E} \sqrt{2E'}} \sum_{\sigma, \tau} \sum_{\sigma', \tau'} \times \quad (5.337) \\ & [e^{ip \cdot y} u_\alpha(\mathbf{p}, \sigma, \tau) a(\mathbf{p}, \sigma, \tau) + e^{-ip \cdot y} v_\alpha(\mathbf{p}, \sigma, \tau) b^\dagger(\mathbf{p}, \sigma, \tau)] \times \\ & [e^{-ip' \cdot x} \bar{u}_\beta(\mathbf{p}', \sigma', \tau') a^\dagger(\mathbf{p}', \sigma', \tau') + e^{ip' \cdot x} \bar{v}_\beta(\mathbf{p}', \sigma', \tau') b(\mathbf{p}', \sigma', \tau')] | 0 \rangle. \end{aligned}$$

The only term which does not get annihilated by either the vacuum bra or the vacuum ket is

$$\langle 0 | \int \int \frac{d^3\mathbf{p} d^3\mathbf{p}'}{(2\pi)^6 \sqrt{2E} \sqrt{2E'}} \sum_{\sigma, \tau} \sum_{\sigma', \tau'} \times \quad (5.338)$$

$$[e^{i(p \cdot y - p' \cdot x)} u_\alpha(\mathbf{p}, \sigma, \tau) \bar{u}_\beta(\mathbf{p}', \sigma', \tau') a(\mathbf{p}, \sigma, \tau) a^\dagger(\mathbf{p}', \sigma', \tau') \theta(y^0 - x^0)] | 0 \rangle. \quad (5.339)$$

Noting that

$$\langle 0 | a(\mathbf{p}, \sigma, \tau) a^\dagger(\mathbf{p}, \sigma, \tau) | 0 \rangle = \langle 0 | a^\dagger(\mathbf{p}', \sigma', \tau') a(\mathbf{p}, \sigma, \tau) + (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \delta_{\sigma\sigma'} \delta_{\tau\tau'} | 0 \rangle \quad (5.340)$$

and keeping the surviving second term on the right hand side and evaluating the \mathbf{p}' integral gives

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3 2p_0} \sum_{\sigma, \tau} [e^{ip \cdot (y - x)} u_\alpha(\mathbf{p}, \sigma, \tau) \bar{u}_\beta(\mathbf{p}, \sigma, \tau) \theta(y^0 - x^0)]. \quad (5.341)$$

Turning now to the second term of the vacuum expectation value of the time ordered product, we have

$$\begin{aligned} & \theta(x^0 - y^0) \langle 0 | \int \int \frac{d^3 \mathbf{p} d^3 \mathbf{p}'}{(2\pi)^6 \sqrt{2E} \sqrt{2E'}} \sum_{\sigma, \tau} \sum_{\sigma', \tau'} \times \left(\right. \\ & [e^{-ip' \cdot x} \bar{u}_\beta(\mathbf{p}', \sigma', \tau') a^\dagger(\mathbf{p}', \sigma', \tau') + e^{ip' \cdot x} \bar{v}_\beta(\mathbf{p}', \sigma', \tau') b(\mathbf{p}', \sigma', \tau')] \times \\ & \left. [e^{ip \cdot y} u_\alpha(\mathbf{p}, \sigma, \tau) a(\mathbf{p}, \sigma, \tau) + e^{-ip \cdot y} v_\alpha(\mathbf{p}, \sigma, \tau) b^\dagger(\mathbf{p}, \sigma, \tau)] \right) | 0 \rangle. \end{aligned} \quad (5.342)$$

The only term which survives both the vacuum bra and vacuum ket is

$$\langle 0 | \int \int \frac{d^3 \mathbf{p} d^3 \mathbf{p}'}{(2\pi)^6 \sqrt{2E} \sqrt{2E'}} \sum_{\sigma, \tau} \sum_{\sigma', \tau'} \times \quad (5.343)$$

$$[e^{-i(p \cdot y - p' \cdot x)} v_\alpha(\mathbf{p}, \sigma, \tau) \bar{v}_\beta(\mathbf{p}', \sigma', \tau') b(\mathbf{p}', \sigma', \tau') b^\dagger(\mathbf{p}, \sigma, \tau) \theta(x^0 - y^0)] | 0 \rangle. \quad (5.344)$$

By using the anti-commutator of the creation operator with the annihilation operator again, and evaluating the \mathbf{p}' integral, we get (also when $b^\dagger(\mathbf{p}, \sigma, \tau) = a^\dagger(\mathbf{p}, \sigma, \tau)$)

$$\int \frac{d^3 \mathbf{p}}{(2\pi)^3 2E} \sum_{\sigma, \tau} [e^{-ip \cdot (y-x)} v_\alpha(\mathbf{p}, \sigma, \tau) \bar{v}_\beta(\mathbf{p}, \sigma, \tau) \theta(x^0 - y^0)]. \quad (5.345)$$

Recombining both terms again, we get

$$\begin{aligned} & \int \frac{d^3 \mathbf{p}}{(2\pi)^3 2E} \sum_{\sigma, \tau} \times \left(\right. \\ & e^{ip \cdot (y-x)} u_\alpha(\mathbf{p}, \sigma, \tau) \bar{u}_\beta(\mathbf{p}, \sigma, \tau) \theta(y^0 - x^0) - e^{-ip \cdot (y-x)} v_\alpha(\mathbf{p}, \sigma, \tau) \bar{v}_\beta(\mathbf{p}, \sigma, \tau) \theta(x^0 - y^0) \left. \right). \end{aligned}$$

Evaluating the spin sums simplifies this so that we get

$$\int \frac{d^3 \mathbf{p}}{(2\pi)^3 E} \left(e^{ip \cdot (y-x)} (\gamma_\mu p^\mu)_{\alpha\beta} \theta(y^0 - x^0) - e^{-ip \cdot (y-x)} (\gamma_\mu p^\mu)_{\alpha\beta} \theta(x^0 - y^0) \right). \quad (5.346)$$

To make further progress we take advantage of the integral representations of the θ functions given by [59, p.365]:

$$\theta(y^0 - x^0) = \lim_{\epsilon \rightarrow 0^+} \int \frac{d\omega}{2\pi i} \frac{e^{i\omega(y^0 - x^0)}}{\omega - i\epsilon} \quad (5.347)$$

and

$$\theta(x^0 - y^0) = \lim_{\epsilon \rightarrow 0^+} \int \frac{d\omega}{2\pi i} \frac{e^{i\omega(x^0 - y^0)}}{\omega - i\epsilon}. \quad (5.348)$$

Substituting these in and dropping the writing of the limit, assuming it to be understood, gives

$$\int \frac{d^3 \mathbf{p} d\omega}{i(2\pi)^4 E} \left(\frac{e^{i(E+\omega)(y^0 - x^0)} e^{-i\mathbf{p} \cdot (\mathbf{y} - \mathbf{x})} (\gamma_\mu p^\mu)_{\alpha\beta}}{\omega - i\epsilon} - \frac{e^{-i(E+\omega)(y^0 - x^0)} e^{i\mathbf{p} \cdot (\mathbf{y} - \mathbf{x})} (\gamma_\mu p^\mu)_{\alpha\beta}}{\omega - i\epsilon} \right). \quad (5.349)$$

Now, letting $p_0 = E + \omega$ so that $\omega = p_0 - E$ and substituting this in to the expression immediately above gives

$$\int \frac{d^4 p}{i(2\pi)^4 p_0} \left(\frac{e^{ip \cdot (y-x)} (\gamma_\mu p^\mu)_{\alpha\beta}}{p_0 - E - i\epsilon} - \frac{e^{-ip \cdot (y-x)} (\gamma_\mu p^\mu)_{\alpha\beta}}{p_0 - E - i\epsilon} \right). \quad (5.350)$$

5 Non-Standard Quantum Fields

In order to have a common exponential in both terms, we make the substitution $p \rightarrow -p$ in the second term and pull out the now common exponential to get

$$\int \frac{d^4 p}{i(2\pi)^4 E} e^{ip \cdot (y-x)} \left(\frac{(\gamma_\mu p^\mu)_{\alpha\beta}}{p_0 - E - i\epsilon} + \frac{(\gamma_\mu p^\mu)_{\alpha\beta}}{-p_0 - E - i\epsilon} \right). \quad (5.351)$$

Combining the two terms yields

$$\int \frac{d^4 p}{i(2\pi)^4 E} e^{ip \cdot (y-x)} (\gamma_\mu p^\mu)_{\alpha\beta} \left(\frac{-p_0 - E - i\epsilon + p_0 - E - i\epsilon}{(p_0 - E - i\epsilon)(-p_0 - E - i\epsilon)} \right). \quad (5.352)$$

The part in the brackets expands out to give

$$\frac{-2E - 2i\epsilon}{-p_0^2 - p_0 E - p_0 i\epsilon + E p_0 + E^2 + E i\epsilon + i\epsilon p_0 + i\epsilon E - \epsilon^2}. \quad (5.353)$$

By noting that $|E| \gg |\epsilon|$, the second term in the numerator may be ignored. By noting also that $|2Ei\epsilon| \approx |i\epsilon|$ and also that $\epsilon^2 \approx 0$, the above fraction becomes

$$\frac{-2E}{-p_0^2 + E^2 + i\epsilon}. \quad (5.354)$$

Finally, replacing E^2 with $m^2 + \mathbf{p}^2$ in the denominator yields the propagator to be

$$-i \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot (y-x)} \frac{2(\gamma_\mu p^\mu)_{\alpha\beta}}{p^2 - m^2 - i\epsilon}. \quad (5.355)$$

This is like the Dirac propagator from spin-1/2 massive quantum fields arising from the standard state space except here there is no mass term in the numerator, making a fundamental distinction between standard and non-standard massive spin-1/2 quantum fields. The Dirac propagator is proportional to a Green's function for the Dirac equation [59, p.185]. A similar thing is not the case here however. The numerator of this non-standard propagator does not annihilate the non-standard field operator.

To describe the dynamics of the field we inspect Eqns. (5.318) – (5.326) from which it follows that

$$(i\gamma^\mu \partial_\mu - m\tau)\psi(x) = 0. \quad (5.356)$$

The associated Lagrangian that gives rise to the above dynamical equations of motion, Eqn. (5.356), via a variational principle following steps along the lines described in the first half of Sec. (3.5.1) (see also [69, Sec.3.2]) is

$$\mathcal{L} = \bar{\psi} \left(i\gamma^\mu \overleftrightarrow{\partial}_\mu - m\tau \right) \psi. \quad (5.357)$$

The form of the Lagrangian should not be surprising, given that we have seen that mathematically, it is possible to transform the non-standard Wigner class to look like two copies of the standard Wigner class.

A simple check reveals that the canonically conjugate field momentum is

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{i}{2} \psi^\dagger \quad (5.358)$$

which is consistent with our observations earlier in the section. Following the procedure described in [7, p.140-142] yields the Hamiltonian to be

$$H(x) = \int d^3\mathbf{x} \psi^\dagger(x) i \overleftrightarrow{\partial}_0 \psi(x) \quad (5.359)$$

which has the same form as the Dirac Hamiltonian [7, p.142].

5.7 Darkness and the New Quantum Field

In this section we discuss the conditions under which this new quantum field is a dark matter candidate. For convenience, we display the Lagrangian density explicitly with the adjoint operator $\psi^\dagger(x)$:

$$\mathcal{L} = \psi^\dagger \gamma_0 \left(i \gamma^\mu \overleftrightarrow{\partial}_\mu - m \right) \psi \quad (5.360)$$

where the adjoint field operator $\psi^\dagger(x)$ is

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma, \tau} \left[e^{ip \cdot x} v^\dagger(\mathbf{p}, \sigma, \tau) a(\mathbf{p}, \sigma, \tau) + e^{-ip \cdot x} u^\dagger(\mathbf{p}, \sigma, \tau) a^\dagger(\mathbf{p}, \sigma, \tau) \right]. \quad (5.361)$$

The darkness of this quantum field comes about when there are no conserved quantum numbers so we have here set $b^\dagger(\mathbf{p}, \sigma, \tau) = a^\dagger(\mathbf{p}, \sigma, \tau)$. Making this identification prevents any kind of $SU(N)$ -type gauge interaction from being possible, as we will explain further shortly.

By inspection of the spinors we see that

$$v(\mathbf{p}, \sigma, 1)^* = Eu(\mathbf{p}, \sigma, 1) \quad \text{and} \quad u(\mathbf{p}, \sigma, 1)^* = Ev(\mathbf{p}, \sigma, 1) \quad (5.362)$$

where E is the constant matrix

$$E \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (5.363)$$

For the $\tau = -1$ spinors we have

$$v(\mathbf{p}, \sigma, -1)^* = \gamma_5 v(\mathbf{p}, \sigma, 1)^* = \gamma_5 Eu(\mathbf{p}, \sigma, 1) = \gamma_5 E \gamma_5 u(\mathbf{p}, \sigma, -1) = -Eu(\mathbf{p}, \sigma, -1) \quad (5.364)$$

and

$$u(\mathbf{p}, \sigma, -1)^* = \gamma_5 u(\mathbf{p}, \sigma, 1)^* = \gamma_5 Ev(\mathbf{p}, \sigma, 1) = \gamma_5 E \gamma_5 v(\mathbf{p}, \sigma, -1) = -Ev(\mathbf{p}, \sigma, -1) \quad (5.365)$$

where we have used $\{E, \gamma_5\} = 0$ and $\gamma_5^2 = 1$. We thus have

$$v(\mathbf{p}, \sigma, \tau)^* = \tau Eu(\mathbf{p}, \sigma, \tau) \quad \text{and} \quad u(\mathbf{p}, \sigma, \tau)^* = \tau Ev(\mathbf{p}, \sigma, \tau). \quad (5.366)$$

Substitution into Eqn. (5.361) yields

$$\psi^\dagger(x) = \psi^T(x) \tau E \quad (5.367)$$

5 Non-Standard Quantum Fields

where we have used the fact that $E = E^T$. Substitution into Eqn. (5.360) yields

$$\mathcal{L} = \psi^T E \gamma_0 \tau \left(i \gamma^\mu \overleftrightarrow{\partial}_\mu - m \tau \right) \psi. \quad (5.368)$$

The Lagrangian is not invariant under $U(1)$ gauge transformations since for $\psi' = e^{iq\chi}\psi$, we have

$$(\psi^T)' \psi' = \psi^T e^{iq\chi} e^{iq\chi} \psi = \psi^T e^{2iq\chi} \psi \neq \psi^T \psi. \quad (5.369)$$

The new Majorana-type quantum field is therefore dark with respect to the Standard Model gauge quanta of Quantum Chromodynamics and the Electroweak Theory which demand invariance of the Lagrangians under $SU(3)$ and $SU(2)_L \times U(1)_Y$ respectively. The darkness here is due to the Majorana condition $b(\mathbf{p}, \sigma, \tau) = a(\mathbf{p}, \sigma, \tau)$ rather than the non-standardness of the quantum field.

By writing the field operator with both an $a(\mathbf{p}, \sigma, \tau)$ and an $a^\dagger(\mathbf{p}, \sigma, \tau)$ operator, we have automatically made the new quantum field a Majorana quantum field, which then automatically becomes a dark matter candidate. We have assumed that the associated particle is its own antiparticle, that is, the particle has no conserved quantum numbers. An operator C is defined by

$$C |\mathbf{p}, \sigma, \tau\rangle = \xi |\mathbf{p}, \sigma, \tau\rangle \quad (5.370)$$

with the creation and annihilation operators transforming under C as

$$C a^\dagger(\mathbf{p}, \sigma, \tau) C^{-1} = \xi a^\dagger(\mathbf{p}, \sigma, \tau) \quad (5.371)$$

and

$$C a(\mathbf{p}, \sigma, \tau) C^{-1} = \xi a(\mathbf{p}, \sigma, \tau) \quad (5.372)$$

respectively. An explicit calculation, the steps of which should now be familiar, gives

$$C \psi(x) C^{-1} = \pm \psi(x) \quad (5.373)$$

so long as $\xi = \xi^* = \pm 1$. If this operator C were to be called the charge conjugation operator, in view of the Lee and Wick Theorem and the CPT Theorem, we would expect the operator equation $[CP^{\text{NS}} T^{\text{NS}}]^2 = -1$. Explicit calculation however reveals that

$$[CP^{\text{NS}} T^{\text{NS}}]^2 |\mathbf{p}, \sigma, \tau\rangle = -\tau^2 \xi^* \xi \eta^* \eta (-1)^{\frac{1}{2}+\sigma} (-1)^{\frac{1}{2}-\sigma} |\mathbf{p}, \sigma, \tau\rangle = + |\mathbf{p}, \sigma, \tau\rangle. \quad (5.374)$$

The operator C is related to the operator C^{NS} by

$$C = C^{\text{NS}} S \quad (5.375)$$

where

$$S |\mathbf{p}, \sigma, \tau\rangle = \tau |\mathbf{p}, \sigma, \tau\rangle. \quad (5.376)$$

If we wish to phrase the Majorana condition in terms of the charge conjugation operator C^{NS} itself, we have a modified *non-standard* Majorana condition

$$C^{\text{NS}} \psi(x) C^{\text{NS}-1} = \pm \tau \psi(x). \quad (5.377)$$

In finishing this section we note that we have found distinct operators C and C^{NS} that have the desired properties of satisfying the usual Majorana condition and the CPT Theorem. We have not proved that there is not one operator which satisfies both of these properties at once.

5.8 The New Quantum Field with $b^\dagger(\mathbf{p}, \sigma, \tau) \neq a^\dagger(\mathbf{p}, \sigma, \tau)$

In the previous section the darkness of the new quantum field came about due to the Majorana condition being satisfied. Specifically, we had set $b^\dagger(\mathbf{p}, \sigma, \tau) = a^\dagger(\mathbf{p}, \sigma, \tau)$. In this section we set $b^\dagger(\mathbf{p}, \sigma, \tau) \neq a^\dagger(\mathbf{p}, \sigma, \tau)$ and thus allow the new quantum field to carry conserved quantum numbers making antiparticles distinct from particles. In this section we split the quantum field into left and right-handed components and make some observations that hint at the quantum field's possible involvement in the Standard Model. Specifically, we put forward the identification of this non-standard quantum field with neutrinos. Based on this identification, we also put forward right-handed neutrinos and left-handed antineutrinos as being possible dark matter candidates.

If we define the projection operators

$$P_L \equiv \frac{1}{2}(1 - \gamma_5) \quad \text{and} \quad P_R \equiv \frac{1}{2}(1 + \gamma_5) \quad (5.378)$$

where

$$P_L \psi \equiv \psi_L \quad \text{and} \quad P_R \psi \equiv \psi_R, \quad (5.379)$$

we can write

$$\psi_\ell(x) = \psi_\ell(x)_L + \psi_\ell(x)_R. \quad (5.380)$$

Explicitly, the components $\psi_\ell(x)_L$ and $\psi_\ell(x)_R$ are

$$\psi_\ell(x)_L = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \sum_{\sigma, \tau} \left[e^{ip \cdot x} u_\ell(\mathbf{p}, \sigma, \tau)_L a(\mathbf{p}, \sigma, \tau) + e^{-ip \cdot x} v_\ell(\mathbf{p}, \sigma, \tau)_L b^\dagger(\mathbf{p}, \sigma, \tau) \right] \quad (5.381)$$

and

$$\psi_\ell(x)_R = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \sum_{\sigma, \tau} \left[e^{ip \cdot x} u_\ell(\mathbf{p}, \sigma, \tau)_R a(\mathbf{p}, \sigma, \tau) + e^{-ip \cdot x} v_\ell(\mathbf{p}, \sigma, \tau)_R b^\dagger(\mathbf{p}, \sigma, \tau) \right] \quad (5.382)$$

respectively. Here $u_{L/R} \equiv P_{L/R}u$ and $v_{L/R} \equiv P_{L/R}v$. In terms of these left and right-handed field components the non-standard Lagrangian is

$$\begin{aligned} \mathcal{L} &= i[\bar{\psi}_L + \bar{\psi}_R]\gamma^\mu \partial_\mu [\psi_L + \psi_R] - m[\bar{\psi}_L + \bar{\psi}_R]\tau[\psi_L + \psi_R] \\ &= i[\bar{\psi}_L\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_L\gamma^\mu \partial_\mu \psi_R + \bar{\psi}_R\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R\gamma^\mu \partial_\mu \psi_R] \\ &\quad - m[\bar{\psi}_L\tau\psi_L + \bar{\psi}_L\tau\psi_R + \bar{\psi}_R\tau\psi_L + \bar{\psi}_R\tau\psi_R] \end{aligned} \quad (5.383)$$

5 Non-Standard Quantum Fields

If we remember that $\bar{\psi} = \psi^\dagger \gamma^0$, then $\gamma^0 \gamma^\mu$, when written as a 4×4 matrix composed of 2×2 blocks, is block diagonal so the cross terms

$$\bar{\psi}_L \gamma^\mu \partial_\mu \psi_R + \bar{\psi}_R \gamma^\mu \partial_\mu \psi_L \quad (5.384)$$

vanish^{††}. Similarly, the presence of the off diagonal matrix γ^0 in the mass terms ensures that

$$\bar{\psi}_L \tau \psi_L = \bar{\psi}_R \tau \psi_R = 0. \quad (5.385)$$

The non-standard Lagrangian thus takes the form

$$\mathcal{L} = i [\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R \gamma^\mu \partial_\mu \psi_R] - m [\bar{\psi}_L \tau \psi_R + \bar{\psi}_R \tau \psi_L]. \quad (5.386)$$

We will now examine ψ_L and ψ_R separately and in more detail. Consider part of the left-handed field $\psi(x)_L$:

$$\sum_{\sigma, \tau} u(\mathbf{p}, \sigma, \tau)_L a(\mathbf{p}, \sigma, \tau) = \sum_{\sigma} [u(\mathbf{p}, \sigma, 1)_L a(\mathbf{p}, \sigma, 1) + u(\mathbf{p}, \sigma, -1)_L a(\mathbf{p}, \sigma, -1)]. \quad (5.387)$$

By inspection of Eqn. (5.305) and Eqn. (5.306) we see that

$$u(\mathbf{p}, \sigma, -1)_L = -u(\mathbf{p}, \sigma, 1)_L \quad (5.388)$$

so Eqn. (5.387) becomes

$$\sum_{\sigma, \tau} u(\mathbf{p}, \sigma, \tau)_L a(\mathbf{p}, \sigma, \tau) = \sum_{\sigma} u(\mathbf{p}, \sigma, 1)_L [a(\mathbf{p}, \sigma, 1) - a(\mathbf{p}, \sigma, -1)]. \quad (5.389)$$

Numerically, $u_\ell(\mathbf{p}, \sigma, 1)$ is identical to the Dirac coefficient $u_\ell(\mathbf{p}, \sigma)$ for each fixed ℓ . We will therefore write

$$u(\mathbf{p}, \sigma)_L \equiv u(\mathbf{p}, \sigma, 1)_L \quad (5.390)$$

and also adopt the convenient notation

$$a(\mathbf{p}, \sigma) \equiv a(\mathbf{p}, \sigma, 1) - a(\mathbf{p}, \sigma, -1). \quad (5.391)$$

Similarly, $v(\mathbf{p}, \sigma, -1)_L = -v(\mathbf{p}, \sigma, 1)_L$ so we will write

$$v(\mathbf{p}, \sigma)_L \equiv v(\mathbf{p}, \sigma, 1)_L \quad \text{and} \quad b^\dagger(\mathbf{p}, \sigma) \equiv b^\dagger(\mathbf{p}, \sigma, 1) - b^\dagger(\mathbf{p}, \sigma, -1). \quad (5.392)$$

The left-handed component of the quantum field then takes the form

$$\psi_L(x)_L = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \sum_{\sigma} \left[e^{ip \cdot x} u_\ell(\mathbf{p}, \sigma)_L a(\mathbf{p}, \sigma) + e^{-ip \cdot x} v_\ell(\mathbf{p}, \sigma)_L b^\dagger(\mathbf{p}, \sigma) \right]. \quad (5.393)$$

^{††}To help see this more easily, we point out that the $P_{L/R}$ matrices project out either the top two or the bottom two components of the spinors formed by the coefficient functions. Depending on where the zero slots are, it can be seen at once whether these terms vanish, without having to calculate every detail.

5.8 The New Quantum Field with $b^\dagger(\mathbf{p}, \sigma, \tau) \neq a^\dagger(\mathbf{p}, \sigma, \tau)$

Turning our attention now to the right-handed field $\psi(x)_R$, if we define

$$u(\mathbf{p}, \sigma)_R \equiv u(\mathbf{p}, \sigma, 1)_R \quad \text{and} \quad v(\mathbf{p}, \sigma)_R \equiv v(\mathbf{p}, \sigma, 1)_R \quad (5.394)$$

together with

$$c(\mathbf{p}, \sigma) \equiv a(\mathbf{p}, \sigma, 1) + a(\mathbf{p}, \sigma, -1) \quad \text{and} \quad d^\dagger(\mathbf{p}, \sigma) \equiv b^\dagger(\mathbf{p}, \sigma, 1) + b^\dagger(\mathbf{p}, \sigma, -1) \quad (5.395)$$

the field $\psi_\ell(x)_R$ becomes:

$$\psi_\ell(x)_R = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \sum_{\sigma} \left[e^{ip \cdot x} u_\ell(\mathbf{p}, \sigma)_R c(\mathbf{p}, \sigma) + e^{-ip \cdot x} v_\ell(\mathbf{p}, \sigma)_R d^\dagger(\mathbf{p}, \sigma) \right]. \quad (5.396)$$

The left-handed field $\psi(x)_L$ destroys particles created by the operators $a^\dagger(\mathbf{p}, \sigma, 1) - a^\dagger(\mathbf{p}, \sigma, -1)$ and creates antiparticles with the operators $b^\dagger(\mathbf{p}, \sigma, 1) - b^\dagger(\mathbf{p}, \sigma, -1)$. The right-handed field $\psi(x)_R$ seems to make what we would consider different particles. The right-handed field destroys particles created by the operators $a^\dagger(\mathbf{p}, \sigma, 1) + a^\dagger(\mathbf{p}, \sigma, -1)$ and creates antiparticles with the operators $b^\dagger(\mathbf{p}, \sigma, 1) + b^\dagger(\mathbf{p}, \sigma, -1)$. It is tempting therefore to consider the fields $\psi(x)_L$ and $\psi(x)_R$ to be the left and right-handed components of the two fields $\nu_\ell^1(x)$ and $\nu_\ell^2(x)$ respectively where

$$\nu_\ell^1(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \sum_{\sigma} \left[e^{ip \cdot x} u_\ell(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{-ip \cdot x} v_\ell(\mathbf{p}, \sigma) b^\dagger(\mathbf{p}, \sigma) \right] \quad (5.397)$$

and

$$\nu_\ell^2(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \sum_{\sigma} \left[e^{ip \cdot x} u_\ell(\mathbf{p}, \sigma) c(\mathbf{p}, \sigma) + e^{-ip \cdot x} v_\ell(\mathbf{p}, \sigma) d^\dagger(\mathbf{p}, \sigma) \right]. \quad (5.398)$$

With these definitions we can also write

$$\tau\psi_R \equiv \tau\nu_R^2 = \nu_R^1 \quad \text{and} \quad \tau\psi_L \equiv \tau\nu_L^1 = \nu_L^2. \quad (5.399)$$

The Lagrangian then takes the form:

$$\mathcal{L} = i \left[\bar{\nu}_L^1 \gamma^\mu \partial_\mu \nu_L^1 + \bar{\nu}_R^2 \gamma^\mu \partial_\mu \nu_R^2 \right] - m \left[\bar{\nu}_L^1 \nu_R^1 + \bar{\nu}_R^2 \nu_L^2 \right]. \quad (5.400)$$

In the kinematical part of the Lagrangian only the left-handed component of what we refer to as the $\nu_\ell^1(x)$ field is present and only the right-handed component of what we refer to as the ν^2 field is present. This seems to indicate that we may be able to identify the fields $\nu_\ell^1(x)$ and $\nu_\ell^2(x)$ with neutrinos and antineutrinos respectively since only left-handed neutrinos and right-handed antineutrinos have been observed in nature. If we can indeed make such an identification then we may have part of an explanation for why observed neutrinos are always left-handed and why antineutrinos are always right-handed. In this view, neutrinos are identified as parts of the non-standard massive spin-1/2 quantum field $\psi_\ell(x)$. One obvious thing we need to check if our neutrino identification is to hold, is that the $\nu_\ell^2(x)$ field corresponds to the charge conjugated field $C^{\text{NS}} \nu_\ell^1(x) (C^{\text{NS}})^{-1}$, and similarly that the $\nu_\ell^1(x)$ field corresponds to the charge conjugated field $C^{\text{NS}} \nu_\ell^2(x) (C^{\text{NS}})^{-1}$. We now check this.

5 Non-Standard Quantum Fields

By observing that $\xi_n^* = \xi_{\bar{n}}$ (for Dirac-type fermions) together with

$$C^{\text{NS}} a(\mathbf{p}, \sigma)(C^{\text{NS}})^{-1} = \xi_n^* d(\mathbf{p}, \sigma), \quad C^{\text{NS}} b^\dagger(\mathbf{p}, \sigma)(C^{\text{NS}})^{-1} = \xi_n^* c^\dagger(\mathbf{p}, \sigma) \quad (5.401)$$

$$C^{\text{NS}} c(\mathbf{p}, \sigma)(C^{\text{NS}})^{-1} = \xi_n^* b(\mathbf{p}, \sigma), \quad C^{\text{NS}} d^\dagger(\mathbf{p}, \sigma)(C^{\text{NS}})^{-1} = \xi_n^* a^\dagger(\mathbf{p}, \sigma) \quad (5.402)$$

we see that the charge conjugated field $C^{\text{NS}} \nu_\ell^1(x)(C^{\text{NS}})^{-1}$ is

$$\xi_n^* \left[\int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \left[e^{ip \cdot x} v^*(\mathbf{p}, \sigma) c(\mathbf{p}, \sigma) + e^{-ip \cdot x} u^*(\mathbf{p}, \sigma) d^\dagger(\mathbf{p}, \sigma) \right] \right]^\dagger. \quad (5.403)$$

Finally, we follow [42, p.225] in observing that

$$v^*(\mathbf{p}, \sigma) = \mathcal{C} u(\mathbf{p}, \sigma), \quad \text{and} \quad u^*(\mathbf{p}, \sigma) = \mathcal{C} v(\mathbf{p}, \sigma) \quad (5.404)$$

where

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad (5.405)$$

and arrive at the results

$$C^{\text{NS}} \nu_\ell^1(x)(C^{\text{NS}})^{-1} = \xi_n^* \sum_{\bar{\ell}} \mathcal{C}_{\ell\bar{\ell}} \nu_{\bar{\ell}}^2(x)^* \quad (5.406)$$

and

$$C^{\text{NS}} \nu_\ell^2(x)(C^{\text{NS}})^{-1} = \xi_n^* \sum_{\bar{\ell}} \mathcal{C}_{\ell\bar{\ell}} \nu_{\bar{\ell}}^1(x)^*. \quad (5.407)$$

In Eqn. (5.400) we see the presence of a right-handed “neutrino” field and a left-handed “antineutrino” field in the mass terms of the Lagrangian. There are no corresponding kinematical terms. If we now consider $U(1)$ gauge transformations, we notice that the mass terms are gauge invariant but because there are no kinematical terms associated with ν_R^1 and ν_L^2 that contain derivatives, the right-handed “neutrino” and left-handed “antineutrino” do not couple to any gauge fields, effectively making them dark with respect to the Standard Model. For this reason we suggest that perhaps right-handed neutrinos and left-handed antineutrinos should be regarded as prime dark matter candidates.

The left-handed “neutrino” field ν_L^1 and the right-handed “antineutrino” field ν_R^2 have kinematical parts in the Lagrangian but have no explicit corresponding mass terms. The writing down of a kinematical left-handed neutrino term in the Lagrangian is consistent with the Electroweak Theory, where there is no free particle mass term present. Particles are assumed to be intrinsically massless in the Standard Model and only acquire mass by interacting with the Higgs field.

Following the procedure outlined in Chapter 2, the Standard Model has an obvious problem. Free particle Lagrangians are written down as massless. The problem with this is that the

little group is $ISO(2)$ instead of $SO(3)$ [42, p.66]. The irreducible representations in the massless case do not give Dirac fields because the little group does not give the same class of representations as one gets when the little group is $SO(3)$. However, in our non-standard model, this problem seems to naturally be resolved. These observations also help to clarify the physical interpretation of the non-standard propagator, Eqn. (5.355), which, although derived from a *massive* quantum field, mysteriously does not have an explicit mass term. This may be the key that justifies on purely quantum field theoretic grounds, the writing down of *massless* Lagrangians in the Standard Model. This lack of explicit mass terms for the left-handed “neutrino” field $\nu_L^1(x)$ and right-handed “antineutrino” field $\nu_R^2(x)$ also seems consistent with the observation that neutrinos are apparently almost massless. If neutrinos may indeed be identified with non-standard fields, perhaps most of the mass is contained by the “dark matter” fields $\nu_R^1(x)$ and $\nu_L^2(x)$, leaving very little mass for the kinematical parts of the neutrino fields, namely $\nu_L^1(x)$ and $\nu_R^2(x)$.

One final observation we point out here, is that if the identification of the non-standard quantum field with neutrinos proved to be correct, then the parity and time reversal symmetries actually still hold. The perception that parity symmetries are broken in weak interactions may be due to the neutrino being identified with the wrong quantum field. This misidentification may relate to how the measurement process works. Perhaps the act of observing a non-standard quantum field acts as some sort of projection operator on the state space so that the observer remains unaware of everything that is actually going on.

5.9 Conclusion and Discussion

In this chapter we have looked at the non-standard representations of the Poincaré group on the Hilbert space of physical states. The initial motivation for doing this was that Elko spinors have parity and time reversal commutation relations consistent with non-standard Wigner classes, thus hinting at the possibility of Elko quantum fields finding their proper theoretical context within these classes. Our search has not yielded Elko fields here either. Regardless of this, while undertaking this investigation, we have more thoroughly explored possible avenues for extending the Standard Model within the already existing standard framework of flat spacetime symmetries, which, in itself, is a worthwhile and important thing to do because it seems natural to us that the full extent of mathematical freedom in a theory should be explored in order to not accidentally or carelessly leave out any possible physics. We showed that in two of the three non-standard Wigner classes, no massive spin-1/2 quantum fields were possible and that there was such a quantum field for the remaining non-standard Wigner class, Case 3. Furthermore, we showed under what conditions this reduces to two copies of the standard Wigner class. The new quantum field remains different from the usual quantum field in any case, because the finite-dimensional representation of the Lorentz group is different.

5 Non-Standard Quantum Fields

If $b(\mathbf{p}, \sigma, \tau) = a(\mathbf{p}, \sigma, \tau)$, then we have a natural dark matter candidate that is Majorana in nature. Moreover, this new quantum field dark matter candidate is local by construction, and also a well-defined quantum field in the sense of Weinberg.

We also looked at the case where the new quantum field had one or more conserved quantum numbers. When written in terms of its left and right-handed components, the non-standard quantum field appears as though it is composed of the left-handed components of one field and the right-handed component of another field. Such fields transform into each other by the charge conjugation operation. We made the suggestion that perhaps neutrinos correspond to these non-standard quantum fields, explaining why the left-handed components of neutrinos and the right-handed components of antineutrinos are observed in contrast to the right-handed neutrino and left-handed antineutrino components which are not observed. Furthermore, since the right-handed components of such particles and left-handed components of such antiparticles have a mass term present, and there are no derivatives in the mass term (and hence no coupling to gauge fields by writing down covariant derivatives), we suggested that these fields are dark with respect to Standard Model gauge interactions. We also suggested that on account of neutrinos being identified with non-standard quantum fields, parity might not be violated in weak interactions after all, but that it may only appear that way, based on our expectations on account of identifying neutrinos with standard quantum fields with standard transformation properties under standard parity and time reversal operators.

In the next chapter we take a different approach to the problem of how to fit Elko into the context of the Standard Model. We explore the concept of mass dimension transmutation in the context of the Electroweak Theory.

6 Elko Fields and the Electroweak Theory: an Elko-Dirac Connection

6.1 Introduction

In this chapter we take another approach to the problem of fitting Elko into the context of the Standard Model. The idea is motivated by the general concept in [39] of a mass dimensionality transmuting operation. The context of the concept expressed in these citations is that of Classical Spinor Theory. No attempt at a quantum theory was given. In this chapter, we take the concept of mass dimensionality transmutation and apply it on the Hilbert space of physical states in the most direct and simple way we can, and explore some possible consequences of such an action, which may help to explain why only left-handed fermion field components participate in electroweak interactions. In Chapter 5 we suggested that perhaps neutrinos were non-standard quantum fields, and a natural consequence of this was getting particles that were only left-handed and getting antiparticles that were only right-handed. At least, as far as Standard Model interactions are concerned (We also suggested that right-handed particles and left-handed antiparticles may be present but they may be dark matter candidates, not having Standard Model interactions arising from a gauge principle). However, in nature, when it comes to other particles like electrons, both left-handed and right-handed components are observed which does not seem to fit the non-standard paradigm so attributing neutrino properties to being represented by non-standard quantum fields does not seem to offer a full explanation for why for example, only the left-handed components of quarks interact electroweakly. Perhaps a fuller analysis might reveal that the identification of neutrinos with non-standard quantum fields somehow forces only the left-handed components of other Dirac-type fermions to interact electroweakly. Such an investigation goes beyond the intended scope of this thesis and so is not pursued here.

The appeal of relating Elko fields to the electroweak sector of the Standard Model resides in the fact that this sector involves spontaneously broken symmetries and does not respect the parity symmetry. Such a broken symmetry situation might suit Elko fields better.

6.2 Elko's Possible Involvement in Electroweak Interactions with Standard Model Matter

We here consider the concept of mass dimensionality transmutation of fermions in the context of the electroweak sector of the Standard Model.

The first thing we here consider is the left-handed aspect of the weak interactions. For standard Dirac type fermions, the wave equation for a free Dirac field ψ is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \quad (6.1)$$

If we apply the projection operator P_L to the left side of the Dirac equation, and note that $\{\gamma^\mu, \gamma_5\} = 0$, we see that [67, p.464]

$$i\gamma^\mu \partial_\mu \psi_R = m\psi_L. \quad (6.2)$$

Similarly, if we apply the projection operator P_R to the Dirac equation, we get

$$i\gamma^\mu \partial_\mu \psi_L = m\psi_R. \quad (6.3)$$

Given that

$$m\psi = m\psi_L + m\psi_R, \quad (6.4)$$

it follows that the Dirac equation cannot be locally invariant under the *left-handed* gauge group $SU(2)_L$. The standard solution to this problem for standard Dirac fermions is to say that free fermions are actually massless and that these massless fermion fields acquire mass by interacting with the Higgs field via the *Yukawa* interactions (see, for example, [67, p.465]):

$$i\gamma^\mu \partial_\mu \psi_R = g_f \phi^\dagger l, \quad i\gamma^\mu \partial_\mu l = g_f \phi \psi_R, \quad (6.5)$$

where l is a doublet with non-zero isospin and g_f is a coupling constant. For example, for leptons, the relevant parts of the Lagrangian take the form

$$\mathcal{L} = -g_e(\bar{l}_e \phi e_R + \bar{e}_R \phi^\dagger l_e) - g_\mu(\bar{l}_\mu \phi \mu_R + \bar{\mu}_R \phi^\dagger l_\mu) - g_\tau(\bar{l}_\tau \phi \tau_R + \bar{\tau}_R \phi^\dagger l_\tau). \quad (6.6)$$

Inserting the vacuum expectation value $\frac{f}{\sqrt{2}}$ of ϕ into the Lagrangian gives

$$\mathcal{L}^{\text{VAC}} = \frac{-g_e f}{\sqrt{2}}(\bar{e}_L e_R + \bar{e}_R e_L) - \frac{g_\mu f}{\sqrt{2}}(\bar{\mu}_L \mu_R + \bar{\mu}_R \mu_L) - \frac{g_\tau f}{\sqrt{2}}(\bar{\tau}_L \tau_R + \bar{\tau}_R \tau_L). \quad (6.7)$$

Making the identifications

$$m_e = \frac{g_e f}{\sqrt{2}}, \quad m_\mu = \frac{g_\mu f}{\sqrt{2}}, \quad m_\tau = \frac{g_\tau f}{\sqrt{2}} \quad (6.8)$$

allows us to identify the terms in \mathcal{L}^{VAC} with the standard Dirac mass terms [20, p.359].

In this way, the Higgs boson takes on an additional very important role in the Standard Model. In addition to giving mass to the weak force gauge quanta, the Higgs field also gives mass to all of the Dirac type fermions of the Standard Model.

6.2 Elko's Possible Involvement in Electroweak Interactions with Standard Model Matter

Now consider the Elko field. At the spinor level, breaking the spinors up into left and right-handed components allows us to more easily directly relate Elko and Dirac spinors via their left and right-handed components. This provides us with a possible clue as to how the Elko field might be related to the Standard Model.

If we make the assumption that the weak force only interacts with left-handed fermions regardless of whether they are Standard Model fermions, or non-Standard Model fermions like Elko, then we need to examine Elko fields to see whether left and right-handed components get mixed like they were in the Dirac field. The dynamical equation of motion for the Elko field is Klein-Gordon in form [34, p.5]:

$$(\partial_\mu \partial^\mu + m^2)\Lambda(x) = 0. \quad (6.9)$$

Applying the projection operators to this equation gives

$$(\partial_\mu \partial^\mu + m^2)\Lambda_L = 0 \quad \text{and} \quad (\partial_\mu \partial^\mu + m^2)\Lambda_R = 0 \quad (6.10)$$

so we see that Elko left and right components are *not* mixed up as they are in the Dirac case. We therefore come to a simple but profound difference between *Elko*-type fermionic fields and standard *Dirac*-type fermionic fields. Elko free particle states might intrinsically have mass and may not need to acquire mass from interacting with the Higgs boson, in contrast to Dirac free particle states, that are intrinsically massless (in the theory anyway) and need to interact with the Higgs field in order to acquire what is normally regarded as *their* rest mass. If Elko particles were to acquire their mass from the Higgs field also, the form of the interaction would be fundamentally different.

A fundamental question now arises. Elko fields are defined with spinors that are dual helicity in nature. The top two components of Elko spinors are eigenstates of the helicity operator $\frac{1}{2}\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}$ with one eigenvalue and the bottom two components of Elko spinors are eigenstates of the helicity operator with opposite eigenvalue. If we take projections and take either the left-handed component or the right-handed component, the question automatically arises whether we could conceivably end up with something that looks like the standard Dirac left and right-handed fields $\psi_L(x)$ and $\psi_R(x)$. We will now explore this by starting with the Dirac field and asking the question *can we make the Dirac field look like either the Elko field, or some combination of left and right-handed components of the Elko field?* Before doing this, we start by choosing a particular basis for Elko. The rest spinors are of the form given

by Eqn. (3.7). We choose the four linearly independent Elko rest spinors to be*

$$\xi\left(\mathbf{0}, \frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \\ 1 \\ 0 \end{pmatrix}, \quad \xi\left(\mathbf{0}, \frac{-1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (6.11)$$

$$\zeta\left(\mathbf{0}, \frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \zeta\left(\mathbf{0}, \frac{-1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -i \\ 1 \\ 0 \end{pmatrix}. \quad (6.12)$$

For ease of comparison, the Dirac rest spinors are [42, p.224]:

$$u\left(\mathbf{0}, \frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u\left(\mathbf{0}, \frac{-1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad (6.13)$$

$$v\left(\mathbf{0}, \frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v\left(\mathbf{0}, \frac{-1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad (6.14)$$

The left and right projection matrices P_L and P_R are respectively

$$P_L = \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & 1_2 \end{pmatrix} \quad \text{and} \quad P_R = \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix}. \quad (6.15)$$

If we define the notation

$$\xi_L(\mathbf{p}, \sigma) \equiv P_L \xi(\mathbf{p}, \sigma), \quad \xi_R(\mathbf{p}, \sigma) \equiv P_R \xi(\mathbf{p}, \sigma), \quad \zeta_L(\mathbf{p}, \sigma) \equiv P_L \zeta(\mathbf{p}, \sigma), \quad \zeta_R(\mathbf{p}, \sigma) \equiv P_R \zeta(\mathbf{p}, \sigma), \quad (6.16)$$

by direct calculation we see that[†]:

$$u_L(\mathbf{p}, \sigma) = \xi_L(\mathbf{p}, \sigma), \quad u_R(\mathbf{p}, \sigma) = i(-1)^{\frac{1}{2}-\sigma} \xi_R(\mathbf{p}, -\sigma) \quad (6.17)$$

and

$$v_L(\mathbf{p}, \sigma) = \zeta_L(\mathbf{p}, \sigma), \quad v_R(\mathbf{p}, \sigma) = i(-1)^{\frac{1}{2}-\sigma} \zeta_R(\mathbf{p}, -\sigma). \quad (6.18)$$

*We are absorbing the \sqrt{m} factor into the boost operator and canceling it with the \sqrt{m} contained in the Elko quantum field integration measure $\frac{1}{\sqrt{2mp_0}}$ so that the Elko spinors and Dirac spinors may be more easily compared, and also so that the Elko and Dirac quantum field operator expressions may be more easily compared since now they have the same explicitly displayed integration measure, namely $\frac{1}{\sqrt{2p_0}}$.

[†]The boost operator commutes with P_L and P_R .

At this point we define a linear and unitary mass dimensionality transmuting operator \mathcal{S} on the state space that changes the mass dimensionality of Dirac-type fermion states of species n from three halves ($[m] = \frac{3}{2}$) to one ($[m] = 1$), the particle species of which, we will denote by n' , so we define[†]

$$\mathcal{S} |\mathbf{p}, \sigma, n\rangle \equiv |\mathbf{p}, \sigma, n'\rangle, \quad \mathcal{S} |0\rangle \equiv |0\rangle. \quad (6.19)$$

This implies that

$$\mathcal{S} a^\dagger(\mathbf{p}, \sigma, n) \mathcal{S}^{-1} = a^\dagger(\mathbf{p}, \sigma, n') \quad (6.20)$$

and

$$\mathcal{S} a(\mathbf{p}, \sigma, n) \mathcal{S}^{-1} = a(\mathbf{p}, \sigma, n'). \quad (6.21)$$

With this most direct approach to defining mass dimensionality transmutation on the state space, we now examine the effect of this operation on the Dirac field. In what follows we abuse notation and write $a(\mathbf{p}, \sigma)$ instead of $a(\mathbf{p}, \sigma, n)$ and $a(\mathbf{p}, \sigma, n')$. For the left-handed component $\psi_L(x)$ of the Dirac field $\psi(x)$ we have $\mathcal{S}\psi_L(x)\mathcal{S}^{-1}$ being

$$\begin{aligned} \mathcal{S} \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma} [e^{ip \cdot x} u_L(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{-ip \cdot x} v_L(\mathbf{p}, \sigma) b^\dagger(\mathbf{p}, \sigma)] \mathcal{S}^{-1} \\ = \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma} [e^{ip \cdot x} \xi_L(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{-ip \cdot x} \zeta_L(\mathbf{p}, \sigma) b^\dagger(\mathbf{p}, \sigma)] \end{aligned} \quad (6.22)$$

so we have

$$\mathcal{S}\psi_L(x)\mathcal{S}^{-1} = \Lambda_L(x). \quad (6.23)$$

Transforming the right-handed component $\psi_R(x)$ to calculate $\mathcal{S}\psi_R(x)\mathcal{S}^{-1}$ yields

$$i \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{(-1)^{\frac{1}{2}-\sigma}}{\sqrt{2p_0}} \sum_{\sigma} [e^{ip \cdot x} \xi_R(\mathbf{p}, -\sigma) a(\mathbf{p}, \sigma) + e^{-ip \cdot x} \zeta_R(\mathbf{p}, -\sigma) b^\dagger(\mathbf{p}, \sigma)]. \quad (6.24)$$

Making the change of variables $\mathbf{p} \rightarrow -\mathbf{p}$, $\sigma \rightarrow -\sigma$ yields

$$-i \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{(-1)^{\frac{1}{2}-\sigma}}{\sqrt{2p_0}} \sum_{\sigma} [e^{i\mathcal{P}p \cdot x} \xi_R(-\mathbf{p}, \sigma) a(-\mathbf{p}, -\sigma) + e^{-i\mathcal{P}p \cdot x} \zeta_R(-\mathbf{p}, \sigma) b^\dagger(-\mathbf{p}, -\sigma)]. \quad (6.25)$$

By recalling that

$$(-1)^{\frac{1}{2}-\sigma} a(-\mathbf{p}, -\sigma) = \zeta_n T a(\mathbf{p}, \sigma) T^{-1} \quad \text{and} \quad (-1)^{\frac{1}{2}-\sigma} b^\dagger(-\mathbf{p}, -\sigma) = \zeta_n^* T b^\dagger(\mathbf{p}, \sigma) T^{-1} \quad (6.26)$$

we see that $\mathcal{S}\psi_R(x)\mathcal{S}^{-1}$ becomes

$$-i \zeta_n T \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \sum_{\sigma} [e^{-i\mathcal{P}p \cdot x} \xi_R^*(-\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{i\mathcal{P}p \cdot x} \zeta_R^*(-\mathbf{p}, \sigma) b^\dagger(\mathbf{p}, \sigma)] T^{-1} \quad (6.27)$$

[†]Mass dimensionality is a property of the field operators. What we mean here is that the creation operator $a^\dagger(\mathbf{p}, \sigma, n)$ belongs to the expression for the field that has mass dimension three halves and similarly the creation operator $a^\dagger(\mathbf{p}, \sigma, n')$ belongs to the expression for the field that has mass dimension one.

where $\zeta_n = \zeta_n^*$. Explicitly, the spinors $\xi_R^*(-\mathbf{p}, \sigma)$ and $\zeta_R^*(-\mathbf{p}, \sigma)$ read:

$$\begin{aligned} \xi_R^*\left(-\mathbf{p}, \frac{1}{2}\right) &\propto \begin{pmatrix} i(p_x + ip_y) \\ -i(m + p_0 + p_z) \\ 0 \\ 0 \end{pmatrix}, & \xi_R^*\left(-\mathbf{p}, -\frac{1}{2}\right) &\propto \begin{pmatrix} i(m + p_0 - p_z) \\ -i(p_x - ip_y) \\ 0 \\ 0 \end{pmatrix} \\ \zeta_R^*\left(-\mathbf{p}, \frac{1}{2}\right) &\propto \begin{pmatrix} i(m + p_0 - p_z) \\ -i(p_x - ip_y) \\ 0 \\ 0 \end{pmatrix}, & \zeta_R^*\left(-\mathbf{p}, -\frac{1}{2}\right) &\propto \begin{pmatrix} -i(p_x + ip_y) \\ i(m + p_0 + p_z) \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

In order to write the expression, Eqn. (6.27) in terms of the right-handed Elko field Λ_R , we would need some matrix A such that

$$\xi_R^*(\mathbf{p}, \sigma)_\ell = \sum_{\bar{\ell}} A_{\ell\bar{\ell}} \xi_R^*(\pm\mathbf{p}, \sigma)_{\bar{\ell}} \quad \text{and} \quad \zeta_R^*(\mathbf{p}, \sigma)_\ell = \sum_{\bar{\ell}} A_{\ell\bar{\ell}} \zeta_R^*(\pm\mathbf{p}, \sigma)_{\bar{\ell}}. \quad (6.28)$$

By inspection of the spinors

$$\begin{aligned} \xi_R\left(\pm\mathbf{p}, \frac{1}{2}\right) &\propto \begin{pmatrix} \pm i(p_x - ip_y) \\ i(m + p_0 \mp p_z) \\ 0 \\ 0 \end{pmatrix}, & \xi_R\left(\pm\mathbf{p}, -\frac{1}{2}\right) &\propto \begin{pmatrix} -i(m + p_0 \mp p_z) \\ \mp i(p_x + ip_y) \\ 0 \\ 0 \end{pmatrix} \\ \zeta_R\left(\pm\mathbf{p}, \frac{1}{2}\right) &\propto \begin{pmatrix} -i(m + p_0 \pm p_z) \\ \mp i(p_x + ip_y) \\ 0 \\ 0 \end{pmatrix}, & \zeta_R\left(-\mathbf{p}, -\frac{1}{2}\right) &\propto \begin{pmatrix} \mp i(p_x - ip_y) \\ -i(m + p_0 \mp p_z) \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

we see that the $(p_x \pm ip_y)$ terms would need to be $(p_x \mp ip_y)$ terms so we can not relate $\mathcal{S}\psi_R(x)\mathcal{S}^{-1}$ to $\Lambda_R(x)$. We will now try to relate $\mathcal{S}\psi_R(x)\mathcal{S}^{-1}$ to the left-handed Elko field $\Lambda_L(x)$. We first observe that

$$u_R(\mathbf{p}, \sigma) = \gamma_0 \xi_L(-\mathbf{p}, \sigma) \quad \text{and} \quad v_R(\mathbf{p}, \sigma) = -\gamma_0 \zeta_L(-\mathbf{p}, \sigma). \quad (6.29)$$

Transforming the right-handed component $\psi_R(x)$ yields

$$\begin{aligned} &\gamma_0 \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma} [e^{ip \cdot x} \xi_L(-\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) - e^{-ip \cdot x} \zeta_L(-\mathbf{p}, \sigma) b^\dagger(\mathbf{p}, \sigma)] \\ &= \gamma_0 \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma} [e^{iPp \cdot x} \xi_L(\mathbf{p}, \sigma) a(-\mathbf{p}, \sigma) - e^{-iPp \cdot x} \zeta_L(\mathbf{p}, \sigma) b^\dagger(-\mathbf{p}, \sigma)]. \end{aligned} \quad (6.30)$$

6.2 Elko's Possible Involvement in Electroweak Interactions with Standard Model Matter

Given that

$$a(-\mathbf{p}, \sigma) = \eta_n P a(\mathbf{p}, \sigma) P^{-1}, \quad b^\dagger(-\mathbf{p}, \sigma) = \eta_n^* P b^\dagger(\mathbf{p}, \sigma) P^{-1} \quad \text{and} \quad \eta_n^* = -\eta_n \quad (6.31)$$

we obtain

$$\eta_n \gamma_0 P \gamma_0 \int \frac{d^3 \mathbf{p}}{(2\pi)^3 \sqrt{2p_0}} \sum_{\sigma} [e^{i\mathcal{P}p \cdot x} \xi_L(\mathbf{p}, \sigma) a(\mathbf{p}, \sigma) + e^{-i\mathcal{P}p \cdot x} \zeta_L(\mathbf{p}, \sigma) b^\dagger(\mathbf{p}, \sigma)] P^{-1} \quad (6.32)$$

so

$$\mathcal{S} \psi_R(x) \mathcal{S}^{-1} = \eta_n \gamma_0 P \Lambda_L(\mathcal{P}x) P^{-1}. \quad (6.33)$$

We therefore arrive at the transformation law for $\psi(x)$ under \mathcal{S} , which is:

$$\mathcal{S} \psi(x) \mathcal{S}^{-1} = \Lambda_L(x) + \eta_n \gamma_0 P \Lambda_L(\mathcal{P}x) P^{-1}. \quad (6.34)$$

Interestingly, the right-handed components of the Elko field do not appear in this expression, but only the left-handed components. Since it is only the left-handed components of Standard Model fermion fields that carry weak isospin, we conjecture that perhaps electroweak interactions involve the mass transmutation of the quarks and leptons from mass dimension three halves to mass dimension one. If the physical scenario during the process of an electroweak interaction involves this mass dimension change of the fermions, and if the fermions take on an Elko form, then locality would be compromised in two spatial directions since Elko preserves an axis of locality. This would have the immediate consequence that the parity space inversion symmetry would automatically be violated, which, along with left-component non-zero isospin, is also a key aspect of electroweak interactions.

The masses of the gauge bosons arise dynamically, generated based on the particular physical vacuum, which, when used to take the vacuum expectation value of the scalar field $\phi(x)$ yields $\frac{f}{\sqrt{2}}$ for some f . If a Standard Model doublet were to be affected by the vacuum in such a way that the mass dimensionality was changed from three halves to one for each fermion in the doublet during the process of the electroweak interaction, the change in mass dimensionality would have to be accounted for. Perhaps the weak gauge bosons acquire their mass dimensionality of one from the fermions in the Standard Model doublet which each had mass dimensionality three halves prior to the electroweak interaction commencing. After electroweak interactions, only the usual Standard Model local mass dimension three halves fermions are observed, so if fermionic mass transmutation does occur during an electroweak interaction, then the mass transmutation would have to reverse by the end of the electroweak interaction. Perhaps the vacuum conditions created during an electroweak interaction are the things which trigger the fermionic mass transmutation in the first place, and this unstable situation soon stabilizes and returns back to normal, restoring the original mass dimensionality of any fermionic states at the end of the electroweak interaction.

Thus far in this section, we have explored a natural first approach to the concept of fermionic mass dimensionality transmutation from three halves to one in the space inversion

symmetry broken, gauge symmetry hidden setting of the electroweak sector of the Standard Model. Such a concept seems to work well in this environment, naturally being consistent with what goes on physically during an electroweak interaction. We will now finish off this chapter by considering Elko fields in their original context, namely that of having stable Elko states that can exist independently of special circumstances (for example, existing only fleetingly during an electroweak interaction). In the preceding discussion, we moved from the setting of the Standard Model in a well defined way with the mass dimensionality transmutation operator \mathcal{S} . The Elko-like states that resulted had clear origins, i.e, they came about as a result of the mass dimension changing operation. In what follows, we consider Elko independently of a theoretical origin so once again will just assume their existence, without worrying about how we might formally derive them on the Hilbert space of physical states. We have also commented on some of the things which follow in [55].

Elko left and right components can be separated leaving the left-handed components to transform differently under $SU(2)_L \times U(1)$ from the right-handed components. Thus, Elko passes a key test concerning its likelihood of being able to interact electroweakly with Standard Model matter. The right-handed components will transform as *singlets* under the electroweak gauge group where as the left-handed components will transform as *doublets* under the electroweak gauge group. We now explore left-handed Elko doublets having non-zero isospin.

In order for an Elko doublet to interact with a W^+ or W^- particle, we require that there be an electric charge difference of unity between the top and bottom components of the weak isospinor. The Elko Lagrangian accommodates global $U(1)$ gauge symmetries (see Sec. (3.3.1)) so Elko particles can, in principle, carry electric charge. Hence there is no problem in writing down an Elko doublet \mathcal{E} of the form

$$\mathcal{E} = \begin{pmatrix} \Lambda^0 \\ \Lambda^- \end{pmatrix}_L \quad (6.35)$$

where the superscript “0” denotes an electrically neutral Elko field and the superscript “-” denotes an electrically charged Elko quantum field differing from the neutral Elko field by one unit of electric charge.

A possible Elko-Standard Model electroweak interaction might therefore look something like

$$\Lambda^- \rightarrow \Lambda^0 + W^- + X$$

where X denotes other decay products adding no net charge to the interaction that might not necessarily consist of purely Standard Model particles. If it turns out that Elko, on account of violating rotational symmetries with its axis of locality, is dark with respect to usual direct detection methods, a possible decay like the one above gives us a possible way of inferring Elko’s presence *indirectly* by studying the Standard Model component of the interactions that we can detect directly.

6.2 Elko's Possible Involvement in Electroweak Interactions with Standard Model Matter

The particular example of a possible interaction may very well come across as the *appearance* of the violation of conservation of electric charge if the special locality attributes of Elko complicate how and where the electromagnetic fields are produced. For example, if an Elko particle was distributed over a plane the size of the galaxy,[§] and if we look for an Elko particle in a very small region of space, the amount of apparent charge in that region might appear as vanishingly small. If electric charge conservation ever did appear to be broken by particle interactions involving W vector bosons, we would have evidence, not necessarily that electric charge conservation has failed to be a universal law of physics, but that Elko particles might be making their presence felt. The incoming Elko particle might have its charge spread over the extent of the particle which is non point-like in at least two spatial directions. One thing that should be studied more in its own right, whether in the context of Elko or more generally, is the physical properties of non-local objects and how non-locality would affect the physical observables like electric charge. We will not further pursue these issues here as they go outside the intended scope of this thesis.

At presently accessible energies it may turn out that such an interaction as that stated above might not ever be observed. This would suggest that either Elko does not exist, or that there is some, as yet unexplored physical mechanism (perhaps some sort of spontaneously broken symmetry mechanism) which prevents Elko symmetry currents from coupling to the charged electroweak symmetry currents, or if they do couple, that the coupling strength is vanishingly small for all practical purposes. If Elko particles do constitute at least a large part of the dark matter sector, then there is evidence [50] that Elko should already be comfortably within the energy range accessible to experiments. Another possibility is that there is some mechanism that restricts the Elko electroweak symmetry currents to only couple with the neutral electroweak symmetry currents involving the Z^0 vector boson.

The first experimental evidence for the Z^0 particle was found using the scattering reaction $\bar{\nu}_\mu + e \rightarrow \bar{\nu}_\mu + e$ [80]. The Z^0 particle can interact with any Standard Model particles except for gluons and photons. We take the view that since Elko particles could, in principle, carry non-zero isospin, they should also be able to interact electroweakly by coupling to the neutral currents. Elko particles should be able to scatter off each other and exchange a Z^0 particle.

[§]The Elko field is non-local in two spatial directions. The extent of the non-locality is unknown. It is also unknown what this may mean for efforts to detect Elko particles. If these locality issues are not of importance when it comes to matters of Elko detection, then this, combined with a lack of observational evidence for Elko particles may put question marks as to the existence of Elko particles.

7 Conclusion

We started this thesis by introducing the Elko quantum field dark matter candidate and bringing to the reader's attention that Elko spinors enjoy a variety of applications. They have become well established in the literature, solidifying the claim that they form an important branch of physics worth studying. The Elko quantum field operators formed by writing down mode expansions using Elko spinors instead of Dirac spinors have not had theoretical origins as clear as the spinors themselves. The Elko theory was clear at the classical spinor level but the Elko Quantum Field Theory at the state space level was more obscure. We wanted to explore Elko at the level of quantum fields on the state space, broadly to:

1. Try to understand the theoretical setting at the level of the Hilbert space of physical states that could give rise to the Elko quantum field dark matter candidate.
2. Explore the Elko Quantum Field Theory to deepen our understanding of Elko fields and their possible interactions with things we can observe.

We made a number of attempts at deriving the Elko quantum field operator on the state space in a well-defined way, encompassing both the standard and non-standard Wigner classes. We reached the general conclusion that Elko fields are not quantum fields in the sense of Weinberg, a result first announced at the Heidelberg International Dark Matter Conference in January 2009. We published a conference proceedings paper soon afterwards [47]. The result of not finding a well defined Elko quantum field operator on the state space led us to consider the possibility that perhaps Elko's theoretical origins might be found among the remaining distinct representations of the state space - the non-standard Wigner classes. No Elko field turned up there either. This, in turn, led us to consider Elko's involvement in the part of the Standard Model that deals with broken symmetries, namely the Electroweak Theory. We considered what might be the possible effects of the mass dimensionality of the Standard Model doublets being changed from three halves to one for the duration of an electroweak interaction.

In Chapter 1 we showed that Elko research is an established and important area in theoretical physics to make it clear that Elko is worth studying. We proceeded in the same chapter to outline how we would add our own significant contribution to the Elko landscape, by deepening and extending our understanding of Elko at the quantum field theoretic level. We then introduced the fundamentals of Quantum Field Theory in Chapter 2 and highlighted Wein-

7 Conclusion

berg's way of setting up the quantum field formalism which has a distinct advantage in the sense that it makes explicit the interplay between the infinite-dimensional representation of the Poincaré group on the state space and the finite-dimensional Lorentz group on the space of spinors. We wrote the review with a chain of concepts following the logical development as set forth by Weinberg.

In Chapter 3 we explored possible Elko field interactions with Standard Model gauge fields. We gave the conditions necessary for Elko fields to admit gauge interactions, and pointed out important areas where Elko Field Theory is incomplete. We proposed what we believe to be the most natural ways of completing the Elko Field Theory and arrived at results that are in sharp contrast to those in the literature concerning the darkness of Elko fields with respect to the Standard Model.

In Chapter 4 we turned our attention from the question: *What can we do with Elko quantum fields?* to the question: *How can the Elko quantum field be made to emerge in some clear well-defined way on the state space?* We started this task by asking the question: *Is the Elko field a quantum field in the sense of Weinberg?* If the answer to this question had have been *yes*, we would have automatically answered the main question because Weinberg quantum fields arise under well-defined and well known conditions. The answer however, turned out to be *no*. The failing of Elko fields to be Weinberg quantum fields comes about because since the elements of the Elko spinors are used as coefficient functions for a quantum field operator instead of the usual Dirac coefficient functions, the Hamiltonian density built from these field operators does not transform under rotations as a Lorentz scalar. This led to the idea (Ahluwalia et al.) that perhaps Elko might be restricted to a plane, which itself was somehow free to move around in ordinary four-dimensional Minkowski spacetime. This also seemed consistent with the observation that Elko has an axis of locality. Only requiring Elko to transform properly under rotations in just one plane initially looked as though Elko spinors could be incorporated in a well-defined way into quantum fields but this also had a serious problem. Without the demand that Elko fields must transform in a certain well-defined way under all rotations, it is only possible to derive a formula which relates $\xi((0, p_y, 0), \sigma)$ and $\zeta((0, p_y, 0), \sigma)$ to the general spinors $\xi((p_x, p_y, p_z), \sigma)$ and $\zeta((p_x, p_y, p_z), \sigma)$. Without additional criteria for Elko fields to satisfy, there was no concrete way to transform the spinors $\xi(\mathbf{0}, \sigma)$ and $\zeta(\mathbf{0}, \sigma)$ to $\xi((0, p_y, 0), \sigma)$ and $\zeta((0, p_y, 0), \sigma)$. After discovering this problem, we decided to examine the issue of Elko quantum field causality more closely. The anti-commutator between the Elko field operator $\Lambda_\ell(x)$ evaluated at spacetime coordinate x and its adjoint field operator $\Lambda_\ell^\dagger(y)$ evaluated at spacelike separated spacetime point y , in general fails to vanish. This is so, even if we choose an axis that is aligned with the axis of locality, so even along the axis of locality there is still some sort of non-locality or non-causality present.

Next we looked at a quantum field based on spinors that have the same form as Elko spinors when $\mathbf{p} = 0$ [50]. They transform under the VSR group. We showed that at the level of the state space, the unitary irreducible representations of the VSR group are one-dimensional,

with the little group being $SO(2)$, so the states labeled by this symmetry group, do not have two-valued discrete indices so it is not clear how this quantum field can emerge from the state space either. This is also a very significant result as this quantum field was written down as a reaction to our result that Elko fields are not quantum fields in the sense of Weinberg, and that the symmetries of the Lorentz group are not respected. We took this observation as a reason why the search for a way to derive the Elko cousin should be extended to include non-commutative spacetime. We proposed (see Sec. (4.5) based on discussions with Butler et al.) that a good first place to look might be the symmetry group associated with the Stabilized Poincaré Heisenberg algebra. In our view, developing a Quantum Field Theory based on this algebra would be a worthwhile pursuit in its own right, independent of Elko-type quantum field searches, and holds the potential to possibly produce a quantum theory that includes gravity.

The most natural attempts to derive the Elko quantum field in the context of the standard representation of the Poincaré group on the state space failed to produce Elko fields. This, combined with the fact that at the level of classical spinors, Elko transforms under the discrete symmetries like the non-standard Wigner classes, led us to extend our search to the non-standard Wigner classes.

In [33, p.1387] Lee and Wick said of the non-standard Wigner classes that:

“...with the added assumption of the local field theory, Wigner’s cases 2,3 and 4 either do not occur or can be reduced to his case 1 by using different operators in the sets $\{SP\}$ and $\{ST\}$.”

By $\{SP\}$ and $\{ST\}$, Lee and Wick mean the sets of operators obtained by combining the space and time inversion operators with internal symmetry operators.

It was not clear to us whether the CPT theorem should be valid for the non-standard Wigner classes since Weinberg only proves the CPT Theorem in the context of the standard Wigner class.

Furthermore, as hinted at in [54], it is not even clear if the CPT symmetry is 100% universally valid in the standard Wigner class in the first place. The other issue motivating the work of Chapter 5 was that it was not entirely clear to what extent the work by Lee and Wick harmonized with the formalism as explained by Weinberg. As mentioned in Chapter 5, Weinberg states in [42, p.104]:

“No examples are known of particles that furnish unconventional representations of inversions, so these possibilities will not be pursued further here.”

We believe that the study of the non-standard Wigner classes, and the search for non-standard quantum fields, is a worthwhile endeavor to embark on, even independently of searching for Elko fields. It is true that exploring new symmetry groups and fundamentally new theories with very different mathematical properties, being guided by experiment is a worthwhile and important thing to do. However, we *also* believe that it is important to fully explore the allowed areas within the mathematical boundaries in already existing theories, as there could

7 Conclusion

be important physics hidden away in these unstudied gaps allowed by the mathematics. Even if such studies yield no new physics in the end, such avenues need to be explored and such conclusions need to be shown rather than assumed, so that we *know* that it is worthwhile moving on and searching for the answers to various physics questions elsewhere.

We have looked for non-standard quantum fields and in Cases 2 and 4 verified the correctness and applicability of the first part of the claim made by Lee and Wick above, at least in the case of massive spin-1/2 fields. For Case 3, we found a non-standard massive spin-1/2 quantum field which is a quantum field in the sense of Weinberg. That is, we found a quantum field that is local, and transforms correctly under the Poincaré group in the sense that the Hamiltonian density built out of such field operators will transform under the full Poincaré group as a Lorentz scalar. This Case 3 quantum field is new. It transforms under a different finite-dimensional representation of the Lorentz group. Even if the non-standard state space is allowed to be transformed to look like two copies of the standard Wigner class, the distinctiveness of the non-standard field to the Dirac field is still preserved.

We then moved on to show that it is possible for the new quantum field to be viewed as a Majorana-type dark matter candidate when assuming that the new quantum field carries no conserved quantum numbers. When conserved quantum numbers are allowed, we brought out some odd properties of the new quantum field, which led us to suggest identifying neutrinos with the non-standard quantum field and also suggest that the right-handed components of what we called neutrinos and left-handed components of what we called antineutrinos may be dark matter candidates. We made this suggestion on the ground that there is no kinematical term for them in the Lagrangian that naturally couples these fields to Standard Model gauge fields by a covariant derivative. If our identification of the non-standard quantum field with neutrinos is correct, then we suggest that right-handed neutrinos and left-handed antineutrinos be prime candidates for dark matter. Furthermore, the strange situation of having kinematical terms in the Lagrangian without corresponding mass terms and having mass terms in the Lagrangian without corresponding kinematical terms leads us to suggest that maybe the mass of a neutrino is not evenly distributed between left and right-handed components. The dark matter components of neutrinos may carry far more mass than the neutrinos that we see that would correspond to kinematical terms in the Lagrangian with no accompanying explicit mass terms. Perhaps this may account for the apparent lightness of observed neutrinos, in contrast to the other much heavier massive particles in the Standard Model.

Turning our attention back now to the question of whether Elko fields are non-standard quantum fields, our search in the non-standard Wigner classes did not turn up Elko fields there either. As we discussed in Sec. (5.9), Elko Field Theory is an incomplete theory. In this thesis we have taken what we believe to be the most natural way of plugging the main holes in this theory. With our way of plugging these holes, Elko does not turn up as a well-defined quantum field in the sense of Weinberg, among any of the Wigner classes, standard

or non-standard.

The issue of the incompleteness of Elko Field Theory has far reaching consequences beyond whether Elko fields can be derived in the non-standard Wigner classes. The incompleteness of the theory regarding the Elko dual operation on operators in Hilbert space greatly obscures whether Elko fields can admit gauge interactions. If Eqn. (3.56) holds, then there is no reason why Elko fields cannot interact with Standard Model gauge quanta, and therefore participate in quantum electrodynamics, quantum chromodynamics and electroweak interactions. The *nature* of the incompleteness of Elko Field Theory therefore constitutes a potentially significant criticism of the theory. The ability for Elko particles, interpreted in what we consider to be the most natural way, given the holes in the theory, to admit such a wide array of interactions with Standard Model particles reduces their suitability as dark matter candidates, but does not rule Elko particles out as dark matter candidates for several reasons. Firstly, since Elko fields have a single axis of locality (at most), the nature of the non-locality may be such that they could be spread out over vast distances, creating all sorts of poorly understood complications when looking for Elko particles in a small area, such as the LHC for example. Secondly, the lack of observations of Elko interacting with the Standard Model gauge quanta could be interpreted as evidence that the holes in Elko Field Theory should be filled in less obvious ways. Specifically, one might decide that we do not want Eqn. (3.56) to hold, not for any theoretical reasons, but simply because of the lack of Elko particle observations in nature.

Darkness aside, and more fundamentally, the suitability of the Elko field as a quantum field is brought into question by a series of observations made in this thesis. Namely, the non-locality/acausality of Elko fields. It may be argued that since Elko is not local/causal, we would not expect it to be a quantum field in the sense of Weinberg, since any Weinberg quantum field is by construction, local (and causal). If we take this view, we are then left with the question of what alternative physically reasonable and mathematically rigorous ways are there to write down non-local new quantum fields? We might take the view that one way to set up alternative quantum fields is to study spinors and then just write down an associated quantum field operator, like was done with Elko fields. However, as we have seen, figuring out all of the mathematical details surrounding spinors in spinor space, including their transformation properties under well-defined symmetry groups, does *not* tell us how we should set up a Quantum Field Theory based around such spinors. If we write down a quantum field based on spinors, as we have seen with Elko fields, the theory is not complete. If this method of coming up with new quantum field operators is going to be useful and physically meaningful, more mathematical axioms and physical assumptions will need to be added in order to have a physical theory that is internally complete.

In Chapter 6 we examined the idea of linking Elko fields in with the Standard Model by considering the concept of mass dimensionality transmutation, together with our observation that the left and right-handed components of Dirac spinors can be related to the left-handed

7 Conclusion

components of Elko spinors. This latter observation led to the suspicion that perhaps the first concept might be involved in electroweak interactions since only the left-handed components of Dirac fields have non-zero weak isospin. The most basic natural consequences of our first and simplest approach to studying mass dimension transmutation seem to harmonize nicely with some of the basic things we know about electroweak interactions. We believe this to be encouraging. It provides justification for pursuing research on this in the future in more detail. We finished off the chapter by once again putting the question of deriving Elko fields on hold, and contemplating the feasibility of having Elko doublets carrying non-zero weak isospin and having electroweak interactions with Elko doublets. We believe that in principle symmetry currents can be written down with such Elko doublets, and that the symmetry currents can be coupled to existing Standard Model electroweak symmetry currents to form a new interaction term, producing electroweak interactions between Elko fields and Dirac fields. We believe that in this sense, Elko is not dark, however, Elko may still *appear* to be dark, on account of the possible difficulty of detecting it, given its non-locality properties. The electroweak Elko symmetry currents should also be able to couple to themselves producing electroweak interactions solely involving Elko fermions without the inclusion of any Dirac fermions. The only Standard Model component to these sorts of interactions would be the gauge bosons mediating the interactions.

We finish Chapter 7 by remarking on something we said near the end of Chapter 1:

“...we believe that in any physical theory, all of the allowed “wriggle room” in the mathematics should be fully explored so that the full extent of the physics may be elucidated and understood. Developing new theories has an important place in physics but we believe it to be *also* important to fully explore the existing physical theories too.”

In this thesis we have seen the importance of this philosophy in the case of Elko Field Theory and the non-standard Wigner classes. By exploring the freedom in the mathematics, we observed significant previously unknown possibilities such as the ability of Elko fields to interact with Standard Model particles other than the Higgs boson and the possible existence of a non-standard massive spin-1/2 quantum field, just to name a few examples.

Bibliography

- [1] D. Griffiths. *Introduction to Elementary Particles*. Vch Verlagsgesellschaft MbH, 1987.
- [2] P.A.M. Dirac. The Quantum Theory of the Emission and Absorption of Radiation. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 114(767):243–265, 1927.
- [3] J.C. Slater. The Development of Quantum Mechanics in the Period 1924-1926. *Wave Mechanics: The First Fifty Years*, page 19, 1973.
- [4] P.A.M. Dirac. The Quantum Theory of The Electron. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 117(778):610–624, 1928.
- [5] H. Goldstein, C. Poole, and J. Safko. *Classical Mechanics, Third Edition*. Addison Wesley, 2002.
- [6] S.M. Carroll. *Spacetime and Geometry. An Introduction to General Relativity*. Addison Wesley, 2004.
- [7] L.H. Ryder. *Quantum Field Theory*. Cambridge University Press, 1996.
- [8] M.A. Srednicki. *Quantum Field Theory*. Cambridge University Press, 2007.
- [9] S. Weinberg. Feynman Rules for any Spin. *Physical Review*, 133:B1318, 1964.
- [10] S. Weinberg. Feynman Rules for any Spin. II. Massless Particles. *Physical Review*, 134(4B):B882–896, 1964.
- [11] E. Wigner. On Unitary Representations of the Inhomogeneous Lorentz group. *Annals of Mathematics*, 40(1):149–204, 1939.
- [12] F. Zwicky. Die Rotverschiebung Von Extragalaktischen Nebeln. *Helvetica Physica Acta*, 6:110–127, 1933.
- [13] K. Freeman and G. McNamara. *In Search of Dark Matter*. Springer Verlag, 2006.
- [14] R. Clausius. XVI. On a Mechanical Theorem Applicable to Heat. *Philosophical Magazine Series 4*, 40(265):122–127, 1870.

Bibliography

- [15] D. Bacon, A. Refregier, and R. Ellis. Detection of Weak Gravitational Lensing by Large-Scale Structure. *Arxiv preprint astro-ph/0003008*, 2000.
- [16] A.A. Penzias and R.W. Wilson. A Measurement of Excess Antenna Temperature at 4080 Mc/s. *The Astrophysical Journal*, 142(1):419–421, 1965.
- [17] G. Hinshaw, J.L. Weiland, R.S. Hill, N. Odegard, D. Larson, C.L. Bennett, J. Dunkley, B. Gold, M.R. Greason, N. Jarosik, et al. Five-Year Wilkinson Microwave Anisotropy Probe Observations: Data Processing, Sky Maps, and Basic Results. *The Astrophysical Journal Supplement Series*, 180:225, 2009.
- [18] R.D. Peccei and H.R. Quinn. Constraints Imposed by CP Conservation in the Presence of Pseudoparticles. *Physical Review D*, 16(6):1791–1797, 1977.
- [19] S. Weinberg. A New Light Boson? *Physical Review Letters*, 40(4):223–226, 1978.
- [20] I.J.R. Aitchison and A.J.G. Hey. *Gauge Theories in Particle Physics, Volume 2, Third Edition*. Institute of Physics Publishing, 2004.
- [21] S. Weinberg. *The Quantum Theory of Fields, Vol. 2*. Cambridge University Press, Cambridge, New York, 1996.
- [22] L. Maiani, R. Petronzio, and E. Zavattini. Effects of Nearly Massless, Spin-Zero Particles on Light Propagation in a Magnetic Field. *Physics Letters B*, 175(3):359–363, 1986.
- [23] E. Zavattini, G. Zavattini, G. Ruoso, E. Polacco, E. Milotti, M. Karuza, U. Gastaldi, G. Di Domenico, F. Della Valle, R. Cimino, et al. Experimental Observation of Optical Rotation Generated in Vacuum by a Magnetic Field. *Physical Review Letters*, 96(11):110406, 2006.
- [24] Y. Fukuda, T. Hayakawa, E. Ichihara, K. Inoue, K. Ishihara, H. Ishino, Y. Itow, T. Kajita, J. Kameda, S. Kasuga, et al. Measurements of the Solar Neutrino Flux From Super-Kamiokande’s First 300 Days. *Physical Review Letters*, 81(6):1158–1162, 1998.
- [25] D.S. Akerib, J. Alvaro-Dean, M.S. Armel, M.J. Attisha, L. Baudis, D.A. Bauer, A.I. Bolozdynya, P.L. Brink, R. Bunker, B. Cabrera, et al. Installation and Commissioning of the CDMSII Experiment at Soudan. *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, 520(1-3):116–119, 2004.
- [26] R. Foot. Relevance of the CDMS II Events for Mirror Dark Matter. *Physical Review D*, 81(8):87302, 2010.
- [27] Y. Mambrini, Y. Cui, T. Gherghetta, J. Stokes, G. Paz, T.H. Wang, G.L. Wang, M. Kurachi, T. Onogi, Y. Takubo, et al. The Kinetic Dark-Mixing in the Light of CoGENT and XENON100. *Arxiv preprint arXiv:1006.3318*, 2010.

- [28] S. Archambault, F. Aubin, M. Auger, E. Behnke, B. Beltran, K. Clark, X. Dai, A. Davour, J. Farine, R. Faust, et al. Dark Matter Spin-Dependent Limits for WIMP Interactions on ^{19}F by PICASSO. *Physics Letters B*, 682(2):185–192, 2009.
- [29] M. Milgrom. A Modification of the Newtonian Dynamics as a Possible Alternative to the Hidden Mass Hypothesis. *The Astrophysical Journal*, 270, 1983.
- [30] De Felice A. Duvvuri V. Easson D.A. Trodden M. Carroll, S.M. and M.S. Turner. Cosmology of Generalised Modified Gravity Models. *Physical Review D*, 71(6), 2005.
- [31] K.A. Bronnikov and V.N. Melnikov. On Observational Predictions From Multidimensional Gravity. *General Relativity and Gravitation*, 33(9):1549–1578, 2001.
- [32] D.V. Ahluwalia and D. Grumiller. Spin-Half Fermions with Mass Dimension One: Theory, Phenomenology, and Dark Matter. *Journal of Cosmology and Astroparticle Physics*, 2005:012, 2005.
- [33] T.D. Lee and G.C. Wick. Space Inversion, Time Reversal, and other Discrete Symmetries in Local Field Theories. *Physical Review*, 148(4):1385–1404, 1966.
- [34] D.V. Ahluwalia, C.Y. Lee, D. Schrittt, and T.F. Watson. Local Fermionic Dark Matter with Mass Dimension One. *Arxiv preprint arXiv:0804.1854v4*, 2008.
- [35] R. da Rocha and W.A. Rodrigues. Where are Elko Spinor Fields in Lounesto Spinor Field Classification? *Modern Physics Letters A*, 21:65–74, 2006.
- [36] P. Lounesto. *Clifford Algebras and Spinors*. Cambridge University Press, 2001.
- [37] C.G. Boehmer. The Einstein-Cartan-Elko System. *Annalen der Physik*, 16(1):38–44, 2007.
- [38] C.G. Boehmer. The Einstein-Elko System—Can Dark Matter Drive Inflation? *Annalen Der Physik*, 16(5-6):325–341, 2007.
- [39] R. Da Rocha and J.M.H. Da Silva. From Dirac Spinor Fields to Eigenspinoren Des Ladungskonjugationsoperators. *Journal of Mathematical Physics*, 48:123517, 2007.
- [40] R. Da Rocha and J.M.H. Da Silva. Elko Spinor Fields: Lagrangians for Gravity Derived From Supergravity. *International Journal of Geometric Methods in Modern Physics*, 6(3):461–477, 2009.
- [41] A. Ashtekar. New Variables for Classical and Quantum Gravity. *Physical Review Letters*, 57(18):2244–2247, 1986.
- [42] S. Weinberg. *The Quantum Theory of Fields, Vol. 1*. Cambridge University Press, Cambridge, New York, 1996.

Bibliography

- [43] M. Dias, F. de Campos, and J.M. da Silva. Exploring Light Elkos Signal at Accelerators. *Arxiv preprint arXiv:1012.4642*, 2011.
- [44] S. Shankaranarayanan. What if the Inflation is a Spinor Condensate? *International Journal of Modern Physics D*, 18(14):2173–2179, 2009.
- [45] L. Fabbri. Causal Propagation for Elko Fields. *Modern Physics Letters A*, 25(15):1295, 2010.
- [46] H. Wei. A Couple of Issues Concerning Spinor Dark Energy. *Arxiv preprint arXiv:1002.4230*, 2010.
- [47] A.B. Gillard and B.M.S. Martin. Proceedings of the 7th International Heidelberg Conference, Dark Matter in Astrophysics and Particle Physics, Dark 2009 *Quantum Fields, Dark Matter and Non-Standard Wigner Classes*. pages 451–456. World Scientific, 2010.
- [48] A. Gillard and B. Martin. Dark Matter, Elko fields and Weinberg’s Quantum Field Theory Formalism. *Reports on Mathematical Physics*, 69(1):113–129, 2012.
- [49] D.V. Ahluwalia, C.Y. Lee, and D. Schrittt. Elko as Self-Interacting Fermionic Dark Matter with Axis of Locality. *Physics Letters B*, 2010.
- [50] D.V. Ahluwalia and S.P. Horvath. Very Special Relativity as Relativity of Dark Matter: the Elko Connection. *Journal of High Energy Physics*, 2010(11):1–19, 2010.
- [51] Lee. C.Y. Ahluwalia, D. and D. Schrittt. Self-interacting Dark Matter with an Axis of Locality. *Physical Revue D*, 83(065017), 2011.
- [52] E.P. Wigner. Unitary Representations of the Inhomogeneous Lorentz Group Including Reflections. *Group Theoretical Concepts and Methods in Elementary Particle Physics: Lectures*, page 37, 1964.
- [53] O.W. Greenberg. CPT Violation Implies Violation of Lorentz Invariance. *Physical Review Letters*, 89(23):231602, 2002.
- [54] A. Kostelecky and N. Russell. Data Tables for Lorentz and CPT Violation. In *Proceedings of the Fourth Meeting on CPT and Lorentz Symmetry, Bloomington, USA, 8-11 August 2007*, page 308. World Scientific Pub Co Inc, 2008.
- [55] A. Gillard. Elko Field Theory and the Nature of Elko Darkness. *Arxiv preprint arXiv:1109.4278*, 2011.
- [56] S. Weinberg. Feynman Rules for any Spin. III. *Physical Review*, 181(5):1893–1899, 1969.
- [57] E.P. Wigner and F.V.S. Gruppentheorie. Braunschweig, 1931; C. Eckart. *Rev. Mod. Phys*, 2:305, 1930.

- [58] J.J. Sakurai. *Modern Quantum Mechanics, Revised Edition*. Addison-Wesley, 1994.
- [59] I.J.R. Aitchison and A.J.G. Hey. *Gauge Theories in Particle Physics, Volume 1, Third Edition*. Institute of Physics Publishing, 2004.
- [60] P.H. Butler, N.G. Gresnigt, and P.F. Renaud. Assumptions and Axioms: Mathematical Structures to Describe the Physics of Rigid Bodies. *Arxiv preprint arXiv:1005.0669*, 2010.
- [61] N.G. Gresnigt, P.F. Renaud, and P.H. Butler. The Stablized Poincaré-Heisenberg Algebra: A Clifford Algebra Viewpoint. *International Journal of Modern Physics D*, 16(9):1519–1529, 2007.
- [62] T.D. Lee and C.N. Yang. Question of Parity Conservation in Weak Interactions. *Physical Review*, 104(1):254, 1956.
- [63] C.S. Wu, E. Ambler, R.W. Hayward, D.D. Hoppes, and R.P. Hudson. Experimental Test of Parity Conservation in Beta Decay. *Physical Review*, 105(4):1413, 1957.
- [64] G.S. Guralnik, C.R. Hagen, and T.W.B. Kibble. Global Conservation Laws and Massless Particles. *Physical Review Letters*, 13(20):585–587, 1964.
- [65] P.W. Higgs. Broken Symmetries and the Masses of Gauge Bosons. *Physical Review Letters*, 13(16):508–509, 1964.
- [66] F. Englert and R. Brout. Broken Symmetry and the Mass of Gauge Vector Mesons. *Physical Review Letters*, 13(9):321–323, 1964.
- [67] I.J.R. Aitchison and A.J.G. Hey. *Gauge Theories in Particle Physics, Second Edition*. Institute of Physics Publishing, 1989.
- [68] G. Hooft. Renormalizable Lagrangians for Massive Yang-Mills Fields. *Nuclear Physics B*, 35(1):167–188, 1971.
- [69] L.H. Ryder. *Quantum Field Theory*. Cambridge University Press, New York, NY, 1985.
- [70] C. Amsler, M. Doser, M. Antonelli, D.M. Asner, K.S. Babu, H. Baer, H.R. Band, R.M. Barnett, E. Bergren, J. Beringer, et al. Review of Particle Physics. *Physics Letters B*, 667(1-5):1–6, 2008.
- [71] Z. Maki, M. Nakagawa, and S. Sakata. Remarks on the Unified Model of Elementary Particles. *Progress of Theoretical Physics*, 28(5):870–880, 1962.
- [72] M.E. Peskin and D.V. Schroeder. *An Introduction To Quantum Field Theory*. Westview Pr, 1995.

Bibliography

- [73] M. Blagojević. *Gravitation and Gauge Symmetries*. Taylor & Francis, 2002.
- [74] B.G. Wybourne. *Classical Groups for Physicists*. Wiley New York, 1974.
- [75] J.B. Zuber and C. Itzykson. *Quantum Field Theory*. McGraw-Hill International Book Co., 1980.
- [76] R.G. Newton. *Scattering Theory of Waves and Particles*. Dover Pubns, 2002.
- [77] W.K. Tung. *Group Theory in Physics*. World Scientific Pub. Co. Inc., 1985.
- [78] M. Tinkham. *Group Theory and Quantum Mechanics. International Series in Pure and Applied Physics*. PhD thesis, McGraw-Hill Book Company, 1964.
- [79] C. Chryssomalakos and E. Okon. Generalized Quantum Relativistic Kinematics: a Stability Point of View. *International Journal of Modern Physics D*, 13(10):2003–2034, 2004.
- [80] U.A. Arnison, A. Astbury, B. Aubert, C. Bacci, G. Bauer, A. Bezaguet, R. Böck, T.J.V. Bowcock, M. Calvetti, P. Catz, et al. Experimental Observation of Lepton Pairs of Invariant Mass Around 95 Gev/c² at the CERN SPS Collider. *Physics Letters B*, 126(5):398–410, 1983.