universe

IMPACT FACTOR 2.5

CITESCORE 4.3

Article

# On the Propagation of Gravitational Waves in the Weyl Invariant Theory of Gravity

Mauro Duarte, Fabio Dahia and Carlos Romero

MDPI

https://doi.org/10.3390/universe10090361 universe

MDPI

*Article*

# On the Propagation of Gravitational Waves in the Weyl Invariant Theory of Gravity

**Mauro Duarte** , **Fabio Dahia and Carlos Romero** *

Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, João Pessoa 58059-970, PB, Brazil; mauro.parnaiba@estudantes.ufpb.br (M.D.); fdahia@fisica.ufpb.br (F.D.)
* Correspondence: cromero@fisica.ufpb.br

**Abstract:** We revisit Weyl's unified field theory, which arose in 1918, shortly after general relativity was discovered. As is well known, in order to extend the program of the geometrization of physics started by Einstein to include the electromagnetic field, H. Weyl developed a new geometry which constitutes a kind of generalization of Riemannian geometry. In this paper, our aim is to discuss Weyl's proposal anew and examine its consistency and completeness as a physical theory. We propose new directions and possible conceptual changes in the original work. Among these, we investigate with some detail the propagation of gravitational waves, and the new features arising in this recent modified gravity theory, in which the presence of a massive vector field appears somewhat unexpectedly. We also speculate whether the results could be examined in the context of primordial gravitational waves.

**Keywords:** Weyl unified theory; gravitational waves; Proca field

**PACS:** 04.20.Jb; 11.10.kk; 98.80.Cq

check for updates

**Citation:** Duarte, M.; Dahia, F.; Romero, C. On the Propagation of Gravitational Waves in the Weyl Invariant Theory of Gravity. *Universe* **2024**, *10*, 361. https://doi.org/10.3390/universe10090361

Academic Editor: Aharon Davidson, Diego Rubiera-Garcia

Received: 22 July 2024
Revised: 30 August 2024
Accepted: 6 September 2024
Published: 9 September 2024



## 1. Introduction

In recent years, there has been a renewed interest in Proca's early theory, which originally appeared in the context of the classical and quantum electrodynamics of a massive photon, a theory proposed to describe the weak interaction and the motion of spin-1 mesons [1,2]. Despite its interesting and original ideas, the model did not survive too long and was subsequently almost forgotten. However, it seems that new motivation has appeared to reconsider the role that the Proca field can play in physical theory. Nowadays, mention of the possible existence of the Proca field mainly appears coming from two distinct contexts. Firstly, the idea of the presence of a massive vector field in the universe has been motivated by current research in astrophysics and cosmology, namely, the dark matter problem [3–6]. Indeed, it has been argued that the massive vector field considered earlier by nuclear physicists can play a role in modeling what is called dark matter. The second motivation comes from the following fact: in standard gravitation theory, i.e., general relativity, the Proca field does not appear in a natural way, and has to be put in by hand as a matter field in much the same way as we do in the case of other physical (i.e, non-geometrical) fields. However, a recent proposed theory of gravity, deeply inspired by Weyl's original unified field theory[1] [7], seems to suggest the appearance of a massive vector field, which has an entirely geometrical nature [8]. Here, we are referring to the so-called Weyl's gauge invariant theory of gravity [9].

Let us recall that, in his attempt to unify gravity with electromagnetism, H. Weyl developed a new geometry, which constitutes one of the simplest generalizations of Riemannian geometry [8]. Recently, Weyl's unified field theory has been significantly reframed into a modified theory of gravity in order to allow matter to couple with the space–time geometry in a gauge-invariant way [9]. This is achieved by strictly following a prescription of "minimum coupling", which complies with the principle of gauge invariance postulated by Weyl[2]. An interesting outcome of this procedure is an unexpected appearance of a vector field in the gravitation sector of the action. As it happens, for some choices of both the values of the cosmological constant $\Lambda$ and $\omega$ (a free parameter of the theory) this vector field may be formally interpreted as a massive vector field satisfying an equation that is identical to the Proca equation. Moreover, unlike the fact that the original Proca field is defined in Minkowski spacetime as not gauge-invariant, a feature that is characteristic of massive fields, here we should emphasize that the gauge invariance of the geometrical vector field is granted by first principles. We shall return to this point later.

Let us remark that the investigation of massive vector fields within the framework of general relativity has been carried out by several authors [10,11]. A particularly interesting development of this line of research has recently shown the possibility of cosmic inflation being driven by a vector field [12].

If we consider a modified gravity theory in which the presence of a massive vector field is necessary, then a number of important questions naturally arise. A particular and relevant question is the following: *how do gravitational waves appear in such a theory?* In the present paper, we approach this problem via linearization of the field equations. As we shall see, in this preliminary step the propagation of the gravitational waves (the metric field waves) appears together (though not simultaneously in time) with the propagation of the Proca field.

The present article is organized as follows. In Section 2, we give a short introduction to the Weyl invariant theory. We recast the field equations of the theory in a form in which the identification of the Weyl vector field with the Proca field becomes apparent. This is simply achieved by absorbing into the energy–momentum tensor corresponding to the electromagnetic field (the latter coming from Weyl's original unified theory) the mass term that naturally appears in the field equations. This procedure leads to the canonical energy–momentum of the Proca field, such as is usually found in any classical field theory textbook [13]. We proceed to Section 3, where we investigate the prediction of gravitational waves in the theory. We then arrive at the interesting result that gravitational waves in this modified theory consist, in fact, of the non-simultaneous propagation of both the metric field $g_{\mu\nu}$ and the vector field $\sigma_\alpha$. In Section 4, we speculate whether the results obtained in the previous section could be examined in the context of primordial gravitational waves, and we hope these considerations may stimulate future research. We conclude with some remarks in Section 5.

## 2. The Weyl Invariant Theory

This theory arose from the attempt to complete Weyl's (unfinished and incomplete) original work [14,15]. To carry out this task we needed (a) to provide a gauge-invariant procedure of how to couple matter with the spacetime geometry; (b) to define proper time in an invariant manner; and (c) to avoid the problem of the so-called "second clock effect", which plagued the theory since its inception. We then realized that one way to sort out these problematic features of the original theory would be to define right from the beginning a gauge-invariant metric tensor. This may be achieved in the following way. We first go to the so-called Weyl *natural gauge* defined by $R = \Lambda \neq 0$. In this gauge, the field equations become rather simplified, while the action becomes linear in $R$ (see [9]). Now, among all elements of the Weyl conformal structure we pick up a pair $(g, \xi)$, where $g$ and $\xi$ represent, respectively, a metric tensor and a 2-$form$ field (the Weyl field) in the natural gauge. With these two elements, we define another metric $\gamma$, given by $\gamma = \frac{R}{\Lambda}g$. We also define the 2-$form$ field $\sigma = \xi + d\ln R$, where $R = g^{\mu\nu}R_{\mu\nu}$ denotes the scalar curvature (recalling that $R_{\mu\nu}$ is defined with respect to the Weyl connection and not with respect to the Christoffel symbols). We now identify $\Lambda$ with the cosmological constant, as did Weyl when he defined his natural gauge. Of course, the numerical value of $\Lambda$ is left to be determined by observation. (Incidentally, Weyl considered that the way of introducing the cosmological constant in his theory was purely geometrical and natural, instead of putting it in the field equations in an ad hoc manner.) We would like stress the fact that $\gamma$ and $\sigma$, defined as above, are invariant under Weyl (gauge) transformations, and therefore are two representatives of the geometry of the whole conformal structure, not dependent on any particular gauge. It is also important to note that the gauge transformations do not affect $\Lambda$, whose value once chosen becomes fixed. Thus, the mass of the Weyl field does not depend on the gauge. In this way, we are able to define a gauge-invariant action leading to gauge-invariant field equations (see [8] for details).

### *2.1. The Field Equations*

Let us start by recalling that the field equations in Weyl's invariant theory are given by [9]

$$\frac{1}{\sqrt{-g}}\partial_\beta\left(\sqrt{-g}F^{\alpha\beta}\right) = \frac{3\Lambda}{2\omega}\sigma^\alpha, \tag{1}$$

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \frac{\Lambda}{4}g_{\alpha\beta} + \frac{3}{2}\left(\sigma_\alpha\sigma_\beta - \frac{1}{2}g_{\alpha\beta}\sigma^\mu\sigma_\mu\right) = \frac{\omega}{\Lambda}T_{\alpha\beta} - \kappa T^{(m)}_{\alpha\beta}, \tag{2}$$

where $R_{\mu\nu}$ and $R$ denote, respectively, the Ricci tensor and the scalar curvature defined with respect to the Riemannian connection, $\sigma$ is a 1-form field, $T_{\alpha\beta} = F_{\alpha\mu}F^{\mu}_{\ \beta} + \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}$ and $T^{(m)}_{\mu\nu}$ represent the energy–momentum tensor of matter, and $\kappa$ is a coupling constant. Let us make a short comment on the role of $\sigma$. In Weyl's original approach, $\sigma$ led naturally to a new notion of curvature, a sort of "length curvature" represented by the 2-form $F = d\sigma$ (*Streckenkrummung*), in addition to the "direction curvature" (*Richtungkrummung*), the latter given by the Riemann tensor [8]. To his amazement, Weyl found that the length curvature $F = d\sigma$ presents striking similarities with the electromagnetic tensor, and it was this discovery, together with the invariance of his modified compatibility condition (between the metric and the affine connection), that led him to the attempt to geometrize the electromagnetic field. It is worth mentioning here that the discovery of this new symmetry, which Weyl called *gauge symmetry*, is now celebrated as one of the most significant facts in the history of modern physics: it represents the birth of modern gauge theories [16].

Let us remark that the above equations may be obtained by varying the action

$$S = \int d^4x\sqrt{-g}[R + \frac{\omega}{2\Lambda}F_{\alpha\beta}F^{\alpha\beta} + \frac{3}{2}\sigma_\alpha\sigma^\alpha - \frac{\Lambda}{2} + \kappa L_m],$$

which is identical to the action of Proca's neutral spin-1 field in curved spacetime with the cosmological constant coupled to gravity [17]. Here, $L_m$ denotes the Lagrangian density of matter with $\kappa$ being a coupling constant. (We recall that here the curvature scalar is calculated with the Riemannian connection.)

By introducing the tensor

$$T^{(P)}_{\alpha\beta} = F_{\alpha\mu}F^{\mu}_{\ \beta} + \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} - \frac{3\Lambda}{2\,\omega}\left(\sigma_\alpha\sigma_\beta - \frac{1}{2}g_{\alpha\beta}\sigma^\mu\sigma_\mu\right), \tag{3}$$

which may be formally considered as the energy–momentum tensor of the Proca field, provided that we define $m = \sqrt{-\frac{3\Lambda}{2\omega}}$ as its mass, it is not difficult to verify that the field equations can be rewritten as

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \frac{\Lambda}{4}g_{\alpha\beta} = \frac{\omega}{\Lambda}T^{(P)}_{\alpha\beta} - \kappa T^{(m)}_{\alpha\beta} \tag{4}$$

$$\frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}F^{\alpha\beta}) = -m^2\sigma^\alpha. \tag{5}$$

By virtue of the analogy with Proca theory, it seems plausible to reinterpret the former Weyl field $\sigma$ not as the electromagnetic field, as Weyl originally did, but as a sort of massive vector field, which enters the theory through a purely geometrical reasoning[3].

*2.2. Energy–Momentum Conservation*

As is well known, in general relativity theory the Bianchi identities imply that the right-hand side of (4) must be divergenceless. On the other hand, with the help of (5) it can easily be shown that $\nabla_\alpha T^{(P)\alpha\beta} = 0$, which in turn leads to $\nabla_\alpha T^{(m)\alpha\beta} = 0$, meaning that the energy–momentum tensor of matter is conserved. In the context of the present theory, this result has an important meaning, namely, that the curves that describe the motion of free-falling particles must be identified with the metrical geodesics, and not the auto-parallels. Thus, the geodesic postulate appears as a consequence of the field equations[4].

## 3. Gravitational Waves

Before their detection, gravitational waves were a general relativistic phenomenon theoretically predicted long ago, at least as early as 1918 by Einstein, who investigated a linearized version of the field equations [18]. A more complete and mathematically rigorous treatment of the subject was given by Choquet-Bruhat in the 1950s based on the fundamental concepts of *global hyperbolicity* attributable to Leray [19]. There is a hyperbolicity property of the Einstein equations, namely the fact that Einstein's equations can be recast in the form

$$\Box g_{\alpha\beta} = N_{\alpha\beta}(g_{\alpha\beta;\partial_\gamma}g_{\alpha\beta}), \tag{6}$$

where $N_{\alpha\beta}$ denote a set of quadratic functions in the derivatives $\partial_\gamma g_{\alpha\beta}$ and $\Box = \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu]$ is the d'Alembertian operator. It should be mentioned that the above equations are supplemented with the so-called *Einstein constraint equations*. The latter amount to giving data on a spacelike hypersurface $\Sigma$, namely a 3-dimensional Riemannian metric $\overline{g}_{ij}$ and a symmetric 2-tensor $K_{ij}$ [20]. The above considerations lead to the following consequence: there must exist gravitational waves propagating in spacetime. Moreover, they also demonstrate that the Einstein field equations can be put in a form where they can be viewed as a well-posed initial problem [21,22]. We now proceed to the investigation of gravitational waves in the Weyl invariant gravity theory. Our treatment is based on the weak field approximation, in which the spacetime metric corresponds to a first-order perturbation of the Minkowski metric in a dimensionless small parameter $\varepsilon$, that is

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \varepsilon h_{\alpha\beta}, \tag{7}$$

where $\eta_{\alpha\beta} = diag(1, -1, -1, -1)$ and $h_{\alpha\beta}$ is a symmetric tensor field. In this approximation, only terms of the first order will be retained. As we shall see, the linearization of the field equations resulting from this approximation will reveal the existence of gravitational waves propagating at the speed of light.

*3.1. The Propagation of Gravitational Waves*

Following the well-known standard procedure, a straightforward calculation gives the linearized expressions for $\Gamma^\alpha{}_{\beta\gamma}$, $R_{\alpha\beta}$ and $R$[5]:

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\varepsilon\left(h^\alpha_{\beta,\gamma} + h^\alpha_{\gamma,\beta} - h_{\beta\gamma,}{}^\alpha\right), \tag{8}$$

$$R_{\alpha\beta} = \frac{1}{2}\varepsilon\left(\Box h_{\alpha\beta} + h_{,\alpha\beta} - h^\gamma{}_{\alpha,\beta\gamma} - h^\gamma{}_{\beta,\alpha\gamma}\right), \tag{9}$$

$$R = \frac{1}{2}\varepsilon\left(\Box h - h^{\gamma\beta}{}_{,\beta\gamma}\right), \tag{10}$$

where in this approximation the indices are raised and lowered with the Minkowski metric $\eta_{\alpha\beta}$ [23]. At this point, let us briefly recall some basic facts of the weak field approximation approach. By coordinating transformations of the type

$$\overline{x}^\alpha = x^\alpha + \varepsilon\,\xi^\alpha(x) \tag{11}$$

it is easy to see that the metric tensor transforms as

$$\overline{g}_{\alpha\beta} = \eta_{\alpha\beta} + \varepsilon\, h_{\alpha\beta} - \varepsilon\, \xi_{\alpha\,,\,\beta} - \varepsilon\, \xi_{\beta_{\,,\,\alpha}}\,, \tag{12}$$

which then leads to the following transformation law for $h_{\alpha\beta}$:

$$\overline{h}_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha\,,\,\beta} - \xi_{\beta_{\,,\,\alpha}}\,, \tag{13}$$

known in the literature as a *gauge transformation*, where the functions $\xi_\alpha$ are arbitrary. An important fact must now be mentioned: the Riemann curvature tensor $R_{\alpha\beta\gamma\eta}$ as well as the Ricci tensor $R_{\alpha\beta}$ (9) and the curvature scalar $R$ (10) are gauge-invariant, i.e., they do not change under (11). We now introduce a new tensor $\psi_{\alpha\beta}$ given by

$$\psi_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h, \tag{14}$$

which is called *the trace-reversed metric tensor*, and is rather useful as it allows the elimination of terms that involve the trace of $h_{\alpha\beta}$ in the expressions of (9) and (10). The Ricci tensor, the scalar of curvature and the Einstein tensor now written in terms of $\psi_{\alpha\beta}$ become, respectively,

$$R_{\alpha\beta} = \frac{1}{2}\varepsilon\left(\Box h_{\alpha\beta} - \psi^{\gamma}{}_{\alpha\,,\,\beta\gamma} - \psi^{\gamma}{}_{\beta\,,\,\alpha\gamma}\right), \tag{15}$$

$$R = \frac{1}{2}\varepsilon\left(\Box h - 2\psi^{\gamma\beta}{}_{,\,\beta\gamma}\right). \tag{16}$$

$$G_{\alpha\beta} = \frac{1}{2}\varepsilon\left(\Box h_{\alpha\beta} + \eta_{\alpha\beta}\psi^{\gamma\delta}{}_{,\gamma\delta} - \psi^{\gamma}{}_{\alpha\,,\,\beta\gamma} - \psi^{\gamma}{}_{\beta\,,\,\alpha\gamma}\right)$$

On the other hand, they can take a simpler form by just choosing a specific gauge, namely, the so-called *Einstein gauge* (also known as the *de Donder gauge*), defined by

$$\partial_\alpha \psi^{\alpha}{}_{\beta} = 0, \tag{17}$$

which is the analog of the Lorenz gauge used in electromagnetism. With the choice of this gauge, Equation (4) becomes

$$\frac{1}{2}\varepsilon\Box\psi_{\alpha\beta} = -\frac{\Lambda}{4}\eta_{\alpha\beta} + \frac{\omega}{\Lambda}T^{(P)}_{\alpha\beta} - \kappa T^{(m)}_{\alpha\beta} \tag{18}$$

(It is clear that the *Einstein gauge* is preserved by the coordinate transformation (11), as long as the condition $\Box\xi_\beta = 0$ holds).

Let us now consider the linearized form of the Weyl field equation. At this point, it would seem quite natural to consider the Proca field also of the same order of approximation assumed for the metric field, that is, $\sigma^\alpha = \varepsilon A^\alpha$, where $A^\alpha$ plays the same role as $h_{\alpha\beta}$. It turns out, however, that in this order of approximation the above field Equations (4) and (5) become uncoupled as $T^{(P)}_{\alpha\beta}$ is of the second order in $\varepsilon$, meaning, in this case, that the metric field and the Proca field would not interact with each other. Therefore, if we want a more realistic situation we must go to a higher order of approximation for $\sigma^\alpha$. Therefore, in what follows we shall take, instead,

$$\sigma^\alpha = \sqrt{\varepsilon}A^\alpha, \tag{19}$$

which then guarantees that the components of the tensor $T^{(P)}_{\alpha\beta}$ are of the first order in $\varepsilon$. We shall also assume that the cosmological constant is a higher-order term in $\varepsilon$ and therefore should be neglected. In this case, the equation for the metric field (18) becomes[6]

$$\Box\psi_{\alpha\beta} = 2\frac{\omega}{\Lambda}\,\theta^{(P)}_{\alpha\beta} - 2\,\kappa T^{(m)}_{\alpha\beta}\,, \tag{20}$$

where $\theta^{(P)}_{\alpha\beta}$ simply denotes the linearized form of $T^{(P)}_{\alpha\beta}$ after taking into account (19).

The above equation describes how perturbations of the metric field $\psi_{\alpha\beta}$ propagate in spacetime. It is important to note the presence of a term representing the energy–momentum tensor of the Weyl field in its linearized form $\theta^{(P)}_{\alpha\beta}$, which is also responsible for the gravitational perturbations. In other words, even in the case where matter is absent there remains the Weyl field acting as a source of the metric field. If the Weyl field is zero and the cosmological constant is taken into account, then the above equation reduces to the well-known linearized Einstein equation in the presence of the cosmological constant, a problem that was considered in [24,25].

### *3.2. The Linearized Equation of the Weyl Field*

It is easy to verify that the Weyl field Equation (5) in its linearized form, i.e., in the weak-field regime, takes the form

$$\left(1-\frac{\varepsilon h}{2}\right)\left\{\partial_\beta\partial^\beta\sigma^\alpha-\partial^\alpha\partial_\beta\sigma^\beta+\partial_\beta\left[\frac{\varepsilon h}{2}\left(\partial^\beta\sigma^\alpha-\partial^\alpha\sigma^\beta\right)\right]\right\}=-m^2\sigma^\alpha.$$

On the other hand, it is not difficult to verify that, in the lower-order approximation in $\varepsilon$, the "Lorenz condition" for the Proca field, i.e., $\partial_\beta\sigma^\beta=0$, is satisfied, which then leads to the following:

$$\Box\, A^\alpha+m^2A^\alpha=0, \tag{21}$$

where $\Box=\eta^{\mu\beta}\partial_\mu\partial_\beta=\frac{\partial^2}{\partial t^2}-\nabla^2$ again denotes the d'Alembertian operator in Minkowski spacetime[7]. This equation reveals that in the weak-field regime, the Weyl field behaves as a wave in massive (or Proca) electrodynamics, and, ipso facto, it does not propagate with the speed of light.

Therefore, we conclude that the Equations (20) and (21) taken together describe the propagation of perturbations in the spacetime geometry in the context of Weyl invariant gravity.

### *3.3. Solving the Weyl Field Equation*

In the previous section, we obtained the Equations (20) and (21), which resulted from a process of linearization of the field equations of Weyl invariant gravity. We shall now consider in detail the equation for the Proca field. Thus, let us start with Equation (21), which gives the dynamics of the Proca field/Weyl field. The solution of this equation is well known. We first write $A^\alpha$ as a superposition of plane waves:

$$A^{\,\alpha}(x^\mu)=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\mathcal{A}^\alpha(k_\mu)e^{i(k_\mu x^\mu)}d^4k \tag{22}$$

where $\mathcal{A}^\alpha$ represents the amplitude of each wave with wave vector $k^\alpha=(\omega_0,\overrightarrow{k})$, which describes the direction of propagation and the frequency of the wave. Following the usual procedure, we substitute (22) into (21), obtaining the dispersion relation $k_\alpha k^\alpha=m^2$, that is

$$-\omega_0^2+k^2+m^2=0, \tag{23}$$

where we denote $k=\left|\overrightarrow{k}\right|$. If we have a packet of Weyl waves, then the phase velocity of each wave is then given by

$$v_p=\frac{\omega_0}{\left|\overrightarrow{k}\right|}=\left(1-\frac{m^2}{\omega_0^2}\right)^{-\frac{1}{2}},$$

whereas the group velocity, that is, the velocity of the envelope, will be given by

$$v_g = \frac{d\omega_0}{dk} = \left(1 - \frac{m^2}{\omega_0^2}\right)^{\frac{1}{2}}.$$

The above equations give the phase velocity and the group velocity in terms of the angular frequency $\omega_0$, the speed of light $c$ and the mass $m$ of the Weyl/Proca field. We see that if the angular frequency increases, then the phase velocity also increases, whereas the group velocity decreases, which means that we shall have dispersion of the wave packet. On the other hand, although the Weyl/Proca waves propagate with a speed less than the speed of light, a wave with a very large frequency propagates with a speed close to the speed of light.

As we have already mentioned, the Weyl field satisfies the condition $\partial_\alpha \sigma^\alpha = 0$, which implies $k_\alpha A^\alpha = 0$, meaning that the Proca field $A^\alpha$ has three degrees of freedom, which implies that we have two transversal polarizations and a longitudinal polarization, the latter meaning that there is a component of the field that varies in the same direction as the wave propagation.

### *3.4. Solving the Metric Field Equation*

In this section, we investigate the propagation of a gravitational field in the absence of baryonic matter, which in this case is given by the equation

$$\Box \psi_{\alpha\beta} = 2\frac{\omega}{\Lambda}\,\theta^{(P)}_{\alpha\beta}, \tag{24}$$

where $\theta^{(P)}_{\alpha\beta}$ is the energy–momentum tensor of the Weyl field $A^\alpha$, which here appears as the source of perturbation of the metric. We already know that the Weyl field $A^\alpha$ propagates as waves in spacetime. We may assume that these waves were possibly produced in the inflation period. In analogy with the cosmic background radiation, one would expect that they should permeate the universe.

In what follows, we present a very simplified model, that is, a *toy model*, so to speak, in which these *primordial Weyl waves* could generate a perturbation in the metric field. Our idea here is that by examining this model, we could take a first step in the further development of a more realistic scenario.

Let us then consider a particular solution of the Equation (21), in which the Weyl field corresponds to the plane wave

$$A^\alpha = \xi^\alpha \cos(k_\rho x^\rho), \tag{25}$$

where the Lorenz gauge $\partial_\alpha A^\alpha = 0$ implies that $k_\alpha \xi^\alpha = 0$. (Clearly, this condition implies that the amplitude $\xi^\alpha$ is a space-like 4-vector).

On the other hand, the energy-momentum tensor is given (to the first order in $\varepsilon$) by

$$\theta^{(P)}_{\alpha\beta} = F_{\alpha\mu}F^{\mu}_{\beta} + \frac{1}{4}\eta_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} + m^2\left(A_\alpha A_\beta - \frac{1}{2}\eta_{\alpha\beta}A^\mu A_\mu\right), \tag{26}$$

where $F_{\alpha\mu} = \left(\partial_\mu A_\alpha - \partial_\alpha A_\mu\right)$. From (25), we obtain the linearized energy–momentum $\theta^{(P)}_{\alpha\beta}$ corresponding to the Weyl plane wave, where we use the dispersion relation $k_\alpha k^\alpha = m^2$ and the Lorenz gauge $k_\alpha \xi^\alpha = 0$, which is given by

$$\theta^{(P)}_{\alpha\beta} = -\frac{1}{2}k_\alpha k_\beta \xi_\mu \xi^\mu + \cos(2k_\rho x^\rho)\left[\frac{1}{2}k_\alpha k_\beta \xi_\mu \xi^\mu + m^2\left(\xi_\alpha \xi_\beta - \frac{1}{2}\eta_{\alpha\beta}\xi_\mu \xi^\mu\right)\right].$$

The above equation can be written as

$$\theta^{(P)}_{\alpha\beta} = -\frac{1}{2}k_\alpha k_\beta \xi_\mu \xi^\mu + \Theta_{\alpha\beta}\cos(2k_\rho x^\rho), \tag{27}$$

where $\Theta_{\alpha\beta} = k_\alpha k_\beta \xi_\mu \xi^\mu + 2m^2\left(\xi_\alpha\xi_\beta - \frac{1}{2}\eta_{\alpha\beta}\xi_\mu\xi^\mu\right)$. (Incidentally, it is not difficult to verify that $\partial_\alpha \theta_\beta^{\alpha(P)} = 0$.)

The equation for the metric field will then become

$$\Box\psi_{\alpha\beta} = -\frac{\omega}{\Lambda}k_\alpha k_\beta \xi^2 + 2\frac{\omega}{\Lambda}\Theta_{\alpha\beta}\cos(2k_\rho x^\rho).$$

The general solution of this equation may be separated into three components:

$$\psi_{\alpha\beta} = \psi_{\alpha\beta}^{(\mathbf{H})} + \psi_{\alpha\beta}^{(\mathbf{P1})} + \psi_{\alpha\beta}^{(\mathbf{P2})}, \tag{28}$$

where $\psi_{\alpha\beta}^{(\mathbf{H})}$ is the solution of the homogeneous equation, $\psi_{\alpha\beta}^{(\mathbf{P1})}$ is a particular solution of the equation associated with the constant term $\frac{\omega}{\Lambda}k_\alpha k_\beta \xi^2$ and $\psi_{\alpha\beta}^{(\mathbf{P2})}$ solves the equation associated with the oscillating term $2\frac{\omega}{\Lambda}\Theta_{\alpha\beta}\cos(2k_\rho x^\rho)$. In other words, we are considering separately the following three equations:

$$\Box\psi_{\alpha\beta}^{(\mathbf{H})} = 0, \tag{29}$$

$$\Box\psi_{\alpha\beta}^{(\mathbf{P1})} = -\frac{\omega}{\Lambda}k_\alpha k_\beta \xi^2, \tag{30}$$

$$\Box\psi_{\alpha\beta}^{(\mathbf{P2})} = 2\frac{\omega}{\Lambda}\Theta_{\alpha\beta}\cos(2k_\rho x^\rho). \tag{31}$$

An interesting solution to the homogeneous Equation (29) is well known and corresponds to gravitational waves propagating at the speed of light. It can be written in the form $\psi_{\alpha\beta}^{(\mathbf{H})} = S_{\alpha\beta}\cos\left(\bar{k}_\mu x^\mu\right)$, where $\bar{k}_\mu = (\bar{\omega}_0, \bar{k}_i)$ is the wave 4-vector, with $\bar{k}_\alpha \bar{k}^\alpha = 0$. This solution may describe the propagation of gravitational waves produced by an astrophysical source (such as a binary system). Therefore, it shows that these gravitational waves will travel with the speed of light, even propagating through the background medium created by the Weyl field, in this order of approximation.

On the other hand, the direct effects of the Weyl plane wave on the geometry can be obtained from the Equations (30) and (31). A solution to (30), which satisfies the Einstein gauge, is given by

$$\psi_{\alpha\beta}^{(\mathbf{P1})} = -\frac{\omega}{\Lambda}k_\alpha k_\beta\left(\frac{(\xi_\mu x^\mu)^2}{2}\right), \tag{32}$$

recalling that $\xi_\alpha k^\alpha = 0$.

Finally, a solution to (31) $\psi_{\alpha\beta}^{(\mathbf{P2})}$ is given by

$$\psi_{\alpha\beta}^{(\mathbf{P2})} = C_{\alpha\beta}\cos(2k_\rho x^\rho), \tag{33}$$

where $C_{\alpha\beta} = -\frac{\omega}{\Lambda}\frac{\Theta_{\alpha\beta}}{2\,m^2}$. As $m^2 = -\frac{3\Lambda}{2\omega}$; this solution can be written as

$$\psi_{\alpha\beta}^{(\mathbf{P2})} = \frac{3\Theta_{\alpha\beta}}{4m^4}\cos(2k_\rho x^\rho).$$

Thus, the general solution (28) will be given by

$$\psi_{\alpha\beta} = S_{\alpha\beta}\ \cos\left(\bar{k}_\mu x^\mu\right) + \frac{3}{4m^2}k_\alpha k_\beta\ (\xi_\mu x^\mu)^2 + \frac{3\Theta_{\alpha\beta}}{4m^4}\cos(2k_\rho x^\rho),$$

recalling that $\Theta_{\alpha\beta} = k_\alpha k_\beta \xi^2 + 2m^2\left(\xi_\alpha\xi_\beta - \frac{1}{2}\eta_{\alpha\beta}\xi^2\right)$. It is worth noting the presence of an oscillating term due to the oscillation of the Weyl field. It is also interesting to remark that the oscillating terms of the this solution have different frequencies, which is to be expected since they correspond to distinct sources. Finally, we note that the *Einstein gauge* implies that

$$\partial_\alpha \psi^\alpha_\beta = \partial_\alpha \left\{ S^\alpha_\beta \, \cos\left(\bar{k}_\mu x^\mu\right) + \frac{3}{4m^2} k^\alpha k_\beta \, \left(\xi_\mu x^\mu\right)^2 + \frac{3}{4m^4}\left[k^\alpha k_\beta \xi^2 + 2m^2\left(\xi^\alpha \xi_\beta - \frac{1}{2}\delta^\alpha_\beta \xi^2\right)\right]\cos\left(2k_\rho x^\rho\right)\right\} = 0,$$

where by using $k_\alpha k^\alpha = m^2$ and $k_\alpha \xi^\alpha = 0$ we obtain $\bar{k}_\alpha S^\alpha_\beta = 0$, meaning that part of the wave has a transversal character.

In terms of $h_{\alpha\beta} = \psi_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\psi$, where $h_{\alpha\beta}$ denotes the perturbation of the metric in the first-order perturbation of the Minkowski metric, we finally write

$$h_{\alpha\beta} = \zeta_{\alpha\beta}\cos\left(\bar{k}_\mu x^\mu\right) + \frac{3}{4m^2} B_{\alpha\beta}\left(\xi_\mu x^\mu\right)^2 + \frac{3}{4m^4} D_{\alpha\beta}\cos\left(2k_\rho x^\rho\right)$$

where $\psi$, the trace of $\psi_{\alpha\beta}$, is given by

$$\psi = S\cos\left(\bar{k}_\mu x^\mu\right) + \frac{3}{4}\,\left(\xi_\mu x^\mu\right)^2 - \frac{3}{4m^2}\xi^2\cos\left(2k_\rho x^\rho\right), \quad (34)$$

where we are defining $\zeta_{\alpha\beta} = \left(S_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}S\right)$, $B_{\alpha\beta} = \left(k_\alpha k_\beta - \frac{1}{2}\eta_{\alpha\beta}m^2\right)$ and $D_{\alpha\beta} = \left[k_\alpha k_\beta \xi^2 + m^2\left(2\xi_\alpha \xi_\beta - \frac{1}{2}\eta_{\alpha\beta}\xi^2\right)\right]$.

## 4. Primordial Gravitational Waves

To date, the gravitational waves we expect to detect with detectors such as LIGO are those which were generated by astrophysical sources (neutron stars, black holes, supernovae, etc.). On the other hand, inflation theory predicts that gravitational waves must have been generated in the primordial universe. It is believed that the origin of these primordial gravitational waves is similar to the origin of primordial density fluctuations. In other words, the primordial gravitational waves would have been produced by quantum fluctuations in the geometry of spacetime, their wavelength being stretched to astronomical sizes by the rapid inflationary expansion [26,27]. According inflation theory, the energy released when a false vacuum decays is converted into a hot fireball of particles. On the other hand, some cosmologists believe that a potential source of dark matter comes from particle physics, such as axions, massive neutrinos, etc. We know that, in modern quantum field theory, the Proca equation describes massive gauge fields, the so-called Z and W bosons. It is hoped that the discovery of these waves could shed some light on the unification of gravitation theory and quantum mechanics.

Of course, the origin of the Proca field in this theoretical setting is still uncertain, although the possibility that it might have been created together with baryonic matter in the inflationary period cannot be ruled out. Moreover, in this hypothetical scenario one would be inclined to associate $T^{(m)}_{\alpha\beta}$ with the energy–momentum of the primordial scalar field during the slow-roll regime.

As is well known, one expects to detect primordial gravitational waves by searching for a possible change in the polarization of cosmic background radiation. In the present theoretical setting, we would also have a new ingredient to be taken into account, namely, the possibility of Proca waves leaving their signature on the metric $h_{\alpha\beta}$. It is plausible to expect that the primordial Proca waves would propagate in all directions in space, and as such could well be modeled as a kind of massive photon [28]. Moreover, it should be mentioned that Proca waves have three polarizations: two transversal and one longitudinal, the latter being a feature to be taken into account in their detection.

With our present knowledge of the primordial universe, it is not easy to devise a concrete model that would take into account the generation of Proca waves. In particular, it is still not clear how to set the initial conditions for the distribution of dark matter before and during the inflation regime. It is this lack of information that still prevent us from providing initial conditions for Equation (22) in order to obtain an explicit solution for the

Proca field. Without such an input, it is difficult to build a tentative model of primordial Proca waves. However, we believe that these initial difficulties should not discourage us from speculating about the possible existence of this kind of wave, which may be predicted in some modified theories of gravity.

## 5. Final Remarks

The history of gravitational waves is very rich and fascinating, with interest in the subject being growing exponentially since the first evidence of their existence given by the discovery of Hulse–Taylor binary pulsars in 1974 [29]. However, direct evidence of them was delayed until 2015, when then the LIGO team announced the detection of a wave which is believed to have been produced by the merging of two binary black holes [30]. In 2017, the LIGO and VIRGO detectors detected gravitational waves after only two seconds of gamma rays and optical telescopes observing signals from the same direction. This almost exact coincidences in the arrival time of the two waves has been interpreted as evidence that the speed of propagation of the gravitational waves is the same as the speed of light. We would like to remark that in the framework of the theory examined here, although they can be produced by the same sources, almost exact simultaneous detection of gravity waves and Proca waves would, in principle, not occur unless for very large frequencies of the latter and short distances from the sources.

It should be noted that the both the energy–momentum tensor $T^{(P)}_{\alpha\beta}$ of the Proca field and the energy–momentum tensor of matter $T^{(m)}_{\alpha\beta}$ are conserved separately, which means that the Proca field does not interact with matter. This simple fact seems to suggest that the Proca field could be a possible candidate to describe dark matter.

Finally, as there are some conjectures regarding the Weyl/Proca field as a possible candidate to account for dark matter, we speculate that primordial gravitational waves may have a component originated by this massive vector field [3].

**Author Contributions:** Conceptualization, M.D., F.D. and C.R.; methodology, M.D. and C.R.; validation, M.D., F.D. and C.R.; formal analysis, M.D., F.D. and C.R.; investigation, M.D., F.D. and C.R.; data curation, F.D.; writing—original draft preparation, M.D.; writing—review and editing, M.D. and C.R.; visualization, M.D., F.D. and C.R.; supervision, C.R.; project administration, C.R.; funding acquisition, M.D. and C.R. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work was partially supported by FAPESQ and CNPq (Brazil).

**Data Availability Statement:** The original contributions presented in the study are included in the article. Further inquiries can be directed to the corresponding author/s.

**Acknowledgments:** The authors would like to thank J. B. Formiga for helpful discussions. We also express our gratitude to the referees for their valuable comments and suggestions.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## Abbreviation

The following abbreviations are used in this manuscript:

| | |
|---|---|
| LIGO | Laser Interferometer Gravitational-Wave Observatory |

## Notes

1 It is important to recall that Weyl theory is invariant under the Weyl transformations, in which, in addition to a conformal transformation in the metric, we have to take into account the gauge transformation of the vector field. It is not conformally invariant in the sense that, for instance, Mannheimer's conformal gravity is (see, for instance, [7]).

2 It is important to recall that the original Weyl theory is invariant under the so-called Weyl transformations, in which, in addition to a conformal transformation in the metric, there is also a gauge transformation of the vector field. These two transformations must be carried out simultaneously. It is not conformally invariant in the sense that, for instance, Mannheimer's conformal gravity is (see, for instance, [7]).

3 If $\Lambda > 0$ is interpreted as the cosmological constant, then we must set $\omega < 0$.

[4] The proof of this assertion follows exactly the same reasoning one uses in general relativity. The simplest way to achieve this is to assume a congruence of a pressureless perfect fluid ("dust") and impose the equation $\nabla_\alpha T^{(m)\alpha\beta} = 0$. It then follows in a straightforward manner that the dust particles follow metric geodesics.

[5] Here, we are adopting the following convention for the Riemann tensor: $\tilde{R}^{\alpha}_{\beta\gamma\eta} = \Gamma^{\alpha}{}_{\gamma\beta,\eta} - \Gamma^{\alpha}{}_{\eta\beta,\gamma} + \Gamma^{\alpha}{}_{\delta\eta}\Gamma^{\delta}{}_{\eta\beta} - \Gamma^{\alpha}{}_{\delta\gamma}\Gamma^{\delta}{}_{\eta\beta}$.

[6] Just for convenience, we now absorb the parameter $\varepsilon$ by redefining $\psi_{\alpha\beta} \to \varepsilon\psi_{\alpha\beta}$.

[7] Here, we are setting $c = 1$.

## References


1. Proca, A. Sur la théorie ondulatoire des électrons positifs et négatifs. *J. Phys. Radium* **1936**, *7*, 347–353. [CrossRef]
2. Greiner, W.; Reinhardt, J. *Field Quantization*; Springer: Berlin/Heidelberg, Germany, 1996.
3. Tucker, R.W.; Wang, C. An Einstein-Proca-fluid model for dark matter gravitational interactions. *Nucl. Phys. B (Proc. Suppl.)* **1997**, *57*, 259–262. [CrossRef]
4. Gabriel Gómez, L.; Rodríguez, Y. Coupled multi-Proca vector dark energy. *Phys. Dark Univ.* **2020**, *31*, 100759. [CrossRef]
5. Loeb, A.; Weiner, N. Cores in dwarf galaxies from dark matter with a Yukawa potential. *Phys. Rev. Lett.* **2011**, *106*, 171302. [CrossRef] [PubMed]
6. Burikan, P.; Harko, T.; Pimsamarn, K.; Shahidi, S. Dark matter as a Weyl geometric effect. *Phys. Rev. D* **2023**, *107*, 064008. [CrossRef]
7. Mannheim, P.D. Making the Case for Conformal Gravity. *Found. Phys.* **2012**, *42*, 388–420. [CrossRef]
8. Weyl, H. Gravitation und Elektrizität. In *Sitzungsberichte der Preussischen Akademie der Wissenschaften*; Springer Spektrum: Berlin/Heidelberg, Germany, 1918; p. 465.
9. Sanomiya, T.A.T.; Lobo, I.P.; Formiga, J.B.; Dahia, F.; Romero, C. Invariant approach to Weyl's unified field theory. *Phys. Rev. D* **2020**, *102*, 124031. [CrossRef]
10. Tauber, G.E. Massive vector meson interacting with the gravitational field. I. General formalism. *J. Math. Phys.* **1969**, *10*, 633–638. [CrossRef]
11. Changsheng, S.; Liu, Z. Proca Effect in Reissner–Nordstrom de Sitter Metric. *Int. J. Theor. Phys.* **2005**, *44*, 303–308. [CrossRef]
12. Ford, L. Inflation driven by a vector field. *Phys. Rev. D* **1989**, *40*, 967. [CrossRef] [PubMed]
13. Năstase, H. *Classical Field Theory*; Cambridge University Press: Cambridge, UK, 2019.
14. Romero, C.; Lima, R.G.; Sanomiya, T.A.T. One hundred years of Weyl's (unfinished) unified field theory. *Sci. Part B Stud. Hist. Philos. Mod. Phys.* **2019**, *66*, 180–185. [CrossRef]
15. Adler, R.; Bazin, M.; Schiffer, M. *Introduction to General Relativity*; McGraw-Hill: New York, NY, USA, 1975.
16. O'Raifeartaigh, L. *The Dawning of Gauge Theories*; Princeton University Press: Princeton, NJ, USA, 1997.
17. Romero, C.; Duarte, M.P. The Coming Back of the Proca Field. *Space Time Fundam. Interact.* **2023**, *3–4*, 247–250.
18. Einstein, A. Über Gravitationswellen. In *Königlich Preußische Akademie derWissenschaften Sitzungsberichte*; Springer Spektrum: Berlin/Heidelberg, Germany, 1918; pp. 154–167.
19. Choquet-Bruhat, Y. *The Cauchy Problem in Gravitation: An Introduction to Current Research*; Witten, L., Ed.; New J. Wiley: New York, NY, USA, 1962.
20. Avalos, R.; Lira, J.H. *The Einstein Constraint Equations*; IMPA: Rio de Janeiro, Brazil, 2021.
21. Wald, R. *General Relativity*; Chicago University Press: Chicago, IL, USA, 1984.
22. Ringström, H. *The Cauchy Problem in General Relativity*; European Mathematical Society: Helsinki, Finland, 2009.
23. Adler, R.J. *General Relativity and Cosmology*; Springer: Berlin/Heidelberg, Germany, 2021.
24. Bernabeu, J.; Espinoza, C.; Mavromatos, N.E. Cosmological constant and local gravity. *Phys. Rev. D* **2010**, *81*, 084002. [CrossRef]
25. Bernabeu, J.; Espriu, D.; Puigdomènech, D. Gravitational waves in the presence of a cosmological constant. *Phys. Rev. D* **2011**, *84*, 063523. Erratum in *Phys. Rev. D* **2012**, *86*, 069904. [CrossRef]
26. Taillet, R. (Ed.) *The Young Universe: Primordial Cosmology*; Wiley: Hoboken, NJ, USA, 2022.
27. Liddle, A.R.; Lyth, D. *Cosmological Inflation and Large-Scale Structure*; Cambridge University Press: Cambridge, UK, 2000.
28. Cuzinatto, R.R.; de Morais, E.M.; Medeiros, L.G.; Naldoni de Souza, C.; Pimentel, B.M. de Broglie-Proca and Bopp-Podolsky Massive Photon Gases in Cosmology. *Eur. Phys. J. C* **2017**, *118*, 19001. [CrossRef]
29. Hulse, R.A.; Taylor, J.H. Discovery of a pulsar in a binary system. *Astrophys. J.* **1975**, *195*, L51–L53. [CrossRef]
30. Abbott, B.P.; Abbott, R.; Abbott, T.D.; Abernathy, M.R.; Acernese, F.; Ackley, K.; Adams, C.; Adams, T.; Addesso, P.; Adhikari, R.X.; et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.* **2016**, *116*, 061102. [CrossRef]