Generalized Symmetries in String Theory Realizations of Quantum Field Theories

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Dedicated to my parents.

Abstract

This thesis contains work from several different projects linked by the common goal of understanding the origin of generalized symmetries and their 't Hooft anomalies in string theory realizations of quantum field theories (QFTs).

Beginning in holography, we study the supergravity origin of the 1-form symmetry of 4d $\mathcal{N} = 1$ $SU(N)$ Super-Yang Mills (SYM). Furthermore we discuss the imprint of a mixed 0-form/1-form symmetry anomaly in the associated holographic geometry.

This holographic work was an early precursor to the notion of the Symmetry Topological Field Theory (SymTFT), which we study in this thesis for 3d QFTs constructed in M-theory. In particular, we derive the SymTFT for 3d supersymmetric QFTs constructed in M-theory either via geometric engineering or holography. The SymTFT encodes the symmetry structures of the QFTs, including their anomalies. We probe our general framework with a variety of examples.

We also present an argument that branes, in a certain topological limit, not only furnish the symmetry generators of generalized symmetries, but also encode the SymTFT. We derive the SymTFT directly from branes, and furthermore demonstrate the central role that Hanany-Witten brane configurations play in this process. After presenting a general analysis, we study various examples of QFTs realized in both geometric engineering and holography.

Statement of Originality

This thesis is based on results from the following publications, to which the author contributed substantially:

[1] **"Holography, 1-Form Symmetries, and Confinement,"**
F. Apruzzi, M. van Beest, D.S.W. Gould, S. Schafer-Nameki,
Phys. Rev. D **104** no. 6, (2021) 066005, [arXiv:2104.12764].

[2] **"Symmetry TFTs for 3d QFTs from M-Theory,"**
M. van Beest, D.S.W. Gould, S. Schafer-Nameki, Y,-N. Wang,
JHEP **02** (2023) 226, [arXiv:221.03703].

[3] **"Aspects of Categorical Symmetries from Branes: SymTFTs and Generalized Charges,"**
F. Apruzzi, F. Bonetti, D.S.W. Gould, S. Schafer-Nameki,
[arXiv:2306.16405].

Versions of [1] have appeared previously in the DPhil thesis of co-author Marieke van Beest. During the DPhil, the author also contributed to [4–6] which are not included in this thesis due to space constraints.

Acknowledgements

I would like to begin by thanking my supervisor, Sakura Schäfer-Nameki. Her guidance and passion for physics have been instrumental in my development. I want to thank her for introducing me to such an exciting area of research, her commitment to the education of her students, and for guiding me personally throughout all aspects of my PhD.

During my time in Oxford I have had the pleasure of collaborating with a number of wonderful people. I would like to thank Fabio Apruzzi, Federico Bonetti, Lea Bottini, Marieke van Beest, Lakshya Bhardwaj, Ludovic Fraser-Taliente, Liam Gladden, Ling Lin, Evyatar Sabag, Hannah Tillim, Arthur Platschorre and Yi-Nan Wang, for their insight, patience, guidance and friendship during our projects together.

Finally, I would like to thank my family, and in particular my parents. Their greatest gift to me has been the freedom to follow my interests along the winding path which has led me to this point. I am eternally grateful for the sacrifices they have made to make all of this a reality. From a young age they instilled in me the importance of my education, a love of reading, a passion for learning, and a commitment to making the most of any opportunities afforded to me. I owe all of my success to these principles, as well as to their unwavering love, support and confidence in me.

Contents

Chapter 1

Introduction

Symmetry plays a central, governing role in the vast majority of physics research. At a coarse level, symmetries can be divided into two categories: *global* and *gauge*. Gauge symmetries are ubiqutous in physical theories: for example, they are the key component in the description of the fundamental forces in the Standard Model. Global symmetries can impose selection rules, be spontaneously broken, have 't Hooft anomalies and can often be gauged. This suite of properties means that studying global symmetries in quantum field theories (QFTs) offers immense power and possibility.

String/M-theory allows computational access to a range of subtle and complex QFTs, in particular strong-coupling regimes and higher-dimensional theories. In this thesis, we are interested in studying the origin, imprint and consequences of symmetries in QFTs from the perspective of their string theory realizations.

Throughout this thesis we will be primarily interested in a new type of global symmetry, called "generalized symmetries" [7], which we review in section 1.1. Furthermore, the central tool we employ throughout this thesis is called the Symmetry Topological Field Theory (SymTFT), which we review in great depth in chapter 2. This is a topological field theory in one dimension higher than the QFT we are studying, and it encodes key symmetry properties which are of interest to us. In this work we explain where the SymTFT sits in string theory realizations of QFTs, how to compute its couplings, and how branes in string theory play a crucial role.

1.1 Generalized Symmetries

In this thesis we are particularly interested in a modern development within the area of *global* symmetries, known as "generalized global symmetries". The concept of generalized symmetries was first introduced in the 2014 paper [7][1], following which there has been a flurry of activity. The central idea of this work can be neatly summarized in the statement:

$$\text{Symmetries in a QFT} \;\leftrightarrow\; \text{Topological Operators} \tag{1.1}$$

This simple relationship, which we will explain in detail below, leads to a range of exciting consequences and new types of symmetries to be studied. Relevant for this thesis are: invertible higher-form symmetries, higher-group symmetries and non-invertible/ categorical higher-form symmetries. We will discuss each of these below.

The first and most natural extension of the notion of global symmetries is to so-called "higher-form" global symmetries. Before we get there, we must first re-cast our understanding of "ordinary" symmetries in a new language.

Ordinary Symmetries. An ordinary (pre-2014) symmetry is traditionally encoded in a unitary operator

$$U(t)\,, \tag{1.2}$$

which acts on the Hilbert space defined at a time t. Such an operator acts on local operators $O(x,t)$ by conjugation

$$U(t)O(x,t)U(t)^{-1} = O'(x,t)\,, \tag{1.3}$$

Here the prime denotes a new local operator defined at the same spacetime location. A key property of such unitary operators U is their commutation with the Hamiltonian:

$$U(t_1) = U(t_2)\,, \quad \forall t_1, t_2\,, \tag{1.4}$$

[1]For greater depth, there are a number of reviews of this topic, including [8–13].

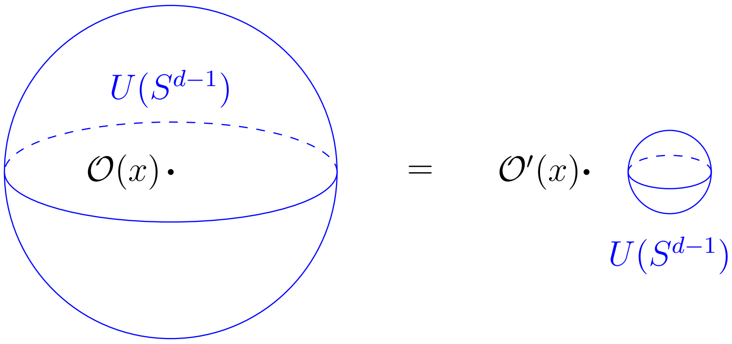


Figure 1.1: On the left we insert an operator U on $\Sigma_{d-1} = S^{d-1}$ which links with the local operator $O(x)$. The operator U is topological, as long as it doesn't cross a charged operator (according to (1.6)), so we can shrink it. Once we cross U with O, this changes $O(x) \to O(x)'$. Finally, once U no longer links with anything charged, it becomes trivial.

A key insight of [7] is to consider placing the operator U on an arbitrary codimension-1 submanifold of d-dimensional spacetime Σ_{d-1}

$$U(\Sigma_{d-1})\,. \tag{1.5}$$

Such an operator is *topological*: meaning that $U(\Sigma_{d-1}) = U(\widetilde{\Sigma}_{d-1})$ for two manifolds $\Sigma_{d-1}, \widetilde{\Sigma}_{d-1}$ which are related by topological manipulation. This is the generalization of (1.4).

The action on local operators is similar:

$$U(\Sigma_{d-1})O(x) = O(x)'U(\widetilde{\Sigma}_{d-1})\,, \tag{1.6}$$

where crucially $O(x) \equiv O(x,t)$ lies in the d-dimensional submanifold connecting Σ_{d-1} and $\widetilde{\Sigma}_{d-1}$. This action can equivalently be represented using "linking", see figure 1.1.

Higher-Form Symmetries. We are now ready to introduce *higher-form* global symmetries. A p-form global symmetry is defined by the existence of codimension-$(p+1)$ operators U which are topological. In other words, they can be inserted on $(d-p-1)$-dimensional submanifolds of spacetime, Σ_{d-p-1}. Owing to their smaller dimension, these operators no longer act on local operators by linking. Instead, they act on p-dimensional *extended* operators. See figure 1.2.

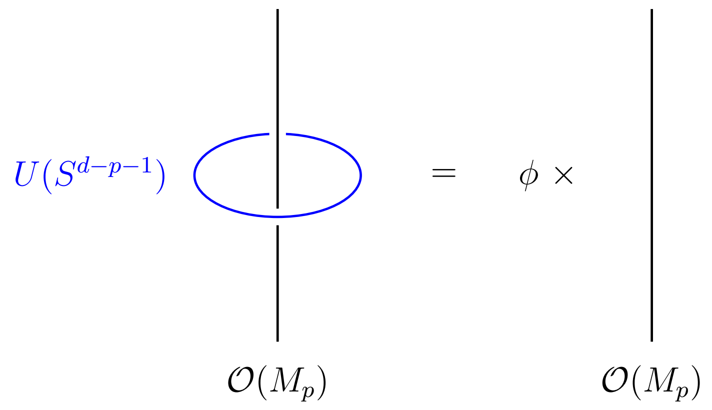


Figure 1.2: Here we demonstrate how a p-form symmetry generator $U(S^{d-p-1})$ acts on an extended operator. Initially, U links with a p-dimensional operator O. U is topological, so we can deform to a configuration where the operators no longer link. However, analogously to the ordinary case, after un-linking, the operator O is modified. This time it picks up a phase, ϕ.

In this language, the ordinary symmetries discussed above are therefore *0-form* symmetries, since they act on 0-dimensional local operators.

Fusion Rules. It is well-known that ordinary global symmetries form groups. In the language of the topological operators, a 0-form symmetry is described by a group G when:

$$U_{g_1}(\Sigma_{d-1})U_{g_2}(\Sigma_{d-1}) = U_{g_1\cdot g_2}(\Sigma_{d-1})\,, \tag{1.7}$$

where $g_i \in G$ are group elements which label the topological operators.

Analogously, codimension-$(p+1)$ operators obeying

$$U_{g_1}(\Sigma_{d-p-1})U_{g_2}(\Sigma_{d-p-1}) = U_{g_1\cdot g_2}(\Sigma_{d-p-1})\,, \tag{1.8}$$

signal the existence of a p-form symmetry group G. Collectively these symmetries come under the title of *invertible* higher-form symmetries. The powerful insights of [7] was that one can use many of the same properties we love about ordinary symmetries in this new paradigm: these symmetries can be gauged, have anomalies, and be spontaneously broken.

It is natural to ask if there exists theories in which such operators do not obey group-law fusion rules. Examples of these symmetries naturally occur in 2d theories (e.g. [14–20]), and have recently been shown to exist in higher-dimensional QFTs [21–23]. Such symmetries are referred to as *non-invertible* or *categorical* symmetries. Schematically, given some generators U_a labelled by $\mathcal{S} = \{a, b, c, \dots\}$, the fusion rules can be more general than the group-theoretic case above:

$$U_a U_b = \sum_i U_i \,. \tag{1.9}$$

Studying these symmetries, their properties, mathematical structure and physical consequences is currently an incredibly active area of research [22, 24–57].

Higher-Group Symmetries. Two or more invertible higher-form symmetries of a QFT can generically combine in a non-trivial way (i.e. not just a direct product) [58–63]. Such symmetries are referred to as *higher-group* symmetries. For example, a 0-form and 1-form symmetry can combine non-trivially in a 2-group symmetry.

The "non-trivial" combination of symmetries is often in the form of a mixing of background fields. Concretely, suppose we begin with a background field B_{p+1} for a $U(1)$ p-form symmetry, with field strength $H_{p+2} = dB_{p+1}$. The presence of a higher-group symmetry means that B_{p+1} may transform under the gauge transformations of some other higher-form symmetry background fields $\{B_{p_1+1}, B_{p_2+1}, \dots, B_{p_n+1}\}$. We can make H_{p+2} gauge-invariant once more by adding extra terms.

$$H_{p+2} = dB_{p+1} + \Theta(B_{p_1+1}, B_{p_2+1}, \dots, B_{p_n+1}) \,, \tag{1.10}$$

where Θ is some function of background fields of the other continuous higher-form symmetries. We say that the theory has a continuous higher-form symmetry group if a relationship of this type exists. We will now explain the notion of background fields and anomalies in more detail below.

1.2 Background Fields and Anomalies

In this section we explain the concept of background fields and anomalies for invertible higher-form symmetries. An ordinary 0-form symmetry background is a connection A on a 0-form symmetry group bundle on spacetime. Locally, we write this as a differential 1-form A_1 with field strength $F_2 = dA_1$.

Continuous p-Form Symmetries. For a continuous $U(1)$ p-form symmetry, the background is a $(p+1)$-form B_{p+1} with field strength $H_{p+2} = dB_{p+1}$. A small gauge transformation of this background field is of the form

$$B_{p+1} \to B_{p+1} + d\lambda_p \,, \tag{1.11}$$

where λ_p is a p-form gauge field.

Finite p-Form Symmetries. For a discrete p-form symmetry, the background gauge field is now a $(p+1)$-cochain which takes its values in the p-form symmetry group. In the presence of no other symmetry backgrounds, this gauge field is flat

$$\delta B_{p+1} = 0 \,. \tag{1.12}$$

A small gauge transformation in this case takes the form

$$B_{p+1} \to B_{p+1} + \delta\lambda_p \,, \tag{1.13}$$

where now λ_p is a group-valued p-cochain. As explained above, in the presence of other symmetries, the flatness relation may not hold: in which case we say we have a higher-group symmetry.

't Hooft Anomalies. 't Hooft anomalies are features of QFTs involving invariance under background gauge transformations of symmetries. It is important to note that the existence of such an anomaly does *not* imply a problem with the QFT, it is still a consistent theory. In particular, a pure 't Hooft anomaly of a p-form symmetry arises when background gauge transformations cause an irreparable change to the partition function

$$Z[B_{p+1}] \neq Z[B_{p+1} + \delta\lambda_p] \,. \tag{1.14}$$

Here "irreparable" means that the change cannot be undone by adding a counter-term which is a function of B_{p+1}. The existence of such an anomaly signals that one *cannot gauge* the p-form symmetry in a consistent way.

Generically, a theory may have two or more higher-form symmetries. Even if these symmetries are non-anomalous on their own, there can be *mixed* 't Hooft anomalies between the symmetries. For example, if

$$Z[B_{p+1}, B_{q+1}] \neq Z[B_{p+1} + \delta\lambda_p, B_{q+1}] \,, \tag{1.15}$$

but

$$Z[B_{p+1}, B_{q+1} = 0] = Z[B_{p+1} + \delta\lambda_p, B_{q+1} = 0] \,, \tag{1.16}$$

for some p- and q-form symmetries, the theory is said to have a mixed 't Hooft anomaly. This anomaly is an obstruction to gauging *both* symmetries simultaneously.

1.3 Outline of Thesis

In Chapter 2 we give a long-form introduction to the topic of Symmetry Topological Field Theories (SymTFT) - the key topic of interest in this thesis. In this chapter we also present the general analysis and structure which underpins all the applications in subsequent chapters.

In Chapter 3 we study a particular holographic description of 4d $\mathcal{N} = 1$ $SU(N)$ Super Yang-Mills, namely the Klebanov-Strassler solution [64]. This field theory has a $\mathbb{Z}_N$ global 1-form symmetry and mixed 0/1-form symmetry anomaly. We use the holographic correspondence to identify the supergravity origin of the global symmetries, as well as the imprint of this mixed anomaly.

In Chapter 4 we study the SymTFT of 3d QFTs realized in M-theory constructions. Using the differential cohomology paradigm of [65] we demonstrate how the global symmetries and 't Hooft anomalies of these QFTs are encoded in the geometries of the string theory constructions.

In Chapter 5 we study a recently discovered relationship between branes and generalized global symmetries. We demonstrate that branes encode not only the symmetry generators of generalized symmetries, but also the SymTFT. We highlight the power of our work in a variety of examples in four spacetime dimensions.

Chapter 2

The Symmetry Topological Field Theory

The symmetry structure of a d-dimensional QFT can be conveniently encoded inside a topological quantum field theory of one dimension higher, called the *Symmetry Topological Field Theory* (*SymTFT* or *Symmetry TFT*) [65–67]. Concretely, the SymTFT encodes the choice of global structure of the QFT gauge group, as well as the possible (mixed) 't Hooft anomalies of the generalized symmetries of the QFT.

In this chapter we describe the general features of the SymTFT, and then give a detailed description of its origin in string theory realizations of QFTs (both in holography and geometric engineering). The content of this chapter represents the backbone of this thesis: it presents an introduction to the ideas and tools applied across various contexts in chapters 3, 4 and 5.

2.1 General Structure

Let us consider a d-dimensional QFT. The associated $(d+1)$-dimensional SymTFT has topological boundary conditions, denoted $\mathcal{B}^{\text{sym}}$, and non-topological boundary conditions denoted $\mathcal{B}^{\text{phys}}$. The latter boundary is generically non-topological, but in instances where the physical d-dimensional theory is topological, it can also be topological. After reduction along an interval, these boundary conditions give rise to the d-dimensional QFT: see figure 2.1. The SymTFT is an extension of the more

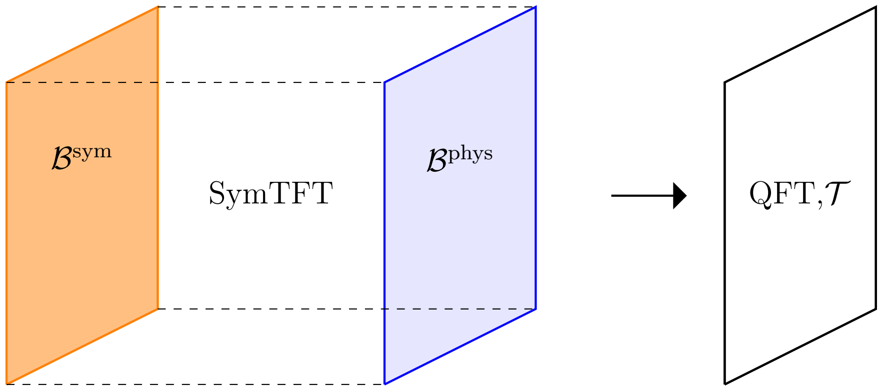


Figure 2.1: Here we depict the SymTFT with its two boundaries $\mathcal{B}^{\text{sym}}, \mathcal{B}^{\text{phys}}$. After interval compactification, we obtain the d-dimensional QFT $\mathcal{T}$.

familiar *anomaly theory* in the following sense. The anomaly theory is an invertible theory whose gauge variation, when placed on a manifold with boundary, exactly cancels that of the partition function of the QFT. This invertible theory assigns a 1-dimensional Hilbert space to closed codimension-1 sub-manifolds in spacetime. This contains information about the phase of the partition function of the QFT $\mathcal{T}$, evaluated on the codimension-1 manifold. Relaxing the invertible condition gives the SymTFT. This allows for the assignment of a larger-than-one dimensional Hilbert space. Now the QFT has a vector of partition functions. This set of distinct partition functions encodes the possible choices of global structures of $\mathcal{T}$. In particular, the QFTs encoded in these choices will have identical local physics but a distinct spectrum of extended operators. Picking a boundary condition here amounts to fixing a global structure, or in other words selecting an *absolute* QFT.

In this thesis, we will focus specifically on generalized symmetries which are abelian. In these cases, the SymTFT has an action formulation in terms of the abelian background fields for these symmetries. The most general symmetry we consider is a product of higher-form symmetry groups:

$$\mathcal{S} = \prod_p G^{(p)} \,, \tag{2.1}$$

Here we assume that the d-dimensional theory, $\mathcal{T}$, is an absolute theory. The background fields for the individual components of the p-form symmetry group $G^{(p)}$ are denoted

$$B^i_{p+1} \in H^{p+1}\left(M_d, \mathbb{Z}_{n^p_i}\right) , \tag{2.2}$$

where $G^{(p)} = \prod_i \mathbb{Z}_{n^p_i}$ for some $n^p_i \in \mathbb{Z}_+$. We encode the possible (mixed) 't Hooft anomalies of these symmetries in a term $\mathcal{A}(\{B_{p+1}\})$. In string theory settings, most generalized symmetries seem to be of this type. For a discussion of the SymTFT in this more general setting, see [68, 69].

One way of viewing the SymTFT is as a gauging of the p-form symmetries in $(d+1)$-dimensions, i.e. coupling the theory to a Dijkgraaf-Witten type dynamical discrete gauge theory. This contains the BF-couplings for the now dynamical fields b^i_{p+1} and the dual fields $\widehat{b}^i_{d-p-1}$, as well as the anomaly term

$$S_{\text{SymTFT}} = \int_{M_{d+1}} \sum_p \sum_{i,j} n^p_{ij}\, b^i_{p+1} \wedge d\widehat{b}^{\,j}_{d-p-1} + \mathcal{A}\left(\{b^i_{p+1}\}\right) . \tag{2.3}$$

Here we use a continuum field formulation. The fields b are $U(1)$-valued, with equation of motion $ndb = 0$.

The SymTFT has proved incredibly powerful in a variety of contexts. See [20–23, 27, 28, 30–45, 47, 68–106] for a selection of recent works.

Topological Defects of the SymTFT. The topological defects of the SymTFT are given in terms of generalized Wilson lines for the gauge fields. Here we denote them by $\boldsymbol{Q}$. Beginning with (2.3), one can determine the defects by exponentiating the Gauss law constraints (see e.g. [41, 107]). To illustrate the setup we begin in the absence of any anomaly couplings $\mathcal{A} = 0$. The topological defects are generated by:

$$\begin{aligned} \boldsymbol{Q}^{(b^i)}_{p+1}(M_{p+1}) &= \exp\left(2\pi i \int_{M_{p+1}} b^i_{p+1}\right) \\ \boldsymbol{Q}^{(\widehat{b}^i)}_{d-p-1}(M_{d-p-1}) &= \exp\left(2\pi i \int_{M_{d-p-1}} \widehat{b}^i_{d-p-1}\right) . \end{aligned} \tag{2.4}$$

These have a non-trivial commutation relation

$$\boldsymbol{Q}^{(b^i)}_{p+1}(M)\boldsymbol{Q}^{(\widehat{b}^i)}_{d-p-1}(M') = \exp\left(2\pi i \frac{L(M_{p+1}, M'_{d-p-1})}{n_i^p}\right)\boldsymbol{Q}^{(\widehat{b}^i)}_{d-p-1}(M')\boldsymbol{Q}^{(b^i)}_{p+1}(M)\,. \quad (2.5)$$

Here $L(M, M')$ is the linking of the two manifolds in the $(d+1)$-dimensional spacetime. In the presence of non-trivial couplings between the fields b^i_{p+1} in $\mathcal{A}(\{b^i_{p+1}\})$, there will be additional terms in the above expressions for the topological defects.

The concept of a SymTFT is in principle completely general and can be applied to capture the global structures of any given QFT.

Condensation Completion of SymTFTs. Whilst the defects in (2.4) represent the core operators in the SymTFT, one should also include condensation defects. These arise by condensing the defects of the dual symmetry, generated by $\boldsymbol{Q}^{(b)}_{p+1}$ on the defect $\boldsymbol{Q}^{(\widehat{b})}_{d-p-1}$ that generates the symmetry $G^{(p)}$:

$$\begin{aligned}&\boldsymbol{C}\left(\boldsymbol{Q}^{(\widehat{b})}_{d-p-1}(M_{d-p-1}), \boldsymbol{Q}^{(b)}_{p+1}(M_{p+1})\right) \\ &\quad= \frac{1}{|H_{p+1}(M_{d-p-1}, \mathbb{Z}_n)|}\sum_{M_{p+1}\in H_{p+1}\left(M_{d-p-1},\mathbb{Z}_n\right)} \boldsymbol{Q}^{(b)}_{p+1}(M_{p+1})\,\boldsymbol{Q}^{(\widehat{b})}_{d-p-1}(M_{d-p-1})\,.\end{aligned} \quad (2.6)$$

We can also condense these on other symmetry generators, up to dimension constraints.

These additional defects can also be realized by introducing localized couplings in the SymTFT, which correspond to coupling lower-dimensional DW type theories to the SymTFT. Taking into account all possible condensations, this is

$$\begin{aligned}S_{\text{SymTFT}} \supset n_p &\int_{M_{d+1}} b_{p+1}\wedge d\,\widehat{b}_{d-p-1} \\ &+ \sum_{k\geq 1}\int_{M_{d-k}} \left(b_{p+1}\wedge a_{d-k-p-1} + n_p a_{d-k-p-1}\wedge d\widehat{a}_{\,p}\right)\,.\end{aligned} \quad (2.7)$$

Symmetries. The SymTFT for the theory $\mathcal{T}$ is constructed in such a way that the symmetry boundary $\mathcal{B}^{\text{sym}}$ has symmetry $\mathcal{S}$ of the theory $\mathcal{T}$. For abelian group-like

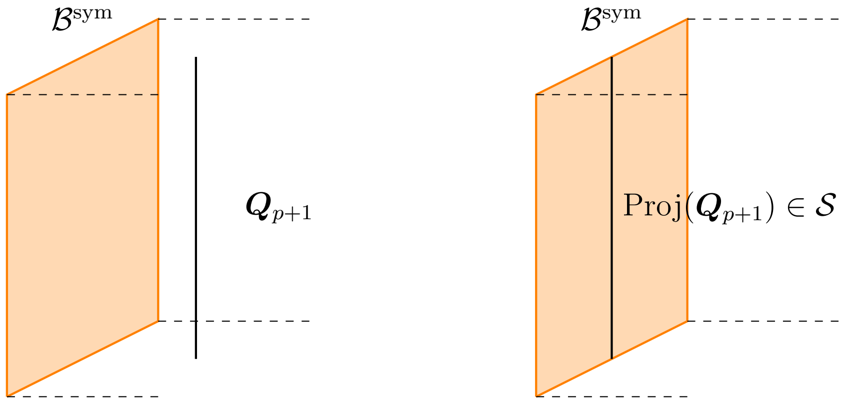


Figure 2.2: Projecting a topological defect $\boldsymbol{Q}$ gives rise to a topological defect on $\mathcal{B}^{\text{sym}}$.

symmetries, given a SymTFT we can recover the symmetry by projecting the bulk topological operators to the symmetry boundary. See figure 2.2.

In the present instance we can simplify the analysis further, by stating that the boundary conditions are specified by a subset $\mathcal{L}$ of topological defects $\boldsymbol{Q}$ of the SymTFT, which have Dirichlet boundary conditions on $\mathcal{B}^{\text{sym}}$. This means the topological defects can end. Furthermore, requiring that this subset is mutually local and maximal defines a so-called "polarization".

All the defects in $\mathcal{L}$ end on the boundary and will define generalized charges – which we will discuss in the next subsection. The symmetry generators are the projections of the bulk topological operators onto the symmetry boundary. An in-depth analysis of all consistency conditions and possibilities in general was undertaken in [68].

Generalized Charges. The charges under generalized symmetries were recently identified as being simply the topological defects of the associated SymTFT [68]. This applies to several invertible and non-invertible symmetries and has been shown to hold in many such instances [103,108,109]. Particularly interesting is the observation that there are generalized charges even for invertible higher-form symmetries. The SymTFT plays the central role in succinctly characterizing these charges.

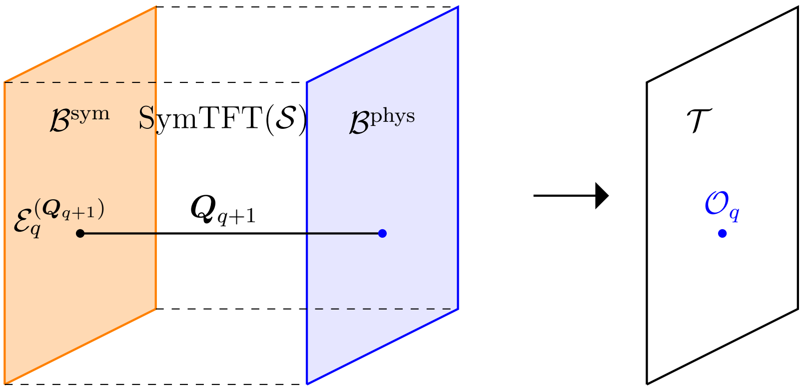


Figure 2.3: Here we consider an operator $\boldsymbol{Q}$ which ends on both boundaries. After interval compactification it gives rise to a genuine q-charge in the absolute theory $\mathcal{T}$.

As proposed in [108], we refer to a q-dimensional, not necessarily topological, defect operator $\mathcal{O}_q$ that is charged under a generalized symmetry as a q-charge. The statement of [108, 109] is that for an invertible higher-form symmetry $G^{(p)}$

$$\text{Genuine } q\text{-charges } \mathcal{O}_q \quad \longleftrightarrow \quad (q+1) - \mathbf{Rep}\left(G^{(p)}\right), \tag{2.8}$$

where the right hand side is the fusion higher-category of higher-representations of $G^{(p)}$ (see e.g. [9, 38, 108] for physics-motivated discussions of these categories). Here $q = 0, \cdots, d-2$. The genuine q-charges are not attached to $(q+1)$-dimensional defects (topological or not), see figure 2.3, and arise after interval compactification as end-points of bulk topological operators $\boldsymbol{Q}_{q+1}$ that end on both physical and topological boundaries[1].

In addition to genuine charges, there can be non-genuine (attached at the end of $\mathcal{O}_{q+1}$) and twisted sector (attached to the end of topological S_{q+1} defects) q-charges. In the SymTFT picture, the twisted sector charges arise from projecting L-shaped bulk topological defects, see figure 2.4: we project a bulk topological defect onto the symmetry boundary in an L-shape, which results in a junction operator $\mathcal{E}_p^{(\boldsymbol{Q}_{q+1})}$ attached to a topological defect $D_{q+1} \in \mathcal{S}$. After interval compactification this is a

[1]Here we employ the standard nomenclature of “genuine” and “non-genuine” operators. A p dimensional non-genuine operator is one which is attached to a collection of $q > p$-dimensional operators. A genuine operator is one which is free from this type of attachment.

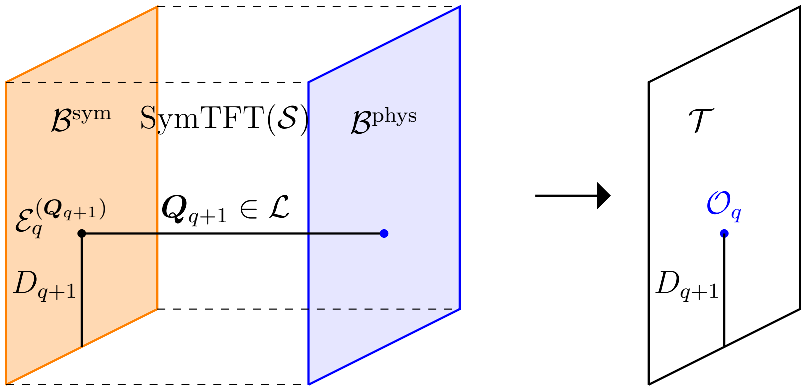


Figure 2.4: Here we consider the L-shape projection of a bulk topological defect $\boldsymbol{Q}_{q+1}$ onto the symmetry boundary. This creates a junction $\mathcal{E}_p$ which is attached to a topological defect D_{q+1}. After interval compactification we obtain a q-charge $\mathcal{O}_q$ attached to the topological defect D_{q+1}.

q-charge $\mathcal{O}_q$, attached to a topological $(q+1)$-dimensional defect D_{q+1}, which is thus a twisted sector operator.

Charges from Linking. Finally let's consider the action of symmetry defects on charges. This arises by computing the linking of bulk topological defects projected onto the symmetry boundary. There is the standard linking action of higher-form symmetry generators on defects, which follows from the linking in (2.4), where the mutually non-local defects are either Neumann or Dirichlet:

$$\text{Proj}(\boldsymbol{Q}^{\widehat{b}}_{d-p-1})(\partial \boldsymbol{Q}_{p+1}|_{\mathcal{B}^{\text{sym}}}) \longrightarrow D_{d-p-1}(\mathcal{O}_p) = q_{\mathcal{O}_p} \mathcal{O}_p \,, \tag{2.9}$$

where the arrow denotes the interval compactification, and q is the charge under the higher-form symmetry. This is shown in figure 2.5. The configuration shown here has various generalizations which were discussed in [68]. For the constructions in string theory, this is the most general setup we will require.

Non-invertible defects can also act by taking an operator in between genuine and twisted sectors. For example, passing the topological defect for a non-invertible symmetry through the end of the bulk topological defect, results in a twisted sector defect, as shown in figure 2.6. This action is well-known in various contexts of non-

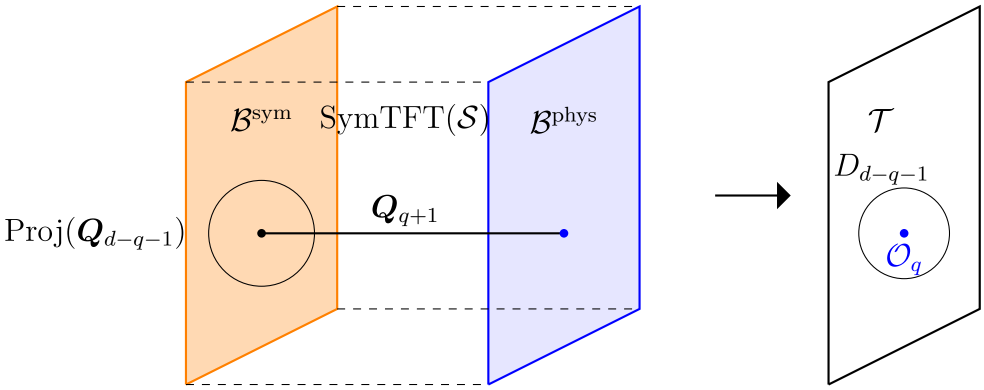


Figure 2.5: Here we demonstrate how a symmetry acts via linking. The topological operator $\boldsymbol{Q}_{p+1}$ ends and gives rise to the (genuine) q-charge $\mathcal{O}_q$ in $\mathcal{T}$. In turn, $\boldsymbol{Q}_{d-q-1}$ projects onto the symmetry boundary and gives rise to a symmetry generator after the interval compactification. The non-trivial linking of these topological defects in the SymTFT results in the generalized charge.

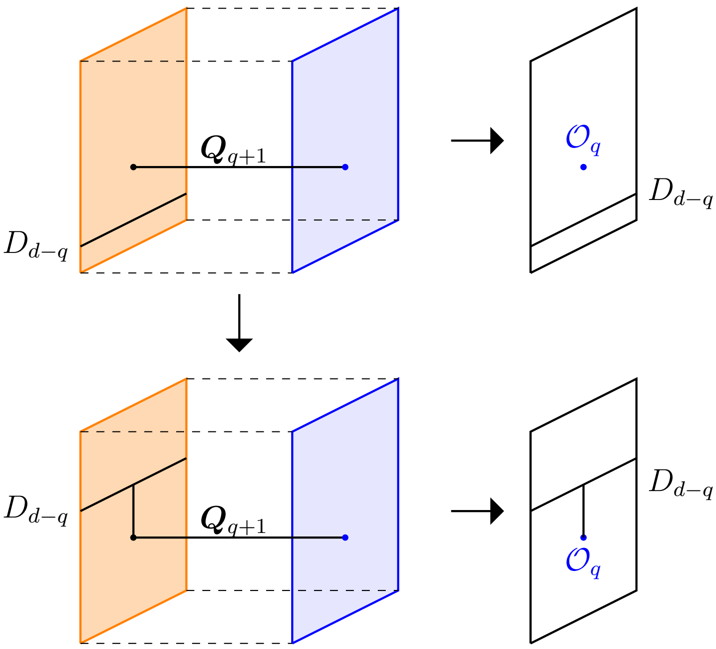


Figure 2.6: Here we demonstrate how a non-invertible symmetry can map a genuine operator to a non-genuine operator. The genuine defect $\mathcal{O}_q$ is acted upon by the topological defect D_{d-q}, which maps it to a non-genuine operator, with an attachment of a $(q+1)$-dimensional operator.

invertible symmetries in a variety of dimensions [23, 110] and was realized in terms of branes as Hanany-Witten moves in [41]. We will provide various generalizations of this in chapter 5.

The SymTFT is a particularly useful notion in the context of string theory since recent progress has shown that the SymTFT can be computed independently using geometric methods. Since their recent inception, generalized symmetries in string theory and related theories have thus been studied extensively[2].This research is most useful within the context of strong-coupled regimes of theories, either in geometric engineering or in holography.

2.2 SymTFTs in String Theory: Generalities

Let us first describe how a holographic or geometric engineering construction in string theory leads to a bulk topological description of the abelian and finite symmetry sector of a relative QFT living at the boundary. In most cases[3] this bulk topological description will play the role of the SymTFT, once suitable boundary conditions are imposed to make the theory absolute. In chapter 5 we incorporate branes that realize symmetry defects and generalized charges into this framework in a very general fashion.

Flux Sector of Supergravity. First of all we focus on the flux sector of 10/11-dimensional supergravity (depending on a string or M-theory starting point, respectively), and in particular on the Bianchi identities and equations of motion that the fluxes will satisfy. In chapter 5 we will also include brane sources, which magnetically charge branes. This is because the bulk gauge potentials will provide background

[2]For a sample list of references see [4, 5, 41–43, 46, 47, 49, 56, 57, 70, 111–150].

[3]In some cases from string theory we obtain a theory that is not a SymTFT, because there are no topological boundary conditions. In these instances one typically cannot have an absolute theory at the boundary. For instance, this is indeed the case for 6d $(2, 0)$ theories, which can be relative theories.

fields for the finite, abelian generalized symmetries. Their holonomies will also give rise to the topological operators defining the generalized symmetries. This information is equivalently encoded in the brane actions [41]. We will not consider the rest of the supergravity action which includes scalar fields, the metric and other modes. This is justified by the fact that we will be interested in the physics of flat discrete gauge fields (vanishing fluxes on-shell), which give non-trivial holonomies. The dilaton and the metric equations of motion will depend only on modes for which the fluxes are non-vanishing. The flux action of 10/11-dimensional supergravity has two pieces, a kinetic term and cubic Chern-Simons topological coupling, which can depend on one or more fluxes. It will be useful to adopt a formulation in which we include both the supergravity fluxes and their Hodge duals in 10/11 dimensions in a democratic way, as detailed below. However before presenting the democratic formulation let us describe the precise relation between the SymTFT and the bulk theory in holographic and geometric/brane engineering setups.

Dimensional Reduction of Flux Sector. The second step consists of dimensionally reducing the flux action with brane sources on the geometry dictated by the holographic description or the boundary at infinity of the geometric engineering setup. Concretely let

$$\partial \boldsymbol{X}_{\mathsf{D}+1} = L_{\mathsf{D}}\,, \tag{2.10}$$

where either the holographic solution is $M_{d+1} \times L_{\mathsf{D}}$ or the geometric engineering corresponds to a compactification on $\boldsymbol{X}_{\mathsf{D}+1}$. This is given by specifying the following geometric background, which is a solution of the supergravity equations of motion

$$M_{\mathsf{D}+d+1} = M_{d+1} \times L_{\mathsf{D}}\,, \tag{2.11}$$

where

$$\mathsf{D} + d + 1 = \begin{Bmatrix} 10 \\ 11 \end{Bmatrix} = D + 1\,, \tag{2.12}$$

depending on string or M-theory. The QFT is d-dimensional, and we start with a $D+1$ dimensional supergravity theory. In the resulting lower-dimensional theory we generically obtain an action that consists of kinetic terms as well as cubic Chern-Simons couplings. This theory is defined on M_{d+1}, which has a d-dimensional boundary. It can be AdS_{d+1} or more general spaces with a boundary which sits at infinity. In chapter 4 we derive a general expression for this reduction in the context of 3d QFTs constructed in M-theory.

Topological Limit. In this dimensionally reduced theory, we are interested in very long-distance regimes, which are realized very close to the boundary at infinity [151]. In this sense, we can implement a derivative expansion of the kinetic and topological couplings of the flux action. The lowest derivatives dominate, which usually consists of topological couplings when they are non-trivial. This is also valid for the dimensionally reduced brane action at very large-distances, where the kinetic terms obtained from expanding the DBI part of the action are subleading with respect to the topological couplings. In addition, the Wess-Zumino part will always provide topological couplings when non-trivial.

The main reason why we can truncate the dimensionally reduced bulk and brane action is that we really want to focus on finite, abelian, global symmetries. The gauge fields of these symmetries in the bulk, which are flat, have vanishing flux on-shell. The modes which we remove in the truncation do not couple to the symmetry sector described by flat fields, and therefore can be ignored. For instance, the kinetic term for the dimensionally reduced fluxes will be non-trivial only for the non-vanishing part of the fluxes and therefore can be ignored [151]. To reconcile the large-distance limit and the truncation to the flat finite abelian gauge fields, we can say that from the point of view of large-distances, i.e. the close to the boundary limit, the modes with non-vanishing flux are massive, and they can be integrated out, effectively leaving only the topological couplings describing the non-trivial fluctuation of flat fields and their non-trivial holonomies.

Finally, having truncated the dimensionally-reduced bulk and brane action to their topological sector, which describes the physics of finite abelian flat gauge fields, we can deform the space without changing its topology such as

$$M_{d+1} \to M_d \times \mathbb{R}\,. \tag{2.13}$$

SymTFT, Boundaries and Holography/String Theory. We can then connect this to the standard notion of SymTFT. The choice of absolute theory, i.e. a choice of polarization, will be implemented by partially compactifying the $\mathbb{R}$ direction, i.e the semi-infinite $[0,\infty)$ interval [65]

$$S_{\text{top}}|_{\text{polarization}} = S_{\text{SymTFT}}\,. \tag{2.14}$$

We can then think of the physical boundary (not necessarily gapped boundary) $\mathcal{B}^{\text{phys}}$ as placed at $r=0$. Likewise, the symmetry boundary, i.e. topological boundary condition, $\mathcal{B}^{\text{sym}}$, is at $r=\infty$.

The position of the two boundaries reflects what appears to be the position of the physical theory and the topological boundary, respectively, both in holography and string theory geometric engineering. The interpretation of the coordinate r and therefore of these precise positions, depends on the metric. For simplicity, for AdS we work in hyper-polar coordinates

$$ds^2(\text{AdS}_{d+1}) \sim r^2 ds^2(\mathbb{R}^{1,d}) + r^{-2}dr^2\,, \tag{2.15}$$

(the conformal boundary is located at $r=\infty$), while for geometric engineering setups we identify r with the real cone direction in the space $\boldsymbol{X}_{\mathsf{D}+1}$

$$ds^2_{\boldsymbol{X}_{\mathsf{D}+1}} = dr^2 + r^2 ds^2_{L_{\mathsf{D}}(\boldsymbol{X}_{\mathsf{D}+1})}\,, \tag{2.16}$$

with the link L_n ($r=0$ is the singularity at the tip of the cone).

In particular, in holographic setups we associate $\mathcal{B}^{\text{phys}}$ with the origin of AdS space ($r = 0$), and not with the conformal boundary ($r = \infty$). This might seem counter-intuitive, since the CFT lives on a spacetime isomorphic to the conformal boundary. Our perspective stems from the fact that the CFT is *dual* to the dynamics of the gravity theory in the bulk of AdS spacetime.

The presence of the conformal boundary on the gravity side teaches us that we need to supplement the bulk action with boundary conditions for the supergravity fields, in order to obtain a gravitational system that is holographically dual to a QFT. This is quantified in the following key relation of the AdS/CFT correspondence [152]

$$Z_{\text{sugra}}[(\text{b.c. for } \varphi) = J_\varphi] = \left\langle e^{\int \mathcal{O}_\varphi J_\varphi} \right\rangle_{\text{CFT}} , \tag{2.17}$$

where $\mathcal{O}_\varphi$ is the gauge-invariant operator dual to the supergravity field φ. In the string theory origin of holography, where we look at the near-horizon limit of some back-reacted brane system, the theory that we are describing is the one living on the stack of branes in some low-energy decoupling limit. Therefore we can practically consider the physical theory to live at $r = 0$ (radial position of the stack of branes) and the boundary of AdS to be at $r = \infty$. Once we truncate to the topological sector the latter becomes $\mathcal{B}^{\text{sym}}$. In geometric engineering, where the compactification space $\boldsymbol{X}_{n+1}$ is a real cone over a link $L_n(\boldsymbol{X}_{n+1})$, this works in a very similar way: $\mathcal{B}^{\text{phys}}$ is placed at $r = 0$ and $\mathcal{B}^{\text{sym}}$ at $r = \infty$.

The real difference between the SymTFT and holography/string theory is that in the latter we cannot really perform the partial compactification of the semi-interval direction between $[0, \infty)$. The main reason is that before truncating to the topological sector, we have gravity and other fields related to the full string theory construction in the bulk, as well as non-local excitations. All these are not necessarily related to the symmetry sector. In particular we cannot deform or compactify the space as we like. Indeed, in holography the geometry of the radial direction provides the non-trivial correspondence between the gravitational theory in the bulk and the QFT.

Rather, we have two different procedures to specify an absolute theory in string theory/holography, and in the SymTFT: in the former, we choose boundary conditions at $r = \infty$; in the latter, we perform the interval compactification. Heuristically, in string theory/holography (quantum) gravity mediates between the topological boundary and the choices of the boundary conditions at infinity with the theory living at $r = 0$ without the need of an actual interval compactification. Let us emphasize, however, that these two procedures can be connected to each other on the string/holography side, if we perform a truncation to the topological sector. Once the truncation is performed, we are indeed free to specify M_{d+1} as in (2.13) and establish a direct link with the SymTFT interval picture.

Singleton Theory. In many string-theory/holographic setups we have to deal with the center of mass degree of freedom. For instance we could consider the stack of branes before taking the near-horizon limit. In this case the theory that is realized on the brane-stack is an absolute theory (with $U(N)$ gauge group for N D-branes). On the gravity side, the $\mathfrak{u}(1)$ in $\mathfrak{u}(N) \cong \mathfrak{su}(N) \oplus \mathfrak{u}(1)$ is described by what is called the singleton mode [107]. This mode has been analyzed in detail in the case of the $\mathcal{N} = 4$ $\mathfrak{su}(N)$ SYM and its holographic construction. In particular, as it was shown by [153], this is a mode in the KK supergravity spectrum (entire supersymmetric multiplet, which contains the two forms B_2, C_2), which comes from an expansion in spherical harmonics of S^5 and satisfies specific conditions, that are different from the other bulk fields. This bulk multiplet is dual to a $U(1)$ gauge field in 4d (the center of mass of the stack of brane). The extra conditions on this bulk multiplet, which we do not repeat here, make the $U(1)$ gauge field pure gauge that is eaten by the combination of B_2, C_2 that has Dirichlet boundary conditions at $r = \infty$. In terms of bulk fields, the BF topological coupling between (B_2, C_2) can be seen as Stückelberg mechanism for the combination of (B_2, C_2) that becomes Dirichlet, in the spirit of [154]. In [154], the standard BF action involving B_2 and C_2 in the presence of a boundary generically gives a $U(1)$ gauge field living at the boundary. For example, giving the field B_2 Dirichlet boundary conditions forces it to become the field stength of the singleton gauge field on the boundary, whilst C_2 is the field strength for the electromagnetic dual. Therefore the singleton mode is pure gauge in the bulk and at the topological boundary, and it is not dual to any propagating physical mode of the $\mathcal{N} = 4$ $\mathfrak{su}(N)$ SYM. The near-horizon limit decouples the center of mass mode of a stack of branes by making it pure gauge and hence non-propagating in the bulk. The mode then localizes on the boundary at $r = \infty$. We expect this to generalize in the context discussed in this work, and it would be insightful to repeat this analysis in other contexts or to generalize it.

2.3 Symmetries from the Bulk in Holography

Having introduced a general framework above, let us first focus on holographic set-ups, where the strong-coupled regime of an SQFT is represented by string/M-theory on $\mathrm{AdS}_{d+1} \times X$ spacetime. Here, the SymTFT action can be found in the topological couplings in the bulk supergravity on AdS_{d+1} (or in more general holographic setups). The most well-studied example of $\mathrm{AdS}_5 \times S^5$ has the bulk coupling $N \int_{\mathrm{AdS}_5} B_2 \wedge dC_2$, which is an example a BF-coupling for the 1-form symmetries of the dual 4d gauge theories (with gauge algebra $\mathfrak{su}(N)$) [107, 151, 154]. In terms of the formulation as generalized symmetries and SymTFTs, there has been much recent interest in the holographic literature [1, 37, 41, 151, 155–161], in particular for $\mathrm{AdS}_4/\mathrm{CFT}_3$ in [157] and for 3d $\mathcal{N} = 6$ SCFTs of ABJM type [162]. The SymTFT also emerges using anomaly inflow methods for QFTs realized with branes [158, 163, 164].

Here we present a short example to demonstrate the salient points of this construction. In chapter 3 we will derive the following 5d bulk supergravity term

$$\frac{S_{\mathrm{BF}}}{2\pi} = \gcd(N, M) \int_{\mathcal{M}_5} b_2 \wedge d\mathcal{C}_2 \,, \tag{2.18}$$

for integers N, M and 2-form fields $b_2, \mathcal{C}_2$. The equations of motion force $b_2, \mathcal{C}_2$ to be flat gauge fields. The topological bulk operators $e^{i2\pi \oint b_2}$, $e^{i2\pi \oint \mathcal{C}_2}$ are mutually non-local due to the BF-action since the two composite fields are canonically conjugate [151]. Now suppose the following boundary conditions are chosen:

$$b_2 \text{ Dirichlet}\,, \quad \mathcal{C}_2 \text{ Neumann}\,. \tag{2.19}$$

Then $e^{i2\pi \oint \mathcal{C}_2}$ are the topological codimension-2 operators in 4d generating a $\mathbb{Z}_{\gcd(N,M)}$ 1-form symmetry with charged lines given by the operators $e^{i2\pi \oint b_2}$ restricted to the boundary. Alternative choices of boundary conditions correspond to different boundary global symmetries or, equivalently, different choices of global form of the boundary field theory gauge group.

2.4 Symmetries in Geometric Engineering

Brane constructions in string theory provide a large class of examples of anomaly theories. Ambient space gauge anomalies are cancelled by worldvolume 't Hooft anomalies via so-called 'anomaly inflow'. In particular, cutting out a neighbourhood around the branes, which act as sources of flux in the ambient string theory background, induces a boundary in the 10/11d geometry, rendering the full effective action no longer gauge invariant. In [158, 163] it was explained that these anomalies, described by a $(d+1)$-dimensional TFT or $(d+2)$-dimensional anomaly polynomial, can be obtained by dimensional reduction of the topological terms of the 10/11d effective action.

In string theory constructions without branes, the notion of inflow becomes less clear. However it was argued in [65] that for compactifications on a $(D-d)$-dimensional cone $\mathcal{C}(Y_{D-d-1})$ (with $D = 10, 11$), dimensional reduction on the link space Y_{D-d-1} remains a powerful tool in determining 't Hooft anomalies. The cases considered in [65] are 7d Yang-Mills and 5d SCFTs obtained from M-theory on singular Calabi-Yau spaces. The SymTFT is derived in both cases from dimensional reduction of the topological terms in the 11d supergravity action and is tested with non-trivial checks with known field theory computations for certain anomalies. For finite group symmetries, we must employ *differential cohomology* to capture the appropriate background fields. Prior applications of differential cohomology to string/M-theory have appeared in [65, 158, 165–168], and for a mathematical review see [169]. This technique is demonstrated in depth in chapter 4, including an introduction to differential cohomology in section 4.2.2

Background Fields from Cohomology. The general-purpose SymTFT introduced in (2.3) is formulated using background fields for the generalized symmetries. In supergravity, the origin of these fields is massless gauge fields. There are two sources of such fields: the reduction of the supergravity gauge potentials C_n on the cohomology of the internal space Y_{D-d-1}, and the gauging of isometries of the geometry. First, considering *continuous* symmetries (and specializing to $D = 11$): expanding the M-theory C_3 field on representatives of the free part of the cohomology $H^p_{\text{Free}}(Y_{10-d}; \mathbb{Z})$ gives rise to massless $(3-p)$-form gauge fields. Schematically, we write

$$G_4 = dc_3 + \sum_i dc_2^i \wedge \omega_1^i + \sum_j dc_1^j \wedge \omega_2^j + \sum_k dc_0^k \wedge \omega_3^k \,, \tag{2.20}$$

where subscripts denote form degrees and the forms ω_p are representatives of the free parts of the p^{th} cohomology group. Superscripts represent various components of the integral cohomology groups $H^p_{\text{Free}}(Y_{10-d}; \mathbb{Z})$. The massless q-form gauge fields c_q furnish background fields for *continuous* $(q-1)$-form symmetries when fixed on the boundary [4].

[4]In recent work [170] where the mathematical framework of SymTFTs has been fleshed out, the authors consider *finite* symmetries only. However here, and in our general SymTFT reduction in section 4.2 we allow for *continuous* global symmetries following the work of [65]. In cases where the continuous fields participate in BF terms, after a choice of consistent boundary conditions these become finite. Where the continuous fields do not participate in a BF term, the SymTFT terms they contribute to represent an additional invertible sector (the anomaly theory for these symmetries). Additionally, in the holographic contexts we consider, although continuous gauge fields admit non-topological kinetic terms, these are sub-leading at large distances (at the boundary where the field theory lives).

Torsional Contributions. An natural extension is to consider *finite* higher-form symmetries that arise from *torsional* contributions to the cohomology of Y_{10-d}. Manifesting the associated discrete background gauge fields requires a reduction of C_3 on torsional cocycles: a problem beyond the scope of ordinary differential forms [5]. This is where the framework of *differential cohomology* $\breve{H}(Y_{10-d})$ can be used to incorporate more general symmetry structures [65]. We include torsional contributions by lifting G_4 to differential cohomology and expanding as follows

$$\breve{G}_4 = \sum_\alpha \breve{B}_3^\alpha \star \breve{t}_1^\alpha + \sum_\beta \breve{B}_2^\beta \star \breve{t}_2^\beta + \sum_\gamma \breve{B}_1^\gamma \star \breve{t}_3^\gamma + \sum_\delta \breve{b}^\delta \star \breve{t}_4^\delta \,. \tag{2.21}$$

Here, $\breve{t}_p^\alpha$ are differential cohomology lifts of generators of $\mathrm{Tor} H^p(Y_{10-d};\mathbb{Z})$ of torsional degree $\ell_p^\alpha \in \mathbb{N}$. We leave a detailed explanation of this notation and technology for chapter 4. Here, we wish only to demonstrate that the notion of 'expanding G_4 in cohomology' is maintained. The fields $\breve{B}_q^\alpha$ represent background fields for $\mathbb{Z}_{\ell_p^\alpha}$ $(q-1)$–form symmetries. Crucially, including gauge fields of this new type allows for a whole new class of SymTFT couplings upon dimensional reduction. It is terms of this type in particular that we explore in this chapter 4.

See [41, 42, 44, 55, 56, 170, 173–178] for a selection of recent applications.

[5] Some attempts towards using standard harmonic forms were made in [171, 172].

Chapter 3

Symmetries in a Confining Theory

In this chapter we study confinement in 4d $\mathcal{N}=1$ $SU(N)$ Super-Yang Mills (SYM) from a holographic point of view, focusing on the 1-form symmetry and its relation to chiral symmetry breaking. In the 5d supergravity dual, obtained by truncation of the Klebanov-Strassler solution, we identify the topological couplings that determine the 1-form symmetry and its 't Hooft anomalies. One such coupling is a mixed 0-form/1-form symmetry anomaly closely related to chiral symmetry breaking in gapped confining vacua. From the dual gravity description we also identify the infra-red (IR) 4d topological field theory (TQFT), which realises chiral symmetry breaking and matches the mixed anomaly.

3.1 Introduction

Global symmetries and their 't Hooft anomalies can highly constrain the dynamics of gauge theories. A prime example is the role of the 1-form symmetry in confinement of $\mathcal{N}=1$ $SU(N)$ super Yang-Mills (SYM) or adjoint QCD theories. In this case the 1-form symmetry $\Gamma^{(1)}=\mathbb{Z}_N$ and corresponds to the center of the gauge group, which acts on line operators [7, 179] and provides a diagnostic of confinement. The order parameter for this symmetry is the vacuum expectation value (vev) of the Wilson line in the fundamental representation, which obeys area law in a confining vacuum. This implies that an infinitely extended Wilson line has vanishing vev, therefore preserving the 1-form symmetry. In addition, $\mathcal{N}=1$ $SU(N)$ SYM also has a 0-form R-symmetry $U(1)^{(0)}_R$. The Adler-Bell-Jackiw (ABJ) or chiral anomaly breaks $U(1)^{(0)}_R$ to $\Gamma^{(0)}=\mathbb{Z}_{2N}$, which by chiral symmetry breaking [180] ($\chi_{\rm SB}$) further breaks to $\mathbb{Z}_2$ in the confining phase

$$U(1) \quad \stackrel{\rm ABJ}{\longrightarrow} \quad \mathbb{Z}_{2N} \quad \stackrel{\chi_{\rm SB}}{\longrightarrow} \quad \mathbb{Z}_2\,. \tag{3.1}$$

There is a 0-/1-form symmetry mixed 't Hooft anomaly

$$\mathcal{A}[b_2, A] = 2\pi\, N^2 \int_{X_5} A\, b_2\, b_2\,, \tag{3.2}$$

where b_2 is the background for $\mathbb{Z}_N^{(1)}$ and A for $\Gamma^{(0)}$, which satisfy $\oint b_2 \in \frac{\mathbb{Z}}{N}$ and $\oint A \in \frac{\mathbb{Z}}{2N}$. This anomaly constrains the IR strongly coupled physics [7,181,182]. In a confining vacuum the 1-form symmetry is unbroken, and the 0-form background has to satisfy $\oint A \in \frac{\mathbb{Z}}{2}$ and $\Gamma^{(0)}$ is broken to $\Gamma^{(0)}=\mathbb{Z}_2$. This breaking indicates N distinct confining vacua, modelled by a gapped TQFT.

The goal of this chapter is to take a holographic perspective, from which we derive the 1-form symmetry and the mixed anomaly, as well as the TQFT describing the IR confining vacua. Higher-form symmetries in the AdS/CFT correspondence were discussed in [151,155,157,183,184]. Our focus here is on holography in a non-conformal setting, where the dual gauge theory is conjectured to be a confining theory related to $\mathcal{N}=1$ $SU(N)$ SYM [185–188]. Concretely, we develop the methods to determine the 1-form symmetry in the Klebanov-Strassler (KS) [64,189–193] solution. The central tool for our analysis is the consistent truncation of supergravity to 5d [194–196]. From this we determine a Stückelberg coupling for the R-symmetry which breaks it to a discrete subgroup as predicted by the ABJ anomaly in field theory, as well as 5d topological couplings from which we identify the 1-form symmetry and anomalies that will be central to $\chi_{\rm SB}$. Finally we show how the 5d supergravity reduction contains as boundary counterterms the 4d TQFT describing the IR vacua of $\mathcal{N}=1$

$SU(N)$ SYM. The approach proposed in this work has a vast number of generalizations, to holographic setups for confining theories, but also to geometric engineering constructions of confining theories e.g. [197]. It provides an exciting opportunity to revisit these setups, and sharpen the predictions, by taking the perspective based on higher-form symmetries.

3.1.1 Holographic Dual to Confinement

One of the most successful holographic realizations of $\mathcal{N} = 1$ $SU(N)$ SYM theory is the KS-solution [64]. This construction is realised in 10d IIB supergravity, and it consists of two main ingredients:

1. N D3-branes probing the conifold $C(T^{1,1})$, which is a conical Calabi-Yau with 5d link $T^{1,1}$, that is topologically $S^2 \times S^3$. The near-horizon of this brane system is $\text{AdS}_5 \times T^{1,1}$ with 5-form flux $\int_{T^{1,1}} F_5 = N$.

2. M D5-branes wrapping the $S^2 \subset T^{1,1}$.

The D5s backreact on the external geometry, modifying the AdS_5 metric. The solution at large radial distances, the KS-solution, is

$$\mathrm{d}s^2_{10} = \mathrm{d}s^2_{\mathcal{M}_5} + \mathcal{R}^2(r)\mathrm{d}s^2_{T^{1,1}}\,, \quad \mathcal{R}(r) \sim \ln\left(\frac{r}{r_s}\right)^{1/4}\,, \tag{3.3}$$

where $\mathrm{d}s^2_{\mathcal{M}_5} = \frac{r^2\mathrm{d}\vec{x}^2}{\mathcal{R}^2(r)} + \frac{\mathcal{R}^2(r)\mathrm{d}r^2}{r^2}$, and $r_s = r_0 e^{-\frac{2\pi N}{3g_s M^2} - \frac{1}{4}}$. At large r, the quantization of fluxes is

$$\begin{aligned} &\int_{S^3} F_3 = M\,, \quad \int_{S^2} B_2 = \mathcal{L}(r)\,, \\ &\int_{T^{1,1}} F_5 = \mathcal{K}(r) = N + M\mathcal{L}\,, \quad \mathcal{L} = \frac{3g_s M}{2\pi}\ln(r/r_0)\,, \end{aligned} \tag{3.4}$$

where r_0 is the UV scale, and we refer to this as the UV KS-solution, valid for r sufficiently large and $g_s\mathcal{K}(r) \gg 1$. Note that F_5 is no longer quantised: its integral over the internal space acquires a radial dependence. The solution has a naked singularity at $\mathcal{R}(r_s) = 0$, and in particular we can consider $r_s \to 0$, when $\frac{N}{M^2} \gg 1$. Due to the naked singularity at small radial distances $r \to r_s$, higher curvature corrections become relevant, and (3.3) is no longer valid. There is a smooth solution describing this regime, and it requires the full warped, deformed conifold solution [64].

Warped, Deformed Conifold. At small radial distances $r \to r_s$ of the KS-solution, higher curvature corrections cause the UV solution (3.3) to break down. This regime, which we call the IR KS-solution, is sensitive to the deformation of the conifold induced by the M D5-branes wrapping $S^2 \subset T^{1,1}$. The non-zero F_3 flux threading the S^3 prevents this cycle from shrinking to zero volume, whereas the S^2 collapses. Here the effective number of D3s is zero and the gauge theory dual is the IR regime of pure $\mathcal{N} = 1$ $SU(M)$ SYM. The warped, deformed conifold is parametrised by a new coordinate τ, which, at large τ, is related to r by $r^2 = 32^{-5/3}\epsilon^{4/3}e^{2\tau/3}$. Near $\tau \to 0$ the metric is approximately $\mathbb{R}^{3,1}$ times the deformed conifold [64]. For the sake of illustration we include the shrinking S^2 in the degenerate metric

$$\begin{aligned} ds_{10}^2 &= c_1\epsilon^{-4/3}(g_s M l_s^2)^{-1}\mathrm{d}\vec{x}^2 + c_2 g_s M l_s^2 ds_6^2\,, \\ ds_6^2 &= \frac{1}{2}d\tau^2 + \frac{1}{2}(g^5)^2 + (g^3)^2 + (g^4)^2 + \frac{1}{4}\tau^2[(g^1)^2 + (g^2)^2]\,, \end{aligned} \tag{3.5}$$

where $\{g^i\}$ are the standard basis of 1-forms on $T^{1,1}$ [198] and c_i are numerical constants [199]. For $g_s M \ll 1$ the curvatures are small everywhere, even in the far IR, such that the supergravity approximation is always reliable. At $\tau = 0$ the flux background has significant simplifications

$$F_5 = 0\,, \qquad H_3 = 0\,, \qquad F_3 = \frac{l_s^2 M}{2} g^5 \wedge g^3 \wedge g^4\,. \tag{3.6}$$

Dual Field Theory. The dual field theory description is given by $SU(N+M) \times SU(N)$ gauge theory and bifundamental matter in $(\mathbf{N}+\mathbf{M}, \overline{\mathbf{N}}) \oplus (\overline{\mathbf{N}+\mathbf{M}}, \mathbf{N})$, where a combination of the two gauge couplings has flown to strong coupling regimes. In particular, this theory is not conformal and the gauge couplings of the two factors run in opposite directions. E.g. when $SU(N+M)$ with $N_F = 2N$ becomes strongly coupled, we apply Seiberg duality [200], resulting in $SU(N-M)$ with $N_F = 2N$. This process perpetuates with the new gauge couplings flowing in opposite directions, giving rise to a 'duality cascade'. For $N = \kappa M$, $\kappa \in \mathbb{N}$, the endpoint is $\mathcal{N} = 1$ $SU(M)$ SYM at strong coupling.

The RG-flow of the gauge theory cascade is mirrored explicitly in the dual gravity background. Moving from large r to $r \to re^{-\frac{2\pi}{3g_sM}}$, $\int_{S^2} B_2$ and $\int_{T^{1,1}} F_5$ change by $\mathcal{L}(r) \to \mathcal{L}(r)-1$, $\mathcal{K}(r) \to \mathcal{K}(r)-M$. At the special slices with fixed $r = r_k = r_0 e^{-\frac{2\pi k}{3g_sM}}$, where $\mathcal{L}, \mathcal{K}$ are integer, the supergravity background is dual to the $SU(N-(k-1)M)\times SU(N-kM)$ gauge theory in the baryonic branch [201]. Alternatively we can work in terms of Page charges defined in [202], where $\widehat{F}_5 = F_5 - B_2 F_3$ is always integrally quantized $\int_{T^{1,1}} \widehat{F}_5 = N-kM$, due to large gauge transformations of $\int_{S^2} B_2 \to \mathcal{L}(r)+k$. For $N = \kappa M$ the endpoint is reached at a value r_κ where there are only M units of F_3-flux and no F_5-flux. In this regime, $r \sim r_\kappa$, the solution (3.3) breaks down before reaching the $r \to r_s$ limit, since $g_s\mathcal{K}(r_\kappa) = 0$, and the metric in (3.3) is not smooth. We therefore have the following hierarchy of scales $r_0 \gg r_k \gg r_\kappa > r_s$. The smooth supergravity solution for $r \leq r_\kappa$ is instead provided by the warped deformed conifold [64]. We denote this by the IR KS-solution, which describes the IR regimes of $\mathcal{N} = 1$ $SU(M)$ SYM.

3.2 1-Form Symmetries from Supergravity

The global form of the gauge group, or put differently, the set of mutually local line operators, can be determined in holography by considering boundary conditions (b.c.) of Chern-Simons-like couplings [151, 183, 184]. Put in a more modern language, the 2-form backgrounds for 1-form symmetries of the holographic field theory are determined by topological couplings in the bulk and specific b.c.s yield absolute theories (i.e. definite spectra of line operators). On the gauge theory side of the duality cascade, the $\mathbb{Z}_{N+M} \times \mathbb{Z}_N$ center symmetry of $SU(N+M)\times SU(N)$ is broken by the matter to

$$\Gamma^{(1)} = \mathbb{Z}_{\gcd(N,N+M)} = \mathbb{Z}_{\gcd(N,M)} \,. \tag{3.7}$$

This remains constant through each step of the cascade.

In order to derive the 1-form symmetry holographically we study fluctuations around the UV KS-solution, when r is sufficiently large and $g_s\mathcal{K}(r) \gg 1$, which describes each step of the cascade until $r \sim r_\kappa$. The latter corresponds to the end of the cascade and the smooth gravity dual is the warped deformed conifold, which we will investigate momentarily.

We reduce IIB supergravity on $T^{1,1}$ and study the topological couplings of the 5d 2-form gauge fields on this background. The strings which couple to these fields induce line operators on the r_k slices, which in turn furnish the 1-form symmetry of the boundary theory, also known as a 'singleton theory' [107,154]. In particular the 1-form symmetry is deduced from the bulk couplings by imposing a set of consistent b.c.s. In the continuous supergravity formalism, we obtain discrete higher-form symmetry groups by fixing subsets of the 2-form gauge potentials within subgroups of $U(1)$.

We derive the equations of motion of the 5d effective theory, obtained from compactifying 10d IIB supergravity on $T^{1,1}$. We isolate dominant topological couplings and determine an effective 5d action, which governs them. To identify the topological couplings, we expand field strengths F_q along $\omega_p \in H^p(T^{1,1},\mathbb{Z})$ as $F_q = \sum_p f_{q-p} \wedge \omega_p$, and insert these into the Type IIB equations

$$\begin{aligned} dH_3 &= 0\,, & d *_{10} H_3 &= -g_s^2 F_5 \wedge F_3\,, \\ dF_3 &= 0\,, & d *_{10} F_3 &= F_5 \wedge H_3\,, \\ dF_5 &= H_3 \wedge F_3\,, & *_{10} F_5 &= F_5\,. \end{aligned} \tag{3.8}$$

The couplings obtained in this way can equally be thought of as embedded into some consistent truncation (e.g. [194]). We will find the following topological term in the 5d reduction in the KS-solution

$$S_{\text{top}} = 2\pi \int_{\mathcal{M}_5} b_2 \wedge (N dc_2 - M da_2)\ . \tag{3.9}$$

In this section we focus on the UV regime: large $r \sim r_0$, dual to the top of the cascade where both cycles $S^2 \times S^3 \subset T^{1,1}$ are non-degenerate. We expand the fluctuations along the volume forms $\omega_{2,3} \in H^*(T^{1,1}, \mathbb{Z})$ (see e.g. [191] for conventions and an explicit parametrization)

$$\begin{aligned} \delta F_3 &= g_3 + \pi l_s^2 \omega_2 \wedge g_1 \,, \qquad \delta H_3 = h_3 \,, \\ \delta F_5 &= \pi l_s^2 \omega_2 \wedge f_3 + \frac{2\pi l_s^2}{6} \mathcal{R} \omega_3 \wedge * f_3 \,. \end{aligned} \tag{3.10}$$

Here, $h_3, g_{1,3}, f_3$ are all external fields and in this section we restore pre-factors for completeness. Self-duality of δF_5 implies a choice of frame: we can fix one expansion component in terms of the other. We use the 3-form piece, since the operators of the boundary 1-form symmetry are manifest in this frame. The Bianchi identities for H_3, F_3 imply that the corresponding 5d fields are closed, so we write $h_3 = db_2, g_3 = dc_2, g_1 = dc_0$. We interpret c_0 as an axion, whereas b_2, c_2 couple to F1s and D1s, respectively. The Bianchi identity for δF_5 implies that f_3 is not closed

$$df_3 = \mathrm{d}\mathcal{L} \wedge g_3 + h_3 \wedge g_1 \,. \tag{3.11}$$

As such, we shift the field to obtain closure and define a new gauge potential $da_2 = f_3 - \mathcal{L} dc_2 - b_2 dc_0$, which couples to D3s wrapping $S^2 \subset T^{1,1}$. The 5d equations of motion are

$$\begin{aligned} d(\mathcal{R} *_5 f_3) &= \frac{3}{2\pi} M db_2 \\ d(\mathcal{R}^5 *_5 db_2) &= -27\pi l_s^4 g_s^2 (\mathcal{K} dc_2 - M f_3 + \frac{2\pi}{3} \mathcal{R} *_5 f_3 \wedge dc_0) \\ d(\mathcal{R}^5 *_5 dc_2) &= 27\pi l_s^4 (\mathcal{K} db_2 + \frac{2\pi}{3} \mathcal{R} *_5 f_3 \wedge \mathrm{d}\mathcal{L}) \\ d(\mathcal{R} *_5 dc_0) &= -\mathcal{R} *_5 f_3 \wedge db_2 \,. \end{aligned} \tag{3.12}$$

From these equations of motion we extract leading topological contributions

$$N db_2 = 0 \,, \qquad M db_2 = 0 \,, \qquad N dc_2 - M da_2 = 0 \,, \tag{3.13}$$

where we ignore c_0, which can be gauged away via a Stückelberg mechanism. We re-package the leading contributions in terms of a field $\mathcal{C}$:

$$\gcd(N, M) d\mathcal{C} = 0 \,, \qquad \gcd(N, M) db_2 = 0 \,, \tag{3.14}$$

where $\mathcal{C} = q_1 c_2 - q_2 a_2$, with $\gcd(N,M)q_1 = N$, $\gcd(N,M)q_2 = M$, and whereby we decoupled the center of mass $U(1)^{(1)}$, corresponding to the 1-form symmetry of the collective motion of the D3s. The couplings are embedded into the consistent truncation of [194]. One can compare by varying their topological action, changing duality frame and restricting to the relevant fields. We find that the following topological term in the 5d supergravity reduction on the UV KS-solution at $r = r_k \gg r_\kappa$ dominates over higher derivative couplings,

$$S_{\text{5d}} \supset 2\pi \gcd(N,M) \int_{\mathcal{M}_5} b_2 \wedge d\mathcal{C} \,. \tag{3.15}$$

1-form symmetries are generated by *topological* surface operators [7], which here are $U_b(M_2) = e^{2\pi i \oint_{M_2} b_2}$ and $U_c(M_2) = e^{2\pi i \oint_{M_2} \mathcal{C}}$, where M_2 are closed surfaces, $\partial M_2 = \emptyset$. Generically, due to non-commutativity of fluxes, these do not commute [151]

$$U_b(M_2) U_c(N_2) = U_c(N_2) U_b(M_2) e^{\frac{2\pi i L(M_2,N_2)}{N}} \,, \tag{3.16}$$

where L is the linking of the surfaces. These *charge* operators generate a 1-form symmetry, which acts on *charged* line operators in the 4d field theory. We find these charged line operators by considering operators of the form $U_b(\Sigma)$ with $\partial\Sigma \subset \mathcal{M}_5|_{r_k}$. Similarly, the line operators $U_b(\Sigma)$ and $U_c(\Sigma)$ are not mutually local due to their linking. At each r_k slice, a maximal set of mutually local line operators corresponds to b.c.s of b_2 and $\mathcal{C}$.

A possible choice of b.c. for (3.15) is b_2 Dirichlet and $\mathcal{C}$ Neumann. Since $\mathcal{C}$ is free to vary at the boundary, U_c will correspond to the topological charge operator for the 1-form symmetry. By varying the topological action we find a condition $\gcd(N,M) b_2 \wedge \delta\mathcal{C}|_{r_k} = 0$, which forces b_2 to take fixed values at the boundary. This implies that we can define a flat connection b_1 in 4d taking values in $\mathbb{Z}_{\gcd(N,M)}$, i.e. $\gcd(N,M) b_2 = db_1 = 0$ at the slice $r = r_k$. Therefore, U_b restricted to $\partial\Sigma \subset \mathcal{M}_5|_{r_k}$ corresponds to the charged line operators of the field theory. As is well known, the fundamental strings, carrying world-volume b_2, ending on the boundary indeed give rise to Wilson lines in the 4d theory. They generate the 1-form symmetry $\Gamma^{(1)}$ of (3.7). The screening can equally be seen by considering the analog of the 'baryon vertex' [203] in this setup: integrating the Bianchi identities for D5s on $T^{1,1}$ and D3s on S^3 yield

$$\begin{aligned} \int_{T^{1,1}} dF_7 &= \int_{T^{1,1}} H_3 \wedge F_5 = (N - kM) H_3 \\ \int_{S^3} dF_5 &= \int_{S^3} H_3 \wedge F_3 = M H_3 \,. \end{aligned} \tag{3.17}$$

Thus D5s on $T^{1,1}$ provide the 'baryon vertex' that screens $N - kM$ F1s, while D3s on S^3 screen M F1s. In particular, $\gcd(N, M)$ F1s is the minimal configuration of strings that are screened. Alternative b.c.s can be studied[1], the IR will fix the one above.

From here onwards, we consider $N = \kappa M$, which allows us to connect to confinement. In this case the deformed conifold IR solution holographically describes the bottom of the cascade for the confining phase of SYM with $SU(M)$ simply-connected gauge group.

In [41] the authors carefully show that the supergravity solution contains bulk terms which can be used to describe both the $SU(M)$ and $PSU(M)$ global forms in the IR. In particular, they also derive the TQFT(s) which describe the IR of these scenarios.

3.3 Mixed Anomaly and χ_{SB} from Holography

All things are now in place to see holographically the mixed anomaly (3.2) and χ_{SB} (3.1). To do this, we need to study the rest of the topological couplings in the 5d bulk supergravity action. In particular we need to include the R-symmetry of the dual field theory, which is realized in terms of the $U(1)$-isometry (Reeb-vector) of the $T^{1,1}$ solution. This can be described by a $U(1)$ 1-form gauge field A, which enters the

[1]Note that there are other b.c. e.g. $\mathcal{C}$ Dirichlet/b_2 Neumann, which result in the same 1-form symmetry. For $\gcd(M, N) = pq$ it is also possible to consider mixed b.c. which yield $\Gamma^{(1)} = \mathbb{Z}_p \times \mathbb{Z}_q$.

metric of $T^{1,1}$ as $\mathrm{d}\beta \to \mathrm{d}\beta - A$, where β is the coordinate of the Hopf fiber of the S^3. The breaking by the ABJ anomaly to $\mathbb{Z}_{2M}$ is realized holographically by a Stückelberg coupling in the 5d consistent truncation. In addition, we argue that chiral symmetry breaking is consistent with the mixed 0-/1-form symmetry anomaly, which we derive from the 5d supergravity, and the KS-solution. In the IR we also derive the TQFT proposed in [7] which matches the mixed anomaly.

The additional 5d topological terms in the action are (see appendix A.1)

$$S_{5\mathrm{d}} \supset 2\pi \int \frac{\mathcal{R}}{2}|dc_0 + 2MA|^2 - M^2 b_2^2\, A + \frac{M}{2} b_2^2\, dc_0\,. \tag{3.18}$$

The first term is the kinetic term for the axion. Since it contains two derivatives it is subleading when evaluated on the UV KS-solution, when r is large, with respect to the topological terms. On the other hand its effect is important, since it realises the Stückelberg mechanism for the $U(1)_R$ gauge field A. The shift symmetry of the axion, $c_0 \sim c_0 + 2\pi$, is gauged by the $U(1)_R$ symmetry, so that the action is invariant under the non-linear transformation

$$A \to A + \mathrm{d}\alpha\,, \qquad c_0 \to c_0 - 2M\alpha\,. \tag{3.19}$$

We can use this symmetry to completely gauge away the axion, leaving only a mass term for the gauge field. Fixing $c_0 = 0$, there is still a residual discrete symmetry generated by $\alpha \in \frac{\pi}{M}\mathbb{Z}$. This is the direct way to identify the breaking of $U(1)_R \to \mathbb{Z}_{2M}^{(0)}$, as required by the ABJ anomaly.

The second term in (3.18) corresponds to the anomaly between the 0-form background A for $\mathbb{Z}_{2M}^{(0)}$, $\oint A \in \frac{\mathbb{Z}}{2M}$, and b_2 for $\mathbb{Z}_M^{(1)}$, $\oint b_2 \in \frac{\mathbb{Z}}{M}$

$$\mathcal{A}[b_2, A] = -2\pi M^2 \int_{\mathcal{M}_5} b_2\, b_2\, A\,, \tag{3.20}$$

which is a mixed 0-/1-form symmetry anomaly [2]. As expected it does not depend on the energy scale, and therefore this term will survive in the IR. In the IR we expect

[2] I.e. $b_2 \in H^2(\mathcal{M}_5, \mathbb{Z}_M)$ and $A \in H^1(\mathcal{M}_5, \mathbb{Z}_{2M})$, and using the cup-product instead of wedge

the theory to be dual to a confining vacuum of $SU(M)$ SYM, so the $\mathbb{Z}_M^{(1)}$ should be unbroken, and this gapped phase should be described by a 4d TQFT. Since A is a $\mathbb{Z}_{2M}$ background, (3.20) does not become integral in general. It was proven in [182] that unless this term is integral there cannot be a 4d TQFT with $\Gamma^{(0)} = \mathbb{Z}_{2M}$ and $\Gamma^{(1)} = \mathbb{Z}_M$ symmetries in the IR that saturates all the anomalies of the theory in the UV. On the other hand, integrality of (3.20) and an unbroken $\mathbb{Z}_M^{(1)}$ implies

$$\oint A \in \frac{\mathbb{Z}}{2} : \qquad \mathbb{Z}_{2M} \ \rightarrow \ \mathbb{Z}_2 \, , \tag{3.21}$$

implying chiral symmetry breaking in the IR vacuum of $SU(M)$ SYM. We stress that our analysis shows that the presence of this topological coupling in the UV KS supergravity solution is already preempting and consistent with the chiral symmetry breaking in the gapped confining vacuum with $\mathbb{Z}_2^{(0)}$ and $\mathbb{Z}_M^{(1)}$ symmetries.

3.4 4d IR TQFT from Holography

Finally, we now turn to the IR description of the theory. 5d supergravity contains topological terms leading to the IR 4d TQFT, which matches (3.20), and realises a spontaneous chiral symmetry breaking, $\mathbb{Z}_{2M} \ \rightarrow \ \mathbb{Z}_2$. From field theory, the IR theory that matches the anomalies of the UV gauge theory was proposed to be [7]

$$S_{\mathrm{TQFT}_{4\mathrm{d}}} = \int M\phi \left(dc_3 + \frac{M}{2} b_2^2 \right) = \int M\phi F_4 \, . \tag{3.22}$$

The M vacua, labelled by $\langle e^{i\phi} \rangle = e^{\frac{2\pi i \ell}{M}}$, $\ell = 0, 1, \ldots M-1$, are separated by domain walls (DWs) [204], $e^{i\oint c_3}$.

The smooth IR gravity dual background is the deformed conifold solution, where $\tau \to 0$. In this regime there is no hierarchy between the 5d bulk kinetic and topological terms, and the former need to be taken into account. Before the S^2 degenerates, the D5s source $C_6 = \omega_3 \wedge c_3$, and in addition since $F_7 = *F_3$ we consider $dc_3 = \frac{\mathcal{R}}{2} *_5 (dc_0 + 2MA)$. Therefore, in the IR the dynamics of c_0 becomes relevant. Since

A corresponds to the true isometry of the IR KS-solution, that is a $\mathbb{Z}_2$ gauge field, c_0 does not shift under a gauge transformation of A. The 5d IR topological action becomes

$$S'_{\rm 5d} \supset 2\pi \int 2MAdc_3 + dc_0 dc_3 + \frac{M}{2} b_2^2 \, dc_0 \,. \qquad (3.23)$$

We first notice that the mixed anomaly between $\Gamma^{(0)} = \mathbb{Z}_{2M}$ and $\Gamma^{(1)} = \mathbb{Z}_M$ has disappeared. This is due to an additional topological term $|c^{\Omega}(r)|^2 b_2^2 A$ [194, 205, 206], which for the UV KS-solution depends on the UV scale r_0, but is constant in the IR $c^{\Omega} = M$. This is consistent with anomaly matching since the IR theory has $\Gamma^{(0)} = \mathbb{Z}_2$, which is not anomalous on spin manifolds. The third term is a total derivative, and varying by c_0 implies $Mdb_2 = 0$. When this condition is satisfied, the last two terms give rise to topological counterterms for the 4d theory living at the boundary. This implies that they are not anomalies, but rather the imprint of the TQFT (3.22) in the IR, which is precisely obtained by identifying $c_0 \leftrightarrow M\phi$ and evaluating these terms at the boundary. In particular, c_0 is related to the presence of DWs given by D5s wrapping S^3. These source $\int_B F_3$, where B is the Poincaré dual cycle in the deformed conifold with S^2 boundary at infinity. This entails that $\int_B F_3 \sim \int_{S^2} c_0 \omega_2$, and because of the presence of the D5 DWs, $c_0 = \ell$ is quantized and corresponds to the number of D5s. Each vacua is labelled by $\langle e^{i\phi} \rangle = e^{\frac{2\pi i \ell}{M}}$, $\ell = 0, 1, \dots M-1$. Therefore, under the identification $c_0 \leftrightarrow M\phi$, we observe the above result in each vacua. The UV anomaly (3.20) in the IR is realised by the action of $\Gamma^{(0)} = \mathbb{Z}_{2M}$, $\ell \to \ell + 1$, which is however not a symmetry of the IR vacuum.

The IR theory proposed in [7] is furthermore invariant under 1-form symmetry transformation $B_2 \to B_2 + d\lambda$, and this implies that $c_3 \to c_3 - NB_2\lambda - \frac{N}{2}\lambda d\lambda$. The transformation of the 1-form symmetry enters in the shift of c_3, which signals the presence of a 3-group [7, 207]. This is again supported by the string theory realisation of these DWs in terms of D5s wrapped on S^3 in the deformed conifold. The CS-action of the D5 is $\mathcal{L}_{CS} = \sum_p C_p \wedge e^{-B}$. The DWs extend in the 4d spacetime such that

$F_4 = \int_{S^3} d\mathcal{L}_{CS} = dc_3 + \frac{M}{2} B_2^2$, the 3-group follows from the gauge invariance of the world-volume action of the D5 and it is analogously consistent with gauge invariance of (3.23).

Chapter 4

Symmetry TFTs for 3d QFTs from M-Theory

In this chapter we derive the Symmetry Topological Field Theories for 3d supersymmetric quantum field theories constructed in M-theory either via geometric engineering or holography. These 4d SymTFTs encode the symmetry structures of the 3d QFTs, for instance the generalized global symmetries and their 't Hooft anomalies. Using differential cohomology, we derive the SymTFT by reducing M-theory on a 7-manifold Y_7, which either is the link of a conical Calabi-Yau four-fold or part of an $\mathrm{AdS}_4 \times Y_7$ holographic solution. In the holographic setting we first consider the 3d $\mathcal{N} = 6$ ABJ(M) theories and derive the BF-couplings, which allow the identification of the global form of the gauge group, as well as 1-form symmetry anomalies. Secondly, we compute the SymTFT for 3d $\mathcal{N} = 2$ quiver gauge theories whose holographic duals are based on Sasaki-Einstein 7-manifolds of type $Y_7 = Y^{p,k}(\mathbb{CP}^2)$. The SymTFT encodes 0- and 1-form symmetries, as well as potential 't Hooft anomalies between these. Furthermore, by studying the gapped boundary conditions of the SymTFT we constrain the allowed choices for $U(1)$ Chern-Simons terms in the dual field theory.

4.1 Introduction

The goal of this chapter is to determine the SymTFT for 3d QFTs which either have a realization as geometric engineering in M-theory on an 8-manifold, or in terms of $\text{AdS}_4/\text{CFT}_3$ holographic setups in M-theory. These two constructions are closely related and we provide a systematic computational approach to determining the SymTFT in both cases. The main focus will be on conical 8-manifolds (with special holonomy) $\mathcal{X}_8 = \mathcal{C}(Y_7)$ in setups with and without branes. Using differential cohomology in the supergravity reduction allows us to take into account the effects of torsion in the homology of Y_7, which is associated with a new set of background fields for finite higher-form symmetries.

For Y_7 a Sasakian 7-manifold we provide a prescription for computing the SymTFT coefficients explicitly, which correspond to secondary invariants in differential cohomology, from the intersection theory in the non-compact complex 4-fold $\mathcal{X}_8$ We give detailed examples when the cone $\mathcal{X}_8$ is toric, in particular for $\mathcal{X}_8 = \mathbb{C}^4/\mathbb{Z}_k$ and $\mathcal{X}_8 = \mathcal{C}(Y^{p,k}(\mathbb{CP}^2))$, where combinatorial formulas for intersection numbers can be explicitly computed. As such, we explain how physical anomaly coefficients and BF-terms are encoded in the geometric information of the toric diagram. In summary, we will derive the SymTFT and give a procedure for computing the coefficients for

1. Geometric engineering: M-theory on a singular, non-compact Calabi-Yau 4-fold $\mathcal{X}_8 = \mathcal{C}(Y_7)$, i.e. Y_7 is a Sasaki-Einstein 7-manifold.

2. Holography: $\text{AdS}_4 \times Y_7$ solutions of M-theory, which are dual to M2-branes probing $\mathcal{X}_8 = \mathcal{C}(Y_7)$, where Y_7 is a Sasakian 7-manifold (Sasaki-Einstein when $\mathcal{X}_8$ is a Calabi-Yau 4-fold).

For concrete applications, we will mostly focus on the holographic setups, leaving the exploration of geometrically engineered 3d QFTs for future work. We first compute the SymTFT in the M-theory models dual to ABJM and ABJ theories. This relatively simple holographic setup is well-suited to demonstrate these new refined geometric methods while, at the same time, allowing for a match with known results from type IIA [157] in the case where discrete background torsional flux is turned off. Finally, we apply this machinery in a much more subtle (and not completely fixed) duality of 3d $\mathcal{N}=2$ theories realized on M2-branes probing $\mathcal{C}(Y^{p,k}(\mathbb{CP}^2))$ [208–210]. By computing the SymTFT from the geometry, we obtain previously unknown anomalies for these theories. Furthermore, we will see that analysing consistent gapped boundaries of the SymTFT provides some further checks and balances to the proposed dictionary, coming from the spectrum of extended operators.

Generalized symmetries and their 't Hooft anomalies have a rich structure that has been studied field-theoretically from various angles in e.g. in [137,211–214]. Some of these results will be used later on to cross-check against our string theoretic results.

The structure of this chapter is as follows: In section 4.2 we provide some background on differential cohomology and compute a general expression for the SymTFT for 3d QFTs which can be constructed from M-theory on $\mathcal{X}_8 = \mathcal{C}(Y_7)$ with and without branes. We then explain how to compute the coefficients in the SymTFT in section 4.3, in particular in the case of toric $\mathcal{X}_8$. In section 4.4 we apply the above technology to our first example: the 3d $\mathcal{N}=6$ $((U(N+b)_k \times U(N)_{-k})/\mathbb{Z}_m$ ABJ(M) theories [162, 215, 216]. We next apply our technology to the $Y^{p,k}(\mathbb{CP}^2)$ 3d $\mathcal{N}=2$ quiver gauge theories of [208–210] in section 4.5. In section 4.6 we discuss matching with field theory results of [210]. Finally, in section 4.7 we highlight various possible future directions. We also provide some appendices. In appendix B.1 we use type IIA to conjecture the existence of an additional BF-term in the $Y^{p,k}$ case.

4.2 SymTFT from M-Theory on Y_7

We derive the SymTFT of any 3d QFT that arises in M-theory, either as compactification on $\mathbb{R}^{1,2} \times \mathcal{C}(Y_7)$, or holographically dual to $\text{AdS}_4 \times Y_7$. This is achieved by reducing the topological terms of 11d supergravity on both the free and torsional parts of the cohomology of Y_7. A caveat in this analysis is that the symmetries we will capture from this approach need to be manifest within the geometric realization. We focus our main attention on the dimensional reduction using differential cohomology.

4.2.1 Reduction using the Free Part of Cohomology

Let us start by performing the reduction of M-theory on $\mathcal{M}_{11} = \mathcal{M}_4 \times Y_7$, using only the free part of the cohomology $H^p_{\text{Free}}(Y_7;\mathbb{Z})$ which gives rise to continuous gauge fields in the effective 4d theory. As discussed above, these massless modes are obtained by a Kaluza-Klein expansion of the 4-form flux G_4 on representatives of the cohomology of the internal space with integral periods. Their topological couplings arise from the 11d supergravity term

$$\frac{S_{\text{11d}}}{2\pi} = \int_{\mathcal{M}_{11}} \left[-\frac{1}{6} C_3 \wedge G_4 \wedge G_4 - C_3 \wedge X_8 \right] . \tag{4.1}$$

The 8-form characteristic class X_8 is constructed from the Pontryagin classes of the tangent bundle

$$X_8 = \frac{1}{192} \left(p_1(T\mathcal{M}_{11}) \wedge p_1(T\mathcal{M}_{11}) - 4p_2(T\mathcal{M}_{11}) \right) . \tag{4.2}$$

To derive the 4d topological couplings we consider the gauge invariant 5-form I_5, on an auxiliary 5d space, which is the derivative of the 4d topological Lagrangian

$$I_5 = dI_4 \,, \quad S_{4d} = 2\pi \int_{\mathcal{M}_4} I_4 \,. \tag{4.3}$$

We identify I_5 as

$$I_5 = \int_{Y_7} I_{12} = \int_{Y_7} \left(-\frac{1}{6} G_4 \wedge G_4 \wedge G_4 - G_4 \wedge X_8 \right) . \tag{4.4}$$

Assuming Y_7 is connected, the betti numbers $b^r(Y_7) = \text{dim} H^r(Y_7, \mathbb{R})$ satisfy $b^i(Y_7) = b^{7-i}(Y_7)$. We denote the associated closed p-forms by

$$\omega_p^i \,, \qquad p = 0, \ldots, 7 \,, \qquad i = 0, \ldots, b^p(Y_7) \,, \tag{4.5}$$

with $\omega_0 \equiv 1$. We expand the 4-form flux using these forms

$$G_4 = \sum_{p=0}^{4} \sum_{i=0}^{b^p(Y_7)} g^i_{4-p} \wedge \omega^i_p \,. \tag{4.6}$$

When considering particular solutions it will be convenient to have separated the background $G_4^{\rm bg}$ supporting the vacuum from the dynamical fluctuations G'_4 around the solution:

$$G_4 = G'_4 + G_4^{\rm bg} \,. \tag{4.7}$$

Imposing the Bianchi identity, we find that g^i_q can locally be written as

$$g^i_0 \equiv \mathcal{N}^i \,, \qquad g^i_q = dc^i_{q-1} \,, \quad q = 1,2,3 \,, \qquad g_4 = dc_3 + \mathcal{L}\mathrm{vol}_{\mathcal{M}_4} \,, \tag{4.8}$$

with the background parametrised by

$$\int_{\mathcal{C}^i} G_4^{\rm bg} = \mathcal{N}^i \in \mathbb{Z} \,, \qquad \int_{\mathcal{M}_4} G_4^{\rm bg} = \mathcal{L} \in \mathbb{Z} \,, \tag{4.9}$$

where $\mathcal{C}^i$ is a basis of 4-cycles in Y_7. It is important to note that in this expression we are including all possible terms, however we will never have non-trivial $\mathcal{L}$ and $\mathcal{N}$ concurrently, as this would amount to imposing quantization conditions on G_4 and G_7 simultaneously. For all examples in this work, we will consider non-trivial $\mathcal{L}$ only. We can therefore write the fluctuations

$$G'_4 = \sum_{p=0}^{3} \sum_{i=0}^{b^p(Y_7)} dc^i_{3-p} \wedge \omega^i_p \,, \tag{4.10}$$

and background

$$G_4^{\rm bg} = \mathcal{L}\mathrm{vol}_{\mathcal{M}_4} + \sum_{i=0}^{b^4(Y_7)} \mathcal{N}^i \omega^i_4 \,. \tag{4.11}$$

In the reduction of the CS-term G_4^3, the background flux over the external space will contribute metric-dependent terms (which belong to the scalar potential) that we neglect. Performing the reduction we find

$$\int_{Y_7} -\frac{1}{6} G_4^3 = \sum_{ijk} \left(-\frac{1}{2} K^{ijk} dc^i_2 \wedge dc^j_0 \wedge dc^k_0 + \frac{1}{2} \mathcal{K}^{ijk} dc^i_1 \wedge dc^j_1 \wedge dc^k_0 + \mathbb{K}^{ijk} \mathcal{N}^i dc^j_1 \wedge dc^k_2 \right) \tag{4.12}$$

where the intersection numbers are given by

$$\begin{aligned} K^{ijk} &= \int_{Y_7} \omega_1^i \wedge \omega_3^j \wedge \omega_3^k \,, & \mathcal{K}^{ijk} &= \int_{Y_7} \omega_2^i \wedge \omega_2^j \wedge \omega_3^k \,, \\ \mathfrak{K}^{ij} &= \int_{Y_7} \omega_4^i \wedge \omega_3^j \,, & \mathbb{K}^{ijk} &= \int_{Y_7} \omega_4^i \wedge \omega_2^j \wedge \omega_1^k \,. \end{aligned} \tag{4.13}$$

To factorise the characteristic class X_8, we can employ the Whitney sum formula for Pontryagin classes defined on a product manifold [217]. Assuming the external space $\mathcal{M}_4$ is orientable and spin, we obtain

$$\begin{aligned} p_1(T\mathcal{M}_{11}) &= p_1(T\mathcal{M}_4) + p_1(TY_7) \,, \\ p_2(T\mathcal{M}_{11}) &= p_2(T\mathcal{M}_4) + p_2(TY_7) + p_1(T\mathcal{M}_4) \smile p_1(TY_7) \,. \end{aligned} \tag{4.14}$$

The second Pontryagin classes vanish on dimensional grounds. We can therefore write the 8-form characteristic class

$$X_8 = -\frac{1}{96} p_1(T\mathcal{M}_4) \smile p_1(TY_7) \,. \tag{4.15}$$

Together with the expansion (4.6) we find

$$-\int_{Y_7} G_4 \wedge X_8 = -\frac{1}{96} \sum_{i=1}^{b^3(Y_7)} \left[\int_{Y_7} \omega_3^i \wedge \; p_1(TY_7) \right] dc_0^i \wedge p_1(T\mathcal{M}_4) \,. \tag{4.16}$$

Defining

$$C^i = \frac{1}{96} \int_{Y_7} \omega_3^i \wedge \; p_1(TY_7) \,, \tag{4.17}$$

the gauge invariant 5-form is

$$\begin{aligned} I_5 = & \sum_{ijk} \left(-\frac{1}{2} K^{ijk} dc_2^i \wedge dc_0^j \wedge dc_0^k + \frac{1}{2} \mathcal{K}^{ijk} dc_1^i \wedge dc_1^j \wedge dc_0^k + \mathbb{K}^{ijk} \mathcal{N}^i dc_1^j \wedge dc_2^k \right) \\ & + \sum_{ij} \mathfrak{K}^{ij} \mathcal{N}^i dc_0^j \wedge dc_3 - \sum_i C^i dc_0^i \wedge p_1(T\mathcal{M}_4) \,. \end{aligned} \tag{4.18}$$

Acting with an anti-derivative, we find

$$\begin{aligned} I_4 = & \sum_{ijk} \left(\frac{1}{2} K^{ijk} dc_2^i \wedge c_0^j \wedge dc_0^k + \frac{1}{2} \mathcal{K}^{ijk} dc_1^i \wedge c_1^j \wedge dc_0^k + \mathbb{K}^{ijk} \mathcal{N}^i c_1^j \wedge dc_2^k \right) \\ & - \sum_{ij} \mathfrak{K}^{ij} \mathcal{N}^i dc_0^j \wedge c_3 - \sum_i C^i c_0^i p_1(T\mathcal{M}_4) \,. \end{aligned} \tag{4.19}$$

Notice in particular the single derivative terms

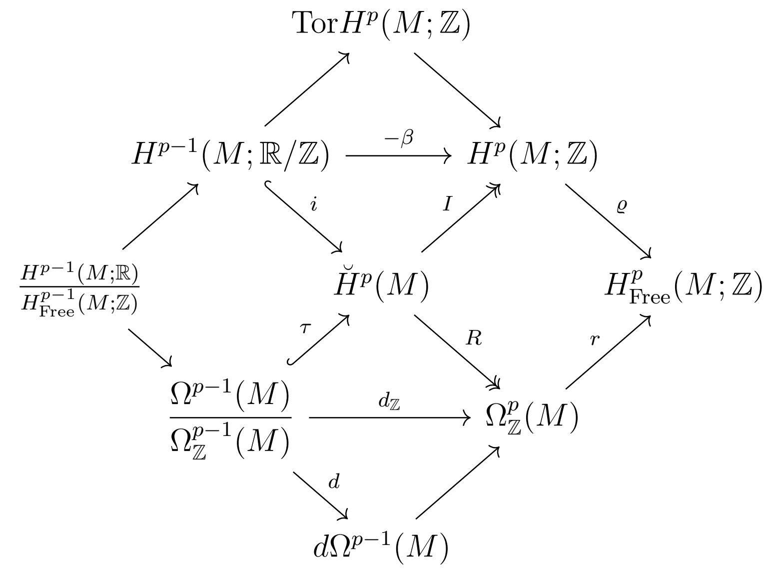
$$I_4 \supset \sum_{ijk} \mathbb{K}^{ijk} \mathcal{N}^i c_1^j \wedge dc_2^k - \sum_{ij} \mathfrak{K}^{ij} \mathcal{N}^i dc_0^j \wedge c_3 \,. \tag{4.20}$$

Such BF-terms constrain the possible boundary conditions that can be imposed on the pairs (c_1^j, c_2^k) and (c_0^j, c_3), which in turn dictates the global symmetries of the resulting field theory. For example, if $\mathbb{K}^{xyz}\mathcal{N}^x \equiv m \neq 0$, giving c_1^y Neumann (free) boundary conditions implies that c_2^z must be fixed to a background value in $\mathbb{Z}_m$ in the boundary theory, giving rise to a $\mathbb{Z}_m^{(1)}$ global 1-form symmetry in the 3d theory. Exchanging the boundary conditions corresponds to gauging the full $\mathbb{Z}_m$ 1-form symmetry, and we obtain a $\mathbb{Z}_m$ 0-form symmetry instead. After a choice of boundary conditions consistent with the BF terms, the other terms in (4.18)/(4.19) give rise to mixed anomalies between the resulting finite higher-form symmetries.

4.2.2 Review of Differential Cohomology

In [65] it was shown, by employing a description in terms of *differential cohomology*, that torsion in $H^p(Y_7;\mathbb{Z})$ may give rise to additional couplings in the SymTFT. In this section we recap the introduction of [65] on differential cohomology in order to introduce both the notation and some of the mathematical machinery we use throughout this work. For further mathematical details and implementations of differential cohomology in string/M-theory see e.g. [65, 165–169].

Differential cohomology combines information about the characteristic class of the gauge bundle and the connection. The p^{th} differential cohomology group $\breve{H}^p(M)$ of an n-dimensional manifold M is a differential refinement of the ordinary integral cohomology group $H^p(M;\mathbb{Z})$. Denote by Ω^p closed p-forms, and by $\Omega^p_{\mathbb{Z}}$ the subset of those with integral periods. The differential cohomology class takes part in the commutative diagram, whose diagonals are all short exact sequences [1]:



(4.22)

Differential cohomology is endowed with a bilinear product

$$\star : \breve{H}^p(M) \times \breve{H}^q(M) \to \breve{H}^{p+q}(M)\,, \tag{4.23}$$

with the properties

$$\breve{a} \star \breve{b} = (-1)^{pq} \breve{b} \star \breve{a}\,, \qquad I(\breve{a} \star \breve{b}) = I(\breve{a}) \smile I(\breve{b})\,, \qquad R(\breve{a} \star \breve{b}) = R(\breve{a}) \wedge R(\breve{b})\,, \tag{4.24}$$

for $\breve{a} \in \breve{H}^p(M)$ and $\breve{b} \in \breve{H}^q(M)$. It has two non-trivial integration maps, namely:

- The *primary invariant* of a differential cohomology class of degree $n = \dim(M)$

$$\int_M \breve{a} = \int_M I(\breve{a}) = \int_M R(\breve{a}) \in \mathbb{Z}\,, \qquad \breve{a} \in \breve{H}^n(M)\,, \tag{4.25}$$

- The *secondary invariant* of a differential cohomology class of degree $n+1$ (see e.g. [218])

$$\int_M \breve{a} = \int_M w \bmod 1 = \int_M u \in \mathbb{R}/\mathbb{Z}\,, \qquad \breve{a} \in \breve{H}^{n+1}(M)\,, \tag{4.26}$$

[1] An alternative way of describing differential cohomology is as follows. Differential cohomology is useful to describe the non-trivial topological structure of higher-form gauge fields. A representative $\breve{A}$ of a class $[\breve{A}] \in \breve{H}^p(M)$ is specified by a tuple [168]

$$\breve{A} = (N, A, F)\,. \tag{4.21}$$

Here F is the field strength and is a closed $(p+1)$-form. A and N are maps from $C_p(M)$, the space of p-chains, to $\mathbb{R}$ and $\mathbb{Z}$ respectively. They encode holonomies and the non-trivial interplay between these holonomies and the field strength. See [168] for more information on differential cohomology phrased in this way.

with

$$\tau(w) = \breve{a}\,, \quad w \in \frac{\Omega^n(M)}{\Omega^n_{\mathbb{Z}}(M)}\,, \qquad \text{and} \qquad \breve{a} = i(u)\,, \quad u \in H^n(M;\mathbb{R}/\mathbb{Z})\,. \quad (4.27)$$

In the setting of M-theory, we can write the action (4.1) as the secondary invariant of a class $\breve{I}_{12} \in \breve{H}^{12}(\mathcal{M}_{11})$,

$$\frac{S}{2\pi} = \int_{\mathcal{M}_{11}} \breve{I}_{12} \bmod 1\,, \quad (4.28)$$

where

$$\breve{I}_{12} = -\frac{1}{6}\breve{G}_4 \star \breve{G}_4 \star \breve{G}_4 - \breve{G}_4 \star \breve{X}_8\,, \quad (4.29)$$

with $\breve{G}_4 \in \breve{H}^4(\mathcal{M}_{11})$ and $\breve{X}_8 \in \breve{H}^8(\mathcal{M}_{11})$.

For a given 7-manifold Y_7, the generators of $H^p(Y_7;\mathbb{Z})$, $p = 0,\ldots,7$ are denoted as follows:

- free generators of $H^p(Y_7;\mathbb{Z})$: $r(\omega^i_p) \equiv v^i_p$, $i = 1,\ldots,b^p(Y_7)$ with $\omega^i_p \in \Omega^p_{\mathbb{Z}}(Y_7)$,
- torsion generators of $H^p(Y_7;\mathbb{Z})$: t^α_p, $\alpha \in A_p$ for some set of superscripts A_p.

For each torsion generator, there exists a minimal positive number $\ell^\alpha_p \in \mathbb{N}$, such that

$$\ell^\alpha_p t^\alpha_p = 0\,. \quad (4.30)$$

We will be particularly interested in the secondary invariant of $\breve{I}_{12}$ on a product space, which is the compactification space of M-theory. For a set of differential cohomology classes $\breve{a} \in H^p(\mathcal{M}_4)$, $\breve{b} \in H^q(Y_7)$ with $p + q = 12$ we have

$$\int_{\mathcal{M}_4\times Y_7} \breve{a} \star \breve{b} = \begin{cases} \left(\int_{\mathcal{M}_4} u\right)\left(\int_{Y_7} R(\breve{b})\right) & \text{if} \quad p = 5 \\ \left(\int_{\mathcal{M}_4} R(\breve{a})\right)\left(\int_{Y_7} s\right) & \text{if} \quad p = 4 \\ 0 & \text{otherwise} \end{cases} \quad (4.31)$$

where

$$i(u) = \breve{a}\,, \qquad i(s) = \breve{b}\,. \quad (4.32)$$

Now, we can choose the differential cohomology uplifts of the torsion generators $\breve{t}$ to be flat [65]

$$R(\breve{t}) = 0\,, \tag{4.33}$$

which implies that for terms in $\breve{I}_{12}$ involving the torsion generators $\breve{t}$, only those with 8 internal components will contribute (i.e. with $p = 4$ in (4.31))

4.2.3 Accounting for Torsion using Differential Cohomology

In this section we expand the differential refinement of G_4, $\breve{G}_4 \in \breve{H}^4(\mathcal{M}_{11})$, on the product space $\mathcal{M}_{11} = \mathcal{M}_4 \times Y_7$ and derive the topological sector of the effective 4d supergravity theory, including torsion contributions.

We will take $\mathcal{M}_4$ to be connected, so $H^0(\mathcal{M}_4;\mathbb{Z}) = \mathbb{Z}$, and assume vanishing torsion $\mathrm{Tor}H^\bullet(\mathcal{M}_4;\mathbb{Z}) = 0$. We will furthermore assume that Y_7 is closed, connected and orientable, so that [219]

$$H^0(Y_7;\mathbb{Z}) = \mathbb{Z}\,, \qquad \mathrm{Tor}H^1(Y_7;\mathbb{Z}) = 0\,. \tag{4.34}$$

Thus, we take $v_0 \equiv 1$ as the generator of $H^0(Y_7)$. We can expand the ordinary cohomology class[2] $G_4 \in H^4(\mathcal{M}_{11};\mathbb{Z})$ as

$$G_4 = \sum_{p=0}^{4} \sum_{i=1}^{b^p(Y_7)} F^i_{4-p} \smile v^i_p + \sum_{p=2}^{4} \sum_{\alpha \in A_p} B^\alpha_{4-p} \smile t^\alpha_p\,. \tag{4.35}$$

Here, $F^i_q \in H^q(\mathcal{M}_4;\mathbb{Z})$ are a set of field strengths related to g^i_q in (4.6) by

$$\varrho(F^i_q) = r(g^i_q)\,, \tag{4.36}$$

and $B^\alpha_q \in H^q(\mathcal{M}_4;\mathbb{Z})$ model a set of closed q-form gauge fields. In particular, let us comment on the 0-forms F^i_0 and B^α_0. Due to flux quantisation (over ordinary and torsional cycles, respectively), F^i_0 and B^α_0 are in fact integers. For F^i_0, we have $\varrho(F^i_0) = r(\mathcal{N}^i)$. We will simply write

$$F^i_0 = \mathcal{N}^i \in \mathbb{Z}\,, \tag{4.37}$$

[2]We use G_4 both for the cohomology class and for the differential form representing the free part.

which is background flux over internal 4-cocycles supporting the vacuum. For $B_0^\alpha \in H^0(\mathcal{M}_4;\mathbb{Z})$, commutativity of the righthand diagram in (4.22) for $p=0$ implies the existence of a set of integers $b^\alpha \in \Omega^0_{\mathbb{Z}}$, such that $\varrho(B_0^\alpha) = r(b^\alpha)$, parametrizing the background flux over torsion 4-cocycles in Y_7. We write

$$B_0^\alpha = b^\alpha \in \mathbb{Z}\,. \tag{4.38}$$

In order to distinguish this background flux from the fluctuating fields, we will use b^α below.

The uplift to differential cohomology $\breve{G}_4 \in \breve{H}^4(\mathcal{M}_{11})$ is performed using the surjective map $I\,:\,\breve{H}^p(M) \twoheadrightarrow H^p(M;\mathbb{Z})$ in (4.22), which implies the existence of differential cohomology classes $\breve{F}^i_{4-p}, \breve{B}^\alpha_{4-p} \in \breve{H}^{4-p}(\mathcal{M}_4)$ and $\breve{v}^i_p, \breve{t}^\alpha_p \in \breve{H}^p(Y_7)$ such that

$$F^i_{4-p} = I(\breve{F}^i_{4-p})\,, \qquad B^\alpha_{4-p} = I(\breve{B}^\alpha_{4-p})\,, \qquad v^i_p = I(\breve{v}^i_p)\,, \qquad t^\alpha_p = I(\breve{t}^\alpha_p)\,. \tag{4.39}$$

We therefore can write the differential cohomology uplift

$$\breve{G}_4 = \sum_{p=0}^{4} \sum_{i=1}^{b^p(Y_7)} \breve{F}^i_{4-p} \star \breve{v}^i_p + \sum_{p=0}^{4} \sum_{\alpha\in A_p} \breve{B}^\alpha_{4-p} \star \breve{t}^\alpha_p\,, \tag{4.40}$$

such that

$$G_4 = I(\breve{G}_4)\,. \tag{4.41}$$

The map I only determines $\breve{G}_4$ up to a topologically trivial element. However the contribution from this element is accessible through the ordinary cohomology formulation, so we set it to zero in the following.

Dimensional reduction of the CS-term, using the expansion of $\breve{G}_4$ in (4.40) (and flatness of $\breve{t}$) yields a significant number of potential topological couplings, which we organise by the number of continuous, respectively discrete, gauge fields (i.e. into four types of the form F^3, F^2B, FB^2 and B^3). Furthermore, we denote the 8-dimensional secondary invariants of $\breve{H}^8(Y_7)$ over the internal space by[3]

$$\begin{aligned}
\Lambda^{ijk}_{nm} &\equiv \int_{Y_7} \breve{v}^i_n \star \breve{v}^j_{8-n-m} \star \breve{v}^k_m \,, \\
\Lambda^{ij\alpha}_{nm} &\equiv \int_{Y_7} \breve{v}^i_n \star \breve{v}^j_{8-n-m} \star \breve{t}^\alpha_m \,, \\
\Lambda^{i\alpha\beta}_{nm} &\equiv \int_{Y_7} \breve{v}^i_n \star \breve{t}^\alpha_{8-n-m} \star \breve{t}^\beta_m \,, \\
\Lambda^{\alpha\beta\gamma}_{nm} &\equiv \int_{Y_7} \breve{t}^\alpha_n \star \breve{t}^\beta_{8-n-m} \star \breve{t}^\gamma_m \,.
\end{aligned} \tag{4.42}$$

For the F^3 component we obtain

$$\begin{aligned}
\int_{\mathcal{M}_{11}} -\frac{1}{6}\breve{G}^3_4\Big|_{F^3} =& \sum_{ijk}\Big[-\frac{K^{ijk}}{2}\int_{\mathcal{M}_4} \breve{F}^i_3 \star \breve{F}^j_1 \star \breve{F}^k_1 + \frac{\mathcal{K}^{ijk}}{2}\int_{\mathcal{M}_4} \breve{F}^i_2 \star \breve{F}^j_2 \star \breve{F}^k_1 + \mathbb{K}^{ijk}\mathcal{N}^i \int_{\mathcal{M}_4} \breve{F}^j_2 \star \breve{F}^k_3 \\
&+ \frac{\Lambda^{ijk}_{23}}{2}\int_{\mathcal{M}_4} \breve{F}^i_2 \star \breve{F}^j_1 \star \breve{F}^k_1 + \frac{\Lambda^{ijk}_{24}}{2}\mathcal{N}^k \int_{\mathcal{M}_4} \breve{F}^i_2 \star \breve{F}^j_2 + \frac{\Lambda^{ijk}_{14}}{2}\mathcal{N}^k \int_{\mathcal{M}_4} \breve{F}^i_3 \star \breve{F}^j_1\Big] \\
&+ \sum_{ij}\Big[\mathfrak{K}^{ij}\mathcal{N}^i \int_{\mathcal{M}_4} \breve{F}^j_1 \star \breve{F}_4 - \frac{\Lambda^{ij}_{40}}{2}\mathcal{N}^i\mathcal{N}^j \int_{\mathcal{M}_4} \breve{F}_4\Big] \,.
\end{aligned} \tag{4.43}$$

Here we notice that the four terms with primary invariants on Y_7 are precisely those captured by the ordinary cohomology reduction (and we therefore use the previous notation for the coefficients). Using the definition of the primary invariant (4.25) we have for $n, m = 0, \dots, 4$ and $3 \le n + m \le 7$

$$\begin{aligned}
\int_{Y_7} \breve{v}_n \star \breve{v}_{7-n-m} \star \breve{v}_m &= \int_{Y_7} R(\breve{v}_n) \wedge R(\breve{v}_{7-n-m}) \wedge R(\breve{v}_m) \\
&= \int_{Y_7} \omega_n \wedge \omega_{7-n-m} \wedge \omega_m \,.
\end{aligned} \tag{4.44}$$

By comparison with (4.13) we conclude that

$$\begin{aligned}
K^{ijk} &= \int_{Y_7} \breve{v}^i_1 \star \breve{v}^j_3 \star \breve{v}^k_3 \,, \qquad & \mathcal{K}^{ijk} &= \int_{Y_7} \breve{v}^i_2 \star \breve{v}^j_2 \star \breve{v}^k_3 \,, \\
\mathfrak{K}^{ij} &= \int_{Y_7} \breve{v}^i_4 \star \breve{v}^j_3 \,, & \mathbb{K}^{ijk} &= \int_{Y_7} \breve{v}^i_4 \star \breve{v}^j_2 \star \breve{v}^k_1 \,.
\end{aligned} \tag{4.45}$$

Furthermore, using (4.26) we have (again, for $n, m = 0, \dots, 4$ and $3 \le n + m \le 7$)

$$\int_{\mathcal{M}_4} \breve{F}_{4-n} \star \breve{F}_{n+m-3} \star \breve{F}_{4-m} = \int_{\mathcal{M}_4} w_{nm} \bmod 1 \,, \qquad w_{nm} \in \frac{\Omega^4(\mathcal{M}_4)}{\Omega^4_{\mathbb{Z}}(\mathcal{M}_4)} \,, \tag{4.46}$$

[3]Note that the Λ's containing $v^i_0 \equiv 1$ will have one less i, j, k index. E.g. we write $\Lambda^{ijk}_{0m} \equiv \Lambda^{jk}_{0m}$.

where

$$d_{\mathbb{Z}} w_{nm} = R(\breve{F}_{4-n} \star \breve{F}_{n+m-3} \star \breve{F}_{4-m}) = g_{4-n} \wedge g_{n+m-3} \wedge g_{4-m} \,. \tag{4.47}$$

From this we see that (4.43) reproduces all the couplings from the CS-term in I_5 (4.18). The other four terms are new compared to the ordinary cohomology reduction. The F^2B contribution is

$$\begin{aligned}\int_{\mathcal{M}_{11}} -\frac{1}{6}\breve{G}_4^3\bigg|_{F^2B} =& \sum_{ij\alpha}\bigg[\Lambda_{33}^{ij\alpha}\int_{\mathcal{M}_4}\breve{F}_1^i \star \breve{F}_2^j \star \breve{B}_1^\alpha + \Lambda_{43}^{ij\alpha}\mathcal{N}^i\int_{\mathcal{M}_4}\breve{F}_3^j \star \breve{B}_1^\alpha \\ &- \Lambda_{42}^{ij\alpha}\mathcal{N}^i\int_{\mathcal{M}_4}\breve{F}_2^j \star \breve{B}_2^\alpha + \Lambda_{34}^{ij\alpha}b^\alpha\int_{\mathcal{M}_4}\breve{F}_1^i \star \breve{F}_3^j \\ &- \frac{\Lambda_{24}^{ij\alpha}}{2}b^\alpha\int_{\mathcal{M}_4}\breve{F}_2^i \star \breve{F}_2^j + \frac{\Lambda_{32}^{ij\alpha}}{2}\int_{\mathcal{M}_4}\breve{F}_1^i \star \breve{F}_1^j \star \breve{B}_2^\alpha\bigg] - \sum_{i\alpha}\Lambda_{44}^{i\alpha}\mathcal{N}^i b^\alpha\int_{\mathcal{M}_4}\breve{F}_4 \,.\end{aligned} \tag{4.48}$$

Finally, the FB^2 and B^3 terms are respectively

$$\begin{aligned}\int_{\mathcal{M}_{11}} -\frac{1}{6}\breve{G}_4^3\bigg|_{FB^2} =& \sum_{i\alpha\beta}\bigg[\Lambda_{32}^{i\alpha\beta}\int_{\mathcal{M}_4}\breve{F}_1^i \star \breve{B}_1^\alpha \star \breve{B}_2^\beta + \frac{\Lambda_{23}^{i\alpha\beta}}{2}\int_{\mathcal{M}_4}\breve{F}_2^i \star \breve{B}_1^\alpha \star \breve{B}_1^\beta \\ &- \Lambda_{22}^{i\alpha\beta}b^\alpha\int_{\mathcal{M}_4}\breve{F}_2^i \star \breve{B}_2^\beta + \Lambda_{13}^{i\alpha\beta}b^\alpha\int_{\mathcal{M}_4}\breve{F}_3^i \star \breve{B}_1^\beta \\ &- \frac{\Lambda_{42}^{i\alpha\beta}}{2}\mathcal{N}^i\int_{\mathcal{M}_4}\breve{B}_2^\alpha \star \breve{B}_2^\beta\bigg] - \sum_{\alpha\beta}\frac{\Lambda_{04}^{\alpha\beta}}{2}b^\alpha b^\beta\int_{\mathcal{M}_4}\breve{F}_4 \,,\end{aligned} \tag{4.49}$$

and

$$\int_{\mathcal{M}_{11}} -\frac{1}{6}\breve{G}_4^3\bigg|_{B^3} = \sum_{\alpha\beta\gamma}\bigg[\frac{\Lambda_{23}^{\alpha\beta\gamma}}{2}\int_{\mathcal{M}_4}\breve{B}_2^\alpha \star \breve{B}_1^\beta \star \breve{B}_1^\gamma - \frac{\Lambda_{24}^{\alpha\beta\gamma}}{2}b^\gamma\int_{\mathcal{M}_4}\breve{B}_2^\alpha \star \breve{B}_2^\beta\bigg] \,. \tag{4.50}$$

Finally, we wish to account for the higher derivative contribution from the M-theory effective action given by $\int_{\mathcal{M}_{11}} C_3 \wedge X_8$ with $X_8 \in H^8(\mathcal{M}_{11};\mathbb{Z})$ in (4.15). We can promote X_8 (equivalently, the Pontryagin classes) to a differential cohomology class $\breve{X}_8 \in \breve{H}^8(\mathcal{M}_{11})$ as described in [220]. We have

$$\breve{X}_8 = -\frac{1}{96}\breve{p}_1(T\mathcal{M}_4) \star \breve{p}_1(TY_7) \,. \tag{4.51}$$

Then

$$\begin{aligned}\int_{\mathcal{M}_{11}} -\breve{G}_4 \star \breve{X}_8 = &\sum_\alpha \left[\frac{1}{96}\int_{Y_7} \breve{t}_4^\alpha \star \breve{p}_1(TY_7)\right] \int_{\mathcal{M}_4} b^\alpha \breve{p}_1(T\mathcal{M}_4) \\ &+ \sum_i \left[\frac{1}{96}\int_{Y_7} \breve{v}_3^i \star \breve{p}_1(TY_7)\right] \int_{\mathcal{M}_4} \breve{F}_1^i \star \breve{p}_1(T\mathcal{M}_4) \\ &+ \sum_i \left[\frac{1}{96}\int_{Y_7} \breve{v}_4^i \star \breve{p}_1(TY_7)\right] \int_{\mathcal{M}_4} \mathcal{N}^i \breve{p}_1(T\mathcal{M}_4)\,.\end{aligned} \tag{4.52}$$

The second term again reproduces what we found using the ordinary cohomology reduction. For the remainder of this work we ignore such contributions. Notice that the first and third terms above contain no dynamical fields. The second term may in principle contribute non-trivially, but for all examples we consider these terms are absent.

Application: Holographic AdS$_4$ Backgrounds. We now turn to AdS$_4$/CFT$_3$ holographic setups, where the supergravity background is supported by $\mathcal{L}$ units of G_4 background flux over AdS$_4$ and the internal space has torsion cycles. In this case the background flux that we have parametrized by $\mathcal{N}^i$ in the above will not be turned on. In addition, all the examples we consider satisfy

$$H^1(Y_7;\mathbb{Z}) = 0\,, \qquad H^3(Y_7;\mathbb{Z}) = 0\,, \tag{4.53}$$

for which the topological action in (4.28) simplifies significantly to

$$\begin{aligned}\frac{S_{\rm top}}{2\pi} = &-\sum_{ij\alpha} \frac{\Lambda_{24}^{ij\alpha}}{2} b^\alpha \int_{\mathcal{M}_4} \breve{F}_2^i \star \breve{F}_2^j - \sum_{i\alpha\beta} \Lambda_{22}^{i\alpha\beta} b^\alpha \int_{\mathcal{M}_4} \breve{F}_2^i \star \breve{B}_2^\beta \\ &- \sum_{\alpha\beta} \frac{\Lambda_{04}^{\alpha\beta}}{2} b^\alpha b^\beta \int_{\mathcal{M}_4} \breve{F}_4 - \sum_{\alpha\beta\gamma} \frac{\Lambda_{24}^{\alpha\beta\gamma}}{2} b^\gamma \int_{\mathcal{M}_4} \breve{B}_2^\alpha \star \breve{B}_2^\beta\,.\end{aligned} \tag{4.54}$$

We briefly comment on the roles of each term in the above expression. The BF-term $b^\alpha \int_{\mathcal{M}_4} \breve{F}_2^i \star \breve{B}_2^\beta = b^\alpha \int_{\mathcal{M}_4} B_2^\beta \smile F_2^i$ encodes non-commutativity of certain extended operators and enforces the requirement to pick a polarization in order to obtain an absolute QFT. After picking a polarization, in certain circumstances terms of this type can correspond to a mixed 't Hooft anomaly polynomial between a discrete 1-form symmetry $\mathbb{Z}_{\ell_\beta}$ with 2-form background gauge field B_2^β and a $U(1)$ 0-form symmetry with field stength F_2^i. The BB-term $b^\gamma \int_{\mathcal{M}_4} \breve{B}_2^\alpha \star \breve{B}_2^\beta = b^\gamma \int_{\mathcal{M}_4} B_2^\alpha \smile B_2^\beta$ is a 't Hooft anomaly for the discrete 1-form symmetries $\mathbb{Z}_{\ell_\alpha}$ and $\mathbb{Z}_{\ell_\beta}$. Note that the presence of discrete background flux $b^\alpha \neq 0$ for some α is essential for the existence of the anomalies. We will not discuss the physical effects of the θ-term $b^\alpha \int_{\mathcal{M}_4} \breve{F}_2^i \star \breve{F}_2^j$ or $b^\alpha b^\beta \int_{\mathcal{M}_4} \breve{F}_4$ in this work[4].

4.3 SymTFT Coefficients from Geometry

A crucial aspect of the above analysis is the coefficients Λ. Clearly, the numerical value of these $\int_{Y_7}$ integrals is important: a value of zero implies the absence of a particular term in the anomaly polynomial, whilst a non-zero coefficient contains physical information. In this section we determine these explicitly in the case of toric Calabi-Yau 4-folds.

4.3.1 SymTFT Coefficients from Intersection Theory

The coefficients of the 4d topological action resulting from (4.43), (4.48)-(4.50) are given by the primary/secondary invariants of elements of $\breve{H}^p(Y_7)$ with $p = 7, 8$ over Y_7. In the case where the integrand is an element of $\breve{H}^7(Y_7)$ we showed that these are simply the intersection numbers (4.45) that we also obtain from the ordinary cohomology reduction. On the other hand, when the integrand is an element of $\breve{H}^8(Y_7)$ as in (4.42) there is no analogue in ordinary cohomology. Still, we would like a convenient way to evaluate these coefficients, which, it turns out, can be accessed by considering a space $\mathcal{X}_8$ of which Y_7 is the boundary. In holography this notion is quite natural since the duality is precisely between supergravity compactified on Y_7 and branes probing the tip of the cone over the compactification space, i.e. we

[4] Such terms in the anomaly polynomial are gauge invariant by themselves, and they do not change the equation of motion for the bulk gauge fields in the AdS/CFT interpretation.

can take $\mathcal{X}_8 = \mathcal{C}(Y_7)$ to be this cone. It should be clear that the Λ's in (4.42) are defined purely in terms of the geometry of Y_7, and we resort to the space $\mathcal{X}_8$ only for computational convenience.

In this section we present an extension to the arguments of section 3.3 in [65], where the coefficients (4.42) are derived from an intersection number computation on the resolved space $\widetilde{\mathcal{X}_8}$. In the geometric engineering set-up, we will assume that $\widetilde{\mathcal{X}_8}$ is a non-compact Calabi-Yau 4-fold. However, the Calabi-Yau condition can be relaxed in the holography setups, such as the ABJ(M) theories in section 4.4.

We will make use of the long exact sequence

$$\cdots \to H_p(\widetilde{\mathcal{X}_8};\mathbb{Z}) \to H_p(\widetilde{\mathcal{X}}_8, Y_7;\mathbb{Z}) \to H_{p-1}(Y_7;\mathbb{Z}) \to H_{p-1}(\widetilde{\mathcal{X}}_8;\mathbb{Z}) \to \dots . \tag{4.55}$$

Note in particular that elements of $H_p(\widetilde{\mathcal{X}}_8;\mathbb{Z})$ are *compact* p-cycles in $\widetilde{\mathcal{X}}_8$ and elements of $H_p(\widetilde{\mathcal{X}}_8, Y_7;\mathbb{Z})$ are *non-compact* p-cycles in $\widetilde{\mathcal{X}}_8$. We assume that there are no compact $(7-n)$-cycles in $\widetilde{\mathcal{X}}_8$

$$H_{7-n}(\widetilde{\mathcal{X}}_8;\mathbb{Z}) = 0\,, \tag{4.56}$$

for a specific $n \in \{0,\dots,6\}$.

This implies that any $(7-n)$-cycle in Y_7 can be realised as the boundary of an $(8-n)$-chain in $\widetilde{\mathcal{X}}_8$. In the examples we consider, $\widetilde{\mathcal{X}}_8$ is a non-compact toric 4-fold, and these have no non-trivial odd-dimensional cycles, $H_{2k-1}(\widetilde{\mathcal{X}}_8;\mathbb{Z}) \equiv 0$ for $k \in \mathbb{N}$. Using Poincaré duality in Y_7, from (4.55) we find

$$H_{8-n}(\widetilde{\mathcal{X}}_8;\mathbb{Z}) \xrightarrow{A} H_{8-n}(\widetilde{\mathcal{X}}_8, Y_7;\mathbb{Z}) \xrightarrow{f} H^n(Y_7;\mathbb{Z}) \to 0\,. \tag{4.57}$$

Since f is surjective, we conclude that *every n-cocycle in Y_7 can be mapped to a non-compact $(8-n)$-cycle D in $\widetilde{\mathcal{X}}_8$*. Furthermore, a torsion class $t_n \in H^n(Y_7;\mathbb{Z})$ satisfies

$$\ell t_n = 0\,, \tag{4.58}$$

for some (minimal) $\ell \in \mathbb{N}$. Then exactness of (4.57) implies that there exists a compact $(8-n)$-cycle $Z \in H_{8-n}(\widetilde{\mathcal{X}}_8;\mathbb{Z})$ such that

$$A(Z) = \ell T\,, \qquad f(T) = t_n\,, \tag{4.59}$$

which we use to *map a torsion class $t_n \in H^n(Y_7;\mathbb{Z})$ to a compact $(8-n)$-cycle Z in $\widetilde{\mathcal{X}}_8$*. Taking A to be the intersection pairing in $\widetilde{\mathcal{X}}_8$, the coefficients (4.42) of the SymTFT can be computed as follows.

We associate to t_n^α of torsional degree ℓ_n^α a compact $(8-n)$-cycle Z_α^{8-n} in $\widetilde{\mathcal{X}}_8$, and to v_m^i a non-compact $(8-m)$-cycle D_i^{8-m} in $\widetilde{\mathcal{X}}_8$. The coefficients are then given by

$$\begin{aligned}
\Lambda_{nm}^{ijk} &= \left[D_i^{8-n}\cdot D_j^{n+m}\cdot D_k^{8-m}\right]_{\text{mod }1}\,,\\
\Lambda_{nm}^{ij\alpha} &= \left[\frac{D_i^{8-n}\cdot D_j^{n+m}\cdot Z_\alpha^{8-m}}{\ell_m^\alpha}\right]_{\text{mod }1}\,,\\
\Lambda_{nm}^{i\alpha\beta} &= \left[\frac{D_i^{8-n}\cdot Z_\alpha^{n+m}\cdot Z_\beta^{8-m}}{\ell_{8-n-m}^\alpha\ell_m^\beta}\right]_{\text{mod }1}\,,\\
\Lambda_{nm}^{\alpha\beta\gamma} &= \left[\frac{Z_\alpha^{8-n}\cdot Z_\beta^{n+m}\cdot Z_\gamma^{8-m}}{\ell_n^\alpha\ell_{8-n-m}^\beta\ell_m^\gamma}\right]_{\text{mod }1}\,,
\end{aligned} \tag{4.60}$$

where $\cdot$ denotes intersections in $\widetilde{\mathcal{X}}_8 = \mathcal{C}(Y_7)$.

We must take extra care with the terms in the 4d effective action which come with a factor of a half. That is, the relevant object to compute is not Λ but rather $\Omega \equiv \Lambda/2$. However multiplication by $1/2$ is not a well-defined operation due to the mod 1 in (4.60). We use the approach by Gordon and Litherland [221] employed in [65] to deal with the refinement by a factor of $1/2$. Concretely, this approach allows for the computation of these secondary invariants - as performed for 3- and 5-dimensional links in [65]. The applicability of this analysis to the current case is borne out by our matching with known field theory results. We compute these terms as follows:

$$
\begin{aligned}
\Omega_n^{ij} &= \frac{1}{2}\int_{Y_7} \breve{v}_n^i \star \breve{v}_n^i \star \breve{v}_{8-2n}^j = \left[\frac{D_i^{8-n}\cdot D_i^{8-n}\cdot D_j^{2n}}{2}\right]_{\text{mod }1}, \\
\Omega_n^{i\alpha} &= \frac{1}{2}\int_{Y_7} \breve{v}_n^i \star \breve{v}_n^i \star \breve{t}_{8-2n}^\alpha = \left[\frac{D_i^{8-n}\cdot D_i^{8-n}\cdot Z_\alpha^{2n}}{2\ell_{8-2n}^\alpha}\right]_{\text{mod }1}, \\
\Omega_n^{i\alpha} &= \frac{1}{2}\int_{Y_7} \breve{v}_{8-2n}^i \star \breve{t}_n^\alpha \star \breve{t}_n^\alpha = \left[\frac{D_i^{2n}\cdot Z_\alpha^{8-n}\cdot Z_\alpha^{8-n}}{2(\ell_n^\alpha)^2}\right]_{\text{mod }1}, \\
\Omega_n^{\alpha\beta} &= \frac{1}{2}\int_{Y_7} \breve{t}_n^\alpha \star \breve{t}_n^\alpha \star \breve{t}_{8-2n}^\beta = \left[\frac{Z_\alpha^{8-n}\cdot Z_\alpha^{8-n}\cdot Z_\beta^{2n}}{2(\ell_n^\alpha)^2\ell_{8-2n}^\beta}\right]_{\text{mod }1},
\end{aligned}
\tag{4.61}
$$

which are $\mathbb{R}/\mathbb{Z}$-valued quantities.

4.3.2 Intersection Numbers of Toric 4-Folds

The above subsection explained that the computation of the SymTFT coefficients reduces to a computation of intersection numbers in $\mathcal{X}_8$. In this section we focus on *toric* 4-folds. It will become apparent in later sections that the non-trivial coefficients we are particularly interested in are those involving $\breve{t}_2^\alpha, \breve{t}_4^\alpha$ and $\breve{v}_2^i$. The key identifications to make are therefore the *compact* divisor Z^6 corresponding to $\breve{t}_2^\alpha$, the *compact* 4-cycles Z_β^4 corresponding to $\breve{t}_4^\beta$ and the *non-compact* divisor D^6 corresponding to $\breve{v}_2^i$. We will address the identifications in turn. First however, we introduce the technology required to compute intersection numbers of toric 4-folds.

Quadruple Toric Intersections. All non-zero integrals of the type we wish to consider reduce to a sum of quadruple intersections of toric divisors T_i in the Calabi-Yau

$$T_i \cdot T_j \cdot T_k \cdot T_l\,. \tag{4.62}$$

We begin with a toric fourfold $\widetilde{\mathcal{X}}_8$, described by a toric diagram with set of rays $\{v_i\}$,

$$v_i = (v_i^x, v_i^y, v_i^z, v_i^w)\,. \tag{4.63}$$

Each ray corresponds to a toric divisor $v_i \leftrightarrow T_i$, amongst which there exists a set of linear relations

$$\sum_i v_i^x T_i = 0\,, \quad \sum_i v_i^y T_i = 0 \quad \sum_i v_i^z T_i = 0\,, \quad \sum_i v_i^w T_i = 0\,. \tag{4.64}$$

Furthermore, we triangulate the toric diagram with a set of 4d cones

$$\{v_a v_b v_c v_d\}\,, \tag{4.65}$$

which restrict the non-zero quadruple intersections in the following way. The intersection of four distinct toric divisors is given by the volume bounded by the rays (we denote this region V_{ijkl})

$$T_i \cdot T_j \cdot T_k \cdot T_l = \frac{1}{\mathrm{vol}(V_{ijkl})}\,, \quad i \neq j \neq k \neq l\,. \tag{4.66}$$

The quadruple intersection numbers involving self-intersections can be computed using (4.66) and the linear equivalence relations (4.64) [5][6].

Now we consider the case of a toric Calabi-Yau 4-fold $\widetilde{\mathcal{X}}_8$, such that the boundary 7-manifold Y_7 is Sasaki-Einstein. The Calabi-Yau condition forces the rays $\{v_i\}$ to lie in a plane. We enforce this in coordinates by choosing the fourth coordinate of all rays to be 1

$$v_i = (\underline{v_i}, 1) = (v_i^x, v_i^y, v_i^z, 1)\,. \tag{4.67}$$

For a 4d cone $v_i v_k v_k v_l$, the volume of V_{ijkl} takes the form of

$$\mathrm{vol}(V_{ijkl}) = \det\left(\underline{v}_j - \underline{v}_i \quad \underline{v}_k - \underline{v}_i \quad \underline{v}_l - \underline{v}_i\right)\,. \tag{4.68}$$

[5] In order to perform such calculations in practise, a computer code is necessary for larger toric diagrams. Assuming we have computed $T_i \cdot T_j \cdot T_k \cdot T_l$ for all distinct $i \neq j \neq k \neq l$ we can compute the following intersections in turn

1. $T_i \cdot T_i \cdot T_j \cdot T_k \quad i \neq j \neq k$
2. $T_i \cdot T_i \cdot T_i \cdot T_k\,, \quad i \neq k$ and $T_i \cdot T_i \cdot T_k \cdot T_k\,, \quad i \neq k$
3. $T_i \cdot T_i \cdot T_i \cdot T_i$.

In each step we use the intersections computed in the step prior.

[6] Note that in the cases of a non-compact toric 4-fold, the quadruple intersection numbers only involving non-compact divisors are usually not well-defined. Nonetheless, we do not encounter this issue as we only use the intersection numbers which involve at least one compact divisor, see analogous computations in the case of CY3 [222, 223].

4.3.3 Differential Cohomology Generators and Toric Divisors

$\breve{t}_2$ generators. From the set of divisors $\{T_i\}$ and the linear relations between them, we can obtain a set of *linearly indepent* divisors. We denote them $C_a\,, D_a$ for compact and non-compact respectively. From these, we can construct a basis of compact curves

$$\{\mathcal{N}_k\} = \{C_a \cdot D_b \cdot D_c\ ,\ C_a \cdot C_b \cdot D_c\ ,\ C_a \cdot C_b \cdot C_c\}\,. \tag{4.69}$$

In general these curves are not linearly independent. For example, for the case $Y^{p,k}(\mathbb{CP}^2)$, the curves $C_a \cdot D_b \cdot D_c$ already form a complete basis of compact curves.

In order to obtain the central divisors Z^6, we compute the SNF of the intersection matrix $\mathcal{N}_k \cdot C_a$

$$\mathrm{SNF}(\mathcal{N}_k \cdot C_a) = \begin{pmatrix} \Gamma_1 & 0 & \dots & 0 \\ 0 & \Gamma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Gamma_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = A \cdot (\mathcal{N}_k \cdot C_a) \cdot B\,, \tag{4.70}$$

where A and B are matrices and Γ_α are a set of integers. The group

$$\Gamma = \oplus_{\alpha=1}^{n} \mathbb{Z}_{\Gamma_\alpha}\,, \tag{4.71}$$

is generated by a set of linear combinations of divisors given by the matrix B.

Thus the change of basis matrices used in the SNF procedure can be used to find explicit expressions for the compact divisor dual to $\breve{t}_2$ in terms of the basis elements C_a. For each $\Gamma_\alpha > 1$, there is a differential cohomology class $\breve{t}_2^\alpha$ with torsion degree Γ_α. Furthermore, it is clear that the group Γ is in fact equal to the 1-form symmetry

$$\Gamma^{(1)} = \Gamma\,. \tag{4.72}$$

In particular, we can read off the generators as follows. The linear combination of divisors generating the factor Γ_α is given by

$$Z^6_\alpha = \sum_i B_{i\alpha} C_i \,. \tag{4.73}$$

$\breve{t}_4$ generators. Analogously to the above procedure, we wish to identify the appropriate linear combination of 4-cycles Z^4_α dual to $\breve{t}^\alpha_4$. We can construct a basis for compact 4-cycles by $\{\mathcal{S}_k\} = \{C_a \cdot D_b,\ C_a \cdot C_b\}$. Once again we take the SNF of the intersection matrix

$$\text{SNF}\{\mathcal{S}_j \cdot \mathcal{S}_k\} = \text{diag}(\Gamma'_1, \Gamma'_2, \dots) = A' \cdot \{\mathcal{S}_j \cdot \mathcal{S}_k\} \cdot B' \,. \tag{4.74}$$

We derive that the group

$$\text{Tor} H^4 = \oplus_\alpha \Gamma'_\alpha \,, \tag{4.75}$$

is generated by the linear combinations

$$Z^4_\alpha = \sum_i B'_{i\alpha} \mathcal{S}_i \,. \tag{4.76}$$

We observe that a consistent choice must be made of ordering of columns in the SNF process when two different $Y^{p,k}$ models are compared. A change of basis of the matrix corresponds to choosing different diagonal combinations of symmetries inside the group $\oplus_i \Gamma'_i$.

$\breve{v}_2$ generators. In general, the number of independent $\breve{v}_2$ generators is equal to $b_3(Y_7)$. One can pick any of the $b_3(Y_7)$ linearly independent non-compact divisors as $\breve{v}_2$ generators, and they will give the same physical results.

4.4 SymTFT for Holography: ABJ(M)

We now employ the geometric tools developed in the previous sections to derive from M-theory the global structure and higher-form symmetries for the 3d $\mathcal{N} = 6$

$U(N{+}b)_k{\times}U(N)_{-k}$ ABJ(M) theories [162,215]. In the brane picture, the theories arise on N M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity, together with b fractional M2-branes localised at the orbifold singularity. The 11d supergravity dual is $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ with N units of G_4 flux over AdS_4 and b units of torsion flux. The 7-manifold $S^7/\mathbb{Z}_k$ is generally a tri-Sasakian manifold, and it is Sasaki-Einstein only when $k = 4$. All 3d $\mathcal{N} = 6$ ABJM type theories were classified, up to discrete quotients of the gauge group, in [224], which was subsequently extended to account for all global forms in [216]. In [157] it was shown how to realise different global forms of the gauge group holographically from type IIA supergravity (in the absence of background torsion, i.e. for $b = 0$) in the regime $k \ll N \ll k^5$ where the M-theory circle is small. We reproduce these results, taking the perspective of 11d supergravity, where the technology presented in previous sections is crucial to understand the geometric origin of the symmetry background fields. Moreover, with b turned on, we determine a 't Hooft anomaly for the 1-form symmetry

$$-\frac{b}{2k}\int_{\mathrm{AdS}_4} B_2 \smile B_2\,. \tag{4.77}$$

We derive the anomaly from torsional geometric data, and match with field theory results [216]. The SymTFT, computed using differential cohomology, is precisely the tool suited to pick up such a torsional effect.

4.4.1 Global Form of the Gauge Group

The global form of the gauge group is associated with a choice of boundary conditions for the gauge fields of the 4d bulk theory [65, 113–115, 151, 183]. This choice is constrained by the fact that, in the presence of torsion in the homology of the internal space $Y_7 = S^7/\mathbb{Z}_k$, the G_4 and G_7 fluxes do not commute at the boundary [218, 225]. In the SymTFT this non-commutativity of fluxes shows up as a set of BF-couplings that constrain the consistent set of boundary conditions which can be imposed on the participating 4d gauge fields. The BF-terms arise from the differential cohomology reduction of the kinetic term in the 11d supergravity action, see [174] for a derivation[7]. In the following however, we will instead take an operator perspective and derive the commutation relation.

In holography the procedure for choosing asymptotic values of the fields at the conformal boundary of $\mathcal{M}_{11} = \mathcal{M}_4 \times Y_7$ is to quantize the theory on $\mathcal{M}_{11} = \mathbb{R}_t \times \mathcal{M}_{10}$, by identifying the radial direction with time, and choosing a state in the Hilbert space of $\mathcal{M}_{10} = M_3 \times Y_7$, where M_3 is the conformal boundary of $\mathcal{M}_4$ at infinity [151, 183]. Consider the operators $\Phi(\mathcal{T}_3)$ and $\Phi(\mathcal{T}_6)$ which detect the periods of the M-theory gauge potential C_3 and the electric-magnetic dual potential C_6 over 3- and 6-cycles $\mathcal{T}_3, \mathcal{T}_6$ defining torsion homology classes in $\mathcal{M}_{10}$. As shown in [218,225], these operators pick up a phase under commutation

$$\Phi(\mathcal{T}_3)\Phi(\mathcal{T}_6) = \Phi(\mathcal{T}_6)\Phi(\mathcal{T}_3)e^{2\pi i \mathsf{L}(\mathcal{T}_3,\mathcal{T}_6)}\,, \tag{4.78}$$

where L is the linking pairing. The homology of $S^7/\mathbb{Z}_k$ is

$$H_\bullet(S^7/\mathbb{Z}_k;\mathbb{Z}) = \{\mathbb{Z}, \mathbb{Z}_k, 0, \mathbb{Z}_k, 0, \mathbb{Z}_k, 0, \mathbb{Z}\}\,. \tag{4.79}$$

Since we are assuming $\mathrm{Tor}_\bullet(M_3;\mathbb{Z}) = 0$, we can apply the Künneth formula to obtain

$$\begin{aligned}\mathrm{Tor}H_3(\mathcal{M}_{10};\mathbb{Z}) &= H_2(M_3;\mathbb{Z}) \otimes H_1(S^7/\mathbb{Z}_k;\mathbb{Z}) \oplus H_0(M_3;\mathbb{Z}) \otimes H_3(S^7/\mathbb{Z}_k;\mathbb{Z})\,,\\ \mathrm{Tor}H_6(\mathcal{M}_{10};\mathbb{Z}) &= H_3(M_3;\mathbb{Z}) \otimes H_3(S^7/\mathbb{Z}_k;\mathbb{Z}) \oplus H_1(M_3;\mathbb{Z}) \otimes H_5(S^7/\mathbb{Z}_k;\mathbb{Z})\,.\end{aligned} \tag{4.80}$$

This implies that the torsional 3- and 6-cycles of $\mathcal{M}_{10}$ must be of the form

$$\mathcal{T}_3 = \Sigma_2 \times T_1\,, \qquad \mathcal{T}_3 = \Sigma_0 \times T_3\,, \tag{4.81}$$

and

$$\mathcal{T}_6 = \Sigma_3 \times T_3\,, \qquad \mathcal{T}_6 = \Sigma_1 \times T_5\,, \tag{4.82}$$

where Σ_p generates $H_p(M_3;\mathbb{Z})$ and T_q generates $H_q(S^7/\mathbb{Z}_k;\mathbb{Z}) = \mathbb{Z}_k$ for $q < 7$ odd. Consider the expansion on cohomology of G_4 around the ABJ(M) background

$$G_4 = N\mathrm{vol}_{\mathrm{AdS}_4} + B_2 \smile t_2 + b \smile t_4\,, \tag{4.83}$$

[7]We thank Iñaki García Etxebarria and Saghar Sophie Hosseini for explaining this to us, and refer the reader to their upcoming work [174] for more details.

where t_2 and t_4 are both torsional generators of degree k. The differential cohomology uplift is

$$\breve{G}_4 = N\breve{\mathrm{vol}}_{\mathrm{AdS}_4} + \breve{B}_2 \star \breve{t}_2 + \breve{b} \star \breve{t}_4 \,, \tag{4.84}$$

Here, $\breve{B}_2$ represents a dynamical $\mathbb{Z}_k$ 2-form gauge field, whereas

$$I(\breve{b}) = b \in \mathbb{Z} \,, \tag{4.85}$$

is an integer parametrizing background flux over torsion 4-cocyles, as argued around (4.38). This discrete flux is associated to b M5-branes wrapping the torsion 3-cycle $H_3(S^7/\mathbb{Z}_k;\mathbb{Z}) = \mathbb{Z}_k$. In the ABJ paper [215] it was conjectured that we must have $b \leq k$ for the superconformal $U(N+b)_k \times U(N)_{-k}$ theories to exist as unitary theories. As was also noted in [215], this restriction is consistent with the interpretation of b as discrete $\mathbb{Z}_k$ torsion. Without imposing a relation between G_4 and G_7, we can make a corresponding cohomology expansion of the latter

$$G_7 = N\mathrm{vol}_{S^7/\mathbb{Z}_k} + B_3 \smile t_4 + B_1 \smile t_6 \,. \tag{4.86}$$

In order to quantize on $\mathcal{M}_{11} = \mathbb{R}_t \times \mathcal{M}_{10}$, we consider a gauge as in [151] where the form representatives of the B_p classes do not have components along the radial direction (or time direction, in terms of the quantization scheme), i.e. they can be taken to define either degree-p cohomology classes in $\mathcal{M}_4$, as in (4.83), (4.86), or in M_3. Then, by Poincaré duality, the integral homology classes Σ_q are dual to B_{3-q} in the (torsion-free) cohomology of $M_3 = \mathbb{R}^{1,2}$.

For the present, let us ignore the discrete background flux and study the ABJM theories whose type IIA duals were studied in [157]. Then the non-commuting operators are $\Phi(\Sigma_2)$ and $\Phi(\Sigma_1)$, which are push-forwards of the 11d operators $\Phi(\Sigma_2 \times T_1)$ and $\Phi(\Sigma_1 \times T_5)$. Their commutation relation is determined using

$$\mathsf{L}(\Sigma_p \times T_q, \Sigma_{p'} \times T_{q'}) = (\Sigma_p \cdot \Sigma_{p'})\mathsf{L}_{S^7/\mathbb{Z}_k}(T_q, T_{q'}) \,, \tag{4.87}$$

with $\Sigma_p \cdot \Sigma_{p'}$ the intersection in M_3 and [114]

$$\mathsf{L}_{S^7/\mathbb{Z}_k}(T_1, T_5) = \frac{1}{k}\,. \tag{4.88}$$

Hence, we have

$$\Phi(\Sigma_2)\Phi(\Sigma_1) = \Phi(\Sigma_1)\Phi(\Sigma_2)e^{2\pi i(\Sigma_1\cdot\Sigma_2)/k}\,. \tag{4.89}$$

If we consider the form representatives of B_2, B_1, and abuse notation by denoting them the same as their corresponding cohomology classes, this commutation relation is encoded in a BF-coupling

$$\boxed{\frac{S_{\rm BF}}{2\pi} = k\int_{\rm AdS_4} B_2 \wedge dB_1\,.} \tag{4.90}$$

As we remarked above, this coupling can alternatively be derived by considering the differential cohomology reduction of the kinetic part of the 11d supergravity action, see [174].

The symmetries of the 3d field theory are determined by imposing boundary conditions on B_2, B_1 consistent with the commutation relation (4.89), or equivalently the action (4.90). Fixing B_2 to a background value as we approach the conformal boundary is associated with a 1-form symmetry, whereas fixing B_1 would furnish a background gauge field for an ordinary 0-form symmetry.

First however, we must also take into account the additional global 0-form symmetries that can arise from gauging the isometry group of the internal space. The isometries of $S^7/\mathbb{Z}_k$ are $U(1) \times SU(4)_R$, with the latter realising the R-symmetry of the 3d $\mathcal{N} = 6$ theory. We can describe $S^7/\mathbb{Z}_k$ as a circle bundle over $\mathbb{CP}^3$ with metric [162]

$$ds^2_{S^7/\mathbb{Z}_k} = \frac{1}{k^2}(\mathrm{d}\varphi + kw)^2 + ds^2_{\mathbb{CP}^3}\,, \tag{4.91}$$

with $\varphi \sim \varphi + 2\pi$ parametrizing the M-theory circle, and $dw = J$ with J the Kähler form on $\mathbb{CP}^3$. The $\mathbb{Z}_k$ quotient is simply making the M-theory circle smaller, and ∂_φ is generating the $U(1)$ isometry. We can gauge the isometries by lifting the 4-form flux to equivariant cohomology, which gives rise to a single 1-form gauge field A_1 for the $U(1)$ isometry

$$d\varphi \to d\varphi + A_1 \,, \tag{4.92}$$

and a set of 15 gauge fields for the $SU(4)_R$. We are interested in the fate of the $U(1)$ global symmetry, and whether it couples to the B_2, B_1 gauge fields in (4.90). We will answer this question by conjecturing a map to the type IIA description.

When $N \gg k^5$ the appropriate supergravity description is 11-dimensional. On the other hand, when $k \ll N \ll k^5$ the M-theory circle becomes very small (in Planck units) and the relevant description is type IIA supergravity on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ with N units of F_6-flux over $\mathbb{CP}^3$ and k units of F_2-flux over $\mathbb{CP}^1 \subset \mathbb{CP}^3$. (In the presence of background torsion flux $b \neq 0$ the NSNS 2-form on $\mathbb{CP}^1 \subset \mathbb{CP}^3$ has a discrete holonomy b/k.) Consider the type IIA supergravity analysis in [157], where a topological term

$$\frac{S_{\mathrm{IIA}}}{2\pi} = \int_{\mathrm{AdS}_4} B_{\mathrm{NS}} \wedge d(kA_{\mathrm{D4}} + NA_{\mathrm{D0}}) \,, \tag{4.93}$$

was identified and the consistent boundary conditions were studied in detail. Here B_{NS} is the NS-NS 2-form, and A_{D4}, A_{D0} are $U(1)$ 1-form gauge fields that couple electrically respectively to D4-branes wrapping $\mathbb{CP}^2 \subset \mathbb{CP}^3$ and D0-branes. Under dimensional reduction from 11d supergravity to type IIA (see e.g. [226]), the $U(1)$ gauge field A_1 associated with the isometry generated by the M-theory circle direction ∂_φ gives rise to the 1-form gauge field A_{D0} sourced by D0-branes. The 1-form gauge field B_1 couples electrically to M5-branes wrapping the torsional 5-cycle, which descends to D4-branes wrapping $\mathbb{CP}^2$ coupled electrically to the 1-form A_{D4}. Finally, B_2 couples electrically to M2-branes wrapping the torsional 1-cycle associated with the M-theory circle. Under dimensional reduction these M2-branes become fundamental strings coupling to the NS-NS 2-form. Therefore, we conjecture a map

$$A_1 \leftrightarrow A_{D0} \,, \qquad B_1 \leftrightarrow A_{D4} \,, \qquad B_2 \leftrightarrow B_{\mathrm{NS}} \,. \tag{4.94}$$

Using this map implies the existence of a topological coupling between B_2 and $F = dA_1$ in M-theory, to which either equivariant or differential cohomology are not sensitive by themselves[8]. The 11d kinetic term is then

$$\boxed{\frac{S_{\text{kin}}}{2\pi} = \int_{\text{AdS}_4} kB_2 \wedge dB_1 + NB_2 \wedge F\,.} \tag{4.96}$$

The different global forms of the gauge group are realised holographically by imposing boundary conditions consistent with this BF-coupling. This part of the analysis is now completely analogous to [157]. For convenience, we here give a brief summary of one extreme possibility, namely $(U(N)_k \times U(N)_{-k})/\mathbb{Z}_k$.

Suppose we apply the conditions

$$A_1, B_1 \text{ Dirichlet}\,, \qquad B_2 \text{ Neumann}\,, \tag{4.97}$$

which constrains the boundary values of the 1-forms to satisfy $kB_1 + NA_1 = 0$. Hence, the 1-form background gauge field we can specify at the boundary gives rise to $\Gamma^{(0)} = U(1) \times \mathbb{Z}_{\gcd(N,k)}$, where the $U(1)$ is supplied by the diagonal combination $(B_1, A_1) = (p\mathcal{A}, -q\mathcal{A})$, with $p \cdot \gcd(N,k) = N$ and $q \cdot \gcd(N,k) = k$, which decouples from the action. If $N = nn'$ for some integers n, n' more complicated boundary conditions are possible. These realise the gauging of a subgroup of the 1-form symmetry.

4.4.2 't Hooft Anomaly for the 1-Form Symmetry

We now turn on the torsion flux $b \neq 0$ and see how the theory is modified. From the differential cohomology reduction of the 11d Chern-Simons term we determine the Symmetry TFT coupling

$$\frac{S_{\text{top}}}{2\pi} = -\Omega \int_{\text{AdS}_4} \breve{B}_2 \star \breve{B}_2 \star \breve{b} \mod 1\,, \tag{4.98}$$

[8]We propose that a full equivariant differential cohomology treatment of the problem will give rise to an improvement of G_4^{eq} by a term involving the 2-form B_2 and the $U(1)$ field strength F like

$$G_4^{\text{eq}} = N\text{vol}_{\text{AdS}_4} + B_2 \wedge F + \cdots\,. \tag{4.95}$$

with

$$\Omega = \frac{1}{2}\int_{S^7/\mathbb{Z}_k} \breve{t}_2 \star \breve{t}_2 \star \breve{t}_4 = \left[\frac{Z^6 \cdot Z^6 \cdot Z^4}{2k^3}\right]_{\text{mod } 1}\,, \tag{4.99}$$

The primary invariant over AdS_4 gives

$$\frac{S_{\text{top}}}{2\pi} = -\Omega b \int_{\mathrm{AdS}_4} B_2 \smile B_2\,, \tag{4.100}$$

which signals an anomaly in the $\mathbb{Z}_k$ 1-form symmetry of $U(N+b)_k \times U(N)_{-k}$, determined by the coefficient Ω. The 4d term (4.100) constrains the possibility of gauging a $\mathbb{Z}_m$ subgroup, with $k = mm'$, of the $\mathbb{Z}_k$ 1-form symmetry. That is, the anomaly is only consistent with B_2 having periodicity $\mathbb{Z}_{m'}$ for $\Omega b m'^2 = 0 \bmod \frac{1}{2}$ (since $B_2 \smile B_2$ is even on a spin manifold) or, equivalently,

$$\frac{2\Omega k^2 b}{m^2} = 0 \bmod 1\,. \tag{4.101}$$

To compute the coefficient Ω geometrically from (4.99), let us consider the resolution $\widetilde{\mathcal{X}}_8$ of $\mathbb{C}^4/\mathbb{Z}_k$. Note that the singularity $\mathbb{C}^4/\mathbb{Z}_k$ has a toric description, with the rays

$$v_1 = (1,0,0,0)\ ,\ v_2 = (0,1,0,0)\ ,\ v_3 = (0,0,1,0)\ ,\ v_4 = (-1,-1,-1,k) \tag{4.102}$$

and the 4d cone $v_1v_2v_3v_4$. It has a unique toric resolution $\widetilde{\mathcal{X}}_8 = \widetilde{\mathbb{C}^4/\mathbb{Z}_k}$, where the compact exceptional divisor C corresponds to the new ray $v_5 = (0,0,0,1)$[9]. The new set of 4d cones in $\widetilde{\mathbb{C}^4/\mathbb{Z}_k}$ is $\{v_1v_2v_3v_5,\ v_1v_2v_4v_5,\ v_1v_3v_4v_5,\ v_2v_3v_4v_5\}$. Denote the non-compact divisor corresponding to v_1 by D (which is linearly equivalent to v_2, v_3 and v_4). We can compute the following intersection numbers

$$D^4 = 0\,,\quad C\cdot D^3 = 1\,,\quad C^2\cdot D^2 = -k\,,\quad C^3\cdot D = k^2\,,\quad C^4 = -k^3\,, \tag{4.103}$$

from which we obtain the generators $Z^4 = C\cdot D$, $Z^6 = C$. Note that we need not define the SymTFT in a supersymmetric way, and one can use any (possibly non-crepant) resolution for the purposes of computing the SymTFT action. Here we

[9]Restricting to toric varieties, $\widetilde{\mathbb{C}^4/\mathbb{Z}_k}$ is the unique resolution because $v_5 = (0,0,0,1)$ is the only primitive ray inside the cone $v_1v_2v_3v_4$.

validate this approach by matching with known field theory results. Hence we can plug (4.103) into (4.99) to get

$$\Omega = \frac{1}{2k} \bmod 1\,. \tag{4.104}$$

Recalling the condition (4.101), the gauging is consistent for

$$\frac{kb}{m^2} = 0 \bmod 1\,. \tag{4.105}$$

The anomaly thus implies that gauging a $\mathbb{Z}_m$ subgroup of the 1-form symmetry of $U(N+b)_k \times U(N)_{-k}$ is consistent only for certain choices of m. That is, compared to the analysis at the end of the previous section, when b is turned on, certain global forms of the 3d gauge group are no longer consistent. E.g. in the presence of this anomaly we can only gauge the full 1-form symmetry $m = k$, if $b/k \in \mathbb{Z}$.

Note that this anomaly was also determined from the field theory point of view in [216], where the authors show that, the anomaly can be measured by the topological spin of a line of charge m', where $k = mm'$. In [216] the anomaly free lines were determined to be exactly the ones satisfying (4.105).

4.5 SymTFT for Holography: $\mathbf{AdS}_4 \times Y^{p,k}(\mathbb{CP}^2)$

We now apply the Symmetry Topological Field Theory technology to a class of holographic 3d $\mathcal{N} = 2$ QFTs. We study the theories living on the worldvolume of a stack of M2-branes probing the cone $\mathcal{C}(Y^{p,k}(\mathbb{CP}^2))$ with torsional G_4 flux turned on [210]. The latter phenomenon arises from wrapped M5-branes on the torsional elements of the third homology group of $Y^{p,k}$ (which is non-trivial). The purpose of this section is to put into practise the machinery developed in the preceding sections of this chapter in an intricate holographic setup, where the SymTFT can be used to derive new constraints on the 3d field theory. In particular, we derive SymTFT terms via M-theory reduction and compute the relevant coefficients using the toric CY_4 methods explained in section 4.3.

The BF-terms we obtain are

$$\frac{S_{\text{BF}}}{2\pi} = \int B_2 \wedge \left(N f_2 + \gcd(p,k) \widetilde{g}_2 + \Omega^{p,k}_{n_0,n_1} g_2 \right) \,, \tag{4.106}$$

with integral coefficients $\Omega^{p,k}_{n_0,n_1}$ which depend on p, k as well as the G_4 torsion flux parameterized by two integers (n_0, n_1). Furthermore, in many cases we derive new 1-form symmetry anomalies of the form

$$\Omega_{BB} \int B_2 \smile B_2 \,, \tag{4.107}$$

for 1-form symmetry background fields B_2 and some coefficient Ω_{BB} which we compute.

The dual field theories described in [210] are subtle and furthermore not completely constrained. We discuss the matching of our results with this work in section 4.6, and comment on how the SymTFT could be used to solve some ambiguities.

It should be noted that more generally one could consider a stack of M2-branes probing the cone $\mathcal{C}(Y^{p,k}(B))$ for more generic base B [227, 228]. For example, for $B = \mathbb{CP}^1 \times \mathbb{CP}^1$ the SymTFT is almost identical in structure, differing only in the number of $\breve{v}_2$ generators, and therefore $\breve{F}_2$ background fields.

4.5.1 SymTFT for General p, k

In this section we perform the torsional reduction detailed in section 4.2 here for

$$\mathcal{M}_{11} = \text{AdS}_4 \times Y^{p,k}(\mathbb{CP}^2) \,. \tag{4.108}$$

The cohomology groups for the 7-dimensional space are

$$H^\bullet(Y^{p,k}(\mathbb{CP}^2); \mathbb{Z}) = \{\mathbb{Z}\,, 0\,, \mathbb{Z} \oplus \mathbb{Z}_{\gcd(p,k)}\,, 0\,, \Gamma\,, \mathbb{Z}\,, \mathbb{Z}_{\gcd(p,k)}\,, \mathbb{Z}\} \,. \tag{4.109}$$

where Γ is a finite group given by

$$\Gamma \cong \mathbb{Z}^2 / \langle (3k, k), (k, p) \rangle \,. \tag{4.110}$$

We expand $\breve{G}_4$ on generators for each of these non-trivial elements

$$\breve{G}_4 = N\breve{\text{vol}}_{\text{AdS}_4} + \breve{F}_2 \star \breve{v}_2 + \breve{B}_2 \star \breve{t}_2 + \sum_{\alpha=1}^{2} \breve{b}^\alpha \star \breve{t}_4^\alpha \,, \tag{4.111}$$

where we have included the flux of the M2-branes in the first term and the parameters $\breve{b}^\alpha$ represent the torsion G_4 flux.

$\breve{t}_2$ generators. We use methods described in section 4.3.3 to compute the 1-form symmetry generator (for cases with non-trivial $\breve{B}_2$ field, so $\gcd(p,k) \neq 1$)

$$Z^6 = \sum_{a=1}^{p-1} aC_a \,, \tag{4.112}$$

where C_a are compact toric divisors associated to the points $(0,0,a,1)$ in the toric diagram.

$\breve{v}_2$ generators. We have $b_2(Y^{p,k}(\mathbb{CP}^2)) = 1$, and we can use any one of the non-compact divisors to represent the single $\breve{v}_2$ generator.

$\breve{t}_4^\alpha$ generators. We follow the prescription in section 4.3.3 and construct a basis of compact 4-cycles. We wish to compute the torsional components of Γ. Again focusing on cases where $\gcd(p,k) \neq 1$, we obtain the following formula from the Smith decomposition of the lattice $\langle (3k,k),(k,p) \rangle$

$$H^4(Y^{p,k}(\mathbb{CP}^2);\mathbb{Z}) = \Gamma = \mathbb{Z}_{\gcd(p,k)} \oplus \mathbb{Z}_{\frac{k(3p-k)}{\gcd(p,k)}} \tag{4.113}$$

We generically denote these torsional components

$$\Gamma = \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \,. \tag{4.114}$$

We can independently turn on G_4 flux of varying amounts in both directions. These flux numbers are denoted (b^1, b^2), along the directions given in (4.113).

In [210] the authors parametrize G_4 flux along the Γ directions with two integers (n_0, n_1), which differ from (b^1, b^2) by a basis change. To make contact with their results we require a mapping between (n_0, n_1) and the torsional flux parameters introduced in the differential cohomology language above. This basis change can be read off by the column entries in the matrix B defined as

$$\begin{pmatrix} \gcd(p,k) & 0 \\ 0 & \frac{k(3p-k)}{\gcd(p,k)} \end{pmatrix} = \mathrm{SNF}\begin{pmatrix} 3k & k \\ k & p \end{pmatrix} = A \cdot \begin{pmatrix} 3k & k \\ k & p \end{pmatrix} \cdot B \,. \tag{4.115}$$

We are able to provide a general expression for $Y^{p,p/c}$ for some $c \in \mathbb{Z}$ which divides p

$$b^1 = 1 \times n_0 + 0 \times n_1 \,, \quad b^2 = -c \times n_0 + 1 \times n_1 \,, \tag{4.116}$$

but give a selection of numerical values in table B.3.

SymTFT Coefficients. We derive the following Symmetry TFT for general $Y^{p,k}(\mathbb{CP}^2)$ geometries with $\gcd(p,k)$ non-trivial [10]

$$\begin{aligned} \frac{S_{\text{top}}}{2\pi} = &+ \alpha_{FF(k_1)} \int_{\mathcal{M}_4} \breve{F}_2 \star \breve{F}_2 \star \breve{b}^1 + \alpha_{FF(k_2)} \int_{\mathcal{M}_4} \breve{F}_2 \star \breve{F}_2 \star \breve{b}^2 \\ &- \alpha_{FB(k_1)} \int_{\mathcal{M}_4} \breve{F}_2 \star \breve{b}^1 \star \breve{B}_2 - \alpha_{FB(k_2)} \int_{\mathcal{M}_4} \breve{F}_2 \star \breve{b}^2 \star \breve{B}_2 \\ &- \alpha_{BB(k_1)} \int_{\mathcal{M}_4} \breve{b}^1 \star \breve{B}_2 \star \breve{B}_2 - \alpha_{BB(k_2)} \int_{\mathcal{M}_4} \breve{b}^2 \star \breve{B}_2 \star \breve{B}_2 \,. \end{aligned} \tag{4.117}$$

Given the non-trivial way the $\breve{t}_4$ dual 4-cycles are determined, we do not expect a nice closed-form expression for general p, k for all coefficients. In table B.2 we summarize the α coefficients for a large set of values of (p, k).

Since the background torsion flux participates in all the above couplings, this general SymTFT only contains terms of three types: FF, BF and BB – we are particularly interested in the latter two[11]. The BB term is a 1-form symmetry anomaly, whilst the BF term will be crucial in understanding possible global forms of the gauge group.

[10]Note here we do note include $\breve{F}_4 \star \breve{b} \star \breve{b}$ and $\breve{p}_1$ terms as these contain only one or zero dynamical fields. The coefficients α are expressible in terms of the $\Lambda^{\cdots}_{\cdots}$ of (4.42), but we choose this notation from now on for compactness.

[11]With a choice of boundary conditions the FF terms are background Chern-Simons terms for 0-form global symmetry background gauge fields.

4.5.2 The BF-Term

In this section we focus in particular on terms of BF type, which govern the choice of gauge group in the 3d SCFT. These terms come from two sources: the first is torsion in the geometry, as introduced via non-commuting flux operators in section 4.4.1. The second is from background flux, both continous and discrete. The first type is standard, appearing already in $\mathrm{AdS}_5 \times S^5$ with N units of F_5 flux over the external space [151], as well as in (2.18) [1]. The latter appears via terms in the differential cohomology reduction of the 11d topological terms of the form

$$\int_{Y_7} \breve{t}_4 \star \breve{t}_2 \star \breve{v}_2 \int_{\mathcal{M}_4} \breve{F}_2 \star \breve{B}_2 \star \breve{b}\,. \tag{4.118}$$

BF-terms from Non-Commuting Fluxes. We follow the procedure outlined in [65] which we applied in section 4.4.1 to derive a new BF term:

$$\frac{S_{\mathrm{BF}}}{2\pi} = \gcd(p,k) \int_{\mathrm{AdS}_4} B_2 \wedge dB_1\,. \tag{4.119}$$

The origin of the field B_1 is exactly the same as that presented in section 4.4.1, and the derivation of the coefficient follows analogously with minor modifications.

BF-terms from 11d CS-term. From reduction of the 11d CS-term we obtain the following 4d term of BF-type

$$\frac{S_{\mathrm{top}}}{2\pi} \supset \gcd(p,k) \left(\alpha_{FB(k_1)} b^1 + \alpha_{FB(k_2)} b^2\right) B_2 \wedge g_2\,. \tag{4.120}$$

We denote this coefficient as

$$\Omega^{p,k}_{n_0,n_1} \equiv \gcd(p,k) \left(\alpha_{FB(k_1)} b^1 + \alpha_{FB(k_2)} b^2\right)\,, \tag{4.121}$$

and give values of $\Omega^{p,k}_{n_0.n_1}$ for certain (p,k) in table B.4. The coefficient depends implicitly on the integers (n_0, n_1), which are linear combinations of (b^1, b^2) determined by (4.115).

Full BF-term. Reduction of the 11d supergravity action on the cohomology of $Y^{p,k}(\mathbb{CP}^2)$, with discrete background flux parametrized by (n_0, n_1), thus yields

$$\frac{S_{\rm BF}}{2\pi} = \int B_2 \wedge d\left(\gcd(p,k)B_1 + \Omega^{p,k}_{n_0,n_1} c_1\right) + \cdots . \tag{4.122}$$

Here we have left open the possibility for further BF-type terms arising when we turn on background gauge fields for the isometry group of $Y^{p,k}(\mathbb{CP}^2)$ which is $SU(3) \times U(1)^2$. In section 4.4 we used a reduction to type IIA to conjecture that gauging the M-theory circle direction would furnish a new coupling with the discrete 2-form B_2. In appendix B.1 we derive the analogous coupling from reduction to type IIA for these geometries. The result is

$$\boxed{\frac{S_{\rm BF}}{2\pi} = \int B_2 \wedge \left(N f_2 + \gcd(p,k) dB_1 + \Omega^{p,k}_{n_0,n_1} g_2\right) \,,} \tag{4.123}$$

where $f_2 = da_1$ is the field strength of the $U(1)$ 1-form gauge field associated with the M-theory circle direction.

Let us consider the field theory interpretation of the bulk gauge fields at the level of the SymTFT – i.e. before imposing boundary conditions consistent with the BF-coupling (4.123), which realise a particular global form of the 3d gauge group. The gauge fields in this EFT arise respectively from a reduction of C_3 on the free part, c_1 (with $g_2 = dc_1$), and torsion part, B_1, B_2, of $H^\bullet(Y^{p,k};\mathbb{Z})$, and from gauging the $U(1)$ isometry of the M-theory circle, a_1. At the boundary, the 2-form gauge field may give rise to a background for a 1-form symmetry which is a subgroup of $U(1)_{B_2}$. Fixing the 1-form gauge fields at the boundary we may realise a 0-form symmetry that sits inside $U(1)^3 = U(1)_{a_1} \times U(1)_{B_1} \times U(1)_{c_1}$. In particular, we can parametrize a set of 1-form gauge fields A, A' for $U(1)^2 \subset U(1)^3$ defined by

$$(a_1, B_1, c_1) = (-yA - zA', xA, xA') \,, \tag{4.124}$$

with

$$\frac{N}{x} = \frac{\gcd(p,k)}{y} = \frac{\Omega^{p,k}_{n_0,n_1}}{z} = \gcd(N, p, k, \Omega^{p,k}_{n_0,n_1}) \,, \tag{4.125}$$

which decouple entirely from the action (4.123) and so can always be fixed at the boundary, giving rise to a $U(1)^{(0)} \times U(1)^{(0)}$ global symmetry of the dual field theory. The isometry group of $Y^{p,k}$ is $SU(3) \times U(1)^2$. In the UV, we can identify one of the $U(1)$ factors with the topological $U(1)$ symmetry of the field theory and the other with the R-symmetry. However, as is well-known, this R-symmetry mixes with the other $U(1)$ global symmetries at the SCFT fixed point to give the superconformal R-symmetry. (The exact IR superconformal R-charge can be determined by extremization of the 3-sphere partition function [229].) The $SU(3)$ isometry group of the base $\mathbb{CP}^2$ corresponds to the baryonic $SU(3)$ that rotates the bifundamental matter in the quiver. In addition to the M-theory $U(1)$ circle direction associated with the gauge field c_1, we therefore have an $SU(3) \times U(1)$ isometry for which we do not turn on gauge fields.

4.5.3 Boundary Conditions and Global Symmetries

Given the SymTFT, we can now realise a selection of the possible global forms of the gauge group of the boundary theory (up to 't Hooft anomalies which obstruct certain gaugings, which we discuss shortly). From a supergravity perspective, determining the *complete* set of boundary conditions would amount to enumerating all boundary conditions consistent with the bulk topological terms. Imposing a particular set of boundary conditions on the gauge fields, consistent with the action (4.123), picks out a specific global structure for the 3d $\mathcal{N} = 2$ quivers.

Standard Boundary Conditions. First consider Dirichlet boundary conditions on a_1, B_1 and c_1, and Neumann on B_2. The action forces the boundary constraint

$$\left(Na_1 + \gcd(p,k)B_1 + \Omega^{p,k}_{n_0,n_1} c_1\right) = 0\,. \tag{4.126}$$

This corresponds to a 0-form symmetry

$$G^{(0)} \sim U(1) \times U(1) \times \mathbb{Z}_{\gcd(N,p,k,\Omega^{p,k}_{n_0,n_1})}\,. \tag{4.127}$$

This global 0-form symmetry sits inside the $U(1)^3 = U(1)_{a_1} \times U(1)_{B_1} \times U(1)_{c_1}$. The two $U(1)$'s in $G^{(0)}$ can be parametrized by A and A' as in (4.124). There is no 1-form symmetry with this choice of boundary conditions.

Mixed Boundary Conditions. Consider fixing c_1 and a_1, but letting B_1 be free within $\mathbb{Z}_n \subset \mathbb{Z}_{\gcd(p,k)}$, with $\gcd(p,k) = nn'$. This is equivalent to saying that B_1 is free in $\mathbb{Z}_{\gcd(p,k)}$ modulo the relation $n'B_1 = 0$. The global symmetries of this choice are therefore

$$G \sim U(1)^{(0)} \times U(1)^{(0)} \times \mathbb{Z}^{(0)}_{\gcd(N,\Omega^{p,k}_{n_0,n_1},n')} \times \mathbb{Z}^{(1)}_n \,. \tag{4.128}$$

Clearly the special case where $(n, n') = (1, \gcd(p, k))$ is the 'standard' choice given above. Another special case is $(n, n') = (\gcd(p, k), 1)$, which realizes the largest possible 1-form symmetry group. In this case, the global symmetries are

$$G \sim U(1)^{(0)} \times U(1)^{(0)} \times \mathbb{Z}^{(1)}_{\gcd(p,k)} \,. \tag{4.129}$$

Here, since B_2, a_1 and c_1 are fixed at the boundary, the following BF term

$$\int B_2 \wedge d\left(Na_1 + \Omega^{p,k}_{n_0,n_1} c_1\right) \,, \tag{4.130}$$

corresponds to a 3d mixed anomaly.

General Boundary Conditions. We describe a subset of the allowed boundary conditions and the resulting global symmetries in tables 4.1 and 4.2 respectively.

BC	a_1	B_1	c_1
1	D	D	N/D; Free mod $\mathbb{Z}_{n'} \subset \mathbb{Z}_{\Omega^{p,k}_{n_0,n_1}}$
2	D	N/D; Free mod $\mathbb{Z}_{m'} \subset \mathbb{Z}_{\gcd(p,k)}$	D
3	N/D; Free mod $\mathbb{Z}_{l'} \subset \mathbb{Z}_N$	D	D

Table 4.1: A selection of the possible boundary conditions consistent with the BF-action (4.123), where D: Dirichlet and N: Neumann. We take $\Omega^{p,k}_{n_0,n_1} = nn'$, $\gcd(p,k) = mm'$ and $N = ll'$. '*N/D*; Free mod $\mathbb{Z}_q$' for a field A means that the field is free to fluctuate modulo the relation $qA = 0$.

Boundary Condition	$G^{(0)}$	$G^{(1)}$
1	$U(1)^2 \times \mathbb{Z}_{\gcd(N,p,k,n')}$	$\mathbb{Z}_n$
2	$U(1)^2 \times \mathbb{Z}_{\gcd(N,m',\Omega^{p,k}_{n_0,n_1})}$	$\mathbb{Z}_m$
3	$U(1)^2 \times \mathbb{Z}_{\gcd(l',p,k,\Omega^{p,k}_{n_0,n_1})}$	$\mathbb{Z}_l$

Table 4.2: 0- and 1-form symmetries for the boundary conditions in table 4.1.

4.5.4 1-Form Symmetry Anomaly

Using the definition of the primary invariant, the BB terms of (4.117) evaluate to

$$\left.\frac{S_{\text{top}}}{2\pi}\right|_{BB} = \Omega_{BB} \int_{\mathcal{M}_4} B_2 \smile B_2 \,, \tag{4.131}$$

with

$$\Omega_{BB} = -\alpha_{BB(k_1)} b^1 - \alpha_{BB(k_2)} b^2 \,. \tag{4.132}$$

The coefficients Ω_{BB} depend on p, k, b^1 and b^2 (or, equivalently p, k, n_0 and n_1). This term is a 1-form symmetry anomaly: it presents an obstruction to gauging certain subgroups of the 1-form symmetry. In other words, it is an obstruction to selecting certain boundary conditions of the BF-action (4.123). Suppose $\gcd(p,k) = mm'$ and we consider gauging a subgroup $\mathbb{Z}_m \subset \mathbb{Z}_{\gcd(p,k)}$ of the 1-form symmetry with background B_2. The anomaly free condition is that

$$2\Omega_{BB} m'^2 = 0 \mod 1 \,. \tag{4.133}$$

Specific coefficients of this anomaly can be computed using table B.2 (for a parametrization in terms of (n_0, n_1) we make use of table B.3 as well).

For example, $Y^{2,2}$ with (n_0, n_1) flux numbers has $\Omega_{BB} = -\frac{3}{4} n_1$. If we consider gauging the $\mathbb{Z}_2$ 1-form symmetry, the anomaly free condition is that $\frac{3}{2} n_1 = 0 \mod 1$. Hence, we can only gauge the 1-form symmetry if n_1 is even. In this way the torsion flux influences the possible choices of gauge group one can have for a given theory. Furthermore, we should highlight the presence of many mixed 0-/1-form symmetry anomalies of the type demonstrated in (4.130).

4.6 Comparison to the Field Theory dual to $Y^{p,k}(\mathbb{CP}^2)$

We now compare our results with the field theory of [210], which is subtle for several reasons. The proposed quiver gauge theories initially have a parity anomaly which much be quenched to ensure consistency. The authors provide several mechanisms through which this could occur. We show that ambiguity in this anomaly resolution permeates into the global symmetries of the theory: in particular the 1-form symmetry is sensitive to the anomaly cancellation mechanism one chooses. In this section we discuss how the SymTFT can be used to constrain this problem.

4.6.1 Quiver Gauge Theories

The quiver gauge theories dual to the $\mathrm{AdS}_4 \times Y^{p,k}(\mathbb{CP}^2)$ M-theory backgrounds [210] are defined for three 'windows' of parameter values of the G_4 torsion flux, parametrized by integers (n_0, n_1).

1. $-k \leq n_0 \leq 0\,, \quad 0 \leq 3n_1 - n_0 \leq 3p - k$
2. $0 \leq n_0 \leq k\,, \quad 0 \leq 3n_1 - n_0 \leq 3p - k$
3. $k \leq n_0 \leq 2k\,, \quad 0 \leq 3n_1 - n_0 \leq 3p - k$

The field theories for these three cases are

1. $U(N + n_1 - p - n_0)_{-n_0+\frac{3}{2}n_1} \times U(N)_{\frac{1}{2}n_0-3n_1+\frac{3}{2}p-k} \times U(N - n_1)_{\frac{1}{2}n_0+\frac{3}{2}n_1-\frac{3}{2}p+k}$
2. $U(N + n_1 - p)_{-n_0+\frac{3}{2}n_1} \times U(N)_{2n_0-3n_1+\frac{3}{2}p-k} \times U(N - n_1)_{-n_0+\frac{3}{2}n_1-\frac{3}{2}p+k}$
3. $U(N+n_1-p)_{\frac{1}{2}n_0+\frac{3}{2}n_1-\frac{3}{2}q} \times U(N)_{\frac{1}{2}n_0-3n_1+\frac{3}{2}p+\frac{1}{2}k} \times U(N-n_1+n_0-k)_{-n_0+\frac{3}{2}n_1-\frac{3}{2}p+k}$

with bi-fundamental matter content arranged in a quiver structure shown in figure 4.1.

The theories as they are presented above suffer from a $\mathbb{Z}_2$ parity anomaly. The authors of [210] suggest that there are several mechanisms through which this residual

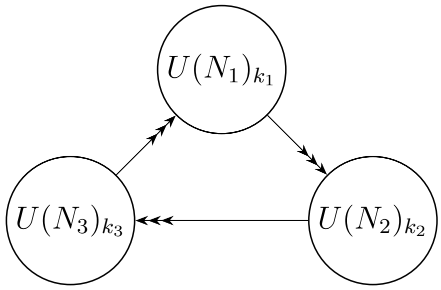


Figure 4.1: Quiver diagram for theory with gauge group $\Pi_{i=1}^{3} U(N_i)_{k_i}$. The triple arrows denote the fact that the bi-fundamental matter fields transform in the fundamental representation of a flavor $SU(3)$.

anomaly could be cancelled. They highlight the simplest: the addition of mixed Chern-Simons couplings between the $U(1)$ pieces of different $U(N_i), U(N_j)$ factors, with levels Λ_{ij} such that [230–232]

$$k_i + \frac{1}{2}\sum_j A_{ij} N_j \in \mathbb{Z}\,, \quad \Lambda_{ij} - \frac{1}{2} A_{ij} \in \mathbb{Z}\,. \tag{4.134}$$

Here k_i are the Chern-Simons levels given above, A_{ij} is the quiver adjacency matrix

$$A_{ij} = \begin{pmatrix} 0 & 3 & -3 \\ -3 & 0 & 3 \\ 3 & -3 & 0 \end{pmatrix}\,. \tag{4.135}$$

The first condition is satisfied by the above, but the second is not since we have so far set $\Lambda_{ij} = 0$. In [210], for theories with $(n_0, n_1) = (0, 0)$, the authors quote a sufficient choice

$$\Lambda_{ij} = \begin{pmatrix} 0 & \frac{3}{2} & -\frac{3}{2} \\ \frac{3}{2} & -3 & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} & 0 \end{pmatrix}\,. \tag{4.136}$$

which does not spoil the matching of the moduli space with the geometry. Considering only the spectrum of local operators this appears to be an ambiguity in the AdS/CFT correspondence. We now return to study this ambiguity from the point of view of the 1-form symmetry, which is sensitive to Λ_{ij}.

4.6.2 1-Form Symmetry of the Quivers

We now compute the 1-form symmetry of these field theories. The key subtlety in this computation is the presence of monopole operators which can screen Wilson lines. A monopole operator in this theory is specificed by its magnetic charges under the $U(1)$ elements of the Cartan subgroup of each $U(N_i)$ factor

$$H_i = (m_{i,1} \dots m_{i,N_i}) \,. \tag{4.137}$$

Crucially, the choice of Chern-Simons levels Λ_{ij} can influence the gauge charges of monopoles and therefore the 1-form symmetry.

Electric Charge of Monopoles. In a vacuum where the gauge group $\Pi_i U(N_i)$ is broken to its maximal abelian subgroup, the Lagrangian becomes [210]

$$\mathcal{L}_{\text{CS}} = \sum_{i,m}\sum_{j,n} \frac{k_i \delta_{ij}\delta_{mn} + \Lambda_{ij}}{4\pi} A_{i,m} \wedge dA_{j,n} \,. \tag{4.138}$$

Suppose we put the theory (4.138) on $\mathbb{R} \times S^2$ and integrate over the S^2:

$$\int_{S^2} \mathcal{L}_{\text{CS}} = \left(\sum_{i,m} \frac{k_i}{4\pi} m_{i,m} + \sum_{i,m}\sum_{j,n} \frac{\Lambda_{ij}}{4\pi} m_{j,n} \right) \int A_{i,m} \,. \tag{4.139}$$

From this we observe that a monopole acquires electric gauge charge under each $U(1)$ Cartan of each $U(N_i)$:

$$g_{i,m} = k_i m_{i,m} + \sum_{j,l} \Lambda_{ij} m_{j,l} \,. \tag{4.140}$$

One-Loop Monopole Charge Modifications. This expression is modified at 1-loop [210] due to integrating out bifundamental matter X_{ij} [12]:

$$\boxed{g_{i,k} = k_i m_{i,k} + \delta g_{i,k} + \sum_{l}\sum_{j} \Lambda_{ij} m_{j,l} \,,} \tag{4.141}$$

with

$$\delta g_{i,k} = -\frac{1}{2} \sum_{X_{ij}} \sum_{l=1}^{N_j} |m_{i,k} - m_{j,l}| + \frac{1}{2} \sum_{X_{ji}} \sum_{l=1}^{N_j} |m_{i,k} - m_{j,l}| \,. \tag{4.142}$$

[12]Note that the formula in [210] contains a third correction term, which for our case of a circular quiver vanishes.

Charge under the Center. The charge of a monopole under the central $U(1)^3 = Z(\mathcal{G})$ is

$$q_i = \sum_k^{N_i} g_{i,k}\,, \tag{4.143}$$

The bifundamental matter breaks this to a diagonal $U(1) \subset U(1)^3$ under which the monopole has charge

$$q_{\text{diag}} = \sum_{i=1}^{3} q_i\,. \tag{4.144}$$

It can be checked explicitly that one can drop the 1-loop correction ($\delta g_{i,k}$) contribution from q_{diag} due to the quiver's shape:

$$q_{\text{diag}} = \sum_{i=1}^{3} q_i = \sum_i^3 \sum_{k=1}^{N_i} \left(k_i m_{i,k} + \sum_l \sum_j \Lambda_{ij} m_{j,l} \right)\,. \tag{4.145}$$

Denoting the topological $U(1)$ charges $m_i = \sum_l m_{i,l}$, we write this as

$$\boxed{q_{\text{diag}} = \sum_{i=1}^{3} q_i = \sum_i^3 \left(k_i m_i + N_i \sum_j \Lambda_{ij} m_j \right)\,.} \tag{4.146}$$

$Y^{p,k}$ **without torsion flux.** We now explicitly compute q_{diag} for an arbitrary monopole in a general $Y^{p,k}$ theory without torsion flux $(n_0, n_1) = (0,0)$, with gauge group

$$U(N-p)_0 \times U(N)_{\frac{3}{2}p-k} \times U(N)_{-\frac{3}{2}p+k}\,. \tag{4.147}$$

Furthermore, we consider *arbitrary* Λ_{ij} which obeys both the parity anomaly condition and the moduli space matching condition [210]

$$\Lambda_{ij} - \frac{1}{2}A_{ij} \in \mathbb{Z}\,, \quad \sum_{j=1}^{3} \Lambda_{ij} = 0\,. \tag{4.148}$$

For a monopole with charge (m_1, m_2, m_3), we obtain

$$q_{\text{diag}} = m_1 \left(-p\Lambda_{11}\right) + m_2 \left(\frac{3}{2}p - k - p\Lambda_{12} \right) + m_3 \left(-\frac{3}{2}p + k - p\Lambda_{13} \right)\,. \tag{4.149}$$

In the triangular quiver in question, the adjacency matrix is given in (4.135). Using the parity anomaly condition, we can rewrite ($\lambda_i \in \mathbb{Z}$)

$$\Lambda_{11} = \lambda_1\,, \quad \Lambda_{12} = \frac{3}{2} + \lambda_2\,, \quad \Lambda_{13} = -\frac{3}{2} + \lambda_3\,, \tag{4.150}$$

with the futher condition $\sum_i \lambda_i = 0$. We therefore have

$$q_{\text{diag}} = -p\lambda_1 m_1 - (\lambda_2 p + k)m_2 + (k + (\lambda_1 + \lambda_2)p)m_3 \,. \tag{4.151}$$

The final 1-form symmetry of the field theory is the subgroup of the diagonal $U(1) \subset Z(\mathcal{G})$ which leaves *all* monopoles invariant:

$$\boxed{\Gamma^{(1)} = \mathbb{Z}_{\text{gcd}(p\lambda_1, \lambda_2 p + k)} \,.} \tag{4.152}$$

For example, picking $\lambda_1 = 1$ and leaving λ_2 arbitrary gives $\Gamma^{(1)} = \mathbb{Z}_{\text{gcd}(p,k)}$. Furthermore, the choice of λ_i must be compatible with supersymmetry. We have a supersymmetric solution when the effective FI parameters satisfy [210]

$$\xi_1^{\text{eff}} = 0 \,, \quad \xi_2^{\text{eff}} = -\xi_3^{\text{eff}} \,. \tag{4.153}$$

Since $\xi_1^{\text{eff}} \propto \lambda_1$, the authors of [210] suggest that a *convenient* choice is $\lambda_1 = 0$, and $\lambda_2 = 0$. In this case, there is an enhancement of the above 1-form symmetry to $\Gamma^{(1)} = \mathbb{Z}_k$. We emphasise that this solution is far from unique. Picking $\lambda_1 \neq 0$ means that we must introduce bare FI parameters ξ_i^{bare} to fulfill the SUSY requirement.

$Y^{p,k}$ with torsion flux. Now consider the general $Y^{p,k}$ with torsion flux $(n_0, n_1) \neq (0,0)$. For demonstration we consider the first window of torsion space, where the gauge group is

$$U(N + n_1 - p - n_0)_{-n_0 + \frac{3}{2}n_1} \times U(N)_{\frac{1}{2}n_0 - 3n_1 + \frac{3}{2}p - k} \times U(N - n_1)_{\frac{1}{2}n_0 + \frac{3}{2}n_1 - \frac{3}{2}p + k} \,. \tag{4.154}$$

We use the parameterization (for $i \neq j$)

$$\Lambda_{ij} = A_{ij} + \lambda_{ij} \,, \quad \lambda_{ij} \in \mathbb{Z} \,, \tag{4.155}$$

to derive

$$\begin{aligned} q_{\text{diag}} &= m_1 \left(-(1 + \Lambda_{11})n_0 + (\Lambda_{11} - \lambda_{31})n_1 - p\Lambda_{11}\right) \\ &+ m_2 \left((-1 - \lambda_{12})n_0 + (\lambda_{12} - \lambda_{32})n_1 + p(-\lambda_{12}) - k\right) \\ &+ m_3 \left((-1 - \lambda_{13})n_0 + (3 + \lambda_{13} - \Lambda_{33})n_1 + p(-\lambda_{13}) + k\right) \\ &\equiv \sum_{i=1}^{3} m_i h_i \,. \end{aligned} \tag{4.156}$$

Once again we have the condition $\sum_j \Lambda_{ij} = 0$ which enforces some redundancies in the parameters λ_{ij} via $\sum_j \lambda_{ij} = 0$. Since all parameters $m_i, \lambda_{ij}, \Lambda_{ii}$ are integers, the trivially acting subgroup of $U(1) \subset Z(\mathcal{G})$ is

$$\boxed{\Gamma^{(1)} = \mathbb{Z}_{\gcd(h_1,h_2,h_3)}\,.} \tag{4.157}$$

Again, one must check that any particular choice of Λ_{ij} preserves supersymmetry.

4.6.3 A Check on the Holographic Dictionary

In this subsection we aim to demonstrate how consistency with the SymTFT can be used to constrain the field theory. In particular, since the 1-form symmetry is sensitive to the $U(1)$ CS-levels Λ_{ij}, the SymTFT predicts that only certain sets of Λ_{ij} can potentially be realised. This illustrates in a concrete problem how the study of higher-form symmetries in AdS/CFT refines the dictionary. We focus on $Y^{p,p}$ with all G_4 torsion flux turned off. The SymTFT is

$$\frac{S_{\mathrm{BF}}}{2\pi} = \int pB_2 \wedge dB_1 + NB_2 \wedge da_1\,. \tag{4.158}$$

The gauge group of the 3d field theory [210] with which we would like to match a boundary condition of the SymTFT is

$$U(N-p)_0 \times U(N)_{\frac{1}{2}p} \times U(N)_{-\frac{1}{2}p}\,, \tag{4.159}$$

with 1-form symmetry given by

$$\Gamma^{(1)} = \mathbb{Z}_{p\cdot\gcd(\lambda_1,\lambda_2+1)}\,. \tag{4.160}$$

We want to show that not every choice of λ_1, λ_2 is consistent with (4.158). In more precise terms, imposing boundary conditions consistent with the BF-term can give rise to a restricted set of 1-form symmetries. We show that not all values of λ_1, λ_2 corresponds to field theories whose 1-form symmetry belongs to this set.

If we pick the Dirichlet boundary condition for a_1 and Neumann for B_1, the boundary field theory has 1-form symmetry $\Gamma^{(1)} = \mathbb{Z}_p$. Swapped boundary conditions would give $\Gamma^{(1)} = \mathbb{Z}_N$ 1-form symmetry, whilst any mixed condition would give $\Gamma^{(1)} \subseteq \mathbb{Z}_{\gcd(p,N)} \subseteq \mathbb{Z}_p$. It is clearly not possible therefore to pick a boundary condition with $\Gamma^{(1)} = \mathbb{Z}_{l\cdot p}$, for some $l \in \mathbb{N}$ for all N. Noticing that the field theory result (4.160) is valid for all N, we can therefore constrain Λ_{ij} to be such that

$$l = \gcd(\lambda_1, \lambda_2 + 1) = 1 \,. \tag{4.161}$$

Thus compatiblity between the SymTFT and field theory computations can be used to constrain the $U(1)$ Chern-Simons levels conjectured to resolve the known parity anomalies of these theories. Note that here we take the perspective where the holographic background and field theory dual are given and fixed - and wish to find consistent choices of CS levels which retain this duality. Our proposal here is that there are certain choices of CS levels which would yield slightly different holographic dualities to field theories which are not contained in [210]. We have focused here on a simple $Y^{p,p}$ model without torsion flux for concreteness, but claim that this technique is generically applicable to a broader class of examples. For general (p, k, n_0, n_1) the coefficients $\Omega^{p,k}_{n_0,n_1}$ are given in table B.4 and the 1-form symmetry is given by (4.157): with this information one can run a similar analysis in any case of interest.

4.7 Outlook

There are several possible avenues of future work, some of which we summarize now:

The utility of SymTFTs is only being uncovered, and much remains to be understood, both field theoretically, but also in the realization of SymTFTs from string/M-theory. In this work we derived SymTFT terms from two sources: the differential cohomology reduction of C_3, and from gauged isometries of the internal space Y_7. However, these two sources were kept distinct, whereas, ideally, they would be treated in a unified manner. If one could identify the appropriate framework, we expect that this could yield new topological couplings in many interesting setups.

Although the examples that we considered were based on Calabi-Yau cones and associated Sasakian 7-manifolds (as well as their holographic counterparts), the methods should equally apply to other string/M-theory compactifications with special or exceptional holonomy. A natural extension of the work in this chapter is to consider Spin(7) holonomy spaces, which often are quotients by orientation-reversal of Calabi-Yau fourfolds.

In view of holography, our main example was to study duals to 3d $\mathcal{N} = 2$ SCFTs, and we focused on the addition of extra $U(1)$ Chern-Simons terms as a resolution to the parity anomaly of the $\mathcal{N} = 2$ quiver gauge theories of [210]. One other possibility is to restrict the gauge group to $\mathcal{G}' = (\Pi_{i=1}^{3} SU(N_i)) \times U(1)$. It would be interesting to examine the full scope of the SymTFT in terms of its constraining power with regards to the consistency of these field theories.

Another possible future direction is to consider the SO-Sp type 3d SUSY gauge theories, which are defined as the worldvolume theory of N M2-branes probing a $\mathbb{C}^4/\widehat{D}_k$ singularity [162]. Here $\widehat{D}_k$ is the binary dihedral group with order $4k$, and the singularity $\mathbb{C}^4/\widehat{D}_k$ is the anti-holomorphic involution of the toric singularity $\mathbb{C}^4/\mathbb{Z}_{2k}$. To compute the SymTFT in this case, one needs to work out a real resolution of $\mathbb{C}^4/\widehat{D}_k$. It would be interesting to compare the geometric results with the expected higher-form, higher-group and non-invertible symmetries from field theory [133, 233].

The SymTFT is a powerful tool. In our analysis we have focused on two of its key features: encoding the choice of the global form of the gauge group, and the presence of 't Hooft anomalies for higher-form symmetries. However, by definition, the SymTFT in all its generality should encode all symmetry information about its associated QFT(s). Developing this further, field theoretically, and in conjunction with string/M-theory/holography provides a very exciting future research direction.

Chapter 5

SymTFTs and Generalized Charges from Branes

Recently it has been observed that branes in geometric engineering and holography have a striking connection with generalized global symmetries. In this chapter we argue that branes, in a certain topological limit, not only furnish the symmetry generators, but also encode the so-called Symmetry Topological Field Theory (or SymTFT). In this work we derive the SymTFT and its topological defects directly from branes. Central to the identification of these are Hanany-Witten brane configurations, which encode both topological couplings in the SymTFT and the generalized charges under the symmetries. We exemplify the general analysis with examples of QFTs realized in geometric engineering and holography.

5.1 Introduction

Branes play a central role in string/M-theory: as carriers of gauge degrees of freedom, non-perturbative defects, and as the origin of holographic dualities when back-reacted. Recently it has been observed that in a particular, non-dynamical, limit they give rise to generators of generalized global symmetries (topological defects) in holography [41, 42] and in geometric engineering of QFTs [43, 83].

Much is known about higher-form and higher-group symmetries in the context of geometric/brane engineering and holography [4, 5, 70, 111–122, 124–127, 129, 133–

139, 141, 142, 144–146]. However, most of this work focuses on the (not necessarily topological) extended defect operators, i.e. the generalized (or higher-) charges, which are constructed by wrapping branes on non-compact cycles. These extended objects then mimic infinitely heavy probes in space-time. In turn, relatively little had been known about the symmetry generators in the context of geometric constructions – see however [113, 114] for some discussion in terms of flux operators. The recent identification of symmetry generators with branes [41–43, 83] in a topological limit[1] provides a systematic way to study the symmetries of a given theory.

SymTFTs and Generalized Charges. Recent developments in the realm of generalized symmetries have lead to the idea that separating symmetries from physical theories can be insightful. The structure that allows for this is the SymTFT. The SymTFT is invariant under gauging of global symmetries (i.e. symmetries that are related by gauging have the same SymTFT), and perhaps physically most relevant, its topological defects encode the generalized charges [68, 108]. The separation that seems to have emerged in string theory constructions, into symmetry generators and generalized charges is therefore somewhat artificial. There should be a unified prescription that derives from the string theory construction (geometric engineering or holography) of the SymTFT directly.

SymTFT and Supergravity. In string/M-theory the initial constructions of the SymTFT were closely related to various topological sectors of dimensionally reduced supergravity theories, as explained in chapter 2.

Given the recent proposal [41–43, 83] relating symmetry generators and branes in string theoretic settings, it is natural to ask whether branes also provide a realization of the SymTFT, in particular the topological defects of the SymTFT, as well as

[1] We will equivalently use both "topological limit" or "topological truncation". The procedure we adopt is a truncation to the topological sector [151], which will be explained in more details in section 5.2.

the generalized charges. In this work we propose a general framework for this, and substantiate it in various setups in both geometric engineering and holography. This connects also to the general philosophy, that the symmetry and generalized charges should all have a unified construction in terms of the SymTFT topological defects.

Summary of Results. In this work we will argue that branes (in a certain topological limit) encode the SymTFT of QFTs that are realized in terms of geometric engineering or in holographic dualities. As geometric engineering and holographic theories mostly admit abelian generalized symmetries, we will focus on these symmetries. Restricting to these symmetries, this amounts to showing that branes give rise to BF-terms and (mixed) 't Hooft anomalies at the level of SymTFT topological couplings. At the level of defects of the SymTFT, we use brane effects to determine generalized charges of higher-form symmetries (which are the topological defects of the SymTFT). In the process we also identify condensation defects in terms of branes.

BF-terms for abelian finite higher-form symmetries will be shown to be encoded in the topological sector of 10/11d supergravity once we also include *source terms for wrapped branes*. These have the interpretation of generating the associated symmetries. Terms of this type derive from two origins: either from Chern-Simons terms or kinetic terms in the supergravity action. By including sources, these terms describe how the geometric linking of wrapped branes in the bulk corresponds to the action of symmetry generators, i.e. topological defects, on (extended) charged operators of the QFT.

Including brane sources induces further topological couplings in the SymTFT, which in some global forms can have the interpretation of (mixed) 't Hooft anomalies. We will refer to these topological couplings in the SymTFT as *anomaly couplings*. They are encoded in various linking configurations of branes in the bulk. We first give a general procedure for deriving anomaly couplings from the 10/11d supergravity topological sector and Bianchi identities in terms of background fields. Re-phrasing these relations in terms of brane sources allows us to re-write anomaly couplings as linking configurations of the branes which generate the associated symmetries.

An important aspect of the categorical description of symmetries is the notion of condensation completion [234]: i.e. all symmetries can be condensed on topological defects that are generically defined on submanifolds of spacetime (as opposed to the whole spacetime). So far the conjectured identification of branes with symmetry generators [41,42] does not incorporate condensation defects. In this work we argue that *condensation defects* can be constructed from a "condensation completed" SymTFT, where in addition to the BF-couplings in the $(d+1)$-dimensional spacetime of the SymTFT, we also include couplings to either lower dimensional discrete gauge theories (possibly with theta angles), which realize standard condensation defects, or more generally lower-dimensional TQFTs which give rise to so-called (twisted) theta defects [68].

In the string theoretic setting we will obtain such couplings by considering *brane-anti-brane pairs* (in a topological limit), where the standard Dp-brane charge cancels out, but topological couplings on the world-volumes survive, which live in lower than $p+1$ dimensions.

The topological defects of the SymTFT encode the *generalized charges* of a categorical symmetry [68]. In particular the linking in the SymTFT (for abelian symmetries) provides a way to compute the charges. It was already shown in [41] that in a specific 4d $\mathcal{N}=1$ Super-Yang-Mills theory setting, the action of generalized symmetries on branes can be realized in terms of Hanany-Witten moves on brane-configurations. This realizes the action of the non-invertible symmetries on 't Hooft lines in the $PSU(N)$ SYM theory. In this work we show that more generally, the action of generalized symmetries on generalized charges has as its origin the *Hanany-Witten* configuration and moves for branes.

The general considerations of this work will be illustrated with numerous examples, both in geometric engineering and holography.

In 4d $\mathcal{N}=4$ SYM with algebra $\mathfrak{su}(N)$ we show how Hanany-Witten configurations can be used to diagnose the intrinsic/ non-intrinsic nature of non-invertible symmetries. The brane mechanism imposes a simple constraint which allows a classification of the type of these non-invertible symmetries for arbitrary gauge group rank, extending previous results [73].

5.2 SymTFT and its Topological Defects from Branes

5.2.1 A Democratic Formulation for Fluxes and Brane Sources

It is convenient to work in a democratic formulation, including both the fluxes and their Hodge duals. Let us describe our notation in a general setting, for a theory defined in $D+1$ dimensions. Here $D+1=10,11$ depending on string or M-theory.

Magnetic Sources. Let $F^{(i)}$ denote the entire collection of fluxes in $D+1$ dimensions, labeled by (i) (unrelated to the form degree). In order to avoid redundancies, we consider only magnetic sources. (An electric source for a given flux $F^{(i)}$ is a magnetic source for the Hodge dual of $F^{(i)}$.) We denote the magnetic source for $F^{(i)}$ by $J^{(i)}$. If the source is localized, $J^{(i)}$ is delta-function supported on a submanifold $\mathcal{W}^{(i)}$. The magnetic source modifies the Bianchi identity for $F^{(i)}$,

$$dF^{(i)}=J^{(i)}=\delta(\mathcal{W}^{(i)})\ , \qquad D+1-\dim\mathcal{W}^{(i)}=\deg J^{(i)}=\deg F^{(i)}+1\,. \tag{5.1}$$

At the moment, we are neglecting the effect of possible Chern-Simons terms. Those will be considered below.

Linking. The $(D+1)$-dimensional linking number between two magnetic sources $J^{(i)}=\delta(\mathcal{W}^{(i)})$ and $J^{(j)}=\delta(\mathcal{W}^{(j)})$ is defined as

$$\mathcal{L}_{D+1}(\mathcal{W}^{(i)},\mathcal{W}^{(j)})=\int_{M_{D+1}} J^{(i)}\wedge d^{-1}J^{(j)}=\int_{M_{D+1}} dF^{(i)}\wedge F^{(j)}\ , \tag{5.2}$$

where the dimensions of $\mathcal{W}^{(i)}$, $\mathcal{W}^{(j)}$ inside the total $(D+1)$-dimensional space satisfy

$$\dim \mathcal{W}^{(i)} + \dim \mathcal{W}^{(j)} = D \ . \tag{5.3}$$

The linking number has the symmetry property

$$\mathcal{L}_{D+1}(\mathcal{W}^{(i)}, \mathcal{W}^{(j)}) = (-1)^{1+\dim \mathcal{W}^{(i)} \dim \mathcal{W}^{(j)}} \, \mathcal{L}_{D+1}(\mathcal{W}^{(j)}, \mathcal{W}^{(i)}) \ , \tag{5.4}$$

which follows from integration by parts in the last integral in (5.2), together with (5.3).

The compact notation with the symbol d^{-1} introduced in (5.2), and used below, is understood as follows. We assume that $\mathcal{W}^{(i)}$, $\mathcal{W}^{(j)}$ are homologically trivial, $\mathcal{W}^{(i)} = \partial \mathcal{S}^{(i)}$, $\mathcal{W}^{(j)} = \partial \mathcal{S}^{(j)}$. The chains $\mathcal{S}^{(i)}$, $\mathcal{S}^{(j)}$ are usually called Seifert (hyper)surfaces [235]. We can then write $\delta(\mathcal{W}^{(i)}) = d\delta(\mathcal{S}^{(i)})$, $\delta(\mathcal{W}^{(j)}) = d\delta(\mathcal{S}^{(j)})$ and interpret (5.2) as

$$\mathcal{L}_{D+1}(\mathcal{W}^{(i)}, \mathcal{W}^{(j)}) = \int_{M_{D+1}} d\delta(\mathcal{S}^{(i)}) \wedge \delta(\mathcal{S}^{(j)}) = \int_{M_{D+1}} \delta(\mathcal{W}^{(i)}) \wedge \delta(\mathcal{S}^{(j)}) = \mathcal{W}^{(i)} \cdot_{M_{D+1}} \mathcal{S}^{(j)} \ , \tag{5.5}$$

where $\mathcal{W}^{(i)} \cdot_{M_{D+1}} \mathcal{S}^{(j)}$ is the number of intersection points of $\mathcal{W}^{(i)}$ and $\mathcal{S}^{(j)}$ inside M_{D+1}, counted with signs depending on orientation.

It is also useful to consider a slight generalization of the notion of linking discussed above, along the following lines. Let's suppose that the supports $\mathcal{W}^{(i)}$, $\mathcal{W}^{(j)}$ span some common directions along some space $\mathcal{V}$, while extending in distinct directions in the rest of spacetime, i.e. $\mathcal{W}^{(i,j)} = \mathcal{V} \times \mathcal{U}^{(i,j)}$. We can define the linking of $\mathcal{W}^{(i)}$ and $\mathcal{W}^{(j)}$ using the same formula as above, but focusing on the $\mathcal{U}^{(i)}$ and $\mathcal{U}^{(j)}$ parts, whose dimensions are such that

$$\dim(\mathcal{W}^{(i)} \cup \mathcal{W}^{(j)}) = D \ . \tag{5.6}$$

This notion of linking is naturally associated to Hanany-Witten moves in string/M-theory, as we discuss in greater detail in section 5.3.

Topological Action in $D+2$ dimensions. The Bianchi identities (5.1) can be derived from a topological action in $D+2$ dimensions,

$$S_{D+2} = \sum_{i,j} \int_{M_{D+2}} \left[\frac{1}{2} \kappa_{ij} F^{(i)} \wedge dF^{(j)} - \kappa_{ij} F^{(i)} \wedge J^{(j)} \right] . \tag{5.7}$$

This is regarded as a functional of the fluxes $F^{(i)}$ (as opposed to the associated gauge potentials). Extending from $D+1$ to $D+2$ dimensions allows us to deal efficiently with gauge invariance. The quantity κ_{ij} is a constant non-degenerate matrix, satisfying

$$\begin{aligned} \kappa_{ij} &= 0 \quad \text{if } \deg F^{(i)} + \deg F^{(j)} \neq D+1 \\ \kappa_{ij} &= (-1)^{[\deg F^{(i)}+1][\deg F^{(i)}+1]} \kappa_{ji} \, . \end{aligned} \tag{5.8}$$

The symmetry property in the second equality reflects the freedom to integrate by parts in the $F^{(i)} \wedge dF^{(j)}$ term. It also ensures that, upon variation of $F^{(i)}$, the topological action yields

$$\sum_j \kappa_{ij} \big(dF^{(j)} - J^{(j)} \big) = 0 \; , \tag{5.9}$$

which, by non-degeneracy of κ_{ij}, is equivalent to $dF^{(j)} = J^{(j)}$, in agreement with our parametrization of magnetic sources.

Let us stress that the relations obtained upon variation of the $(D+2)$-dimensional topological action are still to be supplemented by Hodge duality relations in $D+1$ dimensions. This is illustrated in the following example.

Example: Generalized Maxwell in $D+1$ dimensions. Let us consider a generalized Maxwell theory in $D+1$ dimensions, with a single p-form gauge potential a, with field strength f. In the absence of sources, the action reads

$$S_{D+1} = \int_{M_{D+1}} \frac{1}{2e^2} f \wedge * f \; , \qquad \deg f = p+1 \; . \tag{5.10}$$

The Bianchi identity and equation of motion read

$$df = * \mathcal{J}^{(\mathrm{m})} \; , \qquad e^{-2} d * f = * \mathcal{J}^{(\mathrm{e})} \; , \tag{5.11}$$

where e is the gauge coupling, $*$ is the Hodge star in $D+1$ dimensions, $\mathcal{J}^{(\mathrm{e})}$ is the electric source for a, and $\mathcal{J}^{(\mathrm{m})}$ is the magnetic source for a. In the democratic formulation we introduce two field strengths and two magnetic currents,[2]

$$F^{(1)} = f \ , \qquad F^{(2)} = e^{-2} * f \ , \qquad J^{(1)} = * \mathcal{J}^{(\mathrm{m})} \ , \qquad J^{(2)} = * \mathcal{J}^{(\mathrm{e})} \ . \tag{5.12}$$

The topological action in $d+2$ dimensions is of the form quoted above, with labels i, j ranging from 1 to 2, and with κ_{ij} matrix

$$\kappa_{ij} = \begin{pmatrix} 0 & 1 \\ (-)^{(p+2)(D+1-p)} & 0 \end{pmatrix} \ . \tag{5.13}$$

More explicitly,

$$S_{D+2} = \int_{M_{D+2}} \Big[F^{(1)} \wedge dF^{(2)} - F^{(1)} \wedge J^{(2)} - (-)^{(p+2)(D+1-p)} F^{(2)} \wedge J^{(1)} \Big] \ , \tag{5.14}$$

which, upon variation with respect to $F^{(1)}$, $F^{(2)}$ reproduces $dF^{(1)} = J^{(1)}$, $dF^{(2)} = J^{(2)}$. The $(D+1)$-dimensional Hodge duality relation that has to be supplemented is

$$*F^{(1)} = e^2 F^{(2)} \ . \tag{5.15}$$

Note that in the treatment of [236] such duality relations follow automatically.

Inclusion of Chern-Simons terms. We can include non-trivial Chern-Simons terms by modifying the $(D+2)$-dimensional topological action. The required modification is a polynomial in the $F^{(i)}$ fluxes, denoted $\mathrm{CS}(\{F^{(i)}\})$,

$$S_{D+2} = \int_{M_{D+2}} \bigg[\frac{1}{2} \sum_{i,j} \kappa_{ij} F^{(i)} \wedge dF^{(j)} + \mathrm{CS}(\{F^{(i)}\}) - \sum_{i,j} \kappa_{ij} F^{(i)} \wedge J^{(j)} \bigg] \ . \tag{5.16}$$

Note that here, the terminology "CS" stems from the fact that $\mathrm{CS}(\{F^{(i)}\})$ is a closed $(D+2)$-form which is related to the physical Chern-Simons couplings in $D+1$ dimensions by descent,

$$S^{\mathrm{top}}_{D+1} = \int_{M_{D+1}} I^{(0)}_{D+1} \ , \qquad dI^{(0)}_{D+1} = \mathrm{CS}(\{F^{(i)}\}) \ . \tag{5.17}$$

[2]Notice in particular that in our notation the currents $J^{(i)}$ are closed forms, as opposed to co-closed forms, which is perhaps a more common convention for conserved currents in the literature.

Remember here that $F^{(i)}$ are the full set of magnetic fluxes $dF^{(i)} = J^{(i)}$. Varying (5.16) with respect to $F^{(i)}$ we get

$$\sum_j \kappa_{ij}(dF^{(j)} - J^{(j)}) + \frac{\partial \mathrm{CS}(\{F^{(k)}\})}{\partial F^{(i)}} = 0 \ . \tag{5.18}$$

The notation introduced in this section is summarized in table 5.1.

Actions for Type II and M-theory. To make the above discussion more concrete, let us describe the $(D+2)$-dimensional topological actions for type II ($D+1=10$) and M-theory ($D+1=11$). They are of the form (5.16) with

$$\text{IIA:} \begin{cases} F^{(i)} = (F_0, F_2, F_4, F_6, F_8, F_{10}, H_3, H_7) \ , \\ S_{11} = \int_{M_{11}} \left[F_0 dF_{10} - F_2 dF_8 + F_4 dF_6 + H_3 dH_7 - H_3 \left(F_0 F_8 - F_2 F_6 + \frac{1}{2} F_4^2 + X_8 \right) \right] \ , \end{cases}$$

$$\text{IIB:} \begin{cases} F^{(i)} = (F_1, F_3, F_5, F_7, F_9, H_3, H_7) \ , \\ S_{11} = \int_{M_{11}} \left[F_1 dF_9 - F_3 dF_7 + \frac{1}{2} F_5 dF_5 + H_3 dH_7 + H_3 (F_1 F_7 - F_3 F_5) \right] \ , \end{cases} \tag{5.19}$$

$$\text{M:} \quad \begin{cases} F^{(i)} = (G_4, G_7) \ , \\ S_{12} = \int_{M_{12}} \left[G_4 dG_7 - \frac{1}{6} G_4^3 - G_4 X_8 \right] \ . \end{cases}$$

For simplicity, we have recorded the actions without the source terms. We refer the reader to the appendix of [3] for further details and for a discussion of the sources $J^{(i)}$. The 8-form X_8 is a higher-derivative correction constructed with the curvature form [237,238]. The topological actions (5.19), in $D+2$ dimensions are supplemented by Hodge duality relations in $D+1$ dimensions:

$$\begin{aligned} \text{Type II:} \quad & H_7 = e^{-2\phi} *_{10} H_3 \ , \quad F_p = (-1)^{\lfloor \frac{p}{2} \rfloor} *_{10} F_{10-p} \\ \text{M-theory:} \quad & G_7 = - *_{11} G_4 \ , \end{aligned} \tag{5.20}$$

where in Type II, ϕ is the dilaton and we work in string frame in natural units. A democratic formulation for type II based on 11d Chern-Simons theories is presented in [239]; a democratic formulation based on a non-topological 10d (pseudo)action is presented in [240].

Dimensions	QFT dim $= d$; SymTFT dim $= d+1$; sugra dim $= D+1 = 10,\ 11$ internal space dim $= \mathsf{D}$; $D+1 = \mathsf{D}+d+1$
Top Action	$S = \int_{M_{D+2}} \left[\frac{1}{2} \sum_{i,j} \kappa_{ij} F^{(i)} \wedge dF^{(j)} + \mathrm{CS}(\{F^{(i)}\}) - \sum_{i,j} \kappa_{ij} F^{(i)} \wedge J^{(j)} \right]$
Magnetic Sources	$dF^{(i)} - \sum_j (\kappa^{-1})^{ij} \frac{\mathrm{CS}(\{F^{(k)}\})}{\partial F^{(j)}} = J^{(i)}\ , \qquad J^{(i)}$ supported on $\mathcal{W}^{(i)}$ $D+1-\dim \mathcal{W}^{(i)} = \deg J^{(i)} = \deg F^{(i)} + 1$
Linking	$\mathcal{L}_{D+1}(\mathcal{W}^{(i)}, \mathcal{W}^{(j)}) = \int_{M_{D+1}} J^{(i)} \wedge d^{-1} J^{(j)} = \int_{M_{D+1}} dF^{(i)} \wedge F^{(j)}$ $\dim \mathcal{W}^{(i)} + \dim \mathcal{W}^{(j)} = D, \quad \big(\dim(\mathcal{W}^{(i)} \cup \mathcal{W}^{(j)}) = D$ for HW linking$\big)$

Table 5.1: Summary of Notation: we consider a supergravity theory in $D+1 = 10, 11$, dimensions with fluxes $F^{(i)}$. The auxiliary topological action is formulated in $D+2$ dimensions. The magnetic source for $F^{(i)}$ is denoted by $J^{(i)}$.

$(D+2)$-dimensional Topological Action and Dimensional Reduction. Recall that we are interested in studying setups in which the physical spacetime in 10/11 dimensions is of the form (2.11). This corresponds to $D = d + \mathsf{D}$. The auxiliary topological action in $D+2$ dimensions is formulated on a spacetime of the form

$$M_{D+2} = M_{d+2} \times L_{\mathsf{D}}\ , \tag{5.21}$$

where the external spacetime M_{d+1} has been extended to an auxiliary M_{d+2}, while the internal geometry remains L_{D}. Our task is to integrate S_{D+2} on L_{D} to obtain a topological action S_{d+2}. Next, we reconstruct the physical SymTFT action S_{d+1} from S_{d+2},

$$\text{auxiliary top. action} \quad S_{d+2} = \int_{L_{\mathsf{D}}} S_{D+2} \qquad \longrightarrow \qquad \text{SymTFT action} \quad S_{d+1}\ . \tag{5.22}$$

This arrow represents the descent process demonstrated around (5.17). These steps are exemplified below in a variety of setups.

5.2.2 BF-Terms from Branes

The first aspect to address is how to generate the BF terms of the SymTFT.[3] We adopt the strategy described at the end of the previous section: we work on a $(D+2)$-dimensional spacetime of the form (5.21). The fluxes can have background values on topologically non-trivial cohomologies of the internal manifold L_{D}, and in the reduction ansatz they are generically expanded along representatives of these cohomologies.

Generically we face two possibilities that generate BF-terms:

1. BF-terms from the term $\mathrm{CS}(\{F^{(i)}\})$ in (5.16)

2. BF-terms from the term $\kappa_{ij} F^{(i)} \wedge dF^{(j)}$ in (5.16).

Let us analyze each possibility in turn.

BF-terms from $\mathrm{CS}(\{F^{(i)}\})$**.** The first possibility is when the Chern-Simons functional descend to a non-trivial quadratic wedge product of two fluxes upon compactification on L_{D}. For instance this is the case when there are non-trivial background fluxes on L_{D} or $F^{(i)}$ are expanded on non-trivial cohomologies representatives.

Let us describe schematically the general mechanism for generating BF-terms in this case. The relevant ansatz for the higher-dimensional fluxes and sources reads

$$F^{(i)} = F^{(i)}_{\text{bkrg}} + \sum_a f^{(ia)} \wedge \omega^{(a)} \ , \qquad J^{(i)} = \sum_a j^{(ia)} \wedge \omega^{(a)} \ . \tag{5.23}$$

Here $\omega^{(a)}$ are closed internal forms, with integral periods, representing cohomology classes on L_{D}, enumerated by the label a. The quantities $f^{(ia)}$ are external fluxes, while $F^{(i)}_{\text{bkrg}}$ denotes a possible non-zero background value for the $F^{(i)}$ flux. The latter is also proportional to the volume form of cycles of L_{D} and always integrates to an integer on these cycles because of flux quantization. Finally, we use $j^{(ia)}$ to denote the external parts of the higher-dimensional sources $J^{(i)}$.

[3] In some cases these can be pure Chern-Simons term like $c_3 dc_3$ in $7d$. When this happens the theory is not properly defined as a SymTFT, because of the absence of gapped boundary conditions. An example of this is provided by 6d $(2,0)$ theories.

If we start from the topological action (5.16) in $D+2$ dimensions and integrate over L_{D}, we obtain a topological action in $d+2$ dimensions with couplings of the form (we suppress wedge products for brevity)

$$S_{d+2} = \int_{M_{d+2}} \sum_{i,j,a,b} \left[\frac{1}{2} \kappa_{(ia)(jb)} f^{(ia)} df^{(jb)} - \kappa_{(ia)(jb)} f^{(ia)} j^{(jb)} + \alpha_{(ia)(jb)} f^{(ia)} f^{(jb)} \right] + \cdots . \tag{5.24}$$

On the one hand, the constants $\kappa_{(ia)(ib)}$ are determined from the original constants κ_{ij} in (5.16) and the intersection pairing of the $\omega^{(a)}$ forms on L_{D}. On the other hand, the terms $\alpha_{(ia)(jb)} f^{(ia)} f^{(jb)}$ originate from cubic terms in $\mathrm{CS}(\{F^{(i)}\})$ in (5.16), with the constants $\alpha_{(ia)(jb)}$ determined as integrals over L_{D} of internal top forms constructed with the background fluxes $F^{(i)}_{\mathrm{bkgr}}$ and with the forms $\omega^{(a)}$. Integrality of $F^{(i)}_{\mathrm{bkgr}}$, $\omega^{(a)}$ implies integrality of $\alpha_{(ia)(jb)}$.

The first two terms in the auxiliary $(d+2)$-dimensional action (5.24) correspond to kinetic terms in the physical action on M_{d+1}. As we describe at the beginning of this section, the kinetic terms of the gauge potential do not capture the fluctuations of finite discrete Abelian gauge field, and therefore can be ignored in the truncation to the topological sector. We are left with

$$S^{\mathrm{BF+sources}}_{d+2} = \int_{M_{d+2}} \sum_{i,j,a,b} \left[\alpha_{(ia)(jb)} f^{(ia)} \wedge f^{(jb)} - \kappa_{(ia)(jb)} f^{(ia)} \wedge j^{(jb)} \right] . \tag{5.25}$$

This action reproduces the physical consequences of BF term in $(d+1)$-dimensions.

Example. For illustration purposes, let us consider the simple case in which we only have two relevant external fluxes, denoted $f^{(1)}$, $f^{(2)}$, and one α constant,

$$S^{\mathrm{BF+sources}}_{d+2} = \int_{M_{d+2}} \left[\alpha f^{(1)} \wedge f^{(2)} - f^{(1)} \wedge j^{(1)} - f^{(2)} \wedge j^{(2)} \right] , \tag{5.26}$$

where we have performed a linear redefinition of the external currents to reabsorb the κ constants. An action of this form appears for example in many holographic setups like $\mathrm{AdS}_5 \times S^5$ or $\mathrm{AdS}_7 \times S^4$ in IIB or M-theory respectively, where the we have background fluxes such that $\alpha = \int_{S^5} F_5 = N$ and $\alpha = \int_{S^4} G_4 = N$. The Bianchi identities read

$$\alpha f^{(2)} = j^{(1)} \,, \qquad (-1)^{\deg f^{(1)} \deg f^{(2)}} \alpha f^{(1)} = j^{(2)} \,. \tag{5.27}$$

Plugging this back into the action and evaluating the exponential of the action with brane sources we get

$$\begin{aligned} \langle e^{2\pi i \int_{\mathcal{M}^{(1)}} f^{(1)}} e^{2\pi i \int_{\mathcal{M}^{(2)}} f^{(2)}} \rangle &= \exp\left(\frac{2\pi i}{\alpha} \int_{M_{d+1}} j^{(1)}(\Sigma^{(1)}) \wedge d^{-1} j^{(2)}(\Sigma^{(2)}) \right) \\ &= \exp\left(\frac{2\pi i}{\alpha} \mathcal{L}_{d+1}(\Sigma^{(1)}, \Sigma^{(2)}) \right) \,, \end{aligned} \tag{5.28}$$

where $\Sigma^{(1)} = \partial \mathcal{M}^{(1)}$, $\Sigma^{(2)} = \partial \mathcal{M}^{(2)}$ are (homologically trivial) cycles in M_{d+1} and we have used (5.2) with D replaced by d.

Flux Non-Commutativity. We can now relate the above to flux non-commutativity. By inserting the above operators on alternative $\widetilde{\Sigma}^{(1)} = \partial \widetilde{\mathcal{M}}^{(1)}, \widetilde{\Sigma}^{(2)} = \partial \widetilde{\mathcal{M}}^{(2)}$:

$$\begin{aligned} \left\langle e^{2\pi i \int_{\widetilde{\mathcal{M}}^{(1)}} f^{(2)}} e^{2\pi i \int_{\widetilde{\mathcal{M}}^{(2)}} f^{(1)}} \right\rangle &= \exp\left(\frac{2\pi i}{\alpha} \int_{M_{d+1}} j^{(2)}(\widetilde{\Sigma}^{(1)}) \wedge d^{-1} j^{(1)}(\widetilde{\Sigma}_2) \right) \\ &= \exp\left(\frac{2\pi i}{\alpha} \mathcal{L}_{d+1}(\widetilde{\Sigma}^{(2)}, \widetilde{\Sigma}^{(1)}) \right) \,. \end{aligned} \tag{5.29}$$

This implies that

$$\left\langle e^{2\pi i \oint_{\Sigma_1} f^{(1)}} e^{\oint_{\Sigma_2} f^{(2)}} \right\rangle = \left\langle e^{\oint_{\widetilde{\Sigma}_1} f^{(2)}} e^{\oint_{\widetilde{\Sigma}_2} f^{(1)}} \right\rangle e^{2\pi i \frac{\mathcal{L} - \widetilde{\mathcal{L}}}{\alpha}} \,, \tag{5.30}$$

where $\mathcal{L}, \widetilde{\mathcal{L}}$ are short-hand for the linking numbers $\mathcal{L}_{d+1}$ for $\Sigma^{(i)}$ and $\widetilde{\Sigma}^{(i)}$ respectively.

If the two branes *unlink* in the second configuration, i.e. $\widetilde{\mathcal{L}} = 0$, we exactly get flux non-commutativity as a consequence of the two branes linking in $d+1$ dimensions.

Example: 4d $\mathcal{N} = 1$ SYM with $\mathfrak{g} = \mathfrak{su}(M)$. To exemplify this, consider the holographic realization of pure super-Yang-Mills (SYM). We first consider the Klebanov-Strassler solution [64] dual to 4d $\mathfrak{g} = \mathfrak{su}(M)$ $\mathcal{N} = 1$ SYM. This is the back-reacted configuration of N D3-branes probing the resolved conifold, i.e. the cone of the $T^{1,1}$

Sasaki-Einstein space, $C(T^{1,1})$ and M D5-branes wrapping $S^2 \subset T^{1,1}$. The relevant flux quantization is

$$\int_{S^3} F_3 = M \,. \tag{5.31}$$

The ansatz for the higher-dimensional fluxes is

$$\begin{aligned} F^{(1)} &= F_3 = M \mathrm{vol}_{S^3} + f^{(1,1)} \wedge \mathrm{vol}_{S^2} + f^{(1,2)} \,, \\ F^{(2)} &= H_3 = f^{(2)} \,, \\ F^{(3)} &= F_5 = F^{(3)}_{\mathrm{bkrg}} + f^{(3,1)} \wedge \mathrm{vol}_{S^2} + f^{(3,2)} \wedge \mathrm{vol}_{S^3} \,. \end{aligned} \tag{5.32}$$

From the IIB topological action, we derive the term

$$S_{d+2} = \int_{M_{d+2}} M f^{(2)} \wedge f^{(3,1)} \,. \tag{5.33}$$

where $d = 4$. We must also include sources for the external fluxes

$$\begin{aligned} dH_7 = J^{(2)} : \quad & J^{(2)} = j^{(2,1)} \wedge \mathrm{vol}_{T^{1,1}} \dots \\ dF_5 = J^{(3)} : \quad & J^{(3)} = j^{(3,1)} \wedge \mathrm{vol}_{S^3} + \dots \,, \end{aligned} \tag{5.34}$$

where we recall that the labels on top of the $F^{(i)}$, f and j do not reflect the form degrees, which can instead be read off from the identification with the IIB fluxes F_3, H_3, F_5 in (5.32) and from their derivatives in (5.34). We then obtain

$$S^{\mathrm{BF+sources}}_{d+2} = \int_{M_{d+2}} M f^{(2)} \wedge f^{(3,1)} - j^{(2,1)} \wedge f^{(2)} - j^{(3,1)} \wedge f^{(3,1)} \,. \tag{5.35}$$

This is the IR BF term which describes flux non-commutativity [1].

BF-terms from $\kappa_{ij} F^{(i)} \wedge dF^{(j)}$. The second case is when the BF-terms or quadratic terms in the topological action, after compactification on L_{int}, are generated by one of the $F^{(i)} \wedge dF^{(j)}$ terms in (5.16). For ease of exposition, instead of considering the general action (5.16) we can consider a simpler action in $D+2$ dimensions, with only two fluxes and no Chern-Simons interactions,

$$S_{D+2} = \int_{M_{D+2}} \left[F^{(1)} \wedge dF^{(2)} + (\mathrm{sources}) \right] . \tag{5.36}$$

For simplicity we also assume that there is only one pair of relevant cycles onto which $F^{(1)}$, $F^{(2)}$ are expanded, so that the relevant terms in the reduction ansatz are

$$F^{(1)} = f \wedge \phi \ , \qquad F^{(2)} = \tilde{f} \wedge \tilde{\phi} \ . \tag{5.37}$$

In the previous expressions the internal forms ϕ, $\tilde{\phi}$ are not closed. Rather, they represent torsional cohomology elements in $H^{\bullet}(L_{\text{int}}, \mathbb{Z})$ [171, 172], as explained in greater detail below. The degrees of the forms f, $\tilde{f}$, ϕ, $\tilde{\phi}$ must satisfy

$$\deg f + \deg \tilde{f} = d + 2 \ , \qquad \deg \phi + \deg \tilde{\phi} = \mathsf{D} - 1 \ . \tag{5.38}$$

The reduction of (5.36) yields a $(d+2)$-dimensional action of the form

$$S_{d+2}^{\text{BF+sources}} = \int_{M_{d+2}} \left[\alpha f \wedge \tilde{f} - f \wedge j - \tilde{f} \wedge \tilde{j} \right] . \tag{5.39}$$

Here the coefficient α is given by

$$\alpha = (-1)^{(1+\deg \tilde{f})(1+\deg \phi)} \int_{L_{\mathsf{D}}} d\phi \wedge \tilde{\phi} \ . \tag{5.40}$$

The external source terms j, $\tilde{j}$ originate from the source terms in (5.36). From $dF^{(2)}$ we also get a term in which the derivative acts on $\tilde{f}$. This term, however, yields $f \wedge d\tilde{f}$ in $d+2$ dimensions. Such terms will lead to kinetic terms in the $(d+1)$-dimensional action that we ignore once we consider the theory of flat gauge potentials only, as it was for the first case.

The integral in the α coefficient can be interpreted as linking in the internal space: this is illustrated below. We then see how the BF-terms as well as flux non-commutativity are directly equivalent to the brane and its magnetic dual brane linking also in external space as it was for the first case. Notice here the branes link doubly, i.e. both internally and externally. The internal linking of the branes leads to the coefficient of the BF-term. On top of this the external linking leads to flux non-commutativity.

As anticipated above, the non-closed forms ϕ, $\tilde{\phi}$ encode torsional cohomology classes. More precisely, let us consider the pairs (ϕ, Φ), $(\tilde{\phi}, \tilde{\Phi})$ with [171, 172]

$$\ell\Phi = d\phi \ , \qquad \tilde{\ell}\tilde{\Phi} = d\tilde{\phi} \ , \tag{5.41}$$

where ℓ, $\tilde{\ell}$ are positive integers. [4] The pair (ϕ, Φ) models an element of $H^{\deg\phi+1}(L_{\mathsf{D}}, \mathbb{Z})$ of torsional degree ℓ, while the pair $(\tilde{\phi}, \tilde{\Phi})$ models an element of $H^{\deg\tilde{\phi}+1}(L_{\mathsf{D}}, \mathbb{Z})$ of torsional degree $\tilde{\ell}$. The relation $\ell\Phi = d\phi$ corresponds to a statement of the form $\ell\Sigma = \partial\mathcal{M}$, where Σ is the cycle dual to the closed form Φ, hence of dimension $\mathsf{D} - \deg\phi - 1$, and $\mathcal{M}$ is a chain of dimension $\mathsf{D} - \deg\phi$. Similarly, $\tilde{\ell}\tilde{\Phi} = d\tilde{\phi}$ translates to $\tilde{\ell}\tilde{\Sigma} = \partial\tilde{\mathcal{M}}$, where $\tilde{\Sigma}$ is a cycle of dimension $\mathsf{D} - \deg\tilde{\phi} - 1$. The torsional pairing of the cycles Σ, $\tilde{\Sigma}$ can be computed by taking the intersection number between Σ and $\tilde{\mathcal{M}}$ and dividing by $\tilde{\ell}$, the torsional order of $\tilde{\Sigma}$,

$$\mathcal{T}_{L_{\mathsf{D}}}(\Sigma, \tilde{\Sigma}) = \frac{1}{\tilde{\ell}}\,\Sigma \cdot_{L_{\mathsf{D}}} \tilde{\mathcal{M}} \mod \mathbb{Z} \ . \tag{5.42}$$

Using $\Sigma \cdot_{L_{\mathsf{D}}} \tilde{\mathcal{M}} = \int_{L_{\mathsf{D}}} \Phi \wedge \tilde{\phi}$ and (5.40), (5.41), we can write

$$(-1)^{(1+\deg\tilde{f})(1+\deg\phi)}\alpha = \int_{L_{\rm int}} d\phi \wedge \tilde{\phi} = \ell\,\Sigma \cdot_{L_{\mathsf{D}}} \tilde{\mathcal{M}} = \ell\tilde{\ell}\,\mathcal{T}_{L_{\mathsf{D}}}(\Sigma, \tilde{\Sigma}) \ . \tag{5.43}$$

We have thus established a general relation between the BF coefficient α, the torsional pairing $\mathcal{T}_{L_{\rm int}}(\Sigma^{(1)}, \tilde{\Sigma}^{(1)})$, and the torsional orders ℓ, $\tilde{\ell}$. We also confirm the integrality of the BF coefficient α.

To make this approach more concrete, consider the simple example of $S^3/\mathbb{Z}_k$. We write the metric in Hopf coordinates as follows:

$$g = \frac{1}{k^2}(d\psi + A)^2 + \frac{1}{4}(d\beta^2 + \sin^2\beta d\theta^2)\,, \tag{5.44}$$

[4] Note that this prescription is less geometrically interpretable than the differential cohomology framework. We employ it here to highlight the physical interpretation of brane linking associated with various couplings. However, when directly computing integrals we will map such expressions into linking numbers or differential cohomology integrals which are easier to calculate.

where both ψ and θ have period 2π and $A = \frac{k}{2}\cos\beta d\theta$. In the above notation, we construct two forms

$$\phi = \frac{1}{2\pi}(d\psi + A)\,, \quad \Phi = -\frac{1}{4\pi}\sin\beta d\beta d\theta\,, \tag{5.45}$$

such that

$$d\phi = k\Phi\,. \tag{5.46}$$

We can then compute integrals directly, such as

$$\int_{S^3/\mathbb{Z}_k} \phi \wedge d\phi = k\,. \tag{5.47}$$

BF-terms from Both Mechanisms. Finally, we can consider cases where both situations show up, namely where we have non-zero background fluxes as well as non-trivial torsional pairings. Examples are furnished by $\mathrm{AdS}_5 \times \mathbb{RP}^5$ in type IIB [90] or $\mathrm{AdS}_7 \times \mathbb{RP}^4$ in M-theory [241] (the fluxes are F_5 and G_4, respectively). In the standard setting (case 2 above), torsional flux non-commutativity applies to branes that are electro-magnetic duals in the original $D+1$ dimensional theory. In the hybrid case, the non-zero background flux induces torsional flux non-commutativity for branes that are not duals in $D+1$ dimensions. As a result, some technical aspects of the computation of BF couplings in this class of scenarios require a more refined analysis; we refer the reader to the references above.

5.2.3 Topological Couplings in the SymTFT from Branes

So far we have been focusing on how to get the BF-terms of the SymTFT from the branes in the holographic or geometric constructions. Dimensional reduction of the 10/11-dimensional flux sector with brane sources can lead to additional topological couplings, which, upon choosing suitable gapped boundary conditions, provides an invertible topological theory. This corresponds to the anomalies involving finite, abelian symmetries of an absolute QFT at the boundary. We refer to such couplings in the SymTFT as anomaly couplings (with the understanding that these couplings result in 't Hooft anomalies after certain choices of boundary conditions).

The strategy to obtain these extra topological (non BF-type) couplings is similar to the one implemented for identifying the BF-terms as linking of branes. It consists of dimensionally reducing the action (5.16) on L_{D} and then applying the (dimensionally reduced) Bianchi identities, with sources, (5.18)[5]. By substituting the fluxes in terms of brane sources, we can directly connect the extra topological couplings to brane linking. We now describe how this works in general, where we limit though to quadratic or cubic couplings, which are the cases of interest for us. On the other hand, extending to higher topological coupling is straightforward.

The extra topological (non BF-type) couplings of interest are couplings in the SymTFT in $d+1$ dimensions. As in the previous section, however, we find it convenient to describe these couplings using an action in $d+2$ dimensions, since this is what we naturally get from (5.16). We use $j^{(i)}$ to denote external currents. Their form degrees are left unspecified. In each case, it is assumed that they are such that the integrals we write can be non-zero.

5.2.3.1 Quadratic Couplings

For the quadratic couplings there are two types of extra topological couplings, depending on their expression in terms of the brane sources.

Quadratic Couplings 1. The first case is

$$S_{\text{extra}} = a \int_{M_{d+2}} j^{(1)} \wedge d^{-1} j^{(2)} \, . \tag{5.48}$$

[5] Alternatively one can reduce the Bianchi identities with brane sources directly and construct the lower-dimensional action from this. The two procedures are equivalent.

These sorts of terms in $d+2$ dimensions are expected to combine into a total derivative, which we can rewrite as an integral in M_{d+1}, of the form

$$S_{\text{extra}} = a \int_{M_{d+1}} d^{-1} j^{(1)} \wedge d^{-1} j^{(2)} \quad . \tag{5.49}$$

In this case the branes can link in M_d, where the QFT lives.

Example. A Pontryagin square $\mathfrak{P}(b_2)$ coupling in the 4d SymTFT for a QFT in 3d is an example of such a coupling. The finite 2-form gauge field b_2 is BF-dual to $\widehat{b}_1$ in 3d. The topological operators realized by the dimensionally reduced branes, are identified with the holonomies of $\widehat{b}_1$. They are lines that can link in the 3d spacetime, where the QFT lives. For example, this coupling can be found in setups where a stack of M5-branes is wrapped on a compact 3-manifold Σ_3 with an appropriate topological twist to preserve 3d $\mathcal{N}=2$ or $\mathcal{N}=1$ supersymmetry. In this case, the link geometry L_7 is an S^4 fibration over Σ_3. Depending on the geometry of Σ_3, L_7 can admit non-trivial torsional cohomology classes of degree 2. Expansion of G_4 onto such classes yields both discrete gauge fields 2-forms as b_2, and $\mathfrak{P}(b_2)$ couplings in the SymTFT, by applying the techniques of [65], see [242] where this will be utilized. Couplings of the form $\mathfrak{P}(b_2)$ are also found in the SymTFTs of supersymmetric 3d QFTs realized in M-theory using geometric engineering or M2-branes [2].

Quadratic couplings 2. The second case is

$$S_{\text{extra}} = a \int_{M_{d+2}} j^{(1)} \wedge j^{(2)} = (-)^{\deg j^{(1)}} a \int_{M_{d_{\text{ext}}}} j^{(1)} \wedge d^{-1} j^{(2)} \, . \tag{5.50}$$

This is instead a case where the two branes link in M_{d+1}, where the SymTFT lives.

Example. A coupling

$$b_2 \cup \text{Bock}(b_2') \, , \tag{5.51}$$

in a 5d SymTFT for a 4d QFT is of this type. We explain how Bockstein terms in anomalies appear in our brane-linking picture around (5.69). Here, we motivate this term on physical grounds - the associated topological generators have dimensions such that their linking pairing must be of this form. Here b_2, b'_2 are $\mathbb{Z}_N$ 2-form fields, and Bock is the Bockstein homomorphism associated to the short exact sequence

$$0 \to \mathbb{Z}_N \xrightarrow{\times N} \mathbb{Z}_{N^2} \to \mathbb{Z}_N \to 0\,. \tag{5.52}$$

The dimensionally reduced branes are identified with the holonomies of $\widehat{b}_2$, $\widehat{b}'_2$, the 5d BF-duals of b_2, b'_2. Thus the brane sources are 2d surfaces, and indeed can link in the 5d spacetime where the SymTFT lives.

In both these quadratic couplings, a is an integer constant coupling coefficient which is determined by an integral over L_{D} of non-trivial fluxes components over the internal space. It depends on the representatives of non-trivial cohomology or non-trivial geometric linking of cycles in the internal space, wherever we face the first or the second situation described in the previous section, respectively. In addition a is an integer because of flux quantization.

5.2.3.2 Cubic Couplings

For cubic couplings we face three distinct possibilities.

Cubic Couplings 1. The first case is

$$S_{\text{extra}} = a \int_{M_{d+2}} j^{(1)} \wedge d^{-1} j^{(2)} \wedge d^{-1} j^{(3)}\ . \tag{5.53}$$

Terms of this form correspond to an integral in M_{d+1} of the form

$$S_{\text{extra}} = a \int_{M_{d+1}} d^{-1} j^{(1)} \wedge d^{-1} j^{(2)} \wedge d^{-1} j^{(3)}\ . \tag{5.54}$$

This is a triple linking configuration (cf. Milnor's triple intersection number [243]). We can formally recast it as a standard linking in M_{d+1}, namely a quantity of the form $\int_{M_{d+1}} j^{(12)} \wedge d^{-1} j^{(3)}$, with the identification

$$j^{(12)} = d^{-1} j^{(1)} \wedge d^{-1} j^{(2)}\ . \tag{5.55}$$

Recall that $j^{(i)}$ is supported on a cycle $\Sigma^{(i)}$ that is the boundary of a chain $\mathcal{M}^{(i)}$, which is usually referred to as Seifert (hyper)surface [235]. Then the RHS of (5.55) represents the intersection inside M_{d+1} of the Seifert surfaces $\mathcal{M}^{(1)}$, $\mathcal{M}^{(2)}$ associated to $j^{(1)}$, $j^{(2)}$.

Example: B^3 Anomaly in 5d. A cubic coupling $b_2 b_2 b_2$ in the 6d SymTFT of a QFT in $d = 5$ dimensions, such as have appeared in [65, 244]. Here b_2 is a $\mathbb{Z}_N$ discrete 2-form. The dimensionally reduced branes are again identified with the holonomies of the BF-dual field $\widehat{b}_3$ in six dimensions, which arise from M5-branes wrapping torsional 3-cycles of L_n. They are therefore 3d surfaces, which in six dimensions can form a non-trivial triple linking configuration as in (5.54).

Example: 4d $\mathcal{N} = 1$ SYM with $G = SU(M)$. This theory has mixed 't Hooft anomaly

$$\mathcal{A} = -2\pi \frac{1}{M} \int A_1 \cup \frac{\mathfrak{P}(B_2)}{2}\,, \tag{5.56}$$

where B_2 is the background for a $\mathbb{Z}_M^{(1)}$ 1-form symmetry and A_1 is the background for a $\mathbb{Z}_{2M}^{(0)}$ 0-form symmetry. Using the Klebanov-Strassler solution, a detailed supergravity origin of this anomaly is given in [1, 41].

We continue with the field notation introduced around (5.35). The generator of the 0-form symmetry was identified with a D5-brane wrapped on $S^3 \subset T^{1,1}$ in [41]. We introduce a source for the external field corresponding to background for this symmetry via

$$dF_3 = J^{(1)}\,, \qquad J^{(1)} = j^{(1,1)} \wedge \mathrm{vol}_{S^2} + \dots\,, \tag{5.57}$$

where we use the expansion in (5.32), and with (5.34) we can rewrite the anomaly as follows

$$\frac{1}{2M} \int_{M_{d+2}} j^{(1,1)} \wedge d^{-1} j^{(3,1)} \wedge d^{-1} j^{(3,1)}\,, \tag{5.58}$$

where $d = 4$ and the form degrees of $j^{(3,1)}$ and $j^{(1,1)}$ are 3 and 2 respectively, and they can be read off from (5.32), (5.34) and (5.57).

Cubic Couplings 2. The second case is

$$S_{\text{extra}} = a \int_{M_{d+2}} j^{(1)} \wedge j^{(2)} \wedge d^{-1} j^{(3)} \ , \tag{5.59}$$

with corresponding term in M_{d+1} of the form

$$S_{\text{extra}} = a \int_{M_{d+1}} j^{(1)} \wedge d^{-1} j^{(2)} \wedge d^{-1} j^{(3)} \ . \tag{5.60}$$

Again this is interpreted as suitable triple linking configuration. We can formally recast it as a standard linking in M_{d+1}, namely a quantity of the form $\int_{M_{d+1}} j^{(12)} \wedge d^{-1} j^{(3)}$, with the identification

$$j^{(12)} = j^{(1)} \wedge d^{-1} j^{(2)} \ . \tag{5.61}$$

The RHS represent the intersection inside M_{d+1} of the cycle associated to $j^{(1)}$ with the Seifert surface associated to $j^{(2)}$.

Example. The coupling $b_2 b_2 \text{Bock}(a_1)$ in the 6d SymTFT of a QFT in $d = 5$ dimensions. Here b_2 is a $\mathbb{Z}_N$ 2-form field, a_1 a $\mathbb{Z}_M$ 1-form field, and the Bockstein homomorphism is analogous to the one introduced above in the $b_2 \text{Bock}(b'_2)$ example.[6] The dimensionally reduced branes provide the holonomies of the BF-dual fields in six dimensions, $\widehat{b}_3$ and $\widehat{a}_4$, and are therefore 3d and 4d surfaces, respectively. Such 3d-3d-4d system in 6d can exhibit the triple linking described in (5.60).

[6] This example may be realized using 5d gauge theories. More precisely, we may start with a 5d gauge theory in which the $U(1)$ instanton 0-form symmetry and the center 1-form symmetry have a mixed anomaly, encoded in a coupling $b_2 b_2 f_2$ in the 6d SymTFT, where f_2 is the field strength of a continuous 1-form gauge field. If we restrict to a $\mathbb{Z}_M$ subgroup, this coupling becomes $b_2 b_2 \text{Bock}(a_1)$.

Cubic Couplings 3. The third case is

$$S_{\text{extra}} = a \int_{M_{d+2}} j^{(1)} \wedge j^{(2)} \wedge j^{(3)} \ , \tag{5.62}$$

corresponding to the following in M_{d+1},

$$S_{\text{extra}} = a \int_{M_{d+1}} j^{(1)} \wedge j^{(2)} \wedge d^{-1} j^{(3)} \ , \tag{5.63}$$

This is again a suitable triple linking configuration. We can formally recast it as a standard linking in M_{d+1}, namely a quantity of the form $\int_{M_{d+1}} j^{(12)} \wedge d^{-1} j^{(3)}$, with the identification

$$j^{(12)} = j^{(1)} \wedge j^{(2)} \ . \tag{5.64}$$

The RHS represent the intersection inside M_{d+1} of the cycles associated to $j^{(1)}$, $j^{(2)}$.

Example. An example is the coupling $a_1 \text{Bock}(a_1)\text{Bock}(a_1)$ in the 5d SymTFT of a 4d QFT. Here a_1 is a $\mathbb{Z}_N$ 1-form gauge field, and Bock is the same Bockstein homomorphism as in the previous examples of this section. The dimensionally reduced branes give the holonomies of $\widehat{a}_3$, the BF-dual of a_1 in 5d, and are therefore 3d surfaces. Three such branes can link as in (5.63).

Example: $G^{(0)}$ Anomaly of 4d $\mathcal{N}=1$ SYM with $\mathcal{G} = SU(M)$. There is a pure 0-form symmetry anomaly [41]

$$S_{\text{top}} \supset \int -\frac{\kappa^2 M^2}{4} \mathcal{F}_2^3 \, , \tag{5.65}$$

where $\mathcal{F}_2 = dA_1$ is the field strength of a background gauge field for the 0-form symmetry and $N = \kappa M$ is the number of D3-branes. In terms of brane sources, this is a pure 0-form symmetry anomaly of the third type

$$\frac{\kappa^2}{32M} j^{(1,1)} \wedge j^{(1,1)} \wedge j^{(1,1)} \, , \tag{5.66}$$

where using (5.32) and with $df^{(1,1)} = 2M\mathcal{F}_2$ (see Appendix A of [41]), we find

$$\mathcal{F}_2 = \frac{j^{(1,1)}}{2M} \,. \tag{5.67}$$

It would be interesting to compare this with a direct field theory analysis, in the large rank limit.

For the cubic coupling the coefficient, a, is integer and constant because of flux quantization and depends on the internal sources and how they integrate on L_{D} non-trivially. In particular, a can originate from either situation listed in the previous section depending on how the fluxes and their Bianchi identities get compactified. We notice that the three cases 1, 2, 3 listed above correspond to triple linkings of type 0, 1, 2, respectively, in the notation of [87], see also [235].

5.2.4 Anomaly Couplings as Charges of Defect Junctions from Branes

These extra topological couplings in the SymTFT can lead to anomalies of an absolute QFT once suitable boundary conditions are chosen. This is also true for the BF-couplings, there are some choice of boundary condition for which some left over BF-couplings can lead to a quadratic anomaly. The choice of boundary condition depends on the specific theory itself. In terms of branes these corresponds to picking a radial direction of the SymTFT and understanding how the branes providing topological and charged defects can extend in M_{d+1}. For instance charged defects come from branes extended in the radial direction (the direction perpendicular to the boundary where the relative QFT lives), i.e. the field electrically charging the branes has Dirichlet boundary condition.

Whereas topological operators comes from branes parallel to the boundary, i.e. the field which electrically charge the brane is freely varying. From this point of view it is easy to interpret the quadratic anomaly as the charge of a brane, corresponding to a topological defect, with respect to the same or a different brane, corresponding to the same or a different topological defect. These correspond to a pure or a mixed 't Hooft anomaly, respectively. A cubic anomaly can be interpreted as charges of a brane intersection, corresponding to a junction of topological defects (equal or different, for pure or mixed 't Hooft anomalies respectively), with respect to the same or another brane, corresponding to the same or another topological defect (for pure or mixed 't Hooft anomalies respectively). The charges computed here correspond to the number a which is an integral over the internal manifold times the linking of the branes in the external space as specified for the different cases of quadratic and cubic extra topological couplings above. The anomalies can be interpreted as an ambiguity of the topological defects whenever they link or unlink in the radial direction.

Example 1. We now illustrate the above ideas for two concrete classes of mixed anomalies for discrete p-form symmetries. Firstly, let us consider an anomaly action of the schematic form

$$\mathcal{A} = \alpha \int_{M_{d+1}} A_{p_1}^{(1)} \dots A_{p_n}^{(n)} \,, \quad \sum_{j=1}^{n} p_j = d+1 \,. \tag{5.68}$$

Each $A_{p_j}^{(j)}$ is a discrete background field for a global symmetry, a cohomology class of degree p_j. The constant α is the anomaly coefficient. We denote the topological defect implementing the global symmetry associated to $A_{p_j}^{(j)}$ as $D_{d-p_j}^{(j)}$. (Notice that these topological defects live in d dimensions.) Let us consider topological defects $D_{p_2}^{(2)}$, ..., $D_{p_n}^{(n)}$ in generic positions in d-dimensional spacetime. They intersect along a locus of dimension $p_1 - 1$. The mixed anomaly (5.68) means that this intersection has non-zero charge (proportional to α) under the topological operator $D_{d-p_1}^{(1)}$. For ease of discussion, we have singled out the symmetry associated to $A_{p_1}^{(1)}$, but clearly analogous statements can be made by singling out any other $A_{p_j}^{(j)}$.

Example 2. Next, let us consider a mixed anomaly for two finite global symmetries, of the form

$$\mathcal{A} = \alpha \int_{W_{d+1}} A_{d-p} \mathrm{Bock}(B_p) \ . \tag{5.69}$$

We denote the topological defects operator generating the global symmetries associated to the background fields A_{d-p}, B_p as $D_p^{(A)}$, $D_{d-p}^{(B)}$, respectively. For simplicity, we assume that B_p is associated to a $\mathbb{Z}_k$ $(p-1)$-form symmetry. Then Bock denotes the Bockstein homomorphism associated to the short exact sequence $0 \to \mathbb{Z} \xrightarrow{k} \mathbb{Z} \to \mathbb{Z}_k \to 0$. This anomaly can only be detected on spacetimes with torsion, because $\mathrm{Bock}(B_p)$ lies in the torsion subgroup of $H^{p+1}(W_{d+1}, \mathbb{Z})$. An interpretation in terms of junctions of topological defects can be given along the lines of appendix F of [7] (and many subsequent works). The relevant torsion in d dimensions is in $H^{p+1}(W_d, \mathbb{Z})$, or $H_{d-p-1}(W_d, \mathbb{Z})$ by Poincaré duality. We thus consider a torsional $(d-p-1)$-dimensional cycle M_{d-p-1} in d dimensions, satisfying $rM_{d-p-1} = \partial N_{d-p}$. We may insert a topological defect $D_{d-p}^{(B)}$ supported on N_{d-p}, with M_{d-p-1} regarded as a codimension one junction inside $D_{d-p}^{(B)}$. The anomaly (5.69) means that this junction on M_{d-p-1} has non-zero charge (proportional to α) under the topological defects $D_p^{(A)}$ (indeed, they link in d dimensions). In the action (5.69) of the anomaly theory, the Bockstein map can be "integrated by parts" and the roles of A and B in the previous discussion can be exchanged. This sort of anomaly can be found, for example, in 4d gauge theory with gauge algebra $\mathfrak{su}(N)$, with $N = \ell\ell'$ for integers $\ell > 1$, $\ell' > 1$. We can specify a global form of the theory with both non-trivial electric and magnetic 1-form symmetries. The mixed anomaly between the latter is of the form (5.69) with $d = 4$, $p = 2$.

5.2.5 Condensation Defects from Branes

From a symmetry categorical point of view the condensation completion (or Karoubi completion) corresponds to adding all possible condensation defects. We have seen how this is realized in terms of the SymTFT by including the couplings to lower-dimensional DW-theories in (2.7).

We now turn to the string theory interpretation. For definiteness we work in type II, but similar remarks apply to M-theory. Let us consider a Dp-brane on M_{p+1}. We are interested in writing down a topological action, formulated on an auxiliary manifold M_{p+2} in one dimension higher, that captures the topological couplings on the Dp-brane. Moreover, we also want to capture the kinetic term $f_2 \wedge * f_2$ from the DBI action using the auxiliary topological action. We propose the following,

$$S_{p+2}^{\mathrm{D}p} = \int_{M_{p+2}} \left[\widehat{f}_{p-1} \wedge df_2 + \left(e^{f_2} \sum_q F_q \sqrt{\frac{\widehat{A}(T)}{\widehat{A}(N)}} \right)_{p+2} \right] . \tag{5.70}$$

Here f_2 is the field strength of the gauge field on the Dp-brane and $\widehat{f}_{p-1}$ is its Hodge dual in M_{p+1}. The quantities F_q are the RR fluxes, pulled back from the bulk, and we have also included the standard A-roof terms from the Wess-Zumino couplings, for the tangent and normal bundles, respectively. If we consider an anti Dp-brane, we flip the sign of action (5.70)[7].

We want to argue that considering a combined Dp/$\overline{\mathrm{D}p}$ system provides a possible stringy origin for the condensation-completed SymTFT action (2.7). We proceed by considering a couple of illustrative examples.

Example: 4d $\mathcal{N} = 1$ Holographic dual. We continue with the Klebanov-Strassler example 5.2.2. For this we need to consider the action of D5-branes. Their action is given by

$$\begin{aligned} S_7^{\mathrm{D5}} = \int_{M_7} \Bigg[& \widehat{f}_4 \wedge df_2 + F_7 + f_2 F_5 + \left(\frac{1}{2} f_2^2 + \frac{p_1(N) - p_1(T)}{48} \right) F_3 \\ & + \left(\frac{1}{3!} f_2^3 + f_2 \frac{p_1(N) - p_1(T)}{48} \right) F_1 \Bigg] . \end{aligned} \tag{5.71}$$

[7] The sign of the DBI term for a brane and an antibrane is the same. The flip in sign in the BF term reformulation of the DBI kinetic term is compensated by a flip in sign in the Hodge duality relation between f_2 and $\widehat{f}_{p-1}$.

As a result, the combined action for a D5-brane/anti-D5-brane reads

$$S_7^{\mathrm{D5}/\overline{\mathrm{D5}}} = \int_{M_7} \Big[\widehat{f}_4 \wedge df_2 - \widehat{f}_4' \wedge df_2' + (f_2 - f_2')F_5 + \frac{1}{2}(f_2 - f_2')(f_2 + f_2')F_3 \\ + (f_2 - f_2')\Big(\frac{f_2^2 + f_2 f_2' + f_2'^2}{6} + \frac{p_1(N) - p_1(T)}{48} \Big) F_1 \Big] \ . \tag{5.72}$$

We have used a prime to denote the gauge field f_2' on the anti-D5-brane and its partner $\widehat{f}_4'$.

In the Klebanov-Strassler holographic setup, the D5/$\overline{\mathrm{D5}}$ system is wrapped on $M_7 = M_4 \times S^3$ with M units of F_3 through the S^3. Our task is to integrate the 7d topological action on S^3. The discussion parallels exactly the two cases discussed in section 5.2.2. The terms $\widehat{f}_4 \wedge df_2$ would correspond to kinetic terms in the lower-dimensional theory on M_4, which we neglect because we are studying the topological sector. Next, the terms quadratic in f_2, f_2' yield a 4d description of a set of abelian CS-terms in 3d. Moreover, F_5 admits a non-trivial component along S^3: from $(f_2 - f_2')F_5$ on S^3 we get a coupling of $(f_2 - f_2')$ to a 2-form bulk field, denoted g_2. In summary, the relevant terms are

$$S^{\mathrm{D5}/\overline{\mathrm{D5}}} = \int_{M_4} \Big[\frac{M}{2}(f_2 - f_2')(f_2 + f_2') + (f_2 - f_2')g_2 \Big] \ . \tag{5.73}$$

We suggest the following interpretation, making contact with the general expression (2.7) for the condensation-completed SymTFT. The combination $f_2 - f_2'$ is identified with the localized field a_1 in the lower-dimensional DW type theory in the SymTFT that accounts for a class of condensation defects. The combination $f_2 + f_2'$ corresponds instead to $\widehat{a}_1$, the BF-dual to a_1 in the lower-dimensional DW type theory, Finally, g_2 corresponds to one of the bulk fields b_{p+1} (here $p = 1$). This can be seen more explicitly in the case of M odd. We perform a field redefinition implemented by an integral, unimodular matrix, and we rename g_2,

$$\begin{pmatrix} f_2 \\ f_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} da_1 \\ d\widehat{a}_1 \end{pmatrix} \ , \qquad g_2 = b_2 \ . \tag{5.74}$$

We get the action

$$\int_{M_3} \left[M\widehat{a}_1 \wedge da_1 + a_1 \wedge b_2 + \frac{K}{2} a_1 \wedge da_1 \right] , \qquad K = M \ . \tag{5.75}$$

The Lagrangian in bracket describes the $(\mathcal{Z}_M)_K$ discrete gauge theory [212], coupled to the bulk field b_2. On a spin manifold, K can be any integer and the periodicity is $K \sim K + 2M$ if M is even and $K \sim K + M$ if M is odd. We then see that, for M odd, the Lagrangian (5.75) on a spin manifold is equivalent to

$$\int_{M_3} \left[M\widehat{a}_1 \wedge da_1 + a_1 \wedge b_2 \right] , \tag{5.76}$$

which matches (2.7). For M even, (5.75) still describes a condensation defect, but with non-trivial discrete torsion $K = M$, see e.g. appendix B of [32].

Example: 4d $\mathcal{N} = 4$ $\mathfrak{so}(4n)$ SYM. Let us now discuss an example that illustrates the importance of the terms $\widehat{f}_{p-1} df_2$ in (5.70) in the presence of torsion. The action (5.70) for a D3-brane reads

$$S_5^{\text{D3}} = \int_{M_5} \left[\widehat{f}_2 \wedge df_2 + F_5 + f_2 F_3 + \left(\frac{1}{2} f_2^2 + \frac{p_1(N) - p_1(T)}{48} \right) F_1 \right] , \tag{5.77}$$

and therefore a D3/$\overline{\text{D3}}$ system is described by

$$S_5^{\text{D3}/\overline{\text{D3}}} = \int_{M_5} \left[\widehat{f}_2 \wedge df_2 - \widehat{f}_2' \wedge df_2' + (f_2 - f_2')F_3 + \frac{1}{2}(f_2 - f_2')(f_2 + f_2')F_1 \right] . \tag{5.78}$$

We consider the holographic dual setup to 4d $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{so}(4N)$. In this case $M_5 = M_4 \times \mathbb{RP}^1$, with $\mathbb{RP}^1$ regarded as an element of $H_1(\mathbb{RP}^5, \mathbb{Z}) \cong \mathbb{Z}_2$. From the point of view of M_5, the $\mathbb{RP}^1$ factor provides a torsional class of degree one t_1, of torsional order 2. Following the approach of [171, 172], we can model this by introducing a pair of differential forms on $\mathbb{RP}^1$,

$$2\Phi_1 = d\phi_0 \ , \tag{5.79}$$

see (5.41). We expand f_2 and $\widehat{f}_2$ onto ϕ_0,

$$f_2 = \mathcal{F}_2 \phi_0 + \dots \ , \qquad \widehat{f}_2 = \widehat{\mathcal{F}}_2 \phi_0 + \dots \tag{5.80}$$

and similarly for the primed fields, and we have

$$\int_{M_4\times\mathbb{RP}^1}\left[\widehat{f}_2 df_2-\widehat{f}_2' df_2'\right]=\left[\int_{\mathbb{RP}^1} d\phi_0\phi_0\right]\int_{M_4}\left[\widehat{\mathcal{F}}_2\mathcal{F}_2-\widehat{\mathcal{F}}_2'\mathcal{F}_2'+\dots\right] . \tag{5.81}$$

As in section 5.2.2, the terms where the derivative acts on the $\mathcal{F}$'s can be neglected, because they describe kinetic terms after integrating over $\mathbb{RP}^1$. The integral of $d\phi_0\phi_0$ encodes the torsional self-pairing of t_1,

$$\int_{\mathbb{RP}^1} d\phi_0\phi_0 = 2 . \tag{5.82}$$

We compute this integral, and similar expressions throughout this work, using the methodology described around (5.43) by mapping to a linking number computation, or equivalently using the known result in differential cohomology. Finally, we also expand the bulk field F_3 onto (Φ_1,ϕ_0): the relevant term is $F_3 = g_2\wedge\Phi_1$. As a result, the term $(f_2-f_2')F_3$, after integration on $\mathbb{RP}^1$, yields a term $(\mathcal{F}_2-\mathcal{F}_2')g_2$. In conclusion, we arrive at the following set of couplings,

$$S^{\mathrm{D3}/\overline{\mathrm{D3}}}=\int_{M_4}\left[2\widehat{\mathcal{F}}_2\wedge\mathcal{F}_2-2\widehat{\mathcal{F}}_2'\wedge\mathcal{F}_2'+(\mathcal{F}_2-\mathcal{F}_2')\wedge g_2\right] . \tag{5.83}$$

Because of S-duality, however, we know that the presence of a coupling of f_2 to C_2 implies the presence of a $\widehat{f}_2$ (which is the electromagnetic dual of f_2) to B_2. We then expect the full set of relevant couplings to be

$$S^{\mathrm{D3}/\overline{\mathrm{D3}}}=\int_{M_4}\left[2\widehat{\mathcal{F}}_2\wedge\mathcal{F}_2-2\widehat{\mathcal{F}}_2'\wedge\mathcal{F}_2'+(\mathcal{F}_2-\mathcal{F}_2')\wedge g_2+(\widehat{\mathcal{F}}_2-\widehat{\mathcal{F}}_2')\wedge h_2\right] . \tag{5.84}$$

Here we have expanded H_3 onto Φ_1 as $H_3=h_2\wedge\Phi_1$. The full action might be derived using the $SL(2,\mathbb{Z})$-covariant formulation of [245].

To make contact with (2.7) we perform a redefinition implemented a matrix in $GL(4,\mathbb{Z})$, and we rename g_2 and h_2,

$$\begin{pmatrix}\mathcal{F}_2\\ \mathcal{F}_2'\\ \widehat{\mathcal{F}}_2\\ \widehat{\mathcal{F}}_2'\end{pmatrix}=\begin{pmatrix}1&0&1&1\\0&0&1&1\\1&1&0&0\\1&1&-1&0\end{pmatrix}\begin{pmatrix}da_1\\ d\widehat{a}_1\\ da_1'\\ d\widehat{a}_1'\end{pmatrix} , \qquad g_2=b_2 , \qquad h_2=\widehat{b}_2 . \tag{5.85}$$

We obtain the action

$$\int_{M_3}\left[\left(2\widehat{a}_1 da_1+a_1b_2+\frac{K}{2}a_1da_1\right)+\left(2\widehat{a}_1'da_1'+a_1'\widehat{b}_2+\frac{K'}{2}a_1da_1'\right)\right]\,,\qquad \begin{matrix} K=4\,, \\ K'=4\,. \end{matrix} \tag{5.86}$$

We recognize two copies of a $(\mathcal{Z}_M)_K$ discrete gauge theory [212] with $M=2$, $K=4$. On a spin manifold with M even, $K\sim K+2M$ and hence the above action is equivalent to

$$\int_{M_3}\left[\left(2\widehat{a}_1 da_1+a_1b_2\right)+\left(2\widehat{a}_1'da_1'+a_1'\widehat{b}_2\right)\right]\,, \tag{5.87}$$

matching with (2.7).

5.2.6 Non-Genuine and Twisted Sector Operators

Non-genuine or twisted sector operators arise from branes that couple to backgrounds which cannot necessarily end on the boundary, i.e. do not have Dirichlet boundary conditions.

Lets see how this is encoded in terms of the branes. If we were to end a brane that is electrically charged under a $(p+q+1)$-form field C_{p+q+1} on the boundary, then imposing Neumann boundary conditions on $\mathcal{B}^{\text{sym}}$ reads

$$\text{Neumann:}\qquad \partial_{[r}C_{ij\dots]}\big|_{\mathcal{B}^{\text{sym}}}\wedge\omega(\Sigma_q)=0\,, \tag{5.88}$$

where r is the direction transverse to the boundary (i.e. the radial direction), $i,j=1\dots p+1$ denote the direction parallel to $\mathcal{B}^{\text{sym}}$, the brane can also wrap internal submanifold of $\Sigma_q\subset L(\boldsymbol{X})$, and $\omega(\Sigma_q)$ is transverse to the SymTFT. The internal manifold Σ_q is important to determine properties of the $(p+1)$-dimensional defect of the SymTFT, but it is a spectator with respect to the boundary conditions, hence we can put $\omega(\Sigma_q)$ aside for the moment.

Expanding out the Neumann boundary conditions along the direction of the SymTFT and restricting to the symmetry boundary we obtain

$$(\partial_r C_{ij\dots}-\partial_i C_{rj\dots})|_{\mathcal{B}^{\text{sym}}}=0\,. \tag{5.89}$$

There are various configurations we can consider:

- Symmetry generators: for a brane without a radial component, this means we simply have

$$\partial_r C_{ij\dots} = 0\,, \tag{5.90}$$

which corresponds to the projection in figure 2.2 of the brane parallel to the boundary. This gives rise to $(p+1)$-dimensional (topological) symmetry defects.

- Twisted Sector: if the second term in (5.89) is present, it electrically charges a $(p+q)$-brane ($(p+1)$-dimensional operator when integrated on $\omega(\Sigma_q)$) extended along the radial direction ending at the boundary in a p-dimensional operator, which forms a junction with a $(p+q)$-brane ($(p+1)$-dimensional operator when integrated on $\omega(\Sigma_q)$) extending along $\mathcal{B}^{\text{sym}}$. When we consider the first term as well, this correspond exactly to the L-shaped configuration, where the gauge transformation of the first term is cancelled by the gauge transformation of the second, in figure 2.4.

Example: BF-couplings in AdS$_5$. The simplest example to consider is the BF-theory for $\mathbb{Z}_N$ 2-form fields in 5d, which is the SymTFT for the 4d $SU(N)$ maximal SYM theory

$$S_{\text{SymTFT}} = N \int_{M_5} b_2 \wedge dc_2\,. \tag{5.91}$$

For example, imposing the boundary conditions

$$b_2 \;\; \text{Dirichlet}\,, \qquad c_2 \;\; \text{Neumann}\,, \tag{5.92}$$

the topological defects $\boldsymbol{Q}_2^{(b)}$, which are realized in terms of F1-strings, can end on the physical boundary and give rise to line operators in the gauge theory. On the other hand the D1-strings, which give rise to the bulk topological defects $\boldsymbol{Q}_2^{(c)}$, cannot end. There are two configurations:

- $\boldsymbol{Q}_2^{(c)}$ project parallel to the boundary as in figure 2.2 and give rise to the topological defects D_2 that generate the $\mathbb{Z}_N^{(1)}$ 1-form symmetry.

- $\boldsymbol{Q}_2^{(c)}$ project in an L-shape as in figure 2.4, and give rise (after interval compactification) to twisted sector 't Hooft lines, i.e. non-genuine, in this case, twisted sector, line operators that are attached to a topological surface.

This is of course well-known in the context of this standard holographic setup [7,151] and was recently expanded upon in [246].

5.2.7 Example: 4d $\mathcal{N} = 4$ $\mathfrak{so}(4n)$ SYM

It is known that theories with an array of global structures based on the algebra $\mathfrak{so}(4n)$ contain non-invertible topological operators [28]. In this section we will use the holographic dual of these theories to study the SymTFT, in particular the BF terms.

Holographic Dual. The holographic solution relevant for these theories is IIB on $\text{AdS}_5 \times \mathbb{RP}^5$ [203]. The various global forms of the gauge group correspond to different choices of boundary conditions for various bulk gauge fields [246].

We refer the reader to [90] for more details on this setup. For convenience we collect the co/homology groups of the internal space $\mathbb{RP}^5$ with un/twisted coefficients below

$$\begin{aligned} &H^\bullet(\mathbb{RP}^5, \mathbb{Z}) = \{\mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, \mathbb{Z}\}\,, \quad H^\bullet(\mathbb{RP}^5, \widetilde{\mathbb{Z}}) = \{0, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2\} \\ &H_\bullet(\mathbb{RP}^5, \mathbb{Z}) = \{\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \mathbb{Z}\}\,, \quad H_\bullet(\mathbb{RP}^5, \widetilde{\mathbb{Z}}) = \{\mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0\}\,. \end{aligned} \tag{5.93}$$

For $\mathfrak{so}(4n)$ the dual supergravity solution contains 5-form flux

$$\int_{\mathbb{RP}^5} F_5 = 2n\,. \tag{5.94}$$

BF Terms. Before we begin, we introduce notation for the forms on which we will be expanding fluxes and sources

$$\begin{aligned} &H_i(\mathbb{RP}^5, \widetilde{\mathbb{Z}}): \quad (\widetilde{\phi}_i, \widetilde{\Phi}_i), \quad d\widetilde{\phi}_i = 2\widetilde{\Phi}_i\,, \quad i \in \{0, 2, 4\} \\ &H_i(\mathbb{RP}^5, \mathbb{Z}): \quad (\phi_i, \Phi_i), \quad d\phi_i = 2\Phi_i\,, \quad i \in \{1, 3\} \end{aligned} \tag{5.95}$$

The BF terms come from more than one source in this case since we have both flux and torsion in the internal space.

First, we consider terms coming from the IIB Chern-Simons term. For this we require the fluxes:

$$\begin{aligned} F^{(1)} &= F_3 = f^{(1)} \wedge \widetilde{\phi}_4 + \dots , \\ F^{(2)} &= H_3 = f^{(2)} \wedge \widetilde{\phi}_4 + \dots . \end{aligned} \tag{5.96}$$

Due to (5.94), we obtain a term

$$S^{\rm BF}_{d+2} \supset \int_{M_{d+2}} 2n f^{(1)} \wedge f^{(2)} \,. \tag{5.97}$$

Including sources

$$\begin{aligned} dF_7 = J^{(1)} : &\quad J^{(1)} = j^{(1)} \wedge \widetilde{\Phi}_0 + \dots , \\ dH_7 = J^{(2)} : &\quad J^{(2)} = j^{(2)} \wedge \widetilde{\Phi}_0 + \dots . \end{aligned} \tag{5.98}$$

We recall that the form degrees of $F^{(i)}$, f and j are not specified by their labels on top, but they can be read off from (5.96) and (5.98), as well as from (5.100), (5.102) and (5.106) for what follows, once identified with the IIB fluxes H_3, F_3, F_5, F_7, H_7 and derivatives thereof. We then have

$$S^{\rm BF+sources}_{d+2} \supset \int_{M_{d+2}} 2n f^{(1)} \wedge f^{(2)} - f^{(1)} \wedge j^{(1)} - f^{(2)} \wedge j^{(2)} \,, \tag{5.99}$$

where $d = 4$. Now we look to BF terms coming from $\kappa_{ij} F^{(i)} dF^{(j)}$ terms. There are three such terms. We reduce the IIB kinetic terms $H_3 \wedge dH_7$, $F_3 \wedge dF_7$ and $F_5 \wedge dF_5$ in turn.

Beginning with the first,

$$F^{(3)} = H_7 = \widetilde{f}^{(3,\widetilde{\phi}_0)} \wedge \widetilde{\phi}_0 + \dots , \tag{5.100}$$

The new BF term coefficient comes from the integral identity

$$\int_{\mathbb{RP}^5} d\widetilde{\phi}_0 \wedge \widetilde{\phi}_4 = 2 \,. \tag{5.101}$$

Including the F1 string source for H_7

$$dH_3 = J^{(3)} : \quad J^{(3)} = j^{(3)} \wedge \widetilde{\Phi}_4 \,. \tag{5.102}$$

Finally, we obtain the new contributions:

$$S_{d+2}^{\text{BF+sources}} \supset \int_{M_{d+2}} 2f^{(2)} \wedge \widetilde{f}^{(3,\widetilde{\phi})} + \widetilde{f}^{(3,\widetilde{\phi})} \wedge j^{(3)} \,. \tag{5.103}$$

For F_3 there is also the Bianchi identity

$$dF_3 = J^{(4)}: \quad J^{(4)} = j^{(4)} \wedge \widetilde{\Phi}_4 \,. \tag{5.104}$$

There is an identical contribution from the $F_3 \wedge F_7$ term which we denote with (4) superscripts

$$S_{d+2}^{\text{BF+sources}} \supset -\int_{M_d+2} 2f^{(1)} \wedge \widetilde{f}^{(4,\widetilde{\phi})} + \widetilde{f}^{(4,\widetilde{\phi})} \wedge j^{(4)} \,. \tag{5.105}$$

Lastly we also consider the $dF_5 \wedge F_5^D$ term:

$$\begin{aligned} F^{(5)} &= F_5 = f^{(5,1)} \wedge \phi_1 + f^{(5,3)} \wedge \phi_3 \,, \\ dF_5 &= J^{(5)}: \quad J^{(5)} = j^{(5,1)} \wedge \Phi_3 + j^{(5,3)} \wedge \Phi_1 \,, \end{aligned} \tag{5.106}$$

such that we obtain

$$S_{d+2}^{\text{BF+sources}} \supset \int_{M_{d+2}} 2f^{(5,1)} \wedge f^{(5,3)} - f^{(5,3)} \wedge j^{(5,3)} - f^{(5,1)} \wedge j^{(5,1)} \,. \tag{5.107}$$

Putting all of these pieces together, we match the BF terms of [90, 246].

The $\mathfrak{so}(4n)$ theory also has an additional topological coupling, which depending on boundary conditions lead to a mixed anomaly,

$$\mathcal{A} = \frac{1}{2} \int_{M_{d+1}} A_1 C_2' B_2 \,, \tag{5.108}$$

where A_1 is a background for $\mathbb{Z}_2^{(0)}$ and C_2', B_2 are both $\mathbb{Z}_2^{(1)}$ backgrounds. We can re-write this coupling in terms of sources using the identifications (e.g. using table 5.3)

$$f^{(2)} \leftrightarrow dB_2 \,, \quad f^{(1)} \leftrightarrow dC_2' \,, \quad f^{(5,1)} \leftrightarrow dA_1 \,. \tag{5.109}$$

The anomaly term comes from the IIB cubic Chern-Simons coupling which by using the Bianchi identities (5.102), (5.104) and (5.106) can be re-written in terms of brane sources as

$$\mathcal{A} = \frac{1}{2} \int_{M_{d+1}} d^{-1} j^{(3)} \wedge d^{-1} j^{(4)} \wedge d^{-1} j^{(5,3)} \,, \tag{5.110}$$

where the coefficient is given by the following integration on $\mathbb{RP}^5$, [8]

$$
\frac{1}{8}\int_{\mathbb{RP}^5} d\tilde{\phi}_4 \wedge d\tilde{\phi}_4 \wedge \phi_1 = \int_{\mathbb{RP}^5} \tilde{\Phi}_4 \wedge \tilde{\Phi}_4 \wedge d^{-1}\Phi_1 = \frac{1}{2} \tag{5.111}
$$

This is cubic coupling of type 1 (5.54) coming from three type of brane sources: NS5 on $\mathbb{RP}^4$, D5 on $\mathbb{RP}^4$ and D3 on $\mathbb{RP}^1$, which model the topological defects once properly compactified on the torsional cycles.

5.2.8 Example: Duality and Triality Defects for $\mathcal{N} = 2$ $[A_2, D_4]$ Theory

In this section we use our general setup to construct symmetry defects as branes in the isolated hypersurface singularity (IHS) (Calabi-Yau threefold) describing the 4d $\mathcal{N} = 2$ $[A_2, D_4]$ SCFT in IIB string theory. This theory admits generalized symmetries [93, 117, 120, 129, 247]. In particular, we will propose a new construction of symmetry defects as lower-dimensional branes induced by world-volume flux for a higher-dimensional brane. The singularity $\boldsymbol{X}$ is described by the following hypersurface equation [117],

$$
x_1^2 + x_2^3 + x_3^3 + x_4^3 = 0 \subset \mathbb{C}^4 \,. \tag{5.112}
$$

We construct now the symmetry defects wrapping topological cycles of the link geometry, $\partial\boldsymbol{X} = L(\boldsymbol{X})$. There is no flux in the background, but $L(\boldsymbol{X})$ has non-trivial torsional cycles [117]

$$
H_2(L(\boldsymbol{X}), \mathbb{Z}) = \mathfrak{f} \oplus \mathfrak{f}' = \mathbb{Z}_2 \oplus \mathbb{Z}_2' \,. \tag{5.113}
$$

In the last equality we specialize to $[A_2, D_4]$. Wrapping D3-branes on these torsion cycles results in the topological defects of the SymTFT.

[8] Where $\int_{\mathbb{RP}^5} \tilde{\Phi}_4 \wedge \tilde{\Phi}_4 \wedge d^{-1}\Phi_1$ is identified with the differential cohomology integral of [90], i.e. $\int_{\mathbb{RP}^5} \breve{u}_4 \star \breve{t}_1^{\text{RR}} \star \breve{t}_1^{\text{NSNS}}$. This identification similarly holds for the coefficients of the BF couplings, $\int_{\mathbb{RP}^5} d\widetilde{\phi}_0 \wedge \widetilde{\phi}_4 = 2$ means that $\int_{\mathbb{RP}^5} \widetilde{\Phi}_0 \wedge d^{-1}\widetilde{\Phi}_4 = \frac{1}{2}$, the latter is identified in differential cohomology $\int_{\mathbb{RP}^5} \breve{u}_4 \star \breve{u}_2 = \frac{1}{2}$.

There is a non-trivial linking of the generators obtained by wrapping D3s on the two $\mathbb{Z}_2$ factors

$$\text{Link}_{L(\boldsymbol{X})}(\gamma,\gamma') = \frac{1}{2}\,, \qquad \gamma \in \mathbb{Z}_2\,, \quad \gamma' \in \mathbb{Z}_2'\,. \tag{5.114}$$

Depending on the symmetry boundary conditions on $\mathcal{B}^{\text{sym}}$, these branes become symmetry generators or generalized charges.

We now apply the general procedure described in section 5.2.2 for the case of torsional cycles. For instance we have that

$$J^{\text{D3}} = dF_5 = g_3 \wedge d\phi_{\mathfrak{f}} - g_3' \wedge d\phi_{\mathfrak{f}'} + \dots\,, \tag{5.115}$$

where g_3, g_3' are flat in the space where the SymTFT lives, M_{4+1}, and we describe the torsional cohomology in the continuum as

$$2\Phi_{\mathfrak{f}} = d\phi_{\mathfrak{f}}, \qquad 2\Phi_{\mathfrak{f}'} = d\phi_{\mathfrak{f}'} \tag{5.116}$$

and from (5.38) and (5.39) we get the following BF action

$$S_{BF} = \alpha \int_{M_{4+2}} g_3 \wedge g_3'\,, \tag{5.117}$$

where

$$\alpha = -\int_{L(\boldsymbol{X})} \phi_{\mathfrak{f}} \wedge d\phi_{\mathfrak{f}'} = 2\,, \tag{5.118}$$

which is exactly analogous to the BF-action for the bulk theory of $\mathcal{N}=4$ $\mathfrak{su}(2)$ theory.

Let us now go back to the IHS equation (5.112) and look at the complex structure deformation that corresponds to the marginal coupling of the theory [93]. The deformed equation reads

$$x_1^2 + x_2^3 + x_3^3 + x_4^3 + \tau x_2 x_3 x_4 = 0 \subset \mathbb{C}^4\,, \tag{5.119}$$

where τ corresponds to the marginal coupling of the SCFT and therefore does not desingularize the geometry, as expected when activating the deformation corresponding to marginal couplings of the theory. This τ also corresponds to the complexified gauge coupling of the $\mathfrak{su}(2)$ when we think of this AD theory as a gauging of three AD$[A_1, A_3]$ theories, and it is identified with the complex structure of the torus when the theory is constructed via compactification of the E_6 minimal 6d $\mathcal{N} = (1,0)$ SCFT on T^2 [93]. We now exploit the identification of the complex structure deformation parameter with τ in (5.119) and use how S-duality, S, and the ST transformations act on it, to argue that S and ST are symmetries of the IHS equation, hence of the geometry, when $\tau = e^{i\pi/2}, e^{i\pi/3}$, respectively. For instance, the S-duality action by definition exchanges the magnetic 1-form symmetry with the electric one at an 5d effective topological field theory (BF-theory) level. This is indeed achieved when S and ST act on the torsional two cycles as follows

$$(\Phi_{\mathfrak{f}}, \Phi_{\mathfrak{f}'}) \mapsto M_S(\Phi_{\mathfrak{f}}, \Phi_{\mathfrak{f}'}), \qquad (\Phi_{\mathfrak{f}}, \Phi_{\mathfrak{f}'}) \mapsto M_{ST}(\Phi_{\mathfrak{f}}, \Phi_{\mathfrak{f}'}) \tag{5.120}$$

where M_S and M_{ST} are the monodromies defined by

$$M_S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_{ST} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}. \tag{5.121}$$

At this level the symmetry acts geometrically, and the topological defect generating self-duality and -triality in this frame are hard to engineer as branes [9]. However, we can activate world-volume fluxes on torsional cycles that induces (p,q)-string on the D3-brane.

Induced (p,q)-String Charges on D3-branes and Symmetry Generators. In this section we will draw a connection between two symmetry actions. Above, we demonstrated that there is a geometric symmetry action which acts on the torsional two cycles. Below, we will show that by turning on different (p,q)-string charges on D3-branes wrapping these cycles - there is a second symmetry which acts in the same way, namely the one generated by 7-branes. Combining these two symmetries gives duality/ triality symmetries of the field theory.

[9]See [83], for a geometric construction of these defects as degeneration of the link geometry at the boundary.

Instead of expanding the 5-form fluxes on torsional cycles we consider (p, q)-string charges on the D3-branes. In terms of magnetic sources we have

$$J^{D1} = f^{D3}\delta(\mathrm{D3}), \qquad J^{F1} = (f')^{\mathrm{D3}}\delta(\mathrm{D3})\,, \tag{5.122}$$

and we choose

$$f^{\mathrm{D3}} = d\phi_{\mathfrak{f}}, \qquad (f')^{\mathrm{D3}} = d\phi_{\mathfrak{f}'}\,. \tag{5.123}$$

where the torsional pairs $(\phi_{\mathfrak{f}}, \Phi_{\mathfrak{f}})$ and $(\phi_{\mathfrak{f}'}, \Phi_{\mathfrak{f}'})$ have been introduced in (5.116). This induces the backgrounds for the 3-forms H_3, F_3,

$$H_3 = h_3 + d\phi_{\mathfrak{f}}, \qquad F_3 = f_3 + d\phi_{\mathfrak{f}'}\,, \tag{5.124}$$

where the second identity follows from the $SL(2,\mathbb{Z})$ covariant formalism [245], and we also turned on flat f_3, h_3 in M_{4+1}. Now the magnetically sourced Bianchi identity (5.115) gets modified,

$$J^{\mathrm{D3}} = dF_5 = g_3 \wedge d\phi_{\mathfrak{f}} - g_3' \wedge d\phi_{\mathfrak{f}'} = f_3 \wedge d\phi_{\mathfrak{f}} - h_3 \wedge d\phi_{\mathfrak{f}'}\,. \tag{5.125}$$

This implies that we can identify

$$g_3 \leftrightarrow f_3, \qquad g_3' \leftrightarrow h_3\,. \tag{5.126}$$

It is now easy to verify that in this frame the action of S and ST on the torsional cycles (5.120) is equivalent to the action of the monodromy matrices M_S and M_{ST} on the (f_3, h_3) pair and hence on the electrically charged (p, q)-strings. As we know from $\mathcal{N} = 4$ and its holographic construction, the self-duality and self-triality defects are engineered by 7-branes where the corresponding monodromy matrices act on the (p, q)-strings that generate the 1-form symmetries. In the next section, we will study properties of the SymTFT, the topological defects that generate the 1-form symmetries of the theory at the boundary from (p, q)-strings, and the self-dualities and -trialities topological defects from 7-branes. To summarize and conclude, mapping a

Stringy Origin	Symmetries
D3 on $H_2(L(\boldsymbol{X}),\mathbb{Z})$	Isometry acting on $\mathfrak{f}\oplus\mathfrak{f}'\in H_2(L(\boldsymbol{X}),\mathbb{Z})$
(p,q)-strings induced by $f^{\mathrm{D3}}=d\phi_{\mathfrak{f}},(f')^{\mathrm{D3}}=d\phi_{\mathfrak{f}'}$ on D3	7-branes with monodromies $M_S, M_{ST}\in SL(2,\mathbb{Z})$

Table 5.2: Summary of topological defects construction in $[A_2, D_4]$ via IIB branes on $L(\boldsymbol{X})$ where $\boldsymbol{X}$ is the IHS defined in (5.112). D3-branes with and without world-volume flux provide two alternative but equivalent description of topological defects which generate the same symmetry action.

discrete isometry of the geometry, which generates duality and triality defects for the engineered QFT, to the standard action of $SL(2,\mathbb{Z})$ on (p,q)-strings via monodromy matrices generated by 7-branes wrapping $L(\boldsymbol{X})$ is possible only when a world-volume flux on the D3-brane along torsional cycles is turned on, see table 5.2.

Example: 4d $\mathcal{N}=4$ from Type IIB. In addition, as a cross check of our proposal, we can also apply this construction directly to the 4d $\mathcal{N}=4$ SYM theories engineered in IIB on $\boldsymbol{X}=T^2\times\mathbb{C}^2/\Gamma_{ADE}$, with link $L=T^2\times S^3/\Gamma$. Consider the A-type theories, then $L(\boldsymbol{X})$ has non-trivial torsion link homology

$$\mathrm{Tor}(H_2(L(\boldsymbol{X}),\mathbb{Z}))=\mathfrak{f}\oplus\mathfrak{f}'=\mathbb{Z}_N\oplus\mathbb{Z}'_N\,, \tag{5.127}$$

where

$$\mathfrak{f}=\Sigma_1\otimes\gamma_1,\qquad \mathfrak{f}'=\Sigma'_1\otimes\gamma_1\,, \tag{5.128}$$

with torsional $\gamma_1\in H_1(S^3/\mathbb{Z}_N,\mathbb{Z})$ and $\Sigma'_1\oplus\Sigma'_2=H_1(T^2,\mathbb{Z})$. We can wrap D3-branes to generate topological surface defects of the SymTFT. The action of duality and triality defects corresponds to a finite subset of large diffeomorphisms of the T^2 acting on its complex structure. For fixed values of the complex structure $\tau=e^{i\pi/2},e^{i\pi/3}$ they provide symmetries of the 4d QFT, where the action on the 1-cycles of the torus induces an action on the torsional part of $H_2(L(\boldsymbol{X}),\mathbb{Z})$ via (5.128). We can then turn on fluxes on the D3-brane world-volume to map the topological defect to induced

(p, q)-strings and to 7-branes with monodromies acting on the strings like for the $[A_2, D_4]$ theory case.

We can extend this also to more complicated examples, straightforwardly when the dimension of the conformal manifold is 1, or when we are able to identify the action of S and ST on a 1-dimensional subspace of the conformal manifold, [93]. It would be also interesting to generalize these to theories with a more complicated conformal manifold. We leave this to future work.

5.3 Hanany-Witten Effect: Generalized Charges and Anomalies

We have so far introduced the notion of charges of topological defects in terms of brane linking. In all of the above, we explained the brane origin of the action of codimension-$(q+1)$ topological defects on charged q-dimensional extended operators, i.e. q-charges.

Generalized Charges. It is however also known field theoretically that codimension-$(p + 1)$ topological defects can act on extended operators of dimension $q \neq p$ as higher-representations [108, 109]. In this section we demonstrate how branes know about this generalized concept of q-charges through the so-called Hanany-Witten effect [248]. We will furthermore show that this effect is intimately related to additional couplings in the topological bulk theory, corresponding to 't Hooft anomalies of the symmetries generated by these same branes, depending on the boundary conditions, or leading to a twisted DW theory.

Our earlier notion of charge had two origins: either via the d-dimensional flux sector dimensionally reduced on L_{int} or the Bianchi identities, where we truncate everything to the topological sector that describes the behaviour of finite flat abelian fields.

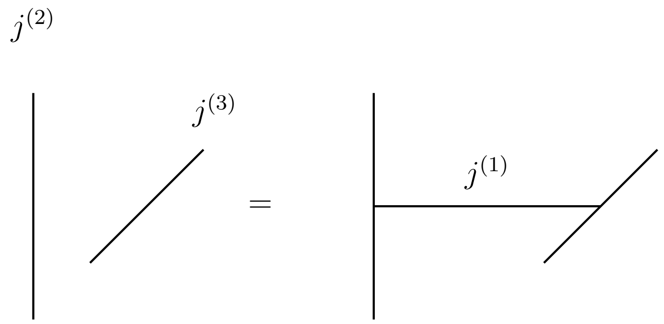


Figure 5.1: Passing two branes corresponding to magnetic sources $j^{(2)}$ and $j^{(3)}$ through each other can result in the creation of a new brane, stretching between them, corresponding to magnetic source $j^{(1)}$, if the currents are linked by relation (5.130).

The starting point of our present discussion is one particular case of interest when the dimensionally reduced Bianchi identities feature three external fluxes satisfying

$$df^{(1)} = f^{(2)} \wedge f^{(3)} + j^{(1)} \,, \qquad df^{(2)} = j^{(2)} \,, \qquad df^{(3)} = j^{(3)} \,, \tag{5.129}$$

with the $f^{(2)} \wedge f^{(3)}$ term in the first relation originates from a non-trivial Chern-Simons term in the original $(D+1)$-dimensional action (5.16). These are exactly the type of Bianchi identities that lead to Hanany-Witten transitions [248]. One can quickly notice the potential for non-trivial physics in this situation by differentiating the above equation

$$0 = j^{(2)} \wedge d^{-1} j^{(3)} + (-1)^{(\deg f^{(2)}+1)(\deg f^{(3)}+1)} j^{(3)} \wedge d^{-1} j^{(2)} + dj^{(1)} \,. \tag{5.130}$$

The first consequence of this relation is that the two branes corresponding to magnetic sources $j^{(2)}$ and $j^{(3)}$ link in the $(d+1)$-dimensional space-time. Exchanging the position of the two branes in the linking direction generates a difference in the total linking number. This number must be fixed, due to the Bianchi identity realizing charge conservation, by the creation of branes corresponding to the $j^{(1)}$ magnetic source extending along the linking direction[10] (see figure 5.1), see [249]. The crucial insight we provide in this work is how to interpret this bulk property of branes in

[10]See [248] for the electric point of view on how the change of linking leads to the creation of a brane.

terms of the symmetry generators which they correspond to in the field theory. The Hanany-Witten (HW) effect can be interpreted in two ways depending on the allowed topological boundary conditions, which concretely means how we place the branes in $M_{d_{\rm QFT}+1}$: this encodes

1. The q-charges (or generalized charges) of a symmetry. This occurs when the branes in the HW-configuration are such that one wraps the radial direction, and the other does not.

2. The (mixed) 't Hooft anomalies or the topological coupling leading to a twisted DW theory: this occurs when none of the branes extend along the radial direction.

These implications will be discussed in subsequent sections.

5.3.1 The Hanany-Witten Effect

We will now discuss the relevant Hanany-Witten (HW) transitions, following the original effect discussed in [248]. For our symmetry considerations we will require various HW-setups, in type II and M-theory.

Before exploring generalizations, we first illustrate the effect in a simple example.

Motivating Example. Consider the following configuration of branes in type IIA on a generic 10d spacetime parameterized by coordinates $\{x_i : i = 0, \ldots, 9\}$.

Brane	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
NS5	X	X	X	X	X	X				
D8	X	X	X	X	X	X	X	X	X	

(5.131)

The NS5-brane is a magnetic source for the NS-NS gauge field B_2 with field strength H_3. Using this fact, one can consider the concept of *linking* between the two branes by computing the flux

$$\int_{x_6,x_7,x_8} H_3 \,. \tag{5.132}$$

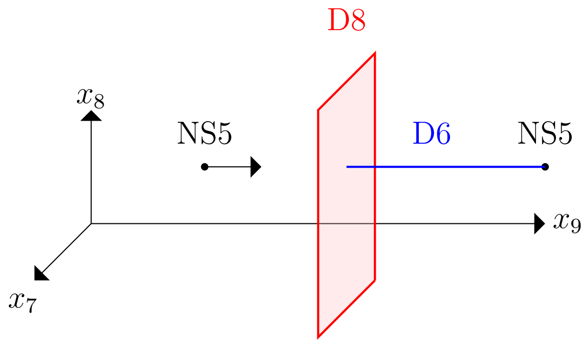


Figure 5.2: The D8/ NS5 Hanany-Witten configuration projected onto $\{x_7, x_8, x_9\}$ directions. The NS5 brane is a point and the D8 is a plane in the $\{x_7, x_8\}$ directions. Passing the NS5 brane through the D8 brane generates a D6 brane attachment (a line along the x_9 direction).

This computes the total linking number of D8-branes with all NS5-branes, in a way we will shortly explain.

The key observation is that the NS-NS 3-form flux H_3, pulled back to the world-volume of the D8-brane, is trivial in cohomology. Indeed, let a_1 denote the $U(1)$ gauge field localized on the D8-brane, and let f_2 denote its field strength. The pullback of the NS-NS 2-form B_2 to the D8-brane world-volume combines with f_2 in the gauge-invariant and globally defined combination $\mathcal{F}_2 = f_2 - B_2$. Making use of the Bianchi identity $df_2 = 0$, we see that $H_3 = -d\mathcal{F}_2$. Naïvely, we may conclude that the linking number defined above is therefore always necessarily zero, if the space spanned by x_6, x_7, x_8 is a closed, compact, oriented 3-manifold. If this were the case, it would not be possible to move the NS5-brane across the D8-brane. Such a move is allowed, however at the cost of creating a D6-brane in the process (see figure 5.2).

Recall that a D6-brane ending on a D8-brane acts as a magnetic source for the a_1 gauge field on the D8-brane, modifying the Bianchi identity for f_2 to $df_2 = \pm\delta_3$, where δ_3 represents the locus inside the D8-brane where the D6-brane ends, and the sign keeps track of orientation.

We will utilize tables of the following type as a compact way of summarizing

Hanany-Witten configurations:

Brane	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D8	X	X	X	X	X	X	X	X	X	
NS5	X	X	X	X	X	X				
D6	X	X	X	X	X	X				X

(5.133)

We now look to find general classes of brane configurations which undergo HW transitions.

Using Brane Linking. A natural language to discuss these transitions in generality is the notion of string theoretic linking of two magnetic sources introduced in section 5.2.1

$$\mathcal{L}_d(\mathcal{W}^{(i)}, \mathcal{W}^{(j)}) = \int_{M_d} J^{(i)} \wedge d^{-1} J^{(i)} = \int_{M_d} dF^{(i)} \wedge F^{(j)} \,. \tag{5.134}$$

Recall that this is a topological property associated to two branes, whose magnetic sources are localized on sub-manifolds $\mathcal{W}^{(i)}, \mathcal{W}^{(j)}$. Notice that since $dF^{(i)} = \delta(\mathcal{W}^{(i)})$, this integral is readily re-written in terms of a lower-dimensional integral as in (5.132)

Let us consider two branes in string/M-theory. We look for configurations in which a subset of the directions in the world-volumes of the branes link (in the above sense) inside a subset of the total directions of spacetime. The general situation we face in this section is indeed such that the dimension formula reads (5.6). We have several cases.

Direct Linking in Spacetime. In the simplest case, all world-volume directions of both branes link inside the entire spacetime. An example is furnished by an NS5-brane and a D2-brane in Type IIA string theory:

Brane	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
NS5	X	X	X	X	X	X				
D2							X	X	X	

(5.135)

The integer linking number for this configuration is

$$\mathrm{Link}_{X_{10}}(M_6^{\mathrm{NS5}}, M_3^{\mathrm{D2}}) \,, \qquad X_{10} = \{x_0, \dots, x_9\}, \quad M_6^{\mathrm{NS5}} = \{x_0, \dots, x_5\} \,, \quad M_3^{\mathrm{D2}} = \{x_6, \dots, x_8\} \,. \tag{5.136}$$

This case is however not relevant for the applications in this work.

HW-Configurations for Symmetries. Next, we have the case in which the two branes are simultaneously extending along a subset of the directions of spacetime. The problem is effectively reduced from $D = 10$ or 11 to a smaller dimensionality D', in which the remaining world-volume directions of the branes link. This type of configuration corresponds to setups of HW type, which we classify below. An example is furnished by the original HW configuration of an NS5-brane and a D5-brane:

Brane	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
NS5	X	X	X	X	X	X				
D5	X	X	X				X	X	X	

(5.137)

The common directions are $x_{0,1,2}$ and the relevant linking number is

$$\text{Link}_{X_7}(M_3^{\text{NS5}}, M_3^{\text{D5}}) \,, \qquad X_7 = \{x_3, \dots, x_9\}, \quad M_3^{\text{NS5}} = \{x_3, \dots, x_5\} \,, \quad M_3^{\text{D5}} = \{x_6, \dots, x_8\} \,. \tag{5.138}$$

HW-Configurations for Generalized Charges. Finally, for completeness we tabulate all possible Hanany-Witten setups in type II and M-theory, which are relevant for computing generalized charges. An example appeared already in [41]. These configurations can be grouped together as follows:

1. The first class is realized in IIB or IIA and is given by the following brane system:

Brane	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
Dp	X	X	X	X	X	X	X	X		
Dp'	X								X	
F1	X									X

(5.139)

where $8 = p + p'$, and we can apply T-duality in the $x_{1,2,3,4,5,6,7,8}$ directions. In addition when $p = 7$ and $p' = 1$, the role of F1 and the D1 can be exchanged, and in generalised to (p, q)-strings and 7-branes.

2. The second class is a special case in IIB given by the following system:

Brane	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$[p, q]$7-brane	X	X	X	X	X	X	X	X		
(p', q')5-brane	X	X	X	X	X				$p'x_8 = q'x_9$	
(r, s)5-brane	X	X	X	X	X				$rx_8 = sx_9$	
(p, q)5-brane	X	X	X	X	X				$px_8 = qx_9$	

(5.140)

where $px_8 = qx_9$ means that the 5-brane extend along this locus. The last 5-brane is the one created once the 7-brane crosses the junction between the (p', q') 5-brane and the (r, s) 5-brane. Finally the total 5-brane charge must be conserved, i.e. $p + p' + r = 0$ and $q + q' + s = 0$.

3. The third class is related to the original Hanany-Witten setup by T-dualities:

Brane	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
Dp	X	X	X	X	X	X				
NS5	X	X	X				X	X	X	
Dp'	X	X	X							X

(5.141)

where $p' = p - 2$ and we can apply T-duality[11] in the $x_{1,2,6,7,8}$ directions. In the case $p = 5$, $p' = 3$ we also have a generalization, with a (p, q) 5-brane in the first row, a (p', q') 5-brane in the second row, and $pq' - p'q$ D3-branes in the third row [251].

4. The fourth class is a single brane system in M-theory:

Brane	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
M5	X	X	X	X	X	X					
M5'	X	X					X	X	X	X	
M2	X	X									X

(5.142)

Details of HW-configurations. We have already discussed class (III) above. We note that similar remarks apply in general to $D(p-2)$ branes ending on Dp-branes, and refer the reader to [251] for a generalization of the setup of class (III), involving the creation of $pq' - p'q$ D3-branes when a (p, q) 5-brane and a (p', q') 5-brane are passed across each other.

Let us now turn to setups of class (IV) in M-theory. This Hanany-Witten setup is discussed in [252] and can be derived in a way analogous to the argument for class (III). In this case, we use the fact that the world-volume of an M5-brane supports a localized 2-form field b_2, with self-dual field strength h_3. The latter combines with the pullback of the M-theory 3-form C_3 into the gauge-invariant and globally defined combination $\mathcal{H}_3 = h_3 - C_3$. As a result, on the world-volume of the M5-brane we have $G_4 = -d\mathcal{H}_3$, where we have made use of the Bianchi identity $dh_3 = 0$. Once again,

[11] Note that T-duality along the NS5-brane world-volume results in another NS5-brane, whereas transverse to it, results in a KK-monopole [250].

this would naïvely suggest the vanishing of the linking number $\int_{M_4} G_4$ computed on the orthogonal directions of the second M5-brane M_4. The correct conclusion is that, when the two M5-branes are passed across each other, an M2-brane is generated. In fact, this is the correct object to modify the Bianchi identity for h_3 from $dh_3 = 0$ to $dh_3 = \pm\delta_4$, where δ_4 represent the locus inside the M5-brane where the M2-brane ends, and the sign keeps track of orientation.

The brane setups of classes (I) may be derived from those of class (III) with the help of S- and T-dualities. Let us start from the class (III) setup with $p = 3$, $p' = 1$, describing a Hanany-Witten move in which a D1-brane is generated when an NS5-brane and a D3-brane are passed across each other. By S-duality, this is mapped to a setup of class (I) with $p = 3$, $p' = 5$: an F1-string is generated when a D5-brane and a D3-brane are passed across each other [252]. This can also be seen as follows. The D5-brane is a magnetic source for the RR 3-form field strength F_3. The relevant linking is then measured by integrating F_3 on the world-volume of the D3-brane. Invariance of the D3-brane under S-duality, however, implies that the electromagnetic dual $\tilde{a}_1$ of the gauge field a_1 on the brane combines with the pullback of the RR 2-form into the gauge-invariant combination $\tilde{\mathcal{F}}_2 = \tilde{f}_2 - C_2$, where $\tilde{f}_2$ is the field strength of $\tilde{a}_1$. Setting $C_0 = 0$ for simplicity, on the world-volume of the D3-brane we have $F_3 = -d\tilde{\mathcal{F}}_2$. The argument then proceeds as for class (III). Once the setup of class (I) is established for $p = 3$, other values for p are derived by T-duality.

5.3.2 Generalized Charges

Let us denote the HW brane pair (brane$_1$, brane$_2$). Passing one through the other generates the third brane brane$_3$. Suppose that we pick brane$_1$ to be parallel to the boundary, and brane$_2$ to wrap the radial direction. Field theoretically, brane$_1$ corresponds to a topological symmetry generator D_p, whilst brane$_2$ is a non-topological (extended) defect $\mathcal{O}_q$.

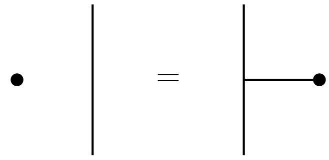


Figure 5.3: In this figure dots and lines are defects in the boundary QFT of interest (this could be thought of as a (1+1)d system or a 2d projection of a higher-dimensional analogue). In the higher-dimensional configuration, dragging the point-like operator through the extended line operator generates a non-genuine operator (or twisted sector, if the line defect is topological).

The HW transition implies that passing $\mathcal{O}_q$ defect through D_p creates a third topological operator D_l (brane$_3$ necessarily does not wrap the radial direction) which is *attached* to $\mathcal{O}_q$. Generically this process maps a genuine operator to non-genuine one (see figure 5.3 for an example of this effect). This is precisely the charge of a non-invertible abelian categorical symmetry on charged defects, which does not preserve the dimensionality of the defects [23, 32, 41].

Example: Klebanov-Strassler. From the Bianchi identity $dF_5 + H_3F_3 = J^{(\mathrm{D3})}$ we learn that there is a Hanany-Witten effect between a D3 brane and a D5 on S^3 which generates an F1 string stretched between the two.

It is known that the $\mathcal{G} = PSU(M)$ theory has a non-invertible 0-form symmetry. In [41] the non-invertible 0-form topological operator was given a string theory origin as a D5-brane wrapped on $S^3 \subset T^{1,1}$. Furthermore, its non-invertible action on 't Hooft lines was explained using the Hanany-Witten effect. Now the wrapped D3 is perpendicular to the boundary (the 1-form symmetry is gauged), and the brane creation turns a genuine line operator into a non-genuine one. We refer the reader to appendix C for a field theory analysis of non-invertible actions on line operators.

Example: Maldacena-Nunez. A second description (MN) of 4d $\mathcal{N} = 1$ SYM is given in [253]. We begin with the 6d $\mathcal{N} = (1, 1)$ LST living on M NS5 branes in IIB. The four-dimensional theory is obtained via a topologically twisted S^2 reduction.

For the sake of brevity, we use the fact that the for the purposes of our computations the above background is S-dual to that of Klebanov Strassler, in the sense that we replace

$$\int_{S^3} F_3 = M \leftrightarrow \int_{S^3} H_3 = M \,. \tag{5.143}$$

The derivations of the BF terms and anomalies proceed identically. We therefore identify the brane responsible for the non-invertible 0-form symmetry as

$$D_3(M_3^{\mathrm{NS5}}) \leftrightarrow \mathrm{NS5}(M_3^{\mathrm{NS5}} \times S^3) \,. \tag{5.144}$$

On the LHS we use the notation for topological defects $D_q(M_q)$, i.e. a q-dimensional topological defect on the spacetime manifold M_q, whereas on the RHS we use the notation of a brane (NS5 or Dp or Mp wrapped on an internal cycle an M_q). Following an analogous procedure as appendix B of [41] it is easy to see that this brane's topological world-volume terms correctly reproduce the expected TQFT stacking and therefore fusion rules known from field theory [22].

Once again we consider the three brane origins of 2-surfaces in the 5d bulk: F1-, D1- and wrapped D3-branes. However, since in this setup there is only H_3 flux over the S^3, the linking configurations are simpler. Only the D1 and D3 wrapped on S^2 link in the 5d bulk. We can therefore identify

$$\begin{aligned} D_2(M_2^{\mathrm{D3}}) &\leftrightarrow D3(M_2^{\mathrm{D3}} \times S^2) \,, \\ \widehat{D_2}(M_2^{\mathrm{D1}}) &\leftrightarrow D1(M_2^{\mathrm{D1}}) \,, \end{aligned} \tag{5.145}$$

as the generators of the electric (magnetic) 1-form symmetries in the $SU(N)(PSU(N))$ theories respectively.

The D3/ D5 Hanany-Witten effect responsible for generalized charges in the Klebanov-Strassler solution has an S-dual partner involving a D3/ NS5 transition.

Consider a boundary condition such that the NS5 and stretched D1-brane are topological, and the D3 is not ($\mathcal{G} = PSU(N)$). The HW transition describes the non-invertible action of D_3 on the charged 't Hooft line (the wrapped D3-brane) by attaching a topological 2-surface (the D1-brane).

Example: M-theory on G_2. M-theory on the singular G_2 holonomy manifold $\mathbb{C}^2/\mathbb{Z}_N \to S^3$ models the UV of 4d $\mathcal{N}=1$ pure SYM [197, 254–256]. The boundary geometry is $S^3/\mathbb{Z}_N \to S_3$. The link L_6 therefore has homology groups

$$H_\bullet(L_6, \mathbb{Z}) = \{\mathbb{Z}, \mathbb{Z}_N, 0, \mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}_N, 0, \mathbb{Z}\}\,. \tag{5.146}$$

We propose that the branes generating the 1- and 0-form symmetries respectively are [12]

$$\begin{aligned} \mathrm{M5}(\Sigma_2 \times \gamma_1 \times \mathrm{S}^3) &\leftrightarrow D_2(\Sigma_2)\,,\\ \mathrm{M5}(\mathrm{M}_3 \times \mathrm{S}^3/\mathbb{Z}_\mathrm{N}) &\leftrightarrow D_3(M_3)\,. \end{aligned} \tag{5.147}$$

From the Bianchi identity

$$dG_7 - \frac{1}{2}G_4^2 = J^{G_7}\,, \tag{5.148}$$

one can see there is a Hanany-Witten transition involving two M5 branes, generating an M2 brane, as demonstrated in (5.142).

The global variant $\mathcal{G} = PSU(N)$ corresponds to picking the M5-brane wrapping the torsional 4-cycle to be perpendicular to the boundary, whilst the other is parallel. In this case, the Hanany-Witten effect produces a topological attachment to the non-topological string charged under the 1-form symmetry: turning it from a genuine to non-genuine line operator.

5.3.3 Hanany-Witten and 't Hooft Anomalies

Now suppose that both (brane$_1$, brane$_2$) are *parallel* to the boundary. They therefore both correspond to topological defects $D_p, D_{p'}$ whose non-trivial linking forces the creation of a brane in the radial direction, corresponding to a non-topological (extended) defect $\mathcal{O}_q$.

We will now argue that such a configuration indicates the existence of certain 't Hooft anomalies using two complimentary approaches.

[12] Identifying geometrically $U(1)_R$ symmetry and its breaking to $\mathbb{Z}_{2N}$ is still a challenge in the geometric engineering in M-theory [257].

1. The first is that the created brane wrapped along the radial direction creates a non-topological ambiguity in terms of how the topological defects are separated in the bulk. When we push these to infinity we argue that this signals the presence of an anomaly.

2. The second involves directly projecting the bulk Hanany-Witten configuration to the boundary. The bulk picture becomes a junction in the boundary, which the Hanany-Witten computation tells us must be charged under certain symmetries. This is another hallmark of a 't Hooft anomaly.

The anomalies are computed using suitable intersections of branes which depend on the spacetime dimension. In particular, we look for intersections such that one of the participating branes links with the intersection of the other two, as discussed in section 5.2.4.

Anomalies from Topological Defects. Coupling a theory to a background for a higher-form symmetry amounts to inserting a mesh of the corresponding topological defects. This mesh contains junctions, inconsistencies of which can signal the presence of anomalies [7].

For example, consider a theory with both a p and $(d-p-1)$-form symmetry. If the codimension-p topological operators generating the former symmetry are charged under the codimension-$(d-p-1)$ topological operators generating the latter, the two symmetries participate in a mixed 't Hooft anomaly [7]. This is because two related meshes of these defects may differ by a phase.

The above is a special case where the two participating symmetries have appropriate dimension such that their operators link in spacetime. However, it is generically possible that a codimension$-(p+1)$ operator can also act on an extended operator of dimension $q \neq p$. In this way, we are able to explore 't Hooft anomalies involving higher-form symmetries of different degrees from the perspective of their topological operators and their junctions.

In general, a mixed 't Hooft anomaly between two (or more) higher-form symmetries is encoded in the junctions of their corresponding defects. We argue that this information is naturally encoded in our understanding of branes. For example, two branes may intersect and generate a third (by Hanany-Witten). If this third brane is *charged* under one of the symmetries generated by one of the intersecting branes - this junction signals a mixed 't Hooft anomaly.

Example: $\mathcal{N} = 1$ SYM. In the three presentations of 4d $\mathcal{N} = 1$ $\mathfrak{su}(M)$ SYM presented earlier, in each case there was a Hanany-Witten configuration of branes which in the $\mathcal{G} = PSU(M)$ variant described a generalized charge. By the above argument, in the frame where we pick boundary conditions such that $\mathcal{G} = SU(M)$, these configurations also signal the mixed 't Hooft anomaly in these models.

5.3.4 Example: 4d $\mathcal{N} = 4$ $\mathfrak{so}(4n)$ SYM

In this section we demonstrate that the HW effect is responsible for generalized charges in several global variants of the $\mathfrak{so}(4n)$ theory, and the mixed anomaly

$$\mathcal{A} = \frac{1}{2} \int A_1 C_2' B_2 \,, \tag{5.149}$$

in the $\mathcal{G} = SO(4n)$ theory, where A_1 is a background for $\mathbb{Z}_2^{(0)}$ and C_2', B_2 are both $\mathbb{Z}_2^{(1)}$ backgrounds. From the SymTFT/ Gauss law perspective we can read off the brane origins of the topological symmetry generators [90]. For convenience we summarize these findings in table 5.3.

$\mathcal{G}$ =SO$(4n)$. For $\mathcal{G}$ =SO$(4n)$, the brane identification is [90]

$$\begin{aligned} D_2(M_2) &\leftrightarrow \text{D5}(M_2 \times \mathbb{RP}^4) \,, \\ D_3(M_3) &\leftrightarrow \text{D3}(M_3 \times \mathbb{RP}^1) \,, \end{aligned} \tag{5.150}$$

	Symmetry	Background Field	Brane Origin of Sym Generator
SO(4n)	$\mathbb{Z}_2^{(0)}$ $\mathbb{Z}_2^{(1)}$ $\mathbb{Z}_2^{(1)}$	A_1 C_2' B_2	D3 on $\mathbb{RP}^1$ D5 on $\mathbb{RP}^4$ NS5 on $\mathbb{RP}^4$
Spin(4n)	$\mathbb{Z}_2^{(0)}$ $\mathbb{Z}_2^{(1,s)}$ $\mathbb{Z}_2^{(1,c)}$ $\mathbb{Z}_2^{(1,v)}$	A_1 C_2 B_2	D3 on $\mathbb{RP}^1$ D1 on pt $\in \mathbb{RP}^5$ NS5 on $\mathbb{RP}^4\oplus$ D1 on pt $\in \mathbb{RP}^5$ NS5 on $\mathbb{RP}^4$
PO(4n)	$\mathbb{Z}_2^{(2)}$ $\mathbb{Z}_2^{(1)}$ $\mathbb{Z}_2^{(1)}$	A_3 C_2' B_2'	D3 on $\mathbb{RP}^3$ D5 on $\mathbb{RP}^4\oplus$F1 on pt $\in \mathbb{RP}^5 + \int A_1 B_2$ F1 on pt $\in \mathbb{RP}^5$
$\text{Pin}^+(4n)$	$\mathbb{Z}_2^{(2)}$ $\mathbb{Z}_2^{(1)}$ $\mathbb{Z}_2^{(1)}$	A_3 C_2 B_2	D3 on $\mathbb{RP}^3$ D1 on pt $\in \mathbb{RP}^5$ NS5 on $\mathbb{RP}^4\oplus$D1 on pt $\in \mathbb{RP}^5 + \int A_1 C_2'$
Sc(4n)	$\mathbb{Z}_2^{(0)}$ $\mathbb{Z}_2^{(1)}$ $\mathbb{Z}_2^{(1)}$	A_1 C_2 B_2'	D3 on $\mathbb{RP}^1 + \int B_2 C_2'$ D1 on pt $\in \mathbb{RP}^5$ F1 on pt $\in \mathbb{RP}^5$

Table 5.3: SymTFT and brane origins of symmetry generators in various global forms of $\mathfrak{so}(4n)$ 4d SYM theories.

where D_2, D_3 are the generators of $\mathbb{Z}_2^{(1,C')} \subset \mathbb{Z}_2^{(1,C')} \times \mathbb{Z}_2^{(1,B)} = \Gamma^{(1)}$ and $\mathbb{Z}_2^{(0)}$ respectively. The charged lines under the $\mathbb{Z}_2^{(1,B)}$ 1-form symmetry factor are

$$\mathcal{O}_1(\Sigma_1) \leftrightarrow \text{F1}(\Sigma_1 \times \mathbb{R}^{>0}) \,. \tag{5.151}$$

These three wrapped branes form a HW configuration, from which we observe that the $\mathbb{Z}_2^{(1,C')}$ and $\mathbb{Z}_2^{(0)}$ symmetry defects intersect in 4d in a line which is charged under $\mathbb{Z}_2^{(1,B)}$ factor: this signals the presence of the mixed 't Hooft anomaly between all three symmetries. One can derive a similar result using the S-dual branes: pulling the NS5-brane, which generates $\mathbb{Z}_2^{(1,B)}$, across D_3 generates a D1-brane which is charged under $\mathbb{Z}_2^{(1,C')}$.

The theory with gauge group $\mathcal{G} = \text{Spin}(4n)$ is related to $\mathcal{G} = \text{SO}(4n)$ via gauging of the 1-form symmetry (for more details also the categorical structure, see [28]). This maps the mixed anomaly to a split 2-group symmetry. In this way the HW brane configuration explained above also encodes this split 2-group global symmetry.

$\mathcal{G}$ =Sc$(4n)$. We now consider how the non-invertible 0-form symmetry in the $\mathcal{G}$ =Sc$(4n)$ variant acts on the charged lines of the theory. The 0-form symmetry is generated by

$$D_3(M_3) \leftrightarrow \text{D3}(M_3 \times \mathbb{RP}^1) \,. \tag{5.152}$$

Meanwhile the invertible 1-form symmetries have non-topological charged lines

$$\begin{aligned} \mathcal{O}_1(\Sigma_1) &\leftrightarrow \text{D5}(\Sigma_1 \times \mathbb{R}^{>0} \times \mathbb{RP}^4) \,, \\ \mathcal{O}_1'(\Sigma_1') &\leftrightarrow \text{NS5}(\Sigma_1' \times \mathbb{R}^{>0} \times \mathbb{RP}^4) \,. \end{aligned} \tag{5.153}$$

If we pass $\mathcal{O}1$ or $\mathcal{O}1'$ through $D_3(M_3)$, there is a non-trivial Hanany-Witten move which generates an F1 or D1 brane respectively. These are *topological* operators which respectively generate the invertible 1-form symmetry which acts on the other charged line. These results agree with the complementary field theory analysis, reported in appendix C.

$\mathcal{G}$ =**PO**$(4n)$**.** Now we look at how the non-invertible 1-form symmetry in the $\mathcal{G}$ =PO$(4n)$ variant acts on the 2-surfaces charged under the invertible 2-form symmetry.

In this case there are a number of non-invertible actions we should consider. The non-invertible 1-form symmetry is generated by

$$D_2(M_2) \leftrightarrow \text{D5}(M_2 \times \mathbb{RP}^4)\,. \tag{5.154}$$

On the other hand, the charged 2-surfaces are given by

$$\begin{aligned} \mathcal{O}2(\Sigma_2) &\leftrightarrow \text{D3}(\mathbb{R}^{>0} \times \Sigma_2 \times \mathbb{RP}^1)\,, \\ \mathcal{O}2'(\Sigma_2) &\leftrightarrow \text{NS5}(\mathbb{R}^{>0} \times \Sigma_2' \times \mathbb{RP}^3)\,. \end{aligned} \tag{5.155}$$

There is a non-trivial Hanany-Witten move for both of these. First, passing $\mathcal{O}_2$ through D_2 generates an F1-string stretched between the two: this is the generator of the invertible 2-form symmetry under which $\mathcal{O}_2'$ is charged. On the other hand, passing $\mathcal{O}_2'$ through D_2 generates a D3-brane: this is the generator of the other invertible 2-form symmetry which acts on $\mathcal{O}_2$.

$\mathcal{G}$ =**Pin**$^+(4n)$**.** The non-invertible 1-form symmetry in this case is generated by

$$D_2(M_2) \leftrightarrow \text{NS5}(M_2 \times \mathbb{RP}^4)\,. \tag{5.156}$$

On the other hand, the charged 2-surfaces are given by

$$\begin{aligned} \mathcal{O}_2(\Sigma_2) &\leftrightarrow \text{D3}(\mathbb{R}^{>0} \times \Sigma_2 \times \mathbb{RP}^1)\,, \\ \mathcal{O}_2'(\Sigma_2) &\leftrightarrow \text{D5}(\mathbb{R}^{>0} \times \Sigma_2' \times \mathbb{RP}^3)\,, \end{aligned} \tag{5.157}$$

There is also a non-trivial Hanany-Witten move for both of these. First, passing $\mathcal{O}_2$ through D_2 generates an D1-string stretched between the two: this is the generator of the invertible 2-form symmetry under which $\mathcal{O}_2'$ is charged. On the other hand, passing $\mathcal{O}_2'$ through D_2 generates a D3-brane: this is the generator of the other invertible 2-form symmetry which acts on $\mathcal{O}_2$.

Outer-Automorphism Action on $\mathcal{G} = \mathbf{Spin}(4n)$. There is also a brane origin to the $\mathbb{Z}_2^{(0)}$ outer-automorphism in the $\mathcal{G} = \text{Spin}(4n)$ theory which exchanges

$$\mathcal{O}_1^{(s)} \leftrightarrow \mathcal{O}_1^{(c)}\,, \tag{5.158}$$

where $\mathcal{O}_1^{(s,c)}$ are the spinor/co-spinor Wilson lines. An equivalent way of describing this action is shown in figure 5.4. In terms of branes, these lines are [42]

$$\begin{aligned} &\mathcal{O}_1^{(s)} \leftrightarrow \text{D5}(\text{M}_1 \times \mathbb{R}^{>0} \times \mathbb{RP}^4)\,, \\ &\mathcal{O}_1^{(c)} \leftrightarrow \text{D5}(\text{M}_1 \times \mathbb{R}^{>0} \times \mathbb{RP}^4) \oplus \text{F1}(\text{M}_1 \times \mathbb{R}^{>0})\,. \end{aligned} \tag{5.159}$$

Furthermore, it is known that the brane generating the outer-automorphism symmetry is

$$D_3(M_3) \leftrightarrow \text{D3}(M_3 \times \mathbb{RP}^1)\,. \tag{5.160}$$

We now discuss the action of this operator at the level of the branes. In the arrangement shown in figure 5.5, we consider what happens when the wrapped D5 pierces through the wrapped D3 representing the outer-automorphism generator. Since the D3-brane is a source for the RR C_4 field, as we pass from left to right there is a flux jump which induces a non-trivial F1 charge via the 6d 5-brane world-volume coupling

$$\int B_2 C_4\,, \tag{5.161}$$

such that an F1 string (which couples to B_2) emanates from the defect. This is exactly the outer-automorphism action we expect from field theory.

Now consider the arrangement in figure 5.6. In this case the F1 string passes through the brane un-changed, there are not enough dimensions to run the same argument as above. This is exactly the invariance of the operator $\mathcal{O}_1^{(v)}$ (the vector Wilson line) under the outer-automorphism.

5.3.5 Example: Generalized Charges for Duality/Triality Defects in 4d

In this section we study duality and triality defects which generate non-invertible symmetries that arise from subgroups of $SL(2,\mathbb{Z})$. They provide 0-form symmetries

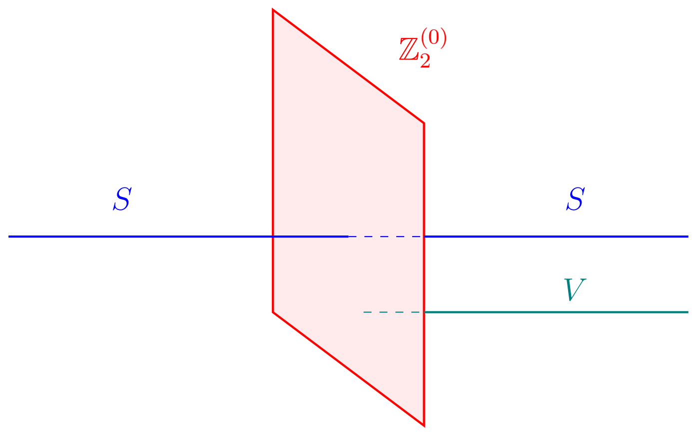


Figure 5.4: $\mathbb{Z}_2^{(0)}$ outer-automorphism action, depicted in terms of defects: the outer automorphism acts as S maps to C. It is useful in the following to write $C = V \otimes S$, as this is how the branes will realize the action.

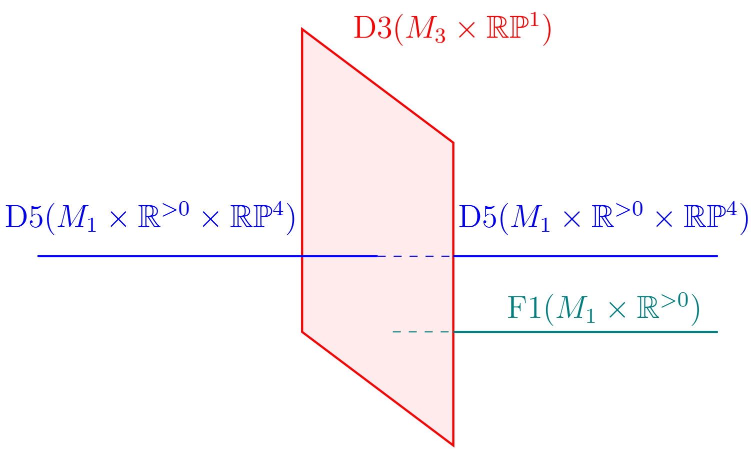


Figure 5.5: The same $\mathbb{Z}_2^{(0)}$ outer-autmorphism action as in figure 5.4, now in terms of branes. The wrapped D3 brane induces a jump in F1 flux which is absorbed by emitting an F1 string.

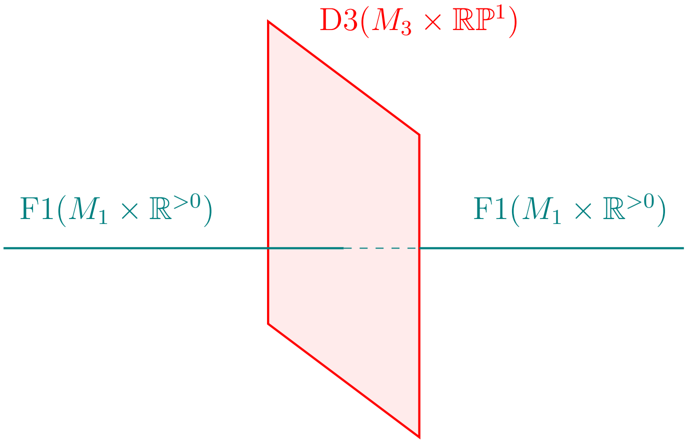


Figure 5.6: The F1-brane, that realizes the vector Wilson line is invariant under the outer-automorphism.

at certain fixed loci under these groups on the conformal manifold of 4d SCFTs. Field theoretically these defects have been studied in [22, 30, 31, 47, 73, 258]. For the bulk theory we refer to [44, 71], and the realization of the topological defects in term of branes to [83].

In string theory the self-duality or triality symmetries are generated by $[p, q]$-7-branes as first observed in [41] and subsequently studied in detail in [83]. We will exemplify this brane-approach for $\mathcal{N} = 4$ $\mathfrak{su}(N)$ SYM. Moreover, since we have analogously constructed topological defects for the $\mathcal{N} = 2$ Argyres Douglas theory of type $[A_2, D_4]$ via geometric engineering in IIB, the properties highlighted in this section will be valid also for that case.

In this section we will put these defects into the context of the SymTFT and derive the generalized charges (in terms of the topological defects of the SymTFT) realized again in terms of "branes" and Hanany-Witten transitions among them.

The (p, q)-strings give rise to topological defects in the SymTFT, which depending on the $\mathcal{B}^{\text{sym}}$ boundary conditions give rise to either topological defects that generate the 1-form symmetry or to the line operators, i.e. generalized charges.

The Hanany-Witten effect between (p, q)-strings and $[p, q]$-7-branes encodes whether the resulting non-invertible symmetry is gauge equivalent to an invertible symmetry or not. In terms of the SymTFT couplings this was analyzed in [44, 71]. This allows the distinction between intrinsic and non-intrinsic non-invertible symmetries – if one wishes to use this formulation. More categorically, the SymTFT is either the same as for an invertible (i.e. higher group) symmetry or not.

We will focus on generalized charges via the Hanany-Witten effect between (p, q)-strings and $[p, q]$-7-branes. In particular, from the brane realization and the Hanany-Witten phenomenon we will be able to provide a diagnostic for intrinsic versus non-intrinsic non-invertible symmetries, even beyond $\mathfrak{su}(p)$ with p prime.

Duality and triality defects for $\mathcal{N} = 4$ SYM arise for fixed values of $\tau = e^{i\pi/2}, e^{i\pi/3}$, respectively, i.e. the values that are invariant under $\mathbb{Z}_4$ or $\mathbb{Z}_6$ subgroups of $SL(2, \mathbb{Z})$ [30, 31, 71, 73, 83]. Our convention for the monodromy matrices labelled by (p, q) charges are

$$M_{p,q} = \begin{pmatrix} pq+1 & p^2 \\ -q^2 & 1-pq \end{pmatrix} \tag{5.162}$$

and we take the basis

$$a = M_{1,0} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad b = M_{1,1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \qquad c = M_{1,-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}. \tag{5.163}$$

We summarize the fixed values of τ and associated monodromy matrices in table 5.4 [13].

Hanany-Witten Setups with $[p, q]$-7-branes and (r, s)-strings

We now describe in more detail the specific Hanany-Witten configuration already introduced in (5.139) that is relevant for this example. Let us consider the original Hanany-Witten brane configuration, consisting of an NS5-brane extended along $x^{0,1,2,3,4,5}$ and a D5-brane extended along $x^{0,1,2,6,7,8}$; when these are moved past each other, a D3-brane extended along $x^{0,1,2,9}$ is created. By applying T-duality in the

[13] We use the conventions of [259], but act on tau from the right as to give rise to the canonical choice of fixed values of τ as e.g. in [260].

Kodaira Type	τ	G	Monodromy Matrix M
II	$e^{\pi i/3}$	$\mathbb{Z}_6$	$ab = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$
II^*	$e^{\pi i/3}$	$\mathbb{Z}_6$	$a^6bcba = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$
III	$e^{\pi i/2}$	$\mathbb{Z}_4$	$a^2b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
III^*	$e^{\pi i/2}$	$\mathbb{Z}_4$	$a^6bcb = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
IV	$e^{\pi i/3}$	$\mathbb{Z}_3$	$a^2ba = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
IV^*	$e^{\pi i/3}$	$\mathbb{Z}_3$	$a^5bcb = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$
I_0^*	τ	$\mathbb{Z}_2$	$a^4bc = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Table 5.4: The Kodaira singularities, associated constant values of τ, the monodromy group and the monodromy matrix M.

$x^{1,2}$ directions, followed by an S-duality S transformation, followed by T-duality in the $x^{6,7}$ directions, we reach a Hanany-Witten setup with a D7-brane extended along $x^{0,...,7}$ and a D1-brane extended along $x^{0,8}$. When these are moved past each other, an F1-string extended along $x^{0,9}$ is created. This configuration conserves both the linking number between the D7-brane and the D1-brane, and the (r, s)-string charge of the system. The latter observation stems from the relation

$$(1\ \ 0)M_{1,0} = (1\ \ 0) + (0\ \ 1)\ . \tag{5.164}$$

In our conventions the charges of an (r, s)-string are collected in the row vector $(s\ \ r)$. Thus, in the above relation, $(1\ \ 0)$ represents the D1-brane, $(0\ \ 1)$ the F1-string, while $M_{1,0}$ is the monodromy matrix of the D7-brane (see figure 5.7).

The generalization of (5.164) is the identity

$$(s\ \ r)M_{p,q} = (s\ \ r) + n(q\ \ p)\ , \qquad n := ps - qr\ . \tag{5.165}$$

We interpret this relation as follows. Start with a configuration with a $[p, q]$-7-brane extended along $x^{0,...,7}$, and an (r, s)-string extended along $x^{0,8}$. If we move the (r, s)-

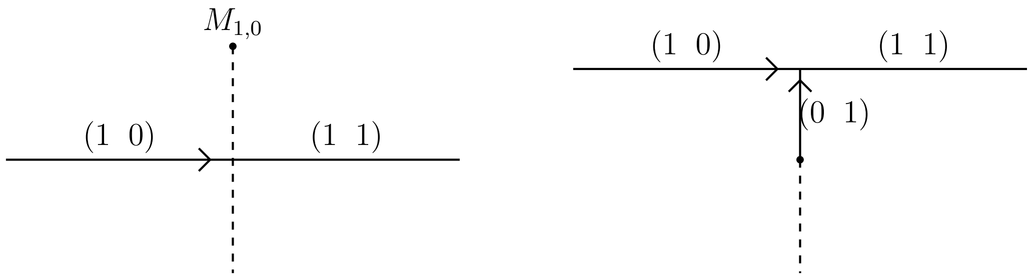


Figure 5.7: Conservation of charge during a Hanany-Witten move involving a D1 and D7 brane. On the left hand side; passing the D1 through the monodromy cut for the D7 brane modifies the charge from $(1,0) \to (1,1)$. On the right hand side; sliding the configuration off the cut must preserve charge, meaning an F1 string is created.

string across the $[p,q]$-7-brane, n copies of a (p,q)-string are generated, extended along $x^{0,9}$. If n is negative, this is understood as $|n|$ copies of a $(-p,-q)$-string. In the special case $n=0$, the (r,s)-string and the $[p,q]$-7-brane are mutually local and the (r,s)-string can end on the $[p,q]$-7-brane; there is no Hanany-Witten brane creation effect if these two objects are passed across each other.

It is important to study the generalized charges, i.e. SymTFT topological defects, coming from (r,s)-strings and the 7-branes with monodromy M, which we take parallel to the boundary. In the spirit of section 5.3.3, instead of being directly related to a mixed 't Hooft anomaly, it has a SymTFT that is a DW theory with twisted cocyles. Let us now consider a 7-brane with monodromy matrix M, written as a product $M = M_{p_1,q_1} M_{p_2,q_2} M_{p_3,q_3} \ldots$. A repeated application of the basic Hanany-Witten move encoded in (5.165) yields the configuration depicted in the figure, where the multiplicities n_1, n_2, n_3, $\ldots$, of the created strings are determined by the charges of the (r,s)-string and by the $[p_k,q_k]$-7-brane labels,

$$n_1 = p_1 s - q_1 r \ , \tag{5.166}$$

$$n_2 = p_2(s + n_1 q_1) - q_2(r + n_1 p_1) \ , \tag{5.167}$$

$$n_3 = p_3(s + n_1 q_1 + n_2 q_2) - q_3(r + n_1 p_1 + n_2 p_1) \ , \tag{5.168}$$

and so on. In general due to $M = M_{p_1,q_1} M_{p_2,q_2} M_{p_3,q_3} \ldots$, we will not have a single

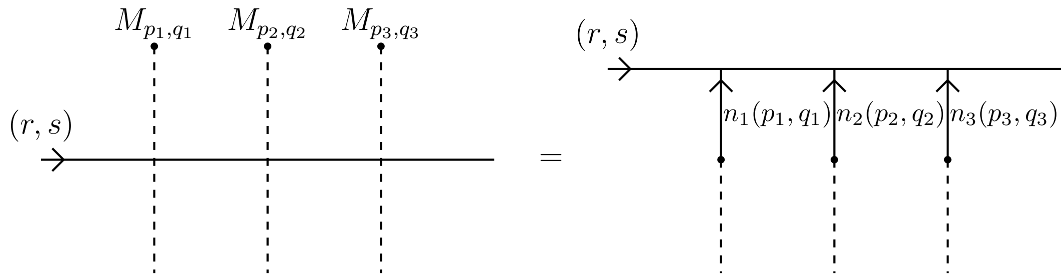


Figure 5.8: Hanany-Witten transitions for a general 7-brane configuration with no fixed $[p,q]$ charge as usually appear in F-theory and (r,s)-strings.

string creation event. If we have multiple string creation events, even modulo N, it signals that something more general than a mixed 't Hooft anomaly is at play. This indeed generically corresponds to a twisted cocycle in the SymTFT.

Examples. Let us consider the Kodaira type IV^* monodromy matrix $M = \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We use the decomposition $M = a^5bcb$, $a = M_{1,0}$, $b = M_{1,1}$, $c = M_{1,-1}$. The multiplicities n_1, ..., n_8 of the created strings are

$$(n_1, \dots, n_8) = (s, s, s, s, s, -r-4s, -r-2s, r) \ . \tag{5.169}$$

Alternatively we can use the decomposition $M = A^5BC^2$, $A = M_{1,0}$, $B = M_{3,1}$, $C = M_{1,1}$. In this case the multiplicities are

$$(n_1, \dots, n_8) = (s, s, s, s, s, -r-2s, r, r) \ . \tag{5.170}$$

Next, let us consider the Kodaira type IV with $M = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$. We can write $M = a^2ba$. The four multiplicities are

$$(n_1, \dots, n_4) = (s, s, -r-s, -r) \ . \tag{5.171}$$

In passing we note that it is not possible to write $M = A^xB^yC^z$ with non-negative x, y, z.

If we consider again Kodaira Type IV, but we work modulo $N = 3$, we can write $M = M_{1,2} = M_{1,1} = c \bmod 3$. In this case there is a single event of string creation, with multiplicity

$$n_1 = r + s \mod 3 \ . \tag{5.172}$$

In this case the HW transition is related to a mixed 't Hooft anomaly between self-triality and the 1-form symmetry.

Intrinsic vs. Non-Intrinsic

Let us recall the notion of intrinsic non-invertible symmetry [31]. Suppose $\mathcal{T}$ is a QFT that admits a non-invertible symmetry. We say that the non-invertible symmetry is of non-intrinsic type if $\mathcal{T}$ can be connected by gauging of a global symmetry to a QFT $\mathcal{T}'$ that only admits invertible symmetries (i.e. higher-form or higher-group symmetries). We say that the non-invertible symmetry of $\mathcal{T}$ is of intrinsic type if such $\mathcal{T}'$ does not exist.

Global Forms of SYM. The global variants of 4d $\mathcal{N} = 4$ $\mathfrak{su}(N)$ SYM are $SU(N)$ and $(SU(N)/\mathbb{Z}_k)_n$, where $k \neq 1$ is a divisor of N and $n = 0, 1, \ldots, k-1$ [179]. Global variants can be acted upon by topological manipulations (gauging 1-form symmetry, stacking with SPT). They all form a unique orbit under topological manipulations. Indeed, we can start from $SU(N)$ and reach $(SU(N)/\mathbb{Z}_k)_n$ by selecting a $\mathbb{Z}_k$ subgroup of the $\mathbb{Z}_N$ center 1-form symmetry of $SU(N)$, and gauging it with a discrete torsion given by n.[14] Let $L_{\mathcal{T}}$ denote the set of line operators of the global variant $\mathcal{T}$. Explicitly [179]

$$\begin{aligned} &\mathcal{T} = SU(N) \ , && L_{\mathcal{T}} = \{a(1,0) \mod N, a \in \mathbb{Z}\} \ , \\ &\mathcal{T} = (SU(N)/\mathbb{Z}_k)_n \ , && L_{\mathcal{T}} = \{a(n, N/k) + b(k,0) \mod N, a, b \in \mathbb{Z}\} \ . \end{aligned} \tag{5.173}$$

In terms of topological defects of the symmetry TFT and their brane realization, these are provided by the full set of (p,q)-strings that can end on the boundary depending on the choice of the $\mathcal{B}^{\text{sym}}$ boundary conditions.

[14] In contrast, the set of global forms can split into disjoint non-empty orbits under the action of the $SL(2,\mathbb{Z})$ duality group, depending on N [7, 179, 246].

We want to study when a duality/triality symmetry associated to one of the monodromy matrices of table 5.4 is of intrinsic/non-intrinsic type, depending on N. This is equivalent to asking: for a given N and a given monodromy matrix, is there a global variant in which the associated duality/triality defect acts invertibly on all line operators?

Hanany-Witten Diagnostic of Intrinsicality. This question can be addressed in terms of Hanany-Witten moves, as follows. The duality/triality defect specified by the monodromy matrix M acts **invertibly on all lines** of the global variant $\mathcal{T}$ if the following condition holds,[15]

$$(M - \mathbb{I}_{2\times 2}) \cdot (r, s) \in L_{\mathcal{T}} \ , \qquad \text{for all } (r, s) \in L_{\mathcal{T}} \ . \tag{5.174}$$

The quantity $(M - \mathbb{I}_{2\times 2}) \cdot (r, s)$ is the total (p, q)-string charge created, when a line operator with charges (r, s) crosses the 7-brane implementing the duality/triality defect. We demand that the total (p, q)-string charge that is created can be written as a combination of the same charges as those of the lines in $L_{\mathcal{T}}$. This is because, in the global variant $\mathcal{T}$, a string with those charges, projected parallel to the boundary, yields the trivial surface defect. As a result, we are guaranteed an invertible action on all line operators of $\mathcal{T}$, as desired.

The condition (5.174) can be analyzed explicitly for each of the monodromy matrices in table 5.4, for some small values of N. We report the results of our analysis in table 5.5. For each monodromy matrix and N, we indicate the global variant(s) that satisfy (5.174); if none is found, the duality/triality symmetry is intrinsically non-invertible. The fact that the global variants indicated in the table are invariant

[15] The notation $(r', s') = M \cdot (r, s)$ stands for the matrix equation $(s' \ r') = (s \ r)M$ in our conventions.

under S or ST can also be checked directly making use of [179]

$$\begin{aligned}
T\ :\ \ & (SU(N)/\mathbb{Z}_k)_n \ \rightarrow\ (SU(N)/\mathbb{Z}_k)_{n+N/k} \\
S\ :\ \ & (SU(N)/\mathbb{Z}_k)_0 \ \rightarrow\ (SU(N)/\mathbb{Z}_{N/k})_0 \\
& (SU(N)/\mathbb{Z}_k)_n \ \rightarrow\ (SU(N)/\mathbb{Z}_{k^*})_{n^*} \qquad (n \neq 0)\,.
\end{aligned} \tag{5.175}$$

In the above relations we let k be any divisor of N, including $k = 1$ and $k = N$. The label n is understood mod k (if $k = 1$, then $n = 0$; this is the $SU(N)$ variant). The new labels k^*, n^* are given by

$$k^* = \frac{N}{\gcd(n,k)}\ , \qquad \alpha k + \beta n = \gcd(n,k)\ , \qquad n^* = -\beta \frac{N}{k} \mod k^*\ . \tag{5.176}$$

The global forms under the second column in table 5.5 are invariant under the action of S, followed by T, in our conventions.

For N a prime number, we reproduce the results of [73]. This can also be seen from table 5.5 and algebraically as follows. If $N = p$ is prime, the global variants are $SU(p)$, $PSU(p)_n$, $n = 0, 1, \ldots, p-1$. For each of them, the corresponding set of lines $L_{\mathcal{T}}$ consists of multiples of a single line: $(1, 0)$ for $SU(p)$ and $(n, 1)$ for $PSU(p)_n$. As a result, the condition (5.174) boils down to the eigenvalue problem

$$M \cdot (r, s) = \lambda(r, s) \mod p\ , \tag{5.177}$$

where $(r, s) = (1, 0)$ for $\mathcal{T} = SU(p)$ and $(r, s) = (n, 1)$ for $\mathcal{T} = PSU(p)_n$. In fact, as soon as (5.177) admits a non-trivial solution (r, s), the latter can be identified as the line generating $L_{\mathcal{T}}$ for one of the global variants $\mathcal{T}$. Thus, for $N = p$ prime, (5.177) is a necessary and sufficient condition for finding a global variant $\mathcal{T}$ in which the duality/triality defects acts invertibly on all lines.

5.4 Conclusions and Outlook

We constructed the SymTFT for QFTs realized either holographically or in geometric engineering, in terms of branes. The main results are as follows: in section 5.2 we

N	$S = III$	$ST = IV$, $(ST)^2 = IV^*$, $S^2(ST) = II^*$, $S^2(ST)^2 = II$
2	$PSU(2)_1$	intrinsic
3	intrinsic	$PSU(3)_2$
4	$(SU(4)/\mathbb{Z}_2)_0$	$(SU(4)/\mathbb{Z}_2)_0$
5	$PSU(5)_{2 \text{ or } 3}$	intrinsic
6	intrinsic	intrinsic
7	intrinsic	$PSU(7)_{3 \text{ or } 5}$
8	$(SU(8)/\mathbb{Z}_4)_2$	intrinsic
9	$(SU(9)/\mathbb{Z}_3)_0$	$(SU(9)/\mathbb{Z}_3)_0$
10	$PSU(10)_{3 \text{ or } 7}$	intrinsic
11	intrinsic	intrinsic
12	intrinsic	$(SU(12)/\mathbb{Z}_6)_4$
13	$PSU(13)_{5 \text{ or } 8}$	$PSU(13)_{4 \text{ or } 10}$
14	intrinsic	intrinsic
15	intrinsic	intrinsic
16	$(SU(16)/\mathbb{Z}_4)_0$	$(SU(16)/\mathbb{Z}_4)_0$
17	$PSU(17)_{4 \text{ or } 13}$	intrinsic
18	$(SU(18)/\mathbb{Z}_6)_3$	intrinsic
19	intrinsic	$PSU(19)_{8 \text{ or } 12}$
20	$(SU(20)/\mathbb{Z}_{10})_{4 \text{ or } 6}$	intrinsic
21	intrinsic	$PSU(21)_{5 \text{ or } 17}$
22	intrinsic	intrinsic
23	intrinsic	intrinsic
24	intrinsic	intrinsic
25	$(SU(25)/\mathbb{Z}_5)_0$, $PSU(25)_{7 \text{ or } 18}$	$(SU(25)/\mathbb{Z}_5)_0$
26	$PSU(26)_{5 \text{ or } 21}$	intrinsic
27	intrinsic	$(SU(27)/\mathbb{Z}_9)_6$
28	intrinsic	$(SU(28)/\mathbb{Z}_{14})_{6 \text{ or } 10}$
29	$PSU(29)_{12 \text{ or } 17}$	intrinsic

Table 5.5: For each monodromy matrix and each N, we indicate the global variant(s) on which the associated duality/triality defect acts invertibly on all line operators. If no such global variant exists, the non-invertible symmetry is of intrinsic type. In labeling global variants, we do not keep track of background fields and their counterterms. We do not include the monodromy $S^2 = I_0^*$ because every global variant is invariant under S^2.

demonstrated that branes encode the topological couplings of the SymTFT, and in section 5.3 we highlighted that our proposal also incorporates a notion of generalized charges via the Hanany-Witten effect.

Whilst presenting a general framework in both cases, we gave evidence for our proposal in various geometric and holographic examples, including 4d SYM theories.

In section 5.2.8 we use our general approach to give a brane origin to the symmetry generators in the 4d $\mathcal{N} = 2$ $[A_2, D_4]$ SCFT and in section 5.3.5 we use the generalized charge/ Hanany-Witten relationship to propose a sharp criterion to distinguish intrinsic and non-intrinsic non-invertible symmetries, for rank beyond $\mathfrak{su}(p = \text{prime})$.

Studying properties of topological symmetry generators from the perspective of branes is a new and exciting area of research. It would be interesting to apply our general approach to more exotic non-Lagrangian QFTs where the use of standard field theory tools to study generalized symmetries is either obstructed or non-existent.

The study of generalized charges is another interesting avenue to pursue. In this work we demonstrated that the Hanany-Witten effect encodes the case where a non-invertible p symmetry acts on extended operators of dimension $q = p + 1$. It would be interesting to explore the full suite of generalized charges for invertible symmetry and non-invertible symmetries, e.g. understanding symmetry fractionalization from a brane perspective, as well as generalized charges for genuine and non-genuine operators, see [68]. The brane-perspective will be key to studying theories at strong coupling and in holographic settings [16].

[16]Note: *A paper [46] by Ibou Bah, Enoch Leung and Thomas Waddleton with some related, but complementary, content was published at the same time as our work. We thank these authors for coordination.*

Appendix A

5d Consistent Truncation

A.1 5d Consistent Truncation

An important component of the analysis presented in chapter 3 is the consistent truncation of IIB supergravity to 5d for conifold solutions. In [194] such a consistent truncation was found which encompasses both the UV and IR KS-solutions, and where we show, the holographic realization of the ABJ anomaly and the mixed 0-/1-form symmetry anomaly are both manifest. In what proceeds, we present the map required to translate between our work and their notation. The KS flux background is parametrised as

$$F_3 = q\Phi \wedge \eta\,, \qquad B_2 = b^{\Phi}\Phi\,, \qquad F_5 = -(k - qb^{\Phi})\Phi \wedge \Phi \wedge \eta\,, \tag{A.1}$$

where Φ, η are left-invariant forms on $T^{1,1}$. They are related to the volume forms of S^2 and S^3 as $\Phi = \frac{1}{3}\omega_2\,,\ \Phi \wedge \eta = -\frac{1}{9}\omega_3$. We rescale the IIB fields by

$$F_3 \to -\frac{9l_s^2}{2}F_3\,, \qquad B_2 \to -3\pi l_s^2 B_2\,, \qquad F_5 \to \frac{27\pi l_2^4}{2}F_5\,, \tag{A.2}$$

which ensures that the background is quantised as

$$\int_{S^3} \frac{F_3}{(2\pi l_s)^2} = q \in \mathbb{Z}\,, \qquad \int_{T^{1,1}} \frac{F_5}{(2\pi l_s)^4} = k \in \mathbb{Z}\,, \tag{A.3}$$

and furthermore gives the identifications

$$q = M\,, \qquad b^{\Phi} = -\mathcal{L}\,, \qquad k = N\,. \tag{A.4}$$

Notice that the rescalings are consistent: they give rise to the same factor on either side of the Bianchi identity for F_5. These normalisations also imply that we should identify the fluctuation $dc^\Phi = -\frac{2\pi}{3}dc_0$. Finally, we rescale the $U(1)$ gauge field in [194] $A \to \frac{4\pi}{3}A$ so that it is normalised as in [261]. The 5d topological couplings obtained in [194] are

$$\mathcal{L}_{5d} = \mathcal{R}|g_1^\Phi|^2 - \frac{1}{2} f_2^\Phi \wedge \left(-qb_2 \wedge A + b_2 \wedge Dc^\Phi\right) \,, \qquad f_2^\Phi \supset qb_2 \,, \quad g_1^\Phi \supset Dc^\Phi \,, \quad Dc^\Phi = dc^\Phi - qA \,. \tag{A.5}$$

Using the map detailed above gives the action (3.18).

Appendix B

BF-Terms from Type IIA

B.1 BF-Terms from Type IIA for $Y^{p,k}$

In this section we utilize a reduction to type IIA to derive an extra BF-term contribution on top of those computed via M-theory methods in section 4.5.2. In [210] the authors reduce the M-theory solution on $\mathrm{AdS}_4 \times Y^{p,k}$ background to IIA along a circle. The IIA supergravity background is

$$\mathrm{AdS}_4 \times_w M_6 \,, \tag{B.1}$$

where M_6 is a S^2 bundle over $\mathbb{CP}^2$. The homology groups of M_6 are

$$H_\bullet = \{\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}^2, 0, \mathbb{Z}\} \,. \tag{B.2}$$

The RR field strengths and Kalb-Ramond field are parametrized as

$$\begin{aligned} [F_2] &= pD^+ - kD \,, \\ [B_{\mathrm{NS}}] &= -b_0 D^- + b^+ D \,, \\ [F_6] &= ND \cdot \mathcal{C}^+ \,. \end{aligned} \tag{B.3}$$

Here, $\{D, D^+, D^-\}$ are an over-complete basis of 4-cycles. There is a dual set of 2-cycles $\{\mathcal{C}, \mathcal{C}^+, \mathcal{C}^-\}$ which is also overcomplete. They are related by

$$D^+ = D^- + 3D \,, \quad \mathcal{C}^+ = \mathcal{C}^- + 3\mathcal{C} \,. \tag{B.4}$$

Their mutual intersections are given in table B.1. We write a set of Poincaré dual 2-

	$\mathcal{C}$	$\mathcal{C}^+$	$\mathcal{C}^-$	D	D^+	D^-
D	0	1	1	$\mathcal{C}$	$\mathcal{C}^+$	$\mathcal{C}^-$
D^+	1	3	0	$\mathcal{C}^+$	$3\mathcal{C}^+$	0
D^-	1	0	-3	$\mathcal{C}^-$	0	$-3\mathcal{C}^-$

Table B.1: Intersections between 4-cycles $\{D, D^+, D^-\}$ and 2-cycles $\{\mathcal{C}, \mathcal{C}^+, \mathcal{C}^-\}$ [210].

and 4-forms

$$\begin{aligned} \{D, D^+, D^-\} &\leftrightarrow \{\omega_2, \omega_2^+, \omega_2^-\}\,, \\ \{\mathcal{C}, \mathcal{C}^+, \mathcal{C}^-\} &\leftrightarrow \{\omega_4, \omega_4^+, \omega_4^-\}\,. \end{aligned} \tag{B.5}$$

Let us consider fluctuations around this background in this basis

$$\begin{aligned} F_2' &= F_2 + f_2 = p\omega_2^+ - k\omega_2 + f_2\,, \\ B_{\rm NS}' &= B_{\rm NS} + b_2 = -b_0\omega_2^- + b^+\omega_2 + b_2 \\ F_6' &= F_6 + f_6 = N\omega_2 \wedge \omega_4^+ + g_4^+ \wedge \omega_2^+ + g_4^{(0)} \wedge \omega_2 + g_2^- \wedge \omega_4^- + g_2^{(0)} \wedge \omega_4\,. \end{aligned} \tag{B.6}$$

We now look for single derivative terms in the IIA equations of motion [1,65], which will dominate at long distances, i.e. near the conformal boundary of AdS. In particular, we are interested in couplings involving b_2 and 1-form gauge fields

$$\begin{aligned} d\star_{10} F_2 &= H_3 \wedge F_6 = db_2 \wedge N\text{vol}(M_6) + \cdots = Ndb_2 \wedge \text{vol}(M_6) + \cdots\,. \\ d\star_{10} H_3 &= F_2 \wedge F_6 = \left(pg_2^{(0)} - kg_2^- + Nf_2\right) \wedge \text{vol}(M_6) + \cdots \\ d\star_{10} F_6 &= H_3 \wedge F_2 = db_2 \wedge (p\omega_2^+ - k\omega_2) + \cdots \end{aligned} \tag{B.7}$$

The Bianchi identities are $dF_6 = H_3 \wedge F_4$, $dF_2 = H_3 \wedge F_0$, $dH_3 = 0$ are trivially satisfied given our expansion. At the boundary, we are left with the following topological equations of motion

$$\begin{aligned} Ndb_2 &= 0 \\ pdb_2 &= 0 \\ kdb_2 &= 0 \\ \left(pg_2^{(0)} - kg_2^- + Nf_2\right) &= 0\,. \end{aligned} \tag{B.8}$$

These equations of motion are reproduced by

$$\frac{S_{\rm IIA}}{2\pi} = \int b_2 \wedge \left(Nf_2 + pg_2^{(0)} - kg_2^-\right)\,. \tag{B.9}$$

If we package up

$$(3p-k)g_2^+ + pg_2^{(0)} \equiv \gcd(p,k)\left(q_1 g_2^- + q_2 g_2^{(0)}\right) \equiv \gcd(p,k)\widetilde{g}_2\,, \tag{B.10}$$

we can rewrite

$$\frac{S_{\rm IIA}}{2\pi} = \int b_2 \wedge \left(N f_2 + \gcd(p,k)\widetilde{g}_2\right)\,. \tag{B.11}$$

Let us compare with the M-theory analysis of section 4.5.2, in particular (4.122). As discussed in section 4.4.1, we conjecture that b_2, which couples electrically to the fundamental string, uplifts to B_2 which couples to M2-branes wrapping the torsional 1-cycle. The field a_1 sourced by D0-branes with field strength f_2 uplifts to the $U(1)$ isometry gauge field A_1 associated with the M-theory circle direction. In IIA, the 1-form gauge field $\widetilde{c_1}$ with field strength $\widetilde{g}_2$ couples electrically to D4-branes wrapping the two 4-cycles in the M_6 geometry. We expect that the linear combination $\widetilde{c_1}$ maps to B_1 upon uplift to M-theory, which couples electrically to M5-branes wrapping the torsional 5-cycle.

The $NB_2 \wedge f_2$ coupling is precisely the one we do not have access to from M-theory. We claim that this would be visible if we combined the equivariant cohomology description with differential cohomology, analogously to the matching we did in the ABJM example. On the other hand, IIA does not see the $\Omega^{p,k}_{n_0,n_1} B_2 \wedge g_2$ term of (4.122), at the level of our analysis. We use this to conjecture an additional term in the M-theory BF-coupling:

$$\frac{S_{\rm BF}}{2\pi} = \int B_2 \wedge \left(N f_2 + \gcd(p,k)dB_1 + \Omega^{p,k}_{n_0,n_1} g_2\right)\,. \tag{B.12}$$

p	k	$\alpha_{BB(k_1)}$	$\alpha_{BB(k_2)}$	$\alpha_{FB(k_1)}$	$\alpha_{FB(k_2)}$
2	2	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	0
3	3	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{3}$	0
4	2	0	$\frac{1}{4}$	0	$\frac{1}{2}$
4	4	$\frac{7}{8}$	$\frac{5}{8}$	$\frac{1}{4}$	0
4	6	0	$\frac{1}{4}$	0	$\frac{1}{2}$
5	5	$\frac{9}{10}$	$\frac{3}{5}$	$\frac{1}{5}$	0
6	2	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
6	3	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
6	4	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0
6	6	$\frac{11}{12}$	$\frac{7}{12}$	$\frac{1}{6}$	0
6	8	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
6	9	0	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
7	7	$\frac{13}{14}$	$\frac{4}{7}$	$\frac{1}{7}$	0
8	2	0	$\frac{3}{4}$	0	$\frac{1}{2}$
8	4	0	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{3}{4}$
8	6	0	$\frac{3}{4}$	0	$\frac{1}{2}$
8	8	$\frac{15}{16}$	$\frac{9}{16}$	$\frac{1}{8}$	0
8	10	0	$\frac{3}{4}$	0	$\frac{1}{2}$
8	12	0	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{3}{4}$

Table B.2: A selection of the $Y^{p,k}(\mathbb{CP}^2)$ SymTFT coefficients obtained for selected p, k values with non-trivial $\gcd(p, k)$. Note we have also not included pairs of $\gcd(p, k)$ values related by $(p, k) \to (p, 3p - k)$.

p	6	6	6	8	8	8	9	9
k	4	8	9	6	10	12	6	12
b^1	$n_1 - n_0$	$3n_1 - 2n_0$	$2n_1 - n_0$	$n_1 - n_0$	$4n_1 - 3n_0$	$2n_1 - n_0$	$n_1 - n_0$	$3n_1 - 2n_0$
b^2	$2n_1 - 3n_0$	$4n_1 - 3n_0$	$3n_1 - 2n_0$	$3n_1 - 4n_0$	$5n_1 - 4n_0$	$3n_1 - 2n_0$	$2n_1 - 3n_0$	$4n_1 - 3n_0$

Table B.3: Mappings from (n_0, n_1) torsion flux numbers to (b^1, b^2) flux numbers. Here we give cases where $k \neq \frac{p}{c}$ for some $c \in \mathbb{Z}$.

p	p	4	4	6	6	6	6
k	p	2	6	2	3	4	8
$\Omega^{p,k}_{n_0,n_1}$	n_0	$-2n_0 + n_1$	$-2n_0 + 3n_1$	$-2n_0 + n_1$	$-2n_0 + 2n_1$	$n_1 - n_0$	$7n_1 - 5n_0$
p	6	8	8	8	8	8	
k	9	2	4	6	10	12	
$\Omega^{p,k}_{n_0,n_1}$	$10n_1 - 6n_0$	$-4n_0 + n_1$	$-4n_0 + n_1$	$3n_1 - 4n_0$	$5n_1 - 4n_0$	$13n_1 - 8n_0$	

Table B.4: Values of $\Omega^{p,k}_{n_0,n_1}$ coefficients for various values of p and k. These were computed by using values in table B.2 and mapping (b^1, b^2) to (n_0, n_1) (table B.3). Notice in the first column we give a general expression for $Y^{p,p}$.

Appendix C

Non-Invertible Symmetry Actions on Line Operators

C.1 Non-invertible Symmetries Acting on Line Operators

To complement the analysis in the main text using branes, we provide a field theoretic alternative to the derivation of the action of non-invertible 0-form symmetries on line operators in 4d QFTs, using half-space gauging as in [22, 32]. We consider two examples: 4d $\mathcal{N}=1$ SYM with gauge algebra $\mathfrak{su}(M)$, and 4d pure YM with gauge algebra $\mathfrak{so}(4n)$, which are discussed in the main text in sections 5.3.2 and 5.3.4, respectively.

C.1.1 4d $\mathcal{N}=1$ SYM with Gauge Algebra $\mathfrak{su}(M)$

We want to study the non-invertible 0-form symmetry of the global variant $PSU(M)_0$. To this end, we use the $SU(M)$ global variant as starting point. It has a $\mathbb{Z}_{2M}$ 0-form symmetry (background field: $A_1 \in H^1(W_4;\mathbb{Z}_{2M})$) and a $\mathbb{Z}_M$ 1-form symmetry (background field: $B_2 \in H^2(W_4;\mathbb{Z}_M)$) with mixed anomaly

$$\mathcal{A} = \exp\left(2\pi i \frac{-1}{M}\int_{W_5} A_1 \cup \frac{\mathfrak{P}(B_2)}{2}\right) \,. \tag{C.1}$$

We introduce the usual stacking operation τ and gauging operation σ

$$\begin{aligned} Z_{\tau\mathcal{T}}[B_2] &= Z_{\mathcal{T}}[B_2] e^{2\pi i \frac{1}{M}\int_{W_4} \frac{\mathfrak{P}(B_2)}{2}} \ , \\ Z_{\sigma\mathcal{T}}[B_2] &= \sum_{b_2 \in H^2(W_4;\mathbb{Z}_M)} Z_{\mathcal{T}}[b_2] e^{2\pi i \frac{1}{M}\int_{W_4} b_2 B_2} \ , \end{aligned} \tag{C.2}$$

where $\mathcal{T}$ denotes a global variant of 4d $\mathcal{N} = 1$ SYM with gauge algebra $\mathfrak{su}(M)$. For simplicity, throughout this appendix we omit normalization factors in partition functions and we work on a Spin manifold up to gravitational counterterms. We also make use of the compact notation

$$\begin{aligned} SU(M)_0 &:= SU(M) \ , & SU(M)_p &:= \tau^p SU(M) \ , \\ PSU(M)_{n,0} &:= PSU(M)_n \ , & PSU(M)_{n,p} &:= \tau^p PSU(M)_n \ . \end{aligned} \tag{C.3}$$

One verifies the following identities,

$$\begin{aligned} (\sigma SU(M)_p)[B_2] &= PSU(M)_{p,0}[B_2] \ , \\ (\sigma PSU(M)_{n,0})[B_2] &= SU(M)_n[-B_2] \ , \\ (\sigma PSU(M)_{n,p})[B_2] &= PSU(M)_{n-p^{-1},-p}[-p^{-1}B_2] \ , \qquad \text{if } p \in \mathbb{Z}_M^\times \ . \end{aligned} \tag{C.4}$$

We notice that, for any integer $M \geq 2$, $\pm 1 \in \mathbb{Z}_M^\times$; if $p = \pm 1$, $p^{-1} = \pm 1$. A special case of the last relation is therefore

$$(\sigma PSU(M)_{n,-1})[B_2] = PSU(M)_{n+1,1}[B_2] \ . \tag{C.5}$$

The anomaly (C.1) implies that (perform a 0-form gauge transformation)

$$Z_{SU(M)}[B_2] = Z_{SU(M)}[B_2] e^{2\pi i \frac{-1}{M}\int_{W_4} \frac{\mathfrak{P}(B_2)}{2}} \ , \qquad \text{i.e.} \qquad SU(M)_0[B_2] = SU(M)_{-1}[B_2] \ . \tag{C.6}$$

By applying τ repeatedly on both sides, we get

$$SU(M)_p[B_2] = SU(M)_{p-1}[B_2] \ . \tag{C.7}$$

We may now apply σ on both sides, followed by repeated applications of τ, and get

$$PSU(M)_{n,p}[B_2] = PSU(M)_{n-1,p}[B_2] \ . \tag{C.8}$$

By combining (C.5) and (C.8), we conclude that

$$(\sigma PSU(M)_{n,-1})[B_2] = PSU(M)_{n,1}[B_2] \ . \tag{C.9}$$

This can also be written as

$$(\tau^{-1}\sigma\tau^{-1} PSU(M)_{n,0})[B_2] = PSU(M)_{n,0}[B_2] \ . \tag{C.10}$$

We conclude that the $PSU(M)_{n,0}$ theory is invariant under the combined operation $\tau^{-1}\sigma\tau^{-1}$.

Half-space gauging and action on lines. As anticipated above, we want to study the global variant $PSU(M)_{0,0}$. This theory has 't Hooft lines, charged under a magnetic 1-form symmetry. Let us now use C_2 for the associated background field, and $D_2^{(k)}(M_2)$ for the topological defects implementing the symmetry. We can describe $D_2^{(k)}(M_2)$ explicitly if we think of $PSU(M)_{0,0}$ as originating from gauging of $SU(M)_0$,

$$Z_{PSU(M)_{0,0}}[C_2] = \sum_{b_2} Z_{SU(M)_0}[b_2] e^{2\pi i \frac{1}{M}\int_{W_4} b_2 C_2} \ , \qquad D_2^{(k)}(M_2) = e^{2\pi i \frac{k}{M}\int_{M_2} b_2} \ . \tag{C.11}$$

We observed above that $PSU(M)_{0,0}$ is invariant under $\tau^{-1}\sigma\tau^{-1}$. We can therefore perform this operation in the half-space region $x > 0$, sschematically

$$\begin{aligned} x<0 \ : & \qquad Z_{PSU(M)_{0,0}}[C_2] \ , \\ x>0 \ : & \qquad e^{2\pi i \frac{-1}{M}\int \frac{\mathfrak{P}(C_2)}{2}} \sum_{c_2} Z_{PSU(M)_{0,0}}[c_2] e^{2\pi i \frac{-1}{M}\int \frac{\mathfrak{P}(c_2)}{2}} e^{2\pi i \frac{1}{M}\int c_2 C_2} \ . \end{aligned} \tag{C.12}$$

We impose Dirichlet boundary conditions for c_2 at $x = 0$. The locus $x = 0$ realizes the topological operators implementing the non-invertible 0-form symmetry of the $PSU(M)_{0,0}$ theory. Next, let $\mathbf{H}(\gamma)$ denote a 't Hooft line of minimal charge, supported on a contractible loop γ bounded by a disk D. In the region $x < 0$, $\mathbf{H}(\gamma)$ is a genuine line operator, but it is not invariant under gauge transformations of the magnetic 1-form symmetry background C_2. The gauge invariant combination is

$$x < 0 \ : \qquad \mathbf{H}(\gamma) e^{-2\pi i \frac{1}{M}\int_D C_2} \ . \tag{C.13}$$

The analog of this quantity in the $x > 0$ region is written with c_2, as opposed to C_2,

$$x > 0 \ : \qquad \mathbf{H}(\gamma) e^{-2\pi i \frac{1}{M} \int_D c_2} \ . \tag{C.14}$$

Let us recast the theory in the region $x > 0$ in terms of $SU(M)_0$,

$$x > 0 \ : \qquad \sum_{b_2, c_2} Z_{SU(M)_0}[b_2] e^{2\pi i \frac{1}{M} \int \left[b_2 c_2 - \frac{\mathfrak{P}(c_2)}{2} + c_2 C_2 - \frac{\mathfrak{P}(C_2)}{2} \right]} \ . \tag{C.15}$$

Upon varying c_2 in the exponent, we get the following on-shell relation in the $x > 0$ region,

$$c_2 = b_2 + C_2 \ . \tag{C.16}$$

If we use this in (C.14), we obtain

$$x > 0 \ : \qquad \mathbf{H}(\gamma) e^{-2\pi i \frac{1}{M} \int_D b_2} e^{-2\pi i \frac{1}{M} \int_D C_2} = \mathbf{H}(\gamma) D_2^{(-1)}(D) e^{-2\pi i \frac{1}{M} \int_D C_2} \ . \tag{C.17}$$

We conclude that the non-invertible defects of the $PSU(M)_{0,0}$ theory act on the minimal-charge 't Hooft line by attaching a 1-form symmetry surface defect to the line. The additional C_2 contribution is a c-number that drops away if we turn off the C_2 background field.

We can also rephrase the argument above in the continuum formulation. The continuum counterpart of (C.15) contains the following topological action,

$$\int \mathcal{D}b_2 \mathcal{D}\beta_1 \mathcal{D}c_2 \mathcal{D}\gamma_1 \exp 2\pi i \int_{W_4} \left[M b_2 d\beta_1 + M c_2 d\gamma_1 + M b_2 c_2 - \frac{M}{2} c_2 c_2 + M c_2 C_2 - \frac{M}{2} C_2 C_2 \right] . \tag{C.18}$$

The quantities b_2, β_1, c_2, γ_1 are p-form gauge fields whose field strengths have integral periods, while C_2 is a closed 2-form with integral periods. In the simpler case in which C_2 is turned off, the gauge transformations are

$$b_2' = b_2 + d\lambda_1 \ , \quad \beta_1' = \beta_1 + d\lambda_0 + \mu_1 \ , \quad c_2' = c_2 + d\mu_1 \ , \quad \gamma_1' = \gamma_1 + d\mu_0 - \lambda_1 - \mu_1 \ . \tag{C.19}$$

The BF pair b_2, β_1 couples to the $SU(M)_0$ theory, while c_2 and γ_1 only enter via the topological terms spelled out above. The equations of motion for γ_1, c_2 read

$$M dc_2 = 0 \ , \qquad M c_2 = M(d\gamma_1 + b_2 + C_2) \ . \tag{C.20}$$

In the normalization relevant for the continuum formulation, the gauge invariant combination in the $x > 0$ region is

$$\mathbf{H}(\gamma)e^{-2\pi i \int_D c_2} \ , \tag{C.21}$$

while the topological defect implementing the magnetic 1-form symmetry of $PSU(M)_{0,0}$ is

$$D_2^{(k)}(M_2) = e^{2\pi i k \int_{M_2} b_2} = e^{2\pi i k \int_{M_2}(b_2 + d\gamma_1)} \ . \tag{C.22}$$

In the second step we have observed that $d\gamma_1$ is a globally defined 2-form with integral periods. We can thus add it in the exponent without affecting the result. We thus see that the continuum formulation confirms (C.17).

C.1.2 4d pure YM with Gauge Algebra $\mathfrak{so}(4n)$

We are interested in studying the non-invertible 0-form symmetry of the global variant $Sc(4n)$. We find it convenient to adopt the $SO(4n)$ variant as our starting point. It has a $\mathbb{Z}_2$ 0-form symmetry and a $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry. We denote the corresponding background fields as $A_1 \in H^1(W_4; \mathbb{Z}_2)$ and $B_2, C_2 \in H^2(W_4; \mathbb{Z}_2)$. The theory has the mixed anomaly

$$\mathcal{A} = \exp 2\pi i \frac{1}{2} \int_{W_5} A_1 \cup B_2 \cup C_2 \ . \tag{C.23}$$

The anomaly implies the following relation,

$$Z_{SO(4n)}[B_2, C_2] = Z_{SO(4n)}[B_2, C_2] e^{2\pi i \frac{1}{2} \int_{W_4} B_2 C_2} \ . \tag{C.24}$$

The theory $Sc(4n)$ is obtained by gauging both B_2 and C_2.

Given any theory $\mathcal{T}$ coupled to two background fields $\widehat{B}_2$, $\widehat{C}_2 \in H^2(W_4; \mathbb{Z}_2)$, we define the following three operations:

$$\begin{aligned}
Z_{\tau\mathcal{T}}[\widehat{B}_2, \widehat{C}_2] &= Z_{\mathcal{T}}[\widehat{B}_2, \widehat{C}_2] e^{2\pi i \frac{1}{2} \int_{W_4} \widehat{B}_2 \widehat{C}_2} \ , \\
Z_{\sigma\mathcal{T}}[\widehat{B}_2, \widehat{C}_2] &= \sum_{\widetilde{B}_2, \widetilde{C}_2} Z_{\mathcal{T}}[\widetilde{B}_2, \widetilde{C}_2] e^{2\pi i \frac{1}{2} \int_{W_4} (\widetilde{B}_2 \widehat{B}_2 + \widetilde{C}_2 \widehat{C}_2)} \ , \\
Z_{K\mathcal{T}}[\widehat{B}_2, \widehat{C}_2] &= Z_{\mathcal{T}}[\widehat{C}_2, \widehat{B}_2] \ .
\end{aligned} \tag{C.25}$$

Making use of the anomaly relation (C.24), one may then verify the identity

$$(K\tau\sigma\tau Sc(4n))[\widehat{B}_2, \widehat{C}_2] = (Sc(4n))[\widehat{B}_2, \widehat{C}_2] \ . \tag{C.26}$$

Indeed, we have (always up to prefactors and gravitational counterterms, working on a Spin manifold)

$$\begin{aligned}
Z_{K\tau\sigma\tau Sc(4n)}[\widehat{B}_2, \widehat{C}_2] &= Z_{\tau\sigma\tau Sc(4n)}[\widehat{C}_2, \widehat{B}_2] = Z_{\sigma\tau Sc(4n)}[\widehat{C}_2, \widehat{B}_2] e^{2\pi i \frac{1}{2}\int_{W_4} \widehat{B}_2\widehat{C}_2} \\
&= \sum_{\widetilde{B}_2, \widetilde{C}_2} Z_{\tau Sc(4n)}[\widetilde{B}_2, \widetilde{C}_2] e^{2\pi i \frac{1}{2}\int_{W_4}(\widetilde{B}_2\widehat{C}_2 + \widetilde{C}_2\widehat{B}_2)} e^{2\pi i \frac{1}{2}\int_{W_4} \widehat{B}_2\widehat{C}_2} \\
&= \sum_{\widetilde{B}_2, \widetilde{C}_2} Z_{Sc(4n)}[\widetilde{B}_2, \widetilde{C}_2] e^{2\pi i \frac{1}{2}\int_{W_4}(\widetilde{B}_2\widetilde{C}_2 + \widetilde{B}_2\widehat{C}_2 + \widetilde{C}_2\widehat{B}_2 + \widehat{B}_2\widehat{C}_2)} \\
&= \sum_{\widetilde{B}_2, \widetilde{C}_2, B_2, C_2} Z_{SO(4n)}[B_2, C_2] e^{2\pi i \frac{1}{2}\int_{W_4}(B_2\widetilde{B}_2 + C_2\widetilde{C}_2 + \widetilde{B}_2\widetilde{C}_2 + \widetilde{B}_2\widehat{C}_2 + \widetilde{C}_2\widehat{B}_2 + \widehat{B}_2\widehat{C}_2)} \ .
\end{aligned} \tag{C.27}$$

To proceed we perform the redefinitions

$$\widetilde{B}_2 \to \widetilde{B}_2 + \widehat{B}_2 + C_2 \ , \qquad \widetilde{C}_2 \to \widetilde{C}_2 + \widehat{C}_2 + B_2 \ . \tag{C.28}$$

We get

$$\begin{aligned}
Z_{K\tau\sigma\tau Sc(4n)}[\widehat{B}_2, \widehat{C}_2] &= \sum_{\widetilde{B}_2, \widetilde{C}_2, B_2, C_2} Z_{SO(4n)}[B_2, C_2] e^{2\pi i \frac{1}{2}\int_{W_4}(B_2C_2 + B_2\widehat{B}_2 + C_2\widehat{C}_2 + \widetilde{B}_2\widetilde{C}_2)} \\
&= \sum_{B_2, C_2} Z_{SO(4n)}[B_2, C_2] e^{2\pi i \frac{1}{2}\int_{W_4}(B_2C_2 + B_2\widehat{B}_2 + C_2\widehat{C}_2)} \ .
\end{aligned} \tag{C.29}$$

Now we make use of the anomaly relation (C.24) inside the sum,

$$\begin{aligned}
Z_{K\tau\sigma\tau Sc(4n)}[\widehat{B}_2, \widehat{C}_2] &= \sum_{B_2, C_2} Z_{SO(4n)}[B_2, C_2] e^{2\pi i \frac{1}{2}\int_{W_4}(B_2\widehat{B}_2 + C_2\widehat{C}_2)} \\
&= Z_{Sc(4n)}[\widehat{B}_2, \widehat{C}_2] \ ,
\end{aligned} \tag{C.30}$$

as claimed above.

Half-space gauging and action on lines. Let us regard the $Sc(4n)$ theory as coming from gauging the $SO(4n)$ theory. This allows us to write the topological defects generating the 1-form symmetries of the $Sc(4n)$ in terms of discrete gauge fields. More precisely,

$$Z_{Sc(4n)}[\widehat{B}_2,\widehat{C}_2]=\sum_{B_2,C_2} Z_{SO(4n)}[B_2,C_2]e^{2\pi i\frac{1}{2}\int_{W_4}(B_2\widehat{B}_2+C_2\widehat{C}_2)}\ , \tag{C.31}$$

where we identify

$$\begin{aligned}\widehat{B}_2 &\leftrightarrow D_2^{(\widehat{B})}(M_2)=e^{2\pi i\frac{1}{2}\int_{M_2}B_2}\\ \widehat{C}_2 &\leftrightarrow D_2^{(\widehat{C})}(M_2)=e^{2\pi i\frac{1}{2}\int_{M_2}C_2}\ .\end{aligned} \tag{C.32}$$

We consider a half-space gauging configuration, in which the region $x<0$ has the $Sc(4n)$ theory, and the region $x>0$ the $K\tau\sigma\tau Sc(4n)$ theory,

$$\begin{aligned} x<0\ : &\qquad Z_{Sc(4n)}[\widehat{B}_2,\widehat{C}_2]\ ,\\ x>0\ : &\qquad \sum_{\widetilde{B}_2,\widetilde{C}_2} Z_{Sc(4n)}[\widetilde{B}_2,\widetilde{C}_2]e^{2\pi i\frac{1}{2}\int_{W_4}(\widetilde{B}_2\widetilde{C}_2+\widetilde{B}_2\widehat{C}_2+\widetilde{C}_2\widehat{B}_2+\widehat{B}_2\widehat{C}_2)}\ .\end{aligned} \tag{C.33}$$

We impose Dirichlet boundary conditions for $\widetilde{B}_2$, $\widetilde{C}_2$ at $x=0$. The locus $x=0$ realizes the codimension-1 topological defects generating the non-invertible symmetry of the $Sc(4n)$ theory.

In the $Sc(4n)$ theory we have line operators of charges $(1,0)$ and $(0,1)$ under the 1-form symmetries generated by $D_2^{(\widehat{B})}(M_2)$ and $D_2^{(\widehat{C})}(M_2)$. In the region $x>0$ we have the gauge invariant combinations

$$x>0\ :\qquad \mathbf{L}^{(1,0)}(\gamma)e^{2\pi i\frac{1}{2}\int_D\widehat{B}_2}\ ,\qquad \mathbf{L}^{(0,1)}(\gamma)e^{2\pi i\frac{1}{2}\int_D\widehat{C}_2}\ , \tag{C.34}$$

where $\partial D=\gamma$. These combinations in the region $x>0$ become

$$x>0\ :\qquad \mathbf{L}^{(1,0)}(\gamma)e^{2\pi i\frac{1}{2}\int_D\widetilde{B}_2}\ ,\qquad \mathbf{L}^{(0,1)}(\gamma)e^{2\pi i\frac{1}{2}\int_D\widetilde{C}_2}\ . \tag{C.35}$$

To proceed, we write the theory in the region $x>0$ in terms of the $SO(4n)$ theory,

$$x>0\ :\ \sum_{\widetilde{B}_2,\widetilde{C}_2,B_2,C_2} Z_{SO(4n)}[B_2,C_2]e^{2\pi i\frac{1}{2}\int_{W_4}(B_2\widetilde{B}_2+C_2\widetilde{C}_2+\widetilde{B}_2\widetilde{C}_2+\widetilde{B}_2\widehat{C}_2+\widetilde{C}_2\widehat{B}_2+\widehat{B}_2\widehat{C}_2)}\ . \tag{C.36}$$

Varying the exponent with respect to $\widetilde{B}_2$, $\widetilde{C}_2$ yields

$$\widetilde{B}_2=\widehat{B}_2+C_2\ ,\qquad \widetilde{C}_2=\widehat{C}_2+B_2\ . \tag{C.37}$$

We can ignore signs because these are $\mathbb{Z}_2$ classes. Using these relations in (C.35) and recalling (C.31), we find

$$x > 0 \ : \qquad \mathbf{L}^{(1,0)}(\gamma) D_2^{(\widehat{C})}(D) e^{2\pi i \frac{1}{2}\int_D \widehat{B}_2} \ , \qquad \mathbf{L}^{(0,1)}(\gamma) D_2^{(\widehat{B})}(D) e^{2\pi i \frac{1}{2}\int_D \widehat{C}_2} \ . \quad \text{(C.38)}$$

We thus learn that, if the line $\mathbf{L}^{(1,0)}$ passes through the non-invertible defect, it emerges attached to a $D_2^{(\widehat{C})}$ surface, and analogously for the other line.

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