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Fermionic topological phases and bosonization in higher dimensions

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We review a recipe to produce a lattice construction of fermionic phases of matter in the presence of time reversal symmetry by extending the fermionization and bosonization known in (1+1) dimensions to various setups including higher spacetime dimensions in the presence of global symmetries. As an application, we provide a state sum lattice path integral for a (1+1)-dimensional topological superconductor with time reversal symmetry generating the $\mathbb{Z}_8$ classification of the symmetry-protected topological phase. We also illustrate a state sum path integral for a (3+1)-dimensional topological superconductor with time reversal symmetry that generates the $\mathbb{Z}_{16}$ classification.

Subject Index B01, B31, B38, B69, I68

1. Introduction

Fermionic topological phases have attracted great interest, since fermionic systems admit novel phases that have no counterpart in bosonic systems [1,2]. For example, let us consider a fermionic system with no global symmetry except for the fermion parity $(-1)^F$. In that case, there is a nontrivial (1+1)-dimensional symmetry-protected topological (SPT) phase realized by a topological superconductor [3], while in the bosonic case there is no nontrivial (1+1)-dimensional invertible phase in the absence of global symmetry. The (1+1)-dimensional topological superconductor is called the Kitaev wire, and generates the $\mathbb{Z}_2$ classification in the absence of global symmetry except for the fermion parity. If we take time reversal (T) symmetry with $T^2 = 1$ (symmetry class BDI) into account, the Kitaev wire instead generates the classification of T-symmetric SPT phase given by $\mathbb{Z}_8$ [4]. As another example, a (3+1)-dimensional SPT phase with time reversal symmetry such that $T^2 = (-1)^F$ (class DIII) is classified by $\mathbb{Z}_{16}$ [5–7], while a (3+1)-dimensional bosonic T-SPT phase has only $\mathbb{Z}_2$ classification [8]. The $\mathbb{Z}_{16}$ classification of the fermionic SPT phase is again generated by a topological superconductor with T symmetry.

In this review article we introduce lattice path integrals that describe the topological superconductors in (1+1) and (3+1) dimensions with T symmetry, in terms of the state sum definition of topological quantum field theory (TQFT), following the author's collaborations [9,10]. This is done by generalizing the bosonization and fermionization well known in (1+1) dimensions for various setups, including higher spacetime dimensions [11–13] and/or in the presence of global symmetry [9,14]. The fermionization is a map transforming a given bosonic theory to a fermionic theory. For example, the celebrated Jordan–Wigner transformation in (1+1) dimensions maps a bosonic spin chain with $\mathbb{Z}_2$ spin-flip symmetry to a chain of fermions. One can generalize the Jordan–Wigner transformation to higher spacetime dimensions, mapping a d-dimensional bosonic theory with a $(d-2)$-form $\mathbb{Z}_2$ symmetry to a fermionic theory intrinsically coupled with the spin structure, which we call the fermionization. By preparing a suitable bosonic theory on a lattice, we construct a topological superconductor by utilizing the fermionization.

This review is organized as follows. In Sect. 2 we introduce the concept of fermionization, and explain a way to obtain a lattice definition of fermionic topological theories in generic spacetime dimensions. Then, in Sect. 3 we construct a theory called the Gu–Wen Grassmann integral which is used to describe the fermionic path integral after the fermionization. After these preparations, we construct a (1+1)-dimensional lattice path integral that generates the $\mathbb{Z}_8$ classification of a topological superconductor in class BDI, and finally sketch the construction of a (3+1)-dimensional lattice path integral that generates the $\mathbb{Z}_{16}$ classification of a topological superconductor in class DIII.

2. Fermionization and bosonization

2.1. *Fermionic phases and spin, pin structure*

Before talking about fermionization, let us clarify what we mean by fermionic field theories. In order to define a Lorentz-invariant field theory that describes a fermionic system, the theory depends on a choice of spin structure η. Mathematically, a spin structure η is a trivialization of the second Stiefel–Whitney class, $\delta\eta = w_2$, and two distinct spin structures on the spacetime manifold M are related to each other by $H^1(M, \mathbb{Z}_2)$. Namely, for a given $\chi \in H^1(M, \mathbb{Z}_2)$, we can shift η by χ to define another spin structure $\eta + \chi$. The need for a spin structure arises for the following reason.

A relativistic quantum field theory in d spacetime dimensions possesses the Lorentz $SO(d)$ symmetry. However, since fermions are spinors, fermions transform according to the double cover of $SO(d)$, which is $Spin(d)$. To define the field theory on a generic spacetime manifold, one needs to consider an $SO(d)$ bundle $\phi\colon M \to BSO(d)$, which can be thought of as the tangent bundle TM of an oriented manifold M. In order to have fermions, the transition functions $\phi_{ij} \in SO(d)$ between overlapping patches U_i and U_j must be lifted to $\widetilde{\phi}_{ij} \in Spin(d)$. Since $Spin(d)$ is the group extension

$$\mathbb{Z}_2 \to Spin(d) \to SO(d) \tag{2.1}$$

whose extension is given by $w_2 \in H^2(BSO(d), \mathbb{Z}_2)$, $Spin(d)$ is identified as $SO(d) \times \mathbb{Z}_2$ *as a set*, so we can express $\widetilde{\phi}_{ij}$ as a pair $(\phi_{ij}, \eta_{ij}) \in SO(d) \times \mathbb{Z}_2$. The nontrivial group extension is reflected in the multiplication law of $\mathbb{Z}_2$ elements η_{ij} twisted by w_2. Namely, for transition functions $\widetilde{\phi}_{ij} \in Spin(d)$, we have the multiplication law

$$\widetilde{\phi}_{ij}\widetilde{\phi}_{jk} = (\phi_{ij}\phi_{jk}, \eta_{ij} + \eta_{jk} + w_2(\phi_{ij}, \phi_{jk})). \tag{2.2}$$

Due to the cocycle condition $\widetilde{\phi}_{ij}\widetilde{\phi}_{jk} = \widetilde{\phi}_{ik}$, we find

$$\eta_{ij} + \eta_{jk} + \eta_{ik} = w_2(\phi_{ij}, \phi_{jk}). \tag{2.3}$$

In coordinate-free notation, this is precisely the equation $\delta\eta = \phi^* w_2 = w_2(TM)$.

When the field theory possesses the time reversal (T) symmetry that reverses the orientation of the spacetime, we may put the theory on an unoriented spacetime manifold. In that case, the Lorentz symmetry is now expressed as $O(d)$, where the time reversal symmetry corresponds to the $\mathbb{Z}_2$ subgroup $\mathbb{Z}_2^T \subset O(d)$ generated by the orientation-reversing element. Then, we have the tangent bundle $\phi\colon M \to BO(d)$, and the transition function $\phi_{ij} \in O(d)$ will be lifted to its double cover via the group extension

$$\mathbb{Z}_2 \to Pin^{\pm}(d) \to O(d), \tag{2.4}$$

whose extension is characterized by the element of $H^2(BO(d),\mathbb{Z}_2)$. Since $H^2(BO(d),\mathbb{Z}_2)=\mathbb{Z}_2\times\mathbb{Z}_2$ generated by w_1^2 and w_2, there are four possible choices of the symmetry extension of $O(d)$ by $\mathbb{Z}_2$. In addition, $SO(d)\subset O(d)$ must be lifted to $Spin(d)$ for fermionic theories, so the extension class in Eq. (2.4) must be w_2 or $w_2+w_1^2$. This amounts to two choices of the double cover, $Pin^+(d)$ or $Pin^-(d)$:

- When the extension class is chosen as w_2, we have the Pin^+ group by the extension in Eq. (2.4). In that case, analogously to Eq. (2.3), the pin$^+$ structure is specified by a choice of $\eta\in C^1(M,\mathbb{Z}_2)$ with $\delta\eta=w_2$.
- When the extension class is chosen as $w_2+w_1^2$, we have the Pin^- group by the extension in Eq. (2.4). In that case, the pin$^-$ structure is specified by a choice of $\eta\in C^1(M,\mathbb{Z}_2)$ with $\delta\eta=w_2+w_1^2$.

Physically, pin$^-$ structure differs from pin$^+$ by the action of time reversal symmetry on a fermion. That is, the subgroup $\mathbb{Z}_2^T\in O(d)$ is extended to $\mathbb{Z}_4$ in the case of $Pin^-(d)$,

$$\mathbb{Z}_2\to\mathbb{Z}_4^T\to\mathbb{Z}_2^T, \tag{2.5}$$

while $\mathbb{Z}_2^T$ is not extended for $Pin^+(d)$. This means that *in the Euclidean spacetime* the time reversal symmetry acts as $T^2=(-1)^F$ for the pin$^-$ structure, while $T^2=+1$ for the pin$^+$ structure.

When we are interested in the action of T in the Minkowski signature, the above T action should be Wick-rotated, meaning $T^2=(-1)^F$ for the pin$^+$ structure, while $T^2=+1$ for the pin$^-$ structure.

2.2. Fermionization: fermion condensation

Now let us illustrate the fermionization. To do this, let us start with a hand-waving description of fermionization to obtain fermionic topological phases in generic spacetime dimensions, starting with a bosonic theory. We consider a bosonic topological phase $\mathcal{T}_b$ with a $(d-2)$-form $\mathbb{Z}_2$ symmetry in d spacetime dimensions; here, "bosonic" means that the theory is independent of the spin structure of the spacetime. The $\mathbb{Z}_2$ symmetry is generated by a one-dimensional line operator, and let us assume that this line operator has fermionic statistics, i.e. it is interpreted as a worldline of a fermionic quasiparticle ψ. Since the bosonic shadow theory supports a nontrivial quasiparticle ψ, $\mathcal{T}_b$ realizes a nontrivial topological ordered state. For example, in the case of a (2+1)-dimensional $\mathbb{Z}_2$ gauge theory (toric code), there is a dyonic line operator generating a $\mathbb{Z}_2$ 1-form symmetry, which is regarded as a worldline of a fermion. $\mathcal{T}_b$ is sometimes called a "bosonic shadow" theory [12].

The fermionization of the bosonic theory $\mathcal{T}_b$ proceeds as follows. We prepare a fermionic theory $\mathcal{T}_c$ that depends on the spin/pin$^\pm$ structure of the spacetime, which has a topologically trivial fermionic excitation c. Then, the fermionization is carried out by stacking $\mathcal{T}_b$ with $\mathcal{T}_c$, and condensing the composite boson ψc, see Fig. 1. The resulting theory is a fermionic theory that is intrinsically coupled with the spin/pin$^\pm$ structure. This fermionization is sometimes termed "fermion condensation" in condensed matter literature [11].

2.3. 't Hooft anomaly of the bosonic theory

Now we want to formulate the above process of fermion condensation in a precise way. To do this, we note that the fermionic statistics of a line operator of $\mathcal{T}_b$ is characterized by a specific 't Hooft anomaly of the $(d-2)$-form $\mathbb{Z}_2$ symmetry generated by the line. Let us start with the case of the spacetime dimension $d=3$ for simplicity. Then, we consider a partition function $Z_b(M^3,f_2)$ of $\mathcal{T}_b$

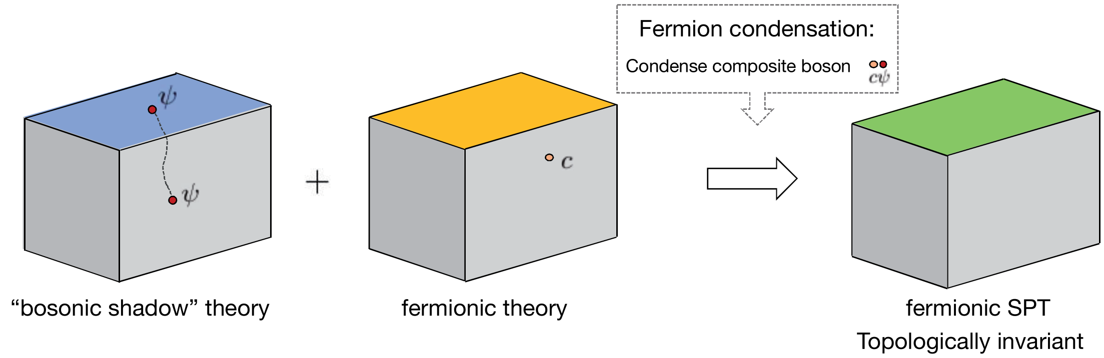

Fig. 1. Fermion condensation to obtain a fermionic topological phase. We prepare a bosonic shadow theory $\mathcal{T}_b$ with a fermionic anyon quasiparticle ψ and a fermionic theory $\mathcal{T}_c$, and then condense the composite boson ψc.

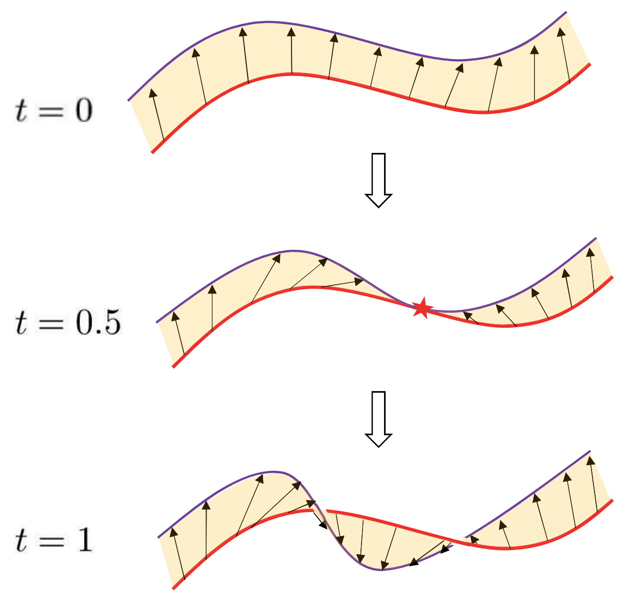

Fig. 2. A fermionic line operator ψ (red curve) and its framing v_1 (blue arrows) for $d = 3$. The figure shows a movie that changes the framing by 1, where the framing is discontinuously changed at the middle of the movie. The framing defines a perturbation of the line (purple curve), and the change of the framing is characterized by the intersection between the line operator and the perturbed one. This intersection effect is understood as a 't Hooft anomaly, Eq. (2.7).

on a 3-manifold M^3, in the presence of the background gauge field $f_2 \in Z^2(M^3, \mathbb{Z}_2)$ of the 1-form $\mathbb{Z}_2$ symmetry. The background gauge field f_2 is realized by an insertion of the line operator $f_2^\vee$ Poincaré dual of f_2 in the spacetime. Then, the partition function in the presence of the line operator depends on a choice of framing of the line operator due to its fermionic statistics; see Fig. 2. The framing of the line is specified by a choice of a vector field v_1 along the line, which is linearly independent of the tangent of the line. When the spacetime M^3 is oriented, this is equivalent to choosing the section of the normal bundle of the line, since one can choose the second vector field v_2 as the tangent of the line, and then v_3 is determined by the global orientation of M^3; v_1, v_3 specify the section of the normal bundle of the line.

To make a connection with the 't Hooft anomaly, suppose that we initially have a framing at the time $t = 0$, and then consider a movie during $0 \leq t \leq 1$ to get a different framing at $t = 1$; see Fig. 2.

For a two-dimensional worldsheet $\widetilde{f}_2^\vee$ of the line $f_2^\vee$, we consider a shifted two-dimensional object $\widetilde{f}_{\mathrm{shift}}^\vee$ by perturbing it along the direction of v_1. When the framing is changed during the movie, we can see that the worldsheet of the line operator $\widetilde{f}_2^\vee$ intersects with the shifted version $\widetilde{f}_{\mathrm{shift}}^\vee$, where the vector fields become degenerate at the intersection point. Then, the fermionic statistics of the line operator is characterized by the phase shift of the partition function by -1 at the intersection,

$$Z_b(M^3, f_2)\big|_{t=0} = (-1)^{\sharp\mathrm{int}(\widetilde{f}_2^\vee, \widetilde{f}_{\mathrm{shift}}^\vee)} \cdot Z_b(M^3, f_2)\big|_{t=1}. \tag{2.6}$$

Physically, the phase shift corresponds to the topological spin of the fermionic quasiparticle. In particular, when $d = 3$ the phase shift is controlled by the intersection of $\widetilde{f}_2^\vee$ with its shifted version, since we do not have a nontrivial thickening process in $d = 3$. In that case, it is well known that the intersection is the Poincaré dual of the cup product of $\widetilde{f}_2$ with itself, $\widetilde{f}_2 \cup \widetilde{f}_2$. Hence, in $d = 3$ we can say that the 't Hooft anomaly in Eq. (2.6) is controlled by the response action $\widetilde{f}_2 \cup \widetilde{f}_2$,

$$Z_b(M^3, f_2)\big|_{t=0} = (-1)^{\int_{M^3\times[0,1]} \widetilde{f}_2 \cup \widetilde{f}_2} \cdot Z_b(M^3, f_2)\big|_{t=1}. \tag{2.7}$$

In higher spacetime dimensions $d > 3$, the framing of $f_{d-1}^\vee$ is specified by the choice of independent $(d-2)$ vector fields $v_1, v_2, \ldots, v_{d-2}$ which are linearly independent of the tangent of $f_{d-1}^\vee$. We think of a $(d-1)$-dimensional object $\widetilde{f}_{\mathrm{thicken,shift}}^\vee$ given by thickening the $\widetilde{f}_{d-1}$ in $(d-3)$ directions $v_1, v_2, \ldots, v_{d-3}$, and then shifting it in the direction of v_{d-2}. Then, the fermionic statistics is again expressed as the anomaly given by the intersection of $\widetilde{f}_{d-1}^\vee$ and $\widetilde{f}_{\mathrm{thicken,shift}}^\vee$,

$$Z_b(M^d, f_{d-1})\big|_{t=0} = (-1)^{\sharp\mathrm{int}(\widetilde{f}_{d-1}^\vee, \widetilde{f}_{\mathrm{shift}}^\vee)} \cdot Z_b(M^d, f_{d-1})\big|_{t=1}. \tag{2.8}$$

It is also known that the intersection of $\widetilde{f}_{d-1}^\vee$ and $\widetilde{f}_{\mathrm{thicken,shift}}^\vee$ admits an expression in terms of the action expressed by the Steenrod square [10,15],

$$Z_b(M^d, f_{d-1})\big|_{t=0} = (-1)^{\int_{M^d\times[0,1]} \mathrm{Sq}^2(\widetilde{f}_{d-1})} \cdot Z_b(M^d, f_{d-1})\big|_{t=1}, \tag{2.9}$$

where $\mathrm{Sq}^2(\widetilde{f}_{d-1}) := \widetilde{f}_{d-1} \cup_{d-3} \widetilde{f}_{d-1}$ using the higher cup product $\cup_i$ reviewed in Appendix A. See also Refs. [11,15] for references on the higher cup product. In particular, since $\cup_0 = \cup$ it reduces to Eq. (2.7) when $d = 3$.

Apart from the framing anomaly given by the response action $\widetilde{f}_2 \cup \widetilde{f}_2$, there can also be a 't Hooft anomaly when the theory has an orientation-reversing symmetry like time reversal, regarded as a mixed anomaly between the orientation-reversing symmetry R and the $(d-2)$-form $\mathbb{Z}_2$ symmetry. This anomaly encodes how the fermionic quasiparticle ψ is acted on by the orientation-reversing symmetry. Concretely, when the ψ acts as the Kramers doublet under the orientation-reversing symmetry $R^2 = (-1)^F$, the corresponding $(d-2)$-form $\mathbb{Z}_2$ symmetry has an anomaly given by the response action [10,16]

$$(-1)^{\int \widetilde{w}_1^2 \cup \widetilde{f}_{d-1}}. \tag{2.10}$$

Here, $\widetilde{w}_1 \in Z^1(W^{d+1}, \mathbb{Z}_2)$ denotes the first Stiefel–Whitney class of the bulk $(d+1)$-dimensional manifold W^{d+1}, which is regarded as the symmetry defect for the orientation-reversing symmetry. That is, the Poincaré dual $w_1^\vee$ of w_1 is a codimension-1 submanifold that reverses the orientation of the spacetime. Accordingly, the fermion ψ is acted on by R when ψ goes across $w_1^\vee$. When the fermion transforms as $R^2 = (-1)^F$ under the orientation reversal, R no longer acts as the $\mathbb{Z}_2$ symmetry on the

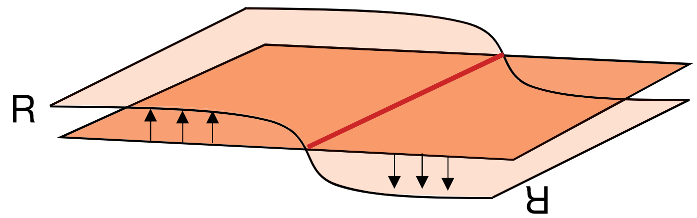

Fig. 3. When the reflection symmetry acts on the fermion ψ as a Kramers doublet $R^2 = (-1)^F$, the symmetry defect $w_1^\vee$ should be directed by a vector perpendicular to the defect, since $R = R\cdot(-1)^F \neq R$. The vector then defines the perturbation $w^\vee_{1,\text{shift}}$, and the intersection between $w_1^\vee$ and $w^\vee_{1,\text{shift}}$ defines $w_1^{2,\vee}$, which is the Poincaré dual of w_1^2. When the fermion ψ goes around $w_1^{2,\vee}$, ψ is acted on by $R^2 = (-1)^F$.

fermion, and R should be distinguished from its inverse $R^{-1} = R\cdot(-1)^F$. Hence, the orientation-reversing defect should be directed by an arrow perpendicular to the defect, to specify whether the defect carries R or R^{-1}; see Fig. 3. The direction of the defect then defines the perturbation $w^\vee_{1,\text{shift}}$ of $w_1^\vee$. The intersection between $w^\vee_{1,\text{shift}}$ and $w_1^\vee$ defines the Poincaré dual of w_1^2. When ψ transforms as $R^2 = (-1)^F$, the phase of the partition function in the presence of the line operator $f^\vee_{d-1}$ is shifted by -1 if $f^\vee_{d-1}$ is moved across $w_1^{2,\vee}$. This phase shift under moving $f^\vee_{d-1}$ is expressed as the 't Hooft anomaly

$$Z_b(M^d, f_{d-1})\big|_{t=0} = (-1)^{\sharp\text{int}(\widetilde{w_1}^{2,\vee}, \widetilde{f}^\vee_{d-1})} \cdot Z_b(M^d, f_{d-1})\big|_{t=1}. \tag{2.11}$$

Hence, the response action is expressed as Eq. (2.10).

Summarizing, the $(d-2)$-form $\mathbb{Z}_2$ symmetry has a 't Hooft anomaly that encodes the fermionic statistics and the transformation property under the orientation-reversing symmetry. If the fermionic quasiparticle ψ transforms as $R^2 = (-1)^F$, then the anomaly is given by the response action

$$(-1)^{\int \mathrm{Sq}^2(\widetilde{f}_{d-1}) + \widetilde{w}_1^2 \cup \widetilde{f}_{d-1}}. \tag{2.12}$$

Instead, in the case of $R^2 = 1$, the anomaly is given by the response action

$$(-1)^{\int \mathrm{Sq}^2(\widetilde{f}_{d-1})}. \tag{2.13}$$

2.4. Fermionization: coupling with the pin structure

We then couple the bosonic theory $\mathcal{T}_b$ with the fermionic theory $\mathcal{T}_c$ and perform the fermion condensation. The partition function of the fermionic theory consists of two parts,

$$z_c(M^d, f_{d-1}, \eta) = \sigma(M^d, f_{d-1})(-1)^{\int_{M^d} \eta \cup f_{d-1}}. \tag{2.14}$$

Here, $\eta \in C^1(M^d, \mathbb{Z}_2)$ is the pin$^\pm$ structure of the spacetime. This gives the trivialization of the obstruction class, and we have

$$\begin{cases} \delta\eta = w_2 & \text{for the pin}^+ \text{ structure,} \\ \delta\eta = w_2 + w_1^2 & \text{for the pin}^- \text{ structure.} \end{cases} \tag{2.15}$$

So, the anomaly of the second term is given by $(-1)^{\int \delta\widetilde{\eta}\cup\widetilde{f}_{d-1}}$. Meanwhile, the first term, $\sigma(M^d, f_{d-1})$, has a 't Hooft anomaly given by

$$(-1)^{\int \mathrm{Sq}^2(\widetilde{f}_{d-1}) + (\widetilde{w}_2 + \widetilde{w}_1^2)\cup\widetilde{f}_{d-1}}. \tag{2.16}$$

The first term, $\sigma(M^d, f_{d-1})$, is a bosonic theory realized by a path integral of Grassmann variables in the spacetime, as reviewed in Sect. 3. We note that due to the Wu relation [17], we have

$$(-1)^{\int_{W^{d+1}} (\mathrm{Sq}^2(\widetilde{f}_{d-1}) + (\widetilde{w}_2 + \widetilde{w}_1^2) \cup \widetilde{f}_{d-1})} = +1 \tag{2.17}$$

when W^{d+1} is a closed manifold and f_{d-1} is a cocycle. This means that $\int_{W^{d+1}} (\mathrm{Sq}^2(\widetilde{f}_{d-1}) + (\widetilde{w}_2 + \widetilde{w}_1^2) \cup \widetilde{f}_{d-1})$ represents a trivial phase in $(d+1)$ dimensions, and therefore there should be a trivial boundary in d dimensions. We can think of the theory $\sigma(M^d, f_{d-1})$ as providing an explicit formula for such a trivial boundary. Combining the anomalies of $(-1)^{\int_{M^d} \eta \cup f_{d-1}}$ and $\sigma(M^d, f_{d-1})$, the response action for the 't Hooft anomaly of $z_c(M^d, f_{d-1}, \eta)$ is given by

$$\begin{cases} (-1)^{\int \mathrm{Sq}^2(\widetilde{f}_{d-1}) + \widetilde{w}_1^2 \cup \widetilde{f}_{d-1}} & \text{for the pin}^+ \text{ structure,} \\ (-1)^{\int \mathrm{Sq}^2(\widetilde{f}_{d-1})} & \text{for the pin}^- \text{ structure.} \end{cases} \tag{2.18}$$

These response actions are identical to those of the bosonic theories in Eqs. (2.12) and (2.13). Thus, if we want a pin$^+$ theory, the fermion condensation is performed by the following procedure:

(1) We prepare a bosonic theory $\mathcal{T}_b$ with a fermionic quasiparticle ψ that transforms as $R^2 = (-1)^F$ under the orientation-reversing symmetry, i.e. with a $(d-2)$-form $\mathbb{Z}_2$ symmetry whose anomaly is expressed by the response action

$$(-1)^{\int \mathrm{Sq}^2(\widetilde{f}_{d-1}) + \widetilde{w}_1^2 \cup \widetilde{f}_{d-1}}. \tag{2.19}$$

(2) Then, we couple $\mathcal{T}_b$ with a pin$^+$ theory $\mathcal{T}_c$, and gauge the diagonal $(d-2)$-form $\mathbb{Z}_2$ symmetry

$$Z(M^d, \eta) \propto \sum_{[f_{d-1}] \in Z^{d-1}(M^d, \mathbb{Z}_2)} Z_b(M^d, f_{d-1}) \sigma(M^d, f_{d-1}) (-1)^{\int_{M^d} \eta \cup f_{d-1}}. \tag{2.20}$$

Note that the 't Hooft anomaly of this combination is vanishing, since those of $Z_b(M^d, f_{d-1})$ and $z_c(M^d, f_{d-1}, \eta)$ are canceled out.

Analogously, the fermion condensation for the pin$^-$ theory proceeds as follows:

(1) We prepare a bosonic theory $\mathcal{T}_b$ with a fermionic quasiparticle ψ that transforms as $R^2 = 1$ under the orientation-reversing symmetry, i.e. with a $(d-2)$-form $\mathbb{Z}_2$ symmetry whose anomaly is expressed by the response action

$$(-1)^{\int \mathrm{Sq}^2(\widetilde{f}_{d-1})}. \tag{2.21}$$

(2) Then, we couple $\mathcal{T}_b$ with a pin$^-$ theory $\mathcal{T}_c$, and gauge the diagonal $(d-2)$-form $\mathbb{Z}_2$ symmetry

$$Z(M^d, \eta) \propto \sum_{[f_{d-1}] \in Z^{d-1}(M^d, \mathbb{Z}_2)} Z_b(M^d, f_{d-1}) \sigma(M^d, f_{d-1}) (-1)^{\int_{M^d} \eta \cup f_{d-1}}. \tag{2.22}$$

3. Grassmann integral

Now let us construct the Grassmann integral $\sigma(M^d, f_{d-1})$ on a d-dimensional manifold M^d which might be unoriented, following Ref. [9]. We construct an unoriented manifold by picking locally oriented patches and then gluing them along codimension-one loci by transition functions. The locus where the transition functions are orientation reversing constitutes a representative $w_1^\vee$ of the Poincaré dual of the first Stiefel–Whitney class w_1. We will sometimes call the locus an "orientation-reversing

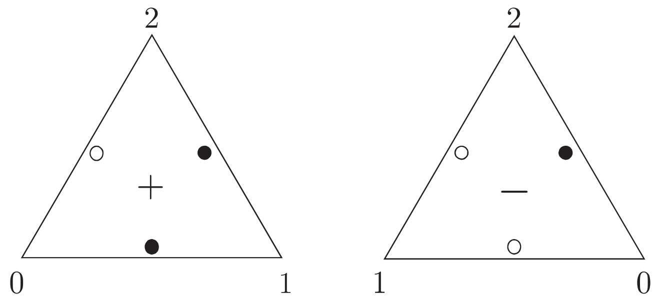

Fig. 4. Assignment of Grassmann variables on 1-simplices in the case of $d = 2$. θ (resp. $\overline{\theta}$) is represented as a black (resp. white) dot.

wall." In addition, we take the barycentric subdivision for the triangulation of M^d. Namely, each d-simplex in the initial triangulation of M^d is subdivided into $(d+1)!$ simplices, whose vertices are barycenters of the subsets of vertices in the d-simplex. We further assign a local ordering to vertices of the barycentric subdivision, such that a vertex on the barycenter of i vertices is labeled by i. Each simplex can then be either a + simplex or a − simplex, depending on whether the ordering agrees with the orientation or not. We assign a pair of Grassmann variables $\theta_e, \overline{\theta}_e$ on each $(d-1)$-simplex e of M^d such that $f_{d-1}(e) = 1$ for a given $f_{d-1} \in Z^{d-1}(M^d, \mathbb{Z}_2)$.

For a d-simplex $t = (01\dots d)$, we label a $(d-1)$-simplex $(01\dots\widehat{i}\dots d)$ (i.e. a $(d-1)$-simplex given by omitting a vertex i) simply as $\widehat{i}$. If the $(d-1)$-simplex e is located away from the orientation-reversing wall, we choose the assignment of θ and $\overline{\theta}$ on each e such that a d-simplex t contains $\overline{\theta}_e$ when e is labeled by an odd (respectively even) number if t is a + (resp. −) simplex; see Fig. 4.

We remark that the above assignment rule of $\theta, \overline{\theta}$ fails when e lies on the orientation-reversing wall. In this case, we would have to assign Grassmann variables of the same color on both sides of e (i.e. both are black (θ) or white ($\overline{\theta}$)), since the two d-simplices sharing e have identical signs when e is on the orientation-reversing wall; see Fig. 5(a). Hence, we need to slightly modify the construction of the Grassmann integral on the orientation-reversing wall. To do this, instead of specifying a canonical rule to assign Grassmann variables on the wall, we just place a pair $\theta_e, \overline{\theta}_e$ on the wall in an arbitrary fashion. We then define the Grassmann integral in the form

$$\sigma(M^d, f_{d-1}) = \int \prod_{e|f_{d-1}(e)=1} d\theta_e d\overline{\theta}_e \prod_t u(t) \prod_{e|\text{wall}} (\pm i)^{f_{d-1}(e)}, \tag{3.1}$$

where t denotes a d-simplex, and $u(t)$ is the product of Grassmann variables contained in t. For instance, for $d = 2$, $u(t)$ on $t = (012)$ is the product of $\vartheta_{12}^{f_{d-1}(12)}$, $\vartheta_{01}^{f_{d-1}(01)}$, and $\vartheta_{02}^{f_{d-1}(02)}$. Here, ϑ denotes θ or $\overline{\theta}$ depending on the choice of the assignment rule, which will be discussed later. The order of Grassmann variables in $u(t)$ will also be defined shortly. We note that $u(t)$ is ensured to be Grassmann-even when f_{d-1} is closed.

Due to the fermionic sign of Grassmann variables, $\sigma(f_{d-1})$ becomes a quadratic function whose quadratic property depends on the order of Grassmann variables in $u(t)$. We will adopt the order used in Ref. [11], which is defined as follows:

- The order of $\vartheta_{\widehat{i}} = \vartheta_{01\dots\widehat{i}\dots d}$ for a + d-simplex t is defined by first assigning even $(d-1)$-simplices in ascending order, then odd simplices in ascending order again:

$$\widehat{0} \to \widehat{2} \to \widehat{4} \to \cdots \to \widehat{1} \to \widehat{3} \to \widehat{5} \to \cdots \tag{3.2}$$

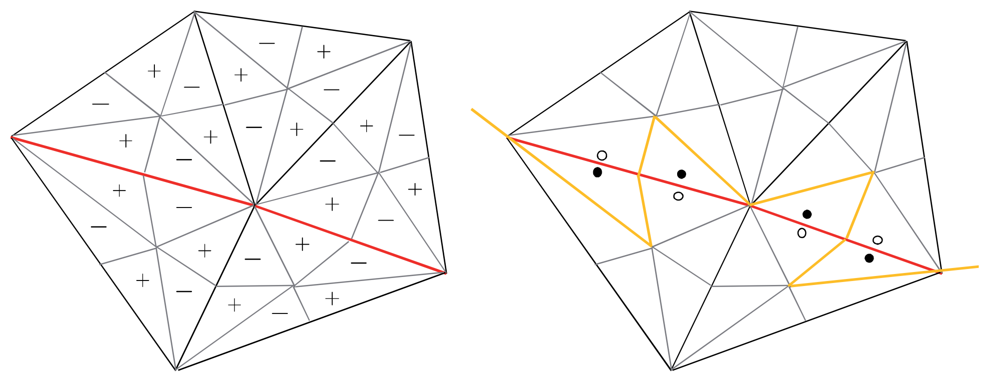

Fig. 5. (a) The signs of d-simplices near the orientation-reversing wall, which is represented as a red line. (b) Assignment of Grassmann variables on the wall specifies a perturbation of the wall that intersects the wall transversally at $(d-2)$-simplices.

- For $-\ d$-simplices, the order is defined in the opposite way:

$$\cdots \to \hat{5} \to \hat{3} \to \hat{1} \to \cdots \to \hat{4} \to \hat{2} \to \hat{0}. \tag{3.3}$$

For example, for $d = 2$, $u(012) = \vartheta_{12}^{f_{d-1}(12)}\vartheta_{01}^{f_{d-1}(01)}\vartheta_{02}^{f_{d-1}(02)}$ when (012) is a $+$ triangle, and $u(012) = \vartheta_{02}^{f_{d-1}(02)}\vartheta_{01}^{f_{d-1}(01)}\vartheta_{12}^{f_{d-1}(12)}$ for a $-$ triangle.

The $\prod_{e|\text{wall}}(\pm i)^{f_{d-1}(e)}$ term in Eq. (3.1) assigns weight $+i^{f_{d-1}(e)}$ (resp. $-i^{f_{d-1}(e)}$) on each $(d-1)$-simplex e on the orientation-reversing wall, when e is shared with $+$ (resp. $-$) d-simplices. There is no ambiguity in such a definition, since both d-simplices on the side of e have the same sign. This factor makes the Grassmann integral a $\mathbb{Z}_4$-valued function of $f_{d-1} \in Z^{d-1}(M^d, \mathbb{Z}_2)$.

3.1. *Properties of the Grassmann integral*

Now we discuss the 't Hooft anomaly of the Grassmann integral under the gauge transformation of the $(d-2)$-form $\mathbb{Z}_2$ symmetry. The effect of the gauge transformation is determined by a couple of key formulae:

- The quadratic property governed by the higher cup product $\cup_{d-2}$,

$$\sigma(f_{d-1})\sigma(f'_{d-1}) = \sigma(f_{d-1} + f'_{d-1})(-1)^{\int f_{d-1}\cup_{d-2}f'_{d-1}}. \tag{3.4}$$

- When $f_{d-1} = \delta\lambda$ for some $\lambda \in C^{d-2}(M, \mathbb{Z}_2)$, the Grassmann integral is explicitly computed as

$$\sigma(\delta\lambda) = (-1)^{\int_{w_2+w_1^2}\lambda}(-1)^{\int_M \lambda\cup_{d-3}\delta\lambda+\lambda\cup_{d-4}\lambda}, \tag{3.5}$$

  where the integral over $w_2 + w_1^2$ means that we sum λ over a $(d-2)$-cycle $S \in Z_{d-2}(M, \mathbb{Z}_2)$ Poincaré dual of the Stiefel–Whitney class $w_2 + w_1^2$.

Here, the $(d-2)$-cycle S Poincaré dual of $w_2 + w_1^2$ in Eq. (3.5) is specified as follows. First, the set of all $(d-2)$-simplices in the barycentric subdivision of the triangulation is known to represent the dual of w_2. Second, w_1^2 is determined via an assignment of the Grassmann variables on the orientation-reversing wall. That is, the choice of the assignment of Grassmann variables on the wall corresponds to choosing a slight perturbation of the wall such that the perturbation intersects transversally with the wall at $(d-2)$-simplices. Concretely, we deform the wall on each $(d-1)$-simplex of the wall to the side where θ (black dot) is contained, see Fig. 5(b). Then, the Poincaré dual of w_1^2 is given by the intersection of the wall $w_1^\vee$ and the perturbed one $w_{1,\text{shift}}^\vee$, $w_1^\vee \cap w_{1,\text{shift}}^\vee$. We then take S as the set of all $(d-2)$-simplices of the barycentric subdivision, plus extra $(d-2)$-simplices given by the intersection $w_1^\vee \cap w_{1,\text{shift}}^\vee$. As such, we provide a combinatorial representation of w_2 and w_1^2, which is also used to formulate a combinatorial pin structure as a choice of a cochain η with $\delta\eta = w_2 + w_1^2$ or $\delta\eta = w_2$.

3.2. *'t Hooft anomaly of the Grassmann integral*

To see the 't Hooft anomaly of $\sigma(M^d, f_{d-1})$, we introduce an expression of $\sigma(M^d, f_{d-1})$ convenient for our purpose. Let us assume that the spacetime manifold M^d equipped with the background gauge field $f_{d-1} \in Z^{d-1}(M, \mathbb{Z}_2)$ is null-bordant, i.e. M^d is a boundary of some $(d+1)$-dimensional manifold K^{d+1} and f_{d-1} is extended to K^{d+1}. Then, we can consider the Wess–Zumino–Witten-like (WZW-like) expression of the Grassmann integral

$$\sigma(M^d, f_{d-1}) = (-1)^{\int_{K^{d+1}} \mathrm{Sq}^2 f_{d-1}} (-1)^{\sum_{S_K} f_{d-1}}, \tag{3.6}$$

where S_K represents the dual of $w_2 + w_1^2$, that is, a set of all $(d-1)$-simplices of K^{d+1} plus extra $(d-1)$-simplices that represent the dual of w_1^2 in K^{d+1}. Due to the Wu relation [17], $\mathrm{Sq}^2(f_{d-1}) + (w_2 + w_1^2) \cup f_{d-1}$ is exact for an arbitrary $(d+1)$-dimensional manifold. Hence, the above expression does not depend on the extending manifold K^{d+1}. We can explicitly check that Eq. (3.6) satisfies the properties of the Grassmann integral in Eqs. (3.4) and (3.5). First, let us check the quadratic property,

$$\begin{aligned} \sigma(f_{d-1})\sigma(f'_{d-1}) &= (-1)^{\int_{K^{d+1}} (f_{d-1} \cup_{d-3} f'_{d-1} + f'_{d-1} \cup_{d-3} f_{d-1})} \sigma(f_{d-1} + f'_{d-1}) \\ &= (-1)^{\int_{M^d} f_{d-1} \cup_{d-2} f'_{d-1}} \sigma(f_{d-1} + f'_{d-1}), \end{aligned} \tag{3.7}$$

where $\cup_i$ is the higher cup product; see Appendix A for a review. Next, when $f_{d-1} = \delta\lambda$ for some $\lambda \in C^{d-2}(K, \mathbb{Z}_2)$, we have

$$\begin{aligned} \sigma'(\delta\lambda) &= (-1)^{\int_K \mathrm{Sq}^2 \delta\lambda} (-1)^{\sum_{S_K} \delta\lambda} \\ &= (-1)^{\int_M \lambda \cup_{d-3} \delta\lambda + \lambda \cup_{d-4} \lambda} (-1)^{\sum_S \lambda}, \end{aligned} \tag{3.8}$$

where we used $\partial S_K = S$, namely the boundary of S_K again gives the dual of $w_2 + w_1^2$ on M^d [18]. Then, the above WZW definition $\sigma(M^d, f_{d-1})$ is identified as $\sigma(M, f_{d-1})$, up to a gauge-invariant counterterm which does not affect the 't Hooft anomaly of the theory.

Based on the WZW expression, we immediately find the formula for the 't Hooft anomaly as follows. Suppose we have two configurations of f_{d-1}, orientation-reversing walls and triangulations on $M^d \times \{0\}$ and $M^d \times \{1\}$ interpolated by $K = M^d \times [0,1]$. Then, according to the WZW expression for $\overline{\sigma}(M^d \times \{0\})\sigma(M^d \times \{1\})$, up to gauge-invariant counterterms, $\sigma(M^d \times \{0\})$ is given by

$$\sigma(M^d \times \{0\}) = (-1)^{\int_K \mathrm{Sq}^2(f_{d-1})} (-1)^{\sum_{S_K} f_{d-1}} \cdot \sigma(M^d \times \{1\}), \tag{3.9}$$

where $K^{d+1} = M^d \times [0,1]$, and f_{d-1} on $M^d \times \{0\}$, $M^d \times \{1\}$ is extended to K^{d+1}. This expression directly shows that the effect of gauge transformation and retriangulation of $\sigma(M^d, f_{d-1})$ is controlled by the bulk response action

$$(-1)^{\int_{K^{d+1}} \mathrm{Sq}^2 f_{d-1}} (-1)^{\sum_{S_K} f_{d-1}}. \tag{3.10}$$

This is the expected anomaly of the Grassmann integral in Eq. (2.16), since S_K represents the Poincaré dual of $w_2 + w_1^2$.

4. (1+1)-dimensional topological superconductor: Arf–Brown–Kervaire invariant

In this section we construct a lattice path integral for a field theory that describes a (1+1)-dimensional topological superconductor called a Kitaev wire. In the presence of a time reversal symmetry with $T^2 = 1$, the Kitaev wire generates the SPT phase classified by $\mathbb{Z}_8$ [4]. The SPT classification corresponds to the pin$^-$ cobordism group $\Omega^2_{\mathrm{pin}^-} = \mathbb{Z}_8$, and its generator is known as an invertible topological field theory whose partition function on a closed manifold becomes the Arf–Brown–Kervaire (ABK) invariant [19].

This is done by fermionizing a bosonic theory by coupling with the Grassmann integral on a lattice, utilizing the fermionization procedure in the pin$^-$ case.

The weight for the bosonic theory on a two-dimensional triangulated manifold M^2 is assigned in the same manner as the case of the Arf invariant for oriented spin manifolds [11], described as follows. For a given configuration $f_1 \in C^1(M^2, \mathbb{Z}_2)$, we assign weight $1/2$ to each 1-simplex e, and also assign weight 2 to each 2-simplex t when $\delta f_1 = 0$ at t, otherwise 0. Let us denote the product of the whole weight as $\widetilde{Z}(f_1)$. Then, we consider a pin$^-$ theory obtained by gauging the $\mathbb{Z}_2$ symmetry,

$$\begin{aligned} Z(M^2, \eta) &= \sum_{f_1 \in Z^1(M^2,\mathbb{Z}_2)} \sigma(M^2, f_1)(-1)^{\int_{M^2} \eta \cup f_1} \widetilde{Z}(f_1) \\ &= 2^{|T|-|E|} \cdot \sum_{f_1 \in Z^1(M^2,\mathbb{Z}_2)} \sigma(M^2, f_1)(-1)^{\int_{M^2} \eta \cup f_1} \\ &= 2^{\chi(M^2)-1} \cdot \sum_{[f_1] \in H^1(M^2,\mathbb{Z}_2)} \sigma(M^2, f_1)(-1)^{\int_{M^2} \eta \cup f_1} \\ &= \sqrt{2}^{\chi(M^2)} \mathrm{ABK}[M^2, \eta], \end{aligned} \tag{4.1}$$

where $|T|$ and $|E|$ denote the number of 2-simplices and 1-simplices in M^2, respectively. $\chi(M^2)$ denotes the Euler number of M^2, and $\mathrm{ABK}[M^2, \eta]$ is the ABK invariant,

$$\mathrm{ABK}[M^2, \eta] = \frac{1}{\sqrt{|H^1(M^2, \mathbb{Z}_2)|}} \sum_{[f_1] \in H^1(M^2,\mathbb{Z}_2)} i^{Q_\eta[f_1]}. \tag{4.2}$$

Here, $i^{Q_\eta[f_1]} = \sigma(M^2, f_1)(-1)^{\int_{M^2} \eta \cup f_1}$ is a $\mathbb{Z}_4$-valued quadratic function that satisfies

$$Q_\eta[f_1] + Q_\eta[f_1'] = Q_\eta[f_1 + f_1'] + 2 \int_{M^2} f_1 \cup f_1'. \tag{4.3}$$

The ABK invariant determines the pin$^-$ bordism class of two-dimensional manifolds $\Omega_2^{\mathrm{pin}^-} = \mathbb{Z}_8$, which is generated by $\mathbb{RP}^2$ [19]. To compute the partition function on $\mathbb{RP}^2$, let f_1 be a nontrivial 1-cocycle that generates $H^1(\mathbb{RP}^2, \mathbb{Z}_2) = \mathbb{Z}_2$. Then, using the quadratic property for $f_1 = f_1'$ in Eq. (4.3), one can see that $Q_\eta[f_1]$ takes values of ± 1, since $Q_\eta[0] = 0$ and $\int_{M^2} f_1 \cup f_1' = 1 \bmod 2$. $Q_\eta[f_1] = \pm 1$ corresponds to two possible choices of pin$^-$ structure on $\mathbb{RP}^2$. Then, the ABK invariant is computed as an eighth root of unity,

$$\mathrm{ABK}[\mathbb{RP}^2, \eta] = \frac{1 \pm i}{\sqrt{2}} = e^{\pm 2\pi i/8}. \tag{4.4}$$

5. (3+1)-dimensional topological superconductor: eta invariant for pin$^+$ Dirac operator

In this section we talk about a lattice state-sum path integral that describes a (3+1)-dimensional topological superconductor with the time reversal symmetry $T^2 = (-1)^F$. The symmetry corresponds to the pin$^+$ structure, and is classified by the pin$^+$ cobordism group $\Omega^4_{\text{pin}^+} = \mathbb{Z}_{16}$. Our path integral produces a (3+1)-dimensional T-SPT phase that corresponds to $3 \in \mathbb{Z}_{16}$ of the classification. As such, our path integral carries the same information as the eta invariant of the pin$^+$ Dirac operator, which also generates the $\mathbb{Z}_{16}$ classification. Here we outline the rough idea of the construction and introduce properties of the path integral. The whole construction of the path integral requires a massive description, which is found in Ref. [10].

The construction of our path integral is based on the fermionization for the pin$^+$ theory in Eq. (2.20), so we need a four-dimensional bosonic theory $Z_b(M^4, f_3)$ coupled with a 3-form $\mathbb{Z}_2$ gauge field $f_3 \in Z^3(M^4, \mathbb{Z}_2)$ with a 't Hooft anomaly

$$(-1)^{\int \mathrm{Sq}^2(\widetilde{f}_3) + \widetilde{w}_1^2 \cup \widetilde{f}_3}. \tag{5.1}$$

This bosonic theory $Z_b(M^4, f_3)$ can be realized by a version of (3+1)-dimensional Crane–Yetter TQFT [20], which is described by a bosonic state-sum path integral based on the input data of a braided fusion category $\mathcal{C}$ [21]. The braided fusion category is an algebraic theory that describes the fusion and braiding of anyons in (2+1)-dimensional TQFT. In particular, for our purpose we need a braided fusion category $\mathcal{C}$ that describes the bosonic dual of the $SO(3)_3$ Chern–Simons theory. The Crane–Yetter TQFT then defines a (3+1)-dimensional bosonic theory with a gapped boundary, and fermionizing the theory yields a (3+1)-dimensional fermionic SPT phase whose gapped boundary is given by the $SO(3)_3$ Chern–Simons theory [5].

For the $SO(3)_3$ Chern–Simons theory, its bosonic dual $\mathcal{C}$ contains four anyons including the trivial one, which we label as $\{1, s, \widetilde{s}, \psi\}$. In particular, since $\mathcal{C}$ describes a bosonic dual of a (2+1)-dimensional fermionic topological phase, it contains a fermionic anyon ψ generating the 1-form $\mathbb{Z}_2$ symmetry as reviewed in Sect. 2.3. This fermionic anyon ψ is transparent, which means that the braiding phases between ψ and all other anyons in $\mathcal{C}$ are trivial. In the presence of such a transparent fermion ψ in $\mathcal{C}$, the (3+1)-dimensional Crane–Yetter TQFT has a single fermionic quasiparticle ψ in the (3+1)-dimensional bulk, and its line operator generates a $\mathbb{Z}_2$ 2-form symmetry with a 't Hooft anomaly

$$(-1)^{\int \mathrm{Sq}^2(\widetilde{f}_3)}, \tag{5.2}$$

which produces the first term of Eq. (5.1).

One can endow the Crane–Yetter TQFT with a global T symmetry by utilizing a (2+1)-dimsensional TQFT $\mathcal{C}$ with a T symmetry. In particular, $SO(3)_3$ Chern–Simons theory has an anomalous T symmetry labeled by $3 \in \mathbb{Z}_{16}$, and this T symmetry is encoded in the braided fusion category $\mathcal{C}$ in terms of an invertible map $\varphi_T : \mathcal{C} \to \mathcal{C}$ that acts on all physical properties of $\mathcal{C}$ (e.g. spins of anyons, braiding, etc.) in anti-unitary fashion [22]. Once we fix the map $\varphi_T : \mathcal{C} \to \mathcal{C}$, we can define the symmetry fractionalization data that dictates how the global symmetry acts on the state with anyons of $\mathcal{C}$. In particular, the symmetry fractionalization controls the T action on the transparent fermion ψ, which can be Kramers singlet or doublet. As reviewed in Sect. 2.3, we want the T action on ψ to be a Kramers doublet $T^2 = -1$, in order to have the desired anomaly, Eq. (5.1).

Then, one can fermionize the Crane–Yetter TQFT Z_b as

$$Z(M^4, \eta) \propto \sum_{[f_3]\in Z^3(M^4,\mathbb{Z}_2)} Z_b(M^4, f_3)\sigma(M^4, f_3)(-1)^{\int_{M^4} \eta\cup f_3}, \tag{5.3}$$

with a pin$^+$ structure η with $\delta\eta = w_2$. This gives a topologically invariant path integral that describes a topological superconductor with the time reversal symmetry $T^2 = (-1)^F$. The $\mathbb{Z}_{16}$ classification of this theory is diagnosed by computing the partition function $Z(M^4, \eta)$ on $\mathbb{RP}^4$, which generates the pin$^+$ bordism group $\Omega_4^{\text{pin}^+} = \mathbb{Z}_{16}$. This is explicitly computed in Ref. [10] as $Z(M^4, \eta) = e^{\pm 3\cdot 2\pi i/16}$, where the $\pm$ signs correspond to two possible choices of the pin$^+$ structure η on $\mathbb{RP}^4$. This implies that the (3+1)-dimensional fermionic theory obtained corresponds to the $3 \in \mathbb{Z}_{16}$ phase.

Finally, with standard mathematical assumptions we note that the above construction gives a state sum for an unoriented pin$^+$ TQFT that can detect certain exotic smooth structure. That is, there exists a manifold Q^4 called a “fake $\mathbb{RP}^4$” which is homeomorphic but not diffeomorphic to $\mathbb{RP}^4$. Q^4 admits a pin$^+$ structure, and then Q^4 is pin$^+$ bordant to nine copies of $\mathbb{RP}^4$ [23,24]. So far we do not have a fully rigorous proof for the bordism invariance of $Z(M^4, \eta)$ (while topological invariance is proven rigorously), but it is strongly expected that our theory $Z(M^4, \eta)$ has the property that distinguishes two manifolds $\mathbb{RP}^4$ and Q^4 in the pin$^+$ bordism group.

6. Discussions

In this review we discussed a way to construct a fermionic topological phase by starting with a bosonic topological phase with a fermionic quasiparticle and then condensing it by coupling with a fermionic theory. The utility of fermion condensation is not limited to the construction of topological superconductors with time reversal symmetry. In Ref. [10], starting with the data $\mathcal{C}$ of a (2+1)-dimensional fermionic topological ordered phase enriched by any ordinary (0-form) global symmetry, we construct a state sum path integral of a (3+1)-dimensional fermionic SPT phase whose boundary is realized by a given fermionic topological order. This is done by constructing a bosonic shadow theory realized by a Crane–Yetter TQFT and then condensing the fermionic quasiparticle of the theory, as illustrated in Sect. 5. This gives us a systematic way to compute the ’t Hooft anomaly of the fermionic topological order by evaluating the path integral of a (3+1)-dimensional SPT phase on an appropriate closed manifold. For example, we derive a complete formula that evaluates the $\mathbb{Z}_{16}$-valued T anomaly of a given (2+1)-dimensional fermionic topological ordered phase. Our construction allows us to obtain an arbitrary (3+1)-dimensional SPT phase as long as it admits a symmetry-preserving gapped boundary realized by a (2+1)-dimensional topological ordered phase.

One possible direction is to extend the prescription of fermion condensation to a more exotic spacetime structure other than spin structure for fermionic systems. For example, it was recently proposed in Ref. [25] that there are nontrivial invertible topological field theories based on the spacetime structure called Wu structure, which is inequivalent to any (possibly twisted) oriented or spin structure previously discussed in the literature, and thus gives a new class of invertible field theories which are phrased as neither bosonic nor fermionic. The topological phases that depend on the spacetime Wu structure are called exotic topological phases. Wu structure in d spacetime dimensions corresponds to the global symmetry given by a specific nontrivial mixture of the spacetime Lorentz symmetry $O(d)$ and the 1-form $\mathbb{Z}_2$ symmetry. Since the Lorentz group is taken as $O(d)$, the exotic topological phases possess time reversal symmetry. Mathematically, the symmetry is described by a specific 2-group that corresponds to a sort of extension of $O(d)$ by the 1-form $\mathbb{Z}_2$ symmetry. This generalizes the spin/pin structure required for fermionic systems, where one extends the Lorentz symmetry $SO(d)$ or

$O(d)$ by the ordinary (0-form) $\mathbb{Z}_2$ symmetry that corresponds to the $\mathbb{Z}_2$ fermion parity. In Ref. [26], we discussed a systematic way to obtain a state sum path integral for such exotic topological phases, by generalizing the fermion condensation in the spin case. It would be interesting to consider such novel spacetime structures realized by the higher group involving Lorentz $O(d)$ symmetry.

Appendix A. Cup product and higher cup product

A branching structure on a triangulation is a local ordering of vertices, which can be specified by an arrow on each 1-simplex $\langle ij\rangle$, such that there are no closed loops on any 2-simplices. This defines a total ordering of vertices on every single d-simplex $\langle 0\dots d\rangle$. In this appendix we review the cochain-level product operation called higher cup product, whose definitions are based on the branching structure of the triangulation. Here we limit ourselves to the $\mathbb{Z}_2$-valued cochains for simplicity. See also Ref. [15] for a nice reference on the higher cup product.

Let M be a triangulated d-dimensional manifold. First, the cup product gives the product of cochains

$$-\cup -: C^k(M,\mathbb{Z}_2)\times C^l(M,\mathbb{Z}_2)\to C^{k+l}(M,\mathbb{Z}_2), \tag{A.1}$$

whose explicit form is written as

$$(\alpha\cup\beta)(0,\dots,k+l)=\alpha(0,\dots,k)\beta(k,\dots,k+l). \tag{A.2}$$

Note that this definition of cup product depends on the branching structure on the triangulation, where the ordering of vertices on each $(k+l)$-simplex is specified as $0\to 1\to\cdots\to k+l$. The cup product satisfies the Leibniz rule at the cochain level,

$$\delta(\alpha\cup\beta)=\delta\alpha\cup\beta+\alpha\cup\delta\beta. \tag{A.3}$$

According to the Leibniz rule, one can show that the cup product defines the product of cohomologies $H^k(M,\mathbb{Z}_2)\times H^l(M,\mathbb{Z}_2)\to H^{k+l}(M,\mathbb{Z}_2)$. Actually, for given $\alpha\in Z^k(M,\mathbb{Z}_2)$, $\beta\in Z^l(M,\mathbb{Z}_2)$, the shift of these cocycles by coboundaries is evaluated as

$$(\alpha+\delta A)\cup(\beta+\delta B)=\alpha\cup\beta+\delta(\alpha\cup B+A\cup\beta+A\cup\delta B), \tag{A.4}$$

so this also shifts $\alpha\cup\beta$ by a coboundary, and thus defines a map between cohomologies. Such a product operation defined on cohomologies is called a cohomology operation.

It is known that the cup product has a geometrical interpretation in the picture of the Poincaré dual. That is, for given cochains $\alpha\in C^k(M,\mathbb{Z}_2)$ and $\beta\in C^l(M,\mathbb{Z}_2)$, the cup product $\alpha\cup\beta$ is the Poincaré dual to the intersection of Poincaré duals $\alpha^\vee\cap\beta^\vee$, for $\alpha^\vee\in C_{d-k}(M^\vee,\mathbb{Z}_2)$, $\beta^\vee\in C_{d-l}(M^\vee,\mathbb{Z}_2)$. This can be understood as follows. Let us consider a shifted version of the Poincaré dual $\beta^\vee_{\text{shift}}$, where the shifting vector is determined by the branching structure. See Fig. 6 for two dimensions. Then, the intersection of $\alpha^\vee$ and $\beta^\vee_{\text{shifted}}$ corresponds to the dual of $\alpha\cup\beta$.

As a generalization of the cup product, the higher cup product $\cup_i$ gives

$$-\cup_i -: C^k(M,\mathbb{Z}_2)\times C^l(M,\mathbb{Z}_2)\to C^{k+l-i}(M,\mathbb{Z}_2), \tag{A.5}$$

whose explicit form is written as

$$(\alpha\cup_i\beta)(0,\dots,k+l-i)=\sum_{0\le j_0<\cdots<j_i\le k+l-i}\alpha(0\to j_0,j_1\to j_2,\dots)\beta(j_0\to j_1,j_1\to j_3,\dots). \tag{A.6}$$

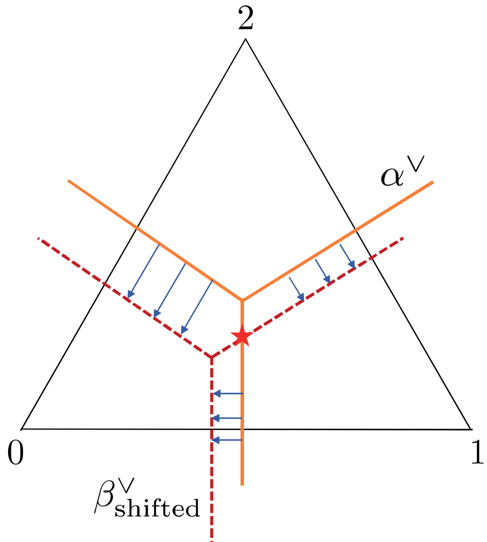

Fig. 6. The cup product $(\alpha \cup \beta)(012) = \alpha(01)\beta(12)$ can be understood as the intersection of the dual chains in the Poincaré dual picture.

Here, the notation $i \to j$ denotes all vertices from i to j, $\{i, i+1, \ldots, i+j\}$. In particular, $\cup_0$ is identified as the cup product $\cup$ defined in Eq. (A.2). The higher cup product is subject to the generalized Leibniz rule,

$$\delta(\alpha \cup_i \beta) = (\delta\alpha) \cup_i \beta + \alpha \cup_i (\delta\beta) + \alpha \cup_{i-1} \beta + \beta \cup_{i-1} \alpha, \tag{A.7}$$

which is regarded as the noncommutative property of $\cup_{i-1}$ being controlled by the $\cup_i$ product. According to the above Leibniz rule, for closed α and β one can see that $\alpha \cup_i \beta$ is not necessarily closed, $\delta(\alpha \cup_i \beta) = \alpha \cup_{i-1} \beta + \beta \cup_{i-1} \alpha$ for $\alpha \in Z^k(M, \mathbb{Z}_2)$, $\beta \in Z^l(M, \mathbb{Z}_2)$. Hence, the product $\cup_i$ does not give a cohomology operation for $i > 0$.

However, it turns out that the map

$$\begin{aligned} \mathrm{Sq}^{d-i}(\alpha) &: Z^k(M, \mathbb{Z}_2) \to Z^{k+d-i}(M, \mathbb{Z}_2), \\ \mathrm{Sq}^{d-i}(\alpha) &:= \alpha \cup_{i+k-d} \alpha \end{aligned} \tag{A.8}$$

does give a cohomology operation. Actually, one can check that $\mathrm{Sq}^{d-i}(\alpha + \delta A) = \mathrm{Sq}^{d-i}(\alpha) + \delta(\alpha \cup_{i+k-d} A + A \cup_{i+k-d} \alpha + A \cup_{i+k-d-1} A + A \cup_{i+k-d} \delta A)$ by using the generalized Leibniz rule. This shows that Sq^{d-i} defines a map $H^k(M, \mathbb{Z}_2) \to H^{k+d-i}(M, \mathbb{Z}_2)$.

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