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Hyperscaling violation and the shear diffusion constant

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ABSTRACT

We consider holographic theories in bulk $(d+1)$-dimensions with Lifshitz and hyperscaling violating exponents z, θ at finite temperature. By studying shear gravitational modes in the near-horizon region given certain self-consistent approximations, we obtain the corresponding shear diffusion constant on an appropriately defined stretched horizon, adapting the analysis of Kovtun, Son and Starinets. For generic exponents with $d-z-\theta > -1$, we find that the diffusion constant has power law scaling with the temperature, motivating us to guess a universal relation for the viscosity bound. When the exponents satisfy $d-z-\theta = -1$, we find logarithmic behaviour. This relation is equivalent to $z = 2 + d_{eff}$ where $d_{eff} = d_i - \theta$ is the effective boundary spatial dimension (and $d_i = d-1$ the actual spatial dimension). It is satisfied by the exponents in hyperscaling violating theories arising from null reductions of highly boosted black branes, and we comment on the corresponding analysis in that context.

1. Introduction and summary

The viscosity bound [1] is a universal feature of large families of strongly coupled quantum field theories arising in investigations using holography [2]. The shear viscosity η satisfies $\frac{\eta}{s} = \frac{1}{4\pi}$ for a wide variety of theories, s being the entropy density. A slightly different approach to studying hydrodynamics and viscosity was studied in [3]. It was observed that metric perturbations governing diffusive charge and shear modes in the near horizon region of the relevant dual black branes simplify allowing a systematic expansion there, resulting in a diffusion equation on a stretched horizon with universal behaviour for the diffusion constant. This is akin to the membrane paradigm [4] for black branes, the horizon exhibiting diffusive properties. This approach does not directly assume any holographic duality per se, although it is consistent with holographic results *e.g.* [5,6] (see *e.g.* [7] for a review of these aspects of hydrodynamics).

In recent years, nonrelativistic generalisations of gauge/gravity duality have been studied extensively. An interesting class of non-relativistic theories exhibits so-called hyperscaling violation. The gravity duals are conformal to Lifshitz spacetimes [8,9]. These hyperscaling violating spacetimes arise in effective Einstein–Maxwell–Dilaton theories *e.g.* [10–21]. Certain gauge/string realisations of these arise in null x^+-reductions of AdS plane waves [22,24], which are large boost, low temperature limits [23] of boosted black branes [25]. Various aspects of Lifshitz and hyperscaling violating holography appear in *e.g.* [19,26–30]. Some of these exhibit novel scaling for entanglement entropy *e.g.* [17–19]: the string realisations above reflect this [31–34], suggesting corresponding regimes in the gauge theory duals exhibiting this scaling.

It is of interest to study hydrodynamic behaviour in these nonrelativistic generalisations of holography: previous investigations include *e.g.* [35,37–42]. In this paper, we study the shear diffusion constant in bulk $(d+1)$-dimensional hyperscaling violating theories (1) with z, θ exponents adapting the membrane-paradigm-like analysis of [3]. As in [3], we map the diffusion of shear gravitational modes on a stretched horizon to charge diffusion in an auxiliary theory obtained by compactifying one of the d_i boundary spatial dimensions exhibiting translation invariance. This gives a near horizon expansion for perturbations with modifications involving z, θ. We find (sec. 2) that for generic exponents with $d-z-\theta > -1$, the shear diffusion constant is $\mathcal{D} = \frac{r_0^{z-2}}{d-z-\theta-1}$, *i.e.* power-law scaling (18) with the temperature $T \sim r_0^z$. Studying various special cases motivates us to guess (22), *i.e.* $\#\mathcal{D}T^{\frac{2-z}{z}} = \frac{1}{4\pi}$ where # is some (d, z, θ)-dependent constant, suggesting that $\frac{\eta}{s}$ has universal behaviour. The condition $z < 2 + d_i - \theta$ representing this universal sector appears related to requiring standard quantisation from the point of view of holography. It would be interesting to understand the hydrodynamics and viscosity here more systematically.

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When the exponents satisfy $d-z-\theta=-1$, the diffusion constant exhibits logarithmic behaviour (sec. 2.2), suggesting a breakdown of some sort in this analysis. This condition appears compatible however with various known constraints on the exponents: it would be interesting to understand this from other considerations. The exponents arising in null reductions of AdS plane waves or highly boosted black branes [22–24] satisfy this condition, which can be written as $z=2+d_{eff}$. It is interesting to note that highly boosted black branes (or AdS plane waves) give rise upon x^+-reduction to hyperscaling violating theories with z,θ exponents leading to novel entanglement scaling, as well as the condition $z=2+d_{eff}$ here. The two appear independent however: the entanglement entropy stems from $d_i-1\le\theta\le d_i$ and does not depend on z while the relation here involves both z,θ. We discuss (sec. 3) the corresponding picture of hydrodynamics that is likely to arise in the null reduction by mapping the perturbations accordingly. Details of the diffusion analysis appear in the Appendix.

2. Shear diffusion on the stretched horizon

We are considering nonrelativistic holographic backgrounds described by a $(d+1)$-dim hyperscaling violating metric at finite temperature,

$$ds^2=r^{2\theta/d_i}\left(-\frac{f(r)}{r^{2z}}dt^2+\frac{dr^2}{r^2 f(r)}+\frac{\sum_{i=1}^{d_i}dx_i^2}{r^2}\right),$$
$$d_i=d-1,\qquad d_{eff}=d_i-\theta\,, \tag{1}$$

where $f(r)=1-(r_0 r)^{d+z-\theta-1}$. d_i is the boundary spatial dimension while d_{eff} is the effective spatial dimension governing various properties of these theories, for instance the entropy density $s\sim T^{d_{eff}/z}$. The temperature of the boundary field theory (*i.e.* Hawking temperature of the black hole) is

$$T=\frac{(d+z-\theta-1)}{4\pi}\,r_0^z\,. \tag{2}$$

These spacetimes are conformal to Lifshitz spacetimes [8,9], and exhibit $t\to\lambda^z t,\ x_i\to\lambda x_i,\ r\to\lambda r,\ ds\to\lambda^{\theta/(d-1)}ds$. They arise in Einstein–Maxwell–Dilaton theories and are sourced by gauge fields and scalars. The window $d_i-1\le\theta\le d_i$ shows novel scaling for entanglement entropy [17–19]: these arise in the string realisations [22–24], with entanglement entropy studies in [31–34]. The null energy conditions following from (1) constrain the exponents, giving

$$(d-1-\theta)\big((d-1)(z-1)-\theta\big)\ge 0\,,$$
$$(z-1)(d-1+z-\theta)\ge 0\,. \tag{3}$$

We want to study the diffusion of shear gravitational modes in these backgrounds as a way of studying shear viscosity. In [3], Kovtun, Son and Starinets formulated charge and shear diffusion for black brane backgrounds in terms of long-wavelength limits of perturbations on an appropriately defined *stretched horizon*, the broad perspective akin to the membrane paradigm [4]. Their analysis, which is quite general, begins with a background metric of the form

$$ds^2=G_{\mu\nu}dx^\mu dx^\nu=G_{tt}(r)dt^2+G_{rr}(r)dr^2+G_{xx}(r)\sum_{i=1}^{d-1}dx_i^2\,, \tag{4}$$

which includes the hyperscaling violating backgrounds (1) as a subfamily. Charge diffusion of a gauge field perturbation A_μ in the background (4) is encoded by the charge diffusion constant D, defined through Fick's Law $j^i=-D\partial_i j^t$, where the 4-current j^μ is defined on the stretched horizon $r=r_h$ (with n the normal) as $j^\mu=n_\nu F^{\mu\nu}|_{r=r_h}$. Then current conservation $\partial_\mu j^\mu=0$ leads to the diffusion equation $\partial_t j^t=-\partial_i j^i=D\partial_i^2 j^t$, with D the corresponding diffusion constant. Fick's law in turn can be shown to apply if the stretched horizon is localised appropriately with regard to the parameters Γ,q,T. Translation invariance along $x\in\{x_i\}$ allows considering plane wave modes for the perturbations $\propto e^{-\Gamma t+iqx}$, where Γ is the typical time scale of variation and q the x-momentum. In the IR regime, the modes vary slowly: this hydrodynamic regime is a low frequency, long wavelength regime.

The diffusion of shear gravitational modes can be mapped to charge diffusion [3]: under Kaluza–Klein compactification of one of the directions along which there is translation invariance, tensor perturbations in the original background map to vector perturbations on the compactified background. Here we carry out a similar analysis for the shear diffusion constant in the backgrounds (1), adapting [3] to the present context. We turn on the metric fluctuations h_{xy} and h_{ty} $(x\equiv x_1,\ y\equiv x_2)$ around (4), depending only on t,r,x, *i.e.* $h_{ty}=h_{ty}(t,x,r)$, $h_{xy}=h_{xy}(t,r,x)$. Other fluctuation modes can be consistently set to zero. There is translation invariance along the y-direction: thus after a y-compactification, the modes h_{xy} and h_{ty} become components of a $U(1)$ gauge field in the dimensionally reduced d-dim spacetime. The components are given by

$$g_{\mu\nu}=G_{\mu\nu}(G_{xx})^{\frac{1}{d-2}}\qquad[\mu,\nu=0,\dots,d-1];$$
$$A_0=(G_{xx})^{-1}h_{ty}\,,\quad A_x=(G_{xx})^{-1}h_{xy}\,, \tag{5}$$

where $G_{\mu\nu}$ is the metric given by (4). A part of the gravitational action contains the Maxwell action with an r-dependent coupling constant, $\sqrt{-G}R\ \to\ -\frac{1}{4}\sqrt{-g}F_{\alpha\beta}F_{\gamma\delta}g^{\alpha\gamma}g^{\beta\delta}(G_{xx})^{\frac{d-1}{d-2}}$.

The gauge field equations following from the action are

$$\partial_\mu\left(\frac{1}{g_{\rm eff}^2}\sqrt{-g}F^{\mu\nu}\right)=0\,,\qquad \frac{1}{g_{\rm eff}^2}=G_{xx}^{\frac{d-1}{d-2}}\,, \tag{6}$$

where we have read off the r-dependent $g_{\rm eff}$ from the compactified action. Analysing these Maxwell equations and the Bianchi identity assuming gauge field ansatze $A_\mu=a_\mu(r)e^{-\Gamma t+iqx}$ and radial gauge $A_r=0$ as in [3] shows interesting simplifications in the near-horizon region. When $q=0$, these lead to $\partial_r\big(\frac{\sqrt{-g}}{g_{\rm eff}^2}g^{rr}g^{tt}\partial_r A_t\big)=0$. We impose the boundary condition that the gauge fields vanish at $r=r_c\sim 0$. As in [3], for q nonzero but small, we assume an ansatz for A_t as a series expansion in $\frac{q^2}{T^{2/z}}$

$$A_t=A_t^{(0)}+A_t^{(1)}+\dots\,,\qquad A_t^{(1)}=O\left(\frac{q^2}{T^{2/z}}\right),$$
$$A_t^{(0)}=Ce^{-\Gamma t+iqx}\int_{r_c}^{r}dr'\,\frac{g_{tt}(r')g_{rr}(r')}{\sqrt{-g(r')}}\cdot g_{\rm eff}^2(r') \tag{7}$$
$$=Ce^{-\Gamma t+iqx}\int_{r_c}^{r}dr'\,\frac{G_{tt}(r')G_{rr}(r')}{G_{xx}(r')\sqrt{-G(r')}}\,,$$

using (5), (6), with C some constant. Making a second assumption

$$|\partial_t A_x|\ll|\partial_x A_t| \tag{8}$$

as in [3], the gauge field component A_x, using the A_t solution, becomes

$$A_x=A_x^{(0)}+A_x^{(1)}+\dots\,,$$

$$A_x^{(0)} = -\frac{i\Gamma}{q} C e^{-\Gamma t + iqx} \int_{r_c}^{r} dr' \frac{g_{xx}(r')g_{rr}(r')}{\sqrt{-g(r')}} \cdot g_{\text{eff}}^2(r') \qquad (9)$$
$$= -\frac{i\Gamma}{q} C e^{-\Gamma t + iqx} \int_{r_c}^{r} dr' \frac{G_{rr}(r')}{\sqrt{-G(r')}} ,$$

again as a series expansion. As for A_t, we impose the boundary condition $A_x \to 0$ as $r \to r_c \sim 0$. In Appendix A, B, we show that the above series expansions are self-consistent provided certain conditions hold on the location r_h of the stretched horizon and the parameters q, Γ and T (equivalently r_0). This enables us to define Fick's law on the stretched horizon, and thereby the diffusion equation. The shear diffusion constant then becomes

$$\mathcal{D} = \frac{\sqrt{-g(r_h)}}{g_{\text{eff}}^2(r_h) g_{xx}(r_h)\sqrt{-g_{tt}(r_h)g_{rr}(r_h)}} \int_{r_c}^{r_h} dr \frac{-g_{tt}(r)g_{rr}(r)g_{\text{eff}}^2(r)}{\sqrt{-g(r)}} , \qquad (10)$$

where r_c is the location of the boundary, and we are evaluating $\mathcal{D}$ at the stretched horizon. The leading solutions $A_{t,x}^{(0)}$ and $\mathcal{D}$ depend on the exponents: we analyse this separately below.

2.1. Shear diffusion constant: $d - z - \theta > -1$ (or $z < 2 + d_{eff}$)

Using (1), the expression (7) for $A_t^{(0)}$ becomes

$$A_t^{(0)} = C e^{-\Gamma t + iqx} \int_{r_c}^{r} dr\, r^{d-z-\theta} . \qquad (11)$$

For generic values

$$d - z - \theta > -1 , \qquad (12)$$

the leading solution (11) for $A_t^{(0)}$ has power law behaviour

$$A_t^{(0)} = \frac{C}{d - z - \theta + 1} e^{-\Gamma t + iqx}\, r^{d-z-\theta+1} . \qquad (13)$$

We expect that the hyperscaling violating phase breaks down close to the boundary at r_c: for our purposes, strictly speaking we will only require that the horizon is well-separated from the boundary, *i.e.* $r_0 r_c \ll 1$, or equivalently that the temperature is sufficiently below the ultraviolet cutoff in the theory. Thus the condition $d - z - \theta > -1$ arises from the boundary condition $A_t^{(0)} = 0$ at $r = r_c \sim 0$. This includes various subfamilies of hyperscaling violating metrics that arise in gauge/string realisations, *e.g.* through dimensional reductions of nonconformal Dp-branes [19] for $p \le 4$ (here $\theta = p - \frac{9-p}{5-p}$, $d_i = p$). The case of $d - z - \theta = -1$, arises in the reductions of various D-brane plane waves [22,23,32]: here the leading solution has logarithmic behaviour, as we describe later.

From (1), the condition (12) can be written as $d_i - \theta - z + 2 > 0$, or $z < 2 + d_{eff}$, *i.e.* the Lifshitz exponent is not too high. For $z > 2 + d_{eff}$, it appears that the perturbations (11) do not die far from the horizon. For $z = 1$, this gives $\theta > d_i + 1$ which arises *e.g.* from the reduction of $D6$-branes: in such cases, it would appear that gravity does not decouple and the asymptotics is not well-defined. It would be interesting to understand the condition (12) better from other considerations *e.g.* holography [26–30]: more comments appear later.

The leading solution for A_x likewise is

$$A_x^{(0)} = -\frac{i\Gamma}{q} C e^{-\Gamma t + iqx} \frac{r_0^{\theta+1-d-z}}{\theta + 1 - d - z} \log\left(1 - (r_0 r)^{d+z-\theta-1}\right) . \qquad (14)$$

Self-consistency of (7), (9), (8), when $d - z - \theta > -1$ leads to

$$e^{-\frac{T^{2/z}}{q^2}} \ll \frac{\frac{1}{r_0} - r_h}{\frac{1}{r_0}} \ll \frac{q^2}{T^{2/z}} \ll 1 . \qquad (15)$$

This means that the stretched horizon has to be sufficiently close to the horizon (to $O(q^2)$) but not exponentially close to it. These conditions can be simultaneously satisfied in a self-consistent manner as we discuss in Appendix A, adapting [3].

Using (5), (6), the shear diffusion constant (10) becomes

$$\mathcal{D} = \frac{\sqrt{-G(r_h)}}{\sqrt{-G_{tt}(r_h)G_{rr}(r_h)}} \int_{r_c}^{r_h} dr \frac{-G_{tt}(r)G_{rr}(r)}{G_{xx}(r)\sqrt{-G(r)}}$$
$$= \frac{1}{r_h^{d-\theta-1}} \int_{r_c}^{r_h} r^{d-z-\theta} dr . \qquad (16)$$

Thus, for a hyperscaling violating theory with $d - z - \theta > -1$, we obtain

$$\mathcal{D} = \frac{r_h^{2-z}}{d - z - \theta + 1} \simeq \frac{r_0^{z-2}}{d - z - \theta + 1} + O(q^2) , \qquad (17)$$

where we have dropped the contribution in the integral from r_c since the UV scale $r_c \ll r_h$ is well-separated from the horizon scale. The diffusion constant in (16), (17), is evaluated at the stretched horizon r_h: however $r_h \sim \frac{1}{r_0} + O(q^2)$ so that to leading order $\mathcal{D}$ is evaluated at the horizon $\frac{1}{r_0}$. It is interesting that θ cancels in the r_0-dependent terms in $\mathcal{D}$, which is essentially the ratio of A_t to a field strength component (both of which have nontrivial θ dependence).

In the present hyperscaling violating case, we have seen that $T \sim r_0^z$ and $\mathcal{D} \sim r_0^{z-2}$ so the product $\mathcal{D}T \sim r_0^{2(z-1)}$ is not dimensionless. Using (2), we have

$$\mathcal{D} = \frac{1}{d - z - \theta + 1}\left(\frac{4\pi}{d + z - \theta - 1}\right)^{\frac{z-2}{z}} T^{\frac{z-2}{z}} , \qquad (18)$$

as the scaling with temperature T of the leading diffusion constant (17). See also *e.g.* [35,36,38,39,41,42] for previous investigations including via holography.

2.1.1. Comments on $\frac{\eta}{s}$

We now make a few comments on (17), (18) towards gaining insight into $\frac{\eta}{s}$:

(1) As a consistency check, we see that for pure AdS with $\theta = 0$, $z = 1$, we obtain $\mathcal{D} = \frac{1}{4\pi T}$. This corresponds to a relativistic CFT: the shear diffusion constant is $\mathcal{D} = \frac{\eta}{\varepsilon + P}$ and thermodynamics gives $\varepsilon + P = Ts$, where ε, P, s are energy, pressure and entropy densities. This gives the relation $\frac{\eta}{s} = \mathcal{D}T$ and thereby $\frac{\eta}{s} = \frac{1}{4\pi}$.

(2) Theories with metric (1) and $\theta = 0$ enjoy the Lifshitz scaling symmetry, $x_i \to \lambda x_i$, $t \to \lambda^z t$: Then the diffusion equation $\partial_t j^t = D\partial_i^2 j^t$ shows the diffusion constant to have scaling dimension $dim[\mathcal{D}] = z - 2$, where momentum scaling is $[\partial_i] = 1$ (or equivalently, $[x_i] = -1$, $[t] = -z$). With temperature scaling as inverse time, we have $dim[T] = z$. Thus on scaling grounds, the temperature scaling in (18), which here is

$$\mathcal{D} = \frac{1}{d - z + 1}\left(\frac{4\pi}{d + z - 1}\right)^{\frac{z-2}{z}} T^{\frac{z-2}{z}} , \qquad (19)$$

is expected, upto the d, z-dependent prefactors. For $z = 2$, the diffusion equation (structurally like a Schrodinger equation) already saturates the scaling dimensions, and $\mathcal{D}$ has apparently no temperature dependence. As T increases, $\mathcal{D}$ decreases for $z < 2$: however $\mathcal{D}$ increases with T for $z > 2$.

Aspects of Lifshitz hydrodynamics have been studied in *e.g.* [37, 40]. As discussed in [40], under the Lifshitz symmetry, we have the scalings $[T]=z$, $[\varepsilon]=z+d-1$, $[P]=z+d-1$, $[s]=d-1$, $[\eta]=d-1$. Indeed for Lifshitz black branes with horizon r_H and temperature (2), the entropy density is $s=\frac{r_H^{d-1}}{4G_{d+1}}=\frac{1}{4G_{d+1}}(\frac{4\pi}{d+z-1}T)^{\frac{d-1}{z}}$. The thermodynamic relations give $\varepsilon+P=Ts$. The shear viscosity [40] is $\eta=\frac{1}{16\pi G_{d+1}}T^{\frac{d-1}{z}}$ satisfying the universal bound $\frac{\eta}{s}=\frac{1}{4\pi}$. For this to arise from (19), we guess that the relation between shear viscosity and the shear diffusion constant is

$$\frac{\eta}{s} = \frac{(d-z+1)}{4\pi}\mathcal{D}r_0^{2-z} = \frac{(d-z+1)}{4\pi}\Big(\frac{4\pi}{d+z-1}\Big)^{\frac{2-z}{z}}\mathcal{D}T^{\frac{2-z}{z}}\,. \tag{20}$$

(3) For $\theta\neq 0$, the scaling analysis of the Lifshitz case is not applicable: however the temperature is θ-independent and the relation (18) continues to hold for generic θ. Towards guessing the hydrodynamics from the diffusion constant in this case, we first recall from [3] that nonconformal branes give $\mathcal{D}=\frac{1}{4\pi T}$, and thereby $\frac{\eta}{s}=\frac{1}{4\pi}$ continues to hold. On the other hand, [19] observed that nonconformal Dp-branes upon reducing on the sphere S^{8-p} give rise to hyperscaling violating theories with $z=1$ and $\theta\neq 0$. It would therefore seem that the near-horizon diffusion analysis continues to exhibit this universal behaviour since the sphere should not affect these long-wavelength diffusive properties.

Happily, we see that (18) for $z=1$ gives $\mathcal{D}=\frac{1}{4\pi T}$, with the θ-dependent prefactors cancelling precisely. Thus all hyperscaling violating theories with $z=1$ appear to satisfy the universal viscosity bound

$$\frac{\eta}{s}=\mathcal{D}T=\frac{1}{4\pi}\,. \tag{21}$$

Putting this alongwith the Lifshitz case motivates us to guess the universal relation

$$\begin{aligned}\frac{\eta}{s} &= \frac{(d-z-\theta+1)}{4\pi}\mathcal{D}r_0^{2-z}\\ &= \frac{(d-z-\theta+1)}{4\pi}\Big(\frac{4\pi}{d+z-\theta-1}\Big)^{\frac{2-z}{z}}\mathcal{D}T^{\frac{2-z}{z}} = \frac{1}{4\pi}\end{aligned} \tag{22}$$

between $\eta, s, \mathcal{D}, T$, for general exponents z,θ. This reduces to (20) for the Lifshitz case $\theta=0$. One might wonder if the prefactors for $\theta\neq 0$ somehow conspire to violate the universal bound: in this regard, it is worth noting that z,θ appear in linear combinations in the prefactors. Alongwith the previous subcases, this suggests consistency of (22).

Finally we know that the entropy density is $s=\frac{r_H^{d_{eff}}}{4G_{d+1}}\sim\frac{1}{4G_{d+1}}T^{\frac{d_i-\theta}{z}}$ in hyperscaling violating theories, with $d_{eff}=d_i-\theta=d-1-\theta$ the effective spatial dimension. Then (22) gives the shear viscosity as $\eta\sim\frac{1}{16\pi G_{d+1}}T^{\frac{d_i-\theta}{z}}$.

It is fair to say that to study this conjecture in detail, it is important to systematically understand the thermodynamic/hydrodynamic relations between the expansion of the energy–momentum tensor, the shear viscosity η and the diffusion constant $\mathcal{D}$. Towards this, it is worth putting the analysis here leading to (17), (18), and the comments above in perspective with the calculation of viscosity via the Kubo formula $\eta=-\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{xy,xy}(\omega)$, with G^R the retarded Green's function [6] (assuming $T_{ij}\sim\eta(\partial_i v_j+\ldots)$ in the dual field theory). Holographically, this is obtained by modelling the h^x_y perturbation as a massless scalar and thereby gleaning the $\langle T_{xy}T_{xy}\rangle$ correlation function: for various subfamilies in (1), this has been carried out in *e.g.* [35,39,41]. For instance in [41], the appropriate solutions at zero momentum $\vec{k}=0$ to the scalar wave equation in the near and far regions are matched to obtain $G^R=-i\frac{\omega}{16\pi G}\frac{R^{d_i}}{r_{hv}^{\theta}}r_0^{d_i-\theta}$ and thereby η, where we have written the metric (1) as $ds^2=R^2(\frac{r}{r_{hv}})^{2\theta/d_i}(-f(r)\frac{dt^2}{r^{2z}}+\ldots)$, explicitly retaining the dimensionful factors R and the scale r_{hv} inherent in these theories [19]. Likewise the entropy density is $s=\frac{1}{4G}\frac{R^{d_i}}{r_{hv}^{\theta}}r_0^{d_i-\theta}$ from the area of the horizon, recovering $\frac{\eta}{s}=\frac{1}{4\pi}$ in agreement with our analysis (and θ cancels[1]).

We have seen the condition $z<2+d_i-\theta$ arising naturally from the perturbations falling off asymptotically (11) in our case. It is worth noting that although we implicitly regard hyperscaling violating theories as infrared phases arising from *e.g.* string realisations in the ultraviolet, $z<2+d_i-\theta$ in the analysis here ensures that the ultraviolet structure of these theories is essentially unimportant: the diffusion constant arises solely from the near horizon long-wavelength modes. The theories satisfying this condition are in some sense continuously connected to AdS-like relativistic theories ($z=1$, $\theta=0$), as the analysis in Appendix A suggests. Identifying this condition from the point of view of holographic calculations appears more subtle. While a detailed analysis is ongoing, we outline a few comments here, in part following discussions in [26] for Lifshitz theories. Bulk field modes have asymptotic fall-offs $\phi\sim r^{\Delta_-}(\phi_-+\ldots)+r^{\Delta_+}(\phi_++\ldots)$ near the boundary $r\to r_c$, where $\Delta_-+\Delta_+=d_i-\theta+z$ [19] (see also [28,30]). If $\Delta_-<\Delta_+$, the ϕ_- fall-off near the boundary $r\to r_c\sim 0$ is slower (thus dominating), leading to fixed ϕ_- boundary conditions relevant for standard quantisation (ϕ_- taken as source). In particular, the momentum density operator $\mathcal{P}^i$ has dimension $d_i-\theta+1$ (while energy density has dimension $d_i-\theta+z$): taking $\Delta_+=d_i-\theta+1$ gives $\Delta_-=z-1$, so that $\Delta_-<\Delta_+$ implies $z<2+d_i-\theta$, which is precisely our condition (12). When this condition is violated in a reasonable theory,[2] it would seem that the analog of alternative quantisation [43] will be applicable, with fixed ϕ_+ boundary conditions. The case $z=2+d_{eff}$ discussed in the remainder of the paper may be interesting, with $\Delta_-=\Delta_+$. In these cases, θ may not disappear. We hope to explore these issues further.

2.2. Shear diffusion constant: $d-z-\theta=-1$ (or $z=2+d_{eff}$)

Now we consider the family of hyperscaling violating backgrounds (1) with $d-z-\theta=-1$. In this case, the leading solution (11) for $A_t^{(0)}$ has logarithmic behaviour,

$$A_t^{(0)}=Ce^{-\Gamma t+iqx}\log\Big(\frac{r}{r_c}\Big)\,,\qquad d-z-\theta=-1\,. \tag{23}$$

Then working through, we have from (16)

$$\mathcal{D}=r_0^{d-\theta-1}\log\Big(\frac{1}{r_0r_c}\Big)=r_0^{z-2}\log\Big(\frac{1}{r_0r_c}\Big)\,. \tag{24}$$

This implies that in the low temperature limit $r_0\to 0$, the diffusion constant becomes vanishingly small if $d_i-\theta>0$, or equivalently $z>2$. The energy conditions (3) eliminating $\theta=d-z+1$ give $(z-2)\big((d-1)z-2d+z\big)\geq 0$, $(z-1)(z-2)\geq 0$. When $\theta=0$, we obtain Lifshitz theories: the energy conditions become $(z-1)(d-1+z)\geq 0$. Then the condition here is $z=d+1=2+d_i$. Since we are considering theories in $d+1\geq 3$ bulk dimensions,

[1] We have seen that θ disappears from the temperature dependence of $\mathcal{D}$ in (18). It would be inconsistent if θ remained, at least in cases where the hyperscaling violating phase arises from string constructions such as nonconformal branes which are known to have universal $\frac{\eta}{s}$ behaviour, as discussed in comment (3) above.

[2] Unlike *e.g.* $d_i=6$, $z=1$, $\theta=9$, arising from the reduction of *e.g.* $D6$-branes where the asymptotics is ill-defined with gravity not decoupling.

$z \geq 3$ consistent with the energy conditions above. With $\theta \neq 0$, we have $z = 2 + d_{eff}$. In [19], it was noted that the entropy scaling $S \sim T^{(d_i-\theta)/z}$ implies that the specific heat is positive if $\frac{d_i-\theta}{z} \geq 0$. Here we have $S \sim T^{\frac{d_{eff}}{2+d_{eff}}}$ so that positivity of the specific heat gives $\frac{d_{eff}}{2+d_{eff}} > 0$ if $d_{eff} > 0$. Relatedly, we recall that entanglement entropy has novel scaling behaviour in the window $d_i - 1 \leq \theta \leq d_i$, which does not involve z. The entangling surface has been observed to have some instabilities for $\theta > d_i$. The present condition $z = 2 + d_{eff}$ is thus distinct from and compatible with them.

We show in Appendix B that the conditions (15) on the stretched horizon now become

$$\exp\Big(-\frac{T^{2/z}}{q^2}\frac{1}{\log\frac{1}{r_0 r_c}}\Big) \ll \frac{\frac{1}{r_0}-r_h}{\frac{1}{r_0}} \ll \frac{q^2}{T^{2/z}}\log^2\frac{1}{r_0 r_c}\,. \tag{25}$$

In the generic cases (15), the power law behaviour ensured that the short distance cutoff decoupled from the near-horizon behaviour (*e.g.* since $\frac{1}{r_0^{\#}} \gg r_c^{\#}$). Here the solution $A_t^{(0)}$ contains a logarithm which requires a scale, which filters through to (25). While it is unusual for the UV cutoff r_c to appear in what is manifestly a hydrodynamic or long-wavelength regime that we have restricted our analysis to, (15) implies (25) since $r_c \ll \frac{1}{r_0}$ implies $\log\frac{1}{r_0 r_c} \gg 1$ so that the window for the stretched horizon is not over-constrained. However, the subleading corrections (48) to the gauge field perturbations again contain terms involving $\log\frac{1}{r_0 r_c}$ factors affecting the validity of the series expansion in $\frac{q^2}{T^{2/z}}$.

It appears reasonable to conclude that the series expansion is perhaps breaking down in this case. Towards gaining some insight into this, it is useful to look for gauge/string embeddings of these effective gravity theories. In this regard, we recall that AdS plane waves (equivalently, highly boosted black branes) upon x^+-reduction give rise to hyperscaling violating spacetimes with certain values for the z, θ-exponents [22–24]. It turns out that the exponents satisfy $d - z - \theta = -1$. We will discuss this in what follows.

3. Diffusion constant: highly boosted black branes

A simple subclass of (zero temperature) hyperscaling violating theories can be constructed from the dimensional reduction of AdS_{d+2} plane wave spacetimes [22,24]

$$\begin{aligned} ds^2 &= \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] \\ &\quad + R^2 Q r^{d-1}(dx^+)^2 + R^2 d\Omega_S^2 \quad \longrightarrow \end{aligned} \tag{26}$$

$$\begin{aligned} ds^2 &= r^{\frac{2\theta}{d_i}}\Big(-\frac{dt^2}{r^{2z}} + \frac{\sum_{i=1}^{d_i} dx_i^2 + dr^2}{r^2}\Big), \\ z &= \frac{d+3}{2}, \quad \theta = \frac{d-1}{2}, \quad d_i = d-1. \end{aligned} \tag{27}$$

These can be obtained from a low-temperature, large boost limit [23] of boosted black branes [25] arising from the near horizon limits of the conformal $D3$-, $M2$- and $M5$-branes. Similar features arise from reductions of nonconformal Dp-brane plane waves [32, 23], with exponents

$$z = \frac{2(p-6)}{p-5}\,, \qquad \theta = \frac{p^2-6p+7}{p-5}\,, \qquad d_i = p-1\,, \tag{28}$$

where the Dp-brane theory after dimensional reduction on the sphere S^{8-p} and the x^+-direction has bulk spacetime dimension $d+1 \equiv p+1$. The holographic entanglement entropy in these theories exhibits interesting scaling behaviour [31–34].

To obtain the finite temperature theory, let us for simplicity consider the AdS_5 black brane

$$ds^2 = \frac{R^2}{r^2}\Big(-(1-r_0^4r^4)dt^2 + dx_3^2 + \sum_{i=1}^{2} dx_i^2\Big) + R^2\frac{dr^2}{r^2(1-r_0^4r^4)}\,, \tag{29}$$

which is a solution to the action $S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g^{(5)}}(R^{(5)} - 2\Lambda)$. Rewriting (29) in lightcone coordinates and boosting as $x^\pm \to \lambda^\pm x^\pm$, we obtain

$$\begin{aligned} ds^2 &= \frac{R^2}{r^2}\Big(-2dx^+dx^- + \frac{r_0^4r^4}{2}(\lambda dx^+ + \lambda^{-1}dx^-)^2 + \sum_{i=1}^{2}dx_i^2\Big) \\ &\quad + \frac{R^2dr^2}{r^2(1-r_0^4r^4)}\,. \end{aligned} \tag{30}$$

Writing in Kaluza–Klein form

$$\begin{aligned} ds^2 &= \frac{R^2}{r^2}\Big[-\frac{(1-r_0^4r^4)}{Qr^4}(dx^-)^2 + dx^2 + dy^2 + \frac{dr^2}{(1-r_0^4r^4)}\Big] \\ &\quad + QR^2r^2\left(dx^+ - \frac{(1-\frac{r_0^4r^4}{2})}{Qr^4}dx^-\right)^2\,, \end{aligned} \tag{31}$$

where $Q = \dfrac{\lambda^2 r_0^4}{2}$ and compactifying along the x^+ direction gives

$$\begin{aligned} ds^2 &= (Q^{1/2}R^3)r\Big[-\frac{(1-r_0^4r^4)}{Qr^6}(dx^-)^2 + \frac{dx^2+dy^2}{r^2} \\ &\quad + \frac{dr^2}{r^2(1-r_0^4r^4)}\Big]. \end{aligned} \tag{32}$$

This is simply the hyperscaling violating metric (1) with $z = 3$, $\theta = 1$, $d_i = 2$, in [22], but now at finite temperature. It is a solution to the equations stemming from the 4-dim Einstein–Maxwell–Dilaton action $S = \frac{1}{16\pi G_4}\int d^4x\sqrt{-g^{(4)}}\,(R^{(4)} - 2\Lambda e^{-\phi} - \frac{3}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}e^{3\phi}F^{\mu\nu}F_{\mu\nu})$ which arises upon dimensional reduction along the x^+-direction of the 5-dim Einstein action $S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g^{(5)}}(R^{(5)} - 2\Lambda)$. The scalar field has the profile $e^{2\phi} = R^2Qr^2$ while the gauge field is $\mathcal{A}_t = -\frac{1+f}{2Qr^4}$, $\mathcal{A}_i = 0$ with $f = 1 - r_0^4r^4$. The finite temperature theory is of course obtained by taking the boost λ to be large but finite, and the temperature r_0 to be small but nonzero, while holding $Q = \frac{\lambda^2r_0^4}{2}$ fixed. The boost simply serves to create a hierarchy of scales in the energy momentum components $T_{++} \sim \lambda^2r_0^4 \sim Q$, $T_{--} \sim \frac{r_0^4}{\lambda^2} \sim \frac{r_0^8}{Q}$, $T_{+-} \sim r_0^4$, $T_{ij} \sim r_0^4\delta_{ij}$, while keeping them nonzero.

The z, θ-exponents (27) arising in these reductions satisfy $d - z - \theta = -1$, coinciding with the special case discussed earlier. This is also true for nonconformal Dp-brane plane waves (28). It is worth noting that this relation between the exponents is distinct from the window $d_i - 1 \leq \theta \leq d_i$ where the holographic entanglement entropy exhibits novel scaling behaviour: in particular the present relation involves the Lifshitz exponent. The diffusion constant for this class of hyperscaling violating theories then has the logarithmic behaviour (24) described earlier, provided we restrict to modes that describe the lower dimensional theory.

In the above x^+-compactification, we see that x^- above maps to the time coordinate t below. Thus mapping the perturbations between the higher dimensional description and the hyperscaling violating one, we see that the metric in KK-form (31) including the shear gravitational perturbations is of the form (with gauge condition $\tilde{h}_{\mu r}, \tilde{h}_{rr} = 0$)

$$ds^2 = \tilde{g}_{--}(dx^-)^2 + \tilde{g}_{ii}dx_i^2 + \tilde{g}_{rr}dr^2 + 2\tilde{h}_{-y}dx^-dy + 2\tilde{h}_{xy}dxdy + \tilde{g}_{++}(dx^+ + A_-dx^-)^2 , \qquad (33)$$

where A_- is the background gauge field in the lower dimensional description. In other words, the perturbations map as $\tilde{h}_{-y} \to h_{ty}$, $\tilde{h}_{xy} \to h_{xy}$, upto the conformal factor arising from the x^+-reduction. In addition, the x^+-reduction requires that the perturbations $\tilde{h}_{-y}, \tilde{h}_{xy}$ are x^+-independent. This in turn translates to the statement that the near horizon diffusive modes are of the form

$$h_{\mu y}(r)e^{-k_-x^- + ik_x x} , \qquad k_+ = 0 , \qquad [\mu = x^-, x] , \qquad (34)$$

i.e. the nontrivial dynamics in the lower dimensional description arises entirely from the zero mode sector $k_+ = 0$ of the full theory.

Likewise, vector perturbations $\delta A_t, \delta A_y$ in the lower dimensional theory arise in (33) as

$$\ldots + g_{++}(dx^+ + A_-dx^- + \tilde{h}_{+-}dx^- + \tilde{h}_{+y}dy)^2 .$$

We see that these arise from gravitational perturbations $\tilde{h}_{+-}, \tilde{h}_{+y}$.

To ensure that the massive KK-modes from the x^+-reduction decouple from these perturbations, it suffices to take the x^+-circle size L_+ to be small relative to the scale set by the horizon, *i.e.* $L_+ \ll \frac{1}{r_0}$: equivalently, the temperature is small compared to the KK-scale $\frac{1}{L_+}$. The ultraviolet cutoff near the boundary is $r_c \sim Q^{-1/4} \ll \frac{1}{r_0}$: the hyperscaling violating phase is valid for $r \gtrsim Q^{-1/4}$.

Finally to map (32) to (1) precisely, we absorb the factors of the energy scale Q by redefining $\tilde{x}^- = \frac{x^-}{\sqrt{Q}}$. Now the shear diffusion constant can be studied as in the hyperscaling violating theory previously discussed, by mapping it to charge diffusion in an auxiliary theory obtained from the finite temperature x^+-compactified theory by compactifying along say the y-direction. This requires mapping the shear gravitational perturbations to the lower dimensional auxiliary gauge fields as $A_t \propto \tilde{h}_{-y}$, $A_y \propto \tilde{h}_{xy}$, which can then be set up in a series expansion in the near horizon region. Thus finally the shear diffusion constant follows from (24) giving $\mathcal{D} = r_0 \log(\frac{1}{r_0 Q^{-1/4}})$. For Q fixed, as appropriate for the lower dimensional theory, we see that the low temperature limit $r_0 \to 0$ gives a vanishing shear diffusion constant suggesting a violation of the viscosity bound. It is worth noting that the diffusion equation here is $\partial_{\tilde{x}^-} j^- = \tilde{\mathcal{D}}\partial_i^2 j^-$ where $\tilde{x}^- = \frac{x^-}{\sqrt{Q}}$ reflecting the Lifshitz exponent $z = 3$.

Noting that $Q \sim \lambda^2 r_0^4$, the diffusion equation and constant in the upstairs theory are

$$\partial_{x^-} j^- \sim \frac{\tilde{\mathcal{D}}}{\sqrt{Q}}\partial_i^2 j^- , \qquad \mathcal{D}r_0 \sim \frac{\tilde{\mathcal{D}}}{\sqrt{Q}} r_0 \sim \frac{1}{\lambda}\log\lambda . \qquad (35)$$

The $r_0 \to 0$ limit of the lower dimensional theory (where $T \sim r_0^3$) implies a highly boosted limit $\lambda \to \infty$ of the black brane for fixed Q: here $\mathcal{D}r_0$ vanishes. However this appears to be a subtle limit of hydrodynamics. From the point of view of the upstairs theory of the unboosted black brane, shear gravitational modes are h_{ty}, h_{xy}. Upon boosting, it would appear that these mix with other perturbation modes as well, suggesting some mixing between shear and bulk viscosity. From the point of view of the boosted frame, this system has anisotropy generated by the boost direction. Previous studies of anisotropic systems and shear viscosity include *e.g.* [44–50]. (See also *e.g.* [51] for a review of the viscosity bound.) In the present case, the shear viscosity tensor can be analysed from a systematic study of the expansion of the energy–momentum tensor of the finite temperature Yang–Mills fluid in the highly boosted regime. However the scaling (35) is likely to be realised only after phrasing the boosted black brane theory in terms of the variables appropriate for the lower dimensional hyperscaling violating theory (which arises in the $k_+ = 0$ subsector as discussed above). It would be interesting to understand the hydrodynamics in the lower dimensional theory better, as a null reduction of the boosted black brane theory, perhaps similar in spirit to nonconformal brane hydrodynamics [52,53] as a reduction of nonlinear hydrodynamics [54] of black branes in M-theory. We hope to explore this further.

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Appendix A. Diffusion analysis details: generic case

In the near-horizon region, the metric (1) is approximated as $g_{tt}(r) \approx -\gamma_0(\frac{1}{r_0} - r)$, $g_{rr}(r) \approx \frac{\gamma_r}{(\frac{1}{r_0} - r)}$, $g_{xx}(r) \approx const$, for constants γ_0, γ_r: the Maxwell equations (6) simplify substantially here, as a series expansion in $\frac{q^2}{\Gamma^{2/z}}$ for the gauge fields. Here we only mention the modifications in the analysis of [3] arising in the present context. An intermediate step in the self-consistent analysis gives

$$F_{tr} \sim r_0^{2(z-1)} \cdot \frac{q}{\Gamma^2} \cdot \frac{(1/r_0) - r_h}{1/r_0} \partial_r F_{tx} , \qquad (36)$$

which is then used to obtain a wave equation for F_{tx}: we choose the solution that is ingoing at the horizon, and then solve for the various gauge field components. Self-consistency constrains the location of the stretched horizon

$$\frac{(1/r_0) - r_h}{1/r_0} \ll \frac{1}{r_0^{2(z-1)}} \cdot \frac{\Gamma^2}{q^2} . \qquad (37)$$

For thermal AdS ($z = 1$), this becomes $\frac{(1/r_0) - r_h}{1/r_0} \ll \frac{\Gamma^2}{q^2}$ as in [3].

With small but nonzero q, we write A_t, A_x as series expansions (7), (9), in $\frac{q^2}{\Gamma^{2/z}}$ and impose (8) and the boundary conditions that $A_t, A_x \to 0$ at $r = r_c \sim 0$. Using the $A_t^{(0)}, A_x^{(0)}$ solutions (7), (9), (13), (14), in the near horizon region $\frac{1}{r_0} - r \ll \frac{1}{r_0}$ shows $\frac{A_x^{(0)}}{A_t^{(0)}} \sim \frac{1}{r_0^{2(z-1)}} \frac{\Gamma}{q} \log(\frac{1/r_0}{1/r_0 - r})$. Thus (8) is valid only if

$$\frac{1/r_0}{(1/r_0) - r_h} \ll e^{\frac{q^2 r_0^{2(z-1)}}{\Gamma^2}} . \qquad (38)$$

Combining (37), (38), gives the window

$$e^{-\frac{q^2 r_0^{2(z-1)}}{\Gamma^2}} \ll \frac{(1/r_0) - r_h}{1/r_0} \ll \frac{1}{r_0^{2(z-1)}} \frac{\Gamma^2}{q^2} , \qquad (39)$$

for the stretched horizon r_h in terms of the perturbation parameters $\frac{\Gamma^2}{q^2} \ll 1$.

We define the following gauge field currents on the stretched horizon

$$j^x = n_r F^{xr} = -\frac{F_{tx}}{g_{xx}\sqrt{-g_{tt}}}\,, \qquad j^t = n_r F^{tr} = -\frac{1}{g_{tt}\sqrt{g_{rr}}}F_{tr}\,. \tag{40}$$

The assumption (8) implies we can approximate the field strength $F_{tx} \approx -\partial_x A_t$, and $\partial_r F_{tx} \approx \partial_x F_{tr}$ in radial gauge. This gives $\partial_x j^t = -\frac{1}{g_{tt}\sqrt{g_{rr}}}\partial_x F_{tr} = -\frac{1}{g_{tt}\sqrt{g_{rr}}}\partial_r F_{tx}$. Further, multiplying and dividing by F_{tx} and using the definition of j^x, we get $\partial_x j^t = -\frac{g_{xx}}{\sqrt{-g_{tt}g_{rr}}}\frac{\partial_r F_{tx}}{F_{tx}} j^x$, *i.e.*

$$j^x = -\frac{\sqrt{-g_{tt}g_{rr}}}{g_{xx}}\frac{F_{tx}}{\partial_r F_{tx}}\partial_x j^t \;\equiv\; -D\partial_x j^t\,, \tag{41}$$

which is Fick's Law.[3] From the solutions to A_t and A_x, we have

$$-\frac{F_{tx}}{\partial_r F_{tx}}\bigg|_{r\approx r_h} \approx -\frac{A_t}{F_{tr}}\bigg|_{r\approx r_h} = \frac{\sqrt{-g(r_h)}}{g_{\rm eff}^2(r_h)g_{tt}(r_h)g_{rr}(r_h)}\int_{r_c}^{r_h}\frac{g_{tt}(r')g_{rr}(r')g_{\rm eff}^2(r')}{\sqrt{-g(r')}}dr'\,. \tag{42}$$

Thus from Fick's Law (41), we read off the shear diffusion constant

$$D = \frac{\sqrt{-g_{tt}(r_h)g_{rr}(r_h)}}{g_{xx}(r_h)}\cdot\frac{A_t}{F_{tr}}\bigg|_{r=r_h}\,, \tag{43}$$

evaluated at the stretched horizon r_h, where the boundary is $r_c \ll r_h$. Using (42) gives (10).

For generic exponents, the diffusion constant (17) becomes $D \sim r_0^{z-2} \sim T^{(z-2)/z}$. On the other hand, the diffusion equation gives $\Gamma = Dq^2$. These give the condition

$$\frac{\Gamma}{q} \sim \frac{q}{T^{\frac{2}{z}-1}}\,. \tag{44}$$

Using this estimate, (39) becomes (15), which is always satisfied for sufficiently small $\frac{q^2}{T^{2/z}}$. In particular for thermal AdS we have $z=1$, $D \sim T^{-1}$ so (15) becomes $e^{-\frac{T^2}{q^2}} \ll \frac{(1/r_0)-r_h}{1/r_0} \ll \frac{q^2}{T^2}$, the condition obtained in [3].

Using the expansion over q^2 for A_t and A_x in the gauge field equations, it can be checked that the leading $O(q^0)$ terms are consistent with the ansatz for $A_t^{(0)}, A_x^{(0)}$ in the regime (15). Likewise the subleading terms can be evaluated: collecting terms of $O(q^2)$ consistently, using (44) and simplifying gives

$$\partial_r A_t^{(1)} \sim r_0\Big[\frac{q^2}{T^{2/z}}\log\Big(\frac{1/r_0}{(1/r_0)-r}\Big) + \frac{q^4}{T^{4/z}}\log^2\Big(\frac{1/r_0}{(1/r_0)-r}\Big)\Big]A_t^{(0)}\,. \tag{45}$$

Using (15), we see that $\partial_r A_t^{(1)} \ll A_t^{(0)}$, and after integrating, that $A_t^{(1)} \ll A_t^{(0)}$, verifying that these are indeed subleading. Likewise, we find

$$A_x^{(1)} \sim \Big[\frac{q^2}{T^{2/z}}\log\Big(\frac{1/r_0}{1/r_0 - r}\Big) + \frac{q^4}{T^{4/z}}\log^2\Big(\frac{1/r_0}{1/r_0 - r}\Big)\Big]A_x^{(0)} \ll A_x^{(0)}\,. \tag{46}$$

[3] The antisymmetry of $F_{\mu\nu}$ implies $n_\mu j^\mu = 0$, *i.e.* the current is parallel to the stretched horizon (only n_r is nonzero with $g^{rr}n_r^2 = 1$). Contracting gives $n_\nu \partial_\mu(\frac{\sqrt{-g}}{g_{\rm eff}^2}F^{\mu\nu}) = 0 \Rightarrow n_r(\partial_M F^{Mr}) = 0$, with $M = t, x_i$. This gives current conservation $\partial_\mu j^\mu = \partial_M(n_\nu F^{M\nu}) = n_r(\partial_M F^{Mr}) = 0$.

Appendix B. Diffusion analysis details: special case

For the case $d-z-\theta=-1$, we obtain $\frac{A_x^{(0)}}{A_t^{(0)}} \sim \frac{1}{r_0^{2(z-1)}}\frac{\Gamma}{q}\frac{\log(\frac{1/r_0}{1/r_0-r})}{\log(\frac{1}{r_0 r_c})}$ so that imposing (8) gives

$$\frac{1}{r_0^{2(z-1)}}\cdot\frac{\Gamma^2}{q^2}\cdot\frac{\log(\frac{1/r_0}{1/r_0-r})}{\log(\frac{1}{r_0 r_c})} \ll 1\,. \tag{47}$$

From the estimates obtained for $\mathcal{D}$ from the diffusion equation and the diffusion integral, we obtain $\frac{\Gamma}{q} \sim \frac{q}{T^{\frac{2}{z}-1}}\log(\frac{1}{r_0 r_c})$ instead of (44). Using this in (37), we obtain the modified bound $\frac{(1/r_0)-r_h}{1/r_0} \ll \frac{q^2}{T^{2/z}}\log^2(\frac{1}{r_0 r_c})$. Likewise, (38) also changes to $\frac{q^2}{T^{2/z}}\log(\frac{1/r_0}{(1/r_0)-r_h})\log\frac{1}{r_0 r_c} \ll 1$. The subleading terms now give

$$\partial_r A_t^{(1)} \sim r_0\Big[\frac{q^2}{T^{2/z}}\log\Big(\frac{1/r_0}{(1/r_0)-r}\Big) + \frac{q^4}{T^{4/z}}\log^2\Big(\frac{1/r_0}{(1/r_0)-r}\Big)\log\Big(\frac{1}{r_0 r_c}\Big)\Big]A_t^{(0)}\,. \tag{48}$$

Within the regime (25), it would appear that $\partial_r A_t^{(1)} \ll A_t^{(0)}$: however $r_0 r_c \ll 1$ implies that $\log(\frac{1}{r_0 r_c})$ is large so that the $O(q^4)$ term need not be small even if $\frac{q^2}{T^{2/z}} \ll 1$, suggesting a breakdown of the series expansion.

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