4. Summary and Outlook of Part I

In this part of the thesis a new solution of the leading order evolution equation for the generalized parton distributions has been presented. The form of the solution is completely fixed by the symmetry properties of the evolution kernels. It has been shown that the evolution of the GPDs should be treated differently in the ERBL and DGLAP kinematical regions. This agrees both with the mathematical structure of the evolution equation and its physical content.

The solution which has been derived can be used for analytical and numerical studies of the GPD evolution, since all quantities entering the derivation and final result are unambiguously defined. The solution reflects in a clear mathematical way the physical properties of the GPD evolution, for instance the phenomenon of parton migration to the ERBL region at high scales. It has been demonstrated that the solution for $\xi \to 0$ and $\xi \to 1$ takes the known form of the DGLAP and ERBL equations, respectively. Further the behavior of the solution has been studied analytically in different limiting cases, namely at large evolution scales and in the limit of small ξ.

A Mathematica computer code has been developed in order to transcribe the analytically derived solution into a numerical algorithm. The program is worked out for the isovector as well as the isosinglet GPDs and has been tested to reproduce the general features of the evolution and works in reasonable time. Several examples have been given in the text.

The current situation in the GPD business is not too satisfactory. This is mainly due to the fact that data with high accuracy is not available by now. Even if high accuracy data were be accessible, it is non-trivial to extract the GPDs from this data. This is due to the fact that GPDs enter at the amplitude level and not on the level of the cross section. Nevertheless, several experimental groups make the effort to perform the relevant experiments and thereby improve the situation on the experimental side. For example, the determination of GPDs is one of the driving forces behind the 12GeV upgrade of the CEBAF facility at the Thomas Jefferson Laboratory (JLAB) in Virginia, USA, as well as the HERMES experiment at Deutsches Elektronen Synchrotron (DESY) in Hamburg, Germany. When these data become available there will be the need to have a suitable tool to perform the evolution and this is what we have at hand.

There are also other applications of the presented approach which one may think of. It would be interesting to analyze the analytic structure of the evolution of twist-3 operators. It is known that the knowledge of the reggeon bound states allows one to extract the anomalous dimensions of the operators at singular points in the j-plane [86]. This correspondence has been checked for twist-2 operators. Recently, the methods for the calculations of the multi-reggeon bound states were developed [87, 88, 89], and the predictions for the anomalous dimensions of higher twist operators at the singular points have been obtained [90, 91]. However, so far it is not known how to solve the problem of the analytical continuation of the anomalous dimensions of higher twist operators. The approach presented here might be helpful to solve this problem – at least for the class of twist-3 operators for which the evolution equation is known to be integrable.

In the refs. [29, 30, 92, 93] it has been observed that the Hamiltonian which governs the scaling behavior of certain QCD operators, can be mapped on a completely integrable system known as non-compact spin chain. This allows to determine the anomalous dimensions of these operators exactly. The key feature of integrability is, again, the collinear conformal symmetry.

In the second part of this thesis the non-compact spin chains are treated within the framework of the Quantum Inverse Scattering Method.

8. Summary and Outlook of Part II

In this part of the thesis we have worked out several new results concerning some particular completely integrable models within the framework of the Quantum Inverse Scattering Method. These models are defined on tensor product spaces of unitary representations of the group $SL(2,\mathbb{R})$. Therefore, we formulated the Yang-Baxter relation on the four possible distinct product spaces and derived explicitly the corresponding solutions. Thereby we obtained two, previously unknown, solutions, namely those on the spaces $D \otimes T$ and $T \otimes T$, where D denotes a representation space of the discrete series and T is one of the continuous series. The solution on the completely continuous space, $T \otimes T$, was then taken to define a completely integrable Heisenberg spin chain model, which, due to its symmetry group $SL(2,\mathbb{R})$ is referred to as a non-compact spin chain. The Hilbert space attached to each lattice site of this model is $L_2(\mathbb{R})$ and the spin operators are realized as the generators of the continuous series representation of $SL(2,\mathbb{R})$. The Hamiltonian of this model is defined as the derivative of the fundamental transfer matrix, which is constructed out of the solution of the Yang-Baxter relation of the corresponding Hilbert space. Taking our result for the $\mathcal{R}$-operator, the Hamiltonian is given by the sum of two-particle Hamiltonians and only involves nearest neighbor interactions. The pairwise Hamiltonians have been shown to possess a discrete and a continuous spectrum, which reflects the pattern of the decomposition of the tensor product of two representations of the principal series into its irreducible subspaces. The eigenstates belonging to the continuous spectrum are labeled by the $sl(2)$ spin $s = 1/2 + i\rho$, $\rho \in \mathbb{R}^+$, and the parity with respect to the permutation of arguments. The energies of the states with the same spin but different parity are different and the gap between them is maximal for $\rho = 0$ and decreases exponentially with ρ.

To solve the spin chain model, the methods of the Baxter $\mathcal{Q}$-operator and Separation of Variables have been applied. The standard technique, the Algebraic Bethe Ansatz (ABA), is not applicable for the model in question, because of the absence of a lowest weight vector in the underlying Hilbert space. After defining the Baxter operator as an integral operator acting on the Hilbert space of the model, we solved the defining equations and obtained the kernel of the $\mathcal{Q}$-operator in an explicit form. This allowed us to determine the analytical properties of the eigenvalues of the $\mathcal{Q}$-operator as functions of the spectral parameter. Then, the eigenvalues of the Baxter operator can be obtained as solutions to the Baxter equation on the appropriate class of functions and this eigenvalues already encode all information about the corresponding eigenstates. We have shown that the Hamiltonian of the model can be obtained as a derivative of the Baxter operator at special values of the spectral parameter. Moreover, the arbitrary transfer matrix factorizes into the product of two Baxter $\mathcal{Q}$-operators at certain values of the spectral parameters. We have also constructed the representation of the separated variables for the model in question. The kernel of the unitary operator, which maps the eigenfunction to the SoV representation, has been obtained in an explicit form. It factorizes into the product of $N-1$ (N is the number of sites) operators each depending on one separated variable only. The kernel of the transition operator can be visualized as a Feynman diagram with a specific pyramidal form. This form of the kernel, first obtained for the $SL(2,\mathbb{C})$ spin chain in [87], is a general feature of all non-compact $SL(2)$ spin magnets [126, 128]. Using the diagram technique we calculated the scalar product of the transition kernels and determined the Sklyanin's integration measure. We have shown that the wavefunction in the separated variables is given by the product of the eigenvalues of the (conjugated) Baxter operator. Therefore the knowledge of the eigenvalue of the Baxter operator allows to restore the eigenfunction.

The interest in non-compact Heisenberg spin chains is originated in the studies of the scaling behavior of certain composite operators in QCD. Nowadays, the focus has moved from QCD to super-symmetric theories, where scaling dimensions of operators are computed in order to support the famous AdS/CFT conjecture. It has been found that the hidden integrability, which has been originally detected in the QCD context, is a generic feature of all Yang-Mills theories and is even enhanced in their super-symmetric extensions. In all cases which have been considered so far, super-symmetric or not, the Heisenberg spin chain Hamiltonian on which a particular evolution operator could be mapped, is diagonalizable with the ABA method. For the spin chains from the $SL(2)$ sector this originated from the fact that they belong to the discrete series representations. If one looks at the non-compact spin chains from a more general, mathematical point of view, then it seems desirable to find a unified picture, in which all non-compact spin chains with $SL(2)$ symmetry can be treated on the same footing. But, since this necessarily involves also the continuous series representation, one has to rely on the alternative approach within the QISM, namely Baxter $\mathcal{Q}$-operators and SoV representation. From this viewpoint it was natural to consider the spin chain on the space $T \otimes T$ as we did here. The spin chain model on the space $D \otimes T$, which can be constructed from the $\mathcal{R}$-operator given in section 6.3, is the last in line which is missing. It will be an interesting question for the future, whether it is possible to construct a universal solution to the Yang-Baxter relation on the space $V_1 \otimes V_2$, where V_j is a unitary $SL(2,\mathbb{R})$ representation, discrete or continuous. Then, one possibly can treat all the different spin chains on the same footing and identify their common basis, although from the current knowledge, they seem to exhibit such different behaviors. The application of the thereby achieved knowledge should be helpful for the construction and solution of so-called graded, or super-symmetric non-compact spin chains. This, as a final aim, may shed some light on the mathematical nature of the hidden integrability in four dimensional gauge theories.