



# Geometric aspects of three dimensional $\mathcal{N} = 4$ gauge theories

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## DECLARATION

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This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. It is not substantially the same as any that I have submitted, or am concurrently submitting, for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or is being concurrently submitted, for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. This dissertation does not exceed the prescribed limit of 60 000 words.

Samuel Crew  
January, 2021



# ABSTRACT

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## Geometric aspects of three dimensional $\mathcal{N} = 4$ gauge theories

Samuel Crew

We study geometric aspects of three dimensional  $\mathcal{N} = 4$  gauge theories. We focus mainly on the factorisation property of supersymmetric partition functions of these theories and introduce hemisphere blocks that precisely realise the factorisation. We define these blocks as UV partition functions on a hemisphere  $S^1 \times H^2$  with an exceptional Dirichlet boundary condition and demonstrate that the resulting object is determined by the enumerative geometry of the Higgs branch.

The partition function on the hemisphere is closely related to a half-index that counts local operators of the theory on a flat spacetime with boundary. In this context, we show that the hemisphere blocks realise characters of lowest weight Verma modules of the Higgs and Coulomb branch chiral rings acting on boundary local operators.

We study the geometric interpretation of the twisted index factorisation in particular and demonstrate a relationship between the twisted index and the Hilbert series of a 3d  $\mathcal{N} = 4$  theory. We then use factorisation to provide a novel geometric expression for the Coulomb branch Hilbert series in terms of invariants of moduli spaces of quasimaps to the Higgs branch.

Finally, we apply these ideas to a particularly rich example of a non-abelian gauge theory with adjoint matter whose Higgs branch coincides with a moduli space of instantons. We compute hemisphere blocks for the theory and explicitly recover Verma module characters of the Coulomb branch chiral ring. In this example, the blocks have interesting combinatorial content and can be related to generating functions of reverse plane partitions—we discuss the interpretation of 3d mirror symmetry in this context. We also study line operators in this theory and show that half indices in the presence of a line operator exhibit an integrable structure. Along the way we find interesting connections between the twisted index gluing of hemisphere blocks and related calculations in topological string theory.

For my mum and dad.

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*If the doors of perception were cleansed every thing  
would appear to man as it is, Infinite.*

WILLIAM BLAKE



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# CHAPTER 1

## INTRODUCTION

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Quantum field theory is a theoretical framework that can be used to describe fundamental particle physics. The theory successfully explains a wide variety of physical phenomena in fields ranging from condensed matter physics to cosmology and the development of quantum field theory is one of the great achievements of twentieth century physics. In the form of the standard model of particle physics, quantum field theory is arguably the most precisely tested theory in the history of science—for example the latest measurements [1] of the anomalous magnetic dipole moment of the electron agree with the theoretical prediction to one part in a trillion. One of the crowning achievements of quantum field theory has been the prediction in 1964 [2, 3, 4] and eventual discovery almost half a century later of the Higgs boson [5, 6], completing the experimental verification of the standard model.

Despite these successes, many basic aspects of quantum field theory are still poorly understood. Indeed, even in ‘simple’ examples a precise definition is lacking; one of the millennium problems of the Clay mathematics institute [7] promises a prize of one million dollars to formulate quantum Yang-Mills theory in four dimensional spacetime and prove the existence of a mass gap. Many physicists<sup>1</sup> go further still and argue that even our current formulations of quantum field theory have serious deficiencies. For example, there exist quantum field theories with no semi-classical description or obvious fundamental degrees of freedom, and even if a theory does admit a Lagrangian description then often it cannot be understood in terms of local operators alone but rather one should include categories of higher dimensional defects. Perhaps this implies that quantum field theory can and should be formulated entirely geometrically in some to-be-determined sense.

Many of the twentieth century successes of quantum field theory are due to the use of perturbation theory and Feynman diagrams to understand weakly coupled physical processes. The difficulties and open problems in quantum field theory arise from trying to understand strongly coupled physics that is largely invisible to these perturbative methods. For example, the strong coupling behaviour of the quantum chromodynamics sector of the

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<sup>1</sup>See, for example, Seiberg’s 2015 talk [8].

standard model remains mysterious. After half a century of working with the standard model, we still do not have an analytical understanding of the confinement of quarks. Greater challenges arise still when trying to incorporate physically relevant models of gravity into the framework of quantum field theory. A first step on this path is surely to better understand and study non-perturbative aspects of quantum field theory.

A powerful organising principle in physics is the notion of symmetry. The Coleman-Mandula theorem [9] asserts that, under reasonable assumptions, the maximal bosonic symmetry group of a quantum field theory is the direct sum of the Poincaré group together with global symmetries. In this thesis we study supersymmetric quantum field theories which enjoy additional spacetime symmetries exchanging bosonic and fermionic particles in their spectrum. Supersymmetry evades the Coleman-Mandula no-go theorem by enlarging the Poincaré group to the super Poincaré group containing fermionic generators and the Haag-Sohnius-Lopuszanski theorem [10] ensures this is the maximal symmetry a theory can admit.<sup>2</sup>

Supersymmetric field theories are most likely not models that describe the world around us, and evidence of supersymmetric particles has not been found in collider experiments.<sup>3</sup> Nonetheless supersymmetry serves as a powerful toy model whose extra symmetries allow us to compute *exact* physical quantities without relying on perturbation theory whilst still describing interesting physics. In this way, supersymmetry offers a rare glimpse into the strongly coupled behaviour of interacting quantum field theory. A stunning early demonstration of the power of supersymmetry is the seminal work of Seiberg and Witten [12, 13] where, using the power of holomorphicity [14], the strongly coupled IR dynamics of a four dimensional theory with  $\mathcal{N} = 2$  supersymmetry is solved and an analytic description of confinement of monopoles is provided.

Supersymmetric quantum field theories have flat directions in their scalar potentials that are generally preserved by quantum corrections. This often leads to moduli spaces of quantum vacua with interesting geometric structures. An important ingredient in the Seiberg-Witten solution is the realisation that, in certain regions of the vacuum moduli space, the UV degrees of freedom are not always an appropriate description of the IR physics. Instead, a version of Montonen and Olive’s electromagnetic duality [15] is used to exchange electrically charged and magnetically charged excitations to describe the effective potential on the entire moduli space. The notion of duality more generally is a cornerstone in the study of quantum field theory and often arises from our ignorance of strongly coupled physics. It is possible for two different UV theories to lie in the same universality class and describe identical physics in the IR. A prototypical example of duality is the sine-Gordon and Thirring duality [16]: at the quantum level topological solitons

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<sup>2</sup>At least with a mass gap, otherwise conformal symmetry is also a possibility.

<sup>3</sup>For recent experimental results see, for example, [11].

in the sine-Gordon model are exchanged with local fermionic excitations in the Thirring model—this is a strong-weak duality and exchanges topological and flavour quantum numbers. The sine-Gordon and Thirring duality is a distant ancestor of a more modern example found in three dimensional  $\mathcal{N} = 4$  gauge theories known as three dimensional mirror symmetry. Three dimensional mirror symmetry is another strong-weak duality that acts on pairs of gauge theories and exchanges two distinct sectors of the vacuum moduli space known as the Higgs and Coulomb branches. It is also a strong-weak duality in the sense that fundamental perturbative excitations are exchanged with monopoles.

Branes are extended supersymmetric objects in string theory that provide boundary conditions for open strings. Supersymmetric quantum field theories can be naturally embedded in string theory by realising them as worldvolume theories on branes in a limit that decouples the gravitational degrees of freedom. Much of the progress in supersymmetry in the 1990s came from studying brane constructions that suggest and provide ‘zeroth order’ checks of supersymmetric dualities descending from the web of string dualities. Three dimensional mirror symmetry is an example for which the relevant gauge theories can be realised as intersections of branes in type IIB string theory [17] and in this context three dimensional mirror symmetry is a consequence of  $S$ -duality.

Supersymmetric theories contain sectors of protected BPS states that preserve some fraction of supercharges and, since they live in shortened multiplets, BPS states are generally invariant under continuous deformations of the theory. Supersymmetric indices count BPS states and the invariance property allows an index to be computed at a convenient point of the RG flow—for example where the theory has a weakly coupled Lagrangian description. Indices are a powerful tool to test dualities more precisely and gain insight into the strongly coupled dynamics of supersymmetric field theories from a purely quantum field theoretic perspective without reference, necessarily, to string theory. Significant progress in supersymmetry in the last two decades has been thanks to the development of localisation methods to compute supersymmetric indices and partition functions. Supersymmetric indices can be computed by a path integral and localisation uses the fermionic symmetries to add exact terms that reduce the path integral to a lower dimensional integral over BPS field configurations—if we are lucky this is a simple matrix model with a finite number of variables. A prototypical example appears for the Witten index in [18] where the index of a quantum mechanical sigma model is shown to localise to fixed points of a group action on the target. The method was further developed by Nekrasov in the work [19] where a four dimensional  $\mathcal{N} = 2$  super Yang-Mills theory is topologically twisted and regularised on an omega background and the path integral is reduced to an equivariant integral over the instanton moduli space. This powerful calculation systematises the results of Seiberg and Witten discussed previously and precisely explains the geometric origin of their solution. More recently, localisation has

been developed further by Pestun in the work [20] where four dimensional  $\mathcal{N} = 2$  super Yang-Mills theories are placed on a compact four sphere and localised to a product of instanton contributions at the north and south poles. In fact, Pestun’s result is a pre-cursor to the notion of holomorphic factorisation in three dimensions that we meet later in this thesis. The techniques and methods of localisation have since been expanded to a diverse range of theories and partition functions, largely due to the systematic methods developed by Festuccia and Seiberg [21] to place theories supersymmetrically on curved spacetimes.

Putting physics to one side, in recent years supersymmetry has, perhaps surprisingly, proved itself an enormously useful tool in mathematics. The last few decades have seen the birth of a new field of ‘physical mathematics’—the term was coined by Moore in his vision talk at Strings 2014 [22]. Whilst physics is well-used to borrowing from mathematics, this transfer of ideas began flowing in the opposite direction towards the end of the 1980s. Raoul Bott summarised the situation in his plenary lecture at the AAAS meeting in 1988:

*“Although we still often do not understand each other, the push and pull relationship of our two points of view has never been stronger and has invigorated both of us. Certainly in mathematics the physically inspired aspects of the Yang-Mills theory has had a profound effect on our understanding of the structure of four-manifolds, and I also think we mathematicians are only now learning to appreciate the rich mathematical structure of the Dirac sea—and indeed of the whole Fermion-inspired world of the physicists, as well as their mystical belief in supersymmetry. And on the other hand the most modern achievements of mathematics—from cobordism to index theory and K theory—have by now made their way into some aspects of present day physics—I think to stay.”*

Localisation is often the technique that allows us to make precise contact between supersymmetric field theory and moduli space geometry and in this way the mystery of the path integral can suggest surprising results in mathematics. The lines between supersymmetry and geometry are increasingly blurred and the fields often evolve in parallel. Physical mathematics is perhaps best described with some examples—we give a limited sample in the following paragraphs.

Mirror symmetry [23, 24, 25] in type IIA/B string theory is the first well-known example of physical mathematics and uses dual pairs of 2d  $\mathcal{N} = (2, 2)$  topological string theories—the A and B models—together with their categories of boundary conditions to predict surprising relations between the enumerative geometry of mirror pairs of Calabi-Yau threefolds. The physically motivated conjecture led to the development of homological mirror symmetry in mathematics [26]. Another prototypical example is the work of Witten in 1988 [27] that introduced topologically twisted 4d  $\mathcal{N} = 2$  theories. The partition functions of these theories realise Donaldson invariants of four manifolds [28] and, together with the Seiberg-Witten solution discussed previously in the introduction, this led to the Witten conjecture relating Donaldson invariants to Seiberg-Witten invariants and the



developments in four manifolds alluded to by Bott in the above quotation. The AGT correspondence [29] is a more recent example that directly exploits localisation methods to make contact with geometry. In its original form it relates the partition function of certain 4d  $\mathcal{N} = 2$  super Yang-Mills theories to the conformal blocks of an auxiliary 2d conformal field theory. The partition function can be interpreted geometrically in terms of instanton moduli space and the correspondence implies a conformal symmetry action on the equivariant homology of the moduli space. The antecedent to the result appears in the mathematical literature [30] where Nakajima constructs Heisenberg algebra actions on the equivariant homology of the Hilbert scheme of points in the plane. The AGT correspondence has been extended in a number of directions in physics and mathematics since and we refer the reader to [31] for a more comprehensive review. Many of the constructions and dualities discussed in this introduction have their origin in M-theory and arise as compactifications or dimensional reductions of systems of M5 and M2 branes. The worldvolume theory on M5 branes in particular, a six dimensional  $\mathcal{N} = (2, 0)$  superconformal field theory, lacks a semi-classical description and it is not even clear exactly what the fundamental degrees of freedom should be. Combined with the vast web of string dualities, it is this difficulty and ambiguity in formulating the theory that often leads to surprises in geometry.

In the spirit of physical mathematics, the broad aim of this thesis is to study three dimensional  $\mathcal{N} = 4$  gauge theories and understand more about the connections between their strongly coupled behaviour and geometry and representation theory. We now introduce the main aspects of these theories and review the relevant areas of geometry and representation theory—we defer technical details to the background chapter that follows.

**3d  $\mathcal{N} = 4$  gauge theory.** We study three dimensional super Yang-Mills theories preserving eight supercharges. The theories are specified by a UV Lagrangian which is determined by a choice of gauge group  $G$ , a matter representation  $\mathcal{R}$ , and gauge couplings  $g^2$ . We deform the theories by turning on masses and FI parameters. Dimension counting tells us that the theory is free in the UV and in many cases we find a strongly coupled conformal field theory in the IR.

The vacua of three dimensional theories were first studied in the work [32] and, as we discuss in more detail in the following chapter, the moduli space of vacua of 3d  $\mathcal{N} = 4$  theories can be described by two branches, the Higgs branch  $\mathcal{M}_H$  parameterised by hypermultiplet scalars and the Coulomb branch  $\mathcal{M}_C$  parameterised by vector multiplet scalars and monopole operators. Supersymmetry constrains these vacuum moduli spaces to be hyperkähler manifolds. In this thesis we focus on a particular class of 3d  $\mathcal{N} = 4$  theories known as quiver gauge theories—the Higgs branch of these theories coincides with a Nakajima quiver variety. The Higgs branch is protected from quantum corrections by non-renormalisation theorems and can be determined classically from the UV Lagrangian

whereas the Coulomb branch is intrinsically non-perturbative and has only recently been constructed to some degree of generality in the works [33, 34, 35]. In the examples we study in this work the Coulomb branch is also described by a Nakajima quiver variety.

As discussed previously, three dimensional  $\mathcal{N} = 4$  theories enjoy a powerful IR duality known as 3d mirror symmetry and the theories often come in mirror dual pairs that flow to the same theory in the IR with their Higgs and Coulomb branches exchanged. Higgs and Coulomb branch operators, denoted  $\mathcal{R}_H$  and  $\mathcal{R}_C$  respectively, form half BPS rings of protected operators which geometrically coincide with the coordinate rings of the Higgs and Coulomb branch. In this work we study the Hilbert series that counts these operators. The spectrum of 3d  $\mathcal{N} = 4$  theories also includes vortices. Vortices are half BPS particles created by monopole operators and the moduli space of vortices can be realised by certain moduli spaces of maps to the Higgs branch  $\mathcal{M}_H$ . In this way vortices connect the study of three dimensional gauge theory to the enumerative geometry of Nakajima quiver varieties.

**3d indices and factorisation.** Supersymmetric indices are useful tools to understand the non-perturbative behaviour of three dimensional gauge theories. It is often possible to compute indices by placing the theory supersymmetrically on a curved three manifold  $\mathcal{M}_3$  and localising the path integral. In many examples three manifold partition functions have been shown to factorise as a sum over Higgs vacua  $\alpha$  in terms of blocks  $H_\alpha$ . The blocks are defined as partition functions on manifolds topologically equivalent to a hemisphere  $S^1 \times H^2$  and the factorisation is of the form

$$\mathcal{Z}_{\mathcal{M}_3} = \sum_{\alpha} H_{\alpha} \bar{H}_{\alpha} . \quad (1.1)$$

In fact, many different three dimensional partition functions can be factorised in this way from the same set of fundamental blocks. When the three manifold is a fibration over  $S^2$ , the Higgs branch localisation scheme introduced in the works [36, 37] provides an interpretation of the factorisation as a localisation to vortices at the north and south poles of the  $S^2$ —this is analogous to Pestun’s factorisation of  $S^4$  partition functions discussed previously in the introduction.

Holomorphic blocks [38] are an elegant approach to understanding the factorisation. They are defined via decomposing the three manifold along a boundary torus and stretching the two halves of the geometry into cigars. The holomorphic blocks are then BPS indices on the cigar geometry and they are almost uniquely determined as solutions to  $q$ -difference equations arising from Ward identities. In chapter 4 we take an alternative approach and define hemisphere blocks with a UV boundary condition that realise the factorisation.

In this work we focus on the twisted indices  $S^2 \times_{A/B} S^1$  which are partition functions of the theory with a partial topological twist on  $S^2$ . In chapter 4 we describe a geometric interpretation of the factorisation of the twisted index as a gluing of vertex functions from

quasimap theory and, in a certain class of examples, we show that this gluing realises the  $\chi_t$  genus of a *global* quasimap space.

**Integrability.** Geometric representation theory studies algebraic objects (often quantum algebras such as Yangians or quantum affine groups that are the symmetry groups of integrable systems) by realising their representation theory in geometry. The first historical example of geometric representation theory is the Borel-Weil-Bott theorem [39] that realises spaces of sections of line bundles over flag varieties as modules of semi-simple Lie algebras.

The most direct connection between geometric representation theory and physics arises from the seminal work of Nakajima [40, 41, 42]. Nakajima shows that the representation theory of quantum affine algebras (or Yangians) associated to ADE quivers can be realised geometrically as actions on the equivariant  $K$ -theory (or equivariant homology) of Nakajima quiver varieties. Physically this is closely related to the gauge-Bethe correspondence of Nekrasov and Shatashvili [43, 44, 45] that relates the ground states of supersymmetric field theories with four supercharges in two or three dimensions (whose Higgs branches are Nakajima quiver varieties) to solutions to the Bethe equations associated with a Yangian or quantum group. The work of Maulik and Okounkov [46, 47] unifies the gauge-Bethe correspondence and geometric representation theory with their geometric  $R$ -matrix formalism. This work uses stable envelopes to directly construct  $R$ -matrices of quantum groups directly on the equivariant homology (or  $K$ -theory) of Nakajima quiver varieties—the Bethe equations appear in this context from counting vortices in the associated gauge theory. These constructions have been extended in many directions since [48, 49, 50]—the most relevant for us being the quantum  $K$ -theory construction of [51] that naturally applies to 3d  $\mathcal{N} = 4$  gauge theory.

**Boundary conditions and symplectic duality.** Symplectic duality is an emerging field in geometric representation theory that relates certain categories of modules associated to pairs of ‘dual’ symplectic resolutions [52, 53]. There is not yet a classification of such pairs but all known examples of symplectic duality arise from pairs of Higgs and Coulomb branches of 3d  $\mathcal{N} = 4$  gauge theories. The relevant theories are ‘good’ theories (in the classification of [54]) that have conventional IR fixed points and admit generic FI and mass parameters to fully resolve and lift the Higgs and Coulomb branches to isolated fixed points.

Physically, the relevant category of modules arises from specifying a boundary condition  $\mathcal{B}$  for the theory in the UV that flows to Lagrangian boundary conditions  $\mathcal{B}_C$  and  $\mathcal{B}_H$  in the Coulomb branch and Higgs branch respectively in the IR. Modules are then constructed from the action of the (quantised) bulk Higgs and Coulomb branch algebras  $\mathcal{R}_H$  and  $\mathcal{R}_C$  acting on operators that survive at the boundary—this physical realisation of symplectic duality is due to the work of Bullimore *et. al.* [55]. Often the Higgs and Coulomb

branches arise as the vacuum moduli spaces of mirror pairs of 3d theories and consequently symplectic duality is closely related to the study of 3d mirror symmetry.

## 1.1 Summary

We now outline the structure of the thesis and summarise the main results from each chapter.

**Quivers, integrability and Macdonald polynomials.** In chapter 3 we study representation theoretic aspects of Nakajima quiver varieties abstractly, mostly deferring physical applications to 3d  $\mathcal{N} = 4$  gauge theory to later in the thesis. In the spirit of geometric representation theory discussed above we prove that certain geometric invariants of a class of quivers, known as handsaw quivers, realise characters of an integrable spin chain. The result is summarised in theorem 10 which states:

**Theorem.** The normalised equivariant Euler characteristic of line bundles over the ‘toothless’ handsaw quiver  $\mathfrak{Q}(\mathbf{v}, \mathbf{w}_N)$  (the quiver diagram is shown in figure 3.9) realise characters of the following spin chain Hilbert space consisting of a tensor product of Yangian modules:

$$\mathcal{H}_{\vec{n}} = \bigotimes_{l=1}^k \mathrm{KR}(lw_1)^{n^{(l)}}. \quad (1.2)$$

In particular, writing  $\lambda = (n^{(1)}, \dots, n^{(k)})$  we have

$$\frac{\chi_{\mathrm{T}}(\mathcal{L}_1^{\zeta_1} \otimes \dots \otimes \mathcal{L}_N^{\zeta_N})}{\chi_{\mathrm{T}}(\mathcal{O}_{\mathfrak{Q}})} = Q'_{\lambda}(X; t), \quad (1.3)$$

where  $Q'_{\lambda}(X; t)$  is a Milne polynomial which coincides with the required character.

In this chapter we also introduce Macdonald polynomial methods to evaluate ‘ $q$ -deformed’ Hilbert series for quiver gauge theories. We use these methods to evaluate the Hilbert series of chainsaw quiver varieties and demonstrate that Macdonald polynomials are a convenient tool to compute large gauge rank limits of the Hilbert series. One of the main theorems of this chapter from which these results follow is theorem 9:

**Theorem.** The  $q$ -deformed Molien integral of the chainsaw quiver  $\mathscr{D}(\mathbf{v}, \mathbf{w})$  shown in figure 3.7 can be written as a sum of skew Macdonald polynomials<sup>4</sup> labelled by partitions

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<sup>4</sup>Our conventions for symmetric polynomials are summarised in appendix B.

$\{\nu^{(a)}, \sigma^{(a)}\}_{a=1, \dots, N}$ . We have

$$\mathcal{I}_{\{\mathbf{w}\}\{\mathbf{v}\}}(X; q, t) = \sum_{\{\nu^{(a)}, \sigma^{(a)}\}} \prod_{a=1}^N \frac{(q; q)_{\infty}^{\nu_a}}{(t; q)_{\infty}^{\nu_a}} \tilde{c}_{\nu_a}(\nu^{(a)}; q, t) P_{\nu^{(a)}/\sigma^{(a)}}\left(\frac{X^{(a)}}{1-t}; q, t\right) Q_{\nu^{(a)}/\sigma^{(a-1)}}\left(\frac{\bar{X}^{(a+1)}}{1-t}; q, t\right), \quad (1.4)$$

where  $X^{(a)}$  are sets of flavor fugacities and  $X^{(N+1)} = X^{(1)}$ .  $\bar{X}$  denotes inverse variables and  $\tilde{c}_{\nu}(\lambda; q, t)$  is the Macdonald integral normalisation constant defined in appendix B.

**Vortex geometry and hemisphere blocks.** In chapter 4 we study the  $A$ - and  $B$ -twisted indices of 3d  $\mathcal{N} = 4$  theories. The first part of the chapter is an expanded version of the publication

- S. Crew, N. Dorey and D. Zhang, *Factorisation of 3d  $\mathcal{N} = 4$  twisted indices and the geometry of vortex moduli space*, *JHEP* **08** (2020) 015.

Firstly, we argue that the  $A$ - and  $B$ -twisted indices essentially coincide with the Hilbert series of either the Coulomb or the Higgs branch respectively. We then introduce an angular momentum refinement that allows us to factorise the indices and interpret the Coulomb branch Hilbert series of a 3d  $\mathcal{N} = 4$  gauge theory in terms of the geometry of vortex moduli space. For theories in the class  $T_{\rho}[SU(N)]$ , we find a particularly compact expression

$$\text{H.S.}[\mathcal{M}_C] = P_{\mathcal{L}_H}(t) P_{\text{QM}}(\zeta; t) P_{\text{QM}}(\zeta^{-1}; t), \quad (1.5)$$

where  $P_{\mathcal{L}_H}(t)$  denotes the Poincaré polynomial of the compact core of the Higgs branch and  $P_{\text{QM}}(\zeta; t)$  is a generating function of Poincaré polynomials of moduli spaces of quasimaps into the Higgs branch.

The second part of the chapter is based on the work

- M. Bullimore, S. Crew and D. Zhang, *Boundaries, Vermas, and Factorisation*, 2010.09741.

We introduce hemisphere partition functions with exceptional Dirichlet boundary conditions and show that these partition functions precisely factorise the twisted index. In certain specialised limits, we demonstrate that the hemisphere partition functions realise characters of lowest weight Verma modules of Higgs and Coulomb branch chiral rings. We compute the partition function explicitly for the example of supersymmetric QED with  $N$  flavours (SQED[ $N$ ]).

Sections 4.3.3 and 4.3.4 are based on the author's currently unpublished work. Inspired by the SQED[ $N$ ] example, we propose a definition of a *hemisphere block* that can be constructed solely in terms of Higgs branch geometry for quiver gauge theories. The

definition of a hemisphere block in a vacuum  $\alpha$  is given by

$$H_\alpha := e^{\phi_\alpha} \text{PE} \left[ \frac{t-q}{1-q} N_\alpha^+ \right] V_\alpha. \quad (1.6)$$

Each factor in the block corresponds to classical, one-loop and vortex contributions and geometrically these contributions are each respectively determined by: line bundles over the Higgs branch; attracting directions<sup>5</sup> of the tangent bundle at the isolated vacuum  $\alpha$ ; and the vertex function of the quasimap moduli space based at  $\alpha$ . We then discuss applications to factorisation and show in particular that hemisphere blocks fuse *exactly* to the twisted index. In this chapter we also discuss the geometric implications of mirror symmetry properties of the hemisphere blocks.

**3d ADHM quiver gauge theory.** Chapter 5 is based on the work

- S. Crew, N. Dorey and D. Zhang, *Blocks and Vortices in the 3d ADHM Quiver Gauge Theory*, 2010.09732. (accepted to JHEP)

In this chapter we apply ideas from the previous parts of the thesis to a particularly rich non-abelian theory with adjoint matter that we refer to as *3d ADHM* (the quiver diagram can be found in figure 3.3). We focus mainly on a particular case where the Higgs branch of the theory coincides with the Hilbert scheme of points in the plane. The vortex partition functions of this theory can be expressed in terms of reverse plane partition fixed points of the relevant quasimap moduli space and we show that 3d mirror symmetry implies interesting combinatorial generating functions in this context. For example, in one particular limit, we use the self-mirror property of the theory to recover a familiar ‘Hook generating function’ of reverse plane partitions<sup>6</sup>

$$\widehat{H}_\lambda^B = \prod_{s \in \lambda} \frac{1}{1 - z^{h_\lambda(s)}} = \sum_{\pi \in \text{RPP}(\lambda)} z^{|\pi|} = \widehat{H}_\lambda^A. \quad (1.7)$$

We show that, in accordance with the general theory of chapter 4, the hemisphere blocks realise Verma module characters of the Coulomb branch chiral ring of the Hilbert scheme of points in the plane.

As an important check on the hemisphere block proposal, we also demonstrate that the hemisphere blocks of the 3d ADHM theory fuse exactly to the twisted index and recover Nekrasov’s partition function [19]. In this example we also find an interesting connection

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<sup>5</sup>The definition of the block also depends on a choices of chamber for masses and FI parameters. The attracting directions are defined with respect to this chamber choice.

<sup>6</sup>Appendices A and B summarise our conventions for partitions and combinatorics.

between the blocks and the topological vertex:

$$\begin{aligned}\mathcal{Z}_{\mathcal{O}(-1)\oplus\mathcal{O}(-1)\rightarrow\mathbb{P}^1} &= \sum_{\lambda} \Lambda^{|\lambda|} C_{\emptyset\emptyset\lambda}(\mathbf{t}, \mathbf{q}) C_{\emptyset\emptyset\lambda^\vee}(\mathbf{q}, \mathbf{t}) \\ &= \sum_{\lambda} \Lambda^{|\lambda|} \left[ H_{\lambda}^B(z, \zeta; q, t) \right] \left[ H_{\lambda}^B(z, \zeta; q^{-1}, t) \right] = I^B, \end{aligned} \tag{1.8}$$

where the first line is a gluing of topological vertices to form the conifold amplitude and the second line is a gluing of hemisphere blocks to compute the twisted index.

Finally, we discuss Neumann boundary conditions and compute the half indices for the 3d ADHM theory in the presence of a line operator—here we make contact with the abstract results of chapter 3 and find that the operator count exhibits an integrable structure. In particular we show that

$$\hat{\mathcal{I}}_{\mathcal{W}_{\mathfrak{n}}}^B(\mathcal{N}) = \frac{1}{(z; z)_N} Q'_{(\mathfrak{n}^N)}(x_1, \dots, x_{N_f}; z), \tag{1.9}$$

where the left hand side denotes the half index in the presence of a Wilson line and the right hand side is a Milne polynomial which, as we show in chapter 3, is the character of a module of the Yangian  $Y(\mathfrak{sl}_{N_f})$ .

We conclude the thesis with a discussion of further directions relating the enumerative geometry of the Hilbert scheme of points to AdS<sub>4</sub> black hole entropy.





## CHAPTER 2

### BACKGROUND

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In this chapter we review the background material on 3d  $\mathcal{N} = 4$  supersymmetric field theories that we use throughout the thesis.

**Overview.** We begin with a review of three dimensional  $\mathcal{N} = 4$  supersymmetric quantum field theory. We review symmetries of the theory, construct UV Lagrangians and discuss various half BPS objects. We then move onto the geometry of the vacuum moduli spaces of 3d  $\mathcal{N} = 4$  theories. We review the definition of Nakajima quiver varieties and discuss their basic geometric properties. We conclude this chapter with an extended example: supersymmetric QED with  $N$  flavours.

### 2.1 3d $\mathcal{N} = 4$ supersymmetric gauge theories

In this section we review basic aspects of the field theories studied throughout this work. We write  $\mathcal{T}$  for a three dimensional super Yang-Mills theory with eight supercharges— $\mathcal{N} = 4$  supersymmetry in three dimensions. Our supersymmetry conventions follow [56, 57].

**Supersymmetry algebra.** We work in three dimensional Euclidean space with Lorentz group  $\text{Spin}(3)_E \cong SU(2)_E$ . The theories we study preserve 8 supercharges  $Q_\alpha^{a\dot{a}}$  where  $\alpha$  is an  $SU(2)_E$  index and  $a$  and  $\dot{a}$  indices for an  $SU(2)_H \times SU(2)_C$   $R$ -symmetry. The supersymmetry algebra satisfied by the supercharges is

$$\{Q_\alpha^{a\dot{a}}, Q_\beta^{b\dot{b}}\} = -\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma_{\alpha\beta}^\mu P_\mu + 2\epsilon_{\alpha\beta}(\epsilon^{ab}Z^{\dot{a}\dot{b}} + \epsilon^{\dot{a}\dot{b}}Z^{ab}), \quad (2.1)$$

where  $(\sigma^\mu)^\alpha_\beta$  are the standard Pauli matrices and spinor indices are raised and lowered with  $\epsilon^{ab}$  with  $\epsilon^{12} = \epsilon_{21} = 1$ . The  $P_\mu$  generate spacetime translations and, as we discuss in more detail momentarily, the central charges act by various combinations of gauge and flavour transformations.

### 2.1.1 UV Lagrangians

We now discuss the construction of UV Lagrangian field theories that realise the supersymmetry algebra (2.1). These field theories were first studied from the perspective of dimensional reduction from  $\mathcal{N} = 2$  theories in four dimensions in the work [32] and realised as worldvolume theories of D3-D5-NS5 brane intersections in type IIB string theory in [58]. A useful review of 3d  $\mathcal{N} = 2$  theories can be found in [59].

A 3d  $\mathcal{N} = 4$  theory  $\mathcal{T}$  is specified by a choice of compact gauge group  $G$  and a quaternionic<sup>1</sup> representation  $\mathcal{R}$  of  $G$ . For us, the quaternionic representation will always be of the form  $\mathcal{R} = R \oplus \bar{R}$  with  $R$  a unitary representation of  $G$ .

Together with dimensionful couplings  $g^2$  for each gauge group factor, the theory is then uniquely specified by familiar kinetic terms and couplings for  $\mathcal{N} = 4$  vector multiplets and hypermultiplets (arising from dimensional reduction of the corresponding  $\mathcal{N} = 2$  multiplets in four dimensions, see *e.g.* [60]).

**Symmetries.** We consider  $\mathcal{N} = 4$  vector multiplets associated to the gauge group  $G$  and  $\mathcal{N} = 4$  hypermultiplets transforming in  $\mathcal{R}$ . We denote the bosonic components of the vector multiplet by a triplet of scalars  $\vec{\phi}$  living in the Lie algebra  $\mathfrak{g}^3$  of the gauge group  $G$  together with the gauge connection  $A_\mu$ . Denoting by  $N$  the quaternionic dimension of the matter representation so that  $\mathcal{R} \cong \mathbb{H}^N$ , the bosonic components of the matter are  $4N$  real scalars in  $\mathbb{R}^{4N}$  with the canonical hyperkähler structure. The  $R$ -symmetry  $SU(2)_C$  acts by rotating the triplet of vector multiplet scalars  $\vec{\phi}$  and  $SU(2)_H$  acts by rotating the hypermultiplet scalars as doublets  $(X, Y)$  under a decomposition  $\mathcal{R} = R \oplus \bar{R}$ .

The theory  $\mathcal{T}$  has a global symmetry group, denoted  $G_H \times G_C$ , that commutes with the supercharges and  $R$ -symmetry.  $G$  acts as a subgroup of the hyperkähler isometry group  $USp(N) = U(2N) \cap Sp(2N, \mathbb{C})$  of  $\mathcal{R}$ , denoting this embedding by  $\mathcal{R}(G)$ ,  $G_H$  acts on the hypermultiplet scalars as

$$G_H = N_{USp(N)}(\mathcal{R}(G)) / \mathcal{R}(G), \quad (2.2)$$

where  $N_{USp(N)}$  denotes the normaliser inside  $USp(N)$ . We call  $G_H$  the *flavour symmetry* group.

In three dimensions, every abelian factor  $A_{U(1)}$  of the gauge group has a conserved current  $J = *dA_{U(1)}$  for which we can introduce a dual photon  $\gamma$  with  $J = d\gamma$  and  $\gamma$  periodic with  $\gamma \sim \gamma + 2\pi g^2$ . Hence for each abelian factor of  $G$  we have a  $U(1)$  global symmetry that acts by shifting the dual photon. This defines the *topological symmetry*  $G_C$  by

$$G_C = \text{Hom}(\pi_1(G), U(1)). \quad (2.3)$$

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<sup>1</sup> $G$  acts as a subgroup of  $USp(2N)$  preserving the hyperkähler structure of  $\mathcal{R}$ .

The topological symmetry is often enhanced to a non-abelian symmetry in the IR [61], we see explicit examples of this later. In the theories we study in this work, the gauge group will be a product of unitary groups  $G = \prod_{i \in I} U(V_i)$  so that  $\pi_1(G) = \mathbb{Z}^{|I|}$  and the topological symmetry is then classically  $G_C = U(1)^{|I|}$ .

**Lagrangians.** We now fix an  $\mathcal{N} = 2$  subalgebra of the supersymmetry algebra and decompose our fields into  $\mathcal{N} = 2$  multiplets. This corresponds to choosing an element of  $\mathbb{P}^1 \times \mathbb{P}^1$  parametrising embeddings  $U(1)_H \times U(1)_C \subset SU(2)_H \times SU(2)_C$ . We denote the corresponding  $U(1)$   $R$ -charges by  $R_H$  and  $R_C$ .

The on-shell fields of a  $\mathcal{N} = 4$  hypermultiplet transforming in  $\mathcal{R}$  are given by a pair of 3d  $\mathcal{N} = 2$  chiral multiplets  $\Phi_X = (X, \psi^X)$  and  $\Phi_Y = (Y, \psi^Y)$ . The  $\mathcal{N} = 4$  vector multiplet consists of a 3d  $\mathcal{N} = 2$  vector multiplet denoted  $\mathcal{V} = (A_\mu, \sigma, \lambda)$  and a 3d  $\mathcal{N} = 2$  chiral multiplet in the adjoint of  $G$  denoted  $\Phi = (\varphi, \eta)$ . The  $R$ -charges for the hypermultiplet fields are

$$\begin{array}{c|cccccc} & X & Y & \psi_+^X & \psi_+^Y & \bar{\psi}_-^X & \bar{\psi}_-^Y \\ \hline R_C & 0 & 0 & -1 & -1 & +1 & +1 \\ R_H & 1 & 1 & 0 & 0 & 0 & 0 \end{array} \quad (2.4)$$

and for the vector multiplet we have

$$\begin{array}{c|ccccccc} & A_\mu & \sigma & \varphi & \lambda_\pm & \bar{\lambda}_\pm & \eta_\pm & \bar{\eta}_\pm \\ \hline R_C & 0 & 0 & 2 & 1 & -1 & 1 & -1 \\ R_H & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{array} \quad (2.5)$$

The kinetic terms for the vector multiplet can be expressed in terms of a linear multiplet<sup>2</sup>  $\Sigma$ , containing the field strength, and the  $\mathcal{N} = 2$  chiral contributions in the vector multiplet. We have

$$\mathcal{L}_{\mathcal{N}=4 \text{ V.M.}} = \frac{1}{g^2} \int d^3x d^4\theta \operatorname{tr} \Sigma^2 + \frac{1}{g^2} \int d^3x d^4\theta \left( \Phi^\dagger e^{-2\mathcal{V}} \Phi e^{2\mathcal{V}} \right). \quad (2.6)$$

The kinetic and  $\mathcal{N} = 4$  superpotential terms for the hypermultiplets are given by

$$\mathcal{L}_{\mathcal{N}=4 \text{ H.M.}} = \int d^3x d^4\theta \left( \Phi_X^\dagger e^{2\mathcal{V}} \Phi_X + \Phi_Y^\dagger e^{-2\mathcal{V}} \Phi_Y \right) - \sqrt{2}i \int d^3x d^2\theta \left( \Phi_Y \Phi \Phi_X + \text{h.c.} \right). \quad (2.7)$$

**Mass and FI deformations.** The theories can be deformed by adding mass and FI parameters. These enter the Lagrangian in the usual way [62] by coupling to non-dynamical background multiplets for global symmetries. We first consider coupling to a background vector multiplet for the  $G_H$  flavour symmetry. Setting the fermion variations to zero allows a triplet (transforming under  $SU(2)_H$ ) of parameters  $\vec{m} \in \mathfrak{t}_H^3$  in the scalar components

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<sup>2</sup>See e.g. the review [59] for more details on 3d  $\mathcal{N} = 2$  multiplets.

and in a fixed complex structure, as we see in (2.7), the deformations enter as real and complex mass parameters  $(m_{\mathbb{R}}, m_{\mathbb{C}}) \in \mathfrak{t} \oplus \mathfrak{t}_{\mathbb{C}}$  where  $\mathfrak{t}$  denotes the Cartan subalgebra of  $G_H$ .

Similarly we can couple the theory to a twisted vector multiplet for the topological symmetry  $G_C$  and in a fixed subalgebra/complex structure we have an  $SU(2)_C$  triplet  $\vec{\xi} \in \mathfrak{g}_C^3$ . The parameters enter the Lagrangian<sup>3</sup> as real and complex FI parameters  $(\xi_{\mathbb{R}}, \xi_{\mathbb{C}}) \in \mathfrak{t} \oplus \mathfrak{t}_{\mathbb{C}}$  where  $\mathfrak{t}$  denotes the Cartan subalgebra of  $G_C$ .

Masses and FI parameters realise the central charges in the supersymmetry algebra (2.1) by

$$Z^{11} \sim \xi_{\mathbb{C}}, \quad Z^{12} \sim i\xi_{\mathbb{R}}, \quad Z^{i1} \sim m_{\mathbb{C}}, \quad Z^{i2} \sim im_{\mathbb{R}}. \quad (2.8)$$

In a fixed  $\mathcal{N} = 2$  subalgebra the diagonal combination of  $U(1)_H \times U(1)_C$  becomes an  $R$ -symmetry whilst the anti-diagonal combination  $\frac{1}{2}(R_H - R_C)$  is a flavour symmetry—we denote this symmetry by  $U(1)_t$ . The theory can be coupled to a background vector multiplet for  $U(1)_t$  with vacuum expectation  $\tau$  for the real scalar and, reading off (2.4) and (2.5) we see this contributes a real mass  $\frac{1}{2}\tau$  to the  $\mathcal{N} = 4$  hypermultiplet and contributes  $-\tau$  to the mass of the  $\mathcal{N} = 2$  chiral in the vector multiplet. This deformation ‘softly’ breaks  $\mathcal{N} = 4$  supersymmetry to  $\mathcal{N} = 2$  and this is often referred to as  $\mathcal{N} = 2^*$  supersymmetry [64].

### 2.1.2 Moduli space of vacua

Three dimensional  $\mathcal{N} = 4$  theories have flat directions in their classical potential that generically lead to a non-compact quantum moduli space of vacua  $\mathcal{M}$ . In the IR the theory flows to a sigma model to the vacuum moduli space and  $\mathcal{N} = 4$  supersymmetry constrains the target to be hyperkähler [65]. In a non-compact three dimensional spacetime a choice of vacuum state is required to define the theory—it is a ‘superselection sector’ [66].

We first discuss the classical vacua of (2.7) and (2.6). Including mass and FI deformations, minimising the scalar potential gives a sum of positive-definite contributions that must all be set to zero in a vacuum. In a fixed  $\mathcal{N} = 2$  subalgebra we find  $[\varphi, \varphi^\dagger] = 0$  and  $[\varphi, \sigma] = 0$  together with

$$\begin{aligned} (\sigma + m_{\mathbb{R}} + \tau) \cdot (X, Y) &= 0, \\ (\varphi + m_{\mathbb{C}}) \cdot (X, Y) &= 0, \\ \mu_{\mathbb{R}} + \xi_{\mathbb{R}} &= 0, \\ \mu_{\mathbb{C}} + \xi_{\mathbb{C}} &= 0. \end{aligned} \quad (2.9)$$

where  $\mu_{\mathbb{R}}$  and  $\mu_{\mathbb{C}}$  realise the real and complex moment maps for the tri-Hamiltonian  $G$  action on  $\mathcal{R}$  with respect to the canonical hyperkähler structure on  $\mathcal{R} \cong \mathbb{H}^N$ . In the above

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<sup>3</sup>See for example [63].

we have used the  $\cdot$  notation to indicate that the vector multiplet scalars and the mass parameters act as infinitesimal gauge and flavour transformations on the hypermultiplet scalars in the appropriate representation. Explicitly the moment maps are given by

$$\mu_{\mathbb{C}} = YTX, \quad \mu_{\mathbb{R}} = X^{\dagger}TX - Y^{\dagger}TY. \quad (2.10)$$

where  $T$  are anti-Hermitian generators of  $\mathfrak{g}$ .

In general the vacuum moduli space consists of mixed branches along which both hypermultiplet and vector multiplet scalars can obtain vevs. In this work we consider the distinct Higgs  $\mathcal{M}_H$  and Coulomb  $\mathcal{M}_C$  branches where vectormultiplet and hypermultiplet scalars separately obtain vevs. These branches are so-named because at a generic point on the Higgs branch the vector multiplet obtains mass via the Higgs mechanism and on the Coulomb branch the gauge group is broken to the maximal torus of the gauge group and generically abelian gauge fields remain in the IR.

**Higgs branch.** When the mass parameters are set to zero the Higgs branch is parametrised by vevs for the hypermultiplet scalars and vanishing vevs for the vector multiplet scalars  $\sigma = \varphi = 0$ . In that case (2.9) is a description of the Higgs branch as a hyperkähler quotient of  $\mathcal{R}$  by the action of  $G$

$$\mathcal{M}_H = \{(X, Y) : \mu_{\mathbb{R}} + \xi_{\mathbb{R}} = 0, \mu_{\mathbb{C}} + \xi_{\mathbb{C}} = 0\}/G. \quad (2.11)$$

The Higgs branch is then a hyperkähler manifold of complex dimension  $\dim \mathcal{M}_H = 2N - 2 \dim \mathcal{R}(G)$ . The moduli space (2.11) is expressed in a fixed  $\mathcal{N} = 2$  subalgebra which we see corresponds to a choice of complex structure on  $\mathcal{M}_H$ . In this fixed complex structure the action of  $G_H$  on the hypermultiplet scalars descends to a Hamiltonian action on  $\mathcal{M}_H$ . Turning on infinitesimal generators  $m_{\mathbb{R}}$  the equations (2.9) show that mass parameters lift the Higgs branch to the fixed points of the one-parameter subgroup in  $G_H$  specified by the  $m_{\mathbb{R}}$ . It is an assumption throughout the work that it is possible to turn on ‘generic’<sup>4</sup> mass parameters such that the theory has isolated massive vacua under the isometry generated by  $m_{\mathbb{R}}$  which coincide with the fixed points  $\mathcal{M}_H^{G_H \times U(1)_t}$ —we denote these isolated vacua by  $\alpha$ .

The non-renormalisation theorems of [67, 59] ensure that this classical description of the Higgs vacua is preserved in the quantum theory. The non-renormalisation theorem can essentially be summarised by noting that the gauge coupling constants  $g$  are real parameters with no natural complexification and can therefore only be promoted to background superfields that cannot correct the effective potential.

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<sup>4</sup>We define this more carefully later in subsection 3.1.4.

**Coulomb branch.** The Coulomb branch is described, in the case of vanishing FI parameters, by  $X = Y = 0$  and  $(\sigma, \varphi)$  obtaining vevs. In this case the gauge group is broken to a maximal torus  $T \subset G$  and after dualising the photons to periodic scalars  $\gamma$  we have

$$\mathcal{M}_C^{\text{class.}} = \left( \mathbb{R}^3 \times S^1 \right)^{\text{rk} G} / \text{Weyl}(G). \quad (2.12)$$

This is a complex manifold with  $\dim \mathcal{M}_C = 2 \text{rk} G$ . The superscript here indicates that the classical description of the Coulomb branch does receive quantum corrections, however supersymmetry ensures the quantum Coulomb branch is not lifted and generically we expect an IR description of the theory on the Coulomb branch as a non-linear sigma model to a hyperkähler cone target [65]. We discuss the Coulomb branch in more detail in the following subsections.

### 2.1.3 Chiral rings

The spectrum of 3d  $\mathcal{N} = 4$  theories includes certain protected operators that form commutative operator algebras known as chiral rings [68]. In this subsection we review the construction of these chiral rings.

We first define the following two Higgs and Coulomb supercharges<sup>5</sup>

$$Q_H = Q_+^{11} + Q_-^{12}, \quad Q_C = Q_+^{11} + Q_-^{21}. \quad (2.13)$$

These supercharges are nilpotent<sup>6</sup> and in the topologically twisted context they are scalars under an improved Lorentz group defined as the diagonal in  $SU(2)_E \times SU(2)_{H/C}$  with  $Q_H$  and  $Q_C$  defining the topological Rozansky-Witten model and its mirror [69, 70]. The chiral rings are defined to be the cohomology rings of  $Q_H$  and  $Q_C$  which we denote by  $\mathcal{R}_H$  and  $\mathcal{R}_C$  respectively. Much of the discussion to follow is the same for either the Higgs or Coulomb supercharge so we denote either by  $Q$ . We now show that operators in the chiral rings are independent of position. Define the supercharges

$$Q_\mu^H = \frac{1}{2} (\sigma^\mu)_a^\alpha Q_\alpha^{2\dot{a}}, \quad Q_\mu^C = \frac{1}{2} (\sigma^\mu)_a^\alpha Q_\alpha^{a\dot{2}}, \quad (2.14)$$

we denote either of these vector supercharges by  $Q_\mu$  and from the supersymmetry algebra (2.1) it follows that in either case we have  $\{Q, Q_\mu\} = P_\mu$ . We then deduce for an operator

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<sup>5</sup>We make an explicit choice of subalgebras—in general one could consider rotations of these supercharges parametrised by  $\mathbb{P}^1 \times \mathbb{P}^1$  choices of complex structure on  $\mathcal{M}_H$  and  $\mathcal{M}_C$ .

<sup>6</sup>Up to gauge transformations and symmetries generated by masses and FI parameters respectively.

$\mathcal{O}$  in  $Q$ -cohomology that

$$\begin{aligned}
P_\mu \cdot \mathcal{O} &= [P_\mu, \mathcal{O}] = [\{Q, Q_\mu\}, \mathcal{O}], \\
&= \{[Q, \mathcal{O}], Q_\mu\} + \{Q, [Q_\mu, \mathcal{O}]\}, \\
&= 0,
\end{aligned} \tag{2.15}$$

where the first term vanishes since  $\mathcal{O}$  is  $Q$ -exact and the second term is  $Q$ -exact, therefore the operator  $\mathcal{O}$  is independent of position. Since the operators are position independent, we can move them far away from each other and apply the cluster decomposition principle [71] to define a product structure on operators  $\mathcal{O}_1$  and  $\mathcal{O}_2$  in the cohomology by  $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle$ . Furthermore, in three dimensional flat space there is no notion of operator ordering and hence the chiral rings are commutative. Later, in chapter 4, we meet the  $\Omega$ -deformation that forces the operators in the cohomology to lie on a line and introduces a notion of operator ordering that quantises the chiral rings. We assume that  $Q_H$  and  $Q_C$  cohomology contains only bosonic operators in the examples that we consider.

**Higgs and Coulomb operators.** Throughout this work we meet several BPS objects of 3d  $\mathcal{N} = 4$  theories annihilated by some fraction of the supercharges  $Q_\alpha^{a\dot{a}}$ . The chiral rings  $\mathcal{R}_H$  and  $\mathcal{R}_C$  are the first of these; they consist of half-BPS Higgs and Coulomb branch operators annihilated by the following supercharges:

	$Q_-^{1\dot{1}}$	$Q_-^{1\dot{2}}$	$Q_-^{2\dot{1}}$	$Q_+^{1\dot{1}}$	$Q_+^{1\dot{2}}$	$Q_+^{2\dot{1}}$
$\mathcal{R}_H$	•	•		•	•	
$\mathcal{R}_C$	•		•	•		•

(2.16)

It is an assumption throughout this work that the theories we study are ‘good’ or ‘ugly’ in the classification of [54]. Such theories flow to a 3d  $\mathcal{N} = 4$  superconformal field theory in the IR with the UV  $R$ -symmetry matching the IR  $R$ -symmetry. In this case we have an  $\mathfrak{osp}(4|4, \mathbb{R})$  superconformal symmetry at our disposal with additional superconformal generators  $S$  providing the Hermitian conjugates  $S = Q^\dagger$  in radial quantisation. We refer the reader to [57] for further details of the superconformal algebra. In this quantisation, local operators inserted at the origin correspond to states on a surrounding  $S^2$ —in chapter 4 we extend this construction to a putative state-operator map for the theory with a boundary.

In the work [57], Dolan classifies the unitary representations of  $\mathfrak{osp}(4|4, \mathbb{R})$ —we closely follow the presentation of [38]. We consider  $Q_H$  cohomology for definiteness, but the arguments below follow for  $Q_C$  identically by appropriately exchanging  $H$  and  $C$ . Firstly, note that states in  $Q_H$ -cohomology manifestly preserve  $Q_+^{1\dot{1}}$  and  $Q_-^{1\dot{2}}$ , and for a unitary

representation we further have the bound

$$\{Q_H, Q_H^\dagger\} = \frac{1}{2}(D - \frac{R_H}{2}) \geq 0, \quad (2.17)$$

where  $D$  denotes the charge under dilations. The classification of  $\mathfrak{osp}(4|4, \mathbb{R})$  multiplets implies that operators saturating this bound are superconformal primaries annihilated by all four supercharges in the first row of (2.16). Furthermore, the unitarity bounds associated to  $Q_+^{11}$  and  $Q_-^{12}$  ensure that  $J + \frac{R_C}{2} = 0$  for these operators.

In fact, the chiral ring  $\mathcal{R}_H$  is generated by local operators in the UV Lagrangian. In particular, solutions to the BPS equations for the supercharges in the first row of (2.16) are solved by gauge invariant fields in bottom components of chiral superfields [72] that satisfy the complex moment map constraint—that is  $\mathcal{R}_H$  coincides with the coordinate ring of the Higgs branch (2.11).

The generators of the Coulomb branch chiral ring  $\mathcal{R}_C$  can also be realised at short distances by scalars in the vector multiplet and certain disorder operators known as monopoles [73, 74, 54]. However the operator product in  $\mathcal{R}_C$  receives quantum corrections—we discuss  $\mathcal{R}_C$  in more detail in the following subsection.

**Chiral rings and holomorphic functions.** In three dimensional flat spacetime the definition of the theory requires specification of a Higgs or Coulomb vacuum state  $|\Omega\rangle$ . Associated to each element of either the Higgs or the Coulomb chiral ring we can define a holomorphic function  $f : \mathcal{M}_H \rightarrow \mathbb{C}$  by

$$f_{\mathcal{O}} : |\Omega\rangle \rightarrow \langle \mathcal{O} \rangle_{|\Omega\rangle}. \quad (2.18)$$

It is a basic assumption of 3d  $\mathcal{N} = 4$  theories that this map is surjective. The chiral rings  $\mathcal{R}_H$  and  $\mathcal{R}_C$  are then identified with the coordinate rings of the Higgs and Coulomb branches denoted  $\mathbb{C}[\mathcal{M}_H]$  and  $\mathbb{C}[\mathcal{M}_C]$  respectively. This assumption becomes the definition of the Coulomb branch  $\mathcal{M}_C$  in the mathematical construction of the Coulomb branch due to Nakajima *et. al.* [33, 34].

**Hilbert series.** The Hilbert series [75, 76, 77] is a count of  $Q_H$ - or  $Q_C$ -cohomology graded by  $R$ -charge and further refined by flavour symmetries  $G_H$  or  $G_C$ . More precisely, the Hilbert series are defined by

$$\text{H.S.}[\mathcal{M}_H] := \text{Tr}_{\mathcal{R}_H} x^{T_H} t^{R_H}, \quad \text{H.S.}[\mathcal{M}_C] := \text{Tr}_{\mathcal{R}_C} \zeta^{T_C} t^{R_C}. \quad (2.19)$$

From the discussion above, the chiral rings coincide with holomorphic functions on the Higgs and Coulomb branches, and under certain conditions, the Hilbert series defined



above coincides with the Hilbert series of an affine variety with a torus action defined as a torus character of the coordinate ring. In the following chapter, under this correspondence, we review how to compute the Hilbert series using localisation methods in geometry.

In chapter 4 we compute twisted indices of 3d  $\mathcal{N} = 4$  theories that can be realised as path integrals on  $S^2 \times S^1$ . We find that these indices receive contributions only from states in either  $\mathcal{R}_H$  or  $\mathcal{R}_C$  and agree with the Hilbert series (2.19). This perspective allows us to relate the Higgs and the Coulomb branch Hilbert series to vortex geometry.

### 2.1.4 Coulomb branch and 3d mirror symmetry

We now discuss the non-perturbative chiral ring  $\mathcal{R}_C$  of the Coulomb branch. The chiral ring determines the complex geometry of the Coulomb branch via

$$\mathcal{M}_C := \text{Spec } \mathbb{C}[\mathcal{M}_C] = \text{Spec } \mathcal{R}_C. \quad (2.20)$$

Classically, the Coulomb branch  $\mathcal{M}_C$  is generated by vector multiplet scalars and the dual photons. This description receives quantum corrections and in addition to  $X = Y = 0$ , the time-independent half-BPS equations for the supercharges in the second row of (2.16) include  $F = \star D\sigma$  and  $D \star \sigma = 0$ . These equations admit 't Hooft monopole solutions [73] which are local disorder operators defined by singular gauge field configurations. Specifically, excising a point  $p \in \mathbb{R}^3$  and writing local spherical coordinates  $(r, \theta, \varphi)$  around  $p$ , then the monopole is determined by a cocharacter  $\mathfrak{m} \in \text{Hom}(U(1), T) = \Lambda_{\mathfrak{h}}$  modulo the Weyl group  $W$  and given by the field configuration

$$A_{\pm} \sim \frac{\mathfrak{m}}{2}(\pm 1 - \cos(\theta))d\varphi, \quad \sigma \sim \frac{\mathfrak{m}}{2r}. \quad (2.21)$$

The BPS equations further admit monopole configurations that are ‘dressed’ by constant configurations for the complex scalar  $\varphi$  compatible with the flux  $\mathfrak{m} \in \Lambda_{\mathfrak{h}}/W$ . It is an assumption, that has been tested in many examples [75, 76, 77, 78, 79], that these *dressed monopole operators* generate the chiral ring  $\mathcal{R}_C$ . Furthermore, the ring relations and full hyperkähler structure have more recently been constructed using abelianisation methods in the work [35], leading to a full non-perturbative definition of the Coulomb branch geometry. The complex geometry of the Coulomb branch has also been rigorously formulated in the mathematical literature in the work of Nakajima *et. al.* [34, 33]. A physical perspective on this constructions is provided in [80], where the Coulomb branch algebra is re-constructed from its action on the cohomology of the vortex moduli space of the theory. In chapter 4 we study a related physical setup and study the action of the Coulomb branch algebra on boundary conditions of 3d  $\mathcal{N} = 4$  theories.

**Coulomb branch Hilbert series.** Classically, monopole operators have a charge  $J(\mathbf{m})$  under the abelian part of the topological symmetry  $G_C$ . In the IR they can acquire non-trivial quantum numbers and the topological symmetry is typically enhanced to a non-abelian symmetry [63]. In particular, the  $R_C$ -charge is given by [54]

$$\Delta(\mathbf{m}) = - \sum_{\alpha \in R_+} |\alpha(\mathbf{m})| + \frac{1}{2} \sum_{\rho \in \mathcal{R}} |\rho(\mathbf{m})|, \quad (2.22)$$

where  $\rho$  are the weights of the matter representation  $\mathcal{R}$  and  $R_+$  denotes the positive roots of  $\mathfrak{g}$ . The ‘good’ or ‘ugly’ assumption (which implies  $\Delta \geq 1/2$ ) is important here since it ensures the unitarity bound is not violated and this formula gives the correct infrared  $R$ -charge of the monopole operator.

The work [77] proposes the following expression for the Coulomb branch Hilbert series (2.19)

$$\text{H.S}[\mathcal{M}_C] = \sum_{\mathbf{m} \in \Lambda_{\mathfrak{h}}/W} \zeta^{J(\mathbf{m})} t^{\Delta(\mathbf{m})} P_G(t, \mathbf{m}). \quad (2.23)$$

where  $P_G(t, \mathbf{m})$  is the contribution of complex scalars dressing the ‘bare’ monopoles and is given by

$$P_G(t, \mathbf{m}) = \prod_{i=1}^r \frac{1}{1 - t^{d_i(\mathbf{m})}}. \quad (2.24)$$

where  $d_i(\mathbf{m})$  with  $i = 1, \dots, r$  are the degrees of the Casimir invariants of the residual gauge group unbroken by  $\mathbf{m}$ .

**3d mirror symmetry.** 3d  $\mathcal{N} = 4$  theories enjoy a powerful duality known as 3d mirror symmetry [63, 81]. A zeroth order version of this duality is as an automorphism of the supersymmetry algebra that exchanges  $SU(2)_H$  with  $SU(2)_C$  and  $G_H$  with  $G_C$ . 3d mirror symmetry is therefore expected to exchange strongly coupled Coulomb branch physics with the classical Higgs branch. The symmetry exchanges masses and FI parameters  $m \leftrightarrow \xi$  and, in a fixed subalgebra, the  $R_H$  and  $R_C$  charges so that  $t \leftrightarrow t^{-1}$ .

3d mirror symmetry is an infrared duality between pairs  $(\mathcal{T}, \mathcal{T}')$  of 3d  $\mathcal{N} = 4$  theories and observables and operators should, in principle, be mapped across the duality. The duality can be motivated for a wide class of theories by a construction in type IIB string theory [17]—in that context the duality arises as  $S$ -duality of the D3, D5 and NS5 system. Since the original conjecture, much progress has been made with physical checks—these are too numerous to comprehensively list here but we note a small sample: In the work of [82] the spectrum of Higgs and Coulomb operators are matched precisely in a large flavour limit; the work of [83] provides a non-trivial check that the IR metric on the Higgs and Coulomb branches of mirror dual theories match; in the work [84] the three sphere partition function of various mirror dual theories are shown to be equal and Dedushenko

*et. al.* [85] check correlation functions and operator products of Coulomb branch operators in the one dimensional topological sector.<sup>7</sup>

3d mirror symmetry is closely related to the notion of *symplectic duality* in geometry [52, 53]. Symplectic duality relates pairs of symplectic resolutions  $X$  and  $X^\dagger$  that can often be realised as the Higgs branch and Coulomb branch of a theory  $\mathcal{T}$ . In particular, symplectic duality relates the categories of modules associated to the quantised coordinate rings (chiral rings  $\mathcal{R}_H$  and  $\mathcal{R}_C$ ) of the resolutions  $X$  and  $X^\dagger$  in a precise way [52]. Physically, this correspondence can be understood as relating dual pairs of boundary conditions for mirror dual theories placed on a spacetime with boundary [55]. Symplectic duality now largely evolves in parallel with the study of 3d mirror symmetry in physics.

The enumerative geometry of pairs  $X$  and  $X^\dagger$  is also mirror dual in a way made precise in the recent works [86, 87, 88, 89, 90, 91]. Specifically, vertex functions<sup>8</sup> of mirror dual theories can be related via the *elliptic stable envelope* [92]. In chapter 4 of the present work, we use these techniques from enumerative geometry and curve counting to interpret the ‘gluing’ of twisted indices in physics, and consequently the Hilbert series (2.19), in terms of vortex geometry.

### 2.1.5 Vortices

The final BPS object that we consider in this background chapter are half-BPS vortex solutions preserving the supercharges  $Q_-^{11}$ ,  $Q_-^{12}$ ,  $Q_+^{21}$  and  $Q_+^{22}$ . We follow closely the presentation of [80]. Vortices are time-independent BPS solitons localised in the  $x^{1,2}$ -plane, we seek a complex algebraic description of the moduli space and introduce complex coordinates  $z = x^1 + ix^2$  and  $\bar{z} = x^1 - ix^2$ . Turning off masses, the BPS equations for the vector and chiral multiplets yield

$$D_{\bar{z}}X = D_{\bar{z}}Y = 0, \quad \mu_{\mathbb{C}} = 0, \quad F_{z\bar{z}} \sim \mu_{\mathbb{R}} + t_{\mathbb{R}}. \quad (2.25)$$

We impose that  $X$  and  $Y$  tend to a Higgs vacuum  $\alpha \in \mathcal{M}_H$  at infinity in  $\mathbb{C}$ . We compactify the complex plane to  $\mathbb{P}^1$  and, as is typical in moduli space constructions [93], the real moment map can be dropped in favour of a quotient by the complex gauge group together with a stability condition. The moduli space of vortices in a vacuum  $\alpha$  can then be expressed as the infinite-dimensional Kähler quotient

$$\mathcal{M}_\alpha = \{D_{\bar{z}}X = D_{\bar{z}}Y = 0, \mu_{\mathbb{C}} = 0 \mid X, Y \xrightarrow{z \rightarrow \infty} \alpha\} / \mathcal{G}_{\mathbb{C}}, \quad (2.26)$$

---

<sup>7</sup>This is the same sector of operators discussed in section 4.4.

<sup>8</sup>The definition of a vertex function will be reviewed in section 4.1.1.

where in the above  $X, Y \rightarrow \alpha$  means that  $X$  and  $Y$  lie in the orbit of  $\alpha$  at infinity and we quotient by gauge transformations constant at  $\infty$ . This data defines a principal  $G_{\mathbb{C}}$ -bundle on  $\mathbb{P}^1$  trivialised at  $\infty$  with holomorphic sections  $(X, Y)$  of associated bundles satisfying the moment map constraint  $\mu_{\mathbb{C}} = 0$ . The moduli space splits into distinct topological sectors  $\mathcal{M}_{\alpha}^{\mathfrak{m}}$  labelled by Chern numbers/fluxes

$$\mathfrak{m} = \frac{1}{2\pi} \int_{\mathbb{P}^1} \text{tr } F, \quad (2.27)$$

we have  $\mathfrak{m} \in \pi_1(G)$  and therefore a natural pairing with exponentiated FI parameters  $\zeta_{\mathbb{R}} \in \text{Hom}(\pi_1(G), U(1))$ , we see this pairing manifested in vortex partition functions later. In the case of a product gauge group  $G = \prod_{i \in I} U(N_i)$  (*e.g.* the quiver field theories we discuss in the following section) the fluxes form a lattice in  $\mathbb{Z}^{|I|}$  corresponding to the non-empty vortex moduli spaces. Later, we see that for Nakajima quiver varieties  $H^2(\mathcal{M}_H, \mathbb{Z})$  can be identified with  $\mathfrak{t}_G$  and so under the above pairing, fluxes can naturally be identified with homology classes of images of maps  $\mathbb{P}^1 \rightarrow \mathcal{M}_H$ . We explore the geometry further in chapter 4 where the relevant moduli spaces are quasimap spaces [94, 95] and in this context we elucidate the precise connection between  $K$ -theoretic vertex functions and partition functions of boundary conditions.

In section 4.2 of chapter 4 we pursue a realisation of vortex moduli spaces as moduli spaces of flags of sheaves on  $\mathbb{P}^1$ . We show for theories of the class  $T_{\rho}[SU(N)]$  the relevant moduli space is a smooth Laumon space [96].

## 2.2 Nakajima quiver varieties

In this section we review the definition and basic properties of Nakajima quiver varieties [41, 40]. Physically, Nakajima quiver varieties describe the Higgs branch geometry (2.11) of a wide class of 3d  $\mathcal{N} = 4$  theories.

We begin with a review of the field content of these theories and then discuss the powerful algebraic realisation of Nakajima quiver varieties as symplectic resolutions. We focus mostly on the physical implications of these constructions and leave detailed proofs to the existing literature.

**Quiver field theories.** In the previous section we considered very general properties of 3d  $\mathcal{N} = 4$  gauge theories. We now turn to a particular class of theories known as *quiver gauge theories* which we denote by  $\mathcal{T}_Q$ . These theories are specified by a quiver  $Q$ —an example of which can be seen in figure 2.1. The Higgs branch of  $\mathcal{T}_Q$  coincides with a Nakajima quiver variety and often the Coulomb branch can also be described by a quiver variety—also known as the symplectic dual (for example the class of  $T_{\rho}^{\sigma}(G)$  theories are

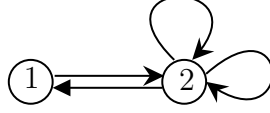


Figure 2.1: A quiver  $Q$  with  $I = \{1, 2\}$  and  $E = \{(1, 2), (2, 2)\}$ . The edges of  $\bar{Q}$  are illustrated together with a choice of orientation  $\bar{Q} = \vec{Q} \sqcup \vec{Q}^{\text{op}}$ .

‘closed’ in this way under mirror symmetry [97, 98], as is the 3d ADHM quiver theory we study in chapter 5).

**Quiver representations.** A quiver is a finite graph  $Q$  with a vertex set  $I$  and edge set  $E$ . We denote by  $\bar{Q}$  the quiver with double the edges of  $Q$  and write  $\bar{E}$  for the doubled edge set.  $\vec{Q}$  denotes a choice of orientation of the edges in  $Q$  and  $\vec{Q}^{\text{op}}$  denotes the opposite choice of orientation so that  $\bar{Q} = \vec{Q} \sqcup \vec{Q}^{\text{op}}$ . See figure 2.1 for an example. We write the adjacency matrix for the oriented graph as  $\vec{Q}_{ij} = |\{\# \text{ edges joining } i \text{ and } j\}|$  for  $(i, j) \in I \times I$ . We specify dimension vectors  $\mathbf{v} = \{\mathbf{v}_i\}_{i \in I}$  and  $\mathbf{w} = \{\mathbf{w}_i\}_{i \in I}$  called *gauge nodes* and *flavour nodes* respectively. A *quiver representation* is an assignment of the following vector spaces to the quiver

$$\begin{aligned} R(\vec{Q}, V) &= \bigoplus_{i, j \in I} \text{Hom}(V_i, V_j) \otimes \vec{Q}_{ij}, & L(V, W) &= \bigoplus_{i \in I} \text{Hom}(V_i, W_i), \\ M(\mathbf{v}, \mathbf{w}) &= R(\vec{Q}, V) \oplus L(V, W). \end{aligned} \quad (2.28)$$

where  $(V_i)_{i \in I}$  and  $(W_i)_{i \in I}$  are complex vector spaces with dimensions determined by the gauge and flavour nodes  $(\mathbf{v}_i)$  and  $(\mathbf{w}_i)$ . A *framed double quiver representation* is defined by

$$\mathcal{R}(\mathbf{v}, \mathbf{w}) := T^*M(\mathbf{v}, \mathbf{w}) = R(\vec{Q}, V) \oplus R(\vec{Q}^{\text{op}}, V) \oplus L(W, V) \oplus L(V, W). \quad (2.29)$$

Since it is a cotangent bundle, the representation  $\mathcal{R}(\mathbf{v}, \mathbf{w})$  can play the role of a quaternionic matter representation  $\mathcal{R}$  for a 3d  $\mathcal{N} = 4$  theory of the type discussed in section 2.1.1. The gauge group  $G$  of the theory is specified by the set of gauge nodes  $(\mathbf{v}_i)$  so that  $G = \prod_{i \in I} U(\mathbf{v}_i)$ . In this way the quiver  $Q$  determines a quiver field theory  $\mathcal{T}_Q$ . In summary:

- To every gauge node  $i \in I$  we associate a  $\mathcal{N} = 4$  vector multiplet with gauge group  $G = U(\mathbf{v}_i)$ .
- To every edge  $(i, j) \in E$  between gauge nodes we associate a bifundamental  $\mathcal{N} = 4$  chiral multiplet.
- To every flavour node we associate  $\mathbf{w}_i$  fundamental  $\mathcal{N} = 4$  hypermultiplets.

**Nakajima quiver varieties.** We now discuss the identification of the Higgs branch of  $\mathcal{T}_Q$  with a Nakajima quiver variety [41, 40]. The matter representation  $\mathcal{R}(\mathbf{v}, \mathbf{w})$  is a flat space of quaternionic dimension  $\sum_{(i,j) \in \bar{E}} \mathbf{v}_i \mathbf{v}_j + \sum_{i \in I} \mathbf{v}_i \mathbf{w}_i$  with a natural hyperkähler structure. We write<sup>9</sup>  $(A, B, I, J)$  for elements of  $\mathcal{R}(\mathbf{v}, \mathbf{w})$ . The quiver representation  $\mathcal{R}(\mathbf{v}, \mathbf{w})$  admits a tri-holomorphic Hamiltonian action by the gauge group  $G$  given by

$$(A, B, I, J) \rightarrow (gAg^{-1}, gBg^{-1}, gI, Jg^{-1}), \quad (2.30)$$

for  $g \in \prod_{i \in I} U(V_i)$ . In the canonical complex structure the associated real and complex moment maps for this action are

$$\begin{aligned} \mu_{\mathbb{R}} &:= [A, A^\dagger] + [B, B^\dagger] + II^\dagger - JJ^\dagger \in \prod_{i \in I} \mathfrak{u}(V_i), \\ \mu_{\mathbb{C}} &:= [A, B] + IJ \in \prod_{i \in I} \mathfrak{u}(V_i) \otimes \mathbb{C}. \end{aligned} \quad (2.31)$$

These are precisely the moment maps appearing in the Higgs branch vacuum equations (2.9) when specialised to a quiver representation.

**Definition 2.1.** (Nakajima [41]) The Nakajima quiver variety  $\mathcal{M}_\xi(\mathbf{v}, \mathbf{w})$  is the Higgs branch  $\mathcal{M}_H(\mathcal{T}_Q)$  and is given by the hyperkähler reduction

$$\mathcal{M}_\xi(\mathbf{v}, \mathbf{w}) = \mu_{\mathbb{R}}^{-1}(\xi_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) / G. \quad (2.32)$$

We defer a discussion of geometrical properties of the quotient, such as smoothness and dimension, until after discussing the construction of Nakajima quiver varieties as quasi-projective varieties. For the remainder of this chapter we turn off complex masses  $m_{\mathbb{C}}$  and FI parameters  $\xi_{\mathbb{C}}$ . We turn these back on later in section 4.4 of chapter 4 where they give complex structure deformations of the Higgs branch and Coulomb branch respectively.

**Algebraic realisation.** The second of Nakajima's papers on quiver varieties [40] realises these spaces as quasi-projective varieties. They can be constructed as a *geometric invariant theory* (GIT) quotient [99] and in this section we review the essentials of this theory as it applies to quiver varieties. Throughout, we work in a fixed complex structure and mostly drop the  $\mathbb{C}$  subscripts.

Firstly, we note that  $\mu^{-1}(0) \subset \mathcal{R}(\mathbf{v}, \mathbf{w})$  is an affine algebraic variety and admits an action of the complexified gauge group  $G_{\mathbb{C}} = \prod_{i \in I} GL(V_i)$ . We can therefore form the affine algebraic quotient

$$\mathcal{M}_o(\mathbf{v}, \mathbf{w}) = \text{Spec } \mathbb{C}[\mu^{-1}(0)]^{G_{\mathbb{C}}}. \quad (2.33)$$

---

<sup>9</sup>We use a slightly different notation to the  $(X, Y)$  splitting of the previous section and now further distinguish between fields between gauge nodes and flavour nodes.

This space is, by construction, an affine algebraic variety and generally singular. The right hand side is simply a more formal version of the discussion in subsection 2.1.3—the Higgs branch chiral ring is formed from gauge invariant hypermultiplet scalars. It is a standard fact of GIT theory that the geometric points of the affine quotient correspond to closed  $G$ -orbits in  $\mathcal{R}(\mathbf{v}, \mathbf{w})$ . Under certain conditions that we discuss in the following, the quiver variety  $\mathcal{M}(\mathbf{v}, \mathbf{w})$  will be a resolution of the singular affine space  $\mathcal{M}_o(\mathbf{v}, \mathbf{w})$ .

The construction of  $\mathcal{M}(\mathbf{v}, \mathbf{w})$  as a GIT quotient depends on a *stability parameter*  $\theta \in \mathbb{Z}^{|I|}$ . Associated to  $\theta$  we define a character  $\chi_\theta : G_{\mathbb{C}} \rightarrow \mathbb{C}^\times$  by  $\chi_\theta(g) = \prod_{i \in I} (\det g_i)^{-\theta_i}$  and this allows us to define a graded algebra

$$A := \bigoplus_{n \geq 0} \mathbb{C}[\mu^{-1}(\xi_{\mathbb{C}})]^{G_{\mathbb{C}}, \chi^n}, \quad (2.34)$$

where the graded components are given by the  $\chi$  semi-invariants

$$\mathbb{C}[\mu^{-1}(\xi_{\mathbb{C}})]^{G_{\mathbb{C}}, \chi} := \{f \in \mathbb{C}[\mu^{-1}(\xi_{\mathbb{C}})] : f(g \cdot x) = \chi_\theta(g)f(x)\}. \quad (2.35)$$

We then have the following

**Definition 2.2.** (Nakajima [40]) A Nakajima quiver variety is defined by the GIT quotient construction

$$\mathcal{M}_{\theta, \xi_{\mathbb{C}}}(\mathbf{v}, \mathbf{w}) = \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) //_{\chi_\theta} G_{\mathbb{C}} := \text{Proj}(A). \quad (2.36)$$

This, by definition, realises the Nakajima quiver variety as a quasi-projective variety. The Proj construction comes with a natural morphism induced by the inclusion of the zero component  $\mathbb{C}[\mu^{-1}(0)]^{G_{\mathbb{C}}} \subset A$  which gives a projective morphism

$$\pi : \mathcal{M}_{\theta, \xi_{\mathbb{C}}}(\mathbf{v}, \mathbf{w}) \rightarrow \mathcal{M}_{o, \xi_{\mathbb{C}}}(\mathbf{v}, \mathbf{w}) \quad (2.37)$$

that relates the singular space  $\mathcal{M}_o(\mathbf{v}, \mathbf{w})$  to the resolved Higgs branch  $\mathcal{M}(\mathbf{v}, \mathbf{w})$ .

**Stability.** We now define loci of *stable points* and *semistable points*. The following is due to King [100].

**Definition 2.3.** We say that a quiver representation  $\mathcal{R}(\mathbf{v}, \mathbf{w})$  is *semistable* with respect to the stability parameter  $\theta \in \mathbb{R}^{|I|}$  if the following condition holds: Whenever  $S = (S_i)_{i \in I}$  is a collection of subspaces contained in  $V = (V_i)_{i \in I}$  and preserved by  $A$  and  $B$  we have

- If  $S_i \subset \ker J_i$  for all  $i \in I$  then  $\theta \cdot \dim S \leq 0$ .
- If  $\text{im } I_i \subset S_i$  for all  $i \in I$  then  $\theta \cdot \dim S \leq \theta \cdot \dim V$ .

We further say that a representation  $\mathcal{R}(\mathbf{v}, \mathbf{w})$  is *stable* (with respect to  $\theta$ ) if the above inequalities are strict. We write  $\mu^{-1}(0)^s$  and  $\mu^{-1}(0)^{ss}$  for the locus of stable and semistable

points respectively. The conditions respect the group action (2.30) and therefore the notion of stable and semistable descends to definitions of stable and semistable loci, denoted by  $\mu^{-1}(0)^s/G_{\mathbb{C}}$  and  $\mu^{-1}(0)^{ss}/G_{\mathbb{C}}$  respectively, in the topological quotient  $\mu^{-1}(0)/G_{\mathbb{C}}$ .

In this thesis we always take the stability condition  $\theta = (1, \dots, 1)$  and in that case the second stability condition is automatic and the first simply states that if  $S$  is any subspace of  $\ker J$  that is stable under  $A$  and  $B$  then  $S = \{0\}$ . We discuss the general  $\theta$  case because in a moment we will use this to make contact with the the real FI parameters of the 3d quiver gauge theory  $\mathcal{T}_Q$ .

In terms of stability, we can now give a more intuitive picture of the geometric points of the GIT quotient  $\mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w})$ .

**Theorem 1.** (Nakajima [40]) The set of geometric points of the scheme  $\mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w})$  is topologically equivalent to the quotient

$$\mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0) //_{\chi_{\theta}} G_{\mathbb{C}} = \mu^{-1}(0)^s / \sim, \quad (2.38)$$

where the equivalence relation on the right identifies  $x \sim x'$  if the closure of their  $G_{\mathbb{C}}$  orbits intersect in  $\mu^{-1}(0)^{ss}$ .

Note that this is a coarser condition than the topological quotient  $\mu^{-1}(0)^s/G_{\mathbb{C}}$  which in general is a ‘badly-behaved’ non-Hausdorff space.

**Generic stability parameters.** In fact, the stability parameter  $\theta \in \mathbb{Z}^{|I|}$  can often be chosen so that  $\mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w})$  is a non-singular variety. We define the following disconnected set of chambers

$$\bigcup \mathfrak{C}_C := \mathbb{R}^{|I|} \setminus \bigcup_{\alpha \in R_+(\mathbf{v})} D_{\alpha}, \quad (2.39)$$

where  $R_+(\mathbf{v}) = \{\alpha \in \mathbb{Z}_+^{|I|} : \alpha \neq 0, (\alpha, \alpha) \leq 2, \alpha \leq \mathbf{v}\}$  and  $D_{\alpha} = \{\theta : \theta \cdot \alpha = 0\}$ . When  $Q$  is the Dynkin diagram of a simply-laced Lie algebra  $\mathfrak{g}$  then these are precisely the usual Weyl chambers associated to  $\mathfrak{g}$ . In general, the form  $(, )$  is the symmetrised Euler form associated to  $Q$  defined by<sup>10</sup>

$$\begin{aligned} \langle \mathbf{v}, \mathbf{v}' \rangle &:= \sum_{i \in I} \mathbf{v}_i \mathbf{v}'_i - \sum_{e \in \bar{E}} \mathbf{v}_{h(e)} \mathbf{v}'_{t(e)}, \\ (\mathbf{v}, \mathbf{v}') &:= \langle \mathbf{v}, \mathbf{v}' \rangle + \langle \mathbf{v}, \mathbf{v}' \rangle. \end{aligned} \quad (2.40)$$

We then say that  $\theta \in \mathbb{Z}^{|I|}$  is *generic* if it lies within a chamber  $\mathfrak{C}_C$ . Physically, if the FI parameter<sup>11</sup> lies on a root hyperplane then mixed branches of the moduli space open up as the tension between domain walls separating vacua goes to zero.

<sup>10</sup>Where  $h(e)$  and  $t(e)$  denote the vertex at the head and tail of the edge  $e$  respectively.

<sup>11</sup>We discuss how  $\theta$  relates to the FI parameter in the following.



If  $\theta \in \mathbb{Z}^{|I|}$  is generic then Nakajima shows [40] that the stabiliser in  $G_{\mathbb{C}}$  of a point  $x \in \mu^{-1}(0)^s$  is trivial and therefore all orbits in  $\mu^{-1}(0)^s$  are closed. Under these conditions the geometric points of the quiver variety are given simply by

$$\mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0)^s / G_{\mathbb{C}}. \quad (2.41)$$

Furthermore, under the genericity assumption we have the following

**Theorem 2.** (Nakajima [40]) If  $\theta \in \mathbb{R}^{|I|}$  is generic and  $\mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w})$  is non-empty<sup>12</sup> then we have

- $\mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w})$  is a non-singular connected variety of dimension  $2\mathbf{v} \cdot \mathbf{w} - 2\langle \mathbf{v}, \mathbf{v} \rangle$ .
- If  $\theta, \theta' \in \mathbb{R}^{|I|}$  are in the same chamber  $\mathfrak{C}_C$  then the quiver varieties  $\mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w})$  and  $\mathcal{M}_{\theta'}(\mathbf{v}, \mathbf{w})$  are isomorphic.
- The map  $\pi : \mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w}) \rightarrow \mathcal{M}_o(\mathbf{v}, \mathbf{w})$  is a resolution of singularities.

**The Kempf-Ness theorem.** At the beginning of this chapter we defined the Nakajima quiver variety as a hyperkähler quotient. We now explain the connection between the hyperkähler quotient and algebraic approaches.

The stability parameter  $\theta \in \mathbb{Z}^{|I|}$  can be related to the real FI parameter  $\xi_{\mathbb{R}}$  as follows. Restricting the character  $\chi_{\theta}$  to the real gauge group  $G$  defines a character  $\chi_{\theta} \in \Lambda_G^{\vee}$ . Taking the derivative we obtain a map  $\chi_* : \mathfrak{t}_G \rightarrow \mathfrak{u}(1) \cong \mathbb{R}$  which, via equation (2.31), can be considered an element of  $\mathfrak{t}_G$  and identified with the real FI parameter  $\xi_{\mathbb{R}}$ . By theorem 2, quiver varieties in the same chamber are isomorphic and therefore we can always choose an integer  $\theta$  in the same chamber as  $\xi_{\mathbb{R}}$ .

**Theorem 3.** (Kempf-Ness [101]) Assuming  $\theta \in \mathbb{Z}^{|I|}$  is generic and the FI parameter  $\xi$  is determined in terms of  $\theta$  as above, then the hyperkähler quotient (2.32) is smooth and we have an isomorphism

$$\mu_{\mathbb{R}}^{-1}(\xi_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(0) / G \cong \mu_{\mathbb{C}}^{-1}(0) //_{\chi_{\theta}} G_{\mathbb{C}}. \quad (2.42)$$

We therefore use without ambiguity the notation  $\mathcal{M}_{\theta, \zeta_{\mathbb{C}}}(\mathbf{v}, \mathbf{w})$  to describe a Nakajima quiver variety as either a hyperkähler quotient or the GIT quotient (2.36).

We assume from this point that our physical FI parameters  $\xi_{\mathbb{R}}$  are chosen so that we do not cross root hyperlanes and the Higgs branch  $\mathcal{M}_H$  is fully resolved.

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<sup>12</sup>This innocent constraint is often actually the main obstruction to obtaining a non-trivial variety.

**Flavour symmetries.** There is a natural Hamiltonian action of  $G_W = \prod_{i \in I} GL(W_i)$  on  $\mathcal{R}(\mathbf{v}, \mathbf{w})$  associated to the framing factors  $(\mathbf{w}_i)$ . Physically, this is a flavour symmetry and acts by

$$G_W : (A, B, I, J) \rightarrow (A, B, Ig, g^{-1}J). \quad (2.43)$$

for  $g \in G_W$ . This action commutes with the gauge group action  $G_{\mathbb{C}}$ , preserves the moment map equation (2.31) and respects the stability conditions (2.3) and therefore descends to a Hamiltonian isometry of  $\mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w})$ . Comparing with the vacuum equations (2.9) we see this is the finite version of the infinitesimal transformation generated by the real mass parameters  $m_{\mathbb{R}}$ . We denote the maximal torus of this action by  $\mathbf{A}$ .

It is an assumption throughout this thesis that we work with theories with isolated fixed points under  $G_W$ . For ADE quivers theories, this is equivalent to fixing a chamber  $\mathfrak{C}_H$  for  $m_{\mathbb{R}}$  that lies outside the root hyperplanes:

$$\bigcup \mathfrak{C}_H := \mathbb{R}^{|\mathbf{w}|} \setminus \bigcup_{\alpha \in R_+(\mathbf{v})} D_{\alpha}, \quad (2.44)$$

where the definition of the root hyperplanes is parallel to (2.39) but for the  $G_W$  symmetry. We see that for theories with Higgs branch and Coulomb branch both described by Nakajima quiver varieties, these two constraints are exchanged under 3d mirror symmetry so that fully resolving the Higgs branch corresponds to lifting the Coulomb branch to isolated fixed points and *vice versa*. Our assumption throughout is that our theories admit suitably generic masses  $m_{\mathbb{R}}$  and FI parameters  $\xi_{\mathbb{R}}$  to achieve this.

Nakajima quiver varieties also admit a contracting  $\mathbb{C}_t^{\times}$  action<sup>13</sup> that acts on the linear data by

$$\mathbb{C}_t^{\times} : (A, B, I, J) \rightarrow (tA, B, tI, J). \quad (2.45)$$

The action preserves the moment map constraint and commutes with the gauge group and therefore descends to an algebraic action on  $\mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w})$  that scales the symplectic form by  $t$ . The resolution  $\pi : \mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w}) \rightarrow \mathcal{M}_o(\mathbf{v}, \mathbf{w})$  is equivariant with respect to  $\mathbb{C}_t^{\times}$  and has a unique fixed point  $o \in \mathcal{M}_o(\mathbf{v}, \mathbf{w})$ , such actions are known as contracting. We denote by  $\mathbf{T} := \mathbf{A} \times \mathbb{C}_t^{\times}$  the combined torus action and write coordinates on  $\mathbf{T}$  in terms of the corresponding mass deformations as  $t = e^{-\tau}$  and  $z_a^{(i)} = e^{-m_a^{(i)}}$  for  $i = 1, \dots, |I|$  and  $a = 1, \dots, \mathbf{w}_i$ .

**Symplectic resolutions.** The coordinate ring of the unresolved space  $\mathcal{M}_o(\mathbf{v}, \mathbf{w})$  admits a Poisson algebra structure induced from the canonical symplectic form on  $\mathcal{R}(\mathbf{v}, \mathbf{w})$ . The notion of *symplectic singularity* captures when this algebra coincides with the algebra of

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<sup>13</sup>There is another choice  $(A, B, I, J) \rightarrow (tA, tB, tI, tJ)$  that corresponds to the Higgs branch  $R_H$ -symmetry action. In this section we follow the conventions of the quiver variety literature—in fact for the theories studied in this thesis the two choices are related by a combined gauge and flavour transformation.

holomorphic functions on the resolved Higgs branch  $\mathcal{M}(\mathbf{v}, \mathbf{w})$

**Definition 2.4.** (Beauville [102], Ginzburg [103]) A symplectic resolution is a resolution of singularities  $\pi : \widetilde{X} \rightarrow X$  with  $X$  an irreducible Poisson affine variety and  $\widetilde{X}$  an algebraic symplectic manifold such that  $\pi$  induces a Poisson algebra isomorphism

$$\pi^* : \mathbb{C}[X] \rightarrow \Gamma(\widetilde{X}, \widetilde{\mathcal{O}}_X), \quad (2.46)$$

where  $\widetilde{\mathcal{O}}_X$  denotes the sheaf of holomorphic functions on  $\widetilde{X}$ .

The work of [104] ensures that a wide class of quiver varieties are symplectic resolutions. This means the Poisson algebra structure on the chiral ring  $\mathcal{R}_H$  coincides with the Poisson algebra implied by the quotient construction in theorem 3 on the ring of functions on  $\mathcal{M}(\mathbf{v}, \mathbf{w})$ . In particular, the  $A$ -type quivers and the 3d ADHM quiver studied in this thesis are symplectic resolutions.

Nakajima quiver varieties have the extra structure of a contracting  $\mathbb{C}_t^\times$  action that induces a positive grading on the coordinate ring

$$\mathbb{C}[\mathcal{M}_0(\mathbf{v}, \mathbf{w})] = \bigoplus_{k \geq 0} \mathbb{C}^k[\mathcal{M}_0(\mathbf{v}, \mathbf{w})], \quad (2.47)$$

with  $\mathbb{C}^0[\mathcal{M}_0(\mathbf{v}, \mathbf{w})] = \mathbb{C}$  and grades the Poisson structure with degree two so that for any  $i, j \geq 0$  we have

$$\{\mathbb{C}^i[\mathcal{M}_0(\mathbf{v}, \mathbf{w})], \mathbb{C}^j[\mathcal{M}_0(\mathbf{v}, \mathbf{w})]\} \subset \mathbb{C}^{i+j-2}[\mathcal{M}_0(\mathbf{v}, \mathbf{w})]. \quad (2.48)$$

The work of Kaledin in [105] shows that this structure admits a canonical lift to the ring of holomorphic functions on  $\mathcal{M}(\mathbf{v}, \mathbf{w})$ . Symplectic resolutions have many favourable geometrical properties, such as higher sheaf cohomology vanishing, that allow us to make progress computing geometric invariants—we discuss these in more detail in the following chapter. For now, we focus on the physical interpretation.

Regarding a Nakajima quiver variety as a Higgs branch, then under the identification from the previous section we have  $\mathcal{R}_H = \mathbb{C}[\mathcal{M}_H]$ . The interpretation of the symplectic resolution property in physics is that the chiral ring admits a holomorphic Poisson bracket and the  $\mathbb{C}_t^\times$  action corresponds to the grading of the operators by  $R$ -charge. Finally, since  $\mathcal{M}_H$  is a symplectic resolution, the purely algebraic chiral ring  $\mathcal{R}_H = \mathbb{C}[\mu^{-1}(0)]^{G_c}$  captures the ring of holomorphic functions on the resolved Higgs branch. Conversely, this allows the information of the physical chiral ring to be extracted using geometric localisation methods that receive contributions from the isolated fixed points on the resolved Higgs branch  $\mathcal{M}_H$ —we return to these ideas in chapter 3.

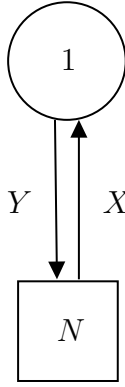


Figure 2.2: SQED[ $N$ ] Quiver

## 2.3 SQED[ $N$ ] example

We now demonstrate the abstract ideas of the previous two sections with a concrete example: supersymmetric QED with  $N$  fundamental hypermultiplets, hereafter denoted SQED[ $N$ ].

This theory is sufficiently complicated to effectively illustrate the main ideas we meet throughout the thesis but simple enough that the geometrical and algebraic aspects can be understood in concrete terms. Consequently, we will return to this example at many points throughout the work, adding layers of complexity as we go. In this section we discuss the Higgs and Coulomb branch geometry in detail. In subsection 4.1.1 we will discuss the vortex moduli space of the theory realised as a quasimap space and compute the *vertex functions*. In subsection 4.3.2 we place the theory on a half space with a boundary and compute the partition function of an *exceptional Dirichlet boundary condition*—later in the same chapter we show that this partition function yields Verma characters of the quantised Higgs and Coulomb chiral rings. We visit SQED[ $N$ ] for a final time in section 5.3.3 where we compute the partition function with a Neumann boundary condition and discuss the geometric interpretation of this object.

**Field content.** SQED[ $N$ ] is an abelian quiver gauge theory described by the quiver in figure 2.2<sup>14</sup>. It has gauge group  $G = U(1)$  and  $N$  fundamental hypermultiplets with a  $G_H = PSU(N)$  flavour symmetry. The topological symmetry is  $G_C = U(1)$ .

We introduce real mass deformations<sup>15</sup>  $m_1, \dots, m_N$  with  $\sum_i m_i = 0$  and an FI parameter  $\xi$ . SQED[ $N$ ] is mirror dual to another  $A$ -type quiver and consequently the condition for both the Higgs and the Coulomb branches to be fully resolved and have isolated singularities is that the masses and FI parameters live in chambers  $\mathfrak{C}_H$  and  $\mathfrak{C}_C$  of  $\mathfrak{t}_H = \mathbb{R}^{N-1}$  and

<sup>14</sup>Note that throughout we use the convention of the mathematical quiver variety literature [41, 40] and draw only the hypermultiplet chirals.

<sup>15</sup>Here and throughout the work masses and FI parameters without  $\mathbb{R}$  or  $\mathbb{C}$  subscripts denote real parameters.

$\mathfrak{t}_C = \mathbb{R}$  respectively. We choose the following chambers for the masses and FI parameters

$$\mathfrak{C}_H = \{m_1 < \dots < m_N\}, \quad \mathfrak{C}_C = \{\xi > 0\}. \quad (2.49)$$

the chamber of the FI parameters<sup>16</sup> corresponds to the stability condition  $\theta = 1$  as in section 2.2. We also turn on a mass deformation  $\tau$ .

**Higgs branch geometry.** The Higgs branch<sup>17</sup> is the Nakajima quiver variety  $\mathcal{M}_H = \mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$  with  $\mathbf{v} = (1)$  and  $\mathbf{w} = (N)$ . Following section 2.2, we denote the scalars in  $\mathcal{R} = R \oplus \bar{R}$  by  $(X, Y)$  and the complex moment map is  $\mu(X, Y) = XY$ . We call  $\mathcal{M}_o(\mathbf{v}, \mathbf{w})$  the unresolved Higgs branch and denote it by  $\mathcal{M}_H^o$ , it has geometric points

$$\mathcal{M}_H^o = \{A \in \text{End}(\mathbb{C}^N) : A^2 = 0, \text{rk} A \leq 1\}, \quad (2.50)$$

where  $A = YX$ . To understand the resolved Higgs branch we note that the stability condition (2.3) with  $\theta = 1$  implies that  $Y$  is injective and therefore specifies a line  $\mathbb{C} \subset \mathbb{C}^N$ . Now  $\text{im } A$  lies inside both  $\text{im } Y$  and  $\ker A$  and therefore  $A$  gives a well-defined element of  $\text{Hom}(\mathbb{C}^N / \text{im } Y, \text{im } Y)$  which can be identified<sup>18</sup> with the cotangent fibre of  $\mathbb{P}^{N-1}$  at the point  $\text{im } Y$ , conversely  $A$  and  $\text{im } Y$  determines  $(X, Y)$  up to the action of the gauge group  $\mathbb{C}^\times$  and we therefore have an isomorphism  $\mathcal{M}_H \cong T^*\mathbb{P}^{N-1}$  with  $\text{im } Y$  identified with projective coordinates  $[Y_1 : \dots : Y_N]$  on  $\mathbb{P}^{N-1}$ . The projective morphism  $\pi : \mathcal{M}_H \rightarrow \mathcal{M}_H^o$  acts by

$$\pi : (\text{im } Y, A) \rightarrow A \quad (2.51)$$

and we see that the pre-image of the singularity  $o \in \mathcal{M}_H^o$  under  $\pi$  is the core  $\mathbb{P}^{N-1}$ . In the case  $N = 2$  the symplectic resolution property is particularly clear as we now show. We choose the following generators for the coordinate ring of  $\mathcal{M}_H^o$

$$h = X_1 Y_1 - X_2 Y_2, \quad e = X_1 Y_2 \quad f = X_2 Y_1. \quad (2.52)$$

The moment map gives the ring relations  $h^2 = 4ef$  and geometrically the unresolved space is a singular curve in  $\mathbb{C}^3$ . The Poisson bracket structure induced from the symplectic form on the pre-quotient space  $\mathcal{R}(1, N)$  is given by

$$\{e, f\} = h, \quad \{h, e\} = 2e, \quad \{h, f\} = -2f, \quad (2.53)$$

this agrees with the smooth symplectic structure on the resolution  $T^*\mathbb{P}^1$ . This is the simplest case of the famous Springer resolution [107]. In chapter 4, we consider a deformation

<sup>16</sup>Likewise the chamber of the mass parameters is the stability condition  $\tilde{\theta} = 1$  on the dual variety.

<sup>17</sup>With masses momentarily turned off.

<sup>18</sup>See, for example, Chriss and Ginzburg [106].

quantisation of this algebra and realise modules by counting boundary operators in the theory.

**Fixed points.** The mass deformation  $\tau$  corresponding to the action  $\mathbb{C}_t^\times$  scales the cotangent directions in  $T^*\mathbb{P}^{N-1}$  since<sup>19</sup>  $(X, Y) \rightarrow (X, tY)$  and therefore  $A \rightarrow tA$ .

Recall that the flavour group acts on the linear data by  $(X, Y) \rightarrow (gX, Yg^{-1})$  for  $g \in GL(\mathbb{C}^N)$ . It is immediate that we must have  $A = \{0\}$  (considered as an element of  $\text{Hom}(\mathbb{C}^N/\text{im } Y, \text{im } Y) = T_Y^*\mathbb{P}^{N-1}$ ) and therefore the fixed points  $\alpha$  lie on the compact core  $\mathbb{P}^{N-1}$ . The fixed point  $\alpha$  can then be labelled by  $\alpha = 1, \dots, N$  depending on which coordinate of  $Y$  is non-zero.

**Coulomb branch geometry.** The Coulomb branch  $\mathcal{M}_C$  for SQED $[N]$  is generated by  $\varphi$  and the bare monopole operators<sup>20</sup>  $v^\pm = e^{\pm \frac{1}{g^2}(\sigma + i\gamma)}$ . Classically, the monopoles satisfy the relation  $v^+v^- = 1$  but, as explained in [55], in the quantum theory the relation is modified to

$$v^+v^- = \varphi^N. \quad (2.54)$$

Together with the symplectic form  $\Omega = d\varphi \wedge d\log v^+$ , this describes the Coulomb branch as the  $A_{N-1}$  singularity  $\mathbb{C}^2/\mathbb{Z}_N$ . The bare monopole operators  $v^+$  and  $v^-$  have topological and  $R$ -charges  $\zeta t^{\frac{N}{2}}$  and  $\zeta^{-1}t^{\frac{N}{2}}$  respectively and the Coulomb branch Hilbert series is given by

$$\text{H.S.}[\mathcal{M}_C] = \frac{1}{1-t} \sum_{m \in \mathbb{Z}} \zeta^m t^{\frac{N}{2}|m|} = \frac{1-t^N}{(1-t)(1-\zeta t^{\frac{N}{2}})(1-\zeta^{-1}t^{\frac{N}{2}})}. \quad (2.55)$$

The Coulomb branch Hilbert series is naturally expressed as a series expansion in  $R$  charge  $1+t+O(t^2)$ . One of the results of chapter 4 is to interpret the right hand side of this expression as a ‘holomorphic factorisation’ and instead expand in the topological fugacity  $\zeta$ . In section 4.3.3 we relate this expansion to the Poincaré polynomial of the vortex moduli space.

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<sup>19</sup>We continue to work with the convention from the quiver variety literature—this is gauge equivalent to the  $R_H$ -symmetry action  $(X, Y) \rightarrow (tX, tY)$

<sup>20</sup>Inserting these operators in the path integral induces the singular configuration (2.21).

# CHAPTER 3

## QUIVERS, INTEGRABILITY AND MACDONALD POLYNOMIALS

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This chapter begins with a review of geometric localisation. We add more detail to the introduction to Nakajima quiver varieties we gave in section 2.2 and discuss equivariant  $K$ -theory and the computation of various geometric invariants. We discuss *Molien integrals* that compute the Hilbert series of 3d  $\mathcal{N} = 4$  theories and introduce a ‘ $q$ -deformed Molien integral’ which is closely related to the kind of integral we meet in section 5.3 that counts operators supported on a Neumann boundary condition. We show that *Macdonald polynomials* can be used as a convenient tool to evaluate such integrals and apply these methods to chainsaw quiver varieties and the ADHM moduli space in particular.

In sections 3.3 and 3.4 we study *handsaw quiver varieties* and show that, in the spirit of the Borel-Weil-Bott theorem for flag varieties, certain Yangian modules can be realised as homology groups of line bundles over the so-called ‘toothless’ handsaw quivers. Later, in chapter 5, we discuss the physical interpretation of this result as counting boundary operators in the presence of a line operator in a certain 3d  $\mathcal{N} = 4$  theory.

**Overview.** We begin with a review of geometric localisation. We focus on physical applications and aim to phrase the background material in a way that lends itself to explicit computations of the physical observables that we study throughout the thesis. In section 3.2 we review Yangian characters and spin chain partition functions; we conclude the section with a new result showing that a certain class of symmetric functions, known as Milne polynomials, realise these spin chain partition functions. The second novel result of the chapter is contained in section 3.3. In this section, we study a particular quiver variety that describes a Lagrangian submanifold of the ADHM moduli space, we show that holomorphic sections of line bundles supported on this submanifold form a particular module of the type discussed in section 3.2—the physical interpretation of these results in 3d  $\mathcal{N} = 4$  gauge theory is deferred to chapter 5. We then further extend the results of section 3.3 to a class of quivers called *chainsaw quiver varieties*—we introduce Macdonald function methods to evaluate their Hilbert series and find that a subset of these quivers realise more general Yangian characters. Finally, in section 3.4.3, we introduce Macdonald

polynomial methods as a convenient tool to evaluate the large gauge rank limit of Hilbert series.

**Publications.** The first two sections of this chapter are mostly review material. Parts of section 3.3 and 3.4 appear in [108]

- S. Crew, N. Dorey and D. Zhang, *Blocks and Vortices in the 3d ADHM Quiver Gauge Theory*, 2010.09732. (accepted to JHEP)

Section 3.4.2 is the author's currently unpublished work.

## 3.1 Preliminaries

We work with quasi-projective varieties  $X$  equipped with the action of an algebraic group  $G$  with an  $N$  dimensional maximal torus  $T \subset G$ . Physically, for the most part,  $X$  will be a Nakajima quiver variety describing the Higgs branch  $\mathcal{M}_H$  of a 3d  $\mathcal{N} = 4$  theory and the algebraic torus action will arise from global symmetries of the gauge theory. Later in this chapter,  $X$  will also stand for resolutions of vortex moduli spaces of 3d  $\mathcal{N} = 4$  theories, similarly realised as quiver varieties and in chapter 4 we will consider more general vortex moduli spaces that do not admit smooth quiver descriptions. In this section we review a suitably general setup to cover these cases. We begin with a lightning review of localisation in equivariant  $K$ -theory—throughout, we take a constructive approach focused on explicit calculations and refer the reader to the mathematical literature for detailed proofs.

**Equivariant  $K$ -theory.** The equivariant  $K$ -theory ring  $K_T(X)$  is the Grothendieck group of  $T$ -equivariant coherent sheaves on  $X$ .<sup>1</sup>  $K_T(X)$  has the structure of a commutative  $K_T(\text{pt})$ -module with the product structure given by the tensor product. The  $K$ -theory of a point  $K_T(\text{pt})$  can be identified with the representation ring of the torus  $\mathbb{Z}[t_1^{\pm 1}, \dots, t_N^{\pm 1}]$ .

If  $f : Y \rightarrow X$  is a closed embedding of varieties then we have a well-behaved<sup>2</sup> pullback map  $f^* : K_T(X) \rightarrow K_T(Y)$ . If  $f : X \rightarrow Y$  is a proper morphism (or a projective map to an affine variety) then we have a well-behaved push forward<sup>3</sup>  $f_* : K_T(X) \rightarrow K_T(Y)$ . We will exclusively consider the case where  $Y = X^T$  is the finite set of isolated fixed points of  $X$  under  $T$  and  $\iota : X^T \rightarrow X$  is the inclusion map.

The torus action  $T$  on  $X$  induces an action on the global sections  $\Gamma(X, E)$  of a sheaf  $E$  and in turn endows the sheaf cohomology groups  $H^i(X; E)$  with a  $T$ -module structure.

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<sup>1</sup>When  $X$  is non-singular we can equivalently work with locally free equivariant coherent sheaves (*i.e.* vector bundles) [109].

<sup>2</sup>Respects the  $K_T(\text{pt})$ -module structure.

<sup>3</sup>Defined as the sum of the images of the higher derived functors  $\sum_i (-1)^i R^i f_*$ .



We are interested in computing the *equivariant Euler characteristic* of an equivariant sheaf on  $X$  defined by

$$\chi_T(E, X) := \sum_{i \geq 0} (-1)^i \text{ch}_T H^i(X; E), \quad (3.1)$$

with  $E \in K_T(X)$ . In later chapters, these invariants will be associated to various supersymmetric observables.

**Localisation.** Equivariant Euler characteristics can be computed using the method of localisation in equivariant  $K$ -theory which we now review. We refer the reader to the excellent reviews [106] and [95] for further technical detail.

We define  $K_T(X)_{\text{loc}}$  to be the *localised  $K$ -theory ring*  $K_T(X) \otimes_{K_T(\text{pt})} \mathbb{Q}(T)$  where  $\mathbb{Q}(T)$  denotes the fraction field of the representation ring of  $T$  *i.e.*  $\mathbb{Q}(T) = \mathbb{Q}(t_1, \dots, t_N)$ . Thomason's localisation theorem [110] tells us that the pushforward  $\iota_*$  is an isomorphism after localising so that

$$\iota_* : K_T(X^T)_{\text{loc}} \cong K_T(X)_{\text{loc}}. \quad (3.2)$$

The isomorphism here respects the  $K_T(\text{pt})$  module structure on both sides. Since for us  $X^T$  will always be a set of isolated fixed points, then the left hand side is simply  $|X^T|$  copies of the representation ring and this allows us to think of the localised equivariant  $K$ -theory of  $X$  as a vector space with a basis labelled by fixed points  $|x\rangle$ .

Furthermore, it is possible<sup>4</sup> to explicitly compute the inverse of  $\iota_*$  by

$$\iota_*^{-1}(E) = \bigoplus_{x \in X^T} \text{ch}_T(E_x) \text{PE}[T_x^\vee X]. \quad (3.3)$$

where  $T_x^\vee X$  is the fibre of the cotangent bundle at  $x \in X^T$  and PE is the plethystic exponential defined in (A.12).

**Symplectic resolutions.** Suppose  $\pi : \widetilde{X} \rightarrow X$  is a symplectic resolution as in definition 2.4. We further suppose there exists a  $\mathbb{C}^\times$  action with a unique fixed point  $o \in X$  and we denote the inclusion of this fixed point by  $\iota_o$ . As discussed around definition 2.4, these conditions are met for the Nakajima quiver varieties discussed throughout this thesis.

Under these conditions, one can show that, from the definition of the push forward and the fact that  $\pi$  is projective, we have  $\pi_* E = \sum (-1)^i H^i(X; E)$ . By localisation on the un-resolved affine space  $X$ , since there is a unique fixed point  $o$ , we have:

$$K_T(X)_{\text{loc}} = K_T(\text{pt}) \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_N^{\pm 1}], \quad (3.4)$$

where the isomorphism is given by  $(\iota_o^*)^{-1} = \text{ch}_T$ . Putting these pieces together gives the

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<sup>4</sup>See, for example, section 4 of [111].

commutative square

$$\begin{array}{ccc} K_T(\widetilde{X})_{\text{loc}} & \xrightarrow{\iota_*^{-1}} & \bigoplus_x K_T(\text{pt}) \\ \pi_* \downarrow & & \downarrow \sum_x \\ K_T(X)_{\text{loc}} & \xrightarrow{\text{ch}_T} & K_T(\text{pt}) \end{array}$$

and we obtain a localisation formula for the equivariant Euler characteristic of a sheaf  $E$ :

$$\begin{aligned} \chi_T(E, \widetilde{X}) &= \sum_{i \geq 0} (-1)^i \text{ch}_T H^i(\widetilde{X}; E) \\ &= \sum_{x \in \widetilde{X}^T} \text{ch}_T(E_x) \text{PE}[T_x^\vee \widetilde{X}]. \end{aligned} \tag{3.5}$$

This is the tool we use to compute equivariant Euler characteristics throughout this work.

**Remark.** When  $X$  is a smooth *projective* variety we are on the more familiar ground of smooth Kähler manifolds. Crucially though, a Nakajima quiver variety is not projective. Following *GAGA* [112],  $X$  can be considered a compact complex manifold with a compact connected lie group action  $G$  (with maximal torus  $T$ ) and coherent sheaves can be identified with holomorphic  $G$ -equivariant vector bundles  $E \rightarrow X$ . The sheaf cohomology groups  $H^i(X, E)$  coincide with ordinary complex sheaf cohomology groups in the analytic topology which can be computed by *e.g.* Čech cohomology.

In this case the equivariant Euler characteristic computes the index of the Dolbeault operator  $\partial$  on  $X$ . Explicitly, we now have a proper map to a point  $\pi : X \rightarrow \text{pt}$  with  $\pi_* E = \sum (-1)^i H^i(X, E)$  and equivariant Grothendieck-Riemann-Roch tells us that the push forward is given by

$$\chi_T(E, X) = \text{ind}(\partial, E) = \int_X \text{td}_G(T_X) \text{ch}_G(E), \tag{3.6}$$

where  $T_X$  denotes the holomorphic tangent bundle on  $X$  and  $\text{td}_G$  and  $\text{ch}_G$  are equivariant Todd and Chern classes respectively. The integral can then be computed using *e.g.* Atiyah-Bott localisation to yield the same formula as (3.5).

### 3.1.1 Nakajima quiver varieties

We now discuss how the general theory of the previous section applies to Nakajima quiver varieties  $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$  in particular. As discussed in section 2.2, Nakajima quiver varieties are equipped with torus actions  $\mathbb{T} = \mathbb{A} \times \mathbb{C}_t^\times$  which we assume have isolated fixed point sets  $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w})^\mathbb{A}$  and  $\mathbb{C}_t^\times$  denotes the contracting action. We consider the equivariant  $K$ -theory  $K_\mathbb{T}(\mathcal{M}_\theta(\mathbf{v}, \mathbf{w}))$ . To save on notational clutter we write  $X = \mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$  in the following.

**Tautological bundles.** Nakajima quiver varieties  $X$  have a set of  $\mathbb{T}$ -equivariant tautological bundles  $\mathcal{V}_i$  associated to each vertex  $i \in I$ , the fibre at an orbit  $(A, B, I, J)$  is the vector space  $V_i$  on which the quiver representation acts.<sup>5</sup> We can similarly define bundles  $\mathcal{W}_i$  associated to the vector spaces  $(W_i)_{i \in I}$  in the same way, these bundles are topologically trivial but admit an action of  $\mathbb{T}$ . Associated to the vector bundles  $\mathcal{V}_i$  we also have tautological line bundles defined by  $\mathcal{L}_i = \det \mathcal{V}_i$  for  $i \in I$ .

**Kirwan surjectivity.** The following is known as the Kirwan surjectivity conjecture and was proved by McGerty and Nevins in the work [113].

**Theorem 4.** If  $X$  is a Nakajima quiver variety then  $K_{\mathbb{T}}(X)$  is generated by the Schur functors of tautological bundles  $\mathcal{V}_i$  with  $i \in I$ .

Consequently, the Picard group of  $X$  is generated by the tautological line bundles  $\mathcal{L}_i = \det \mathcal{V}_i$  for  $i \in I$  and we have  $\text{Pic}(X) \cong \mathbb{Z}^{|I|}$ . We write  $\zeta_{\mathbb{R}}$  for coordinates on  $\text{Pic}(X) \otimes U(1)$ —these are known as *Kähler parameters* in the enumerative geometry literature. Recall the definition of the topological symmetry (2.3); for the gauge group of  $X$  we have  $G_C = U(1)^{|I|}$  thus the parameters  $\zeta_{\mathbb{R}}$  can be identified with exponentiated FI parameters  $\xi \in \mathfrak{t}_C$ .

Furthermore, since the Chern classes of line bundles generate  $H^2(\mathcal{M}_H, \mathbb{Z})$ , we can identify coordinates on the Lie algebra of the topological symmetry with the second homology

$$\mathfrak{t}_C = H^2(\mathcal{M}_H, \mathbb{Z}). \quad (3.7)$$

For symplectic dual pairs we can further identify

$$\mathfrak{t}_H = H^2(\mathcal{M}_C, \mathbb{Z}). \quad (3.8)$$

so that 3d mirror symmetry exchanges Kähler and equivariant parameters.

The Chern roots of tautological bundles  $\mathcal{V}_i$  are denoted by  $w_a^{(i)}$  with  $a = 1, \dots, \text{rk } \mathcal{V}_i$  and  $i \in I$  and then from theorem 4 the (non-localised) equivariant  $K$ -theory can be identified with the ring of symmetric Laurent polynomials

$$K_{\mathbb{T}}(X) \cong \mathbb{Z}[(w_a^{(i)})^{\pm 1}, (z_a^{(i)})^{\pm 1}, t^{\pm 1}]^{\text{Sym}} / R. \quad (3.9)$$

where  $R$  is the ideal of Laurent polynomials that vanish at the fixed points in a sense to be defined below. We denote by  $\mathcal{V}^{\vee}$  the  $K$ -theory dual which inverts the weights of  $\mathcal{V}$ ,  $\vee : \omega \rightarrow \omega^{-1}$ .

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<sup>5</sup>More precisely  $\mu^{-1}(0)^s \rightarrow \mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w})$  is a principle  $G$ -bundle and  $\mathcal{V}_i$  is the associated vector bundle  $\mu^{-1}(0)^s \times_G V_i$  [41].

A particularly useful  $K$ -theory class is the tangent bundle that appears in the localisation formulae (3.5) of the previous section. The following lemma due to [46] gives us this class as an element of the ring (3.9).

**Lemma 1.** The  $K$ -theory class of the tangent bundle  $T\mathcal{M}(\mathbf{v}, \mathbf{w})$  of a Nakajima quiver variety with quiver graph  $Q$  and vertex set  $I$  is determined in terms of the tautological bundles by the following  $2|I| \times 2|I|$  quadratic form

$$\mathbf{C} = \begin{pmatrix} \bar{Q} - 1 - t & t\mathbb{1}_{|I|} \\ \mathbb{1}_{|I|} & 0_{|I|} \end{pmatrix}, \quad (3.10)$$

so that

$$T\mathcal{M}(\mathbf{v}, \mathbf{w}) = (\vec{\mathcal{V}}^\vee, \vec{\mathcal{W}}^\vee) \cdot \mathbf{C} \cdot (\vec{\mathcal{V}}, \vec{\mathcal{W}})^T. \quad (3.11)$$

**Fixed point characters.** In the case of isolated fixed points, to evaluate the character of a bundle at a fixed point  $p \in X^{\mathbb{T}}$  we simply take  $\iota_{p,\star}^{-1}$  as in the discussion above to give an element of  $K_{\mathbb{T}}(p) = \mathbb{Z}[x_a^{(i)}, t^{\pm 1}]$ . This determines a map on the Chern roots

$$w_a^{(i)} \rightarrow w_a^{(i)}(p) \equiv t_a^{(i)} \in K_{\mathbb{T}}(\text{pt}) \quad (3.12)$$

Hence, given the character of the Chern roots, we can deduce the  $\mathbb{T}$  character of any  $K$ -theory class  $\mathcal{V}$  at a fixed point  $p \in X^{\mathbb{T}}$  by evaluating the Laurent polynomial corresponding to  $\mathcal{V}$  at the values (3.12).

### 3.1.2 Hilbert series

The Hilbert series is defined for an affine variety  $X$  with a torus action  $T$  by

$$\text{H.S.}[X] = \text{ch}_T \mathbb{C}[X]. \quad (3.13)$$

When  $X$  is an unresolved Higgs branch  $\mathcal{M}_H^o$ , this coincides with the physical Hilbert series defined in equation (2.19). If  $X$  admits a symplectic resolution  $\pi : \widetilde{X} \rightarrow X$  then, under suitable conditions, the Hilbert series counts holomorphic functions on  $\widetilde{X}$  and can be computed using the localisation formula (3.5).

Let  $\widetilde{X}$  be a symplectic resolution and  $E = \mathcal{O}_{\widetilde{X}}$  be the structure sheaf. Symplectic resolutions with a contracting  $\mathbb{C}^\times$  action have the property that  $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$  for  $i > 0$  [103]. In this case the equivariant Euler characteristic of the structure sheaf counts global sections  $\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$  (*i.e.* holomorphic functions) and the localisation formula (3.5) gives

$$\chi_T(\mathcal{O}_{\widetilde{X}}, \widetilde{X}) = \text{ch}_T H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = \sum_{x \in \widetilde{X}^T} \text{PE}[T_x^\vee \widetilde{X}]. \quad (3.14)$$

For symplectic resolutions,  $\pi$  induces a Poisson algebra isomorphism  $\pi^* : \mathbb{C}[X] \rightarrow \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$  and so we have

$$\chi_T(\mathcal{O}_{\widetilde{X}}, \widetilde{X}) = \text{ch}_T \mathbb{C}[X] =: \text{H.S.}[\widetilde{X}]. \quad (3.15)$$

**Line bundles.** We also consider equivariant counts of sections of line bundles  $\mathcal{L} \subset \text{Pic}(X)$ . Using localisation (3.5) we can compute the equivariant sections of  $\mathcal{L}$  (assuming higher cohomology of the line bundle in question vanishes) in terms of characters at the fixed point by

$$\chi_T(\mathcal{L}, \widetilde{X}) = \text{ch}_T H^0(\widetilde{X}, \mathcal{L}) = \sum_{x \in \widetilde{X}^T} \text{ch}_T(\mathcal{L}_x) \text{PE}[T_x^\vee \widetilde{X}]. \quad (3.16)$$

**Example 3.1.** We return to the SQED[ $N$ ] example with Higgs branch  $\mathcal{M}_H = T^* \mathbb{P}^{N-1}$ . As discussed in section 2.3, the fixed points  $\alpha \in \mathcal{M}_H^T$  are labelled by coordinate lines and correspond to an element  $\alpha \in \{1, \dots, N\}$ . There is one tautological (line) bundle which we express in terms of the Chern root  $\mathcal{V} = w$ .

From the discussion in section 2.3, the flavour group action scales the fibre by  $x_\alpha^{-1}$  at the fixed point  $\alpha$  and the evaluation of the Chern root, as in (3.12), is then simply  $w \rightarrow w(\alpha) = x_\alpha^{-1}$ . The rank  $N$  topologically trivial tautological bundle  $\mathcal{W}$  is given by

$$\mathcal{W} = x_1 + \dots + x_N. \quad (3.17)$$

Following the recipe in lemma 1, the tangent bundle for this quiver can be expressed in terms of the tautological bundles as

$$T\mathcal{M}_H = \mathcal{W}^\vee \otimes \mathcal{V} - \mathcal{V}^\vee \otimes \mathcal{V} + t\mathcal{V}^\vee \otimes \mathcal{W} - t\mathcal{V} \otimes \mathcal{V}^\vee, \quad (3.18)$$

and evaluating this at the fixed point  $\alpha$  we find

$$T_\alpha \mathcal{M}_H = \sum_{j \neq \alpha}^N \left( \frac{x_\alpha}{x_j} + t \frac{x_j}{x_\alpha} \right). \quad (3.19)$$

We now have the ingredients to compute the Hilbert series twisted by a line bundle using equation (3.16) which gives

$$\chi_T(\mathcal{L}^{\otimes \kappa}, \mathcal{M}_H) = \sum_{\alpha=1}^N x_\alpha^\kappa \prod_{j \neq \alpha}^N \frac{1}{(1 - x_\alpha/x_j)(1 - tx_j/x_\alpha)}. \quad (3.20)$$

**Quiver varieties and Molien integrals.** If  $X = \mathcal{M}(\mathbf{v}, \mathbf{w})$  is a quiver variety then we have a concrete algebraic handle on the holomorphic functions since  $\Gamma(X, \mathcal{O}_X) = \mathbb{C}[\mathcal{M}_0(\mathbf{v}, \mathbf{w})]$  and the latter ring is, by definition, given by  $\mathbb{C}[\mu^{-1}(0)]^{G_{\mathbb{C}}}$ .

We have the following due to [114]

**Lemma 2.** Let  $X = \mathcal{M}(\mathbf{v})$  be a double quiver without framing.  $\mathbb{C}[X]$  is generated by elements of the form

$$\mathrm{Tr}_{V_{i_1}} X_{i_1 i_2} X_{i_2 i_3} \cdots X_{i_{n-1} i_n} X_{i_n, i_1}, \quad (3.21)$$

where  $(i_1, \dots, i_n, i_1)$  is an oriented cycle in the quiver graph  $\bar{Q}$  with corresponding operators  $X_{i,j} : V_i \rightarrow V_j$ . The trace is taken over  $V_{i_1}$ .

This lemma together with the trick<sup>6</sup> of Crawley-Boevey [115] then gives the following

**Theorem 5.** Let  $X = \mathcal{M}(\mathbf{v}, \mathbf{w})$  be a framed double quiver variety with adjacency matrix  $Q$ . Let  $p = (i_1, i_2, \dots, i_n, i_1)$  be a closed oriented cycle in  $Q$  and  $q = (j_1, \dots, j_k)$  be a connected path in  $Q$ . Denote by  $X_{ij} \in \mathrm{Hom}(V_i, V_j)$  the corresponding operators. The coordinate ring  $\mathbb{C}[X]$  is generated by elements of the two types

$$\begin{aligned} A_p &= \mathrm{Tr}_{V_i} X_{i_1 i_2} X_{i_2 i_3} \cdots X_{i_{n-1} i_n} X_{i_n, i_1}, \\ B_q &= I_{j_1} X_{j_1 j_2} X_{j_2 j_3} \cdots X_{j_{k-1} j_k} J_{j_k}, \end{aligned} \quad (3.22)$$

where  $I_i : W_i \rightarrow V_i$  and  $J_i : V_i \rightarrow W_i$ .

In plain English, the holomorphic functions on a quiver variety are given by appropriate traces over gauge-invariant polynomials in the scalars that end on either a gauge node or begin and end on a flavour node.

The *Molien-Weyl integral* method is a convenient device to compute the Hilbert series of the coordinate ring. The method belongs to algebraic invariant theory [116] and has been popularised in physics by the plethystic program of Hanany *et. al.* [117]. The Molien integral counts generators in the coordinate ring by

$$\mathrm{H.S.}[\mathcal{M}(\mathbf{v}, \mathbf{w})] = \langle \mathrm{PE} \left[ \mathrm{ch}_{G_{\mathbb{C}} \times G_W \times \mathbb{C}_t^\times} \mathcal{Q}(\mathbf{v}, \mathbf{w}) - \mathrm{ch}_{G_{\mathbb{C}} \times G_W \times \mathbb{C}_t^\times} \mu \right], 1 \rangle_{G_{\mathbb{C}}}. \quad (3.23)$$

The plethystic exponential, defined in (A.12), on the right hand side of this expression is a generating function of all possible ‘words’ from scalars in  $\mathcal{Q}(\mathbf{v}, \mathbf{w})$  quotient the ideal generated by  $\mu$  that implements moment map constraint.  $\langle, \rangle_{G_{\mathbb{C}}}$  is the inner product on the character ring of  $G_{\mathbb{C}}$  and we project onto gauge invariant words by taking the inner product against the identity character. Physically, the  $G_W$  characters refine the Hilbert series by flavour symmetries and the contracting  $\mathbb{C}_t^\times$  action is a refinement by  $R$ -symmetry  $R_H$ . The expression (3.23) can be written in terms of the Haar measure for  $G_{\mathbb{C}}$  which, for

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<sup>6</sup>Crawley-Boevey [115] shows that a framed quiver variety is isomorphic to an un-framed quiver with vertex set  $I \cup \{\infty\}$  and  $\mathbf{w}_i$  extra edges for each  $i \in I$  connectecting  $i \rightarrow \infty$ .

the unitary gauge groups discussed in this work, leads to the Molien integral

$$\text{H.S.}[\mathcal{M}(\mathbf{v}, \mathbf{w})] = \prod_{i \in I} \frac{1}{\mathbf{v}_i!} \prod_{a=1}^{\mathbf{v}_i} \oint_C \frac{dw_a^{(i)}}{2\pi i w_a^{(i)}} \prod_{a \neq b}^{\mathbf{v}_i} \left(1 - \frac{w_a^{(i)}}{w_b^{(i)}}\right) \text{PE}[\mathcal{Q}(\mathbf{v}, \mathbf{w}) - \mu] . \quad (3.24)$$

where the contour  $C$  is the unit torus. The Molien integral can also be understood in terms of geometry and the localisation formula (3.5) in  $K$ -theory. The integration variables  $w_a^{(i)}$  of (3.24) can be interpreted as elements of the  $K$ -theory ring (3.9), specifically the  $a = 1, \dots, \mathbf{v}_i$  Chern roots of the rank  $\mathbf{v}_i$  tautological bundles  $\mathcal{V}_i$ . The terms  $\mathcal{Q}(\mathbf{v}, \mathbf{w})$  and  $\mu$  in the integrand are associated with  $K$ -theory classes in a way that we discuss in the following. The poles of the integral are in 1-1 correspondence with  $\mathbb{T}$ -fixed points of  $\mathcal{M}(\mathbf{v}, \mathbf{w})$  and evaluating the integral by residues corresponds to evaluating the plethystic exponential of the tangent bundle at the Chern roots evaluated at fixed points as in equation (3.12). From this perspective, the Hilbert series twisted by a line bundle (3.16) can also be expressed in a similar integral form by inserting the Chern character of a line bundle  $\mathcal{L}$ :

$$\chi_{\mathbb{T}}(\mathcal{L}, \mathcal{M}(\mathbf{v}, \mathbf{w})) = \prod_{i \in I} \prod_{a=1}^{\mathbf{v}_i} \oint_C \frac{dw_a^{(i)}}{2\pi i w_a^{(i)}} \prod_{a \neq b}^{\mathbf{v}_i} \left(1 - \frac{w_a^{(i)}}{w_b^{(i)}}\right) \text{ch}(\mathcal{L}) \text{PE}[\mathcal{Q}(\mathbf{v}, \mathbf{w}) - \mu] . \quad (3.25)$$

Later, in chapter 4, we show that for a 3d  $\mathcal{N} = 4$  gauge theory the  $B$ -twisted index reproduces the Hilbert series in the form of the Molien integral (3.24), in equation (4.61) of that chapter the integrand is shown explicitly.

The Molien integral can be rephrased in terms of  $K$ -theory classes of the quiver  $\mathcal{M}(\mathbf{v}, \mathbf{w})$  with adjacency matrix  $Q$ . Recall that  $K_{\mathbb{T}}(\mathcal{M}(\mathbf{v}, \mathbf{w}))$  is generated over  $K_{\mathbb{T}}(\text{pt})$  by the Chern roots  $(w_a^{(i)})_{a=1, \dots, \mathbf{v}_i}$  and  $(x_a^{(i)})_{a=1, \dots, \mathbf{w}_i}$  for  $i \in I$ . The classes associated to  $\mathcal{R}(\mathbf{v}, \mathbf{w})$  and  $\mu$  are given by

$$\begin{aligned} \mathcal{Q}(\mathbf{v}, \mathbf{w}) &= \bigoplus_{i,j \in I} \mathcal{V}_i^{\vee} (Q_{ij} + t Q_{ij}^T) \mathcal{V}_j + \bigoplus_{i \in I} t \mathcal{W}_i^{\vee} \otimes \mathcal{V}_i + \bigoplus_{i \in I} \mathcal{W}_i \otimes \mathcal{V}_i^{\vee} , \\ \mu &= \bigoplus_{i \in I} t \mathcal{V}_i^{\vee} \otimes \mathcal{V}_i . \end{aligned} \quad (3.26)$$

**Example 3.2.** We compute the Molien integral for SQED[ $N$ ] with  $\mathcal{M}_H = T^*\mathbb{P}^{N-1}$ . The relevant bundles are discussed in example 3.1. We can then use (3.26) to write down the Molien integral to compute the Hilbert series of the tautological line bundle

$$\begin{aligned} \chi_{\mathbb{T}}(\mathcal{L}^{\kappa}, \mathcal{M}_H) &= \oint_{S^1} \frac{dw}{2\pi i w} w^{-\kappa} \text{PE} \left[ \sum_{i=1}^N t w^{-1} x_i^{-1} + \sum_{i=1}^N w x_i - t \right] , \\ &= (1-t) \oint_{S^1} \frac{dw}{2\pi i w} w^{-\kappa} \prod_{i=1}^N \frac{1}{1 - \frac{tw}{x_i}} \frac{1}{1 - w x_i} . \end{aligned} \quad (3.27)$$

This integral has poles at  $w = x_p^{-1}$  for  $p = 1, \dots, N$  which are precisely the evaluations of the Chern roots (3.12). From the first line above we see that evaluating the residues here reproduces the localisation formula (3.5) and we recover the Hilbert series.

**Remark.** The Molien integral (3.24) is ‘Hall-Littlewood type’ in the heirarchy of symmetric functions (see appendix B) with the contracting  $\mathbb{C}_t^\times$  playing the role of the Hall-Littlewood parameter  $t$ . We note here that there is a natural ‘Macdonald  $q$ -deformation’ of (3.24)

$$\mathcal{I}_q(\mathcal{M}(\mathbf{v}, \mathbf{w})) = \prod_{i \in I} \frac{1}{\mathbf{v}_i!} \prod_{a=1}^{\mathbf{v}_i} \oint_C \frac{dw_a^{(i)}}{2\pi i w_a^{(i)}} \prod_{a \neq b}^{\mathbf{v}_i} (w_a^{(i)} / w_b^{(i)}; q)_\infty \text{PE} \left[ \frac{1}{1-q} (\mathcal{Q}(\mathbf{v}, \mathbf{w}) - \mu) \right] \quad (3.28)$$

We evaluate integrals of this type in chapter 4 where they count local boundary operators supported on a Neumann boundary condition in a 3d  $\mathcal{N} = 4$  gauge theory. It would be interesting to investigate further the geometric interpretation of these  $q$ -deformed ‘Molien integrals’.

### 3.1.3 The twisted $\chi_t$ genus

The Hirzebruch  $\chi_t$  genus [118] is a generalisation of the Hilbert series that, at least in the smooth projective case, can be understood as a count of holomorphic forms  $\alpha \in H_{\bar{\partial}}^{p,0}$  as well as just the holomorphic functions counted by the Hilbert series (2.19). Physically, the  $\chi_t$  genus can realise supersymmetric indices of certain one dimensional non-linear sigma models, for example the one dimensional  $\mathfrak{osp}(4^*|4)$  Higgs branch sigma models of [119, 120]. In the present work, vortex partition functions of certain 3d  $\mathcal{N} = 4$  gauge theories will be realised by  $\chi_t$  genera.

The  $\chi_t$  genus can be computed by geometric localisation. In fact, we consider a mild generalisation of the  $\chi_t$  genus twisted by a line bundle  $\mathcal{L}$ . Later, this line bundle will play the role of topological background flux but it is also important as a technical tool to match to the *symmetrised* virtual structure sheaf localisation as we will see later in section 4.1.1.

**Definition 3.1.** Given a line bundle  $\mathcal{L} \in \text{Pic}(X)$  we define the *twisted  $\chi_t$  genus* by<sup>7</sup>

$$\begin{aligned} \chi_t(X) &\equiv \sum_{j \geq 0} (-t)^j \chi_T(\Omega_X^j \otimes \mathcal{L}, X), \\ &= \sum_{i,j \geq 0} t^{i+j} \text{ch}_T H^i(X, \Omega_X^j \otimes \mathcal{L}), \end{aligned} \quad (3.29)$$

where  $\Omega_X^j = \Lambda^j T^\vee X$  is the sheaf of algebraic  $j$ -differentials.

The  $\chi_t$  genus is a polynomial in  $t$  of at most degree  $\dim X$ . Applying the localisation

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<sup>7</sup>Note that this has  $t \rightarrow -t$  compared to the standard definition of the Hirzebruch genus.



formula (3.5) we have:

$$\begin{aligned}\chi_t(X) &= \sum_{j \geq 0} (-t)^j \sum_{x \in X^T} \text{ch}_T(\mathcal{L}_x) \text{ch}_T(\Lambda^j T_x^\vee X) \text{PE}[T_x^\vee X] \\ &= \sum_{x \in X^T} \text{ch}_T(\mathcal{L}_x) \text{PE}[(1-t)T_x^\vee X] .\end{aligned}\tag{3.30}$$

We note two useful limits in the case  $\mathcal{L} = \mathcal{O}_X$ . Firstly the  $t \rightarrow 1$  limit of (3.30) in which we find:

$$\lim_{t \rightarrow 1} \chi_t(X) = \sum_{x \in X^T} 1 = |X^T| ,\tag{3.31}$$

so that this limit counts fixed points on  $X$ . Secondly the limit  $t \rightarrow 0$  which recovers the Hilbert series (provided the higher cohomology of the structure sheaf vanishes):

$$\begin{aligned}\lim_{t \rightarrow 0} \chi_t(X) &= \sum_{x \in X^T} \text{PE}[T_x^\vee X] \\ &= \text{H.S.}[X] .\end{aligned}\tag{3.32}$$

Finally, we note that Serre duality implies the property

$$\chi_t(X) = t^{\dim X} \chi_{t^{-1}}(X) .\tag{3.33}$$

**Projective varieties.** When  $X$  is projective (*i.e.* a compact Kähler manifold) the sheaf cohomology groups  $H^q(X, \Omega_X^p)$  are identified with the Dolbeault cohomology groups  $H_{\bar{\partial}}^{p,q}(X)$ . In this case, the (untwisted)  $\chi_t$  genus coincides with the Poincaré polynomial of  $X$ .

Denote by  $\lambda(s) \subset T$  a generic<sup>8</sup> circle action. Since  $X$  is projective then (3.29) is a finite Laurent series in  $s$  and the only possible singularities are at  $s = 0$  or  $s = \infty$ . Now we consider taking these limits in the localisation formula (3.30). We have

$$\chi_t(X) = \sum_{x \in X^T} \prod_{i=1}^{\dim X} \frac{1 - t s^{d_i^{(x)}}}{1 - s^{d_i^{(x)}}} ,\tag{3.34}$$

where  $d_i^{(x)}$  are a set of integers (they correspond to the attracting and repelling weights in the following subsection). In particular the exponents of  $s$  in the numerators and denominators are equal and so  $\chi_t(X)$  has finite limits at  $s = 0$  and  $s = \infty$  and must therefore be constant in  $s$ . In particular we can take  $s = 1$  so that the character degenerates to the dimension and then by the Hodge decomposition we recover the Poincaré polynomial.

In summary, the  $\chi_t$  genus of a compact space does not depend on global isometries. Later, in chapter 4, this plays a crucial role for us where we show that the  $A$ -twisted index

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<sup>8</sup>In the sense that  $X$  has isolated fixed points under  $\lambda$ .

of a 3d  $\mathcal{N} = 4$  theory can be realised as the  $\chi_t$  genus of a compact quasimap space and we find significant cancellations of this nature.

### 3.1.4 Poincaré polynomials

In this subsection we discuss a localisation formula for the Poincaré polynomial of a smooth projective  $T$ -variety  $X$  with isolated fixed points. The Poincaré polynomial is defined by

$$P_X(t) := \sum_{i \geq 0} t^i \dim H_i(X, \mathbb{Z}). \quad (3.35)$$

We review the Bialynicki-Birula decomposition [121] which provides a way to compute these Poincaré polynomials from fixed point data. We choose a generic one parameter subgroup  $\lambda(t) \subset T$  and decompose the tangent space at a fixed point  $\alpha \in X^T$  into positive and negative torus weights for  $\lambda$ :

$$T_\alpha X = T_\alpha^+ X \oplus T_\alpha^- X. \quad (3.36)$$

The attracting and repelling cells are defined by

$$Y_\alpha^+ := \{x \in X : \lim_{t \rightarrow 0} t \cdot x = \alpha\}, \quad Y_\alpha^- := \{x \in X : \lim_{t \rightarrow \infty} t \cdot x = \alpha\}. \quad (3.37)$$

The attracting and repelling cells  $Y_\alpha^+$  and  $Y_\alpha^-$  are affine spaces and give cellular decompositions of  $X$  [121] of dimensions  $p_\alpha$  and  $n_\alpha$  respectively so that

$$X = \bigcup_{\alpha} Y_\alpha^+ = \bigcup_{\alpha} Y_\alpha^-. \quad (3.38)$$

The homology basis formula then yields

$$\begin{aligned} H_i(X, \mathbb{Z}) &= \bigoplus_{\alpha \in X^T} H_{i-2p_\alpha}(\{\alpha\}, \mathbb{Z}) \\ &= \bigoplus_{\alpha \in X^T} H_{i-2n_\alpha}(\{\alpha\}, \mathbb{Z}), \end{aligned} \quad (3.39)$$

so that each attracting/repelling cell contributes one generator to the homology. Consequently, the decomposition leads to a convenient expression for the Poincaré polynomial of  $X$  expressed in terms of the dimensions of the positive and negative weight spaces of a generic torus action at a fixed point:

$$P_X(t) = \sum_{\alpha \in X^T} t^{2n_\alpha} = \sum_{\alpha \in X^T} t^{2p_\alpha}. \quad (3.40)$$

In particular, note that  $X$  has no odd homology—we abuse notation throughout the thesis by redefining the Poincaré polynomial to be  $P_X(t) := P_X(t^2)$ .

**Example 3.3.** The tangent space of  $\mathbb{P}^{N-1}$  is given by the  $t \rightarrow 0$  limit of (3.19), that is  $T_\alpha \mathbb{P}^{N-1} = \sum_{j \neq \alpha} x_\alpha / x_j$ . Choosing a chamber  $x_1 < \dots < x_N$  we see that at each fixed point  $\alpha = 1, \dots, N$  we have  $p_\alpha = N - \alpha$  and  $n_\alpha = \alpha$  so that

$$P_{\mathbb{P}^{N-1}}(t) = 1 + t + \dots + t^{N-1}. \quad (3.41)$$

**Connection to  $\chi_t$  genus.** The arguments in the previous section explain that when  $X$  is a projective variety with a torus action then the  $\chi_t$  genus is independent of the torus fugacities. In particular we can send  $s \rightarrow 0$  or  $s \rightarrow \infty$  in the expression (3.34). We then recover either of the localisation formulae (3.40) so that

$$\chi_t(X) = P_X(t). \quad (3.42)$$

**Non-compact case.** Under certain conditions there is a connection between the  $\chi_t$  genus of a non-compact space and the Poincaré polynomial of a compact submanifold due to [122]. Let  $\pi : \widetilde{X} \rightarrow X$  be an equivariant resolution with a contracting  $\mathbb{C}_q^\times$  action and isolated fixed points under a torus action  $T$ . Denote by  $o \in X$  the unique fixed point in the singular space. Under these conditions we have

$$\lim_{q \rightarrow 0} \chi_t(\widetilde{X}) = P_{\pi^{-1}(o)}(t). \quad (3.43)$$

The idea of the proof is that, under these assumptions, we can perturb the  $\mathbb{C}_q^\times$  action to a more general subgroup of  $T$  before sending the corresponding fugacity to zero. Since the  $T$ -fixed points are isolated we then recover the fixed point formula (3.40) and together with the fact that  $\{Y_\alpha^+\}$  is again a cellular decomposition of  $\pi^{-1}(o)$  the result follows.

**Nakajima quiver varieties.** Nakajima quiver varieties  $\mathcal{M}(\mathbf{v}, \mathbf{w})$  are symplectic resolutions and, thought of as a Higgs branch  $\mathcal{M}_H$  of a 3d  $\mathcal{N} = 4$  theory, associated to the flavour symmetry  $G_H$  is a real moment map  $\mu_{\mathbb{R}, H}$ . Turning on real mass parameters  $m_{\mathbb{R}}$  is equivalent to choosing a particular circle action  $\lambda(t) \subset T$  in the above discussion. The fixed points are equivalently given by critical points of the Morse function [123]

$$h_m = \mu_{\mathbb{R}, H} \cdot m_{\mathbb{R}}. \quad (3.44)$$

The condition of generic mass parameters is then equivalent to  $\overline{\exp(\mathbb{R} \cdot m_{\mathbb{R}})} = T$ . In this context, the attracting cells are holomorphic Lagrangians attained by Morse flow [55]. The attracting cells depend on the chamber  $\mathfrak{C}_H$  and are associated to vacua  $\alpha$ , we

denote them by  $\mathcal{L}_\alpha$  and in section 4.3 we realise them as Higgs branch images of certain  $\mathcal{N} = (2, 2)$  boundary conditions. Poincaré polynomials of certain quiver varieties can also be computed using the localisation formula (3.40) since they are homotopic to their central fibre  $\pi^{-1}(o)$ , which is often compact [41, 40].

**Example 3.4.** We compute the twisted equivariant  $\chi_y$  genus<sup>9</sup> for SQED[ $N$ ] using the ingredients from example 3.1 and the localisation formula (3.30). We have

$$\chi_y(T^*\mathbb{P}^{N-1}) = \sum_{\alpha=1}^N x_\alpha^\kappa \prod_{j \neq \alpha}^N \frac{(1 - yx_\alpha/x_j)(1 - tx_j/x_\alpha)}{(1 - x_\alpha/x_j)(1 - tx_j/x_\alpha)}. \quad (3.45)$$

The Hilbert series limit  $y \rightarrow 0$  recovers (3.20). Now  $t$  corresponds to the contracting  $\mathbb{C}_t^\times$  action and the compact core is given by  $\pi^{-1}(o) = \mathbb{P}^{N-1}$ . Now sending  $t \rightarrow 0$  in the above (setting the line bundle charge  $\kappa$  to zero now) we find

$$\begin{aligned} \lim_{t \rightarrow 0} \chi_y(T^*\mathbb{P}^{N-1}) &= \sum_{\alpha=1}^N \prod_{j \neq \alpha}^N \frac{1 - yx_\alpha/x_j}{1 - x_\alpha/x_j} \\ &= 1 + y + \dots + y^{N-1}. \end{aligned} \quad (3.46)$$

which we recognise as the Poincaré polynomial of  $\mathbb{P}^{N-1}$  (3.41). Note that in going from the first to the second line it is not *a priori* obvious that the expression is indeed independent of  $x_1, \dots, x_N$ . This is a toy example of the vortex partition functions that we meet in chapter 4 where it is necessary to use the theoretical framework developed in the present section since it is not as straightforward to compute the relevant limits by hand.

## 3.2 Macdonald polynomials and integrability

In this section we review representation theoretic aspects of spin chain partition functions. We found it useful to review material for Yangian representation theory in a self-contained way since much of this material has either been absorbed into the theory of  $q$ -characters for quantum affine algebras [124] or is somewhat scattered throughout the older literature on the quantum inverse scattering method. We want to focus on the Yangian case since these algebras are of the type relevant for Higgs and Coulomb branch algebras of 3d  $\mathcal{N} = 4$  gauge theories.

**Overview.** We begin in section 3.2.1 with a review of Yangian algebras and a particular class of modules known as *Kirillov-Reshetikhin modules*. In section 3.2.2 we review Knight’s [125] definition of Yangian characters—these allow us to efficiently make contact with the quantum  $Q$ -system of [126]. In section 3.2.3 we discuss the relationship between the

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<sup>9</sup>Note we switch to the  $\chi_y$  notation since  $t$  is typically reserved for the contracting  $\mathbb{C}^\times$  action.

physics of the Heisenberg XXX spin chain and Yangian representation theory. Section 3.2.4 concludes with a discussion of a class of symmetric functions known as *Milne polynomials* that we show coincide with spin chain partition functions. We show later in the thesis that Milne polynomials coincide with equivariant counts of line bundles over a particular quiver variety and the present section provides the background to interpret this result in the context of integrability.

**Conventions.** The conventions for symmetric functions and partitions used in this section are reviewed in appendices A and B. Throughout  $\mathfrak{g}$  denotes a finite dimensional complex semi-simple Lie algebra of rank  $r$ . The Cartan subalgebra is denoted  $\mathfrak{h}$  and the Cartan matrix is denoted by  $A = (A_{ij})_{i,j=1,\dots,r}$ . We denote the simple roots of  $\mathfrak{g}$  by  $(\alpha_i)_{i=1,\dots,r}$  and the fundamental weights by  $(\omega_i)_{i=1,\dots,r}$ . We denote by  $(-, -)$  the standard invariant non-degenerate bilinear form on  $\mathfrak{h}^*$  and let  $d_i = (\alpha_i, \alpha_i)$  for  $i = 1, \dots, r$ .

### 3.2.1 Yangians and Kirillov-Reshetikhin modules

**Yangian.** The Yangian  $Y(\mathfrak{g})$  is an infinite dimensional Hopf algebra over  $\mathbb{C}$  that quantises the universal enveloping algebra of the loop algebra  $U(\mathfrak{g}[u])$ . We begin with Drinfeld's first presentation [127] which exhibits  $Y(\mathfrak{g})$  as an algebra generated by  $x$  and  $J(x)$  for  $x \in \mathfrak{g}$  satisfying the relations

$$\begin{aligned} [x, y]_{Y(\mathfrak{g})} &= [x, y]_{\mathfrak{g}}, \\ [x, J(y)] &= J([x, y]), \end{aligned} \tag{3.47}$$

together with Serre-type relations. We direct the reader to [127] for a full list and note here only that this presentation makes manifest that  $\mathfrak{g}$  is a subalgebra of  $Y(\mathfrak{g})$ . The Yangian is a Hopf algebra equipped with a coproduct structure  $\Delta : Y(\mathfrak{g}) \rightarrow Y(\mathfrak{g}) \otimes Y(\mathfrak{g})$  that allows the construction of tensor product representations. The coproduct is given by<sup>10</sup>

$$\begin{aligned} \Delta(x) &= x \otimes I + 1 \otimes x, \\ \Delta(J(x)) &= J(x) \otimes 1 + 1 \otimes J(x) + \frac{1}{2}[x \otimes 1, \Omega]. \end{aligned} \tag{3.48}$$

The Yangian admits a (pseudo<sup>11</sup>) universal  $R$ -matrix. The  $R$ -matrix is an element  $\mathcal{R}(\lambda) \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})[[\lambda^{-1}]]$  that satisfies

$$\begin{aligned} (\Delta \otimes I)\mathcal{R}(\lambda) &= \mathcal{R}_{13}(\lambda)\mathcal{R}_{23}(\lambda), \\ (1 \otimes \Delta)\mathcal{R}(\lambda) &= \mathcal{R}_{13}(\lambda)\mathcal{R}_{12}(\lambda). \end{aligned} \tag{3.49}$$

<sup>10</sup> $\Omega$  denotes the quadratic Casimir of  $U(\mathfrak{g})$ .

<sup>11</sup>The Yangian is only an *almost* quasi-triangular Hopf algebra in the terminology of [128] and does not admit a universal  $R$ -matrix.

where these equations are in  $Y(\mathfrak{g})^{\otimes 3}$  and  $\mathcal{R}_{ij}$  denotes the action in the  $i, j$  component of the tensor product. The  $R$ -matrix satisfies the quantum Yang-Baxter equation

$$\mathcal{R}_{12}(\lambda_1 - \lambda_2)\mathcal{R}_{13}(\lambda_1 - \lambda_3)\mathcal{R}_{23}(\lambda_2 - \lambda_3) = \mathcal{R}_{23}(\lambda_2 - \lambda_3)\mathcal{R}_{13}(\lambda_1 - \lambda_3)\mathcal{R}_{12}(\lambda_1 - \lambda_2). \quad (3.50)$$

**Highest weight modules.** To understand the representation theory of  $Y(\mathfrak{g})$  it is more convenient to work with an alternative presentation of the Yangian known as Drinfeld's second realisation [129]. In this work  $Y(\mathfrak{g})$  is presented as an associative algebra with generators  $X_{i,k}^\pm$  and  $H_{i,k}$  with  $i = 1, \dots, r$  and  $k = 1, 2, \dots$ . We refer the reader to [129] for a full list of generators but note here that among them are

$$[H_{i,k}, H_{j,l}] = 0, \quad (3.51)$$

so that the elements  $H_{i,k}$  generate the analogue of a Cartan subalgebra for  $Y(\mathfrak{g})$ . The relations also include

$$[H_{i,0}, X_{j,k}^\pm] = \pm d_i A_{ij} X_{j,k}^\pm. \quad (3.52)$$

Heuristically, we see that these relations provide a PBW decomposition of the Yangian into  $Y(\mathfrak{g}) = Y^- \otimes Y^0 \otimes Y^+$  where the factors are constructed from ordered monomials in the generators  $X^-, H$  and  $X^+$  respectively; this allows progress to be made understanding the modules of  $Y(\mathfrak{g})$ . The full details of this construction can be found in [130]. It is possible [129] to write an explicit isomorphism  $\phi : Y(\mathfrak{g}) \rightarrow Y(\mathfrak{g})$  between the two different presentations, we note only that it maps the Cartan elements by

$$\phi(H_i) = d_i^{-1} H_{i,0}, \quad (3.53)$$

where  $H_i$  is an element of  $\mathfrak{g}$ .

We now discuss the representation theory of  $Y(\mathfrak{g})$ . We define weight spaces of a  $Y(\mathfrak{g})$ -module  $V$  by

$$V_{\mathbf{h}} = \{v \in V : H_{i,k} \cdot v = h_{i,k} v, \forall i, k\}. \quad (3.54)$$

We say that  $V$  is highest weight if  $V = Y(\mathfrak{g}) \cdot v$  for some vector  $v \in V_{\mathbf{h}}$  with  $X_{i,k}^+ \cdot v = 0$  for all  $i$  and  $k$ . With these definitions, the following is due to [129]

**Theorem 6.** Every finite dimensional irreducible  $Y(\mathfrak{g})$ -module  $V$  is highest weight.

We denote such highest weight modules by  $V(\mathbf{h})$ . There exists a construction of these modules analogous to semi-simple Lie algebra constructions of highest weight modules via quotients of Verma modules. The following theorem then gives the conditions analogous to the highest weight theorem for a highest weight  $Y(\mathfrak{g})$ -module to be finite dimensional:

**Theorem 7.** (Drinfeld [129]) The irreducible module  $V(\mathbf{h})$  is finite dimensional if and

only if there exist polynomials  $P_i \in \mathbb{C}[\lambda]$  for  $i = 1, \dots, r$  such that

$$\frac{P_i(\lambda + d_i)}{P_i(\lambda)} = 1 + \sum_{k=0}^{\infty} h_{i,k} \lambda^{-k-1}. \quad (3.55)$$

The so-called *Drinfeld polynomials*  $P_i$  are determined up to a constant and so we have a correspondence between finite dimensional irreducible representations of  $Y(\mathfrak{g})$  and  $r$ -tuples of monic polynomials. We now turn to a distinguished class of modules known as *Kirillov-Reshetikhin (KR) modules* [131], or fundamental modules—these modules have the property that any finite dimensional  $Y(\mathfrak{g})$  module is a subquotient of a tensor product of KR modules [128]. Kirillov-Reshetikhin modules are also the class for which the quantum inverse scattering method can be formulated for a Heisenberg spin chain [131]. They are defined as follows

**Definition 3.2.** Kirillov-Reshetikhin modules are finite dimensional irreducible representations  $V(\mathbf{h})$  of  $Y(\mathfrak{g})$  with Drinfeld polynomials satisfying

$$P_j(\lambda) = \begin{cases} \lambda - a & j = i, \\ 1 & j \neq i. \end{cases} \quad (3.56)$$

for some  $i \in \{1, \dots, r\}$ .

Recall that  $\mathfrak{g}$  is a subalgebra of  $Y(\mathfrak{g})$  so  $V(\mathbf{h})$  can be pulled back to a  $\mathfrak{g}$ -module we denote by  $\lambda$  the maximal weight of this module. From the isomorphism (3.53) we have that  $H_{i,0} = d_i \lambda(H)_i$  and then from (3.55) we deduce  $\deg(P_i) = \lambda(H_i)$ . Kirillov-Reshetikhin modules can then be labelled uniquely as  $\text{KR}_\lambda(a)$ .

**Evaluation modules.** We restrict to the case  $\mathfrak{g} = \mathfrak{sl}_{r+1}$ . In that case we have a family, labelled by  $a \in \mathbb{C}$ , of evaluation maps  $\text{ev}_a : Y(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  given by the explicit action on the generators in the presentation (3.47)

$$\begin{aligned} \text{ev}_a(x) &= x, \\ \text{ev}_a(J(x)) &= ax + \frac{1}{4} \sum_{a,b} \text{tr}(x(x_a x_b + x_b x_a)) x_a x_b \end{aligned} \quad (3.57)$$

where  $\{x_a\}$  is an orthonormal basis of  $\mathfrak{g}$ . Such evaluation maps that are the identity on  $\mathfrak{g}$  are not available for more Lie algebra general type. If  $V_\lambda$  is a representation of  $\mathfrak{g}$  then we have a pull-back representation of  $Y(\mathfrak{g})$  denoted  $\text{ev}_a^* V_\lambda$  and when  $\lambda$  is a multiple of a fundamental weight  $\lambda = m\omega_i$  the pull-back representation coincides with the Kirillov-Reshetikhin module

$$\text{KR}_{m\omega_i}(a) \cong \text{ev}_a^* V_{m\omega_i}. \quad (3.58)$$

### 3.2.2 Yangian characters

Yangian characters were introduced by Knight in the work [125] and have since been absorbed into the theory of  $q$ -characters for quantum affine algebras [124, 132]. In this subsection we review Knight's construction of Yangian characters.

The generators  $H_{i,k}$  mutually commute however, in general, they do not act semi-simply on a  $Y(\mathfrak{g})$ -module  $V$ . Instead the module  $V$  can be decomposed into generalised eigenspaces for  $H_{i,k}$ . We write

$$V = \bigoplus_{\beta} V_{\beta}, \quad (3.59)$$

$$V_{\beta} = \{v \in V : (H_{i,k} - d_{i,k}^{\beta})^r v = 0 \text{ for some } r > 0\}.$$

The eigenvalues can then be arranged in generating series  $\beta_i = 1 + \sum_{k=0}^{\infty} d_{i,k}^{\beta} u_i^{-k-1}$ . The Yangian characters are elements of the group algebra  $\mathbb{Z}[\mathcal{L}_N]$  of formal Laurent series where

$$\mathcal{L}_r = \{f(u_1, \dots, u_r) = \prod_{i=1}^r f_i(u_i)\}, \quad (3.60)$$

with Laurent polynomials  $f_i$  of the form  $f_i(u_i) = 1 + \sum_{k=0}^{\infty} a_k u_i^{-k-1}$ . The characters are then defined by

**Definition 3.3.** The character of a Yangian module  $V$  is a map  $\text{ch}_{Y(\mathfrak{g})} : \text{Rep } Y(\mathfrak{g}) \rightarrow \mathbb{Z}[\mathcal{L}_r]$  given by

$$\text{ch}_{Y(\mathfrak{g})} V = \sum_{\beta} \dim V_{\beta} e^{\beta_1} \dots e^{\beta_r}. \quad (3.61)$$

The work of [125] shows that the character is multiplicative on tensor products. For  $A$ -type algebras, inclusion of  $\mathfrak{g}$  in  $Y(\mathfrak{g})$  induces a restriction map on the character and we have  $\text{res } \text{ch}_{Y(\mathfrak{g})} V = \text{ch}_{\mathfrak{g}} V$  which recover the familiar  $\mathfrak{g}$ -character as an element of  $\mathbb{Z}[x_i = e^{u_i}]_{i \in I}$ . Informally speaking, the restriction map corresponds to taking the leading part of the formal Laurent series in (3.61). We finish this section with the following theorem due to [133, 134] that shows that the Yangian characters themselves are solutions to a discrete integrable system.

**Theorem 8.** Let  $T_m^{(a)}(u) := \text{ch}_{Y(\mathfrak{g})} \text{KR}_{m\omega_a}(u)$  be the character of a Kirillov-Reshetikhin module. The variables  $T_m^{(a)}$  are solutions to the T-system which in  $A$ -type is given by

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u), \quad (3.62)$$

with  $a = 1, \dots, r$ ,  $m \geq 1$  and  $T_m^{(r+1)} = 1$ .



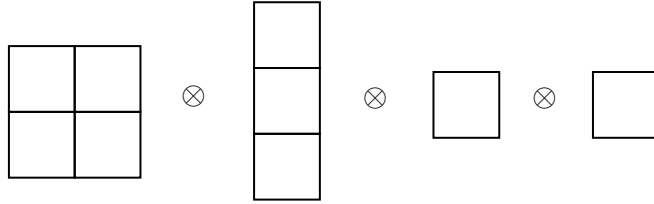


Figure 3.1: An example spin chain Hilbert space  $\mathcal{H}_{\vec{n}}$  with  $n_1^{(1)} = 2$ ,  $n_2^{(2)} = 1$  and  $n_1^{(3)} = 1$ .

### 3.2.3 Heisenberg XXX spin chain

We return to physics and consider the inhomogeneous Heisenberg XXX spin chain with  $N$  sites and periodic boundary conditions—we restrict to the case that the sites are  $\mathfrak{g} = \mathfrak{sl}_{r+1}$  modules. The model is specified by a Hilbert space consisting of a tensor product of Kirillov-Reshetikhin modules:

$$\mathcal{H}_{\vec{n}} = \bigotimes_{1 \leq a \leq r} \bigotimes_{1 \leq l \leq k} \text{KR}_{l\omega_a}^{\otimes n_l^{(a)}} =: \bigotimes_{i=1}^N V_i, \quad (3.63)$$

the modules have generic<sup>12</sup> spectral parameters  $u_1, \dots, u_N$ . The multivector  $\vec{n} = (n_l^{(a)})$  parametrises the modules in the spin chain. As a  $\mathfrak{g}$ -module,  $\mathcal{H}_{\vec{n}}$  is built from a tensor product of rectangular highest weights, an example Hilbert space is shown in figure 3.1. Associated to each pair  $i, j$  of tensor product factors in  $\mathcal{H}_{\vec{n}}$ , the discussion in 3.2.1 ensures that we have  $R$ -matrices  $R_{ij} : V_i \otimes V_j \rightarrow V_j \otimes V_i$  that solve the quantum Yang-Baxter equation (3.50) and after specifying an auxiliary representation  $V_0$  we have the transfer matrix  $t(\lambda) = \text{Tr}_{V_0} R_{01} \dots R_{0N}$ . The transfer matrix satisfies the important relation  $[t(\lambda), t(\mu)] = 0$  that arises from the quantum Yang-Baxter equation. There is then a series of commuting Hamiltonians the spectrum of which can be determined by the quantum inverse scattering method [131]—we return to this construction shortly.

**The partition function.** In this work we are interested in computing the partition function

$$Z_{\vec{n}}(X) := \text{ch}_{\mathfrak{g}} \mathcal{H}_{\vec{n}}, \quad (3.64)$$

where as usual  $X$  denotes the set of variables  $\{x_1, \dots, x_{r+1}\}$  which are fugacities for  $\mathfrak{sl}_{r+1}$  satisfying  $x_1 \dots x_{r+1} = 1$ . Considered a  $\mathfrak{g}$ -module, the Hilbert space decomposes into a sum of highest weight modules  $\mathcal{H}_{\vec{n}} = \bigoplus_{\mu} V_{\mu}$  and the partition function can then be expressed as a sum of Schur polynomials

$$Z_{\vec{n}}(X) = \sum_{\mu} M_{\mu, \vec{n}} s_{\mu}(X). \quad (3.65)$$

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<sup>12</sup>Pairwise not separated by an integer.

$M_{\mu, \vec{n}}$  denote multiplicities of the factor  $V_\mu$  in the Hilbert space. One of the main ideas of the seminal work [131] is that, via the Bethe ansatz, the representation theory of  $Y(\mathfrak{g})$  can teach us about the classical representation theory of  $\mathfrak{g}$ .

**Bethe ansatz.** The completeness conjecture<sup>13</sup> [137] provides an explicit combinatorial description for the coefficients  $M_{\mu, \vec{n}}$  as we now describe.

The eigenstates of the transfer matrix  $t(u)$  are parametrised by solutions to the Bethe equations [138], the relevant equations for the XXX model we study in this section are

$$\prod_{j=1}^N \frac{\lambda_p^{(a)} + u_j + \frac{i}{2}(\tilde{\lambda}_j, \alpha_m)}{\lambda_p^{(a)} + u_j - \frac{i}{2}(\tilde{\lambda}_j, \alpha_m)} = - \prod_{b=1}^r \prod_{q=1}^{m^{(b)}} \frac{\lambda_p^{(a)} - \lambda_q^{(b)} + \frac{i}{2}(\alpha_a, \alpha_m)}{\lambda_p^{(a)} - \lambda_q^{(b)} - \frac{i}{2}(\alpha_a, \alpha_m)}, \quad (3.66)$$

where  $\tilde{\lambda}_j$  denotes the weight of the  $j^{\text{th}}$  site in (3.63). This is a set of equations determined by a choice of integers  $(m^{(a)})_{a=1, \dots, r}$  for the Bethe roots  $\{\lambda_p^{(a)}\}$  with  $p = 1, \dots, m^{(a)}$ . The completeness conjecture states that  $\mathfrak{g}$ -highest weights  $\mu$  are in 1-1 correspondence with solutions to these equations. In particular the multiplicity of the weight  $\mu$ , as we have called  $M_{\mu, \vec{n}}$ , is given by the number of solutions to the equations (3.66) with the  $m^{(a)}$  determined in terms of  $\mu$  by the equation

$$l^{(a)} = \sum_{i=1}^k n_i^{(a)} - \sum_{b=1}^r A_{ab} m^{(b)}, \quad (3.67)$$

where  $l^{(a)}$  denotes the  $a^{\text{th}}$  Dynkin label of  $\mu$ . In brief, the  $m^{(a)}$  are determined by subtracting simple roots from the highest weight in (3.63) until one obtains the weight of interest  $\mu$ .

The string hypothesis<sup>14</sup> [131] provides a combinatorial count of the solutions to the system (3.66) and thereby determines the multiplicities of  $V_\mu$  in  $\mathcal{H}_{\vec{n}}$ . Following [126] the multiplicities are given by

$$M_{\mu, \vec{n}} = \sum_{\vec{m}} \prod_{a=1}^r \prod_{i=1}^k \binom{m_i^{(a)} + P_i^{(a)}}{m_i^{(a)}}, \quad (3.68)$$

where the sum is over all compositions  $m_i^{(a)}$  with  $\sum_{i=1}^k m_i^{(a)} = m^{(a)}$ . The  $P_i^{(a)}$  are *vacancy numbers* defined by

$$P_i^{(a)} := \sum_{j=1}^k \min(i, j) n_j^{(a)} + (B\vec{m})_i^{(a)}, \quad (3.69)$$

with  $B_{i,j}^{(a,b)} = \text{sign}(A_{ab}) \min(|A_{ab}|j, |A_{ba}|i)$ .

<sup>13</sup>Since made into a theorem [135, 136].

<sup>14</sup>The string hypothesis is that solutions to (3.66) form strings in the complex plane. This is not strictly true but nonetheless provides a good count of the solutions [126].

**Quantum  $Q$ -system.** The definition of the spin chain partition function (3.64) gives

$$\mathrm{ch}_{\mathfrak{g}} \mathcal{H}_{\vec{n}} = \prod_{a=1}^r \prod_{m=1}^k (\mathrm{ch}_{\mathfrak{g}} V_{m\omega_a})^{n_m^{(a)}}. \quad (3.70)$$

Setting  $Q_m^{(a)} = \mathrm{ch} V_{m\omega_a}$  then from (3.62), after applying the restriction map, we have that the  $\mathfrak{g}$ -characters solve the  $Q$ -system

$$\left(Q_m^{(a)}\right)^2 = Q_{m-1}^{(a)} Q_{m+1}^{(a)} + Q_m^{(a+1)} Q_m^{(a-1)}, \quad (3.71)$$

where  $a = 1, \dots, r$  and  $m \geq 1$ . In a series of papers [139, 140, 141] Kedem and Di Francesco study a quantisation of the  $Q$ -system. The *quantum  $Q$ -system*<sup>15</sup> is defined by

$$\left(\mathcal{Q}_m^{(a)}\right)^2 = q^a \mathcal{Q}_{m+1}^{(a)} \mathcal{Q}_{m-1}^{(a)} + \mathcal{Q}_m^{(a+1)} \mathcal{Q}_m^{(a-1)}, \quad (3.72)$$

with the initial conditions  $\mathcal{Q}_m^{(0)} = 1$  and  $\mathcal{Q}_m^{(r+1)} = 0$ . The variables  $\mathcal{Q}_m^{(a)}$  are now non-commuting and satisfy

$$\mathcal{Q}_m^{(a)} \mathcal{Q}_{m'}^{(b)} = q^{(m'-m)\min(a,b)} \mathcal{Q}_{m'}^{(b)} \mathcal{Q}_m^{(a)}, \quad \text{when } |m - m'| \leq |a - b| + 1. \quad (3.73)$$

Solutions to the quantum  $Q$ -system lead to a  $q$ -deformed version of the character (3.64) via

$$\begin{aligned} Z_{\vec{n}}(X; q) &:= \prod_{a=1}^r \prod_{l=1}^k (\mathcal{Q}_l^{(a)})^{n_l^{(a)}}, \\ &= \sum_{\mu} M_{\mu, \vec{n}}(q) s_{\lambda}(X). \end{aligned} \quad (3.74)$$

The  $M_{\mu, \vec{n}}(q)$  factor is known as a *fermionic form* [142, 139] and coincides with a  $q$ -deformed version of (3.68). Explicitly this factor is

$$M_{\mu, \vec{n}}(q) = \sum_{\vec{m}} q^{Q(\vec{m}, \vec{n})} \prod_{a=1}^r \prod_{i=1}^k \left[ \frac{m_i^{(a)} + P_i^{(a)}}{m_i^{(a)}} \right]_q, \quad (3.75)$$

where  $Q(m, n) = \frac{1}{2} \vec{m} \cdot P$  and the  $q$ -binomial coefficient is defined in (A.7).

**Remark.** The  $q$ -grading can be interpreted physically in the spin chain as grading the Hilbert space by one of the higher Hamiltonians in the transfer matrix. The representation theoretic interpretation is more clear when the modules  $V_i$  are considered instead as representations of the quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$  or the current algebra  $\mathfrak{g}[t]$ . In the former case  $q$  realises the so called charge function [142] for the crystal limit of the quantum

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<sup>15</sup>This is the *renormalised* quantum  $Q$ -system in the terminology of [141].

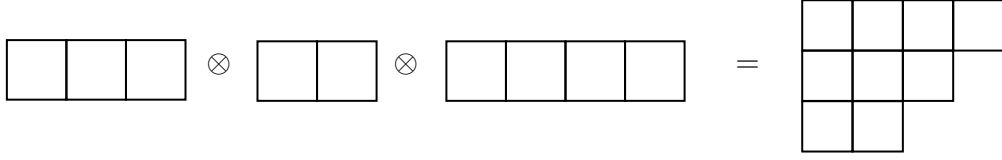


Figure 3.2: Hilbert space of KR modules with  $n_2^{(1)} = 1$ ,  $n_3^{(1)} = 1$  and  $n_4^{(1)} = 1$  associated to a partition  $\lambda$ .

affine algebra. In the latter case  $q$  is associated to the natural grading on fusion products of current algebra modules [143]. We refer the reader to [119] for an excellent review of the relationships between the representation theory of these different algebras.

### 3.2.4 Milne polynomials

We begin this subsection by reviewing the solution of Kedem and Di Francesco [140] to the quantum  $Q$ -system (3.72) by realising it as an algebra acting on symmetric functions in variables  $x_1, \dots, x_{r+1}$ . We first require some notation, let  $I$  be a subset of  $\{1, \dots, r+1\}$  and denote the complement by  $\bar{I}$ . We write  $x_I$  for the product  $\prod_{i \in I} x_i$  and define

$$a_I(x) := \prod_{\substack{i \in I \\ j \in \bar{I}}} \frac{x_i}{x_i - x_j}, \quad \Gamma_I := \prod_{i \in I} \Gamma_{q,i}, \quad (3.76)$$

where  $\Gamma_{q,i}$  is a  $q$ -shift operator acting by  $\Gamma_{q,i}(x_1, \dots, x_{r+1}) = (x_1, \dots, qx_i, \dots, x_{r+1})$ . Now we define difference operators that act on symmetric functions of  $x_1, \dots, x_{r+1}$  by

$$D_{a,m} = \sum_{\substack{I \subset \{1, \dots, r+1\} \\ |I|=a}} x_I^m a_I(x) \Gamma_I. \quad (3.77)$$

Kedem and Di Francesco show that these difference operators solve the quantum  $Q$ -system under the identification  $\mathcal{Q}_m^{(a)} = D_{a,m}$ . The  $q$ -deformed partition function (3.74) can then be constructed from iterating the action of the raising operators.

$$Z_{\vec{n}}(X; q) = \prod_{a=1}^r \prod_{m=1}^k (D_{a,m})^{n_l^{(a)}} \cdot 1. \quad (3.78)$$

**Jing operators.** We now turn to a special case of the spin chain Hilbert space  $\mathcal{H}_{\vec{n}}$ . We consider modules with only  $n_l^{(1)}$  non-zero.<sup>16</sup> The modules are illustrated in figure 3.2. In this case we re-order the modules and associate to  $\vec{n}$  a partition  $\lambda = (n_1^{(1)}, \dots, n_k^{(1)})$ . We write the partition function (3.64) as  $Z_\lambda(x_1, \dots, x_{r+1}; q)$ .

We now turn to Macdonald polynomials  $P_\lambda(X; q, t)$  in variables  $X = \{x_1, \dots, x_{r+1}\}$ .

<sup>16</sup>This is the opposite of the case considered by [140] where spin chain characters are realised by  $q$ -Whittaker functions.

Our conventions for Macdonald polynomials are summarised in appendix B. In this section we consider the alternative Macdonald  $J$  function normalisation of these polynomials together with a plethystic substitution:

$$J_\lambda(X; q, t) := c_\lambda(q, t) P_\lambda \left( \frac{X}{1-t}; q, t \right). \quad (3.79)$$

As shown in the work [144], these renormalised Macdonald polynomials have an expansion in Schur polynomials with coefficients given by the two parameter generalisation of Kostka polynomials  $K_{\mu\lambda}(q, t)$

$$J_\lambda(X; q, t) = \sum_{\mu} K_{\mu\lambda}(q, t) s_{\mu}(X). \quad (3.80)$$

This normalisation also coincides, up to a pre-factor  $t^{n(\lambda)}$ , with Haiman's normalisation [145] where they are shown to be closely related to the geometry of the Hilbert scheme of points. We return to this idea in more detail chapter 5. This correspondence is used in [145] to prove the Macdonald positivity conjecture  $K_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$ .

*Milne polynomials* are defined to be the  $q \rightarrow 0$  limit of the polynomials  $J_\lambda$  and they have a positive expansion in terms of the one parameter Kostka polynomials  $K_{\lambda\mu}(t)$  as follows

$$Q'_\lambda(X; t) := \lim_{q \rightarrow 0} J_\lambda(X; q, t) = \sum_{\mu} K_{\lambda\mu}(t) s_{\mu}(X). \quad (3.81)$$

In theorem 2.1 of [146] it is shown that Milne polynomials can be built iteratively from *Jing raising operators*.

$$Q'_\lambda(x; t) = H_{\lambda_1} \dots H_{\lambda_k} \cdot 1. \quad (3.82)$$

In fact, the Jing operators coincide with the difference operator (3.77) and  $H_m = D_{1,m}$ ; we conclude that the  $q$ -deformed partition function of the spin chain Hilbert space  $\mathcal{H}_\lambda$  in equation (3.78) is given by a Milne polynomial<sup>17</sup> so that

$$Z_\lambda(x_1, \dots, x_{r+1}; q) = Q'_\lambda(X; q). \quad (3.83)$$

Furthermore, the one parameter Kostka polynomials can then be identified with the fermionic forms  $K_{\lambda\mu}(t) = M_{\lambda\mu}(t)$ . In the following section we realise these  $q$ -deformed spin chain partition functions as characters of homology groups of line bundles over a certain quiver variety.

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<sup>17</sup>We relabel the  $t$  parameter of the Milne polynomial to  $q$  here.

### 3.3 Hanany-Tong moduli space

In this section we bring together the methods of the previous two introductory sections and compute geometric invariants of a physically interesting quiver variety. We begin with a discussion of the *Hanany-Tong moduli space*, denoted  $\mathcal{V}_{N,r}$ . This quiver is a Lagrangian submanifold of the *ADHM quiver* and we find that the equivariant Euler characteristics of its tautological line bundles realise modules of the type discussed in the previous section.

In the work [147] the quiver is constructed as the vortex moduli space of a three dimensional  $U(N)$  theory with  $N \mathcal{N} = 4$  hypermultiplets where, using type IIB brane constructions, it is realised as a Lagrangian submanifold of instanton moduli space. In the present work, the quiver plays an alternative physical role. We regard it as a Lagrangian submanifold in the Higgs branch of a 3d  $\mathcal{N} = 4$  theory—we defer a full physical discussion to chapter 5, but the idea is that sections of line bundles over  $\mathcal{V}_{N,r}$  can be interpreted as counting local boundary operators in the 3d theory.

**Outline.** We begin in section 3.3.1 with a review of the ADHM moduli space  $\mathcal{M}_{N,r}$ , this is important background material that we use in this chapter and later in chapter 5—the Hanany-Tong moduli space is a Lagrangian submanifold of the ADHM quiver. In section 3.3.3 we compute the Hilbert series of line bundles of  $\mathcal{V}_{N,r}$  and discuss the connection to the spin chain partition functions of the previous section. The conventions for partitions and polynomials used throughout are summarised in appendix A.

#### 3.3.1 ADHM quiver

We first discuss the *ADHM quiver* (sometimes referred to as the *Jordan quiver*) illustrated in figure 3.3, we denote the quiver variety by  $\mathcal{M}_{N,r}$ . This quiver variety plays an important role in the ADHM construction of instantons [148]. In this thesis it plays a number of other physical roles and in chapter 5 it will be the Higgs branch of a 3d  $\mathcal{N} = 4$  theory. The Hilbert series of this quiver can also be understood as providing the instanton corrections to the partition function of a particular 5d  $\mathcal{N} = 1$  SYM theory, we explore this perspective further in section 5.4.

**ADHM quiver.**  $\mathcal{M}_{N,r}$  is a Nakajima quiver variety of the type discussed in section 2.2 with  $\mathbf{v} = (N)$  and  $\mathbf{w} = (r)$ . We briefly review the construction in this particular case. The gauge nodes and flavour nodes are given by the vector spaces  $V = \mathbb{C}^N$  and  $W = \mathbb{C}^r$  and elements of the quiver representation are denoted

$$(A, B, I, J) \in \text{Hom}(V, V) \oplus \text{Hom}(V, V) \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W). \quad (3.84)$$

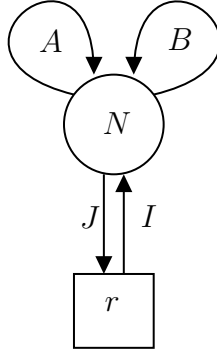


Figure 3.3: ADHM Quiver  $\mathcal{M}_{N,r}$

The flavour group  $GL(V)$  acts by

$$(A, B, I, J) \rightarrow (gAg^{-1}, gBg^{-1}, gI, Jg^{-1}), \quad (3.85)$$

with associated moment map  $\mu = [A, B] + IJ$ . The quiver variety  $\mathcal{M}_{N,r}$  is then defined by

$$\mathcal{M}_{N,r} = \{(A, B, I, J) \in \mu^{-1}(0) : \text{stable}\} / GL(V). \quad (3.86)$$

In the notation of section 2.2 we take the stability parameter  $\theta = 1$  so that the stability condition 2.3 becomes the constraint that there are no non-trivial  $(A, B)$  invariant subspaces of  $V$  contained in  $\ker J$ . The condition  $\theta = 1$  also ensures the conditions of theorem 2 are met, that is to say  $\mathcal{M}_{N,r}$  is a smooth variety of dimension  $2Nr$ .

**Remark.** The ADHM quiver describes the moduli space of rank  $r$  framed  $N$ -instantons on  $\mathbb{R}^4$ . We have

$$\mathcal{M}_{N,r}^{\text{reg.}} = \{\text{isomorphism classes of pairs } (A, \Phi)\}, \quad (3.87)$$

On the right hand side  $A$  is an anti self-dual connection in a rank  $r$  Hermitian vector bundle  $E$  on  $S^4$  with  $c_2(E) = N$  and  $\Phi : E_\infty \rightarrow \mathbb{C}^r$  is a trivialisation of  $E$  at infinity. The left hand side is the regular locus of the Nakajima quiver variety.<sup>18</sup> The proof of this statement is the content of the ADHM construction [148], we thus refer to this quiver as the ADHM quiver and later we talk about the 3d ADHM theory whose Higgs branch  $\mathcal{M}_H$  coincides with  $\mathcal{M}_{N,r}$ .

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<sup>18</sup>The geometric points are given by the points of (3.86) that are in addition *costable*: there is no non-trivial  $(A, B)$ -stable subspace of  $V$  that contains  $\text{im } I$ .

**Flavour symmetry.** The ADHM quiver admits a Hamiltonian group action  $\tilde{T} = \mathbb{C}_{t_1}^\times \times \mathbb{C}_{t_2}^\times \times \mathbf{A}$  given by

$$(A, B, I, J) \rightarrow (t_1 A, t_2 B, I g^{-1}, t_1 t_2 g J). \quad (3.88)$$

where  $g = \text{diag}(e_1, \dots, e_r)$ . The  $\mathbf{A}$  action arises as the maximal torus of the general flavour group action discussed in section 2.2 associated with the framing  $W$ . In the 3d gauge theory context, we realise these symmetries as a combination of the fundamental flavour symmetry and a combination of  $R_H = U(1)$   $R$ -symmetry and a  $U(1)$  flavour symmetry rotating the adjoint fields.

**Torsion-free sheaves.** The ADHM quiver has an equivalent description in terms of a moduli space of torsion-free sheaves on  $\mathbb{P}^2$ :

$$\mathcal{M}_{N,r} = \{\text{isomorphism classes of pairs } (E, \Phi)\}, \quad (3.89)$$

where  $E$  is a torsion free sheaf of rank  $r$  on  $\mathbb{P}^2$  with  $c_2(E) = N$  and, denoting by  $l_\infty$  the line at  $\infty$  in  $\mathbb{P}^2$ ,  $\Phi$  is the *framing*  $\Phi : E|_{l_\infty} \rightarrow \mathcal{O}_{l_\infty}^{\oplus r}$ . We refer the reader to [111] for a proof of the identification with the ADHM data.

**Fixed points.** In this realisation the group action (3.88) acting on pairs  $(E, \Phi)$  can be understood as follows.  $\mathbb{C}_{t_1}^\times \times \mathbb{C}_{t_2}^\times$  naturally acts on  $\mathbb{P}^2$  by  $[z_0 : z_1 : z_2] \rightarrow [z_0 : t_1 z_1 : t_2 z_2]$  leaving the line at infinity invariant; this extends to an action, by pullback, on  $E$ . The torus  $\mathbf{A}$  acts by rotating the framing at infinity.

A pair  $(E, \Phi)$  is then fixed by  $\tilde{T}$  if  $E$  can be decomposed as  $E = I_1 \oplus \dots \oplus I_r$  with each  $I_\alpha \in \mathcal{M}_{N_\alpha, 1}$  and  $\sum N_\alpha = N$  with  $\Phi$  mapping  $I_\alpha|_{l_\infty}$  to the  $\alpha$  factor of  $\mathcal{O}_{l_\infty}^{\oplus r}$ . Later, in section 5.1.1 of chapter 5, we see that when  $r = 1$  there is yet another convenient description of the moduli space in terms of the Hilbert scheme of  $N$  points in the plane  $\mathbb{C}^2$ . In that context, each  $I_\alpha$  is an ideal in  $\mathbb{C}[x, y]$  generated by monomials  $x^i y^j$  that form a partition  $\lambda$  as in figure 5.2. The result is that the  $\tilde{T}$  fixed points of the ADHM quiver are labelled by *coloured Young tableaux*:

$$\mathcal{M}_{N,r}^{\tilde{T}} \leftrightarrow \{(\lambda_1, \dots, \lambda_r) : \sum_{\alpha=1}^r |\lambda_\alpha| = N\}. \quad (3.90)$$

An example of a fixed point is illustrated in figure 3.4. We refer the reader to [111] for a more detailed construction of the fixed points.

**Equivariant  $K$ -theory.** The discussion in section 2.2 explains that  $K_{\tilde{T}}(\mathcal{M}_{N,r})$  is generated by the tautological bundle  $\mathcal{V} = w_1 + \dots + w_N$  and the framing bundle  $\mathcal{W} = x_1 + \dots + x_r$ .



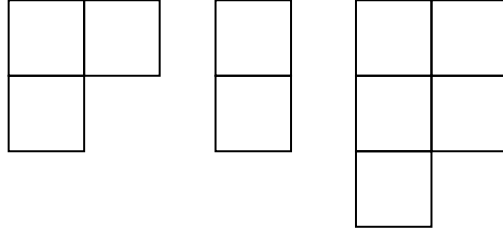


Figure 3.4: A particular fixed point on  $\mathcal{M}_{10,3}$ .

The evaluation of the Chern roots at a fixed point  $\vec{\lambda} = (\lambda_1, \dots, \lambda_r)$  is given by<sup>19</sup>

$$w_s(\vec{\lambda}) = x_{\delta(\alpha,s)} t_1^{-i_s+1} t_2^{-j_s+1}. \quad (3.91)$$

where  $s$  is a box in  $\vec{\lambda}$  and  $\delta(\alpha, s) = \alpha$  where  $\alpha$  is such that  $s \in \lambda_\alpha$ . Using<sup>20</sup> lemma 1 the  $K$ -theory class of the tangent bundle is given by

$$T\mathcal{M}_{N,r} = \mathcal{W}^\vee \otimes \mathcal{V} + \mathcal{W} \otimes \mathcal{V}^\vee t_1 t_2 - \mathcal{V}^\vee \otimes \mathcal{V} (1 - t_1)(1 - t_2), \quad (3.92)$$

and evaluating this at a fixed point  $\vec{\lambda}$  gives<sup>21</sup>

$$T_{\vec{\lambda}} \mathcal{M}_{N,r} = \sum_{\alpha, \beta=1}^r \frac{x_\beta}{x_\alpha} \left( \sum_{s \in \lambda_\alpha} t_1^{-l_{\lambda_\beta}(s)} t_2^{a_{\lambda_\alpha}(s)+1} + \sum_{t \in \lambda_\beta} t_1^{l_{\lambda_\alpha}(t)+1} t_2^{-a_{\lambda_\beta}(t)} \right). \quad (3.93)$$

**Hilbert series.** We compute the twisted Hilbert series of section 3.1.2 for the ADHM quiver. Evaluating the tautological line bundle  $\mathcal{L} = \det \mathcal{V}$  at a fixed point  $\vec{\lambda}$  we find

$$\mathcal{L}_{\vec{\lambda}} = \prod_{\alpha=1}^r \prod_{s \in \lambda_\alpha} x_\alpha t_1^{-i_s+1} t_2^{-j_s+1}. \quad (3.94)$$

We now have the necessary ingredients to compute the (twisted) Hilbert series using the localisation formula (3.16).

$$\begin{aligned} \text{H.S.}[\mathcal{M}_{N,r}] = & \sum_{\substack{\vec{\lambda} \\ |\vec{\lambda}|=N}} \prod_{\alpha=1}^r \prod_{s \in \lambda_\alpha} x_\alpha t_1^{\kappa(-i_s+1)} t_2^{\kappa(-j_s+1)} \\ & \prod_{\alpha, \beta=1}^r \frac{1}{\prod_{s \in \lambda_\alpha} 1 - \frac{x_\beta}{x_\alpha} t_1^{-l_{\lambda_\beta}(s)} t_2^{a_{\lambda_\alpha}(s)+1}} \frac{1}{\prod_{s \in \lambda_\beta} 1 - \frac{x_\beta}{x_\alpha} t_1^{l_{\lambda_\alpha}(t)+1} t_2^{-a_{\lambda_\beta}(t)}}. \end{aligned} \quad (3.95)$$

<sup>19</sup>This can be deduced from the arguments in chapter 5.

<sup>20</sup>Although note the slightly different global symmetries, the  $K$ -theory ring is now a module over  $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, x_1^{\pm 1}, \dots, x_r^{\pm 1}]$  and the contracting action  $\mathbb{C}^\times$  is a combination of  $t_1$  and  $t_2$ . There is also an extra symmetry rotating the adjoint fields which we choose to grade by here.

<sup>21</sup>This follows from some lengthy combinatoric cancellations detailed in [111].

This is the  $K$ -theoretic lift of Nekrasov's partition function [149], and realises the  $N$ -instanton contributions to a 5d  $SU(r)$  gauge theory. The Hilbert series was studied from a mathematical perspective by Nakajima [150, 151]. Recall from section 3.1.2 that it is also possible to express the Hilbert series as a Molien integral

$$\begin{aligned} \text{H.S.}[\mathcal{M}_{N,r}] = \frac{1}{N!} \oint \prod_{a=1}^N \frac{dw_a}{2\pi i w_a} \prod_{a \neq b}^N \left(1 - \frac{w_a}{w_b}\right) \prod_{a,b=1}^N \frac{\left(1 - t_1 t_2 \frac{w_a}{w_b}\right)}{\left(1 - t_1 \frac{w_a}{w_b}\right) \left(1 - t_2 \frac{w_a}{w_b}\right)} \\ \prod_{a=1}^N \prod_{i=1}^r \frac{1}{1 - z_i^{-1} w_a} \frac{1}{1 - t_1 t_2 z_i w_a^{-1}}. \end{aligned} \quad (3.96)$$

One of the advantages of re-writing the Hilbert series in this form is that it allows us to understand the large gauge rank  $N$  limit—this seems very difficult to compute from the expression (3.95). In section 3.4, we evaluate a (generalisation of) this integral using Macdonald symmetric function methods and take the limit  $N \rightarrow \infty$ .

### 3.3.2 Lagrangian subvariety

In this section we consider a Lagrangian subvariety  $\mathcal{V}_{N,r} \subset \mathcal{M}_{N,r}$  of the instanton moduli space fixed by a subtorus of the group action (3.88).

**Hanany-Tong moduli space.** The moduli space is shown in figure 3.5, it corresponds to discarding half of the arrows of figure 3.3. More precisely, setting  $B = J = 0$  we have

$$\mathcal{V}_{N,r} := \{(A, I) \in \text{Hom}(V, V) \oplus \text{Hom}(W, V) : \text{stable}\} / GL(V), \quad (3.97)$$

where the stability condition is that there is no proper  $A$ -stable subspace of  $V$  containing  $\text{im } I$ . This is an example of a handsaw quiver studied by Nakajima [152]. It has complex dimension  $Nr$  and the symplectic form of  $\mathcal{M}_{N,r}$  manifestly vanishes on  $\mathcal{V}_{N,r}$  so that  $\mathcal{V}_{N,r}$  is a Lagrangian subvariety. In particular, from (3.88), we see that  $\mathcal{V}_{N,r}$  is the locus in  $\mathcal{M}_{N,r}$  fixed by the  $\mathbb{C}_{t_1}^\times$  group action. It inherits geometrical properties such as smoothness from  $\mathcal{M}_{N,r}$ .

**Fixed points.** We now consider the fixed points under the residual torus action  $\mathbf{T} = \mathbb{C}_{t_2}^\times \times \mathbf{A}$ . In the description of  $\mathcal{M}_{N,r}$  in terms of torsion-free sheaves  $(E, \Phi)$ , the  $\mathbb{C}_{t_1}^\times$  action scales one of the coordinates on the  $\mathbb{P}^2$  and the fixed sheaves  $E = I_1 \oplus \dots \oplus I_r$  are then tuples of  $\mathbb{C}^\times$ -equivariant sheaves on  $\mathbb{C}$  which can be labelled by integers  $\vec{k} = (k_1, \dots, k_r)$  with  $\sum k_\alpha = N$ —these are the subset of the coloured Young tableaux corresponding to vertical partitions, an example is depicted in figure 3.6.

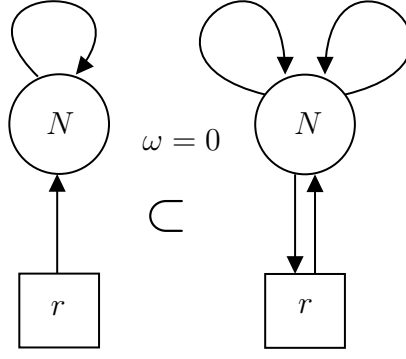


Figure 3.5: Hanany-Tong Moduli space  $\mathcal{V}_{N,r}$  as a Lagrangian submanifold of  $\mathcal{M}_{N,r}$ .

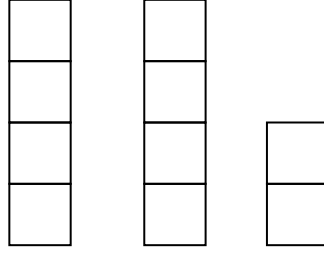


Figure 3.6: An example of a fixed point  $\vec{k}$  on  $\mathcal{V}_{10,3} \subset \mathcal{M}_{10,3}$  with  $k_1 = 4$ ,  $k_2 = 4$  and  $k_3 = 2$ .

**Equivariant  $K$ -theory.** We can deduce the character of the tangent space to  $\mathcal{V}_{N,r}$  at a fixed point  $\vec{k}$  by taking the  $t_1$  invariant part of (3.93). Relabelling<sup>22</sup>  $t_2 \rightarrow q^{-1}$  we have:

$$T_{\vec{k}}\mathcal{V}_{N,r} = \sum_{\alpha,\beta=1}^r \frac{x_\alpha}{x_\beta} \sum_{s=0}^{k_\beta-1} q^{k_\alpha-k_\beta+1+s}. \quad (3.98)$$

Similarly, the tautological line bundle (3.94) restricted to  $\mathcal{V}_{N,r}$  evaluated at a fixed point  $\vec{k}$  is given by

$$\mathcal{L}_{\vec{k}} = \prod_{\alpha=1}^r x_\alpha^{k_\alpha} q^{\frac{1}{2}k_\alpha(k_\alpha-1)}. \quad (3.99)$$

### 3.3.3 Sections of line bundles

We now come to the main result of this section. We find that counts of sections of the line bundle  $\mathcal{L}$  give characters of a quantum group—the Yangian of section 3.2. Precisely, we compute the equivariant Euler characteristic of the line bundle  $\mathcal{L}^{\otimes \zeta}$  with  $\zeta \in \mathbb{Z}_{\geq 0}$  defined by

$$\chi_{\mathbb{T}}(\mathcal{L}^{\otimes \zeta}, \mathcal{V}_{N,r}) := \text{ch}_{\mathbb{T}} H^0(\mathcal{V}_{N,r}, \mathcal{L}^{\otimes \zeta}). \quad (3.100)$$

<sup>22</sup>To match the fermionic form conventions from section 3.2.

Substituting (3.99) and (3.98) into the localisation formula (3.16) we have

$$\chi_{\mathbb{T}}(\mathcal{L}^{\otimes \zeta}, \mathcal{V}_{N,r}) = \sum_{k_1 + \dots + k_r = N} \prod_{\alpha=1}^r x_{\alpha}^{\zeta k_{\alpha}} q^{\frac{\zeta}{2} k_{\alpha}(k_{\alpha}-1)} \prod_{\alpha, \beta=1}^r \frac{1}{(x_{\alpha}/x_{\beta} q^{k_{\alpha}-k_{\beta}+1}; q)_{k_{\beta}}}. \quad (3.101)$$

We then have:

**Proposition 1.** The equivariant Euler characteristic of  $\mathcal{L}^{\otimes \zeta}$  normalised by the Hilbert series of  $\mathcal{V}_{N,r}$  is the  $q$ -deformed character (3.74) of the spin chain Hilbert space

$$\mathcal{H} = \bigotimes_{l=1}^N \text{KR}(l\omega_1)^{\otimes \zeta}. \quad (3.102)$$

The character coincides with the Milne polynomial

$$\frac{\chi_{\mathbb{T}}(\mathcal{L}^{\otimes \zeta})}{\chi_{\mathbb{T}}(\mathcal{O})} = Q'_{(\zeta^N)}(x_1, \dots, x_r; q). \quad (3.103)$$

*Proof.* We write  $\mathcal{Z}_{N,r}$  for  $\chi_{\mathbb{T}}(\mathcal{L}^{\otimes \zeta}, \mathcal{V}_{N,r})$  in equation (3.99) and proceed inductively on  $N$  to show that

$$\mathcal{Z}_{N,r} = (D_{1,\zeta})^N \cdot 1 = Q_{(\zeta^N)}(x_1, \dots, x_r; q), \quad (3.104)$$

where  $D_{1,\zeta}$  are the Jing raising operators of section 3.2.4. When  $N = 1$  the sum over  $\{k_i\}$  in  $\mathcal{Z}_{N,r}$  is a choice of which  $k_i$  is set equal to 1. Furthermore, the first product over  $i$  becomes simply  $x_i^{\zeta}$  for this choice  $k_i$  and the second product receives contributions only from terms involving the non-zero  $k_i$ , and brings out a factor of  $1/(1-q)$ —we have:

$$\mathcal{Z}_{1,r} = \frac{1}{1-q} \sum_{i=1}^r x_i^{\zeta} \prod_{\substack{j=1 \\ j \neq i}}^r \frac{1}{1 - x_i/x_j}. \quad (3.105)$$

This coincides with the raising operator  $D_{1,\zeta}$  divided by  $(q; q)_1$  as required.

Now we act with  $D_{1,\zeta}$  on  $\mathcal{Z}_{N,r}$ . Firstly, consider the action of the shift operator  $\Gamma_{q,i}$  on the summand. We denote the summand by  $\mathcal{Z}_{N,r}^{\{k\}}$  so that:

$$\mathcal{Z}_{N,r} = \sum_{k_1 + \dots + k_r = N} \mathcal{Z}_{N,r}^{\{k\}}. \quad (3.106)$$

The shift operator acts on the summand as follows:

$$\Gamma_{q,i} \cdot \mathcal{Z}_{N,r}^{\{k\}} = x_i^{-\zeta} \mathcal{Z}_{N,r}^{\{\tilde{k}^{(i)}\}} \prod_{j=1}^r \left( 1 - q^{\tilde{k}^{(i)}_j} x_j/x_i \right). \quad (3.107)$$

The set of integers  $\{\tilde{k}^{(i)}\}$  is the same as the set  $\{k\}$  except the  $i^{\text{th}}$  integer is shifted by 1

so that  $\tilde{k}_i = k_i + 1$ . Now applying the whole raising operator we have:

$$D_{1,\zeta} \cdot \mathcal{Z}_{N,r} = \sum_{i=1}^r x_i^\zeta \prod_{\substack{j=1 \\ j \neq i}}^r \frac{1}{1 - x_j/x_i} \sum_{\{k\}} \left[ x_i^{-\zeta} \mathcal{Z}_{N,r}^{\{\tilde{k}^{(i)}\}} \prod_{j=1}^r \left( 1 - q^{\tilde{k}_j^{(i)}} x_j/x_i \right) \right]. \quad (3.108)$$

Now we look to change variable in the sum over  $\{k\}$ . We can re-parametrise the sum as a sum over  $\{k'\}$  with  $\sum_{i=1}^r k'_i = N + 1$  but with  $k'_i \geq 1$ . Now the term in square brackets vanishes if  $k'_i = 0$  so we can write the expression as a sum over all  $\{k'\}$  with  $\sum_i k'_i = N + 1$ . The result is then an expression:<sup>23</sup>

$$D_{1,\zeta} \cdot \mathcal{Z}_{N,r} = \sum_{k_1 + \dots + k_r = N+1} \mathcal{Z}_{N+1,r}^{\{k\}} \left[ \sum_{i=1}^r (1 - q^{k_i}) \prod_{\substack{j=1 \\ j \neq i}}^r \frac{1 - q^{k_j} x_j/x_i}{1 - x_j/x_i} \right]. \quad (3.109)$$

The term in square brackets is in fact independent of  $x_i$  and gives simply  $1 - q^{\sum_{i=1}^r k_i} = 1 - q^{N+1}$  thus completing the proof since

$$D_{1,\zeta} \cdot \mathcal{Z}_{N,r} = \mathcal{Z}_{N+1,r}. \quad (3.110)$$

□

**Remark.** We note that this expression can also be expressed as a Molien integral from section 3.1.2

$$\chi_T(\mathcal{L}^{\otimes \zeta}, \mathcal{V}_{N,r}) = \frac{1}{N!} \oint \prod_{i=1}^N \frac{dw_i}{2\pi i w_i^{\zeta+1}} \prod_{a \neq b} \left( 1 - \frac{w_a}{w_b} \right) \prod_{i,j=1}^N \frac{1}{1 - q w_i/w_j} \prod_{i=1}^N \prod_{a=1}^r \frac{1}{1 - w_i x_a}. \quad (3.111)$$

This integral was evaluated using Milne polynomial methods in the work [153]. In the following section we introduce Macdonald polynomial techniques to evaluate integrals of the kind (3.28). Using these methods, we show that sections of line bundles over more general *toothless handsaw quivers* realise more general KR module characters.

**Twisted  $\chi_t$  genus.** We conclude this subsection with some remarks on the twisted  $\chi_t$  genus of the Hanany-Tong vortex moduli space. We can use the same ingredients to compute the twisted (by  $\mathcal{L}$ )  $\chi_y$  genus of  $\mathcal{V}_{N,r}$ . Substituting (3.98) and (3.99) into the localisation formula (3.30) we find

$$\chi_y(\mathcal{V}_{N,r}) = \sum_{k_1 + \dots + k_r = N} x_\alpha^{\zeta k_\alpha} q^{\frac{\zeta}{2} k_\alpha (k_\alpha - 1)} \prod_{\alpha, \beta=1}^r \frac{(y x_\alpha / x_\beta q^{k_\alpha - k_\beta + 1}; q)_{k_\beta}}{(x_\alpha / x_\beta q^{k_\alpha - k_\beta + 1}; q)_{k_\beta}}. \quad (3.112)$$

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<sup>23</sup>Relabelling  $\{k'\} \rightarrow \{k\}$  for ease of notation.

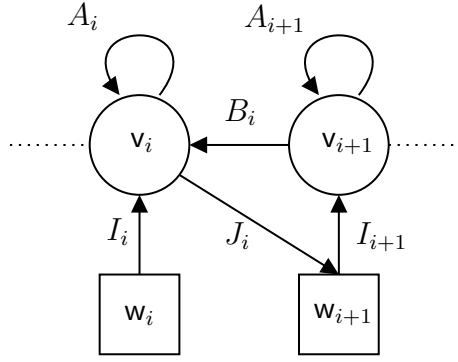


Figure 3.7: A segment of a periodic Chainsaw quiver  $\mathcal{D}(\mathbf{v}, \mathbf{w})$ .

The limit  $y \rightarrow 0$  recovers the twisted Hilbert series of the line bundle above. Another interesting limit is instead  $q \rightarrow 0$ . In the absence of line bundle charge, this is the limit of subsection 3.1.4 and computes the Poincaré polynomial of the compact core. We first re-write the sum over fixed points in terms of the Weyl group  $S_r$  as follows

$$\chi_y(\mathcal{V}_{N,r}) = \sum_{\sigma \in S_r} \sum_{k_1 \geq \dots \geq k_r} x_\alpha^{\zeta k_\alpha} q^{\frac{\zeta}{2} k_\alpha (k_\alpha - 1)} \prod_{\alpha, \beta=1}^r \frac{(yx_\alpha/x_\beta q^{k_\alpha - k_\beta + 1}; q)_{k_\beta}}{(x_\alpha/x_\beta q^{k_\alpha - k_\beta + 1}; q)_{k_\beta}}. \quad (3.113)$$

Now sending  $q \rightarrow 0$  we see from the line bundle term that the only fixed points that contribute are those with  $k_\alpha = 0, 1$  for  $\alpha = 1, \dots, r$ , we thus parametrise the sum by the number,  $k$ , of 1s that appear in  $\vec{k}$  so that

$$\lim_{q \rightarrow 0} \chi_y(\mathcal{V}_{N,r}) = \sum_{\sigma \in S_r} x_1^\zeta \dots x_k^\zeta \prod_{\alpha=1}^k \prod_{\beta=k+1}^r \frac{1 - yx_\alpha/x_\beta}{1 - x_\alpha/x_\beta}, \quad (3.114)$$

up to the normalisation constant  $v_{(\zeta^k)}(y)$  (see appendix B) this is a Hall-Littlewood polynomial B.16

$$\lim_{q \rightarrow 0} \chi_y(\mathcal{V}_{N,r}) = v_{(\zeta^k)}(y) P_{(\zeta^k)}^{\text{HL}}(X; y). \quad (3.115)$$

### 3.4 Chainsaw quiver varieties

In this section we study the chainsaw quiver variety  $\mathcal{D}(\mathbf{v}, \mathbf{w})$  illustrated in figure 3.7.

**Motivation.** The chainsaw quiver was first studied in the mathematical literature by Finkelberg and Rybnikov [154]. Later it was shown by Kanno and Tachikawa in the work [155] that the chainsaw quiver describes the moduli space of instantons in a four dimensional pure  $SU(M)$  gauge theory with  $\mathcal{N} = 2$  supersymmetry in the presence of a surface operator.

The *AGT correspondence* [29] implies, in particular, that the BPS sector of pure  $SU(M)$   $\mathcal{N} = 2$  gauge theories in four dimensions is governed by a  $W_M$ -algebra symmetry.

Often, the partition function (geometrically, the homological limit of the Hilbert series of the relevant quiver variety) coincides with the norm of the Gaiotto state [156] of the  $W_M$ -algebra. In the presence of a surface operator this correspondence has been extended [157, 158, 159] to a particular  $W$ -algebra constructed using the affine symmetry  $\widehat{SU}(M)$ , in this case the relevant instanton moduli space is described by the chainsaw quiver variety [60]. The gauge nodes  $\mathbf{w}_i$  parametrise the choice of surface operator with  $\mathbf{w}_1 + \dots + \mathbf{w}_N = M$  and the flavour nodes  $\mathbf{v}_i$  parametrise the instanton number. In this section we study a  $K$ -theoretic lift (and  $q$ -deformation) of this setup and show that the partition function can be computed using Macdonald polynomial methods that evaluate the associated Molien integral.

**Outline.** We begin in section 3.4.1 where we show that the chainsaw quiver partition function can be evaluated in terms of Macdonald polynomials. In section 3.4.2 we study a special limit of the chainsaw quivers that we call *toothless handsaw quivers*—we find that sections of line bundles supported on these quiver varieties yield the more general spin chain characters discussed in section 3.2. The ADHM moduli space of the previous section is a special case of a chainsaw quiver and we conclude the present section in 3.4.3 by demonstrating that the Macdonald polynomial methods are a convenient tool to compute the large rank limit of the Hilbert series of the ADHM moduli space.

Conventions for Macdonald polynomials and combinatorics used throughout this section are summarised in appendices A and B.

### 3.4.1 Molien integral of chainsaw quiver varieties

**Definition.** The chainsaw quiver is a single framed quiver with gauge nodes and flavour nodes

$$V = \bigoplus_{i=1}^N V_i, \quad W = \bigoplus_{i=1}^N W_i, \quad (3.116)$$

with associated dimension vectors  $(\mathbf{v}_1, \dots, \mathbf{v}_N)$  and  $(\mathbf{w}_1, \dots, \mathbf{w}_N)$ . Quiver representations are specified by the following linear maps for  $i = 1, \dots, N$

$$A_i : V_i \rightarrow V_i, \quad B_i : V_{i+1} \rightarrow V_i, \quad I_i : W_i \rightarrow V_i, \quad J_i : V_i \rightarrow W_{i+1}. \quad (3.117)$$

where we take  $i$  modulo  $N$ . The gauge group  $G = GL(V_1) \times \dots \times GL(V_N)$  acts in the usual way by  $(A, B, I, J) \rightarrow (g^{-1}Ag, g^{-1}Bg, gI, Jg^{-1})$ . The chainsaw quiver is then defined by

$$\mathcal{D}(\mathbf{v}, \mathbf{w}) = \{(A, B, I, J) : \text{stable}\} / GL(V). \quad (3.118)$$

where stability is defined in the sense of definition 2.3. We introduce corresponding Chern roots for the tautological bundles  $i = 1, \dots, N$  denoted  $\{w_{k_i}^{(i)}\}_{k_i=1, \dots, \mathbf{v}_i}$ . We have the usual

framing group action  $G_W = GL(W_1) \times \dots \times GL(W_N)$  for which we introduce fugacities associated to each flavour node  $i = 1, \dots, N$  denoted  $\{x_{k_i}^{(i)}\}_{k_i=1, \dots, \mathbf{w}_i}$ . We also have a contracting  $\mathbb{C}_t^\times$  acting by

$$(A, B, I, J) \rightarrow (tA, B, tI, J), \quad (3.119)$$

we write  $t$  for the corresponding fugacity.<sup>24</sup> We denote the maximal torus of these group actions by  $\mathbb{T}$ . Associated to each gauge node we have the usual tautological line bundles  $\mathcal{L}_i = \det \mathcal{V}_i$  for  $i = 1, \dots, N$  generating the Picard group  $\text{Pic}(\mathcal{D}(\mathbf{v}, \mathbf{w})) \cong \mathbb{Z}^N$ .

**$q$ -deformed Molien integral.** Associated to  $\mathcal{D}(\mathbf{v}, \mathbf{w})$  we consider a  $q$ -deformed Molien integral of the type defined in (3.28). Later in chapter 5 we see that integrals of this type realise counts of local operators of 3d  $\mathcal{N} = 4$  theories. In this chapter they are a technical tool to evaluate the Hilbert series of the chainsaw quiver.

$$\begin{aligned} & \mathcal{I}_{\{\mathbf{w}\}\{\mathbf{v}\}}(X; \zeta; \alpha, \beta, \gamma; q, t) \\ &= \oint \prod_{a=1}^N \frac{1}{\mathbf{v}_a!} \prod_{i=1}^{\mathbf{v}_a} \frac{dw_i^{(a)}}{2\pi i w_i^{(a)}} (w_i^{(a)})^{-\zeta_a} \prod_{a=1}^N \prod_{i,j=1}^{\mathbf{v}_a} \frac{(w_i^{(a)}/w_j^{(a)}; q)_\infty}{(tw_i^{(a)}/w_j^{(a)}; q)_\infty} \\ & \prod_{a=1}^N \prod_{i=1}^{\mathbf{v}_a} \prod_{j=1}^{\mathbf{v}_{a+1}} \frac{(\gamma_a t w_i^{(a)}/w_j^{(a+1)}; q)_\infty}{(\gamma_a w_i^{(a)}/w_j^{(a+1)}; q)_\infty} \prod_{a=1}^N \prod_{i=1}^{\mathbf{v}_a} \prod_{m=0}^{\mathbf{w}_a} \prod_{n=0}^{\mathbf{w}_{a+1}} \frac{1}{(\alpha_a w_i^{(a)} x_m^{(a)}; q)_\infty} \frac{1}{(\beta_a \frac{1}{w_i^{(a)} x_n^{(a+1)}}; q)_\infty}. \end{aligned} \quad (3.120)$$

We note that when  $q \rightarrow 0$  this integral reduces to the Molien integral for the (twisted by the line bundles  $\mathcal{L}$ ) Hilbert series of  $\mathcal{D}(\mathbf{v}, \mathbf{w})$ . Throughout this section we use shorthand for the tensor product of powers of line bundles on  $\mathcal{D}(\mathbf{v}, \mathbf{w})$  and write  $\mathcal{L}^\zeta = \bigotimes_{i=1}^N \mathcal{L}_i^{\otimes \zeta_i}$ . The  $q \rightarrow 0$  limit of the Molien integral (3.120) gives

$$\chi_{\mathbb{T}}(\mathcal{L}^\zeta, \mathcal{D}(\mathbf{v}, \mathbf{w})) = \lim_{q \rightarrow 0} \mathcal{I}_{\{\mathbf{w}\}\{\mathbf{v}\}}(X^{(a)}; \zeta_a; \alpha_a = \beta_a = \gamma_a = 1; q, t). \quad (3.121)$$

We introduce the parameters  $\alpha, \beta$  and  $\gamma$  in the integral (3.120) as convenient book-keeping to take various limits in the following subsection. We now evaluate  $\mathcal{I}_{\{\mathbf{w}\}\{\mathbf{v}\}}$  using the theory of Macdonald polynomials. We denote by  $X^{(a)}$  the set of variables  $(x_1^{(a)}, \dots, x_{\mathbf{w}_a}^{(a)})$ , and  $\bar{X}$  for sets of inverse variables  $\bar{X} = \{x_1^{-1}, x_2^{-1}, \dots\}$ .

**Theorem 9.** The  $q$ -deformed Molien integral of the chainsaw quiver  $\mathcal{D}(\mathbf{v}, \mathbf{w})$  can be written as a sum of skew Macdonald polynomials labelled by partitions  $\{\nu^{(a)}, \sigma^{(a)}\}_{a=1, \dots, N}$

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<sup>24</sup>This is different to the previous section where we called it  $q$ , we want to reserve  $q$  for the Macdonald deformation in this section.



in the flavour fugacities as

$$\mathcal{I}_{\{\mathbf{w}\}\{\mathbf{v}\}}(X; \zeta; \alpha, \beta, \gamma; q, t) = \sum_{\{\nu^{(a)}, \sigma^{(a)}\}} \prod_{a=1}^N \frac{(q; q)_{\infty}^{\mathbf{v}_a}}{(t; q)_{\infty}^{\mathbf{v}_a}} \tilde{c}_{\mathbf{v}_a}(\nu^{(a)}; q, t) \gamma_a^{|\sigma^{(a)}|} P_{\nu^{(a)}/\sigma^{(a)}} \left( \frac{\alpha_a X^{(a)}}{1-t}; q, t \right) Q_{\nu^{(a)}/\tilde{\sigma}^{(a-1)}} \left( \frac{\beta_a \bar{X}^{(a+1)}}{1-t}; q, t \right). \quad (3.122)$$

where the shifted partitions are defined by  $\tilde{\sigma}^{(a)} = \sigma^{(a)} + (\zeta_{a+1}^{\mathbf{v}_{a+1}})$  for  $a = 1, \dots, N$  with  $(\zeta_{N+1}^{\mathbf{v}_{N+1}}) = (\zeta_1^{\mathbf{v}_1})$ . Similarly, the variables  $X^{(N+1)}$  and  $X^{(1)}$  are identified.

*Proof.* Using the Macdonald measure (B.46) and the Macdonald Cauchy identity (B.31)<sup>25</sup>, we can re-write the integrand in terms of symmetric functions:

$$\begin{aligned} \mathcal{I}_{\{\mathbf{w}\}\{\mathbf{v}\}}(X; \zeta; \alpha, \beta, \gamma; q, t) = & \prod_{a=1}^N \frac{(q; q)_{\infty}^{\mathbf{v}_a}}{(t; q)_{\infty}^{\mathbf{v}_a}} \oint \prod_{a=1}^N d\mu[W^{(a)}; q, t] \prod_{i=1}^{\mathbf{v}_a} (w_i^{(a)})^{-\zeta_a} \sum_{\{\lambda^{(a)}\}} P_{\lambda^{(a)}}(W^{(a)}; q, t) Q_{\lambda^{(a)}} \left( \frac{\alpha_a X^{(a)}}{1-t}; q, t \right) \\ & \sum_{\{\mu^{(a)}\}} P_{\mu^{(a)}}(\bar{W}^{(a)}; q, t) Q_{\mu^{(a)}} \left( \frac{\beta_a \bar{X}^{(a+1)}}{1-t}; q, t \right) \\ & \sum_{\{\sigma^{(a)}\}} \gamma_a^{|\sigma^{(a)}|} b_{\sigma^{(a)}}(q, t) P_{\sigma^{(a)}}(W^{(a)}; q, t) P_{\sigma^{(a)}}(\bar{W}^{(a+1)}; q, t). \end{aligned} \quad (3.123)$$

Now we can use (B.33) to absorb the factors of  $(w^{(a)})^{-\zeta_a}$ . Further, using the Macdonald algebra structure constants (B.41) we can write the integral as:

$$\begin{aligned} \mathcal{I}_{\{\mathbf{w}\}\{\mathbf{v}\}} = & \prod_{a=1}^N \frac{(q; q)_{\infty}^{\mathbf{v}_a}}{(t; q)_{\infty}^{\mathbf{v}_a}} \oint \prod_{a=1}^N d\mu[W^{(a)}; q, t] \\ & \sum_{\substack{\{\lambda^{(a)}, \mu^{(a)}, \sigma^{(a)}\} \\ \{\nu^{(a)}, \rho^{(a)}\}}} \prod_{a=1}^N f_{\lambda^{(a)}\sigma^{(a)}}^{\nu^{(a)}}(q, t) P_{\nu^{(a)}}(W^{(a)}; q, t) f_{\mu^{(a)}\tilde{\sigma}^{(a-1)}}^{\rho^{(a)}} P_{\rho^{(a)}}(\bar{W}^{(a)}; q, t) \\ & \gamma_a^{|\sigma^{(a)}|} Q_{\lambda^{(a)}} \left( \frac{\alpha_a X^{(a)}}{1-t}; q, t \right) Q_{\mu^{(a)}} \left( \frac{\beta_a \bar{X}^{(a+1)}}{1-t}; q, t \right) b_{\sigma^{(a)}}(q, t). \end{aligned} \quad (3.124)$$

In the above we write  $\tilde{\sigma}^{(a)}$  to denote the partition shifted by  $(\zeta^{\mathbf{v}})$ , and  $\sigma^{(0)}$  is identified with  $\sigma^{(N)}$ . Precisely:

$$\tilde{\sigma}^{(a)} = \sigma^{(a)} + (\zeta_{a+1}^{\mathbf{v}_{a+1}}), \quad (3.125)$$

where again  $\mathbf{v}_{N+1}$  and  $\zeta_{N+1}$  are identified with  $\mathbf{v}_1$  and  $\zeta_1$  respectively. In the next step of the calculation, we use the orthogonality of the Macdonald polynomials with respect to the inner product (B.45), this introduces a normalisation factor (B.47). Finally, we use

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<sup>25</sup>We use a plethystically substituted form  $X \rightarrow \frac{X}{1-t}$  of this identity for the flavour terms.

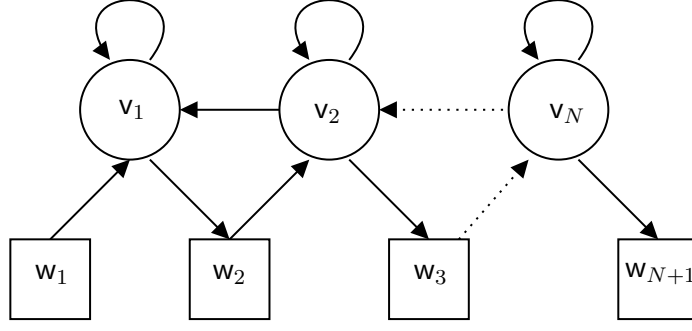


Figure 3.8: Handsaw quiver  $\mathfrak{Q}(\mathbf{v}, \mathbf{w})$ .

the definition of skew Macdonald polynomials (B.42) to write the integral as:

$$\mathcal{I}_{\{\mathbf{w}\}\{\mathbf{v}\}}(X; \zeta; \alpha, \beta, \gamma; q, t) = \sum_{\{\nu^{(a)}, \sigma^{(a)}\}} \prod_{a=1}^N \frac{(q; q)_{\infty}^{\nu_a}}{(t; q)_{\infty}^{\nu_a}} \tilde{c}_{\nu_a}(\nu^{(a)}; q, t) \gamma_a^{|\sigma^{(a)}|} P_{\nu^{(a)}/\sigma^{(a)}} \left( \frac{\alpha_a X^{(a)}}{1-t}; q, t \right) Q_{\nu^{(a)}/\tilde{\sigma}^{(a-1)}} \left( \frac{\beta_a \bar{X}^{(a+1)}}{1-t}; q, t \right). \quad (3.126)$$

In the above, we have also used part of the normalisation of the inner product (the  $b_{\nu}$  term) combined with the  $b_{\sigma}$  term to re-normalise the first skew Macdonald polynomial.  $\square$

### 3.4.2 Handsaw quivers

We now consider some particular examples of chainsaw quivers known as *handsaw quivers*. These quivers are shown in figure 3.8 and are subvarieties of the chainsaw quivers  $\mathcal{D}(\mathbf{v}, \mathbf{w})$ . We denote the handsaw quiver by  $\mathfrak{Q}(\mathbf{v}, \mathbf{w})$ . Later we meet these again as vortex moduli spaces of  $T_{\rho}[SU(N)]$  theories where we study their  $\chi_t$  genera using the fixed point localisation methods of chapter 3.

The Molien integral for these quivers can be computed by setting  $\gamma_N = 0$  which ‘breaks the periodicity’ of the chainsaw quiver. The proof of proposition 9 required  $\gamma_N \neq 0$  and we therefore need to modify the result as follows

**Proposition 2.** The chainsaw quiver  $q$ -deformed Molien integral can equivalently be expressed as

$$\mathcal{I}_{\{\mathbf{w}\}\{\mathbf{v}\}}(X; \zeta; \alpha, \beta, \gamma; q, t) = \sum_{\{\nu^{(a)}, \sigma^{(a)}\}} \prod_{a=1}^N \frac{(q; q)_{\infty}^{\nu_a}}{(t; q)_{\infty}^{\nu_a}} \tilde{c}_{\nu_a}(\nu^{(a)}; q, t) \gamma_a^{|\sigma^{(a)}|} P_{\tilde{\nu}^{(a)}/\sigma^{(a)}} \left( \frac{\alpha_a X^{(a)}}{1-t}; q, t \right) Q_{\nu^{(a)}/\tilde{\sigma}^{(a-1)}} \left( \frac{\beta_a \bar{X}^{(a+1)}}{1-t}; q, t \right), \quad (3.127)$$

where  $\tilde{\nu}^{(1)} = \nu^{(1)} + (\zeta_1^{\nu_1})$  when  $a = 1$  and  $\tilde{\nu}^{(a)} = \nu^{(a)}$  otherwise. Further  $\tilde{\sigma}^{(a)} = \sigma^{(a)} + (\zeta_{a+1}^{k_{a+1}})$

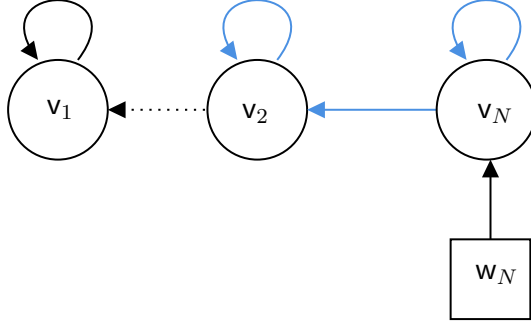


Figure 3.9: Toothless handsaw quiver  $\mathfrak{Q}(\mathbf{v}, \mathbf{w}_N)$  with an example closed oriented cycle highlighted in blue.

when  $a = 1, \dots, N-1$  and  $\tilde{\sigma}^{(N)} = \sigma^{(N)}$ .

*Proof.* The only difference to the proof of proposition 9 is that we multiply  $\prod_{i=1}^{k_1} (w_i^{(1)})^{-\zeta_a}$  with  $P_{\sigma^{(1)}}(W^{(1)})$  in the first line instead of  $P_{\mu^{(1)}}(W^{(1)})$ . We do this because when  $\gamma_N = 0$  we cannot multiply it with  $P_{\sigma^{(N)}}(W^{(1)})$ .  $\square$

Sending  $q \rightarrow 0$  with  $\gamma_N = 0$  and  $\alpha_a = \beta_a = 1$  for  $a = 1, \dots, N$  recovers the Hilbert series of the Handsaw quiver  $\mathfrak{Q}(\mathbf{v}, \mathbf{w})$  of figure 3.8.

$$\chi_{\mathcal{T}}(\mathcal{L}^{\zeta}, \mathfrak{Q}(\mathbf{v}, \mathbf{w})) = \lim_{q \rightarrow 0} \mathcal{I}_{\mathbf{w}, \mathbf{v}}(X; \zeta; \gamma_N = 0). \quad (3.128)$$

**Toothless handsaw.** Now we can consider the case  $\gamma_N = 0$  together with  $\alpha_N = 1$  and  $\alpha_a = 0$  for  $a = 1, \dots, N-1$  and  $\beta_a = 0$  for all  $a = 1, \dots, N$ , the quiver diagram is shown in figure 3.9. We call such quivers *toothless handsaw quiver* and denote them by  $\mathfrak{Q}(\mathbf{v}, \mathbf{w}_N)$ . Using the homogeneity property of the skew Macdonald polynomials in (B.42) we have the following constraints satisfied by the partitions in the sum (3.127)

$$\begin{aligned} \nu^{(1)} + (\zeta_1^{\mathbf{v}_1}) &= \sigma^{(1)}, \quad \nu^{(a)} = \sigma^{(a)}, \quad a \geq 2 \\ \nu^{(1)} &= \sigma^{(N)}, \quad \nu^{(a)} = \sigma^{(a-1)} + (\zeta_a^{\mathbf{v}_a}), \quad a \geq 2. \end{aligned} \quad (3.129)$$

Together with the fact that  $\gamma^N = 0$  implying  $\sigma^{(N)} = \emptyset$  these equations can be solved recursively for  $\nu^{(N)}$  giving a single partition  $\lambda$  that survives in the sum

$$\nu^{(N)} = (\zeta_N^{\mathbf{v}_N}) + \dots + (\zeta_1^{\mathbf{v}_1}) =: \lambda. \quad (3.130)$$

The expression for the  $q$ -deformed Molien integral is then

$$\begin{aligned} \mathcal{I}_{\{\mathbf{w}\}\{\mathbf{v}\}}(X; \zeta; \alpha, \beta, \gamma; q, t) &= \\ \left[ \frac{(q; q)_{\infty}^{\mathbf{v}_1 + \dots + \mathbf{v}_N}}{(t; q)_{\infty}^{\mathbf{v}_1 + \dots + \mathbf{v}_N}} \prod_{a=1}^N \tilde{c}_{\mathbf{v}_a}(\nu^{(a)}; q, t) \right] P_{\lambda} \left( \frac{X}{1-t}; q, t \right). \end{aligned} \quad (3.131)$$

where we have relabelled  $X := X^{(N)}$ . Finally, we consider the Hilbert series limit  $q \rightarrow 0$ . In this limit the Macdonald polynomial becomes a Milne polynomial (see appendix B) and it is a short calculation with the normalisation constant (B.47) to see that

$$\begin{aligned} \lim_{q \rightarrow 0} \mathcal{I}_{\{\mathbf{w}\}\{\mathbf{v}\}}(X; \zeta; t) &= \chi_{\mathcal{T}}(\mathcal{L}_1^{\zeta_1} \otimes \dots \otimes \mathcal{L}_N^{\zeta_N}, \mathfrak{Q}(\mathbf{v}, \mathbf{w}_1)) \\ &= \frac{1}{(t; t)_{\mathbf{v}_1}} \frac{1}{(t; t)_{\mathbf{v}_1 + \mathbf{v}_2}} \dots \frac{1}{(t; t)_{\mathbf{v}_1 + \dots + \mathbf{v}_N}} Q'_\lambda(X; t), \end{aligned} \quad (3.132)$$

where  $\lambda = (\zeta_1^{\mathbf{v}_1}) + \dots + (\zeta_N^{\mathbf{v}_N})$ . The prefactor arises from the more general closed oriented cycles available for the toothless handsaw, these are illustrated on figure 3.9. We therefore have the following generalisation of proposition 1:

**Theorem 10.** The normalised equivariant Euler characteristic of line bundles over  $\mathfrak{Q}(\mathbf{v}, \mathbf{w}_N)$  realise characters of the Kirillov-Reshetikhin module

$$\mathcal{H}_{\vec{n}} = \bigotimes_{l=1}^k \text{KR}(lw_1)^{n_1^{(l)}}. \quad (3.133)$$

Writing  $\lambda = (n^{(1)}, \dots, n^{(k)})$  we have

$$\frac{\chi_{\mathcal{T}}(\mathcal{L}_1^{\zeta_1} \otimes \dots \otimes \mathcal{L}_N^{\zeta_N})}{\chi_{\mathcal{T}}(\mathcal{O}_{\mathfrak{Q}})} = Q'_\lambda(X; t). \quad (3.134)$$

### 3.4.3 Large rank limit

We now compute the large gauge rank limit of the Hilbert series. We focus on the ADHM case  $\mathcal{D}(\mathbf{v}, \mathbf{w}) = \mathcal{M}_{N,r}$ , that is the chainsaw quiver 3.7 with one gauge node. We keep the  $q$ -deformation but turn off line bundle charge and match to the symmetry conventions in the ADHM section 3.3.1 so that

$$\alpha = 1, \quad \beta = t_1 t_2, \quad \gamma = t_2, \quad t = t_1. \quad (3.135)$$

In that case proposition 9 tells us that the  $q$ -deformed Molien integral is given by

$$\begin{aligned} \mathcal{I}_{N,r}(X; q, t_1, t_2) &= \\ \sum_{\{\nu, \sigma\}} \frac{(q; q)_\infty^N}{(t_1; q)_\infty^N} \tilde{c}_N(\nu; q, t_1) t_2^{|\sigma|} P_{\nu/\sigma} \left( \frac{\bar{X}}{1 - t_1}; q, t \right) Q_{\nu/\sigma} \left( \frac{t_1 t_2 X}{1 - t_1}; q, t_1 \right). \end{aligned} \quad (3.136)$$

**Normalisation constant.** A subtle aspect to taking the large  $N$  limit of (3.136) is how to deal with the Macdonald integral normalisation constant  $\tilde{c}_N(\nu; q, t)$ . This constant is

defined in equation (B.47) by

$$\tilde{c}_N(\nu; q, t_1) := \prod_{i=1}^N \frac{\Gamma_q(i\beta)}{\Gamma_q(\beta)\Gamma_q((i-1)\beta+1)} \prod_{s \in \lambda} \frac{1 - q^{a'(s)} t_1^{N-l'(s)}}{1 - q^{a'(s)+1} t_1^{N-l'(s)-1}}, \quad (3.137)$$

where  $\Gamma_q(x)$  is the  $q$ -Gamma function (A.6). We deal with the first factor to begin with. Writing  $t_1 = q^\beta$ , we use the definition of the  $q$ -Gamma function to express the first factor as

$$\prod_{i=1}^N \frac{\Gamma_q(i\beta)}{\Gamma_q(\beta)\Gamma_q((i-1)\beta+1)} = \frac{(t_1; q)_\infty^N}{(q; q)_\infty^N} \prod_{i=1}^N \frac{(qt_1^{i-1}; q)_\infty}{(t_1^i; q)_\infty}. \quad (3.138)$$

This product can be telescoped so that for the entire normalisation constant we have

$$\tilde{c}_N(\nu; q, t_1) = \frac{(t_1; q)_\infty^N}{(q; q)_\infty^N} \frac{(q; q)_\infty}{(t_1; t_1)_N (qt_1^N; q)_\infty} \prod_{s \in \lambda} \frac{1 - q^{a'(s)} t_1^{N-l'(s)}}{1 - q^{a'(s)+1} t_1^{N-l'(s)-1}}. \quad (3.139)$$

Now we assume that  $|t_1| < 1$  so that as  $N \rightarrow \infty$  the second product factor goes to 1 together with  $(qt_1^N; q)_\infty \rightarrow 1$ . Combining this limit with (3.136) we find

$$\lim_{N \rightarrow \infty} \mathcal{I}_{N,r}(X; q, t_1, t_2) = \frac{(q; q)_\infty}{(t_1; t_1)_\infty} \sum_{\nu, \sigma} t_2^{|\sigma|} P_{\nu/\sigma} \left( \frac{\bar{X}}{1 - t_1}; q, t_1 \right) Q_{\nu/\sigma} \left( \frac{t_1 t_2 X}{1 - t_1}; q, t_1 \right). \quad (3.140)$$

To simplify the Hilbert series further we require the following lemma.

**Lemma 3.** Skew Macdonald polynomials satisfy the following Cauchy-like identity:

$$\sum_{\lambda, \mu} \gamma^{|\lambda|} Q_{\mu/\lambda}(X; q, t) P_{\mu/\lambda}(Y; q, t) = \text{PE} \left[ \frac{\gamma}{1 - \gamma} + \frac{1 - t}{(1 - \gamma)(1 - q)} XY \right]. \quad (3.141)$$

*Proof.* The method of proof used here is an adaptation of the Schur case found in exercise (28) of Chapter II.5 in Macdonald [144]. We let

$$F(X, Y; q, t) = \sum_{\lambda, \mu} \gamma^{|\lambda|} Q_{\mu/\lambda}(X; q, t) P_{\mu/\lambda}(Y; q, t). \quad (3.142)$$

Using the identity (B.44) and the fact that Macdonald polynomials are homogeneous we can perform the sum over  $\mu$  to find:

$$F(X, Y; q, t) = \text{PE} \left[ \frac{1 - t}{1 - q} XY \right] \sum_{\lambda, \mu} \gamma^{|\mu|} Q_{\lambda/\mu}(\gamma X; q, t) P_{\lambda/\mu}(Y; q, t). \quad (3.143)$$

In other words:

$$F(X, Y; q, t) = \text{PE} \left[ \frac{1 - t}{1 - q} XY \right] F(\gamma X, Y; q, t). \quad (3.144)$$

Now, provided  $|\gamma| < 1$ , we can iterate this relation to find:

$$F(X, Y; q, t) = F(0, Y; q, t) \text{PE} \left[ \frac{1-t}{(1-\gamma)(1-q)} XY \right]. \quad (3.145)$$

Using the fact that  $P_{\lambda/\mu}$  vanishes unless  $\mu \subset \lambda$  together with the fact  $P_{\lambda/\mu}(0)$  vanishes unless  $\mu = \lambda$  (where it equals 1) we find:

$$F(0, Y; q, t) = \sum_{\lambda} \gamma^{|\lambda|} = \prod_{k=1}^{\infty} \frac{1}{1-\gamma^k} = \text{PE} \left[ \frac{\gamma}{1-\gamma} \right]. \quad (3.146)$$

The lemma then follows.  $\square$

After suitable parameter identifications and making the plethystic substitutions  $X \rightarrow \frac{X}{1-t_1}$ ,  $Y \rightarrow \frac{Y}{1-t_1}$  then we can use this lemma to evaluate (3.140). At large  $N$ , the normalisation constants of (3.140) can also be expressed in a plethystic form and overall we have the relatively compact plethystic expression

$$\lim_{N \rightarrow \infty} \mathcal{I}_{N,r}(X; q, t_1, t_2) = \text{PE} \left[ \frac{1}{1-q} \left( \frac{X\bar{X}}{(1-t_1)(1-t_2)} - q \right) + \frac{t_1}{1-t_1} + \frac{t_2}{1-t_2} \right]. \quad (3.147)$$

Finally, sending  $q \rightarrow 0$  yields a plethystic expression for the large gauge rank Hilbert series of the ADHM quiver

$$\lim_{N \rightarrow \infty} \text{H.S.}[\mathcal{M}_{N,r}] = \text{PE} \left[ \frac{(x_1 + \dots + x_r)(x_1^{-1} + \dots + x_r^{-1})}{(1-t_1)(1-t_2)} + \frac{t_1}{1-t_1} + \frac{t_2}{1-t_2} \right]. \quad (3.148)$$

# CHAPTER 4

## VORTEX GEOMETRY AND HEMISPHERE BLOCKS

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In this chapter we study the geometrical interpretation of holomorphic factorisation of closed three manifold partition functions of 3d  $\mathcal{N} = 4$  theories. We focus on the twisted index and show that this observable coincides with the Hilbert series of the theory. We introduce an angular momentum refinement that allows us to use holomorphic factorisation to understand the Hilbert series in terms of the geometry of vortex moduli space.

We then introduce a UV construction of a *hemisphere block* realised as a partition function of the theory on a hemisphere with an exceptional Dirichlet boundary condition. Inspired by concrete calculations of half indices of these boundary conditions for the SQED[ $N$ ] theory, we propose a geometric prescription for the block for a wide class of quiver gauge theories that depends only on the Higgs branch geometry. We show that the hemisphere block glues *exactly* to reproduce results from the localisation of twisted indices discussed in the first part of the chapter.

We find that in specialised limits the hemisphere blocks have interesting representation theoretic content—they compute lowest weight Verma module characters of the chiral rings of the theory. Combined with the observations in the first parts of the chapter, this allows us to understand holomorphic factorisation of partition functions of 3d  $\mathcal{N} = 4$  theories in terms of ‘gluing’ Verma module characters.

**Overview.** Section 4.1 is a review of background material on vortices, quasimaps and the twisted index that we use throughout the remainder of the thesis. In section 4.2 we investigate the relationships between vortex geometry, holomorphic factorisation and the Coulomb branch Hilbert series. We demonstrate these ideas explicitly for the  $T[SU(N)]$  theory with a novel calculation factorising the topologically twisted index. In section 4.3 we introduce the new notion of *hemisphere blocks*. We provide a prescription to compute these blocks for a wide variety of quiver gauge theories and discuss applications to understanding the geometry of factorisation of the twisted index. Finally, in section 4.4 we discuss the representation theoretic content of hemisphere blocks and meet the final novel result of the chapter where we show that, in certain specialised limits of the block, we recover lowest

weight Verma module characters of the quantised Higgs and Coulomb branch chiral rings. The conventions for partitions and symmetric functions used throughout this chapter are summarised in appendices A and B.

**Publications.** Section 4.2 is based on the work [108] written in collaboration with N. Dorey and D. Zhang

- S. Crew, N. Dorey and D. Zhang, *Factorisation of 3d  $\mathcal{N} = 4$  twisted indices and the geometry of vortex moduli space*, *JHEP* **08** (2020) 015.

Sections 4.3 and 4.4 are based on [160] with M. Bullimore and D. Zhang

- M. Bullimore, S. Crew and D. Zhang, *Boundaries, Vermas, and Factorisation*, 2010.09741.

Sections 4.3.3 and 4.3.4 consist of the author’s currently unpublished work.

## 4.1 Background

**Outline.** In section 4.1.1, we discuss the relationship between *quasimap moduli spaces* and vortex moduli spaces of 3d  $\mathcal{N} = 4$  theories and review the *vertex function* which is the main geometrical object of study in this chapter. In section 4.1.2 we review the definitions of topologically twisted indices and how to compute them using Coulomb branch localisation. We then review the notion of *holomorphic factorisation* of 3d partition functions and how this applies in particular to the topologically twisted indices. Finally, we spend some time introducing the  $T[SU(N)]$  theory which is the main example studied in section 4.2.

### 4.1.1 Vortices and quasimaps

3d  $\mathcal{N} = 4$  theories admit half-BPS vortex solutions. In section 2.1.5 we discussed how the vortex moduli space can be described by the data of a principal  $G_{\mathbb{C}}$ -bundle together with holomorphic sections  $(X, Y)$  in  $\mathcal{R}$  that satisfy the complex moment map constraint  $\mu_{\mathbb{C}} = 0$ . When  $\mathcal{R}$  is a quiver representation and the Higgs branch is the quiver variety  $\mathcal{M}_H = \mathcal{M}_{\theta}(\mathbf{v}, \mathbf{w})$  this construction is realised by the quasimap moduli spaces  $\text{QM}(\mathcal{M}_H)$ —these are fundamental objects in enumerative geometry introduced in the works [94, 95, 161]. The physical observables we are interested are *vortex partition functions*, which in the quasimap literature are known as *vertex functions*. In this subsection we review the construction of quasimap spaces and their vertex functions.



**Quasimap spaces.** Quasimap moduli spaces  $\mathbf{QM}^d(X)$  associated to a Nakajima quiver variety  $X = \mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$  are Deligne-Mumford stacks. Later, in section 4.2, we work with more down-to-earth realisations of particular quasimap moduli spaces as smooth projective varieties known as *Laumon spaces* [96]. However for general quiver gauge theories, such as the 3d ADHM theory studied in chapter 5, we require the more general quasimap machinery.

We begin with an intuitive description. Quasimap moduli spaces parametrise maps  $f : \mathbb{P}^1 \rightarrow X$ . They split into topologically distinct sectors for each map degree which we denote by

$$\mathbf{QM}^d(X) = \{\text{maps of degree } \mathbf{d} : \mathbb{P}^1 \rightarrow X\}. \quad (4.1)$$

The degree  $\mathbf{d}$  describes the image of  $\mathbb{P}^1$  inside  $X$  *i.e.*  $\mathbf{d} = \text{im } f \in H^2(X, \mathbb{Z})$ . Recall from 3.1.1 that a Nakajima quiver variety has tautological bundles  $(\mathcal{V}_i)_{i \in I}$  and  $(\mathcal{W}_i)_{i \in I}$  associated to each node of the quiver. A map  $f$  induces pullback bundles  $\mathcal{V} = f^*\mathcal{V}$  and topologically trivial bundles  $\mathcal{W} = f^*\mathcal{W}$  on  $\mathbb{P}^1$ . We also have a bundle  $\mathcal{R}(\mathbf{v}, \mathbf{w})$  on  $\mathbb{P}^1$  corresponding to the quiver representation  $\mathcal{R}(\mathbf{v}, \mathbf{w})$  defined by

$$\begin{aligned} \mathcal{M}(\mathbf{v}, \mathbf{w}) &= \sum_{i \in I} \text{Hom}(\mathcal{W}_i, \mathcal{V}_i) \oplus \bigoplus_{(i,j) \in E} \vec{Q}_{ij} \otimes \text{Hom}(\mathcal{V}_i, \mathcal{V}_j), \\ \mathcal{R}(\mathbf{v}, \mathbf{w}) &= \mathcal{M}(\mathbf{v}, \mathbf{w}) \oplus \mathcal{M}(\mathbf{v}, \mathbf{w})^\vee. \end{aligned} \quad (4.2)$$

The definition of a quasimap  $f : \mathbb{P}^1 \dashrightarrow X$  involves working with bundle data rather than the map  $f$  itself. The data is

- A collection of rank  $(\mathbf{v}_i)_{i \in I}$  vector bundles  $(\mathcal{V}_i)_{i \in I}$  on  $\mathbb{P}^1$ .
- A section  $f \in H^0(\mathbb{P}^1, \mathcal{R}(\mathbf{v}, \mathbf{w}) \otimes t)$  satisfying  $\mu_{\mathbb{C}} = 0$ .

The degree  $\mathbf{d}$  of a quasimap is defined to be the vector of degrees of the bundles  $(\mathcal{V}_i)_{i \in I}$ , this coincides with the intuitive notion of degree discussed above whenever the quasimap  $f$  is non-singular and matches the first Chern classes in the vortex moduli space description in section 2.1.5. The quasimap moduli space is then defined by

$$\mathbf{QM}^d(X) := \{\text{degree } \mathbf{d} \text{ quasimap data}\} / \cong. \quad (4.3)$$

Moving in the moduli space corresponds to varying the bundles  $(\mathcal{V}_i)_{i \in I}$ , up to isomorphism, and the section  $f$  with the base curve fixed.

**Stable quasimaps.** In general quasimaps define maps from  $\mathbb{P}^1$  to the stack quotient  $\mu^{-1}(0)/G_{\mathbb{C}}$ , the space of *stable quasimaps* is a better behaved space that is a finite type Deligne-Mumford stack with a perfect obstruction theory [94]. Stable quasimaps are defined

to have only finitely many points  $p \in \mathbb{P}^1$  mapping to non-stable points<sup>1</sup> of  $\mu^{-1}(0)/G_{\mathbb{C}}$ . Hereafter we write  $\mathbf{QM}^d(X)$  for the space of stable quasimaps.

**Non-singular quasimaps.** For each point  $p \in \mathbb{P}^1$  we have an associated evaluation map

$$\mathrm{ev}_p : \mathbf{QM}^d(X) \rightarrow \mu^{-1}(0)/G_{\mathbb{C}}, \quad (4.4)$$

the right hand side is a quotient stack that includes the stable points as an open subset  $\mu^{-1}(0)^s/G_{\mathbb{C}} \subset \mu^{-1}(0)/G_{\mathbb{C}}$ . We say that a quasimap  $f$  is *non-singular* at  $p$  if  $f(p)$  lies in the stable locus for all  $p$ —*i.e.*  $f(p)$  is a point in the quiver variety  $X$ . In that case we have a well-defined evaluation map to the Higgs branch

$$\mathrm{ev}_p : \mathbf{QM}_{\text{non. sing. at } p}^d(X) \rightarrow X. \quad (4.5)$$

**Torus actions and based quasimaps.**  $\mathbf{QM}^d(X)$  admits a natural torus action  $\mathbf{T}_q = \mathbb{C}_q^\times \times \mathbf{T}$ . The torus action  $\mathbf{T}$  is inherited from the corresponding action on the quiver variety  $X$ —the fugacities  $x$  of this group action are known as *equivariant parameters* in the curve counting literature, and  $\mathbb{C}_q^\times$  acts by scaling the coordinate on  $\mathbb{P}^1$ . In physical terms, as we saw in section 2.1.5, the fugacities  $x$  on the torus  $\mathbf{T}$  are identified with exponentiated real masses; later we place the theory on an omega-deformed spacetime whose deformation parameter will be associated to rotations of the  $\mathbb{P}^1$  base. We also note from [95] that the quasimap moduli spaces are empty outside a certain cone  $C \subset H^2(X, \mathbb{Z})$  of allowed degrees  $\mathbf{d}$  determined by the stability condition of the quiver  $X$ . In chapter 5 we see the cone condition can give interesting combinatorial constraints on which quasimap moduli spaces are non-empty.

We now fix a point  $p = \infty \in \mathbb{P}^1$  and consider the moduli space of stable quasimaps non-singular at  $\infty$ . If  $\alpha \in X^\mathbf{T}$  is an isolated massive vacuum then we have a well-defined locus of *based quasimaps*  $\mathbf{QM}_\alpha^d(X) \subset \mathbf{QM}_{\text{non. sing. at } \infty}^d$  given by

$$\mathbf{QM}_\alpha^d = \{\text{degree } \mathbf{d} \text{ stable quasimap data non-singular at } \infty : f(\infty) = \alpha\}. \quad (4.6)$$

**Vertex functions.** The main object we are interested in enumerative geometry is known as the *vertex function*  $\mathbf{V}(\zeta, x; q, t)$ . The vertex function can be realised as the equivariant Euler characteristic of a, suitably defined, virtual structure sheaf on  $\mathbf{QM}^d(X)$ . In the work [94] it is shown that the quasimap moduli space admits a perfect obstruction theory and is equipped with a virtual structure sheaf<sup>2</sup>  $\hat{\mathcal{O}}_{\text{vir}}$  and a virtual tangent bundle  $T^{\text{vir}}\mathbf{QM}_\alpha^d$ . We

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<sup>1</sup>In the sense of definition 2.3

<sup>2</sup>The hat denotes the symmetrised virtual structure sheaf which is the virtual structure sheaf twisted by the virtual canonical line bundle, this is analogous to working with the Dirac operator rather than the Dolbeault operator on a Kähler manifold.

do not review the construction in detail here but note only that the perfect obstruction theory allows us to compute Euler characteristics using the localisation formulae of chapter 3.

Consider quasimaps non-singular at  $\infty$ . The evaluation map (4.5) defines a pushforward  $\mathrm{ev}_{p,*}$  with image lying inside  $K_{T_q}(X)$ . The vertex functions are defined by

$$\mathbf{V}(x, \zeta; q, t) := \sum_d \zeta^d \mathrm{ev}_{p,*}(\mathbf{QM}_{\mathrm{non. sing. at } p}^d(X), \hat{\mathcal{O}}_{\mathrm{vir}}) \in K_{T_q}(X)_{\mathrm{loc}}[[\zeta]], \quad (4.7)$$

in the above the *Kähler parameters*  $\zeta$  (FI parameters) are naturally paired with the quasimap degree  $\mathbf{d}$  (vortex number) and  $x$  and  $t$  are coordinates on the torus  $\mathbf{T} = \mathbf{A} \times \mathbb{C}_t^\times$  (mass parameters). Later  $q$  will be identified with the omega background deformation parameter. The vertex function can be expanded in the fixed point basis of the localised  $K$ -theory and, since the fixed locus  $(\mathbf{QM}^d)^{T_q}$  is proper [95], the vertex function at a fixed point computes the equivariant Euler characteristic of the virtual structure sheaf on the moduli space of based quasimaps

$$\mathbf{V}_\alpha(x, \zeta; q, t) = \sum_d \zeta^d \chi(\hat{\mathcal{O}}_{\mathrm{vir}}, \mathbf{QM}_\alpha^d(X)). \quad (4.8)$$

The vertex function can then be computed using the localisation theorems discussed in section 3.1 giving

$$\mathbf{V}_\alpha(x, \zeta; q, t) = \sum_d \zeta^d \sum_{(\mathcal{V}, \mathcal{W})} \hat{a}(T_{(\mathcal{V}, \mathcal{W})}^{\mathrm{vir}} \mathbf{QM}_p^d), \quad (4.9)$$

where the second sum is taken over quasimaps  $(\mathcal{V}, \mathcal{W})$  fixed under the torus action  $T_q$  and  $\hat{a}$  is a symmetric version<sup>3</sup> of the plethystic exponential defined on torus weights by

$$\hat{a}(\omega) = \frac{1}{\omega^{\frac{1}{2}} - \omega^{-\frac{1}{2}}}, \quad \hat{a}(\omega_1 + \omega_2) = \hat{a}(\omega_1) \hat{a}(\omega_2). \quad (4.10)$$

**Remark.** We note that, informally speaking, this index has ‘numerators and denominators’ in the sense that at a fixed point the virtual tangent bundle takes the form of a difference of  $K$ -theory classes

$$T_p^{\mathrm{vir}} = \mathrm{Def}_p - \mathrm{Obs}_p, \quad (4.11)$$

so that the plethystic exponential in the localisation formula (3.5) contributes numerators and denominators at each fixed point. Therefore, the vertex functions have more in common with the  $\chi_t$  genus of section 3.1.3 than the Hilbert series of section 3.1.2.

In section 4.2 we study the  $T[SU(N)]$  theory. In this case, we show that the relevant vertex functions agree with generating functions of  $\chi_t$  genera of *smooth* Laumon spaces, or

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<sup>3</sup>Since we work with the symmetrised virtual structure sheaf  $\hat{\mathcal{O}}_{\mathrm{vir}}$  which carries an extra factor of the the square root of the canonical bundle compared with  $\mathcal{O}_{\mathrm{vir}}$ .

handsaw quiver varieties [152], and the numerators and denominators in (4.9) match the localisation formula for the  $\chi_t$  genus (3.30). In section 4.3 we make the correspondence with enumerative geometry and 3d  $\mathcal{N} = 4$  gauge theory more precise by introducing a particular UV boundary condition in the 3d theory whose partition function realises the vertex functions. In this context we also determine perturbative contributions unambiguously using the formalism of quasimaps.

**SQED[ $N$ ] example.** Following [51], we now show how to compute the relevant vertex functions for our favourite example: SQED[ $N$ ] with Higgs branch  $\mathcal{M}_H = T^*\mathbb{P}^{N-1}$ .

We have two tautological bundles  $\mathcal{V}$  and  $\mathcal{W}$  on  $T^*\mathbb{P}^{N-1}$ . The tangent bundle (3.19) induces a bundle on the base curve denoted  $\mathcal{P} + t\mathcal{P}^\vee$  with  $\mathcal{P} = \mathcal{W}^\vee \otimes \mathcal{V} - \mathcal{V} \otimes \mathcal{V}^\vee$ . For this example, the cone of effective degrees is  $\mathbf{d} = d \in \mathbb{Z}_{\geq 0}$  and the fibre of the (reduced<sup>4</sup>) virtual tangent bundle at a point  $(\mathcal{V}, \mathcal{W})$  is given by

$$T_{(\mathcal{V}, \mathcal{W})}^{\text{vir.}} \text{QM}_\alpha^{\mathbf{d}} = H^\bullet(\mathcal{P} \oplus t\mathcal{P}^\vee) - T_\alpha \mathbb{P}^{N-1}. \quad (4.12)$$

A lemma of Grothendieck says that the bundles  $(\mathcal{V}, \mathcal{W})$  split and in particular at a  $\mathbb{T}_q$  fixed point they are given by

$$\mathcal{V} = \mathcal{O}(d)q^{-d}x_\alpha, \quad \mathcal{W} = \mathcal{O}(1)x_1 + \dots + \mathcal{O}(1)x_N. \quad (4.13)$$

The higher cohomology in the right hand side of (4.12) vanishes and the character of line bundles of this form is given by<sup>5</sup>

$$\text{ch}_{\mathbb{T}_q} H^0(x_\alpha q^{-d} \mathcal{O}(d)) = x_\alpha (1 + q^{-1} + \dots + q^{-d}). \quad (4.14)$$

We now have the ingredients to compute the localisation formula (4.9). We find

$$V_\alpha(x, \zeta; q, t) = \sum_{d \geq 0} \left( \left( q^{\frac{1}{2}} t^{-\frac{1}{2}} \right)^N \zeta \right)^d \prod_{i=1}^N \frac{\left( t \frac{x_\alpha}{x_i}; q \right)_d}{\left( q \frac{x_\alpha}{x_i}; q \right)_d}. \quad (4.15)$$

We reproduce this expression from the  $\chi_t$  genus of the relevant Laumon space in section 4.2

### 4.1.2 The topologically twisted index

In this section we review the definition and localisation computation of one of the main supersymmetric observables studied in this thesis, the topologically twisted index.

<sup>4</sup>Subtracting the tangent bundle of  $\mathbb{P}^{N-1}$  normalises the vertex function to 1 as an expansion in  $\zeta$ .

<sup>5</sup>See the discussion in section 3.4.2.

**Twisted indices.** 3d  $\mathcal{N} = 4$  theories admit two topological twists [162, 163]—the  $A$ -twist and the  $B$ -twist—which are uplifts of the two dimensional  $A$ - and  $B$ -models [24]. The choice of twist corresponds to twisting the  $SU(2)_E$  Lorentz group with either  $SU(2)_H$  or  $SU(2)_C$  respectively. Throughout, we fix a  $U(1)_H \times U(1)_C$  subalgebra of the  $SU(2)_H \times SU(2)_C$   $R$ -symmetry and denote the integer  $R$ -charges by  $R_H$  and  $R_C$ .

In the IR non-linear sigma model context, the  $B$ -twist is known as the Rozansky-Witten twist [70] and preserves the four supercharges:  $Q_+^{11}, Q_+^{21}, Q_-^{12}, Q_-^{22}$  satisfying the supersymmetry algebra

$$\{Q_+^{A1}, Q_-^{B2}\} = 2\epsilon^{AB} E. \quad (4.16)$$

The  $A$ -twist is the mirror of the  $B$ -twist and preserves the supercharges  $Q_+^{11}, Q_-^{22}, Q_+^{12}, Q_-^{21}$  that satisfy the supersymmetry algebra

$$\{Q_+^{1A}, Q_-^{2B}\} = 2\epsilon^{AB} E. \quad (4.17)$$

Both twists preserve the common supercharges  $Q_+^{11}$  and  $Q_-^{22}$  which are compatible with the anti-diagonal  $R$ -symmetry combination  $R_H - R_C$  together with angular momenta  $J + \frac{1}{2}R_H$  and  $J + \frac{1}{2}R_C$ . These are gradings for global symmetries from the perspective of our fixed  $\mathcal{N} = 2$  subalgebra. The twisted indices are then defined by

$$\begin{aligned} I^A &= \text{Tr}_{\mathcal{H}_{S^2}^A} (-1)^F q^{J + \frac{R_C}{2}} t^{\frac{R_H - R_C}{2}} x^{T_H} \xi^{T_C}, \\ I^B &= \text{Tr}_{\mathcal{H}_{S^2}^B} (-1)^F q^{J + \frac{R_H}{2}} t^{\frac{R_C - R_H}{2}} x^{T_H} \xi^{T_H}. \end{aligned} \quad (4.18)$$

where  $T_H$  and  $T_C$  are maximal tori of the flavour and topological symmetries respectively.

**Path integral.** A 3d  $\mathcal{N} = 2$  theory with integer  $R$ -charge assignments can be placed supersymmetrically on a curved  $S^2 \times S^1$  with a non-trivial background connection for the  $R$ -symmetry gauge field  $A^{(R)}$ .

The method to achieve this was pioneered by Seiberg and Festuccia in the work [21]. The particular case of the 3d  $\mathcal{N} = 2$  twisted index has been studied in detail by Benini *et al.* [164, 165, 166] and the  $\mathcal{N} = 4$  case in particular was recently studied by Closset and Kim [167]. We couple the  $\mathcal{R}$  multiplet of our theory to a classical supergravity background. The relevant supergravity theory is known as ‘new minimal supergravity’ [168] and the field content is as follows:

- Metric –  $g_{\mu\nu}$ .
- $R$ -symmetry gauge field –  $A_\mu^{(R)}$ .
- 2-form gauge field –  $B_{\mu\nu}$  (with field strength  $H = \star B$ ).
- Central charge symmetry gauge field –  $C_\mu$  (with field strength  $V = \star C$ ).

- Gravitini –  $\psi_\mu, \tilde{\psi}_\mu$ .

To consistently couple our theory to this background we have to impose that the gravitini variations vanishes  $\delta\psi = 0$ . This yields a pair of generalised Killing spinor equations:

$$\begin{aligned} (\nabla_\mu - iA_\mu^{(R)})\zeta &= -\frac{1}{2}H\gamma_\mu\zeta - iV_\mu\zeta - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta, \\ (\nabla_\mu + iA_\mu^{(R)})\tilde{\zeta} &= -\frac{1}{2}H\gamma_\mu\tilde{\zeta} + iV_\mu\tilde{\zeta} + \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\tilde{\zeta}. \end{aligned} \quad (4.19)$$

A solution to this pair of equations yields, via the supergravity Lagrangian, a rigid supersymmetric Lagrangian that preserves the supercharges  $Q_+^{11}$  and  $Q_-^{22}$ . For the twisted geometry there exists a solution to the Killing spinor equations (4.19) with

$$H = V^\mu = 0, \quad A^{(R)} = \omega_{12}^{S^2}. \quad (4.20)$$

In particular this implies that  $\frac{1}{2\pi} \int_{S^2} dA^{(R)} = -1$  and  $R$ -charges are quantised.

**Coulomb branch localisation.** The twisted indices (4.18) can then be computed by localisation [169] with respect to the supercharge  $\mathcal{Q} = Q_+^{11} + Q_-^{22}$ . We follow the Coulomb branch localisation procedure of [164].  $\mathcal{Q}$ -exact BPS configurations are specified by the following vector multiplet data

- Magnetic flux  $\mathbf{m} = \frac{1}{2\pi} \int_{S^2} F$  co-character living in the coroot lattice  $\Lambda_{\mathfrak{h}}^\vee$ .
- Complex mode  $u = A_t + i\beta\sigma$  here  $A_t$  is the gauge field holonomy along the  $S^1$ ,  $\sigma$  is the real vector multiplet scalar and  $\beta$  is the  $S^1$  radius. We set  $x = e^{iu}$ .

The BPS manifold can then be described by  $\mathfrak{M}_{\text{BPS}} = (H \times \mathfrak{h} \times \Lambda_{\mathfrak{h}}^\vee)/W$  where  $W$  is the Weyl group of the gauge group. The result of the localisation procedure is a Jeffrey-Kirwan contour integral

$$I^{A,B} = \frac{1}{|W|} \sum_{\mathbf{m} \in \Lambda_{\mathfrak{h}}^\vee} \oint_{\text{JK}} \frac{dx}{2\pi i x} Z_{\text{int}}^{A,B}(x, \mathbf{m}). \quad (4.21)$$

In the above  $Z_{\text{int}}^{A,B}(u, \mathbf{m})$  is a meromorphic form determined by the 1-loop determinants around the BPS configurations as computed in the work [164]. We now write down these 1-loop determinants for both the  $A$ - and  $B$ -twist with the angular momentum refinement<sup>6</sup> and  $R$ -charges imposed by  $\mathcal{N} = 4$  supersymmetry. We focus on the multiplets relevant for the unitary quiver gauge theories and write the meromorphic form in terms of the quiver data of section 2.2: the quiver adjacency matrix  $Q$  and dimension vectors  $(\mathbf{v}, \mathbf{w})$ .

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<sup>6</sup>In the following section we show that the indices are actually independent of  $q$ , but introducing  $q$ -refinement is crucial to allow us to factorise.

Firstly, for each edge  $(i, j) \in E$  we have contributions from bifundamental matter:

$$\begin{aligned}
Z_{ij}^{A,\text{bi.}} &= \prod_{a=1}^{v_i} \prod_{b=1}^{v_j} \frac{(w_a^{(i)}/w_b^{(j)})^{\frac{1}{2}(\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(j)})}}{(w_a^{(i)}/w_b^{(j)})t^{\frac{1}{2}}q^{\frac{1}{2}(1-\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(j)})}; q)_{\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(j)}}} \frac{(w_b^{(j)}/w_a^{(i)})^{\frac{1}{2}(-\mathfrak{m}_a^{(i)}+\mathfrak{m}_b^{(j)})}}{(w_a^{(i)}/w_b^{(j)})t^{\frac{1}{2}}q^{\frac{1}{2}(1+\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(j)})}; q)_{-\mathfrak{m}_a^{(i)}+\mathfrak{m}_b^{(j)}}}, \\
Z_{ij}^{B,\text{bi.}} &= \prod_{a=1}^{v_i} \prod_{b=1}^{v_j} \frac{(t^{\frac{1}{2}}w_a^{(i)}/w_b^{(j)})^{\frac{1}{2}(\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(j)}+1)}}{(w_a^{(i)}/w_b^{(j)})t^{\frac{1}{2}}q^{\frac{1}{2}(-\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(j)})}; q)_{\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(j)}+1}} \frac{(t^{\frac{1}{2}}w_b^{(j)}/w_a^{(i)})^{\frac{1}{2}(-\mathfrak{m}_a^{(i)}+\mathfrak{m}_b^{(j)}+1)}}{(w_b^{(j)}/w_a^{(i)})t^{\frac{1}{2}}q^{\frac{1}{2}(\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(j)})}; q)_{-\mathfrak{m}_a^{(i)}+\mathfrak{m}_b^{(j)}+1}}.
\end{aligned} \tag{4.22}$$

Secondly, for each framing factor we have the following contributions from fundamental matter:

$$\begin{aligned}
Z_i^{A,\text{fun.}} &= \prod_{a=1}^{v_i} \prod_{b=1}^{w_i} \frac{(t^{\frac{1}{2}}w_a^{(i)}/x_b^{(j)})^{\frac{1}{2}\mathfrak{m}_a^{(i)}}}{(w_a^{(i)}/x_b^{(j)})t^{\frac{1}{2}}q^{\frac{1}{2}(1-\mathfrak{m}_a^{(i)})}; q)_{\mathfrak{m}_a^{(i)}}} \frac{(t^{\frac{1}{2}}x_b^{(j)}/w_a^{(i)})^{-\frac{1}{2}\mathfrak{m}_a^{(i)}}}{(x_b^{(j)}/w_a^{(i)})t^{\frac{1}{2}}q^{\frac{1}{2}(1+\mathfrak{m}_a^{(i)})}; q)_{-\mathfrak{m}_a^{(i)}}}, \\
Z_i^{B,\text{fun.}} &= \prod_{a=1}^{v_i} \prod_{b=1}^{w_i} \frac{(t^{\frac{1}{2}}w_a^{(i)}/x_b^{(j)})^{\frac{1}{2}(\mathfrak{m}_a^{(i)}+1)}}{(w_a^{(i)}/x_b^{(j)})t^{\frac{1}{2}}q^{-\frac{1}{2}\mathfrak{m}_a^{(i)}}; q)_{\mathfrak{m}_a^{(i)}+1}} \frac{(t^{\frac{1}{2}}x_b^{(j)}/w_a^{(i)})^{\frac{1}{2}(-\mathfrak{m}_a^{(i)}+1)}}{(x_b^{(j)}/w_a^{(i)})t^{\frac{1}{2}}q^{\frac{1}{2}\mathfrak{m}_a^{(i)}}; q)_{-\mathfrak{m}_a^{(i)}+1}}.
\end{aligned} \tag{4.23}$$

Finally, for every node  $i \in I$  we also have a contribution from the vector multiplet:

$$\begin{aligned}
Z_i^{A,\text{vec.}} &= \prod_{a,b=1}^{v_i} \frac{(w_a^{(i)}/w_b^{(i)})^{\frac{1}{2}(\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(i)}-1)}}{(w_a^{(i)}/w_b^{(i)})q^{\frac{1}{2}(2-\mathfrak{m}_a^{(i)}+\mathfrak{m}_b^{(i)})}; q)_{\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(i)}-1}} \frac{(w_a^{(i)}/w_b^{(i)})t^{-\frac{1}{2}})^{\frac{1}{2}(\mathfrak{m}_a^{(i)}+\mathfrak{m}_b^{(i)}+1)}}{(w_a^{(i)}/w_b^{(i)})t^{-1}q^{\frac{1}{2}(-\mathfrak{m}_a^{(i)}+\mathfrak{m}_b^{(i)})}; q)_{\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(i)}+1}}, \\
Z_i^{B,\text{vec.}} &= \prod_{a,b=1}^{v_i} \frac{(w_a^{(i)}/w_b^{(i)})^{\frac{1}{2}(\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(i)}-1)}}{(w_a^{(i)}/w_b^{(i)})q^{\frac{1}{2}(2-\mathfrak{m}_a^{(i)}+\mathfrak{m}_b^{(i)})}; q)_{\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(i)}-1}} \frac{(w_a^{(i)}/w_b^{(i)})t^{-\frac{1}{2}})^{\frac{1}{2}(\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(i)}-1)}}{(w_a^{(i)}/w_b^{(i)})t^{-1}q^{\frac{1}{2}(2-\mathfrak{m}_a^{(i)}+\mathfrak{m}_b^{(i)})}; q)_{\mathfrak{m}_a^{(i)}-\mathfrak{m}_b^{(i)}-1}}.
\end{aligned} \tag{4.24}$$

Each  $U(1)$  factor of  $G_C$ , *i.e.* each vertex of the quiver, also contributes classically to the integrand:

$$Z^{\text{cl.}} = \zeta^{\mathfrak{m}}. \tag{4.25}$$

The Coulomb branch localisation procedure yields a Jeffrey-Kirwan residue prescription for the index<sup>7</sup> to compute (4.18). We have

$$I^{A,B} = \frac{1}{|W|} \sum_{\mathfrak{m} \in \Lambda_{\mathfrak{h}}^{\vee}} \sum_{x_* \in \mathfrak{M}_{\text{sing.}}} \text{JK-Res}_{x=x_*} [Q(x_*, \eta)] Z_{\text{int}}^{A,B}(x, \mathfrak{m}) \tag{4.26}$$

The singular set  $\mathfrak{M}_{\text{sing.}}$  is a subset of the BPS locus  $\mathfrak{M} = H \times \mathfrak{h}$  parametrised by the bosonic zero modes of  $\sigma$  and  $A_t$ .  $\mathfrak{M}$  contains certain hyperlanes  $H_i$  where a chiral multiplet  $\Phi_i$  develops a zero mode, precisely  $H_i = \{u \in \mathfrak{M} : e^{i\rho_i(u)+\rho_f(v)=1}\}$  where  $\rho_i$  and  $\rho_f$  are the gauge and flavour weights respectively of the chiral multiplet—these are poles of the

<sup>7</sup>In this thesis we work with examples where there are no ‘boundary contributions’ in the terminology of [164].

one-loop determinants. We then have

$$\mathfrak{M}_{\text{sing.}}^* = \{u_* \in \mathfrak{M} : \text{at least } r \text{ linearly independent } H_i \text{ meet at } u_*\}. \quad (4.27)$$

Finally  $Q(u_*)$  specifies the charges of the hyperplanes meeting at  $u_*$ . The prescription of [164] can then be stated as if the JK parameter  $\eta > 0$  then we should take residues of  $Z_{\text{int}}^{A,B}(x, \mathbf{m})$  at singular points  $u_*$  with positive charges  $Q(u_*)$ .

### 4.1.3 Holomorphic factorisation

In many examples, partition functions on closed three manifolds  $\mathcal{M}_3$  of theories with  $\mathcal{N} \geq 2$  supersymmetry have been shown to decompose into a set of fundamental blocks  $H_\alpha$ . The blocks can be defined as partition functions on spaces with a boundary topologically equivalent to a hemisphere  $S^1 \times H^2$  and partition functions *factorise* as

$$\mathcal{Z}_{\mathcal{M}_3} = \sum_{\alpha} H_{\alpha} \widetilde{H}_{\alpha}. \quad (4.28)$$

The sum is taken over isolated massive Higgs vacua  $\alpha$  and the  $\sim$  operation implements a conjugation of fugacities corresponding to an element  $g \in SL(2, \mathbb{Z})$  gluing the boundary tori  $\partial(S^1 \times H^2) = T^2$ . Different elements  $g$  correspond to the Heegaard decomposition of the three manifold

$$\mathcal{M}_3 = (S^1 \times H^2) \cup_g (S^1 \times H^2). \quad (4.29)$$

Traditionally, these fundamental blocks have been defined in the IR as *holomorphic blocks* realised as twisted compactifications on a cigar [38]. Later, in section 4.3, we give an alternative definition of the fundamental blocks in the UV as partition functions on a hemisphere.

We review how factorisation is typically observed in the literature. The theory is first placed on the manifold  $\mathcal{M}_3$  by the supergravity method described in section 4.1.2. The partition function can then often be localised on the Coulomb branch to give a contour integral expression for the partition function. Finally, the residues at relevant poles are evaluated and factorised by hand to find formulae of the schematic form

$$\mathcal{Z}_{\mathcal{M}_3} = \sum_{\alpha} \mathcal{Z}_{\text{Perturbative}} ||\mathcal{Z}_{\text{Vortex}}||^2, \quad (4.30)$$

where the particular gluing depends on the choice of three manifold  $\mathcal{M}_3$  and acts by a modular transformation on fugacities. The factorisation has been demonstrated in this way first by [170] and in many examples since [171, 172, 173, 174, 108]. We follow this methodology in section 4.2 to demonstrate the factorisation of the  $A$ -twisted index of the  $T[SU(N)]$  theory.



The *vortex partition function*  $\mathcal{Z}_{\text{Vortex}}$  in the above equation has many different realisations and interpretations, a sample of which are: as partition functions on a hemisphere  $S^1 \times D$  with Neumann boundary conditions [175]; geometrically, as  $J$ -functions in the Gromov-Witten theory of the Higgs branch [176, 177]; and as effective GLSM descriptions of vortex quantum mechanics [178]. Higgs branch localisation [37, 36] offers a more constructive approach to factorisation where the theory is localised on the Higgs branch to vortex configurations. Similarly, recent work [179] gives a rigorous construction of a similar factorisation into three dimensional hemispheres  $HS^3$ .

The holomorphic block approach of [38] gives an elegant IR definition of the block in terms of the  $q$ -difference equations it satisfies, however this includes some ambiguity in the perturbative contributions related to the ability to add  $q$ -constant terms to the block. In section 4.4, we take a constructive approach and define *hemisphere blocks* in terms of a UV boundary condition on a hemisphere  $S^1 \times H^2$ . Our UV prescription resolves ambiguities in the perturbative contributions and fuses *exactly* to three manifold partition functions (4.30). The hemisphere block decomposes into perturbative and vortex contributions

$$\mathcal{Z}_{S^1 \times H^2} = \mathcal{Z}_{\text{Classical}} \mathcal{Z}_{\text{1-loop}} \mathcal{Z}_{\text{Vortex}} , \quad (4.31)$$

and in particular the one-loop and classical pieces here fuse *exactly* to the perturbative contributions to the three manifold partition function (4.30). In this thesis, we discuss the geometrical interpretation of  $\mathcal{Z}_{S^1 \times H^2}$  in terms of the quasimap spaces of the previous subsection. The vortex contributions arise as the vertex functions of the quasimap moduli space and the perturbative contribution are determined by certain Lagrangian submanifolds of the Higgs branch.

**Topologically twisted index.** In this work we focus on  $\mathcal{N} = 4$  theories and the topologically twisted index on the background  $\mathcal{M}_3 = S^2 \times_{A/B} S^1$ . The relevant conjugation is then expected to be [180, 181]

$$I^{A,B} = \sum_{\alpha} H_{\alpha}(q, t) H_{\alpha}(q^{-1}, t) . \quad (4.32)$$

We discuss how to implement the two different twists in section 4.3.

#### 4.1.4 $T[SU(N)]$ theory

The main example we study in the first part of this chapter is the  $T[SU(N)]$  theory depicted in figure 4.1. We review the basic definition and properties of this theory.

**Motivation.** The theories  $T[G]$ , for a more general group  $G$ , were introduced in the work [54] where they are shown to realise the  $S$ -dual of Dirichlet boundary conditions in certain 4d  $\mathcal{N} = 4$  supersymmetric Yang-Mills theories with gauge group  $G$ . In that work,  $T[G]$  theories are shown to be ‘good’ and flow to superconformal field theories in the IR. In the IR the flavour symmetry is  $G$  and the topological symmetry is the Langlands dual group  $G^\vee$ . In this context 3d mirror symmetry [17] can be understood from the IIB  $S$ -duality perspective as exchanging the theories  $T[G]$  and  $T[G^\vee]$  so that  $T[SU(N)]$  is self mirror dual. The  $T[G]$  theories consequently play an important role in the gauge theoretic interpretation of the geometric Langlands correspondence [182, 183, 184].

The  $T[SU(N)]$  theory is also an important ingredient in the finite AGT correspondence. The works [185, 186] show that the holomorphic blocks of the theory realise  $q$ -Toda conformal blocks which can be understood as ‘Higgsing’ the  $q$ -deformed AGT correspondence in five dimensions [187, 188, 189]. These ideas are closely related to the finite AGT correspondence in the mathematical literature [190, 191] where finite  $W$ -algebra modules are realised by the cohomology of quasimap spaces to the Higgs branch of  $T[SU(N)]$ . In the context of section 4.4 of this thesis these actions should be thought of as the Higgs or Coulomb branch chiral ring of  $T[SU(N)]$  acting on the cohomology of its vortex moduli space.

The  $T[SU(N)]$  theory also plays a role in the AdS/CFT correspondence [192, 193] where the theory is dual to a warped  $\text{AdS}_4 \rtimes K$  type IIB supergravity background. Recent work [194] demonstrates that the free energy of the gravity solution can be recovered from a large  $N$  limit of the topologically twisted index—it would be interesting to understand these results from the vortex geometry perspective of this chapter.

In the first part of this chapter we study the moduli space of vortices of the  $T[SU(N)]$  theory. We compute the twisted indices and provide a geometric interpretation of the holomorphic factorisation. Later, after taking a more constructive approach to factorisation, we revisit the  $T[SU(N)]$  example in section 4.3.

**Field content.**  $T[SU(N)]$  is a 3d  $\mathcal{N} = 4$  quiver gauge theory fitting into the framework of chapter 2, the relevant quiver diagram is shown in figure 4.1. The theory has a product gauge group  $U(1) \times \dots \times U(N-1)$  with bifundamental hypermultiplets in  $U(i) \times U(i+1)$  for  $i = 1, \dots, N-1$ . There are  $N$  hypermultiplets in the fundamental of  $U(N)$ . The flavour symmetry is  $G_H = SU(N)$  and the UV topological symmetry is  $G_C = U(1)^N$  which is enhanced to  $G_C = SU(N)$  in the IR. We write  $(x_1, \dots, x_N)$  and  $(\zeta_1, \dots, \zeta_N)$  for the corresponding fugacities and  $t$  for the fugacity associated to the  $U(1)_t$  mass deformation.

**Higgs branch.** The quiver has vertex set  $I = \{1, \dots, N-1\}$  and we write  $V_i = \mathbb{C}^i$  for the corresponding gauge nodes and  $V_N = \mathbb{C}^N$  for the framing node. As usual we denote the bifundamental scalars as  $(A, B)$  with  $A_i : V_i \rightarrow V_{i+1}$  and  $B_i : V_{i+1} \rightarrow V_i$

for  $i = 1, \dots, N - 2$ . The fundamental scalars are denoted  $(I, J)$  and define maps  $I : V_N \rightarrow V_{N-1}$  and  $J : V_{N-1} \rightarrow V_N$ . If we turn on generic FI parameters corresponding to the stability parameter  $\theta = 1$  then the stability condition 2.3 imposes that  $JB_{N-1} \dots B_k$  is injective for any  $k = 1, \dots, N - 2$ . This defines a complete flag in  $\mathbb{C}^N$  given by  $0 \subset V_1 \subset \dots \subset V_{N-1} \subset \mathbb{C}^N$ , the moduli space of these flags is denoted  $\mathcal{B}_N$ . The moment map constraint implies that for each  $i$  we have  $B_i|_{V_i} = B_{i-1}$  and  $I|_{V_N} = B_{N-1}$ , this equivalently defines a map  $X : \mathbb{C}^N \rightarrow \mathbb{C}^N$  with  $X(V_i) \subset V_{i-1}$ . Hence the quiver describes the cotangent bundle to the complete flag variety<sup>8</sup>

$$\mathcal{M}_H = T^* \mathcal{B}_N. \quad (4.33)$$

The flavour group  $G_H = GL(V_N)$  acts on  $V_N$  as in (2.43) and if we fix a basis  $\{e_1, \dots, e_N\}$  then we see that, provided we turn on generic mass parameters, the fixed points under  $G_H$  are labelled by  $0 \subset V_1 \subset \dots \subset V_{N-1} \subset \mathbb{C}^N$  where each  $V_k$  is spanned by a  $k$ -subset of  $\{e_1, \dots, e_N\}$ . Hence the theory has  $N!$  massive vacua parametrised by  $\sigma \in S_N$ . The contracting  $\mathbb{C}_t^\times$  action (2.45) has fixed point subvariety  $\mathcal{B}_N$ .

**Hilbert series.** We now write down the Hilbert series of the Higgs and the Coulomb branches. Evaluating the tangent bundle from lemma 1 at the fixed point  $\sigma$  we have

$$T_\sigma \mathcal{M}_H = \sum_{i < j} \frac{x_{\sigma(i)}}{x_{\sigma(j)}} + t \frac{x_{\sigma(j)}}{x_{\sigma(i)}}. \quad (4.34)$$

The localisation formula (3.14) for the Higgs branch then reads

$$\begin{aligned} \text{H.S.}[\mathcal{M}_H] &= \sum_{\sigma \in S_N} \prod_{i < j} \frac{1}{\left(1 - x_{\sigma(i)}/x_{\sigma(j)}\right) \left(1 - t^{-1}x_{\sigma(j)}/x_{\sigma(i)}\right)} \\ &= (t^{-1}; t^{-1})_N \prod_{i,j=1}^N \frac{1}{1 - t^{-1}x_i/x_j}. \end{aligned} \quad (4.35)$$

For the Coulomb branch, the monopole formula (2.23) reads

$$\text{H.S.}[\mathcal{M}_C] = \sum_{\mathbf{m}_a^{(i)} \in \Lambda_b^\vee / W} t^{\Delta(\mathbf{m})} \prod_{i=1}^{N-1} z_i^{\sum_{a=1}^{N-i} \mathbf{m}_a^{(N-i)}}, \quad (4.36)$$

with the monopole  $R$ -charge given by

$$\Delta(\mathbf{m}) = \frac{1}{2} \left( \sum_{i=1}^{N-1} \sum_{a,b} |\mathbf{m}_a^{(i)} - \mathbf{m}_b^{(i+1)}| - \sum_{i=1}^{N-1} \sum_{a,b} |\mathbf{m}_a^{(i)} - \mathbf{m}_b^{(i)}| \right), \quad (4.37)$$

---

<sup>8</sup>This description of the cotangent bundle appears in for e.g. theorem 4.1.2 of Chriss and Ginzburg [106].

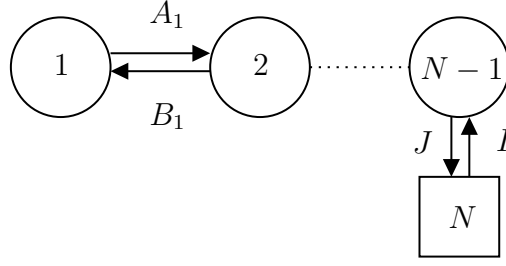


Figure 4.1:  $T[SU(N)]$  quiver diagram.

where  $\mathfrak{m}^{(N)} = 0$ . This expression was computed in [97] and the authors found

$$\text{H.S.}[\mathcal{M}_C] = (t; t)_N \prod_{i,j=1}^N \frac{1}{1 - t\zeta_i/\zeta_j}, \quad (4.38)$$

where  $z_k = \zeta_k/\zeta_{k+1}$  for  $k = 1, \dots, N-1$ . Comparing (4.35) and (4.38) the self-mirror property of  $T[SU(N)]$  is manifest and in the Coulomb branch Hilbert series we see the enhancement of the topological symmetry to the non-abelian group  $G_C = SU(N)$  as expected.

## 4.2 Laumon space and twisted index of $T[SU(N)]$

In this section we focus on the topologically  $A$ -twisted index and give a geometrical interpretation of its factorisation. We show that the  $A$ -twisted index computes the Coulomb branch Hilbert series of the theory and argue that the Coulomb branch localisation procedure for the  $B$ -twisted index yields a Molien integral of the type discussed in section 3.1.2.

We focus on the  $T[SU(N)]$  example and introduce a ‘fictitious’ angular momentum refinement which allows us to factorise the  $A$ -twisted index into vortex partition functions

$$I^A \sim \sum_{\alpha} \mathcal{Z}_{\text{vortex}}^{\alpha} \overline{\mathcal{Z}_{\text{vortex}}^{\alpha}}. \quad (4.39)$$

For the  $T[SU(N)]$  theory, the abstract quasimap moduli spaces of section 4.1.1 coincide with *Laumon spaces* [96], which we denote by  $\mathfrak{Q}_{\alpha}^d$ . The  $\chi_t$  genera of *local* Laumon space coincides with the vortex partition functions of the  $T[SU(N)]$  theory, schematically:

$$\mathcal{Z}_{\text{Vortex}}^{\alpha} = \sum_d \zeta^d \chi_t(\mathfrak{Q}_{\alpha}^d). \quad (4.40)$$

There also exists a notion of a *global Laumon space*  $\mathcal{Q}^d$  that is independent of the particular vacuum  $\alpha$ . The work [191] shows that the  $\chi_t$  genus of global Laumon space can be factorised

$$\chi_t(\mathcal{Q}^d) \sim \sum_{\alpha} \sum_{d=d'+d''} \chi_t(\mathfrak{Q}_{\alpha}^{d'}) \overline{\chi_t(\mathfrak{Q}_{\alpha}^{d''})}. \quad (4.41)$$

We show that the conjugation here coincides precisely with the topologically twisted gluing in equation (4.39). The main result of section 3.1.4 is that, in the absence of external flux, there are surprising cancellations on the right hand side of this equation and in fact the  $A$ -twisted index computes a generating function of the Poincaré polynomials of  $\mathcal{Q}^d$ —this gives a novel geometric interpretation of the Coulomb branch Hilbert series which we write here schematically as

$$\text{H.S.}[\mathcal{M}_C] \sim P_{\text{QM}(\mathcal{M}_H)}(\zeta; t) \overline{P_{\text{QM}(\mathcal{M}_H)}(\zeta; t)}, \quad (4.42)$$

where  $P_{\text{QM}(\mathcal{M}_H)}(\zeta; t)$  denote generating functions of the Poincaré polynomials of quasimap spaces to the Higgs branch  $\mathcal{M}_H$ —we define the conjugation more carefully later. We return to these ideas in greater generality in section 4.3.

**Outline.** We begin in section 4.2.1 by factorising the  $A$ -twisted index of the  $T[SU(N)]$  theory. We show that the index factorises into ‘vortex partition functions’ in the way expected in equation (4.30). In section 4.2.2 we show that the vortex moduli space for the  $T[SU(N)]$  theory is a Laumon space and we show that the vertex functions of Laumon space coincide with the vortex partition functions for  $T[SU(N)]$ . We then show that the factorisation of the topologically twisted index can be interpreted as the factorisation property of the  $\chi_t$  genus of global Laumon space. Finally, in section 3.1.4, we argue that in general the  $A$ -twisted index of a 3d  $\mathcal{N} = 4$  theory coincides with its Coulomb branch Hilbert series.

### 4.2.1 Factorisation

In this section we compute the  $A$ -twisted index of the  $T[SU(N)]$  theory and factorise it into ‘vortex partition functions’. In the following we write  $x = (x_1, \dots, x_N)$  for the set of flavour fugacities and we write  $\zeta = (\zeta_1, \dots, \zeta_N)$  for the topological symmetry fugacities. The action of the symmetric group is denoted by  $\sigma \cdot x = (x_{\sigma(1)}, \dots, x_{\sigma(N)})$ . Following the

prescription to compute the index from section 4.1.2 we find:

$$\begin{aligned}
I^A(x, \zeta; q, t) = & \left( \prod_{i=1}^{N-1} \frac{(-1)^i}{i!} \right) \sum_{\{\mathbf{m}_a^{(i)}\}} \oint_{JK} \prod_{i=1}^{N-1} \prod_{a=1}^i \frac{dw_a^{(i)}}{2\pi i w_a^{(i)}} \left( \prod_{i=1}^{N-1} \left( \frac{\zeta_i}{\zeta_{i+1}} \right)^{\sum_{a=1}^i \mathbf{m}_a^{(i)}} \right) \\
& \frac{\left( \left( \frac{w_a^{(i)}}{w_b^{(i)}} \right)^{\frac{1}{2}} \right)^{(\mathbf{m}_a^{(i)} - \mathbf{m}_b^{(i)} - 1)}}{\left( \left( \frac{w_a^{(i)}}{w_b^{(i)}} \right)^{\frac{1}{2}} t^{-\frac{1}{2}} \right)^{(\mathbf{m}_a^{(i)} - \mathbf{m}_b^{(i)} + 1)}} \\
& \prod_{i=1}^{N-1} \prod_{a,b=1}^i \frac{\left( \frac{w_a^{(i)}}{w_b^{(i)}} q^{\frac{1}{2}(2 - \mathbf{m}_a^{(i)} + \mathbf{m}_b^{(i)})}; q \right)_{\mathbf{m}_a^{(i)} - \mathbf{m}_b^{(i)} - 1}}{\left( \frac{w_a^{(i)}}{w_b^{(i)}} t^{-1} q^{\frac{1}{2}(-\mathbf{m}_a^{(i)} + \mathbf{m}_b^{(i)})}; q \right)_{\mathbf{m}_a^{(i)} - \mathbf{m}_b^{(i)} + 1}} \\
& \prod_{i=1}^{N-1} \prod_{a=1}^i \prod_{b=1}^{i+1} \frac{\left( \left( \frac{w_a^{(i)}}{w_b^{(i+1)}} \right)^{\frac{1}{2}} t^{\frac{1}{4}} \right)^{(\mathbf{m}_a^{(i)} - \mathbf{m}_b^{(i+1)})}}{\left( \left( \frac{w_b^{(i+1)}}{w_a^{(i)}} \right)^{\frac{1}{2}} t^{\frac{1}{4}} \right)^{(\mathbf{m}_b^{(i+1)} - \mathbf{m}_a^{(i)})}} \\
& \frac{\left( \frac{w_a^{(s)}}{w_b^{(i+1)}} t^{\frac{1}{2}} q^{\frac{1}{2}(1 - \mathbf{m}_a^{(i)} + \mathbf{m}_b^{(i+1)})}; q \right)_{\mathbf{m}_a^{(i)} - \mathbf{m}_b^{(i+1)}}}{\left( \frac{w_b^{(i+1)}}{w_a^{(i)}} t^{\frac{1}{2}} q^{\frac{1}{2}(1 + \mathbf{m}_a^{(i)} - \mathbf{m}_b^{(i+1)})}; q \right)_{\mathbf{m}_b^{(i+1)} - \mathbf{m}_a^{(i)}}}.
\end{aligned} \tag{4.43}$$

where  $w_b^{(N)} = x_b$  and  $\mathbf{m}_b^{(N)} = 0$  for  $b = 1, \dots, N$ . Taking the appropriate Jeffrey-Kirwan residues we find the index factorises as follows:

$$I^A(x, \zeta; q, t) = \sum_{\sigma \in S_N} \mathcal{Z}_{\text{Perturbative}}^{A, \sigma} \mathcal{Z}_{\text{Vortex}}^A(\sigma \cdot x, \zeta; q, t) \mathcal{Z}_{\text{Vortex}}^A(\sigma \cdot x, \zeta; q^{-1}, t), \tag{4.44}$$

where the perturbative piece is given by:

$$\mathcal{Z}_{\text{Perturbative}}^{A, \sigma}(x, \zeta; q, t) = \left[ \prod_{i < j}^N (-t)^{-\frac{1}{2}} \frac{\zeta_i}{\zeta_j} \right] \left[ \prod_{i < j}^N \frac{1 - t \frac{x_i}{x_j}}{1 - \frac{x_i}{x_j}} \right], \tag{4.45}$$

with  $\sigma$  acting by permutations on  $\{x\}$ . The vortex partition function is realised as a sum over the relevant poles by

$$\begin{aligned}
\mathcal{Z}_{\text{Vortex}}^A(x, \zeta, q, t) = & \sum_{\{d_a^{(i)}\}} \prod_{i=1}^{N-1} \left( t^{-1} \frac{\zeta_i}{\zeta_{i+1}} \right)^{d_i} \prod_{\substack{a,b=1 \\ a \neq b}}^i \frac{\left( t^{-1} \frac{x_a}{x_b}; q \right)_{d_a^{(i)} - d_b^{(i)}}}{\left( \frac{x_a}{x_b}; q \right)_{d_a^{(i)} - d_b^{(i)}}} \times \prod_{a=1}^i \prod_{b=1}^{i+1} \frac{\left( tq \frac{x_a}{x_b}; q \right)_{d_a^{(i)} - d_b^{(i+1)}}}{\left( q \frac{x_a}{x_b}; q \right)_{d_a^{(i)} - d_b^{(i+1)}}},
\end{aligned} \tag{4.46}$$

where the sum is over the set of vortex numbers  $\{d_a^{(i)}\}$  with  $i = 1, \dots, N-1$  and  $a = 1, \dots, i$ . We write  $d_i = \sum_{a=1}^i d_a^{(i)}$  and  $\mathbf{d} = (d_1, \dots, d_{N-1})$  for the vector of vortex numbers. The

vortex numbers satisfy  $d_a^{(N)} = 0$  for  $a = 1, \dots, N$  and the following constraints:

$$\begin{aligned}
d_1^{(1)} &\geq d_1^{(2)} \geq d_1^{(3)} \geq \dots \geq d_1^{(N-1)} \geq 0 \\
&\quad d_2^{(2)} \geq d_2^{(3)} \geq \dots \geq d_2^{(N-1)} \geq 0 \\
&\quad \quad d_3^{(3)} \geq \dots \geq d_3^{(N-1)} \geq 0 \\
&\quad \quad \quad \ddots \quad \quad \quad \vdots \\
&\quad \quad \quad \quad d_{N-1}^{(N-1)} \geq 0.
\end{aligned} \tag{4.47}$$

As expected, the Coulomb branch localisation calculation yields the twisted index as a sum over Higgs vacua of vortex partition functions. The vortex partition functions agree with localisation on a hemisphere  $S^1 \times D$  of Yoshida and Sugiyama [175]. They also agree with the  $K$ -theoretic vertex functions for  $T^*\mathcal{B}_N$  as computed in the work [49].

**Factorising the perturbative contributions.** One can also factorise the perturbative contributions as follows. We notice that the 1-loop contributions in a vacuum  $\sigma \in S_N$  can be written as

$$\begin{aligned}
\mathcal{Z}_{1\text{-loop}}^A &= \prod_{i < j}^N \frac{1 - t \frac{x_i}{x_j}}{1 - \frac{x_i}{x_j}} = \prod_{i < j}^N \frac{(q \frac{x_i}{x_j}; q)_{-1}}{(tq \frac{x_i}{x_j}; q)_{-1}} \\
&= \left[ \prod_{i < j}^N \frac{(qx_i/x_j; q)_\infty}{(tqx_i/x_j; q)_\infty} \right] \left[ \prod_{i < j}^N \frac{(q^{-1}x_i/x_j; q^{-1})_\infty}{(tq^{-1}x_i/x_j; q^{-1})_\infty} \right],
\end{aligned} \tag{4.48}$$

where we have used the identity (A.5) and the fusion rule (A.10). The classical pieces can also be factorised as we demonstrate in [108] however there is significant ambiguity here in terms of how to resolve these terms—we fix these ambiguities in section 4.3.

Another problem with this factorisation is chamber dependence. Consider, for example, sending<sup>9</sup>  $t \rightarrow q^{-1}$  in (4.48). In the first block we have

$$\mathcal{Z}_{1\text{-loop}} = \prod_{i < j}^N \frac{1}{1 - x_i/x_j}. \tag{4.49}$$

These blocks are then convergent in *different* chambers  $\mathfrak{C}_H$  for each vacuum  $\sigma$ . The blocks should be defined with respect to a fixed chamber choice if they are to arise from a well-defined UV prescription. We thus need to find a chamber independent way to factorise (4.48)—this is one of the goals of section 4.3.

We remark here that these issues are ‘invisible’ from the Coulomb branch localisation procedure since we avoid dealing with the boundary of the hemispheres and instead work directly with  $S^2 \times S^1$ .

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<sup>9</sup>This limit is the ‘ $B$ -limit’ of section 4.3 and restores the  $\mathcal{N} = 2^*$  supersymmetry to  $\mathcal{N} = 4$ .

### 4.2.2 Vortex moduli space of $T[SU(N)]$

We now turn to the geometric interpretation of the factorisation derived in the previous subsection.

For the  $T[SU(N)]$  theory, where the Higgs branch is the cotangent bundle of the complete flag variety, the quasimap spaces  $\mathbf{QM}_\alpha^d$  can be given particularly convenient descriptions as Laumon spaces.

**Laumon spaces.** We now discuss quasimaps for  $T[SU(N)]$ . Firstly, by the general theory of section 4.1.1 the degree  $\mathbf{d}$  is a vector  $\mathbf{d} = (d_1, \dots, d_N)$ . We are interested in quasimaps  $(\mathcal{V}, \mathcal{W})$  fixed by  $\mathbf{T}_q = \mathbf{A} \times \mathbb{C}_t^\times \times \mathbb{C}_q^\times$ . We first consider quasimaps fixed by  $\mathbb{C}_t^\times$ , this forces our quasimaps to lie in the core of  $T^*\mathcal{B}$  i.e. in the complete flag variety  $\mathcal{B}$ . The relevant quasimap data<sup>10</sup>  $(\mathcal{V}_i)_{i \in I}$  is a collection of vector bundles (or locally free sheaves) of  $\deg \mathcal{V}_i = d_i$  and  $\mathrm{rk} \mathcal{V}_i = i$  for  $i = 1, \dots, N-1$  together with a section  $f \in H^0(\mathbb{P}^1, \mathcal{M})$ <sup>11</sup>. A stable section satisfying the moment map constraint  $\mu_{\mathbb{C}} = 0$  is equivalent to imposing that the sheaves form a flag—this is analogous to the discussion in 4.1.4 for the complete flag variety—and we obtain a moduli space of flags

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \dots \subset \mathcal{V}_N = W \otimes \mathcal{O}_{\mathbb{P}^1}, \quad (4.50)$$

where  $W$  is an  $N$ -dimensional complex vector space  $W = \mathrm{sp}\langle x_1, \dots, x_N \rangle$ . This moduli space has an alternative life as the *Laumon space* introduced by Laumon in [96] and further studied in the works [191, 195, 196, 197, 198, 199], we denote the moduli space by  $\mathcal{Q}_d$ . Laumon space predates the more general construction of quasimap moduli spaces discussed in section 4.1.1. The *based Laumon space*  $\mathfrak{Q}_d \subset \mathcal{Q}_d$  is defined by further imposing that  $\mathcal{V}_i \subset \mathcal{V}_N$  is a vector subbundle in a neighbourhood of  $\infty \in \mathbb{P}^1$  and the fibre there is equal to  $\mathrm{sp}\langle x_1, \dots, x_i \rangle$ . Laumon spaces for other vacua  $\alpha \in S_N$  are obtained by the natural  $S_N$  action on  $W$ .

Laumon space is technically easier to work with than a quasimap space:  $\mathcal{Q}_d$  is a smooth projective variety of dimension  $2d_1 + \dots + 2d_{N-1} + \dim \mathcal{B}$  and  $\mathfrak{Q}_d$  is a smooth, non-compact, quasi-projective variety of dimension  $2d_1 + \dots + 2d_N$ .

**Group action.** The fixed points under the  $\mathbf{A}_q := \mathbf{A} \times \mathbb{C}_q^\times$  action are derived in the work [191] and are labelled by the set of integers  $\{d_a^{(i)}\}$  with  $i = 1, \dots, N-1$  and  $a = 1, \dots, i$  satisfying the same constraints as (4.47) with  $\mathbf{d} = \left(\sum_{a=1}^i d_a^{(i)}\right)_{i=1, \dots, N-1}$ . Specifically the

<sup>10</sup>We switch to using  $\mathcal{V}$  notation  $\mathcal{V}$  since hopefully there is no confusion with quiver vector bundles in this section.

<sup>11</sup>Here  $\mathcal{M}$  denotes the bundle associated to the oriented quiver representation.



fixed subsheaves split into sums of line bundles on  $\mathbb{P}^1$  as follows

$$\begin{aligned}
\mathcal{V}_1 &= x_1 q^{d_1^{(1)}} \mathcal{O}(d_1^{(1)}) \\
\mathcal{V}_2 &= x_1 q^{d_1^{(2)}} \mathcal{O}(d_1^{(2)}) \oplus x_2 q^{d_2^{(2)}} \mathcal{O}(d_2^{(2)}) \\
&\dots \\
\mathcal{V}_{N-1} &= x_1 q^{d_1^{(N-1)}} \mathcal{O}(d_1^{(N-1)}) \oplus \dots \oplus x_{N-1} q^{d_{N-1}^{(N-1)}} \mathcal{O}(d_{N-1}^{(N-1)}).
\end{aligned} \tag{4.51}$$

**Quiver realisation.** Nakajima shows in [152] that the (based) Laumon space can also be described by an oriented quiver variety known as a handsaw quiver. We met these quivers already in chapter 3—see figure 3.8. The identification with quasimap data is as follows. For vortices in  $T[SU(N)]$ , we set  $\mathbf{w}_i = 1$  for  $i \in I \cup \{N+1\}$  and the vortex numbers are identified with the gauge nodes  $\mathbf{d} = (\mathbf{v}_i)_{i \in I}$ . The torus action corresponds to the usual action by framing factors and the  $\mathbb{C}_q^\times$  action coincides with the chainsaw action (3.119) restricted to the handsaw quiver. In proposition 2 we computed the Hilbert series in terms of skew Hall-Littlewood polynomials.

**Vertex functions.** We now show that the vertex functions  $V_\alpha$  for the  $T[SU(N)]$  vortex moduli space coincide with generating functions of the  $\chi_t$  genera of the smooth quasi-projective variety  $\mathfrak{Q}_d$ . We follow the construction of the virtual tangent sheaf outlined in section 3.2 of [95]. In the particular case that  $M$  is a smooth variety, it can be considered a zero section  $M = s^{-1}(0)$  of the inclusion  $s : M \rightarrow T^*M$  and the virtual tangent bundle is  $T^{\text{vir.}} = TM - tT^*M$ . Applying this to Laumon space, localisation of the (symmetrised) virtual structure sheaf gives

$$\begin{aligned}
\chi_T(\mathfrak{Q}_d, \hat{\mathcal{O}}_{\text{vir.}}) &= \sum_{p \in \mathfrak{Q}_d^T} \hat{a} \left( T_p \mathfrak{Q}_d - tT_p^\vee \mathfrak{Q}_d \right) \\
&\stackrel{t \rightarrow tq}{=} (t^{-\frac{1}{2}})^{\dim \mathfrak{Q}_d} \sum_{p \in \mathfrak{Q}_d^T} \text{PE} [(1-t) \text{ch}_T T_p \mathfrak{Q}_d] \\
&= (t^{-\frac{1}{2}})^{\dim \mathfrak{Q}_d} \chi_t(\mathfrak{Q}_d),
\end{aligned} \tag{4.52}$$

where in the last line we have used the localisation formula (3.30). The shift  $t \rightarrow tq$  will be explained in the following section, it relates to the fact that the vertex function naturally gives the  $B$ -shifted partition function rather than the  $A$ -shift required in the present section. We conclude that vertex function can be identified with  $\chi_t(\mathfrak{Q}_d)$ .

**Localisation.** We now discuss how to compute the  $\chi_t$  genus using fixed point localisation. The character of the tangent space of  $\mathfrak{Q}_d$  at a fixed point  $\{d_a^{(i)}\}$  according to [200] is given

by

$$\text{ch}_{\mathbf{A}_q} T_{d_a^{(i)}} \mathfrak{Q}_d = \sum_{1 \leq i < j \leq N} \frac{x_j}{x_i} \sum_{l=1}^{d_i^{(j-1)} - d_j^{(j)}} q^l + \sum_{1 \leq i, j \leq N-1} \frac{x_j}{x_i} \sum_{k=\max(i,j)}^{N-1} \sum_{l=d_i^{(k)} - d_j^{(k)} + 1}^{d_i^{(k)} - d_j^{(k+1)}} q^l. \quad (4.53)$$

Now, using the localisation formula (3.30), and the results of the previous paragraph we have the identifications for vacua  $\alpha \in S_N$

$$\begin{aligned} \mathcal{Z}_{\text{Vortex}}^A(x, \zeta; q, t \rightarrow qt) &= \mathcal{V}_\alpha(z, \zeta; q, t) \\ &= \sum_d (t^{-\frac{1}{2}} \zeta)^d \sum_{\{d_a^{(i)}\}} \text{PE} \left[ (1-t) T_{d_a^{(i)}} \mathfrak{Q}_d \right]. \end{aligned} \quad (4.54)$$

It is a non-trivial combinatorial identity that (4.46) can in fact be expressed in the form  $\text{PE}[(1-t)V]$  at each vortex number, the arguments above show this must be the case and we have also verified the resulting identities at fixed finite vortex number with computer checks. Indeed, for a general 3d  $\mathcal{N} = 4$  theory, the vortex partition functions will not be equivariant  $\chi_t$  genera in this way as we will see in the example of chapter 5.

**Example 4.1.** We demonstrate this idea explicitly for the case  $T[SU(2)]$ . There is one vortex number  $d \geq 0$  and one fixed point. The character (4.53) becomes

$$\text{ch}_{\mathbf{A}_q} T_d \mathfrak{Q}_d = \frac{x_1}{x_2} \sum_{l=1}^d q^l + \sum_{l=1}^d q^l. \quad (4.55)$$

In that case the localisation formula (3.30) gives

$$\chi_t(\mathfrak{Q}_d) = \frac{(tq; q)_d (tq \frac{x_1}{x_2}; q)_d}{(q; q)_d (q \frac{x_1}{x_2}; q)_d}. \quad (4.56)$$

We see this agrees with the vortex partition function (4.46) in the case  $N = 2$ . The vacuum dependence arises from permutations  $\sigma$  acting on the flavour fugacities  $(x_1, x_2)$ .

There is a significant advantage in realising the moduli space as the smooth quasi-projective variety  $\mathfrak{Q}_d$  since, as we discussed in 3.1.4, there is a well-defined Poincaré polynomial limit of its  $\chi_t$  genus.

**Global Laumon space.** The global Laumon space also admits an action of the torus  $\mathbb{T}$ . In the work [196] the authors compute the  $\chi_t$  genus of *global* Laumon space. The fixed points of global Laumon space are labelled by a triple  $(\sigma, \mathbf{d}^0, \mathbf{d}^\infty)$  with  $\sigma \in S_N$  and  $\mathbf{d}^0, \mathbf{d}^\infty$  satisfying the same conditions as (4.47). In physical terms, the localisation computation on global Laumon space can be thought of as first localising to vortices at either the north or the south pole of  $\mathbb{P}^1$  and then enumerating the fixed points on the local Laumon spaces

$\mathfrak{Q}_d^\sigma$ , exactly as one would expect from a Higgs branch localisation scheme. The  $\chi_t$  genus of global Laumon space is given by

$$\chi_t(\mathcal{Q}_d) = \sum_{d'+d''=d} \sum_{\sigma \in S_N} \prod_{i < j} \frac{1 - t \frac{x_i}{x_j}}{1 - \frac{x_i}{x_j}} \chi_t(\mathfrak{Q}_{d'}) \overline{\chi_t(\mathfrak{Q}_{d''})}, \quad (4.57)$$

where, as before,  $\sigma \in S_N$  acts on the  $(x_1, \dots, x_N)$  variables. The conjugation on the second factor involves a change of variables  $q \rightarrow q^{-1}$ . The conclusion is that this calculation precisely coincides with the gluing (4.44) and furthermore we can identify the  $A$ -twisted index with the generating function of  $\chi_t$  genera of global Laumon space. Precisely, the identification is

$$I^A(x, \zeta; q, t) = \left[ \prod_{i < j} t^{-1} \frac{\zeta_i}{\zeta_j} \right] \sum_d \prod_{i=1}^{N-1} \left( t^{-1} \frac{\zeta_i}{\zeta_{i+1}} \right)^{d_i} \chi_t(\mathcal{Q}_d). \quad (4.58)$$

We thus realise the  $A$ -twisted index gluing as the  $\chi_t$  genus of Laumon space. In the following sections we study several generalisations of this result. We remark that this is consistent with the recent work of [201], they do not turn on the ‘fictitious’  $q$ -deformation and therefore cannot access the factorisation; nonetheless, they show that the twisted index can be identified with enumerative counts of quasimaps in this way. In fact, we see in the next section that there are dramatic cancellations on the right hand side of this equation and the index is independent of  $q$  and  $x$ , it is in this sense that the  $q$ -deformation is ‘fictitious’.

### 4.2.3 Poincaré polynomial limits and Hilbert series

In this section we consider the relationship between the  $A$ -twisted index  $I^A$  and the Hilbert series of the Coulomb branch  $\text{H.S.}[\mathcal{M}_C]$ . *A priori* the twisted index depends on the parameters  $x, \zeta$  and  $q, t$ —the main result of this section is that in fact the angular momentum refinement  $q$  is an exact deformation and the indices do not depend on  $q$ , it turns out that the index computes the Hilbert series. This implies remarkable cancellations from the enumerative geometry point of view between the equivariant  $K$ -theoretic vertex functions glued in a particular way. However, it is still important to turn on  $q$  because it allows us to factorise the index.

We discuss this from several points of view in the following: 3d mirror symmetry, geometry of the quasimap space and topological quantum field theory.

**3d mirror symmetry.** 3d mirror symmetry exchanges the  $A$ - and the  $B$ -twisted indices [167]. Under the mirror map explained in section 2.1.4,  $R_H$  is exchanged with  $R_C$  and  $G_H$

is exchanged with  $G_C$  so for two mirror dual theories  $\mathcal{T}$  and  $\tilde{\mathcal{T}}$  we expect the relation

$$I^A(\mathcal{T}; x, \zeta, t, q) = I^B(\tilde{\mathcal{T}}; \zeta, x, t^{-1}, q). \quad (4.59)$$

Now we consider the localisation formula (4.21) for the  $B$ -twisted index of a quiver gauge theory  $\mathcal{M}(\mathbf{v}, \mathbf{w})$ . Reading off the one loop determinants from section 4.1.2 we have

$$I^B(x, \zeta; q, t) = \sum_{\mathbf{m} \in \Lambda_{\mathfrak{h}}^{\vee}} \oint \prod_{i \in I} \prod_{a=1}^{\mathbf{v}_i} \frac{1}{\mathbf{v}_i!} \frac{dw_a^{(i)}}{2\pi w_a^{(i)}} Z_i^{B, \text{vec.}} Z_i^{B, \text{fun.}} \prod_{(i,j) \in E} Z_{i,j}^{B, \text{bifun.}}. \quad (4.60)$$

It turns out that in the zero magnetic flux sector  $\mathbf{m} = 0$  dependence on the angular momentum refinement  $q$  drops out and it is an easy exercise with the identity (A.5) to find that the one loop determinants simplify as follows

$$\begin{aligned} I_{\mathbf{m}=0}^B(x, \zeta, q, t) = & \oint \prod_{i \in I} \prod_{i \in I} \prod_{a=1}^{\mathbf{v}_i} \frac{1}{\mathbf{v}_i!} \frac{dw_a^{(i)}}{2\pi w_a^{(i)}} \left(1 - \frac{w_a^{(i)}}{w_b^{(i)}}\right) \left(1 - t \frac{w_a^{(i)}}{w_b^{(i)}}\right) \prod_{b=1}^{\mathbf{w}_i} \frac{1}{(1 - t^{\frac{1}{2}} w_a^{(i)} / x_b^{(i)})} \frac{1}{(1 - t^{\frac{1}{2}} x_b^{(i)} / w_a^{(i)})} \\ & \prod_{(i,j) \in E} \prod_{a=1}^{\mathbf{v}_i} \prod_{b=1}^{\mathbf{v}_j} \frac{1}{1 - t^{\frac{1}{2}} w_a^{(i)} / w_b^{(j)}} \frac{1}{1 - t^{\frac{1}{2}} w_b^{(j)} / w_a^{(i)}}. \end{aligned} \quad (4.61)$$

We see<sup>12</sup> that this is the Molien integral for the quiver  $\mathcal{M}(\mathbf{v}, \mathbf{w})$  discussed in section 3.1.2 and hence reproduces the Hilbert series of the Higgs branch  $\mathcal{M}_H$ . In fact, in the absence of angular momentum refinement, [167] show that in the example of linear quivers this is the only sector that contributes to the Jeffrey-Kirwan residue prescription—it would be interesting to verify this more generally.

3d mirror symmetry then gives us the statements

$$\begin{aligned} I^A(T; x, \zeta, q, t) &= \text{H.S.}[\mathcal{M}_C](\zeta, t), \\ I^B(T; x, \zeta, q, t) &= \text{H.S.}[\mathcal{M}_H](x, t). \end{aligned} \quad (4.62)$$

In the following section we argue that in general the twisted indices can be factorised in terms of the vertex functions of the quiver  $\mathcal{M}(\mathbf{v}, \mathbf{w})$ —schematically we find

$$I^A \sim \sum_{\text{vacua}, \alpha} \mathbf{V}_{\alpha}(x, \zeta; q, t) \mathbf{V}_{\alpha}(x, \zeta, q^{-1}, t), \quad (4.63)$$

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<sup>12</sup>Up to a convention for the contracting  $\mathbb{C}_t^{\times}$  action. Typically in quiver variety literature the contracting action scales half of the quiver arrows (it is an equivalent to an  $R$ -symmetry mixed with a flavour symmetry; in this expression one could redefine  $x \rightarrow tx$ ), however the physical  $R$ -symmetry action scales all of the scalars in the matter multiplets. These different choices are gauge equivalent and lead to an overall pre-factor of a power of  $t$  difference between the twisted index and the Hilbert series.

In this context 3d mirror symmetry provides a non-trivial conjecture that the right hand side of this equation is in fact independent of  $q$ . In the presence of non-trivial background fluxes we should project onto the  $\xi^0$  sector to recover the  $q$ -independent Hilbert series with line bundle charge.

The observation that the twisted index reproduces the Hilbert series was also recently understood in the  $t \rightarrow 1$  limit [201, 160]. In this case the index manifestly becomes  $q$ -independent since the index receives contributions from states annihilated by two additional supercharges *i.e.* Higgs operators. In that case  $\mathcal{N} = 4$  supersymmetry is restored and the twisted index computes the genus zero Rozansky-Witten invariants [70] of  $\mathcal{M}_H$ —these give simply the Hilbert series of the Higgs branch (albeit ungraded by  $R$ -charge  $t$ ).

**Topological state-operator.** We focus on the  $B$ -twisted index, the argument for the  $A$ -twist is identical. The index (4.21) receives contributions from states annihilated by the supercharges  $Q_+^{11}$  and  $Q_-^{22}$ . Quantising in flat space the states annihilated by these supercharges coincide with those in  $Q_H$  cohomology. Hence by the representation theory arguments of section 2.1.3 the  $B$ -twisted index receives contributions only from Higgs operators uncharged under  $J + \frac{R_H}{2}$  and  $T_C$ . It would be interesting to understand the physics of this setup better by a topological state-operator map in three dimensions—we leave such a construction to future work [202].

**Geometrical argument.** In section 4.2.2 we identified the twisted index with the  $\chi_t$  genus of the *global* Laumon space  $\mathcal{Q}_d$ . This space is a projective variety and by the arguments of section 3.1.4 we have that the  $\chi_t$  genus of a compact variety cannot depend on global symmetry fugacities. In particular, we have

$$\chi_t(\mathcal{Q}_d) = P_{\mathcal{Q}_d}(t), \quad (4.64)$$

where the right hand side is manifestly independent of  $q$  and  $x$ .

**$q \rightarrow 0$  limit of blocks.** The above arguments allow us to evaluate the twisted index at any value of  $q$ . We choose to send  $q \rightarrow 0$ .

Realising the moduli space as a handsaw quiver variety, Nakajima [152] shows that the (based) Laumon space is a resolution of singularities<sup>13</sup>  $\pi : \mathfrak{Q}_d \rightarrow \mathfrak{Q}_d^o$ . There is a unique fixed point  $o \in (\mathfrak{Q}_d^o)^{\Gamma_q}$  and the pre-image is a projective variety which we denote by  $\mathcal{O}_d := \pi^{-1}(o)$ . The fugacity  $q$  is conjugate to the group action  $\mathbb{C}_q^\times$  and together these facts imply we are in the setting of the discussion around (3.42) in section 3.1.4 and the

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<sup>13</sup>Although note that it is not a symplectic resolution.  $\mathfrak{Q}_d$  is an oriented quiver and there is no holomorphic symplectic form on the resolution in general.

$q \rightarrow 0$  limit of the  $\chi_t$  genus coincides with the Poincaré polynomial

$$\lim_{q \rightarrow 0} \chi_t(\mathfrak{Q}_d) = P_{\mathcal{O}_d}(t). \quad (4.65)$$

In particular, in this limit the genus becomes independent of the mass parameters  $x$ . Similarly we can send  $q \rightarrow \infty$  using the identity (3.33) to find

$$\begin{aligned} \lim_{q \rightarrow 0} \overline{\chi_t(\mathfrak{Q}_d)} &= t^{\dim \mathfrak{Q}_d} P_{\mathcal{O}_d}(t^{-1}) \\ &= t^{2|d|} P_{\mathcal{O}_d}(t^{-1}). \end{aligned} \quad (4.66)$$

Using the Bialynicki-Birula methods that we reviewed in section 3.1.4, Nakajima [152] computes the generating function for these polynomials which we can write as

$$\sum_d \prod_{s=1}^{N-1} \left( t^{-1} \frac{\zeta_s}{\zeta_{s+1}} \right)^{d_s} P_t(\mathcal{O}_d) = \prod_{i < j}^N \frac{1}{1 - t^{-1} \zeta_i / \zeta_j}. \quad (4.67)$$

**Hilbert series.** We now take this limit through the factorisation (4.44) and recover the Hilbert series. We have

$$\begin{aligned} I^A &= \lim_{q \rightarrow 0} I^A \\ &= \left[ \prod_{i < j}^N (-t)^{-\frac{1}{2}} \frac{\zeta_i}{\zeta_j} \right] \sum_{d', d''} \left[ \sum_{\alpha \in S_N} \frac{1 - tx_i/x_j}{1 - x_i/x_j} \right] \left[ \prod_{i=1}^{N-1} \left( t^{-1} \frac{\zeta_s}{\zeta_{s+1}} \right)^{d'_s} P_{\mathcal{O}_{d'}}(t) \right] \left[ \prod_{i=1}^{N-1} \left( t \frac{\zeta_s}{\zeta_{s+1}} \right)^{d''_s} P_{\mathcal{O}_{d''}}(t^{-1}) \right] \\ &= \left[ \prod_{i < j}^N (-t)^{-\frac{1}{2}} \frac{\zeta_i}{\zeta_j} \right] \left[ \prod_{i=1}^N \frac{1 - t^i}{1 - t} \right] \left[ \prod_{i < j} \frac{1}{1 - t^{-1} \zeta_i / \zeta_j} \right] \left[ \prod_{i < j} \frac{1}{1 - t \zeta_i / \zeta_j} \right] \\ &= t^{\frac{1}{4}N(N-1)} \prod_{i=1}^N (1 - t^i) \prod_{i,j=1}^N \frac{1}{1 - t \zeta_i / \zeta_j} = t^{\frac{1}{4}N(N-1)} \text{H.S.}[\mathcal{M}_C]. \end{aligned} \quad (4.68)$$

In going from the second line to the third line we use the generating function (4.67) and the following lemma

**Lemma 4.**

$$\sum_{\alpha \in S_N} \prod_{i < j}^N \frac{1 - tx_i/x_j}{1 - x_i/x_j} = \prod_{i=1}^N \frac{1 - t^i}{1 - t}. \quad (4.69)$$

*Proof.* Macdonald proves this lemma using combinatorial methods in section III.1 of [144], in that context it is equivalent to Hall-Littlewood polynomials (B.16) being normalised such that  $P_\bullet(X; t) = 1$ .  $\square$

In the last line of (4.68) we recognise the Coulomb branch Hilbert series of  $T[SU(N)]$  from equation (4.38)—up to a power of  $t$  which we will understand in greater generality

in section 4.3. We conclude that the Hilbert series, when expanded in the topological symmetry  $\zeta$ , can be understood in terms of Poincaré polynomials of the vortex moduli space as we advertised in (4.42). Precisely, we can write the penultimate line of (4.68) as

$$\begin{aligned} \text{H.S.}[\mathcal{M}_C] &= \left[ \prod_{i=1}^N \frac{1-t^i}{1-t} \right] \left[ \prod_{i>j} \frac{1}{1-t\zeta_i/\zeta_j} \right] \left[ \prod_{i<j} \frac{1}{1-t\zeta_i/\zeta_j} \right] \\ &= \left[ \prod_{i=1}^N \frac{1-t^i}{1-t} \right] P_{\text{QM}(\mathcal{M}_H)}(\zeta; t) P_{\text{QM}(\mathcal{M}_H)}(\zeta^{-1}; t). \end{aligned} \quad (4.70)$$

We can go further and write the unbroken Casimir factor suggestively as

$$\prod_{i=1}^N \frac{1-t^i}{1-t} = \prod_{i=1}^N \frac{1-t^i}{(1-t^{\text{rk } \mathcal{V}_i - \text{rk } \mathcal{V}_{i-1}})}, \quad (4.71)$$

where  $\text{rk } \mathcal{V}_i$  are the ranks of sheaves in the moduli space construction from section 4.2.2, in this case they are  $\text{rk } \mathcal{V}_i = 1$ , but this expression generalises to  $T_\rho[SU(N)]$  theories. We see later that this computes the Poincaré polynomial of the central fibre  $\pi^{-1}(o)$  of the Higgs branch which, in this example, is the complete flag variety  $\mathcal{L} = \mathcal{B}_N$  which is a compact Lagrangian. The result is then

$$\text{H.S.}[\mathcal{M}_C] = P_{\mathcal{L}}(t) P_{\text{QM}(\mathcal{M}_H)}(\zeta; t) P_{\text{QM}(\mathcal{M}_H)}(\zeta^{-1}; t), \quad (4.72)$$

and we note that this expression is determined in terms of the Higgs branch geometry only. The arguments of this section apply to theories in the class  $T_\rho[SU(N)]$  whose vortex moduli spaces, since the Higgs branches are partial flag varieties, can be realised by the general handsaw quivers of [152]—or equivalently by suitably relaxing the rank constraints on the sheaves in section 4.2.2. In the next section, after introducing hemisphere blocks, we demonstrate this factorisation for the SQCD $[k, N]$  theory. Later, in chapter 5, we also present similar results for a more complicated non-abelian theory with adjoint matter.

**Remark.** We conclude this section by noting that we are also free to send  $q \rightarrow 1$ . In this limit one can compute the holomorphic block by saddle point methods and make contact with the constructions of [51, 49]. This suggests an interesting connection between Poincaré polynomials of vortex moduli spaces and solutions to Bethe ansatz equations that would be interesting to explore in future work.

## 4.3 Hemisphere blocks

In the previous section we derived ‘vortex partition functions’ by factorising the twisted index by hand. In this section we change gear and instead focus on a first principles derivation of a fundamental block  $H_\alpha$  that realises the factorisation.

We define *hemisphere blocks* in the UV as partition functions on a hemisphere  $S^1 \times H^2$ . The hemisphere has a boundary on which we place a particular  $\mathcal{N} = (2, 2)$  boundary condition known as the *exceptional Dirichlet* boundary condition  $\mathcal{B}_\alpha$ . These boundary conditions are canonically associated to isolated vacua  $\alpha$  and are natural candidates to yield a set of fundamental blocks.

We show that the hemisphere blocks  $H_\alpha := \mathcal{Z}_{S^1 \times H^2}(\mathcal{B}_\alpha)$  are closely related to the half index  $\mathcal{I}(\mathcal{B}_\alpha)$  counting local operators in the flat space theory with a boundary. A putative state-operator correspondence on the hemisphere relates the half index to the hemisphere blocks via a ‘Casimir energy’ term

$$\mathcal{Z}_{S^1 \times H^2}(\mathcal{B}_\alpha) = e^{\phi_\alpha} \mathcal{I}(\mathcal{B}_\alpha). \quad (4.73)$$

This set of blocks have a number of favourable features. The blocks are manifestly dependent on choices of chamber  $\mathfrak{C}_H$  and  $\mathfrak{C}_C$  from the start and as the chambers are varied the blocks will exhibit wall-crossing and transform amongst themselves but gluing in such a way that closed three manifold partition functions, such as the twisted index, are invariant under this wall-crossing. The blocks also resolve the perturbative ambiguities discussed in section 4.2.1 and we find that they fuse *exactly* to various three manifold partition functions.

We find that the hemisphere blocks can be expressed solely in terms of Higgs branch geometry. The partition function on the hemisphere can be separated into perturbative and vortex contributions

$$\mathcal{Z}_{S^1 \times H^2}(\mathcal{B}_\alpha) = \mathcal{Z}_{\text{Pert.}}(\mathcal{B}_\alpha) \mathcal{Z}_{\text{Vortex}}(\mathcal{B}_\alpha), \quad (4.74)$$

where the vortex contribution coincides with the equivariant  $K$ -theoretic vertex function discussed in section 4.1.1 and the perturbative contribution can be realised in terms of the attracting directions and fibres of line bundles over the Higgs branch at a particular vacuum  $\alpha$ .

**Outline.** We begin in section 4.3.1 with a discussion of *exceptional Dirichlet* boundary conditions, we discuss how to compute the half index of these boundary conditions and lift these calculations to a hemisphere partition function. In section 4.3.2 we compute the hemisphere partition function explicitly for the SQED[ $N$ ] example. In section 4.3.3 we extrapolate this construction to a more general quiver gauge theory and propose a definition for the *hemisphere block* in terms of the quiver data. Finally, in section 4.3.4, we check this proposal in examples and show that we can glue hemisphere blocks to form twisted indices.



### 4.3.1 $\mathcal{N} = (2, 2)$ boundary conditions

So far in the thesis we have met BPS Higgs and Coulomb branch operators and BPS vortices. We now introduce the last half-BPS object of 3d  $\mathcal{N} = 4$  theories that we study. We consider boundary conditions  $\mathcal{B}$  that preserve  $\mathcal{N} = (2, 2)$  supersymmetry with the supercharges  $Q_+^{1\dot{1}}, Q_-^{1\dot{2}}, Q_-^{2\dot{1}}$  and  $Q_+^{2\dot{2}}$ . The boundary conditions we study will also preserve a maximal torus  $U(1)_V \times U(1)_A$  of the bulk R-symmetry  $SU(2)_H \times SU(2)_C$ .

In the previous section we placed our theory on  $S^2 \times S^1$  with a topological twist. In this section we, momentarily, put our theory back on flat space  $\{x^1, x^2, x^3\}$  but now with a boundary condition at  $x^3 = 0$ . We write  $z = x^1 + ix^2$  and  $\bar{z} = x^1 - ix^2$  for the coordinates on the boundary. This setup is illustrated in figure 4.5. The boundary condition  $\mathcal{B}$  preserves a subalgebra of  $\mathfrak{osp}(4, 4|\mathbb{R})$  and later we will recover the ‘twisted’ blocks studied in the previous section by a simple  $R$ -symmetry shift.

The half superconformal index was defined in the work [203] for 3d  $\mathcal{N} = 2$  theories. It is a trace over states  $\mathcal{H}_{\mathcal{B}}$  in radial quantisation annihilated by  $Q_+^{1\dot{1}}$  and its superconformal conjugate:

$$\mathcal{I}(\mathcal{B}) := \text{Tr}_{\mathcal{H}_{\mathcal{B}}} (-1)^F q^{J+\frac{R_V-R_A}{4}} t^{\frac{R_V-R_A}{2}} x^{F_H} \zeta^{F_C}. \quad (4.75)$$

The boundary conditions we study preserve a maximal torus  $T_H \times T_C$  of the bulk flavour symmetry  $G_H \times G_C$  for which we write  $F_H$  and  $F_C$  for the Cartan generators in the above.

**Exceptional Dirichlet.** The particular boundary conditions we study in this thesis are known as *exceptional Dirichlet* boundary conditions, we follow closely the presentation of [55] where these boundary conditions were first studied for 3d  $\mathcal{N} = 4$  theories. In a fixed chamber  $\mathfrak{C}_H$ , there are a set of Lagrangian manifolds of the Higgs branch  $\mathcal{M}_H$  associated to each isolated massive vacuum denoted  $\mathcal{L}_{\alpha}^+$ —we met these manifolds already in section 3.1.4 where they are discussed in more detail. Exceptional Dirichlet boundary conditions, denoted  $\mathcal{B}_{\alpha}$ , are engineered to have a Higgs branch image coinciding with  $\mathcal{L}_{\alpha}^+$ .

In the IR the 3d  $\mathcal{N} = 4$  theories we study—together with the boundary condition  $\mathcal{B}_{\alpha}$ —flow to non-linear sigma models on the vacuum moduli space [55]. In this context the IR image of the boundary condition is a Lagrangian submanifold that is expected to realise the thimble boundary conditions for the Higgs branch sigma model. In this way, the exceptional Dirichlet boundary conditions  $\mathcal{B}_{\alpha}$  are in one-to-one correspondence with isolated vacua  $\alpha$ .

The boundary condition includes setting the vector multiplet to Dirichlet so that<sup>14</sup>

$$F_{\mu\nu}|_{\partial} = 0, \quad D_3\sigma = 0, \quad D_{\mu}\varphi = 0, \quad (4.76)$$

---

<sup>14</sup>Together with the supersymmetric completion to the rest of the multiplet.

where  $\mu$  and  $\nu$  are directions parallel to the boundary. In the absence of matter, this boundary condition breaks the gauge symmetry to a global symmetry  $G_\partial$  on the boundary. We consider quiver gauge theories and so we also have to pick a boundary condition for the matter. Boundary conditions for hypermultiplets are specified by a Lagrangian splitting  $\mathcal{R}(\mathbf{v}, \mathbf{w}) = L \oplus \bar{L}$  where, relabelling scalars as  $(X_L, Y_L)$ , fields  $Y_L$  are given Dirichlet boundary conditions and  $X_L$  are given Neumann boundary conditions—together with the supersymmetric completion to the rest of the multiplet.

The splitting of  $\mathcal{R}(\mathbf{v}, \mathbf{w})$  for the exceptional Dirichlet boundary condition  $\mathcal{B}_\alpha$  crucially depends on the vacuum  $\alpha$  and is distinct from the natural splitting associated to the cotangent bundle structure of the quiver variety. It is defined as follows: we choose a constant matrix  $c$  of vevs for the hypermultiplets  $Y_L|_\partial = c$  in such a way that the boundary symmetry  $G_\partial$  is broken but a maximal torus  $T_H \times T_C$  of the bulk flavour symmetry is preserved. The constants should be chosen so that the image of  $\mathcal{B}_\alpha$  on the Higgs branch coincides with  $\mathcal{L}_\alpha^+$ .

In practice this is very difficult to compute for an arbitrary non-abelian 3d  $\mathcal{N} = 4$  theory since it depends on the pre-quotient data  $\mathcal{R}(\mathbf{v}, \mathbf{w})$  and an understanding of the affine spaces that descend to Lagrangian submanifolds in the quotient. We compute the boundary condition explicitly only for SQED[ $N$ ] and show that the end result depends only on the local data of the fixed point  $\alpha$  in the quotient space  $\mathcal{M}(\mathbf{v}, \mathbf{w})$ . In section 4.3.3 we use this to formulate a proposal for the *hemisphere block* of a quiver gauge theory that should arise from the exceptional Dirichlet boundary conditions discussed in the present section.

**Half index.** We first write the half index for a generic Dirichlet boundary condition for an abelian theory—this corresponds to setting the matrix of constants  $c$  to zero. Following [203] we have

$$\mathcal{I}(\mathcal{B}_\alpha) = \text{Tr}_{\mathcal{H}_B}(-1)^F q^{J+\frac{R_V+R_A}{4}} t^{\frac{R_V-R_A}{2}} z^{F_H} \zeta^{F_C} w^{F_G}, \quad (4.77)$$

where  $w$  is a fugacity for the  $F_G$  generator of the  $U(1)_\partial$  symmetry. Turning on vevs for the scalars  $Y|_\partial$  initiates an RG flow. We turn on a vev for a scalar with charges  $+1$  under  $R_V$ ,  $+1$  under  $U(1)_\partial$  and  $Q_H$  under  $T_H$ —a linear combination of  $U(1)_V$ ,  $U(1)_\partial$  and  $T_H$  is spontaneously broken however the combinations

$$R'_V = R_V - F_G, \quad F'_H = F_H - Q_H F_G, \quad (4.78)$$

survive along the RG flow and become the new boundary  $R$ -symmetry and flavour symmetry. The upshot is that in index calculations we eliminate  $w$  via the substitution  $q^{\frac{1}{4}} t^{\frac{1}{2}} z^{Q_H} w = 1$  for the chirals that obtain expectation values.

**Operator counting.** The operators from the matter multiplets that contribute to the index (4.77) in general are given by

	$\partial_z^n X$	$\partial_z^n Y$	$\partial_z^n \bar{\psi}_-^X$	$\partial_z^n \bar{\psi}_-^Y$
$G$	+1	−1	−1	+1
$F_H$	+1	−1	−1	+1
$J$	$n$	$n$	$n + \frac{1}{2}$	$n + \frac{1}{2}$
$R_A$	0	0	+1	+1
$R_V$	+1	+1	0	0

(4.79)

This follows from a specialisation of the results of [203] to  $\mathcal{N} = 4$  theories. We then have that hypermultiplets with the boundary conditions

$$\begin{aligned} \mathcal{B}_X : \partial_\perp X|_\partial = 0, \quad Y|_\partial = 0, \\ \mathcal{B}_Y : \partial_\perp Y|_\partial = 0, \quad X|_\partial = 0, \end{aligned} \tag{4.80}$$

contribute to the index with

$$\mathcal{I}^{\mathcal{N}=4\text{HM}}(\mathcal{B}_X) = \frac{(q^{\frac{3}{4}} t^{-\frac{1}{2}} w x; q)_\infty}{(q^{\frac{1}{4}} t^{\frac{1}{2}} w x; q)_\infty}, \quad \mathcal{I}^{\mathcal{N}=4\text{HM}}(\mathcal{B}_Y) = \frac{(q^{\frac{3}{4}} t^{-\frac{1}{2}} w^{-1} x^{-1}; q)_\infty}{(q^{\frac{1}{4}} t^{\frac{1}{2}} w^{-1} x^{-1}; q)_\infty}. \tag{4.81}$$

where the  $q$ -Pochhammer symbols should be thought of as expansions in spin  $q$ —the definition is in (A.1). The perturbative contributions from the vector multiplet come from the operators [203]

	$D_z^{n+1}(\sigma + i\gamma)$	$D_z^n \eta_-$
$G$	adj	adj
$J$	$n$	$n + \frac{1}{2}$
$R_A$	0	−1
$R_V$	0	+1

(4.82)

so that for a theory with gauge group  $G$  we have the perturbative contribution

$$\mathcal{I}^{\mathcal{N}=4\text{VM}} = \frac{(q^{\frac{1}{2}} t; q)_\infty^{\text{rk } G}}{(q; q)_\infty^{\text{rk } G}} \prod_{\alpha \in \text{rt}(G)} \frac{(q^{\frac{1}{2}} t w^\alpha; q)_\infty}{(q w^\alpha; q)_\infty}. \tag{4.83}$$

The exceptional Dirichlet boundary condition supports boundary monopole operators that must also be included in the trace (4.77). These operators arise from bringing the 't Hooft monopole solutions discussed in section 2.1.4 to the boundary and are again labelled by a cocharacter  $\mathfrak{m} \in \Lambda$ . Operators with electric charge  $\lambda \in \mathfrak{g}$  acquire a shift to their spin of

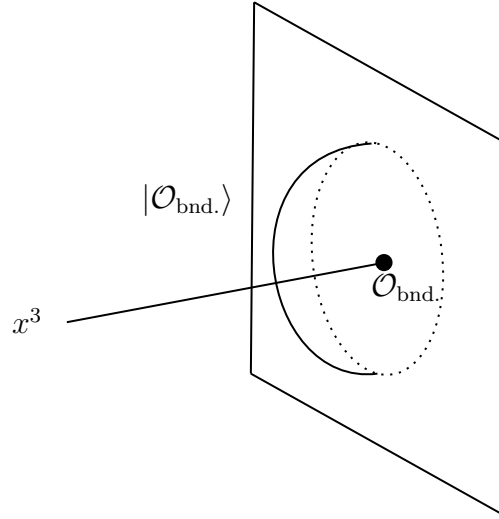


Figure 4.2: Hemisphere state-operator map.

$\lambda(\mathfrak{m})$  and in the  $\mathcal{N} = 4$  case the non-perturbative half index proposed by [203] reads

$$\mathcal{I}(\mathcal{B}) \sim \frac{(q^{\frac{1}{2}}t; q)^{\text{rk } G}}{(q; q)_{\infty}^{\text{rk } G}} \sum_{\mathfrak{m} \in \Lambda} \zeta^{\mathfrak{m}} \left[ \prod_{\alpha \in \text{roots}(G)} \frac{(q^{\frac{1}{2} + \alpha(\mathfrak{m})}tw^{\alpha}; q)_{\infty}}{(q^{1 + \alpha(\mathfrak{m})}w^{\alpha}; q)_{\infty}} \right] [\mathcal{I}_{\text{matter}}^{\mathcal{N}=4}(w \rightarrow q^{\mathfrak{m}}w)] , \quad (4.84)$$

the  $\sim$  here indicates that we have omitted the effective Chern-Simons level since we take an alternative approach and realise this term by a path integral in the next paragraph. We now have the ingredients required to compute the half index of the exceptional Dirichlet boundary conditions.

**State-operator correspondence and Casimir energy.** The above discussion is a count of local operators inserted at the origin of the boundary. In the present superconformal setup with a boundary we expect there to be a 3d state-operator map that relates the operator count to a partition function  $\mathcal{Z}_{S^1 \times H^2}$  counting states on a surrounding hemisphere—this is illustrated in figure 4.2. The hemisphere is the space  $S^1 \times H^2$  with metric

$$ds^2 = d\tau^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) , \quad (4.85)$$

with  $\tau \sim \tau + \beta r$  the  $S^1$  coordinate with  $q = e^{-2\beta}$  and  $0 \leq \theta \leq \pi/2$ . The boundary condition  $\mathcal{B}$  is then imposed at the  $S^1 \times S^1$  boundary  $\theta = \pi/2$ . Typically [204, 205, 206] operator counts and partition functions related via a state-operator map differ by a ‘Casimir energy’ factor that we write as

$$\mathcal{Z}_{S^1 \times D}(\mathcal{B}) = e^{\phi} \mathcal{I}_{\mathcal{B}} . \quad (4.86)$$

The operator counting discussed in the previous paragraph can be directly related to one-loop determinants arising from the localisation computation of the left hand side of (4.86) as we now discuss.

Firstly, we define the following function that arises in the zeta function regularisation of one-loop determinants

$$\mathcal{E}(x) := \frac{\log q}{24} - \frac{x}{4} + \frac{x^2}{4 \log q}. \quad (4.87)$$

In a localisation computation (for example [175]) the  $\mathcal{N} = 4$  vector multiplet contribution (4.83) is modified to

$$Z_{\text{VM}}^{1\text{-loop}} = e^{\mathcal{E}[\log(q^{\frac{1}{2}}t)] - \mathcal{E}[\log(q)]} \frac{(q^{\frac{1}{2}}t; q)_{\infty}^{\text{rk } G}}{(q; q)_{\infty}^{\text{rk } G}} \prod_{\alpha \in \text{rt}(G)} e^{\mathcal{E}[\log(q^{\frac{1}{2}}ts^{\alpha})] - \mathcal{E}[\log(qs^{\alpha})]} \frac{(q^{\frac{1}{2}}ts^{\alpha}; q)_{\infty}}{(qs^{\alpha}; q)_{\infty}}. \quad (4.88)$$

Similarly the  $\mathcal{N} = 4$  hypermultiplet contributions (4.81) are modified to

$$\begin{aligned} Z_{\text{HM}}^{1\text{-loop}}(\mathcal{B}_X) &= e^{\mathcal{E}[\log(q^{\frac{3}{4}}t^{-\frac{1}{2}}wx)] - \mathcal{E}[\log(q^{\frac{1}{2}}t^{\frac{1}{2}}wx)]} \frac{(q^{\frac{3}{4}}t^{-\frac{1}{2}}wx; q)_{\infty}}{(q^{\frac{1}{4}}t^{\frac{1}{2}}wx; q)_{\infty}}, \\ Z_{\text{HM}}^{1\text{-loop}}(\mathcal{B}_Y) &= e^{\mathcal{E}[\log(q^{\frac{3}{4}}t^{-\frac{1}{2}}w^{-1}x^{-1})] - \mathcal{E}[\log(q^{\frac{1}{2}}t^{\frac{1}{2}}w^{-1}x^{-1})]} \frac{(q^{\frac{3}{4}}t^{-\frac{1}{2}}w^{-1}x^{-1}; q)_{\infty}}{(q^{\frac{1}{4}}t^{\frac{1}{2}}w^{-1}x^{-1}; q)_{\infty}}. \end{aligned} \quad (4.89)$$

Finally, we import the on-shell classical contribution to the path integral from [160] where the localisation is performed in greater detail—for an abelian theory we have

$$S_{\text{Cl.}} = \frac{\log \zeta \log(wq^{\mathfrak{m}})}{\log q}. \quad (4.90)$$

The procedure for computing the hemisphere partition function then mirrors (4.84) in that

$$\mathcal{Z}_{S^1 \times D}(z, \zeta; q, t; w) = \sum_{\mathfrak{m} \in \Lambda} \zeta^{\mathfrak{m}} e^{S_{\text{cl.}}} Z_{1\text{-loop}}(w \rightarrow q^{\mathfrak{m}}w). \quad (4.91)$$

The dependence on the gauge fugacity  $w$  is then eliminated from (4.91) by the exceptional Dirichlet procedure of setting the relevant fugacities to one. We compute the hemisphere partition function explicitly for an example in the following section.

**Background flux.** In the topologically twisted setup of [38], we can understand fugacities entering (4.91) as holonomies of background gauge fields coupled to global symmetries. The metric (4.85) becomes a fibred product of  $S^1$  over  $H^2$  with holonomies

$$\nu = \oint_{S^1} A_F + \beta \oint_{\partial H^2} A_F, \quad (4.92)$$

shifting fugacities  $\nu \rightarrow \nu + \mathfrak{n}\beta$  or  $z \rightarrow zq^{\frac{1}{2}\mathfrak{n}}$  then corresponds to introducing  $\mathfrak{n}$  units of background flux on  $H^2$  for the corresponding symmetry. We define the  $A$ - and  $B$ -shifted

hemisphere partition functions by a shift of the  $R$ -symmetry:

$$\begin{aligned}\mathcal{Z}_{S^1 \times H^2}^A(x, \zeta; q, t) &:= \mathcal{Z}_{S^1 \times H^2}(x, \zeta; q, t \rightarrow tq^{\frac{1}{2}}), \\ \mathcal{Z}_{S^1 \times H^2}^B(x, \zeta; q, t) &:= \mathcal{Z}_{S^1 \times H^2}(x, \zeta; q, t \rightarrow tq^{-\frac{1}{2}}).\end{aligned}\tag{4.93}$$

Indeed, in the following sections we show that these hemisphere partition functions, when glued appropriately, reproduce the  $A$ - and  $B$ -twisted indices of the theory.

We similarly write

$$\begin{aligned}\mathcal{I}^A(x, \zeta; q, t) &:= \mathcal{I}(x, \zeta; q, t \rightarrow tq^{\frac{1}{2}}), \\ \mathcal{I}^B(x, \zeta; q, t) &:= \mathcal{I}(x, \zeta; q, t \rightarrow tq^{-\frac{1}{2}}),\end{aligned}\tag{4.94}$$

for  $A$ - and  $B$ -shifted half indices.

Twisting is no longer trivial in this way if the hemisphere is replaced with a more general Riemann surface with boundary.

**Remark.** We conclude this subsection with some remarks about the connection between the hemisphere partition function and enumerative geometry. Firstly, note that in section 3.1 we stressed that the localisation formulae for Nakajima quiver varieties can be understood as evaluating the Chern roots  $w$  associated to tautological bundles  $\mathcal{W}$  at fixed points  $p$  as in (3.12). This precisely mirrors the ‘exceptional Dirichlet procedure’ and indeed there exists [51] methods to evaluate vertex functions of quiver varieties via Jackson  $q$ -integrals that realise the invariants as sums over towers of ‘poles’  $pq^m$ . We will see in the following example that for SQED[ $N$ ] the non-perturbative contributions to (4.91) reproduce the vertex functions for  $T^*\mathbb{P}^{N-1}$  in (4.15) and furthermore these *methods* of evaluation coincide. Secondly, typically ‘holomorphic blocks’ are realised as contour integrals as in, for example, the work [175]. In fact, such ‘holomorphic block integrals’ realise Neumann boundary conditions for the hemisphere partition function and it is not always possible to associate Neumann boundary conditions to vacua in the way that we discussed above for the exceptional Dirichlet boundary conditions (only if their Higgs branch images coincide)—the theory we study in chapter 5 is an example of this. This observation mirrors the fact that Jackson  $q$ -integrals cannot always be represented by a contour integral in the complex plane. Consequently, we propose that the exceptional Dirichlet procedure is the correct notion of a fundamental block and not a Coulomb branch contour integral. We explore this idea in more detail in the following sections.

### 4.3.2 Example: SQED[ $N$ ]

We turn again to supersymmetric QED with  $N$  flavours. The exceptional Dirichlet boundary conditions  $\mathcal{B}_\alpha$  associated to the  $\alpha = \{1, \dots, N\}$  Higgs vacua of SQED[ $N$ ] are

specified by<sup>15</sup>

$$\begin{aligned} \partial_{\perp} Y_j &= 0 & X_j &= c\delta_{\alpha j} & j &\leq \alpha, \\ \partial_{\perp} X_j &= 0 & Y_j &= 0 & j &> \alpha, \end{aligned} \quad (4.95)$$

where  $c \neq 0$ .

**Attracting submanifolds.** For fixed  $\alpha$ , the fields  $(Y_1, \dots, Y_{\alpha})$  are allowed to fluctuate, these cut out  $\mathbb{P}^{\alpha-1} \subset \mathbb{P}^{N-1}$ . Recall from section 2.3 the description of the cotangent bundle at a point  $Y \in \mathbb{P}^N$  as  $YX \in \text{Hom}(\mathbb{C}^N/\text{im } Y, \text{im } Y)$ , evaluating this on the locus (4.95) we see a natural isomorphism to a direct sum of tautological line bundles  $\bigoplus_{i=1}^{\alpha} \mathcal{O}_{\mathbb{P}^{\alpha-1}}(-1)$  which is isomorphic to the conormal bundle to  $\mathbb{P}^{\alpha-1}$  [207]. The conormal bundles are well-known to describe the attracting Lagrangians of  $T^*\mathbb{P}^{N-1}$ . This method can be adapted to cotangent bundles to flag varieties but it is unclear, at least to the author, how to understand the attracting Lagrangians of general Nakajima quiver varieties in terms of the pre-quotient data in this way.

**Half index.** Now we apply the recipe (4.91) outlined in the previous section to the boundary condition (4.95) for SQED[ $N$ ]. We write the appropriate one-loop determinants from (4.88) and (4.89) together with the classical contribution (4.90) and then specialise fugacities as  $w = x_{\alpha}^{-1}t^{-\frac{1}{2}}q^{-\frac{1}{4}}$  appropriate to each boundary condition  $\mathcal{B}_{\alpha}$ . After repeated use of the identity (A.11) we can write the hemisphere partition function in the following form

$$\mathcal{Z}_{S^1 \times H^2}(\mathcal{B}_{\alpha}) = e^{\phi_{\alpha}} \mathcal{I}_{1\text{-loop}}^{\alpha} \mathcal{I}_{\text{Vortex}}^{\alpha}, \quad (4.96)$$

*i.e.* a contribution from the ‘Casimir energy’ and the half index  $\mathcal{I}(\mathcal{B}_{\alpha}) = \mathcal{I}_{1\text{-loop}}^{\alpha} \mathcal{I}_{\text{Vortex}}^{\alpha}$ . The one-loop and non-perturbative contributions are given by

$$\begin{aligned} \mathcal{I}_{1\text{-loop}}^{\alpha} &= \prod_{j=1}^{i-1} \frac{(q^{\frac{x_i}{x_j}}; q)_{\infty}}{(q^{\frac{1}{2}}t^{\frac{x_i}{x_j}}; q)_{\infty}} \prod_{j=i+1}^N \frac{(q^{\frac{1}{2}}t^{-1}\frac{x_j}{x_i}; q)_{\infty}}{(\frac{x_j}{x_i}; q)_{\infty}}, \\ \mathcal{I}_{\text{Vortex}}^{\alpha} &= \sum_{d \geq 0} \left( (q^{\frac{1}{4}}t^{-\frac{1}{2}})^N \zeta \right)^d \prod_{j=1}^N \frac{(q^{\frac{1}{2}}t^{\frac{x_i}{x_j}}; q)_d}{(q^{\frac{x_i}{x_j}}; q)_d}, \end{aligned} \quad (4.97)$$

and for the Casimir energy we find

$$\begin{aligned} \phi_{\alpha} &= \\ \sum_{j < \alpha} \frac{\log(q^{\frac{1}{4}}t^{-\frac{1}{2}}) \log(x_{\alpha}/x_j t^{\frac{1}{2}} q^{\frac{1}{4}})}{\log q} &+ \sum_{j > \alpha} \frac{\log(q^{\frac{1}{4}}t^{-\frac{1}{2}}) \log(x_j/x_{\alpha} t^{-\frac{1}{2}} q^{-\frac{1}{4}})}{\log q} + \frac{\log(x_{\alpha} t^{\frac{1}{2}} q^{\frac{1}{4}}) \log \zeta}{\log q}. \end{aligned} \quad (4.98)$$

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<sup>15</sup>Together with the Dirichlet boundary condition for the vector multiplet and the supersymmetric completion to the rest of the multiplet.

In the following we discuss the geometrical interpretation of the index where it is convenient to work with the  $A$ - and  $B$ -shifted partition functions discussed in the previous section and defined as

$$\begin{aligned}\mathcal{Z}_{S^1 \times H^2}^A(\mathcal{B}_\alpha) &:= \mathcal{Z}_{S^1 \times H^2}(\mathcal{B}_\alpha)(t \rightarrow tq^{\frac{1}{2}}), \\ \mathcal{Z}_{S^1 \times H^2}^B(\mathcal{B}_\alpha) &:= \mathcal{Z}_{S^1 \times H^2}(\mathcal{B}_\alpha)(t \rightarrow tq^{-\frac{1}{2}}).\end{aligned}\tag{4.99}$$

### 4.3.3 Twisted indices and geometry

In this subsection we discuss one of the main results of this thesis. We first show that the hemisphere partition function for SQED $[N]$  derived in the previous example can be expressed solely in terms of the geometry of the Higgs branch  $\mathcal{M}_H = T^*\mathbb{P}^{N-1}$ . We generalise this construction to a theory with Higgs branch a Nakajima quiver variety  $\mathcal{M}_H = \mathcal{M}(\mathbf{v}, \mathbf{w})$  and define a block in terms of the quiver data. We expect the block to agree with the hemisphere partition function of the exceptional Dirichlet boundary condition—this is a significant simplification since the exceptional Dirichlet boundary condition computation depends intricately on the pre-quotient data. We denote the block by

$$H_\alpha(x, \zeta, q, t) := \mathcal{Z}_{S^1 \times H^2}(x, \zeta, q, t).\tag{4.100}$$

This object is defined in definition 4.1 and we test the proposal in several examples.

**Attracting and repelling directions.** Let  $X = \mathcal{M}(\mathbf{v}, \mathbf{w})$  be a Nakajima quiver variety. Recall that  $X$  has a torus action  $\mathsf{T} = \mathsf{A} \times \mathbb{C}_t^\times$  where the  $\mathbb{C}_t^\times$  scales the symplectic form and  $\mathsf{A}$  leaves it invariant. We select a fixed chamber  $\mathfrak{C}_H$  for  $\mathsf{A}$ . Physically, this is an ordering of the mass parameters—for SQED $[N]$  we took  $m_1 < \dots < m_N$ . The tangent space at a fixed point  $\alpha$  splits into positive and negative weights denoted<sup>16</sup>

$$T_\alpha X = N_\alpha^+ \oplus N_\alpha^-.\tag{4.101}$$

The space  $N_\alpha^+$  coincides locally with the attracting Lagrangians  $\mathcal{L}_\alpha^+$  discussed previously in section 3.1.4.

**SQED $[N]$  example.** We now rewrite the  $B$ -shifted hemisphere partition function (4.99) of supersymmetric QED in a geometric form.

First we note that the vortex contributions agree with the  $K$ -theoretic equivariant vertex function for  $T^*\mathbb{P}^{N-1}$  from equation (4.15)

$$\mathcal{I}_{\text{Vortex}}^{B, \alpha} = \mathsf{V}_\alpha(x, \zeta; q, t).\tag{4.102}$$

---

<sup>16</sup>There are no zero weights since our assumption throughout is that  $X^\mathsf{A}$  is finite.



Now, for  $\mathcal{M}_H = T^*\mathbb{P}^{N-1}$ , the splitting of the tangent bundle (3.19) into attracting and repelling directions of  $\mathbf{A}$  is

$$\begin{aligned} N_\alpha^+ &= \sum_{j<\alpha} \frac{x_\alpha}{x_j} + t^{-1} \sum_{j>\alpha} \frac{x_j}{x_\alpha}, \\ N_\alpha^- &= \sum_{j>\alpha} \frac{x_\alpha}{x_j} + t^{-1} \sum_{j<\alpha} \frac{x_j}{x_\alpha}. \end{aligned} \quad (4.103)$$

In terms of which we identify (the  $B$ -shift of) the one-loop piece (4.97) as

$$\mathcal{I}_{1\text{-loop}}^{B,\alpha} = \text{PE} \left[ \frac{t-q}{1-q} N_\alpha^+ \right]. \quad (4.104)$$

Finally, we can write (the  $B$ -shift of) the classical contribution (4.98) in terms of the quiver data as follows<sup>17</sup>

$$\phi_\alpha^B = \sum_{\omega \in N_\alpha^+} \frac{\log(q^{\frac{1}{2}} t^{-\frac{1}{2}}) \log(t^{\frac{1}{2}} \omega)}{\log q} + \frac{\log \xi \log(t^{\frac{1}{2}} \mathcal{L}_\alpha)}{\log q}, \quad (4.105)$$

where  $\mathcal{L}_\alpha$  is the tautological line bundle on  $T^*\mathbb{P}^{N-1}$  from example 3.1 (not to be confused with the attracting Lagrangian  $\mathcal{L}_\alpha^+$ ) evaluated at the fixed point  $\alpha$ .

**Hemisphere blocks.** We now define the hemisphere block for a ‘good’ theory  $\mathcal{T}_Q$  with isolated vacua with respect to fixed chambers  $\mathfrak{C}_H$  and  $\mathfrak{C}_C$ .

**Definition 4.1.** The hemisphere block in the  $B$ -shifted convention is defined by

$$H_\alpha^B(x, \zeta; q, t) = e^{\phi_\alpha^B} \text{PE} \left[ \frac{t-q}{1-q} N_\alpha^+ \right] \mathcal{V}_\alpha(x, \zeta; q, t), \quad (4.106)$$

where  $\mathcal{V}_\alpha(x, \zeta; q, t)$  are the vertex functions of the quasimap moduli space  $\mathbf{QM}^d(\mathcal{M}_H)$  and  $N_\alpha^+$  denotes the attracting directions of the tangent bundle of the Higgs branch  $\mathcal{M}_H$  at the fixed point  $\alpha$ . The Casimir energy is given by

$$\phi_\alpha^B = \sum_{\omega \in N_\alpha^+} \frac{\log(q^{\frac{1}{2}} t^{-\frac{1}{2}}) \log(t^{\frac{1}{2}} \omega)}{\log q} + \sum_{i \in I} \sum_{a=1}^{\mathbf{v}_i} \frac{\log(\xi_i) \log(t^{\frac{1}{2}} w_a^{(i)}(\alpha))}{\log q}, \quad (4.107)$$

where the second sum is over the vertex set of the quiver and  $w_a^{(i)}(\alpha)$  denotes the evaluation of the Chern roots of the tautological bundle  $\mathcal{V}_i$  at the vacuum  $\alpha$  for  $i \in I$  and  $a = 1, \dots, \mathbf{v}_i$ . The  $A$ -shifted block is defined as the  $B$ -shifted block with the substitution  $t \rightarrow qt$  as in

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<sup>17</sup>The slightly annoying powers of  $t^{\frac{1}{2}}$  can again be traced back to the difference in convention for the contracting action in the mathematical literature compared with the physical interpretation as an  $R$ -symmetry.

(4.94)

$$H_\alpha^A(x, \zeta; q, t) = H_\alpha^B(x, \zeta, q, t \rightarrow qt). \quad (4.108)$$

The three factors in the blocks correspond to classical, one-loop and vortex contributions respectively.

**Remark.** It is well-known that in many examples vertex functions can be ‘identified’ with vortex partition functions. One of the new feature here is the normalisation, that arises from a careful consideration of the UV boundary condition, which makes the chamber dependence manifest—we will see several advantages to this normalisation throughout the thesis. The dependence on  $\mathfrak{C}_H$  enters in the decomposition of the tangent bundle and the dependence on  $\mathfrak{C}_C$  enters as the stability condition for the quasimap moduli spaces. It would be interesting to derive this hemisphere block from a Higgs branch localisation calculation on the hemisphere  $S^1 \times H^2$  where we would expect to recover the Jackson  $q$ -integral method for evaluating the vertex function.

**Factorisation.** The proposal 4.1 has been checked in a number of different gluings—in this thesis we focus on the  $A$ - and the  $B$ -twisted indices. The relevant gluing for the  $B$ -twist is

$$I^B = \sum_\alpha H_\alpha^B(q, t) H_\alpha^B(q^{-1}, t), \quad (4.109)$$

and for the  $A$ -twist we have

$$I^A = \sum_\alpha H_\alpha^A(q, t) H_\alpha^A(q^{-1}, t). \quad (4.110)$$

Recall that the arguments of section 4.2.3 ensure say that both of these indices are in fact independent of  $q$  and either  $x$  or  $\zeta$  respectively. From the enumerative geometry perspective this implies surprising cancellations between the vertex functions on the right hand side of the above equations.

**Geometry of the  $B$ -twisted factorisation.** We now discuss how to write the  $B$ -twisted index in terms of the quiver data and the geometrical interpretation of topological flux. Background flux for the topological symmetry can be introduced by shifting  $\xi_i \rightarrow q^{\frac{1}{2}\mathbf{n}_i^{(T)}} \xi_i$  for  $i = 1, \dots, |I|$ . This modifies the Casimir contribution (4.107) to

$$\phi_\alpha \rightarrow \phi_\alpha + \frac{1}{2} \sum_{i \in I} \mathbf{n}_i^{(T)} \sum_{a=1}^{v_i} \log\left(t^{\frac{1}{2}} w_a^{(i)}(\alpha)\right). \quad (4.111)$$

We now prove that the  $B$ -twisted topological index in general glues to the *twisted* Hilbert series of section 3.1.2, including the presence of background topological flux. In this gluing only the shifted term and the  $q$ -independent part of the Casimir energy survives

to give

$$\begin{aligned}
I^B &= \sum_{\alpha \in X^A} \prod_{i \in I} \left( t^{\frac{1}{2}} \mathcal{L}_\alpha^{(i)} \right)^{\mathbf{n}_i^{(T)}} \left( \prod_{\omega \in N_\alpha^+} t^{\frac{1}{2}} \omega \right) \text{PE} \left[ \frac{t-q}{1-q} N_\alpha^+ \right] \text{PE} \left[ \frac{t-q^{-1}}{1-q^{-1}} N_\alpha^+ \right] \\
&= \sum_{\alpha \in X^A} \prod_{i \in I} \left( t^{\frac{1}{2}} \mathcal{L}_\alpha^{(i)} \right)^{\mathbf{n}_i^{(T)}} \left( \prod_{\omega \in N_\alpha^+} t^{\frac{1}{2}} \omega \right) \text{PE} [t N_\alpha^+] \text{PE} [N_\alpha^+] ,
\end{aligned} \tag{4.112}$$

where  $\mathcal{L}_\alpha^{(i)}$  denotes the evaluation of the tautological line bundle associated to  $i \in I$  at the fixed point  $\alpha$ . In the last line we have used the fusion identity (A.10). We now rewrite the twisted index in terms of the attracting weights

$$\begin{aligned}
I^B &= \sum_{\alpha \in X^A} \prod_{i \in I} \left( t^{\frac{1}{2}} \mathcal{L}_\alpha^{(i)} \right)^{\mathbf{n}_i^{(T)}} (-t)^{-\frac{1}{2}|N_\alpha^+|} \prod_{\omega \in N_\alpha^+} \frac{-\omega}{1-\omega} \prod_{\omega \in N_\alpha^+} \frac{1}{1-\omega} \\
&= (-t)^{\frac{1}{4} \dim(\mathcal{M}_H)} \sum_{\alpha \in X^A} \prod_{i \in I} \left( t^{\frac{1}{2}} \mathcal{L}_\alpha^{(i)} \right)^{\mathbf{n}_i^{(T)}} \prod_{\omega \in N_\alpha^-} \frac{1}{1-\omega} \prod_{\omega \in N_\alpha^+} \frac{1}{1-\omega} .
\end{aligned} \tag{4.113}$$

In the second line we flip the relevant weights and recover, up to an overall power of  $t$ , the localisation formula (3.14) for the Higgs branch Hilbert series, namely

$$I^B = (-t)^{\frac{1}{4} \dim(\mathcal{M}_H)} \sum_{\alpha \in X^A} \prod_{i \in I} \left( t^{\frac{1}{2}} \mathcal{L}_\alpha^{(i)} \right)^{\mathbf{n}_i^{(T)}} \text{PE} [T_\alpha \mathcal{M}_H] . \tag{4.114}$$

In particular, we can give a geometric interpretation to the topological flux  $\mathbf{n}^{(T)}$  as controlling which of the tautological line bundles  $(\mathcal{L}_i)_{i \in I}$  are turned on.

**Comments on the  $A$ -twist.** For the  $A$ -shifted block it is convenient to introduce some extra notation. The tangent bundle of a cotangent type quiver variety  $X$  admits a further splitting by the polarisation  $TX = \mathcal{P} + t^{-1}\mathcal{P}^\vee$  and we define the decomposition of the attracting and repelling directions with respect to the polarisation by

$$N_\alpha^+ = P_\alpha^+ + t^{-1}\bar{P}_\alpha^+ , \quad N_\alpha^- = P_\alpha^- + t^{-1}\bar{P}_\alpha^- . \tag{4.115}$$

This splitting has the property  $(P_\alpha^+)^\vee = \bar{P}_\alpha^-$ . The 1-loop contributions to the  $A$ -shifted block can then be expressed as

$$\mathcal{I}_\alpha^{A,1\text{-loop}} = \text{PE} \left[ \frac{1-t^{-1}}{1-q} \bar{P}_\alpha^+ + \frac{q(t-1)}{1-q} P_\alpha^+ \right] . \tag{4.116}$$

As we argued in section 4.2.3 the  $A$ -twisted index reproduces the Hilbert series of the Coulomb branch  $\mathcal{M}_C$  and, in contrast to the  $B$ -twisted index, vortices now play a crucial

role. The terms that survive in the gluing of the one-loop pieces are given by

$$\mathcal{I}_\alpha^{A,1\text{-loop}}(q,t)\mathcal{I}_\alpha^{A,1\text{-loop}}(q^{-1},t) = \text{PE} \left[ (1-t^{-1})\bar{P}_\alpha^+ \right] \text{PE} \left[ (1-t)P_\alpha^+ \right]. \quad (4.117)$$

Similar to the discussion in the previous paragraph we can invert the weights in the first factor and use  $(P_\alpha^+)^V = \bar{P}_\alpha^-$  (at the expense of a power of  $t$ ) so that

$$\mathcal{I}_\alpha^{A,1\text{-loop}}(q,t)\mathcal{I}_\alpha^{A,1\text{-loop}}(q^{-1},t) = (-t)^{-\frac{1}{4}\dim(\mathcal{M}_H)} \text{PE} [(1-t)\mathcal{P}_\alpha]. \quad (4.118)$$

We conclude that the one-loop piece receives contributions from one half of the tangent bundle  $TX = \mathcal{P} + t^{-1}\mathcal{P}^V$  of the Higgs branch.

For the vortex contributions we are free to send  $q \rightarrow 0$  to recover a product of generating functions of ‘Poincaré polynomials’<sup>18</sup> of the quasimap moduli space. We denote the vortex contributions by

$$\mathcal{I}_\alpha^{A,\text{Vortex}}(q,t)\mathcal{I}_\alpha^{A,\text{Vortex}}(q^{-1},t) = P_{\text{QM}_\alpha}(\zeta;t)\overline{P_{\text{QM}_\alpha}(\zeta;t)}. \quad (4.119)$$

Now in some theories, for example  $T_\rho[SU(N)]$ <sup>19</sup> as we saw explicitly in section 4.2.3, the Poincaré polynomials do not depend on the vacuum  $\alpha$ . In that case we have the following:

**Lemma 5.** The vacuum dependent part of the sum (4.110) for the  $A$ -twisted index of a theory in the class  $T_\rho[SU(N)]$  arises only from the one loop contributions. Then, the sum over vacua gives the Poincaré polynomial of the compact core  $\mathcal{L}_H = \pi^{-1}(0) \subset \mathcal{M}_H$  of the Higgs branch. In summary, the one-loop contribution to the Coulomb branch Hilbert series is

$$\sum_{\alpha \in X^A} \text{PE} [(1-t)\mathcal{P}_\alpha] = P_{\mathcal{L}_H}(t). \quad (4.120)$$

*Proof.* The summand on the left hand side is equation (4.118). When the Higgs branch of a theory is a cotangent bundle then  $\mathcal{P}_\alpha$  is the tangent bundle of the core  $\mathcal{L}_H = \pi^{-1}(0) \subset \mathcal{M}_H$  evaluated at a fixed point  $\alpha$  and then the sum over vacua is the localisation formula (3.30) for the  $\chi_t$  genus of  $\mathcal{L}_H$ . Since this space is compact, the arguments of section 3.1.4 ensure that it is independent of the flavour fugacities and coincides with the Poincaré polynomial.  $\square$

In conclusion, for theories where the Poincaré polynomials of the quasimap moduli spaces do not depend on the vacuum  $\alpha$ , then we have the following formula for the Coulomb

<sup>18</sup>This is strictly only a polynomial when the moduli space is a smooth variety but it is always true, from the localisation formula (4.9), that the vertex functions have equal powers of  $q$  in the numerators and denominators so that this limit is well-defined. In the absence of a better word for the limit, we call it again a Poincaré polynomial.

<sup>19</sup>These are theories whose Higgs branch is a *partial* flag variety.

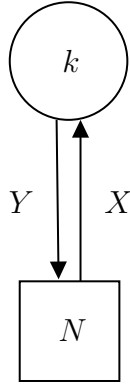


Figure 4.3: SQCD[ $k, N$ ] quiver diagram.

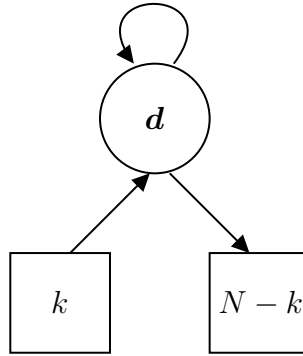


Figure 4.4: Vortex moduli space of SQCD[ $k, N$ ].

branch Hilbert series

$$\text{H.S.}[\mathcal{M}_C] = P_{\mathcal{L}_H}(t) P_{\text{QM}}(\zeta; t) \overline{P_{\text{QM}}(\zeta; t)}. \quad (4.121)$$

where the conjugation inverts  $\zeta$ . This is a generalisation of the  $T[SU(N)]$  result (4.70). We demonstrate this explicitly for the SQCD[ $k, N$ ] example in the following subsection.

In general, some vacuum dependence remains in the Poincaré polynomial. In that case we still have an interpretation of the Coulomb branch Hilbert series in terms of vortex geometry but a non-trivial sum over vacua remains—we see this in the ADHM example of chapter 5.

### 4.3.4 Examples

In this section we focus on the  $A$ -twist since, by the general arguments above, the  $B$ -twist manifestly gives the Higgs branch Hilbert series expressed as a sum over Higgs branch fixed points. The Coulomb branch side is more interesting.

**Example: SQCD[ $k, N$ ].** Supersymmetric QCD with  $k$  colours and  $N$  flavours is a quiver gauge theory described by the quiver in figure 4.3. It has gauge group  $G = U(k)$

and  $N$  hypermultiplets. The Higgs branch can be identified with  $\mathcal{M}_H = T^*\text{Gr}(k, N)$ , the cotangent bundle to the Grassmanian of  $k$ -planes in  $W = \mathbb{C}^N$ .

The torus action  $\mathbf{A}$  has  $\frac{N!}{(N-k)!k!}$  isolated fixed points labelled by injective maps  $\alpha : \{1, \dots, k\} \rightarrow \{1, \dots, N\}$  which, much like the  $T[SU(N)]$  example of section 4.1.4, can be identified with a choice of  $k$ -subset of a fixed basis  $\{e_1, \dots, e_N\}$  of  $W$ .

Following similar arguments to section 4.2.2 the relevant vortex moduli space, here denoted  $\mathfrak{Q}_d^{k,N}$ , describes maps  $\mathbb{P}^1 \rightarrow \text{Gr}(k, N)$  and can be realised by the moduli space of sheaves  $\mathcal{V}$  on  $\mathbb{P}^1$  satisfying

$$0 \subset \mathcal{V} \subset W \otimes \mathcal{O}_{\mathbb{P}^1}, \quad (4.122)$$

with  $\text{rank } \mathcal{V} = k$  and  $\deg \mathcal{V} = d$ . Nakajima [152] realises this moduli space as the handsaw quiver of figure 4.4. The torus  $\mathbf{A}$  acts by permuting the flavour nodes, that is selecting a  $k$ -subset of the flavour fugacities  $\{x_1, \dots, x_N\}$ . The vertex function coincides with the generating function of  $\chi_t$  genera of the Laumon space, and in the  $A$ -shift convention we have

$$\begin{aligned} V_\alpha(z, \zeta; q, t \rightarrow qt) &= \sum_{d \geq 0} \zeta^d t^{-\frac{1}{2} \dim \mathfrak{Q}_d^{k,N}} \chi_t(\mathfrak{Q}_{d,\alpha}^{k,N}) \\ &\stackrel{q \rightarrow 0}{=} \sum_{d \geq 0} \left( t^{-N/2} \zeta \right)^d P_{\mathfrak{Q}_d^{k,N}}(t) \\ &= \prod_{i=1}^k \frac{1}{1 - \zeta t^{-N/2+i-1}}, \end{aligned} \quad (4.123)$$

where in the second line we have used equation (3.43). Crucially this relies on the theory being ‘good’ or ‘ugly’ so that the handsaw quiver is a resolution with a compact core—in the present example we require  $N \geq 2k$ —then we can use Nakajima’s formula [152] for the generating function of the Poincaré polynomials of the handsaw quiver in figure 4.4. We can take  $q \rightarrow 0$  similarly for the conjugate block using (3.33) which sends<sup>20</sup>  $t \rightarrow t^{-1}$ . The theory is cotangent type so we can apply lemma 5 to understand the 1-loop contributions. The Poincaré polynomial of  $\text{Gr}(k, N)$  can be computed using the localisation methods discussed in section 3.1.4 and is given by

$$P_{\text{Gr}(k,N)}(t) = \frac{\prod_{i=1}^N (1 - t^i)}{\prod_{i=1}^k (1 - t^i) \prod_{i=1}^{N-k} (1 - t^i)}. \quad (4.124)$$

Finally, combining with the classical contributions, we have

$$\begin{aligned} I^A &= P_{\mathcal{L}_H}(t) P_{\text{QM}}(\zeta; t) \overline{P_{\text{QM}}(\zeta; t)} \\ &= \frac{\prod_{i=1}^N (1 - t^i)}{\prod_{i=1}^k (1 - t^i) \prod_{i=1}^{N-k} (1 - t^i)} \prod_{i=1}^k \frac{1}{1 - \zeta t^{-N/2+i-1}} \prod_{i=1}^k \frac{1}{1 - \zeta^{-1} t^{-N/2+i-1}}. \end{aligned} \quad (4.125)$$

---

<sup>20</sup>The block gluing is  $q$  to  $q^{-1}$  but amusingly after taking the limit  $q \rightarrow q^{-1}$  this manifests as a  $t \rightarrow t^{-1}$  gluing; when we finally combine with the classical pieces the conjugation is  $\zeta \rightarrow \zeta^{-1}$ !

This matches the Coulomb branch Hilbert series of this theory as computed using the monopole formula in equation (5.3) of [77]. We have given a geometric interpretation of this Coulomb branch Hilbert series purely in terms of the *Higgs branch* geometry.

**Example:**  $T[SU(N)]$ . We have already studied the  $A$ -twist of  $T[SU(N)]$  in detail above. We remark here only that we have now solved the ‘perturbative ambiguity’ problem we discussed towards the end of section 4.2.1 and we demonstrate the ‘correct’ factorisation of the one loop contributions.

Factorising the  $A$ -twisted index we found the perturbative contribution in a vacuum  $\sigma \in S_N$  is given by

$$I_{1\text{-loop}}^{A,\sigma} = \prod_{i < j}^N \frac{1 - tx_{\sigma(i)}/x_{\sigma(j)}}{1 - x_{\sigma(i)}/x_{\sigma(j)}}. \quad (4.126)$$

Lemma 5 shows us that we should recognise (4.126) as the plethystic exponential of the polarisation bundle of  $T^*\mathcal{B}$ , which, up to an overall power of  $t$ , can be factorised into attracting and repelling directions as in (4.117)

$$I_{1\text{-loop}}^{A,\sigma} = \mathcal{I}_{1\text{-loop}}^{A,\sigma}(q, t) \mathcal{I}_{1\text{-loop}}^{A,\sigma}(q^{-1}, t) = \text{PE}[(1-t)\mathcal{P}] = \text{PE}[(1-t)P_\alpha^+] \text{PE}[(1-t^{-1})\bar{P}_\alpha^+]. \quad (4.127)$$

For the  $T[SU(N)]$  theory the factorisation is given explicitly by<sup>21</sup>

$$P_\sigma^+ = \sum_{\substack{i < j \\ \sigma(j) < \sigma(i)}} \frac{x_{\sigma(j)}}{x_{\sigma(i)}}, \quad \bar{P}_\sigma^+ = \sum_{\substack{i < j \\ \sigma(j) > \sigma(i)}} \frac{x_{\sigma(i)}}{x_{\sigma(j)}}. \quad (4.128)$$

Lemma 4 is then lemma 5 in disguise, this is a geometric version of Macdonald’s proof and gives the Poincaré polynomial of the flag variety  $\mathcal{B}$ .

**Remark.** We conclude this chapter with some remarks about the three sphere partition function and offer a preview of the following section. Whilst in this thesis we focus on the twisted index gluing, the hemisphere blocks in definition 4.1 also fuse exactly to other three manifold partitions functions. In particular, the gluing appropriate for the  $S^3$  partition function [170] in the  $\mathcal{N} = 4$  limit can be obtained by gluing the  $t \rightarrow 1$  limit of an  $A$ -shifted block together with the  $t \rightarrow 1$  limit of a  $B$ -shifted block. Focusing on the one-loop and vortex contributions we have

$$\begin{aligned} \mathcal{Z}_{S^3} &\sim \sum_{\sigma \in S_N} \left[ \lim_{t \rightarrow 1} \mathcal{I}^{A,\sigma} \right] \left[ \lim_{t \rightarrow 1} \mathcal{I}^{B,\sigma} \right] \\ &\sim \sum_{\sigma \in S_N} \left[ \lim_{t \rightarrow 1} \mathbb{V}(z, \zeta; q, t) \right] \left[ \text{PE}[N_\alpha^+] \right]. \end{aligned} \quad (4.129)$$

---

<sup>21</sup>This can be deduced from the tangent bundle formula in lemma 1 and the discussion of the fixed points in section 4.1.4.

The first limit counts the fixed points on the quasimap moduli space and the second limit provides the generating function—we revisit these ideas in more detail in the following section. Sending  $t \rightarrow 1$  in (4.46) we find a geometric series that can be resummed. Combining with (4.128) for the attracting directions  $N_\alpha^+ = P_\alpha^+ + t\bar{P}_\alpha^+$  we have

$$\mathcal{Z}_{S^3} \sim \sum_{s \in S_N} \prod_{i < j}^N \frac{1}{1 - z_i/z_j} \frac{1}{1 - \zeta_i/\zeta_j}. \quad (4.130)$$

This recovers the proposed  $S^3$  partition function of  $T[G]$  [208, 209, 210]. This calculation is in general more interesting for non mirror self-dual theories—we see precisely how contributions from both the Higgs branch and the Coulomb branch contribute to the three sphere partition function. Further we note that (4.130) are the ‘Verma denominators’ of  $\mathfrak{g} = \mathfrak{sl}_N$ . Understanding the appearance of Verma characters here was the motivation for the work in the following section where we show that, in general, specialised  $A$ - and  $B$ -shifted hemisphere blocks realise Verma modules of Higgs and Coulomb branch chiral algebras respectively.

## 4.4 Representation theory

In this section we explore representation theoretic aspects of the hemisphere blocks. We discuss how the exceptional Dirichlet boundary conditions yield lowest weight Verma modules of the quantised bulk Higgs and Coulomb chiral rings and explore the implications of 3d mirror symmetry in this context. We demonstrate these ideas explicitly with the SQED[ $N$ ] example.

### 4.4.1 Exceptional Dirichlet and Verma modules

We deform the theory by introducing an  $\Omega$ -background that quantises the ring of bulk chiral operators to form a non-commutative algebra. There is an action of the bulk chiral ring on the operators surviving at the boundary and in the particular case of exceptional Dirichlet, these modules are lowest weight Verma modules [55]. We begin with a review of the  $\Omega$ -background and the module structure.

**$\Omega$  deformations.** 3d  $\mathcal{N} = 4$  theories on  $\mathbb{R}^2 \times \mathbb{R}^{\geq 0}$  admit two omega deformations, denoted  $\Omega_A$  and  $\Omega_B$ , associated to a fixed axis of rotation in  $\mathbb{R}^2 \times \mathbb{R}^{\geq 0}$ . The deformations can be realised in a number of ways—for example as a twisted mass deformation in 3d as in [55] or as dimensional reduction of the Seiberg-Witten-type setup of [19]. However they are realised, the  $\Omega$ -backgrounds deform the Higgs and Coulomb chiral ring supercharges



(2.13) so that<sup>22</sup>  $Q_H^2 = \epsilon \mathcal{L}_V$  and  $Q_C^2 = \epsilon \mathcal{L}_V$  respectively, where  $\mathcal{L}_V$  is the Lie derivative associated to rotations in the  $x^{1,2}$  plane. The  $\Omega$ -deformations are constructed explicitly in the [211, 212] and the same algebras can also be realised by working with a ‘ $Q + S$ ’ cohomology construction as in the works [213, 214, 85, 215]. We take the former perspective in this thesis.

Physically, the local operator count of the previous subsection is unaffected but we now have a notion of operator ordering on the bulk chiral rings  $\mathbb{C}[\mathcal{M}_H]$  and  $\mathbb{C}[\mathcal{M}_C]$ —the cohomology of  $Q_H$  and  $Q_C$  now consists of operators constrained to the line  $x^1 = x^2 = 0$ . We denote these non-commutative rings by  $\hat{\mathcal{R}}_H$  and  $\hat{\mathcal{R}}_C$  respectively.

**Deformation quantisation.** We suppose  $\mathcal{M}_H$  and  $\mathcal{M}_C$  are a dual pair of symplectic resolutions. In that case the quantised chiral rings are quantisations of the coordinate rings  $\hat{\mathcal{R}}_H = \hat{\mathbb{C}}[\mathcal{M}_H]$  and  $\hat{\mathcal{R}}_C = \hat{\mathbb{C}}[\mathcal{M}_C]$ . The  $\Omega$ -deformations can then be understood from the perspective of the geometry of symplectic resolutions [53, 52]—we follow closely the presentation of [216].

We denote by  $\mathcal{A}$  either of the coordinate rings  $\mathbb{C}[\mathcal{M}_H]$  or  $\mathbb{C}[\mathcal{M}_C]$ . Recall from section 2.2 that the coordinate rings are Poisson algebras graded, with degree  $-2$ , by the  $\mathbb{C}_t^\times$  action—we write  $\mathcal{A} = \bigoplus_i \mathcal{A}_i$ . A *deformation quantisation* of a Poisson algebra  $\mathcal{A}$  consists of an associative product  $\star$  on formal power series  $\mathcal{A}[[\epsilon]]$

$$f \star g = \sum_{k=0}^{\infty} C_k(f, g) \epsilon^k, \quad (4.131)$$

and ask that to leading order  $\star$  agrees with the original commutative product of  $\mathcal{A}$ . Furthermore, we impose that  $\star$  is compatible with the Poisson structure<sup>23</sup> of  $\mathcal{A}$  in the sense that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [f, g] = \{f, g\}, \quad (4.132)$$

where the commutator on the left is defined by  $[f, g] := f \star g - g \star f$ . We further ask that the deformation quantisation respects the  $\mathbb{C}_t^\times$  grading so that  $C_k(\mathcal{A}_i, \mathcal{A}_j) \subset \mathcal{A}_{i+j-2k}$  which in particular implies that the products (4.131) are polynomials in  $\epsilon$ . Deformation quantisations for  $\mathcal{M}_H$  or  $\mathcal{M}_C$  can be classified [53, 52] and they are determined uniquely by specifying an element of  $H^2(\mathcal{M}_H, \mathbb{C})$  or  $H^2(\mathcal{M}_C, \mathbb{C})$  respectively. As we saw in section 3.1.1, these homology groups can be identified with complex masses and FI parameters. The upshot is that the quantised Higgs and Coulomb branch algebras  $\hat{\mathcal{R}}_H$  and  $\hat{\mathcal{R}}_C$  are determined uniquely by  $m_{\mathbb{C}}$  and  $\xi_{\mathbb{C}}$ . Often these algebras can be realised by the process of quantum symplectic reduction [217] and in some cases the operator product can also

<sup>22</sup>Masses and FI parameters generating flavour symmetries and gauge transformations can also enter the right hand side of these equations.

<sup>23</sup>Associativity ensures that the leading part of  $[f, g]$  is a well-defined Poisson bracket.

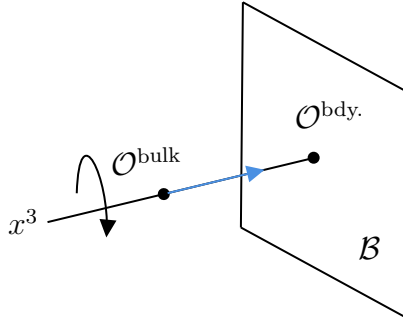


Figure 4.5:  $\mathbb{R}^2 \times \mathbb{R}^{\geq 0}$  with the bulk-boundary action illustrated.

be verified with localisation computations [85]. We construct the non-commutative chiral rings for SQED[ $N$ ] explicitly in the following section.

**Modules.** Bringing bulk elements of  $\hat{\mathcal{R}}_H$  or  $\hat{\mathcal{R}}_C$  to the boundary defines an action of the bulk chiral algebras on local operators inserted at  $z = 0$  that satisfy the boundary condition, we illustrate this action schematically in figure 4.5. The action of  $\hat{\mathcal{R}}_H$  and  $\hat{\mathcal{R}}_C$  on boundary local operators is described in detail in [55]. The Higgs branch action is purely classical and the module structure can be defined as a quotient of the bulk algebra. The Coulomb branch on the other hand receives quantum corrections and an ansatz has to be made for the action of bulk monopole operators. In this section we focus on the half index operator count and show that, in certain specialised limits, we can recover characters of the modules detailed in the work [55].

Both of the algebras  $\hat{\mathcal{R}}_H$  and  $\hat{\mathcal{R}}_C$  contain operators  $J_H$  and  $J_C$  that measure  $T_H$  or  $T_C$  charge [55]. For example, in the case of Higgs operators

$$[J_H, \mathcal{O}_\gamma] = \gamma \mathcal{O}_\gamma, \quad (4.133)$$

where  $\gamma$  lives in the character lattice  $\Lambda_H^\vee$ . This grades the algebras  $\hat{\mathcal{R}}_H$  and  $\hat{\mathcal{R}}_C$  by the character lattices  $\Lambda_H^\vee$  and  $\Lambda_C^\vee$  and ensures that modules generated by  $T_H$  or  $T_C$  preserving boundary conditions in the sense above have a weight decomposition.

Consider, in the presence of real masses and FI parameters, the operators  $J_m = m \cdot J_H$  and  $J_\xi = \xi \cdot J_C$ . Boundary conditions compatible with  $m$  and  $\xi$ , in fixed chambers  $\mathfrak{C}_H$  and  $\mathfrak{C}_C$ , are lowest weight modules for  $J_\xi$  and  $J_m$ . In particular, the exceptional Dirichlet boundary conditions  $\mathcal{B}_\alpha$  associated to vacua are expected to yield lowest weight Verma modules for  $\hat{\mathcal{R}}_H$  and  $\hat{\mathcal{R}}_C$ .

**Example 4.2.** A toy example is a free hypermultiplet  $(X, Y)$ . In this case the classical Poisson bracket on the chiral ring is  $\{X, Y\} = 1$  which we canonically quantise by  $[\hat{Y}, \hat{X}] = \epsilon$ . Consider the boundary condition  $\mathcal{B}_X$  given by  $\partial_\perp X|_\partial = 0$  and  $Y|_\partial = 0$ , this preserves  $T_H = U(1)$ . The module has lowest weight state  $|0\rangle$  satisfying  $\hat{Y}|0\rangle = 0$  and is generated

by  $|n\rangle = \hat{X}^n|0\rangle$ .

The operator  $J_H$  is given by the normal ordered flavour symmetry moment map  $J_H = \frac{1}{\epsilon}\hat{X}\hat{Y} + \frac{1}{2}$  which gives

$$J_H|n\rangle = \left(n + \frac{1}{2}\right)|n\rangle. \quad (4.134)$$

Intuitively one can view this construction as quantising the Higgs branch  $\mathbb{C}^2$  as a phase space by selecting canonical momenta  $p = \hat{Y}$  and position  $x = \hat{X}$  and building polynomial wave functions.

#### 4.4.2 Specialised limits and mirror symmetry

We now consider two supersymmetry enhancing limits of the half index. We call these limits the  $B$ -limit which corresponds to  $t \rightarrow q^{-\frac{1}{2}}$  and the  $A$ -limit which corresponds to sending  $t \rightarrow q^{\frac{1}{2}}$ . Alternatively, these limits correspond to specialising  $t \rightarrow 1$  in either the  $A$ -shifted or  $B$ -shifted indices defined in (4.94).

**$B$ -limit.** In the  $B$ -limit,  $t \rightarrow q^{-\frac{1}{2}}$ , the generator  $J + \frac{R_A}{2}$  that now grades the index commutes with an additional supercharge  $Q_-^{12}$ . By the unitarity bound arguments of section 2.1.3 we then have that the half index receives contributions from the (bosonic) Higgs chiral ring operators with  $D - \frac{R_V}{2} = 0$  and  $J + \frac{R_A}{2} = 0$  so that the indices are independent of  $q$  and  $\xi$  (Higgs operators are uncharged under  $T_C$ ). We write

$$\hat{\mathcal{I}}^B(\mathcal{B}) := \lim_{t \rightarrow q^{-\frac{1}{2}}} \mathcal{I} = \text{Tr}_{\mathcal{H}_B} x^{F_H}. \quad (4.135)$$

In the limit only the one-loop contributions from (4.106) survive.

**Example 4.3.** The  $B$ -limit of SQED[ $N$ ] is obtained by sending  $t \rightarrow q^{-\frac{1}{2}}$  in (4.97)—only the one-loop piece survives and we have

$$\hat{\mathcal{I}}_\alpha^B = \prod_{j=1}^{\alpha-1} \frac{1}{1 - x_\alpha/x_j} \prod_{j=\alpha+1}^N \frac{1}{1 - x_j/x_\alpha}. \quad (4.136)$$

We note that this limit respects the chamber structure  $\mathfrak{C}_H$  in that it is convergent for  $\{x\}$  with respect to the chamber choice.

**$A$ -limit.** The  $A$ -limit corresponds to setting  $t \rightarrow q^{\frac{1}{2}}$ . In the  $A$ -limit we receive contributions only from Coulomb branch operators  $D - \frac{R_A}{2} = 0$  that are uncharged under  $T_H$  and satisfy  $J + \frac{R_V}{2} = 0$  so that

$$\hat{\mathcal{I}}^A(\mathcal{B}) := \lim_{t \rightarrow q^{\frac{1}{2}}} \mathcal{I} = \text{Tr}_{\mathcal{H}_A} \xi^{F_C}. \quad (4.137)$$

In this case only the non-perturbative terms of (4.108) survive.

**Example 4.4.** The  $A$ -limit of  $\text{SQED}[N]$  is obtained by sending  $t \rightarrow q^{\frac{1}{2}}$  in (4.97)—we find

$$\hat{\mathcal{I}}_\alpha^A = \sum_{m \geq 0} \zeta^m = \frac{1}{1 - \zeta}. \quad (4.138)$$

**Casimir energy and highest weights.** We denote the  $A$ - and  $B$ -limits of the Casimir energy term similarly by  $\hat{\phi}_\alpha^A$  and  $\hat{\phi}_\alpha^B$  respectively. The limits above count local boundary operators in modules of  $\hat{\mathcal{R}}_H$  and  $\hat{\mathcal{R}}_C$  respectively. By the general theory reviewed above, these modules are lowest weight Verma modules. We find that the Casimir energy term (4.107) plays a crucial role and realises the lowest weight of these modules so that

$$\mathcal{X}^H(\mathcal{B}_\alpha) = e^{\phi_\alpha^B} \hat{\mathcal{I}}_\alpha^B = \lim_{t \rightarrow 1} H_\alpha^B, \quad \mathcal{X}^C(\mathcal{B}_\alpha) = e^{\phi_\alpha^A} \hat{\mathcal{I}}_\alpha^A = \lim_{t \rightarrow 1} H_\alpha^A. \quad (4.139)$$

We check this explicitly for  $\text{SQED}[N]$  in the following section, and later in chapter 5 for a more complicated non-abelian theory.

Combined with the notion of factorisation, these arguments lead to formulae for closed three manifold partition functions as sums over Verma characters

$$\mathcal{Z}_{\mathcal{M}_3} = \sum_\alpha \mathcal{X}_\alpha^{H/C} \bar{\mathcal{X}}_\alpha^{H/C}, \quad (4.140)$$

where the conjugation corresponds to the particular gluing as reviewed in section 4.1.3. Formulae for the  $S^3$  partition function were recently derived from factorising pole contributions of the specialised Coulomb branch localisation integral by Okazaki and Gaiotto in [208], the present work explains this observation in terms of modules induced by boundary conditions.

**3d mirror symmetry.** We denote by  $\mathcal{T}$  and  $\mathcal{T}'$  mirror dual theories with mirror dual boundary conditions  $\mathcal{B}$  and  $\tilde{\mathcal{B}}$ . Mirror symmetry exchanges  $R_A \leftrightarrow R_V$  and  $F_H \leftrightarrow F_C$  and we expect  $A$ -shifted and  $B$ -shifted half indices of mirror dual theories should be related. In general, it should be possible to expand the mirror dual of an exceptional Dirichlet boundary condition linearly in the exceptional Dirichlet basis  $\mathcal{B}'$  of  $\mathcal{T}'$ , giving schematically

$$\begin{aligned} \mathcal{I}_\mathcal{T}(\mathcal{B}_\alpha) &= \mathcal{I}_{\mathcal{T}'}(\tilde{\mathcal{B}}_\alpha), \\ &= \sum_\beta U_{\beta\alpha} \mathcal{I}_{\mathcal{T}'}(\mathcal{B}'_\beta). \end{aligned} \quad (4.141)$$

Similar transformations appear in the symplectic duality literature [92] where  $U_{\alpha\beta}$  can be realised as the upper-triangular<sup>24</sup> elliptic stable envelopes, in our context this is a basis change of boundary conditions. We find, in examples, that in the two supersymmetry enhancing  $A$ - and  $B$ -limits discussed above the transformation  $U_{\alpha\beta}$  becomes diagonal. This implies that there is a straightforward map between boundary Higgs and Coulomb branch operators in two mirror dual theories. We expect in general

$$\hat{\mathcal{I}}^A(\mathcal{B}_\alpha) = \hat{\mathcal{I}}^B(\mathcal{B}_\alpha), \quad (4.142)$$

up to the appropriate mirror dual parameter swaps  $z \rightarrow \zeta$  and  $t \rightarrow t^{-1}$  and an appropriate matching of isolated vacua/fixed points in the two theories.

The geometric proposal in definition 4.1 implies that the  $A$ - and  $B$ -limits of the half index of a theory  $\mathcal{T}$  receive contributions from only the one-loop or the vortex contributions respectively. In the latter case, the vertex functions degenerate to a sum over the fixed points of the vortex moduli space. The statement of mirror symmetry is then that the generating function of fixed points of the quasimap moduli space based at  $\alpha$  is given by the plethystic exponential of the attracting directions at  $\alpha$  of the Higgs branch of the mirror theory.<sup>25</sup>

**Proposition 3.** Let  $\mathcal{T}$  and  $\tilde{\mathcal{T}}$  be a pair of ‘good’ mirror dual 3d quiver  $\mathcal{N} = 4$  gauge theories that admit generic mass and FI parameters to give fully resolved Higgs and Coulomb branches  $\mathcal{M}_H$  and  $\mathcal{M}_C$  with isolated fixed points. Then

$$\hat{\mathcal{I}}_\alpha^A(\mathcal{T}) = \text{PE} \left[ N_\alpha^+(\mathcal{M}_C) \right] = \sum_d \zeta^d \{ \# \text{fixed points } \text{QM}_\alpha^d(\mathcal{M}_H) \} = \hat{\mathcal{I}}_\alpha^B(\tilde{\mathcal{T}}). \quad (4.143)$$

In the above proposition the  $\mathfrak{C}_H$  chamber dependence is manifest on the left hand side as it determines the attracting directions at a fixed point. For the right hand side: recall from the discussion in section 4.1.1 that the stability condition of the Nakajima quiver variety (which is equivalent to the chamber of FI parameters) determines the effective degrees, hence the right hand side will yield different vertex functions in different chambers  $\mathfrak{C}_C$  *i.e.* the chamber dependence is also to be matched across this equality. The left hand side of (4.143) is a generating function of the right hand side and so this identity can generate interesting combinatorial formulae. We discuss an example of this in the following chapter where the Higgs branch is the Hilbert scheme of points in the plane and we find generating functions of reverse plane partitions.

**Example 4.5.** Consider the  $T[SU(N)]$  theory of section 4.1.4 with fixed chambers  $\mathfrak{C}_H = \{m_1 < \dots < m_N\}$  and  $\mathfrak{C}_C = \{\xi_1 < \dots < \xi_N\}$ . Recall that  $T[SU(N)]$  is mirror self dual.

<sup>24</sup>With respect to the ordering associated to vacua arising from the moment map in section 3.35.

<sup>25</sup>In this identification we make the usual mirror symmetry transformation  $\zeta \leftrightarrow z$ .

The vacua are described by  $\sigma \in S_N$  and the relevant contributions to the attracting directions are given by<sup>26</sup>

$$P_\sigma^+ = \sum_{\substack{i < j \\ \sigma(j) < \sigma(i)}} \frac{x_{\sigma(j)}}{x_{\sigma(i)}}, \quad \bar{P}_\sigma^+ = \sum_{\substack{i < j \\ \sigma(j) > \sigma(i)}} \frac{x_{\sigma(i)}}{x_{\sigma(j)}}, \quad (4.144)$$

hence the  $B$ -limit gives

$$\hat{\mathcal{I}}^B = \text{PE}[P_\sigma^+ + \bar{P}_\sigma^+] = \prod_{i < j}^N \frac{1}{1 - x_i/x_j}. \quad (4.145)$$

On the other hand, in the  $A$ -limit of the mirror dual, the vortex sum degenerates and we find

$$\hat{\mathcal{I}}^A = \sum_{\{d_a^{(s)}\}} \prod_{s=1}^{N-1} \left( \frac{\zeta_s}{\zeta_{s+1}} \right)^{d_s}. \quad (4.146)$$

This is a geometric series that sums to (4.145). We see that the identity in proposition 3 is verified after appropriate identification of masses and FI parameters.

In this case there is a special feature in that the generating functions are vacuum independent (*i.e.* the number of fixed points of each based quasimap space is the same)—this is not always the case, and the theory we study in the following chapter does not have this property.

### 4.4.3 SQED[ $N$ ] example

In this subsection we discuss the quantisations of the Higgs and Coulomb branch algebras for SQED[ $N$ ] and compute the corresponding Verma module characters. We show that they reproduce the specialised limits of the half index and therefore verify (4.139). Throughout we work in fixed chambers for the real mass and real FI parameters given by

$$\mathfrak{C}_H = \{m_1 < \dots < m_N\}, \quad \mathfrak{C}_C = \{\xi > 0\}. \quad (4.147)$$

**Higgs branch characters.** We compute the quantised Higgs branch algebra for SQED[ $N$ ] by quantum symplectic reduction.  $\hat{\mathcal{R}}_H$  is generated by  $N$  copies of the Heisenberg algebra

$$[\hat{X}_i, \hat{Y}_j] = \epsilon \delta_{ij}, \quad i = 1, \dots, N. \quad (4.148)$$

We then restrict to gauge invariant combinations and imposing the, normal ordered, moment map constraint  $\sum_{i=1}^N :\hat{X}_i \hat{Y}_i: = \xi_{\mathbb{C}}$  where the normal ordering is defined by  $:\hat{X}_i \hat{Y}_i: =$

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<sup>26</sup>This can be deduced from the tangent space formula in lemma 1.

$\hat{X}_i \hat{Y}_i + \frac{\epsilon}{2} = \hat{Y}_i \hat{X}_i - \frac{\epsilon}{2}$  we have that the algebra is generated by

$$\begin{aligned} h_j &= \hat{X}_j \hat{Y}_j - \hat{X}_{j+1} \hat{Y}_{j+1}, \\ e_j &= \hat{X}_j \hat{Y}_{j+1}, \\ f_j &= \hat{X}_{j+1} \hat{Y}_j, \end{aligned} \tag{4.149}$$

for  $j = 1, \dots, N-1$ . The relations (4.148) imply that these generators satisfy the usual Chevalley-Serre relations for  $\mathfrak{sl}_N$  including

$$[e_i, f_j] = \epsilon \delta_{ij} h_j, \quad [h_i, e_j] = \epsilon A_{ij} e_j, \quad [h_i, f_j] = -\epsilon A_{ij} f_j, \tag{4.150}$$

where  $A_{ij}$  is the Cartan matrix of  $\mathfrak{sl}_N$ . Note that this algebra quantises the Higgs branch algebra discussed in section 2.3. The generator  $J_m$  associated to the global symmetry  $U(1)_m \subset T_H$  specified by the real mass parameters is given by

$$J_m = \sum_{i=1}^N m_i : \hat{X}_i \hat{Y}_i :. \tag{4.151}$$

Now setting  $e_{ij} = \hat{X}_i \hat{Y}_j$  for  $i < j$  and  $f_{ij} = \hat{X}_j \hat{Y}_i$  for  $i > j$  we see

$$\begin{aligned} [J_m, e_{ij}] &= \epsilon(m_i - m_j) e_{ij} \quad \text{for } i < j, \\ [J_m, f_{ij}] &= \epsilon(m_i - m_j) f_{ij} \quad \text{for } i > j, \end{aligned} \tag{4.152}$$

hence  $e_{ij}$  and  $f_{ij}$  are lowering and raising operators respectively. The module associated to  $\mathcal{B}_\alpha$  is generated by acting with the raising operators on the vacuum state  $|\mathcal{B}_\alpha\rangle$  which satisfies

$$\begin{aligned} \hat{X}_i |\mathcal{B}_\alpha\rangle &= \delta_{i\alpha} c |\mathcal{B}_\alpha\rangle \quad \text{for } i = 1, \dots, \alpha, \\ \hat{Y}_i |\mathcal{B}_\alpha\rangle &= 0 \quad \text{for } i = \alpha + 1, \dots, N. \end{aligned} \tag{4.153}$$

Using (4.151) we can compute the weight of the vacuum state

$$J_m |\mathcal{B}_\alpha\rangle = \left[ \frac{\epsilon}{2} \left( \sum_{j>\alpha} m_j - \sum_{j<\alpha} m_j \right) + \left( \xi_{\mathbb{C}} - \frac{N-2\alpha+1}{2} \epsilon \right) m_\alpha \right] |\mathcal{B}_\alpha\rangle. \tag{4.154}$$

The operators that do not annihilate the vacuum state, namely  $f_{\alpha j}$  for  $j < \alpha$  and  $f_{j\alpha}$  for  $j > \alpha$ , generate the module. Hence, using the relations (4.152), the character is given by

$$\begin{aligned}
\mathcal{X}_\alpha^H &:= \text{Tr } e^{-\frac{J_m}{\epsilon}} \\
&= x_\alpha^{\frac{\xi_{\mathbb{C}}}{\epsilon}} \prod_{j < \alpha} \left( \frac{x_\alpha}{x_j} \right)^{\frac{1}{2}} \prod_{k > \alpha} \left( \frac{x_k}{x_\alpha} \right)^{\frac{1}{2}} \prod_{j < \alpha} \left( 1 + \frac{x_\alpha}{x_j} + \frac{x_\alpha^2}{x_j^2} + \dots \right) \prod_{k > \alpha} \left( 1 + \frac{x_k}{x_\alpha} + \frac{x_k^2}{x_\alpha^2} + \dots \right) \\
&= x_\alpha^{\frac{\xi_{\mathbb{C}}}{\epsilon}} \prod_{j < \alpha} \frac{\left( \frac{x_\alpha}{x_j} \right)^{\frac{1}{2}}}{1 - \frac{x_\alpha}{x_j}} \prod_{k > \alpha} \frac{\left( \frac{x_k}{x_\alpha} \right)^{\frac{1}{2}}}{1 - \frac{x_k}{x_\alpha}}.
\end{aligned} \tag{4.155}$$

Now we compute the  $B$ -limit of the Casimir energy to find

$$\hat{\phi}_\alpha^B = \frac{1}{2} \sum_{j < \alpha} \log \frac{x_\alpha}{x_j} + \frac{1}{2} \sum_{j > \alpha} \log \frac{x_j}{x_\alpha} + \frac{\log x_\alpha \log \xi}{\log q}, \tag{4.156}$$

Combining with the  $B$ -limit of the half index (4.97) and assuming the state operator correspondence of section 4.3.1 identifies parameters as

$$\xi_{\mathbb{C}} \leftrightarrow -\log \zeta, \quad \epsilon \leftrightarrow -\log q. \tag{4.157}$$

then we have

$$\mathcal{X}_\alpha^H = e^{\hat{\phi}_\alpha^B} \hat{\mathcal{I}}^B. \tag{4.158}$$

with the Casimir energy providing the lowest weight term.

**Coulomb branch characters.** The quantised Coulomb branch chiral ring of supersymmetric QED is generated by<sup>27</sup> the complex scalar  $\varphi$  and the monopole operators  $v^\pm$  subject to

$$\begin{aligned}
[\hat{\varphi}, \hat{v}_\pm] &= \pm \epsilon \hat{v}_\pm, \\
\hat{v}_+ \hat{v}_- &= \prod_{i=1}^N \left( \varphi + m_{i,\mathbb{C}} - \frac{\epsilon}{2} \right), \\
\hat{v}_- \hat{v}_+ &= \prod_{i=1}^N \left( \varphi + m_{i,\mathbb{C}} + \frac{\epsilon}{2} \right),
\end{aligned} \tag{4.159}$$

this is a deformation by complex mass parameters and a quantisation of the Coulomb branch algebra (2.54). The topological global symmetry generated by a real FI parameter  $\xi \in \mathbb{R}$  is generated by the operator  $J_\xi = -\xi \hat{\varphi}$  so that

$$[J_\xi, \hat{v}_\pm] = \mp \epsilon \xi \hat{v}_\pm. \tag{4.160}$$

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<sup>27</sup>See [55].



This means that in our chamber  $\mathfrak{C}_H = \{\xi > 0\}$ , the monopole operator  $\hat{v}_+$  is a lowering operator and  $\hat{v}_-$  is a raising operator with respect to  $J_\xi$ .

Modules of the algebra  $\mathcal{R}_C$  are generated by boundary states  $|\mathcal{B}_\alpha\rangle$  satisfying

$$\left(\hat{\varphi} + m_{\alpha,\mathbb{C}} + \frac{\epsilon}{2}\right)|\mathcal{B}_\alpha\rangle = 0, \quad \hat{v}_+|\mathcal{B}_\alpha\rangle = 0. \quad (4.161)$$

These expressions arise from the analysis of boundary monopole operators performed in the work [55]. The boundary condition therefore generates a lowest weight Verma module by acting with  $\hat{v}_-$ . We compute the character of this module

$$\begin{aligned} \text{Tr } e^{-\frac{J_\xi}{\epsilon}} &= \xi^{\frac{m_{\alpha,\mathbb{C}}}{\epsilon} + \frac{1}{2}} (1 + \xi + \xi^2 + \dots) \\ &= \xi^{\frac{m_{\alpha,\mathbb{C}}}{\epsilon}} \frac{\xi^{\frac{1}{2}}}{1 - \xi}, \end{aligned} \quad (4.162)$$

and see that this converges for our chamber choice  $\mathfrak{C}_C$ . Now taking the  $A$ -limit of the Casimir energy (4.98) we have

$$\hat{\phi}_\alpha^A = \frac{1}{2} \log \xi + \frac{1}{2} \frac{\log x_\alpha \log \xi}{\log q}, \quad (4.163)$$

so that, comparing with the  $A$ -limit of the index (4.97), we have the identification

$$\mathcal{X}_\alpha^C = e^{\hat{\phi}_\alpha^A} \hat{\mathcal{I}}^B. \quad (4.164)$$



# CHAPTER 5

## 3D ADHM QUIVER GAUGE THEORY

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This chapter is an extended example of some of the ideas we have met throughout the thesis. We study a particularly rich non-abelian theory with adjoint matter that we refer to as the *3d ADHM quiver gauge theory*.

We compute the hemisphere blocks defined in the previous chapter and show that they glue correctly to the twisted indices—this is an important check on definition 4.1. We find that the representation theoretic content of the blocks is more interesting than the simple Lie algebras studied thus far in the cotangent type examples. We also study the half index of *Neumann boundary conditions*. In the ADHM example we find interesting connections to the geometric results on handsaw quiver varieties that we studied in chapter 3.

The theory has deep connections to the enumerative geometry of Hilbert schemes of points in the plane; in this chapter we turn the correspondence between physics and geometry around and use 3d mirror symmetry to predict a combinatorial expression for the ‘Poincaré polynomial’ of the relevant quasimap space.

One of the main motivations for studying the 3d ADHM quiver gauge theory is that it flows in the IR to a theory with an  $\text{AdS}_4$  holographic dual description. This chapter lays the ground work for understanding geometric and representation theoretic aspects of this correspondence and we conclude with some further directions related to  $\text{AdS}_4$  black hole entropy.

**Overview.** We begin in section 5.1 with background material relating the theory to the geometry of the Hilbert scheme of points in the plane. In section 5.2 we compute the hemisphere blocks introduced in the previous chapter using the prescription of definition 4.1. In the first new result of the chapter we show that these blocks compute the Verma characters of the Coulomb branch algebra  $\hat{\mathcal{R}}_C$  and provide a check of (a refined version of) the 3d mirror symmetry identity of proposition 3. In the second result of the chapter we compute the twisted index in section 5.4—we argue that for the ADHM theory there is some extra structure and the twisted index gluing coincides with gluing vertices in topological string theory. In section 5.3 we consider Neumann boundary conditions and

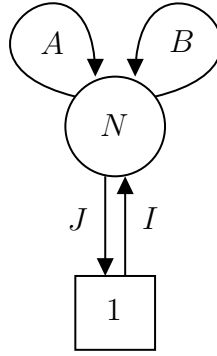


Figure 5.1: Quiver description of  $\text{Hilb}^N(\mathbb{C}^2)$ .

find a new interesting connection to the Hanany-Tong moduli space of chapter 3. Finally in section 5.5 we review the recent work of [218] on the 3d ADHM theory and  $\text{AdS}_4$  black hole entropy and discuss how this work can be interpreted in enumerative geometry.

**Conventions.** We refer the reader to appendix A for a review of the conventions for partitions and combinatorics used throughout this section.

**Publications.** This chapter is mostly based on the author’s contributions to [219]. However, the approach to hemisphere blocks using the definition 4.1 does not feature in this paper—[219] takes an alternative approach and constructs the boundary condition from first principles, we do not review that construction here.

- S. Crew, N. Dorey and D. Zhang, *Blocks and Vortices in the 3d ADHM Quiver Gauge Theory*, 2010.09732. (accepted to JHEP)

Section 5.3 consists of the author’s currently unpublished work.

## 5.1 Preliminaries

We focus on a particular 3d gauge theory living on  $N$  D2-branes on top a single D6-brane in type IIA string theory [220, 84]. The theory has a Lagrangian description as a 3d  $\mathcal{N} = 4$  theory with  $G = U(N)$  gauge symmetry.

The theory has a vector multiplet, one hypermultiplet in the fundamental representation  $(I, J)$  and an adjoint hypermultiplet  $(A, B)$ . The Higgs branch  $\mathcal{M}_H$  is described by the Nakajima quiver variety encoded by the data in figure 5.1 and coincides with the ADHM construction of the instanton moduli space [148]—we thus refer to this theory as 3d ADHM.

Turning on a real mass  $z = e^{-m}$  for the symmetry rotating the adjoint field and the usual axial mass deformation  $t = e^{-\tau}$  generates the group action on the quiver described

in section 3.3.1 and gives rise to isolated massive vacua. We also turn on a positive real FI parameter  $\zeta = e^{-\xi}$ . The Coulomb branch has recently been constructed in the work of Nakajima and Kodera [221] where it is shown that the quantised Coulomb branch is a certain deformation of a quotient of the affine Yangian  $Y(\hat{\mathfrak{gl}}(1))$  known as a spherical cyclotomic rational Cherednik algebra. The theory is expected to be self mirror dual as argued in the works [222, 223].

### 5.1.1 Hilbert schemes

In this chapter we predominantly focus on the theory with one fundamental flavour. In this case the resolved Higgs branch of the theory can be described by the Hilbert scheme of points in the plane  $\mathcal{M}_H = \text{Hilb}^N(\mathbb{C}^2)$ . The unresolved Higgs branch is the symmetric product  $\mathcal{M}_H^0 = \text{Sym}^N(\mathbb{C}^2)$  and the resolution is the Hilbert-Chow morphism  $\pi : \text{Hilb}^N(\mathbb{C}^2) \rightarrow \text{Sym}^N(\mathbb{C}^2)$ . We begin with a brief review of the Hilbert scheme of points in the plane following [123].

**Definition.** The Hilbert scheme of points in the plane parametrises ideals of codimension  $N$  in  $\mathbb{C}[x, y]$

$$\text{Hilb}^N(\mathbb{C}^2) := \{\mathcal{J} \subset \mathbb{C}[x, y] : \dim(\mathbb{C}[x, y]/\mathcal{J}) = N\}. \quad (5.1)$$

As we now show following [103, 224], this space is isomorphic to the Jordan quiver variety of section 3.3.1. When  $N_f = 1$  the stability condition of definition 2.3 is equivalent for  $\theta > 0$  to<sup>1</sup>

$$\mu^{-1}(0)^s = \{(A, B, I, J) : J = 0, I \text{ is cyclic for } (A, B)\}. \quad (5.2)$$

Then, given a point in  $\mu^{-1}(0)^s$ , we can define a polynomial ideal in (5.1) invariant under the  $G$ -action (3.85) by

$$\mathcal{J}_{A,B,I} = \{p(x, y) \in \mathbb{C}[x, y] : p(x, y)I = 0\}. \quad (5.3)$$

This map is 1-1.

**Fixed points.** There is a natural torus action  $\mathbf{T}$  on  $\text{Hilb}^N(\mathbb{C}^2)$  induced by the following action on  $\mathbb{C}^2$

$$(x, y) \rightarrow (xt_1^{-1}, yt_2^{-1}). \quad (5.4)$$

This action coincides with the group actions of section 3.3.1 under the identification  $(t_1, t_2) = (zt^{\frac{1}{2}}, z^{-1}t^{\frac{1}{2}})$  so that the action corresponding to  $t_1 t_2$ , denoted  $\mathbb{C}_t^\times$ , scales the symplectic form and the action associated to  $t_1/t_2$ , denoted  $\mathbf{A}$ , leaves it invariant. We write the combined torus action as  $\mathbf{T} = \mathbf{A} \times \mathbb{C}_t^\times$ . We work with the  $(z, t)$  fugacities since

---

<sup>1</sup>A cyclic vector  $I$  for  $A, B$  satisfies  $\mathbb{C}\langle A, B \rangle I = V$ .

1	$x$
$y$	$xy$
$y^2$	

Figure 5.2: Fixed point in  $\text{Hilb}^N(\mathbb{C}^2)$  corresponding to the ideal generated by monomials  $\{x^2, x^2y, xy^2\}$ .

these correspond to the physical flavour symmetry and axial  $R$ -symmetry. The torus fixed points are labelled by partitions  $\lambda$  of  $N$  and correspond to ideals generated by the monomials

$$\mathcal{J}_\lambda = \{x^{\lambda_i}y^i : i = 1, \dots, l(\lambda)\}. \quad (5.5)$$

An example is shown in figure 5.2.

**Tautological bundle.** According to the general arguments of section 3.1.1, there is one tautological vector bundle  $\mathcal{V}$  of rank  $N$  associated to the single gauge node of figure 5.1. In the Hilbert scheme description (5.1) the fibre of this bundle at a fixed point  $\lambda$  is given by

$$\mathcal{V}|_\lambda = \mathbb{C}[x, y]/\mathcal{J}_\lambda. \quad (5.6)$$

In the  $K$ -theory ring  $K_T(\text{Hilb}^N(\mathbb{C}^2))$  the bundle  $\mathcal{V}$  is described by Chern roots  $\mathcal{V} = w_1 + \dots + w_N$  and from (5.5) the evaluation at a fixed point  $\lambda$  is

$$\mathcal{V}_\lambda = \sum_{(i,j) \in \lambda} z^{i-j} t^{\frac{1}{2}i + \frac{1}{2}j-1}. \quad (5.7)$$

Hence the evaluation of the Chern roots (3.12) is given by  $w_s(\lambda) = z^{i_s-j_s} t^{\frac{1}{2}i_s + \frac{1}{2}j_s-1}$  where  $s$  runs over the boxes in  $\lambda$ . We deduce that the tautological bundle  $\mathcal{L} = \det \mathcal{V}$  at a fixed point  $\lambda$  is then given by

$$\mathcal{L}_\lambda = z^{-\sum_{s \in \lambda} c_\lambda(s)} t^{\sum_{s \in \lambda} \frac{1}{2}h_\lambda(s) + \frac{1}{2}}, \quad (5.8)$$

where we used the definition of the hook length  $h_\lambda(s)$  and content  $c_\lambda(s)$  (A.21). This agrees with equation (3.94) in the case  $r = 1$ .

**Tangent bundle.** Specialising (3.93) to the case  $r = 1$  we have that the evaluation of the tangent bundle at a fixed point  $\lambda$  is given by

$$T_\lambda \mathcal{M}_H = \sum_{s \in \lambda} z^{-l_\lambda(s) - a_\lambda(s) - 1} t^{-\frac{1}{2}l_\lambda(s) + \frac{1}{2}a_\lambda(s) - \frac{1}{2}} + z^{a_\lambda(s) + l_\lambda(s) + 1} t^{-\frac{1}{2}a_\lambda(s) + \frac{1}{2}l_\lambda(s) - \frac{1}{2}}. \quad (5.9)$$

Throughout we fix chambers  $\mathfrak{C}_H = \{z > 0\}$  and  $\mathfrak{C}_C = \{\xi > 0\}$  and in that case the tangent bundle splits into positive and negative weights  $T_\lambda \mathcal{M}_H = N_\lambda^+ + N_\lambda^-$  with

$$\begin{aligned} N_\lambda^+ &= \sum_{s \in \lambda} z^{a_\lambda(s) + l_\lambda(s) + 1} t^{\frac{1}{2}(l_\lambda(s) - a_\lambda(s) - 1)} , \\ N_\lambda^- &= \sum_{s \in \lambda} z^{-a_\lambda(s) - l_\lambda(s) - 1} t^{-\frac{1}{2}(a_\lambda(s) + l_\lambda(s) + 1)} . \end{aligned} \quad (5.10)$$

**$K$ -theory.** Recall from section 3.1.1 that the *localised*  $K$ -theory  $K_{\mathsf{T}}(X)_{\text{loc}}$  of a Nakajima quiver variety is a vector space (over the representation ring) with a basis labelled by skyscraper sheaves  $I_\alpha$  at fixed points. In the case of the Hilbert scheme of points we have [145]

$$K_{t_1, t_2}(\text{Hilb}^N(\mathbb{C}^2))_{\text{loc}} \cong \mathbb{Z}[x_1, x_2, \dots, x_N] \otimes \mathbb{Z}[t_1, t_2^{-1}] , \quad (5.11)$$

with the isomorphism mapping Macdonald polynomials (B.23) to the skyscraper sheaves of fixed points

$$I_\lambda \leftrightarrow \widetilde{H}_\lambda(t_1, t_2^{-1}) , \quad (5.12)$$

with  $|\lambda| = N$ . The normalisation of the Macdonald polynomial is Haiman's modified Macdonald polynomial given by

$$\widetilde{H}_\lambda(X; t_1, t_2^{-1}) = t_1^{n(\lambda)} J_\lambda \left( \frac{X}{1 - t_1^{-1}}; t_1, t_2 \right) . \quad (5.13)$$

The relationship between  $J_\lambda$  and  $P_\lambda$  is reviewed in appendix B. In particular, the product structure in the localised  $K$ -theory is determined by the Macdonald structure constants  $f_{\lambda\mu}^\nu(t_1, t_2^{-1})$  defined in (B.41) and the bilinear pairing coincides with the inner product (B.45).

## 5.2 Hemisphere blocks

We now compute the hemisphere blocks for the 3d ADHM theory. Physically we expect these blocks coincide with hemisphere partition functions of the theory with a  $\mathcal{N} = (2, 2)$  exceptional Dirichlet boundary condition on the  $S^1 \times S^1$  boundary. Following the prescription proposed in definition 4.1 the  $B$ -shifted hemisphere block can be written as

$$H_\lambda^B(\zeta, z; q, t) = e^{\phi_\lambda^B} \mathcal{I}_{1\text{-loop}}^{B, \lambda} \mathcal{I}_{\text{Vortex}}^{B, \lambda} . \quad (5.14)$$

As usual, we denote by  $\mathcal{I}_\lambda^B = \mathcal{I}_{1\text{-loop}}^{B,\lambda} \mathcal{I}_{\text{Vortex}}^{B,\lambda}$  the contributions from the half index part. Using (5.8) and (5.10), the classical contribution is given by

$$\begin{aligned} \phi_\lambda^B = & - \left( \sum_{s \in \lambda} c_\lambda(s) \right) \frac{\log \zeta \log z}{\log q} + \left( \sum_{s \in \lambda} h_\lambda(s) \right) \frac{\log(t^{-\frac{1}{2}} q^{\frac{1}{2}}) \log z}{\log q} + \\ & \left( \sum_{s \in \lambda} h_\lambda(s) \right) \frac{\log(t^{\frac{1}{2}}) \log \zeta}{\log q} - \left( \sum_{s \in \lambda} c_\lambda(s) \right) \frac{\log(t^{\frac{1}{2}}) \log(t^{-\frac{1}{2}} q^{\frac{1}{2}})}{\log q}. \end{aligned} \quad (5.15)$$

The one-loop piece follows using (5.10)

$$\mathcal{I}_{1\text{-loop}}^{B,\lambda} = \text{PE} \left[ \frac{t-q}{1-q} N_\lambda^+ \right] = \prod_{s \in \lambda} \frac{(qz^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)-1)}; q)_\infty}{(tz^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)-1)}; q)_\infty}. \quad (5.16)$$

The vortex contributions are given by the vertex functions for  $\text{Hilb}^N(\mathbb{C}^2)$

$$\mathcal{I}_{\text{Vortex}}^{B,\lambda} = \mathcal{V}_\lambda(\zeta, z; q, t) = \sum_{\mathbf{d}} \zeta^{\mathbf{d}} \chi(\hat{\mathcal{O}}_{\text{vir.}}, \mathbf{QM}_\lambda^{\mathbf{d}}), \quad (5.17)$$

where  $\mathbf{QM}_\lambda^{\mathbf{d}}$  denotes the based quasimap space for  $\text{Hilb}^N(\mathbb{C}^2)$ . These functions were recently computed by [86]<sup>2</sup> and are given by

$$\begin{aligned} & \mathcal{V}_\lambda(\zeta, z; q, t) \\ &= \sum_{\pi \in \text{RPP}(\lambda)} \left( \zeta t^{-\frac{1}{2}} q^{\frac{1}{2}} \right)^{|\pi|} \prod_{s \in \lambda} \frac{(w_s(\lambda)^{-1}; q)_{-\pi_s}}{(qtw_s(\lambda)^{-1}; q)_{-\pi_s}} \prod_{\substack{s, t \in \lambda \\ s \neq t}} \frac{(qt \frac{w_t(\lambda)}{w_s(\lambda)}; q)_{\pi_t - \pi_s}}{\left( \frac{w_t(\lambda)}{w_s(\lambda)}; q \right)_{\pi_t - \pi_s}} \frac{(zt^{-\frac{1}{2}} \frac{w_t(\lambda)}{w_s(\lambda)}; q)_{\pi_t - \pi_s}}{(qzt^{\frac{1}{2}} \frac{w_t(\lambda)}{w_s(\lambda)}; q)_{\pi_t - \pi_s}}, \end{aligned} \quad (5.18)$$

where  $w_s$  denotes the evaluation of Chern roots (5.7). *A priori* the vortex sum should be over all degrees  $\mathbf{d} = (d_1, \dots, d_N) \in \mathbb{Z}^N$  however the same vortex partition function was recently obtained via factorising the superconformal index [218] and there it is argued that only the  $d_i$  that weakly increase along the rows and columns of  $\lambda$  contribute *i.e.*  $\{d_i\}$  form a *reverse plane partition*.<sup>3</sup> Our conventions for reverse plane partitions are summarised in appendix A. In the geometrical context of section 4.1.1 this can be interpreted as the fact that each quasimap  $f : \mathbb{P}^1 \rightarrow \text{Hilb}^N(\mathbb{C}^2)$  with  $f(\infty) = \lambda$  is labelled by a single non-negative integer degree  $d$  and the fixed points on the moduli space are labelled by reverse plane partitions  $\pi$  of  $\lambda$  with  $|\pi| = d$ .

As in section 4.3, the  $A$ -shifted hemisphere block is given by the parameter redefinition

$$H_\lambda^A = H_\lambda^B(t \rightarrow tq). \quad (5.19)$$

<sup>2</sup>There are some differences of convention:  $q \rightarrow q^{-1}$  and their partitions are rotations of ours by  $90^\circ$ .

<sup>3</sup>This is also shown explicitly in appendix D of [219].



### 5.2.1 Verma character limits

In the previous chapter we discussed how the  $A$ -limit yields characters of Verma modules of the Coulomb branch chiral ring. This limit is realised by sending  $t \rightarrow q$  of equation (5.14) and, in agreement with the general arguments of section 4.4, we find that only the vortex contributions survive giving

$$\widehat{\mathcal{I}}_\lambda^A := \lim_{t \rightarrow 1} \mathcal{I}_\lambda^A = \lim_{t \rightarrow q} \mathcal{I}_\lambda^B = \sum_{\pi \in \text{RPP}(\lambda)} \zeta^{|\pi|}. \quad (5.20)$$

The  $A$ -limit of the Casimir term is

$$\hat{\phi}_\lambda^A = - \left( \sum_{s \in \lambda} c_\lambda(s) \right) \frac{\log \zeta \log z}{\log q} + \frac{1}{2} \left( \sum_{s \in \lambda} h_\lambda(s) \right) \log \zeta. \quad (5.21)$$

**Verma characters.** The quantised Coulomb branch algebra  $\widehat{\mathcal{R}}_C$  has recently been studied from the physical [225] and geometric [221] points of view. We briefly review this construction and discuss the conjectural Verma module characters.

The 3d ADHM theory with one flavour is self-mirror dual. We explain in the following how both the  $A$ -limit and the  $B$ -limit give the same character. To the authors knowledge, the Verma modules of the quantised Higgs branch algebra  $\widehat{\mathcal{R}}_H$  have not been studied in the mathematical literature and in this section we focus on the Coulomb branch side.

The algebra  $\widehat{\mathcal{R}}_C$  can be realised as an algebra of difference operators acting on functions on  $\mathbb{C}^N$ . We denote by  $w_a$  variables on  $\mathbb{C}^N$  and  $v_a$  the shift operator

$$v_a : w_a \rightarrow w_a + \log q. \quad (5.22)$$

The difference (monopole) operators are then defined by

$$\begin{aligned} E_t &= \sum_{a=1}^N \prod_{b \neq a}^N \frac{w_a - w_b - \log z}{w_a - w_b} w_a^{t+1} v_a, \\ F_t &= \sum_{a=1}^N \prod_{b \neq a}^N \frac{w_a - w_b + \log z}{w_a - w_b} w_a^t v_a. \end{aligned} \quad (5.23)$$

where  $t$  is a non-negative integer.

**Remark.** We note that these operators look similar to those of section 3.2.4—they are *rational* versions of those operators. We expect that this is not a coincidence: Operators of this type can be collected as generating functions (see [226]) of the quantum toroidal algebra  $U_t(\widehat{\mathfrak{gl}}_1)$  which acts in the quantum  $K$ -theory of  $\text{Hilb}^N(\mathbb{C}^2)$  [227, 228, 50]. We expect this correspondence to be an incidence of a  $K$ -theoretic lift of the quantum Hikita conjecture [229].

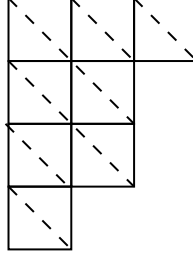


Figure 5.3: The diagonals of  $\lambda = (3, 2, 2, 1)$  are  $\{1, 1, 2, 2, 1, 1\}$ .

In each vacuum  $\lambda$  the action of  $\hat{\mathcal{R}}_C$  can be decomposed in terms of the diagonals  $\{N_k\}$  of  $\lambda$  with  $\sum N_k = N$ —the definition of the diagonals of a partition is shown in figure 5.3. The variables  $w_a$  and shift operators  $v_a$  are then split into groups  $w_{k;a}$  and  $v_{k;a}$  of  $N_k$  variables with corresponding splittings of the monopole operators  $E_{k,t}$  and  $F_{k,t}$ . The Hamiltonians are given by the  $w_{k;a}$  variables and the Verma modules are generated from the action of  $E_{k,t}$  on a lowest weight state annihilated by the  $F_{k,t}$ . We refer the reader to [225] for a detailed construction of the module but note here that the lowest weight state has<sup>4</sup>

$$w_{k;a}|\mathcal{B}_\lambda\rangle = [(\log z - \log q)j - (\log q + \log z)i]|\mathcal{B}_\lambda\rangle, \quad (5.24)$$

where  $(i, j)$  are the coordinates of the box associated to the  $a^{\text{th}}$  element on the  $k^{\text{th}}$  diagonal. Higher weight states have eigenvalues

$$w_{k;a} = -\log q n_{k;a} + (\log z - \log q)j - (\log q + \log z)i, \quad (5.25)$$

where the  $n_{k;a}$  are positive integers non-decreasing along the rows and columns of  $\lambda$ . This is precisely the data of a reverse plane partition  $\pi \in \text{RPP}(\lambda)$  and the module is generated by  $\{n_{k;a} : n_{k;a} \in \text{RPP}(\lambda)\}$ . We can now write down the Verma character<sup>5</sup>

$$\mathcal{X}_\lambda^C(\zeta) = \text{Tr} \prod_{k,a} \zeta^{\frac{w_{k;a}}{\log q}} = e^{-(\sum_{s \in \lambda} c_\lambda(s)) \frac{\log \zeta \log z}{\log q} + \frac{1}{2} (\sum_{s \in \lambda} h_\lambda(s)) \log \zeta} \sum_{\pi \in \text{RPP}(\lambda)} \zeta^{|\pi|}. \quad (5.26)$$

We see this matches the  $A$ -limit in (5.20) and (5.21) and, in accordance with the general arguments of section 4.4, we have

$$\mathcal{X}^C(\mathcal{B}_\lambda) = \lim_{t \rightarrow 1} H_\lambda^A = e^{\hat{\phi}_\lambda^A} \hat{\mathcal{I}}^A. \quad (5.27)$$

<sup>4</sup>We have rescaled variables from [225] and performed a triality transformation  $\epsilon_2 \rightarrow -\epsilon_1 - \epsilon_2$  in their notation to match our conventions.

<sup>5</sup>The Verma characters were also recently computed with an additional  $R$  symmetry twist in [208]. In this equation we have used the identity (A.23).

### 5.2.2 Specialised limits and combinatorics

In this section we verify the mirror symmetry property of the specialised blocks (proposition 3 of section 4.3) for  $\text{Hilb}^N(\mathbb{C}^2)$ . Firstly, we compute the  $B$ -limit of the index and find that only the 1-loop contributions survive, equation (5.16) telescopes to give

$$\hat{\mathcal{I}}_\lambda^B = \lim_{t \rightarrow 1} \mathcal{I}_\lambda^B = \frac{1}{1 - z^{h_\lambda(s)}}. \quad (5.28)$$

This is the well-known Hook generating function [230] for reverse plane partitions. Indeed, computing the  $A$ -limit, the vortex sum in (5.18) degenerates to a count of fixed points giving

$$\hat{\mathcal{I}}_\lambda^B = \frac{1}{1 - z^{h_\lambda(s)}} = \sum_{\pi \in \text{RPP}(\lambda)} \zeta^{|\pi|} = \hat{\mathcal{I}}_\lambda^A, \quad (5.29)$$

with the second equality under the mirror identification  $\zeta \leftrightarrow z$ . We thus verify proposition 3 in a non-trivial example. Geometrically we have that the generating function of the fixed point count on the quasimap space  $\text{QM}_\lambda$  is given by the attracting directions of the tangent bundle evaluated at  $\lambda$ .

**Refined limit.** We now consider two refined limits of the half index. Recall the half index of an exceptional Dirichlet boundary condition  $\mathcal{B}$  is given by

$$\mathcal{I}_\alpha = \text{Tr}_{\mathcal{H}_\mathcal{B}}(-1)^F q^{J + \frac{R_V + R_A}{4}} t^{\frac{R_V - R_A}{2}} z^{F_H} \zeta^{F_C}. \quad (5.30)$$

We consider the two limits  $tq^{\pm \frac{1}{2}} \rightarrow 0$  with  $tq^{\mp \frac{1}{2}}$  fixed. Similar limits were discussed for the full superconformal index in [231]. In the  $A$ - and  $B$ -shifted convention these are the limits

$$\lim_{tq^{\frac{1}{2}} \rightarrow 0} \mathcal{I}_\alpha = \lim_{q \rightarrow 0} \mathcal{I}_\alpha^A, \quad \lim_{tq^{-\frac{1}{2}} \rightarrow 0} \mathcal{I}_\alpha = \lim_{q \rightarrow 0} \mathcal{I}_\alpha^B. \quad (5.31)$$

The first of these was considered in chapter 3 where this limit took us from the  $\chi_t$  genera of vortex moduli space to the Poincaré polynomials. Using essentially identical unitarity arguments to those of section 2.1.3, these limits select operators with either  $J + \frac{R_V}{2} = 0$  and  $D = \frac{R_A}{2}$  or  $J + \frac{R_A}{2} = 0$  and  $D = \frac{R_V}{2}$ . That is to say they are traces over Higgs or Coulomb operators but now refined by  $R$  charge.

In the present example of  $\text{Hilb}^N(\mathbb{C}^2)$  we first compute the refined  $B$ -limit of equation (5.16) where we find

$$\lim_{q \rightarrow 0} \mathcal{I}_\lambda^B = \frac{1}{1 - z^{a_\lambda(s) + l_\lambda(s) + 1} t^{\frac{1}{2}(l_\lambda(s) - a_\lambda(s) + 1)}}. \quad (5.32)$$

The geometrical interpretation of the  $q \rightarrow 0$  limit of the  $A$ -twist is less clear in comparison to the  $T[SU(N)]$  example discussed in chapter 3. In particular, the quasimap space is no longer a non-singular quasi-projective variety and the limit is not strictly a Poincaré polynomial. However, we note that under certain conditions<sup>6</sup> at each fixed point in equation (5.18) the numerator and denominators have matching powers of  $q$  so at least this limit is well-defined. We abuse terminology and still refer to this limit as the ‘Poincaré polynomial limit’. We return to this point after computing the  $A$ - and  $B$ -twisted indices in section 5.4.

## 5.3 Neumann boundary conditions

So far in the thesis we have focused on exceptional Dirichlet boundary conditions. In this section we discuss particular *Neumann boundary conditions*, denoted  $\mathcal{N}$ , and their Higgs branch images. Again there is an action of the bulk chiral ring on the image and the general arguments of [55] show that  $\mathcal{N}$  should yield simple modules of  $\hat{\mathcal{R}}_H$ .

**Overview.** We begin with a discussion of Neumann boundary conditions in general and the computation of their half indices. We study a particular boundary condition, denoted  $\mathcal{N}$ , for the SQED[ $N$ ] and SQCD[ $k, N$ ] examples and discuss the geometric interpretation of  $\mathcal{N}$  as counting sections of line bundles over the Higgs branch. Finally, we study the 3d ADHM theory with  $N_f > 1$  flavours and discuss the relationship between characters of  $\hat{\mathcal{R}}_H$  and the Hanany-Tong moduli space of section 3.3.

### 5.3.1 Half index

The Neumann boundary condition imposes the following conditions for the  $\mathcal{N} = 4$  vector multiplet<sup>7</sup>

$$\mathcal{N}: \quad F_{3\mu}|_{\partial} = 0, \quad D_3\varphi|_{\partial} = 0, \quad D_\mu\sigma|_{\partial} = 0, \quad (5.33)$$

where  $\mu$  are directions parallel to the boundary. In contrast to the exceptional Dirichlet boundary conditions discussed in section 4.3.1, the gauge symmetry is preserved at the boundary. In particular, this means that the half index can be expressed as an integral projecting onto gauge invariant states with the Haar measure for  $G$  [170].

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<sup>6</sup>This condition, together with the limit  $q \rightarrow \infty$  also being well-defined, is equivalent to ‘large frame vanishing’ of the bare vertex—in fact  $\text{Hilb}^N(\mathbb{C}^2)$  has this property for any  $N$  [232]. We expect this to be related to the ‘good’ and ‘ugly’ assumption. We thank Hunter Dinkins for discussions on large frame vanishing.

<sup>7</sup>Together with the supersymmetric completion to the rest of the multiplet.

The operators in the vector multiplet that survive (5.33) at the boundary together with their charges are given by

	$D_z^n \varphi$	$D_z^n \lambda_-$
$G$	adj	adj
$J$	$n$	$n + \frac{1}{2}$
$R_A$	2	-1
$R_V$	0	+1

(5.34)

Hence for a  $G = U(N)$  gauge group the  $\mathcal{N} = 4$  vector multiplet contributes to the half index (4.84) the following

$$\mathcal{I}(\mathcal{N}) = \frac{1}{N!} \frac{(q; q)_\infty^N}{(q^{\frac{1}{2}} t^{-1}; q)_\infty^N} \oint \prod_{i=1}^N \frac{dw_i}{2\pi i w_i} \prod_{i \neq j}^N \frac{\left(\frac{w_i}{w_j}; q\right)_\infty}{\left(q^{\frac{1}{2}} t^{-1} \frac{w_i}{w_j}; q\right)_\infty}. \quad (5.35)$$

The contour here and throughout this section is a product of unit circles appropriate for the Haar measure for  $SU(N)$ . As in section 4.3.1, we again have a choice of Dirichlet or Neumann for each  $\mathcal{N} = 2$  chiral comprising the  $\mathcal{N} = 4$  hypermultiplet. This determines a Lagrangian splitting of  $(X, Y)$  and throughout this section we take the splitting associated to the natural polarisation of a Nakajima quiver variety. That is

$$\mathcal{N} : \quad X|_\partial = 0, \quad D_3 Y|_\partial = 0. \quad (5.36)$$

The matter contributions to the index are the same as in section 4.3.1 except now we integrate over the gauge fugacities  $w$  since gauge symmetry is preserved at the boundary.

The Higgs branch image of  $\mathcal{N}$  is formed of gauge invariant polynomials in  $Y$  that we denote by  $p(Y)$ . We have

$$\mathbb{C}[\mathcal{N}] = \mathbb{C}[\mathcal{M}_H] / \mathcal{I}, \quad (5.37)$$

for the Higgs branch image where  $\mathcal{I}$  is the ideal of functions vanishing on  $\mathcal{N}$ . The action of the quantised bulk chiral algebra  $\hat{\mathcal{R}}_H = \mathbb{C}[\mathcal{M}_H]$  on the Higgs branch image follows from the quantum symplectic reduction and is explicitly

$$\hat{X} \cdot p(Y) = \epsilon \partial_Y p(Y), \quad \hat{Y} \cdot p(Y) = Y p(Y). \quad (5.38)$$

We note that geometrically boundary local operators correspond to holomorphic functions supported on  $\mathcal{N}$  and therefore when  $\mathcal{M}_H$  is a co-tangent bundle then  $\mathcal{N} = \pi^{-1}(o)$  is the compact core so that only the unit operator contributes to the index. We enhance this setup in the following with a line operator so that the boundary chiral ring is less trivial.

**Remark.** We note that it is possible, for example in the class of co-tangent type theories,<sup>8</sup> that the image of Neumann boundary conditions coincides with the image of exceptional Dirichlet boundary conditions on the Higgs branch. In such cases it is possible to define a ‘holomorphic block’ with a contour integral prescription.

### 5.3.2 Line operators

In this section we consider the insertion of a Wilson line at the origin  $z = \bar{z} = 0$  and extending perpendicularly out of the boundary.

$$\mathcal{W}_{\mathcal{R}} = \mathcal{P} \exp i \int_{\substack{x^3 \leq 0 \\ z = \bar{z} = 0}} (A_3 - i\sigma) dx^3. \quad (5.39)$$

The boundary local operators that contribute to the half index then transform in the representation  $\bar{\mathcal{R}}$  and the projection onto gauge invariants is modified accordingly so that

$$\mathcal{I}(\mathcal{N}) \sim \oint d\mu_G \chi_{\mathcal{R}}(w) \mathcal{I}_{\text{Matter}}. \quad (5.40)$$

In terms of the polynomials  $p(Y)$  the gauge invariance condition is now modified to

$$\hat{\mu}_{\mathbb{C}} \cdot p(Y) = \bar{\mathcal{R}} \cdot p(Y). \quad (5.41)$$

Geometrically this implies that we are counting sections of bundles over  $\mathcal{N} \subset \mathcal{M}_H$ . We show this explicitly in the following example.

### 5.3.3 Warm-up: SQED[ $N$ ]

We return to supersymmetric QED with  $N$  flavours for the last time. Recall that  $\mathcal{M}_H = T^*\mathbb{P}^{N-1}$  and now the Higgs branch image of the  $\mathcal{N}$  boundary condition is  $\mathcal{N} = \mathbb{P}^{N-1}$ . We consider the half index count of local operators in the presence of a Wilson line of positive charge  $\mathfrak{n}$ .

**Boundary local operators.** The ring  $\mathbb{C}[\mathcal{N}]$  contains only the identity operator however, in the presence of  $\mathcal{W}_{\mathfrak{n}}$ , the boundary operators are given by polynomials in  $(Y_1, \dots, Y_N)$  satisfying the condition (5.41). The boundary module is then generated by

$$\mathcal{N} : Y_1^{i_1} Y_2^{i_2} \dots Y_N^{i_N} |\mathcal{N}\rangle, \quad (5.42)$$

where  $i_1 + \dots + i_N = \mathfrak{n}$  and  $|\mathcal{N}\rangle$  denotes the unit operator annihilated by all the ‘derivatives’  $X$ . That is, the boundary module is generated by homogeneous polynomials of degree

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<sup>8</sup>Theories whose Higgs branch  $\mathcal{M}_H$  is a cotangent bundle, for example theories in the class  $T_{\rho}[SU(N)]$ .

$\mathfrak{n}$  which can be associated to sections of  $\mathcal{O}(\mathfrak{n}) \rightarrow \mathbb{P}^{N-1}$ . The action of<sup>9</sup>  $\hat{\mathcal{R}}_H = U(\mathfrak{sl}_N)/Z$  realises  $H^0(\mathbb{P}^{N-1}, \mathcal{O}(\mathfrak{n}))$  as a finite dimensional module as in the Borel-Weil-Bott theorem for this simple partial flag variety.

**Half index.** We now compute the half index of  $\mathcal{N}$ . Following the recipe for the vector multiplet in the previous section and the matter recipe from section 4.3.1, keeping track of the gauge fugacity now, we find

$$\mathcal{I}_{\mathcal{W}_n}(\mathcal{N}) = \frac{(q; q)_\infty}{(q^{\frac{1}{2}}t^{-1}; q)_\infty} \oint \frac{dw}{2\pi i w} w^{-n} \prod_{i=1}^N \frac{(q^{\frac{3}{4}}t^{-\frac{1}{2}}wx_i; q)_\infty}{(q^{\frac{1}{4}}t^{\frac{1}{2}}wx_i; q)_\infty}. \quad (5.43)$$

Taking the  $B$ -limit we find the integrand telescopes to

$$\hat{\mathcal{I}}_{\mathcal{W}_n}^B(\mathcal{N}) := \lim_{t \rightarrow q^{\frac{1}{2}}} \mathcal{I}_{\mathcal{N}, \mathcal{W}_n} = \oint \frac{dw}{2\pi i w} w^{-n} \prod_{i=1}^N \frac{1}{1 - wx_i}. \quad (5.44)$$

We note this is the Molien integral from example 3.2 that computes sections of the line bundle  $\mathcal{O}(\mathfrak{n})$  over  $\mathbb{P}^{N-1}$ , or equivalently the boundary operators (5.42), so that<sup>10</sup>

$$\hat{\mathcal{I}}_{\mathcal{W}_n}^B(\mathcal{N}) = \chi_{T_H}(\mathcal{O}(\mathfrak{n}), \mathbb{P}^{N-1}) = \text{ch}_{T_H} H^0(\mathbb{P}^{N-1}, \mathcal{O}(\mathfrak{n})). \quad (5.45)$$

It is straightforward to evaluate the integral explicitly by residues but to illustrate the more general principle we can evaluate it using the symmetric function methods from chapter 3. The Schur measure (B.15) is particularly simple for one variable and we write:

$$\begin{aligned} \hat{\mathcal{I}}_{\mathcal{W}_n}^B(\mathcal{N}) &= \oint d\mu_S[w] s_{(\mathfrak{n})}(w^{-1}) \sum_{\lambda} s_{\lambda}(w) s_{\lambda}(x_1, \dots, x_N), \\ &= s_{(\mathfrak{n})}(x_1, \dots, x_N), \end{aligned} \quad (5.46)$$

where in the second line we use orthogonality of the Schur polynomials under the measure  $d\mu_S[w]$ —see appendix B. Indeed we find the appropriate character for the totally symmetric module  $H^0(\mathbb{P}^{N-1}, \mathcal{O}(\mathfrak{n}))$ .

**SQCD** $[k, N]$ . We briefly discuss the non-abelian theory SQCD $[k, N]$ . The details of this theory were reviewed in example 4.3.4. Recall that  $\mathcal{M}_H = T^*\text{Gr}(k, N)$  and we have  $\mathcal{N} = \text{Gr}(k, N)$ . The flavour symmetry is  $G_H = GL(\mathbb{C}^N)$  and the gauge symmetry is  $G_{\mathbb{C}} = GL(\mathbb{C}^k)$ . We again insert a Wilson line in the totally symmetric representation so

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<sup>9</sup>The arguments of section 4.4 show that the quantised Higgs branch algebra of SQED $[N]$  is a central quotient of  $U(\mathfrak{sl}_N)$  by Casimirs.

<sup>10</sup>The second equality follows since the higher cohomology of line bundles on  $\mathbb{P}^{N-1}$  vanishes.

that the half index is given by

$$\mathcal{I}_{\mathcal{W}_n}(\mathcal{N}) = \frac{1}{k!} \frac{(q; q)_\infty^k}{(q^{\frac{1}{2}} t^{-1}; q)_\infty^k} \oint \prod_{i=1}^k \frac{dw_i}{2\pi i w_i} w_i^{-n} \prod_{i \neq j}^k \frac{\left(\frac{w_i}{w_j}; q\right)_\infty}{\left(q^{\frac{1}{2}} t^{-1} \frac{w_i}{w_j}; q\right)_\infty} \prod_{i=1}^k \prod_{j=1}^N \frac{(q^{\frac{3}{4}} t^{-\frac{1}{2}} w_i x_j; q)_\infty}{(q^{\frac{1}{4}} t^{\frac{1}{2}} w_i x_j; q)_\infty}. \quad (5.47)$$

Before taking the  $B$ -limit, we can evaluate the integral using the Macdonald polynomial methods of chapter 3 to give<sup>11</sup>

$$\begin{aligned} \mathcal{I}_{\mathcal{W}_n}(\mathcal{N}) &= \frac{(q; q)_\infty^k}{(q^{\frac{1}{2}} t^{-1}; q)_\infty^k} \oint d\mu_M^{(k)}[W; q, q^{\frac{1}{2}} t^{-1}] P_{(\mathbf{n})}(W^{-1}) \sum_{\lambda} Q_{\lambda}(W) P_{\lambda}(q^{\frac{1}{4}} t^{\frac{1}{2}} X) \\ &= \frac{(q; q)_\infty^k}{(q^{\frac{1}{2}} t^{-1}; q)_\infty^k} \mathcal{N}_{(\mathbf{n})}(q, q^{\frac{1}{2}} t^{-1}) P_{(\mathbf{n})}(q^{\frac{1}{4}} t^{\frac{1}{2}} X; q, q^{\frac{1}{2}} t^{-1}). \end{aligned} \quad (5.48)$$

In the first line we rewrite the integral in terms of the Macdonald measure (B.46) and use the Cauchy identity (B.31). In the second line we use the orthogonality of Macdonald polynomials (B.24). The  $B$ -limit  $t \rightarrow q^{-\frac{1}{2}}$  is then the limit that sets equal the parameters of the Macdonald polynomial—in this limit the normalisation constant (B.47) simplifies to one and the Macdonald polynomial degenerates to a Schur polynomial so that

$$\widehat{\mathcal{I}}_{\mathcal{W}_n}(\mathcal{N}) = s_{(\mathbf{n})}(x_1, \dots, x_N). \quad (5.49)$$

Again, the count of boundary operators in the presence of  $\mathcal{W}_n$  realises the Borel-Weil-Bott theorem, in this case for the line bundle  $\mathcal{O}(\mathbf{n}) \rightarrow \text{Gr}(k, N)$  *i.e.*

$$\widehat{\mathcal{I}}_{\mathcal{W}_n}(\mathcal{N}) = \text{ch}_{T_H} H^0(\text{Gr}(k, N), \mathcal{O}(\mathbf{n})). \quad (5.50)$$

**Remark.** We note that Macdonald polynomials with arbitrary partition labels can be realised by  $T_\rho[SU(N)]$  theories, generalising equation (5.48). It would be interesting to understand if there is an algebra whose characters are realised by these line operator half indices with the full  $q$  and  $t$  dependence.

### 5.3.4 Hanany-Tong moduli space

We now apply the ideas of the previous section to the theory studied in this chapter and compute the Neumann boundary condition for 3d ADHM. We momentarily consider the theory with  $N_f > 1$  hypermultiplets described by the quiver in figure 3.5 ( $r = N_f$  in that figure). The fields  $(I, J)$  now transform in the fundamental of the flavour group  $G_H = SU(N_f)$  for which we introduce associated mass parameters  $x_i = e^{-m_i}$  with  $i = 1, \dots, N$ . The group action on the quiver data is as in (3.88).

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<sup>11</sup>The dependence of the Macdonald polynomials on the parameters  $(q, q^{\frac{1}{2}} t^{-1})$  is left implicit in the first line.



The ADHM theory is an upgrade from the previous SQCD $[k, N]$  example in that we now add adjoint matter. The boundary condition on the vector multiplet is again given by (5.33) and for the matter fields we take the obvious Lagrangian associated to the quiver in figure 3.3

$$\begin{aligned}\mathcal{N}: \quad J|_{\partial} &= 0, \quad D_3 I|_{\partial} = 0, \\ A|_{\partial} &= 0, \quad D_3 B|_{\partial} = 0.\end{aligned}\tag{5.51}$$

The Higgs branch image of this boundary condition is then the Hanany-Tong moduli space of section 3.3.

**Half index.** Following the previous section we can write down the half index for the boundary condition  $\mathcal{N}$  together with a Wilson line  $\mathcal{W}_n$ . This is the same as the half index for SQCD $[k, N]$  in (5.47) with an extra contribution from the adjoint matter  $B$ .

$$\begin{aligned}\mathcal{I}_{\mathcal{W}_n}(\mathcal{N}) &= \frac{1}{N!} \frac{(q; q)_{\infty}^N}{(q^{\frac{1}{2}} t^{-1}; q)_{\infty}^N} \oint \prod_{i=1}^N \frac{dw_i}{2\pi i w_i} w_i^{-n} \prod_{i \neq j}^N \frac{\left(\frac{w_i}{w_j}; q\right)_{\infty}}{\left(q^{\frac{1}{2}} t^{-1} \frac{w_i}{w_j}; q\right)_{\infty}} \prod_{i,j=1}^N \frac{\left(q^{\frac{3}{4}} t^{-\frac{1}{2}} z \frac{w_i}{w_j}; q\right)_{\infty}}{\left(q^{\frac{1}{4}} t^{\frac{1}{2}} z \frac{w_i}{w_j}; q\right)_{\infty}} \\ &\quad \prod_{i=1}^N \prod_{j=1}^{N_f} \frac{\left(q^{\frac{3}{4}} t^{-\frac{1}{2}} w_i x_j; q\right)_{\infty}}{\left(q^{\frac{1}{4}} t^{\frac{1}{2}} w_i x_j; q\right)_{\infty}}.\end{aligned}\tag{5.52}$$

Now in the  $B$ -limit we find that the integrand telescopes to<sup>12</sup>

$$\begin{aligned}\hat{\mathcal{I}}_{\mathcal{W}_n}^B(\mathcal{N}) &= \lim_{t \rightarrow q^{-\frac{1}{2}}} \mathcal{I}_{\mathcal{W}_n}(\mathcal{N}) \\ &= \frac{1}{N!} \oint \prod_{i=1}^N \frac{dw_i}{2\pi i w_i} w_i^{-n} \prod_{i \neq j}^N \left(1 - \frac{w_i}{w_j}\right) \prod_{i,j=1}^N \frac{1}{1 - z w_i / w_j} \prod_{i=1}^N \prod_{j=1}^{N_f} \frac{1}{1 - w_i x_j}.\end{aligned}\tag{5.53}$$

We evaluated this integral in proposition 1 of chapter 3 (see the remark following the proof for the integral formulation) where we found

$$\hat{\mathcal{I}}_{\mathcal{W}_n}^B(\mathcal{N}) = \frac{1}{(z; z)_N} Q'_{(n^N)}(x_1, \dots, x_{N_f}; z).\tag{5.54}$$

We thus expect Milne polynomials  $Q'_{(n^N)}(x, z)$  to yield simple module characters of the Higgs branch algebra  $\hat{\mathcal{R}}_H$  of the ADHM quiver. We note that in the proof of (5.54), we made use of the raising operators  $D_{1,n}$  that act by

$$D_{1,n} Q'_{(n^N)}(x_1, \dots, x_{N_f}; z) = Q'_{(n^{N+1})}(x_1, \dots, x_{N_f}; z),\tag{5.55}$$

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<sup>12</sup>We note that this integral also arises as the matrix model of a gauged quantum mechanics as in [153]. It would be interesting to investigate the relationship between this model and the one dimensional quantum mechanics of Higgs operators in the  $\Omega$  background.

from this perspective we can view these operators as adding Wilson line charge  $\mathbf{n}$ .

**Geometric interpretation.** In section 3.3 of chapter 3 we realised this half index as computing the equivariant Euler characteristic of powers of the tautological line bundle over the Hanany-Tong moduli space  $\mathcal{V}_{N,N_f}$  so that

$$\widehat{\mathcal{I}}_{\mathcal{W}_{\mathbf{n}}}^B(\mathcal{N}) = \text{ch}_{T_H} H^0(\mathcal{V}_{N,N_f}, \mathcal{L}^{\otimes \mathbf{n}}). \quad (5.56)$$

We note a difference with the SQCD $[k, N]$  example in that the Hanany-Tong moduli space is non-compact. This is reflected in the above in that in the absence of Wilson line charge  $\mathbf{n} = 0$  we still have the contribution  $(z; z)_N^{-1}$  since there are non-trivial holomorphic functions given by  $\text{Tr } B^i$ . Hence the modules generated by local operators in the presence of  $\mathcal{W}_{\mathbf{n}}$  are not finite dimensional.

## 5.4 Twisted index factorisation

In this section we discuss the twisted index factorisation for the 3d ADHM theory. We return to the case of  $N_f = 1$  so that the Higgs branch is  $\text{Hilb}^N(\mathbb{C}^2)$ .

**Outline.** We first verify that the  $A$ - and  $B$ -twisted index gluing yields the Hilbert series. This is an important check of our block definition 4.1. We then discuss 3d mirror symmetry in this self mirror dual context and use it to conjecture an expression for the ‘Poincaré polynomials’ of  $\text{QM}_{\lambda}^d$ . Finally we note some special features of the 3d ADHM theory. The vortex contributions to the block  $H_{\lambda}$  coincide with the 1-leg PT vertex and we make the connection between gluing vertices to form the conifold amplitude in topological string theory and the twisted index factorisation in 3d.

### 5.4.1 $A$ - and $B$ -twisted indices

We first compute the indices by gluing the hemisphere blocks (5.14).

**$A$ -twist.** Recall that the  $A$ -shifted blocks are related to the  $B$ -shifted blocks (5.14) by the following redefinition

$$H_{\lambda}^A(\zeta, z; q, t) = H_{\lambda}^B(\zeta, z, q, t \rightarrow tq). \quad (5.57)$$

The vacua of the 3d ADHM theory with  $N_f = 1$  and gauge group rank  $N$  are labelled by plane partitions  $\lambda$  with  $|\lambda| = N$ . The relevant gluing for the  $A$ -twisted half index is

$$I^A = \sum_{|\lambda|=N} H_\lambda^A(\zeta, z; q, t) H_\lambda^A(\zeta, z; q^{-1}, t). \quad (5.58)$$

Firstly note that contributions from the classical pieces that survive in this gluing are given by

$$I_{\text{Classical}}^{A,\lambda} = \zeta^{\sum_{s \in \lambda} h_\lambda(s)} t^{\frac{1}{2} \sum_{s \in \lambda} c_\lambda(s)}. \quad (5.59)$$

The gluing of the one-loop contributions (5.16) using the fusion identity (A.10) yields

$$I_{1\text{-loop}}^{A,\lambda} = \prod_{s \in \lambda} \frac{\left( z^{a_\lambda(s)+l_\lambda(s)+1} (tq)^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)-1)} q; q \right)_{a_\lambda(s)-l_\lambda(s)}}{\left( z^{a_\lambda(s)+l_\lambda(s)+1} (tq)^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)+1)}; q \right)_{a_\lambda(s)-l_\lambda(s)}}. \quad (5.60)$$

Recall from the general arguments of chapter 3 that the  $A$ -twisted index is independent of the fugacities  $q$  and  $z$ . In particular, we are free to send  $z \rightarrow 0$  in the above which leaves  $I_{1\text{-loop}}^{A,\lambda} = 1$ . The conclusion is that the  $A$ -twisted index only receives contributions from the classical piece and the vertex functions so that

$$I^A = \sum_{|\lambda|=N} \zeta^{\sum_{s \in \lambda} h_\lambda(s)} t^{\frac{1}{2} \sum_{s \in \lambda} c_\lambda(s)} \mathbf{V}_\lambda(\zeta, z; q, tq) \mathbf{V}_\lambda(\zeta, z; q^{-1}, tq). \quad (5.61)$$

We are also free to send  $q \rightarrow 0$  in the above expression. As noted in section 5.2, this limit is well-defined but is not strictly a Poincaré polynomial limit. We take  $q \rightarrow 0$  and write:

$$\text{H.S.}[\text{Hilb}^N(\mathbb{C}^2)] = I^A = \sum_{|\lambda|=N} \zeta^{\sum_{s \in \lambda} h_\lambda(s)} t^{\frac{1}{2} \sum_{s \in \lambda} c_\lambda(s)} \sum_{d,d'} \zeta^{d+d'} t^{\frac{1}{2}(d-d')} P_{\text{QM}_\lambda^d}(t) P_{\text{QM}_\lambda^{d'}}(t^{-1}). \quad (5.62)$$

*A priori* it seems difficult to obtain a closed form expression for  $P_{\text{QM}_\lambda^d}(t)$ , however 3d mirror symmetry provides an answer as we discuss further after computing the  $B$ -twisted index.

**$B$ -twist.** We now compute the  $B$ -twisted index. In this case only the perturbative pieces contribute and fusing (5.15) we have for the classical piece

$$I_{\text{Classical}}^{B,\lambda} = z^{\sum_{s \in \lambda} h_\lambda(s)} t^{-\frac{1}{2} \sum_{s \in \lambda} c_\lambda(s)}. \quad (5.63)$$

Using the fusion identity (A.10) with the one-loop terms (5.16) we find

$$I_{1\text{-loop}}^{B,\lambda} = \prod_{s \in \lambda} \frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)+1)}} \frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)-1)}}, \quad (5.64)$$

so that for the  $B$ -twisted index in total we have

$$\begin{aligned} I^B &= \sum_{|\lambda|=N} H_\lambda^B(\zeta, z; q, t) H_\lambda^B(\zeta, z; q^{-1}, t) \\ &= \sum_{|\lambda|=N} \prod_{s \in \lambda} \frac{z^{h_\lambda(s)} t^{-\frac{1}{2}c_\lambda(s)}}{(1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)+1)}) (1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)-1)})}. \end{aligned} \quad (5.65)$$

We recognise this as Nakajima's formula for the Hilbert series of  $\text{Hilb}^N(\mathbb{C}^2)$  up to an overall power of  $t^{-\frac{1}{2}|\lambda|}$  which agrees with general arguments of section 4.3.3. As expected, (5.65) reproduces the expression (3.95) from chapter 3 in the case  $r = 1$ . In section 3.4.3 we computed the large gauge group rank limit of the Hilbert series/ $B$ -twisted index and found the plethystic form

$$\lim_{N \rightarrow \infty} I^B = \text{PE} \left[ \frac{zt^{\frac{1}{2}}}{1 - zt^{\frac{1}{2}}} + \frac{z^{-1}t^{\frac{1}{2}}}{1 - z^{-1}t^{\frac{1}{2}}} + \frac{1}{(1 - zt^{\frac{1}{2}})(1 - z^{-1}t^{\frac{1}{2}})} \right], \quad (5.66)$$

hence in the large  $N$  limit the Higgs branch chiral ring is freely generated by the operators  $\text{Tr } A^i$ ,  $\text{Tr } B^i$  and  $IA^i B^j J$  for  $i, j \geq 0$ .

**Refined 3d mirror symmetry.** Recall the mirror symmetry statement from proposition 3 which exchanges the one-loop and vortex contributions in the  $A$  and  $B$  limits. If we compare equations (5.62) and (5.65) we see the classical pieces are identified and we now have a refined version of proposition 3. This is refined in the sense that the one-loop contributions to the  $B$ -index can be expressed as

$$\lim_{q \rightarrow 0} \mathcal{I}_{1\text{-loop}}^{B,\lambda} = \lim_{q \rightarrow 0} \text{PE} \left[ \frac{t - q}{1 - q} N_\lambda^+ \right] = \frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)+1)}}, \quad (5.67)$$

which is a refined version of the  $B$ -limit (5.28). We then have the identification

$$\lim_{q \rightarrow 0} \mathcal{I}_{1\text{-loop}}^{B,\lambda} = \lim_{q \rightarrow 0} \mathcal{I}_{\text{Vortex}}^{A,\lambda} = \sum_d (t^{-\frac{1}{2}} \zeta)^d P_{\text{QM}_\lambda^d}(t), \quad (5.68)$$

where the two sides are equal up to the usual mirror symmetry transformations  $z \leftrightarrow \zeta$  and  $t \leftrightarrow t^{-1}$ . Indeed the Hillman-Grassl correspondence [233] provides an expansion of the refined one-loop piece as a sum over reverse plane partitions<sup>13</sup>

$$\frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)+1)}} = \sum_{\pi \in \text{RPP}(\lambda)} (\zeta t^{-\frac{1}{2}})^{|\pi|} t^{\text{ht}(\pi)}, \quad (5.69)$$

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<sup>13</sup>We thank Gjergji Zaimi for pointing this out to us in the MathOverflow post [mathoverflow.net/questions/354618/](https://mathoverflow.net/questions/354618/).

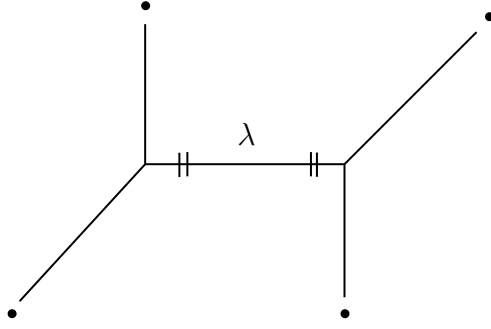


Figure 5.4: Conifold toric diagram. We glue along the preferred edge.

where  $\text{ht}(\pi)$  is a statistic on reverse plane partitions defined in appendix A. Note that when  $t \rightarrow 1$  this becomes the unrefined Hook generating formula (5.29). In particular 3d mirror symmetry now predicts a closed expression for the ‘Poincare polynomials’ of the quasimap moduli spaces.

$$P_{\text{QM}_\lambda^d}(t) := \lim_{q \rightarrow 0} \chi(\hat{\mathcal{O}}_{\text{vir.}}, \text{QM}_\lambda^d) = \sum_{\substack{\pi \in \text{RPP}(\lambda) \\ |\pi|=d}} t^{\text{ht}(\pi)}. \quad (5.70)$$

### 5.4.2 String theory interpretation

As we discussed in section 3.3.1 of chapter 3, the Hilbert scheme is also a rank one instanton moduli space. Specifically, the Hilbert series of the Hilbert scheme of  $N$  points on  $\mathbb{C}^2$  computes the  $N$ -instanton contributions to Nekrasov’s partition function [19] of a five dimensional  $U(1)$  gauge theory on  $\Omega$ -deformed  $\mathbb{C}_{\mathfrak{q}, \mathfrak{t}}^2 \times S^1$ . In this context,  $(t_1, t_2)$  are identified with the 5d  $\Omega$ -deformation parameters  $(\mathfrak{q}, \mathfrak{t})$ .

This 5d gauge theory can be *geometrically engineered* in string theory [234, 235]. We consider M-theory compactified on a non-compact toric Calabi-Yau threefold  $X$ . In this case the relevant geometry is the resolved conifold  $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$  whose (dual) toric diagram we present in figure 5.4. Edges in the toric diagram represent volumes of  $\mathbb{P}^1$  and vertices represent fixed points of the torus action.

The refined topological string  $A$ -model with target  $X$  is known to compute the 5d partition function [236]. The topological  $A$ -model counts holomorphic maps  $f : \Sigma_g \rightarrow X$  and computes the refined Gromov-Witten invariants of  $X$  (for a review see [237]). The refined topological vertex provides a powerful combinatorial realisation of topological string amplitudes. Heuristically speaking, the topological vertex slices the diagram into  $\mathbb{C}^3$  patches, described by tri-valent vertices, by slicing edges with Lagrangian branes and anti-branes. The count of holomorphic maps is then localised to torus fixed points in  $X$ . The result is a Feynman diagram-like method to compute topological  $A$ -model amplitudes. In the case of interest, figure 5.4, the slicing is such that only vertices with one non-trivial

edge are required and the relevant refined vertex is given by [236]

$$\begin{aligned} C_{\emptyset\emptyset\lambda}(\mathbf{t}, \mathbf{q}) &= \mathbf{q}^{\frac{1}{2}\|\lambda\|^2} P_{\lambda^\vee}(\mathbf{t}^{-\rho}; \mathbf{q}, \mathbf{t}) \\ &= \left(zt^{\frac{1}{2}}\right)^{\|\lambda^\vee\|^2} \prod_{s \in \lambda} \frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)-1)}}. \end{aligned} \quad (5.71)$$

The conjugate vertex is

$$\begin{aligned} C_{\emptyset\emptyset\lambda^\vee}(\mathbf{q}, \mathbf{t}) &= \mathbf{t}^{\frac{1}{2}\|\lambda^\vee\|^2} P_\lambda(\mathbf{q}^{-\rho}; \mathbf{t}, \mathbf{q}) \\ &= \left(zt^{-\frac{1}{2}}\right)^{\|\lambda\|^2} \prod_{s \in \lambda} \frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)+1)}}. \end{aligned} \quad (5.72)$$

In the above equations we have used the principal specialisation identity for Macdonald polynomials (B.34) and the notation  $q^{-\rho} = q^{\frac{1}{2}}, q^{\frac{3}{2}}, q^{\frac{5}{2}} \dots$ . Now we consider the  $B$ -shifted hemisphere blocks (5.14). We have

$$\begin{aligned} \lim_{q \rightarrow 0} H_\lambda^B(z, \zeta; q, t) &= z^{\frac{1}{2} \sum_{s \in \lambda} h_\lambda(s)} t^{-\frac{1}{4} \sum_{s \in \lambda} c_\lambda(s)} \prod_{s \in \lambda} \frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)+1)}}, \\ \lim_{q \rightarrow 0} H_\lambda^B(z, \zeta; q^{-1}, t) &= z^{\frac{1}{2} \sum_{s \in \lambda} h_\lambda(s)} t^{-\frac{1}{4} \sum_{s \in \lambda} c_\lambda(s)} \prod_{s \in \lambda} \frac{1}{1 - z^{a_\lambda(s)+l_\lambda(s)+1} t^{\frac{1}{2}(l_\lambda(s)-a_\lambda(s)-1)}}. \end{aligned} \quad (5.73)$$

Hence the refined  $B$ -shifted blocks coincide with the topological vertices (5.71) and (5.72). In particular, using the identity (A.17), the Casimir energy terms coincide with the so-called ‘framing factors’ of the topological vertex—in fact the  $B$ -twisted index suggests an alternative, more symmetric, distribution of the framing factors between the vertices. Furthermore, the topological vertex gluing for the conifold amplitude (section 5.1 of [236]) coincides precisely with the  $B$ -twisted index gluing (5.65). Introducing a fugacity  $\Lambda$  for instanton number we write

$$\begin{aligned} \mathcal{Z}_{\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1} &= \sum_{\lambda} \Lambda^{|\lambda|} C_{\emptyset\emptyset\lambda}(\mathbf{t}, \mathbf{q}) C_{\emptyset\emptyset\lambda^\vee}(\mathbf{q}, \mathbf{t}) \\ &= \sum_{\lambda} \Lambda^{|\lambda|} \left[ \lim_{q \rightarrow 0} H_\lambda^B(z, \zeta; q, t) \right] \left[ \lim_{q \rightarrow 0} H_\lambda^B(z, \zeta; q^{-1}, t) \right] \\ &= \sum_{\lambda} \Lambda^{|\lambda|} \left[ H_\lambda^B(z, \zeta; q, t) \right] \left[ H_\lambda^B(z, \zeta; q^{-1}, t) \right]. \end{aligned} \quad (5.74)$$

The conifold amplitude is well-known [238] to compute the Hilbert series of the instanton moduli space, in agreement with the calculation of section 5.4. This suggests a novel physical interpretation of the refined topological vertex as counting boundary local operators in a three dimensional gauge theory.

**3d mirror symmetry.** Since the theory is self mirror dual, we can equivalently write the  $B$ -twisted index in terms of the  $A$ -twisted index and from the arguments in chapter 4 only the vertex pieces contribute so that (up to relabelling  $z \rightarrow \zeta$  and  $t \rightarrow t^{-1}$ )

$$I^B = I^A = \sum_{\lambda} \zeta^{\sum_{s \in \lambda} h_{\lambda}(s)} t^{\frac{1}{2} \sum_{s \in \lambda} c_{\lambda}(s)} \left[ \sum_d \left( t^{-\frac{1}{2}} \zeta \right)^d P_{\mathbf{QM}_{\lambda}^d}(t) \right] \left[ \sum_d \left( t^{\frac{1}{2}} \zeta \right)^d P_{\mathbf{QM}_{\lambda}^d}(t^{-1}) \right]. \quad (5.75)$$

**Remark.** We conclude this section with remarks on how we expect to prove this correspondence and the physical interpretation. If we consider further compactifying the system on the M-theory circle, then we find type IIA string theory with D2-D0 bound states and a single D6 brane. These bound states are counted by PT invariants [239, 240]—the correspondence with the topological string computation arises from the PT/GW correspondence [241, 242].

The PT invariants of  $X$  can be computed using localisation to the torus fixed points  $X^T$  described by the tri-valent vertices in the toric diagram 5.4. It is argued in [243] that the one-legged PT vertex  $\mathcal{Z}_{\lambda}^{\text{PT}}$  (the torus contributions at the fixed points) are realised by the virtual Euler characteristics of quasimaps to  $\text{Hilb}^N(\mathbb{C}^2)$  so that

$$\mathcal{Z}_{\lambda}^{\text{PT}} = \sum_d \zeta^d \chi_{\tau}(\hat{\mathcal{O}}_{\text{vir}}, \mathbf{QM}_{\lambda}^d). \quad (5.76)$$

We note that it is also possible to take a more pedestrian approach and, in an appropriate chamber, directly identify the box counting formulation of the PT vertex [244, 245] with the sum over reverse plane partitions in equation (5.18). In this way, we can identify the topologically twisted gluing with the localisation calculation of PT invariants of  $X$ . In particular this applies to the full equivariant vertex including  $q$  and  $\zeta$  dependence. By the arguments of section 3.1.4 of chapter 3 the topologically twisted index is independent of these parameters and an exactly analogous cancellation in the conifold gluing of equivariant PT vertices was recently observed in [246].

It would be interesting to complete this circle of ideas physically by realising the twisted compactification on  $\mathbb{P}^1$  of the D2 world-volume theory in this type IIA context along the lines of [247].

## 5.5 Further directions: $\text{AdS}_4$ black holes

In this section we review recent progress in understanding black hole microstate counting from vortex partition functions [248, 218]. We discuss how this relates to our work and place these recent calculations in a geometrical framework.

In the IR our theory flows to a  $\mathcal{N} = 8$  superconformal field theory on the world-volume theory on  $N$  M2 branes [220], known as the ABJM theory. This theory has a Lagrangian

description as a  $U(N) \times U(N)$  gauge theory with opposite Chern-Simons levels  $\kappa = -\tilde{\kappa} = 1$ . ABJM is dual under the AdS/CFT correspondence to eleven dimensional supergravity on  $\text{AdS}_4 \times S^7$  (for a review see [249]). The gravitational theory admits a class of dyonic rotating black holes with  $\text{AdS}_4$  boundary conditions and in many examples indices of the dual field theory have been shown to account for the entropy [250, 251, 252].

Recent work [248, 218] computes holomorphic blocks of the UV theory in a large angular momentum *Cardy limit* and shows that a particular vacuum contribution dominates. Schematically they realise the vortex partition function as a contour integral

$$\mathcal{Z}_{S^1 \times D}(z, \zeta; q, t) \sim \oint \prod_{i=1}^N \frac{dw_i}{2\pi i w_i} w_i^{-\zeta} \prod_{i \neq j}^N \frac{\left(\frac{w_i}{w_j}; q\right)_\infty}{\left(q^{\frac{1}{2}} t^{-1} \frac{w_i}{w_j}; q\right)_\infty} \prod_{i,j=1}^N \frac{\left(q^{\frac{3}{4}} t^{-\frac{1}{2}} z \frac{w_i}{w_j}; q\right)_\infty}{\left(q^{\frac{1}{4}} t^{\frac{1}{2}} z \frac{w_i}{w_j}; q\right)_\infty} \prod_{i=1}^N \frac{\left(q^{\frac{3}{4}} t^{-\frac{1}{2}} w_i; q\right)_\infty}{\left(q^{\frac{1}{4}} t^{\frac{1}{2}} w_i; q\right)_\infty}. \quad (5.77)$$

This is the contour integral discussed in section 5.3 that computes the half index of a Neumann boundary condition for the theory on  $\mathbb{R}^2 \times \mathbb{R}^{\geq 0}$  in the presence of a Wilson line. For cotangent type theories there exist Neumann boundary conditions whose Higgs branch images coincide with the attracting submanifolds associated to the exceptional Dirichlet condition, however for the ADHM theory there is no consistent set of contours for (5.77) that account for all the vacua.

The Cardy limit corresponds to sending  $q \rightarrow 1$  in the partition function (5.77). In fact in a number of examples this limit is not a simplification and at large  $N$  the Cardy limit captures the correct black hole entropy even at finite  $q$ . The integral can be evaluated by a saddle point approximation, schematically we have

$$\mathcal{Z}_{S^1 \times D} \sim \sum_* \frac{1}{\sqrt{\partial^2 \mathcal{W}^*}} e^{\frac{1}{\log q} \mathcal{W}^*} + \dots, \quad (5.78)$$

where  $\dots$  indicate terms subleading in  $\log q$ . The saddle point equations can be re-expressed in the Cardy limit in the following way:

$$\zeta = \frac{1 - w_i t^{\frac{1}{2}}}{1 - w_i t^{-\frac{1}{2}}} \prod_{j \neq i}^N \frac{w_i - t w_j}{w_i - t^{-1} w_j} \frac{w_i - z^{-1} t^{-\frac{1}{2}} w_j}{w_i - z t^{\frac{1}{2}} w_j} \frac{w_i - z t^{-\frac{1}{2}} w_j}{w_i - z^{-1} t^{\frac{1}{2}} w_j}, \quad i = 1, \dots, N. \quad (5.79)$$

The authors of [218, 248] carefully analyse these equations in the large  $N$  limit and argue that a single vacuum  $\lambda = (1^N)$  dominates—we expect this to be related to the fact that, as discussed in section 5.3, the Neumann boundary condition is supported on the Hanany-Tong moduli space which only contains these column fixed points.<sup>14</sup> The dominant

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<sup>14</sup>In the  $N_f = 1$  case there is only one column fixed point.



block is known as a *Cardy block* and scales like

$$\log \mathcal{C}_{S^1 \times D} \sim -\frac{i\sqrt{3}}{2\log q} N^{\frac{3}{2}} \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}, \quad (5.80)$$

where  $\Delta_i$  are constrained combinations of the log fugacities  $\log \zeta$ ,  $\log t$ ,  $\log z$ —we refer the reader to [218] for the precise range of validity of these parameters. The Cardy blocks can be glued to determine the large  $N$  behaviour of the superconformal index

$$\begin{aligned} \mathcal{Z}_{\text{S.C.}} &= \mathcal{C}(q, t) \mathcal{C}(q^{-1}, t^{-1}), \\ \log \mathcal{Z}_{\text{S.C.}} &\sim -\frac{i\sqrt{3}}{\log q} N^{\frac{3}{2}} \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}. \end{aligned} \quad (5.81)$$

We see that the Cardy block gluing reproduces the entropy functional for  $\text{AdS}_4$  black holes at *finite* angular momentum [253].

**Geometric interpretation.** We now discuss how this result fits into the geometry of the Hilbert scheme of  $N$  points in the plane and the hemisphere blocks introduced in this thesis. Instead of writing holomorphic blocks as a contour integral, in section 4.3 we advocated the exceptional Dirichlet prescription for the hemisphere partition function associated to all vacua  $\lambda$  which leads to the hemisphere block

$$H_\lambda := e^{\phi_\lambda} \text{PE} \left[ \frac{t-q}{1-q} N_\lambda^+ \right] \mathbf{V}_\lambda(\zeta, x; q, t). \quad (5.82)$$

The Cardy limit corresponds to sending  $q \rightarrow 1$ . The perturbative contributions are relatively straightforward and we deal with those first. The  $q \rightarrow 1$  behaviour of the Casimir energy is immediate and we consider only the Cardy limit of the one loop contribution. For an analytic function the plethystic exponential (A.12) can be expressed as

$$\text{PE}[f(x)] = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} f(x^n) \right). \quad (5.83)$$

We re-write the one-loop piece in this way and set  $q = e^\epsilon$  with  $\epsilon \rightarrow 0$  to find

$$\begin{aligned} \lim_{q \rightarrow 1} \mathcal{I}_{1\text{-loop}}^\lambda &= \lim_{q \rightarrow 1} \text{PE} \left[ \sum_{n=1}^{\infty} \frac{1}{n} \frac{t^n - 1}{1 - q^n} N_\lambda^+(t^n, z^n) \right] \\ &= \exp \left( -\frac{1}{\epsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} (1 - t^n) \left( z^{a_\lambda(s) + l_\lambda(s) + 1} \right)^n \left( t^{\frac{1}{2}(l_\lambda(s) - a_\lambda(s) - 1)} \right)^n + \dots \right) \\ &= \exp \left( \frac{1}{\epsilon} \left[ \text{Li}_2(z^{a_\lambda(s) + l_\lambda(s) + 1} t^{\frac{1}{2}(l_\lambda(s) - a_\lambda(s) + 1)}) - \text{Li}_2(z^{a_\lambda(s) + l_\lambda(s) + 1} t^{\frac{1}{2}(l_\lambda(s) - a_\lambda(s) - 1)}) \right] + \dots \right). \end{aligned} \quad (5.84)$$

It remains to compute the  $q \rightarrow 1$  limit of the bare vertex  $V_\lambda$  for  $\text{Hilb}^N(\mathbb{C}^2)$ . The asymptotics of this type of limit have been studied extensively in the enumerative geometry literature in the context of quantum  $K$ -theory [51, 49, 50].

**Integrability of  $\text{Hilb}^N(\mathbb{C}^2)$ .** Recall from section 5.1 that the localised equivariant  $K$ -theory of  $\text{Hilb}^N(\mathbb{C}^2)$  is a ring of symmetric functions where skyscraper sheaves of fixed points can be identified with Macdonald polynomials labelled by  $\lambda$ . The geometric  $R$ -matrix construction<sup>15</sup> of Maulik and Okounkov [46] realises the localised  $K$ -theory as a Fock module of the quantum toroidal algebra  $U_t(\hat{\mathfrak{gl}}_1)$ :

$$K_{t_1, t_2} \left( \bigcup_N \text{Hilb}^N(\mathbb{C}^2) \right)_{\text{loc}} \cong \mathcal{F}(a). \quad (5.85)$$

These modules are of a similar class to those studied for the Yangian in section 3.2. We refer the reader to [255] for more details. The module structure can also be understood explicitly as an action on Macdonald polynomials without the full machinery of the geometric  $R$ -matrix [232].

The study of the *quantum  $K$ -theory* of Nakajima quiver varieties was initiated in [51]. We refer the reader to the work [50] for detailed constructions in the ADHM case and collect only some basic facts here. The quantum  $K$ -theory ring  $QK_\tau(\text{Hilb}^N(\mathbb{C}^2))$ , deforms the product on  $K_{t_1, t_2}(\text{Hilb}^N(\mathbb{C}^2))$  with additional contributions from curves in  $\text{Hilb}^N(\mathbb{C}^2)$ . The classical  $K$ -theory is generated by the rank  $N$  tautological bundle  $\mathcal{V}$ , this bundle is deformed in  $QK_\tau(\text{Hilb}^N(\mathbb{C}^2))$  to a quantum tautological bundle, denoted by  $\hat{\mathcal{V}}$ , which generates the quantum  $K$ -theory ring. By the construction of the localised  $K$ -theory, there is a natural action of the quantum  $K$ -theory ring on the localised classical  $K$ -theory that is diagonal in the basis of fixed points  $|\lambda\rangle = \mathcal{O}_\lambda$ . Specifically, if  $\hat{\mathcal{V}} \in QK_\tau(\text{Hilb}^N(\mathbb{C}^2))$  is the rank  $N$  quantum tautological bundle then

$$\hat{\mathcal{V}} \otimes |\lambda\rangle = \hat{\mathcal{V}}(w_1, \dots, w_N) |\lambda\rangle, \quad (5.86)$$

where on the right hand side we evaluate the Chern roots on the solution corresponding to  $\lambda$  in the following set of *Bethe equations*

$$\hat{a} \left( w_i \frac{\partial}{\partial w_i} T \text{Hilb}^N(\mathbb{C}^2) \right) = \zeta, \quad i = 1, \dots, N. \quad (5.87)$$

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<sup>15</sup>The geometric  $R$ -matrix is a systematic construction to understand extended quantum group actions on geometric invariants of Nakajima quiver varieties. It is a culmination of the work initiated in physics by [43, 44] known as the gauge-Bethe correspondence. We refer the reader to [254] for a review of more recent developments.

Using the tangent space formula in lemma 1 these equations are explicitly<sup>16</sup>

$$\zeta = \frac{1 - w_i t^{\frac{1}{2}}}{1 - w_i t^{-\frac{1}{2}}} \prod_{j \neq i}^N \frac{w_i - t w_j}{w_i - t^{-1} w_j} \frac{w_i - z^{-1} t^{-\frac{1}{2}} w_j}{w_i - z t^{\frac{1}{2}} w_j} \frac{w_i - z t^{-\frac{1}{2}} w_j}{w_i - z^{-1} t^{\frac{1}{2}} w_j}, \quad i = 1, \dots, N. \quad (5.88)$$

These are precisely the equations arising from the saddle point evaluation (5.79) of the holomorphic block integral and the Bethe ansatz equations for the quantum toroidal algebra action (5.85). These equations were also expressed using the contour integral form of the block in [50] but we note here that it is not necessary to formulate the block as a contour integral for the geometric argument to hold.

Now the important theorem that we can use to relate enumerative geometry and the black hole entropy story is the following due to [51]

**Theorem 11.** The  $q \rightarrow 1$  limit of the bare vertex is dominated by the eigenvalue of multiplication by the quantum tautological line bundle  $\hat{\mathcal{L}} := \det \hat{\mathcal{V}}$ . Precisely,

$$\lim_{q \rightarrow 1} \mathbf{V}_\lambda(\zeta, z; q, t) = \exp \left( \frac{1}{1 - q} \int d_q \zeta \log(l_\lambda(\zeta)) + \dots \right), \quad (5.89)$$

where  $l_\lambda(\zeta)$  are the (normalised) eigenvalues of multiplication by the quantum tautological line bundle and  $\int d_q \zeta$  denotes the Jackson  $q$ -integral.

**Remark.** Setting  $N = 1$  we have a tractable toy model of this theorem. The theory becomes SQED[1] with a decoupled adjoint hypermultiplet. We can then use the  $q$ -binomial theorem (A.8) to write the vertex function in this case as

$$\mathbf{V}(z, \zeta; q, t) = \sum_{d \geq 0} \left( \zeta t^{-\frac{1}{2}} q^{\frac{1}{2}} \right)^d \frac{(t; q)_d}{(q; q)_d}. \quad (5.90)$$

We can use the  $q$ -binomial theorem to ‘resum’ the vortices

$$\begin{aligned} \mathbf{V}(z, \zeta; q, t) &= \frac{(t \zeta t^{-\frac{1}{2}} q^{\frac{1}{2}}; q)_\infty}{(\zeta t^{-\frac{1}{2}} q^{\frac{1}{2}}; q)_\infty} \\ &= \text{PE} \left[ \frac{1 - t}{1 - q} \zeta t^{-\frac{1}{2}} q^{\frac{1}{2}} \right]. \end{aligned} \quad (5.91)$$

The asymptotics of which can be analysed in the same way as the one-loop contributions discussed above.

We conclude that the recent Cardy limit black hole entropy calculations of [248, 218] can be interpreted in the context of enumerative geometry as finding the dominant eigenvalue of multiplication by the quantum tautological line bundle  $\hat{\mathcal{L}}$ . We leave to future work

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<sup>16</sup>Recall  $(t_1, t_2) = (z t^{\frac{1}{2}}, z^{-1} t^{\frac{1}{2}})$ .

the interpretation of this calculation in terms of the integrable system associated to the quantum toroidal algebra  $U_t(\hat{\mathfrak{gl}}_1)$ .

# CHAPTER 6

## SUMMARY AND OUTLOOK

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In this thesis we have studied geometric and representation theoretic aspects of three dimensional  $\mathcal{N} = 4$  gauge theories. Many of the ideas presented were centred on understanding and developing the factorisation property of supersymmetric indices and partition functions.

In chapter 3 we studied the Hilbert series of Nakajima quiver varieties. We showed that Macdonald polynomials are a convenient tool to evaluate the Hilbert series and compute large gauge rank limits. We then considered handsaw quiver varieties in particular and showed that equivariant counts of line bundles over these quivers realise Yangian characters.

In chapter 4 we presented two main ideas. Firstly, we showed that twisted indices can be reformulated in terms of vortex geometry—in particular we showed that the twisted index coincides with the Hilbert series and can be factorised into a product of Poincaré polynomials of the vortex moduli space. We demonstrated this factorisation explicitly for the  $T[SU(N)]$  and  $\text{SQCD}[k, N]$  examples. Secondly, in this chapter we introduced hemisphere blocks defined in the UV with a particular boundary condition. The blocks realise factorisation of three manifold partition functions without perturbative ambiguities. We showed that these blocks are also determined solely in terms of the Higgs branch geometry of a theory and demonstrated that in certain supersymmetry enhancing limits they realise lowest weight Verma module characters of Higgs and Coulomb branch chiral rings.

In chapter 5 we studied the rich example of the 3d ADHM quiver gauge theory. This example is a non-trivial check on the hemisphere block proposal of chapter 4 and the theory also has some unique features of its own. We showed that the hemisphere blocks of the 3d ADHM theory are related to the topological vertex and the gluing of blocks corresponds to gluing vertices to compute the conifold amplitude. The blocks also have interesting combinatorial properties relating to counting reverse plane partitions and we investigated the implications of 3d mirror symmetry in this context. Finally, in this chapter we studied Neumann boundary conditions, and using the results of chapter 3, we showed that the half index of the 3d ADHM theory in the presence of a line operator exhibits an integrable

structure.

## 6.1 Further directions

We conclude with some directions for further research.

**3d mirror symmetry and quasimaps.** We expect it is possible to prove 3d mirror symmetry of the Hilbert series in generality by using the connection to the twisted index discussed in this thesis. Roughly speaking, we have shown that one can write the Hilbert series of the Higgs branch of a theory  $\mathcal{T}$  in terms of vertex functions as

$$\text{H.S.}[\mathcal{M}_H(\mathcal{T})] = \sum_{\alpha} \mathbf{V}_{\alpha}(x, \zeta; q, t) \mathbf{V}_{\alpha}(x, \zeta; q^{-1}, t). \quad (6.1)$$

This expression already implies surprising cancellations between the vertex functions that would be interesting to investigate from an enumerative geometry perspective. The recent works [86, 87, 88, 89] make explicit that symplectic duality acts on the vertex function by linear transformations induced by the elliptic stable envelope. Schematically we can write

$$\text{H.S.}[\mathcal{M}_H(\mathcal{T})] = \sum_{\alpha} U_{\alpha\beta} U_{\alpha\gamma} \tilde{\mathbf{V}}_{\beta}(x, \zeta; q, t) \tilde{\mathbf{V}}_{\gamma}(x, \zeta; q^{-1}, t), \quad (6.2)$$

where  $U_{\alpha\beta}$  is the elliptic stable envelope evaluated in the fixed point basis and  $\tilde{\mathbf{V}}$  are vertex functions of the symplectic dual variety. On the other hand, provided certain assumptions are met, 3d mirror symmetry allows us to write the Hilbert series in terms of the mirror dual variety as

$$\begin{aligned} \text{H.S.}[\mathcal{M}_H(\mathcal{T})] &= \text{H.S.}[\mathcal{M}_C(\tilde{\mathcal{T}})] \\ &= \sum_{\alpha} \tilde{\mathbf{V}}_{\alpha}(x, \zeta; q, t) \tilde{\mathbf{V}}_{\alpha}(x, \zeta; q^{-1}, t). \end{aligned} \quad (6.3)$$

This implies a ‘unitarity’ property of the elliptic stable envelope matrix  $U_{\alpha\beta} U_{\alpha\gamma} = \delta_{\beta\gamma}$  and it would be interesting to investigate this property from the perspective of quasimap moduli spaces.

**Characters beyond the specialised limits.** It would be interesting to investigate the representation theoretic interpretation of half indices and bulk algebra actions away from the specialised limits of section 4.4 in chapter 4. For example, we showed in that section that in some cases the general half index in the presence of a line operator is given by a Macdonald polynomial. Recent work [256] constructs boundary chiral algebras that categorify the half indices of 3d  $\mathcal{N} = 2$  theories. However, at least naively, this structure

is broken by the omega deformation and is not compatible with the perspective of section 4.4.

**Higgs branch localisation systematics.** In section 4.3 of chapter 4 we showed in the SQED[ $N$ ] example that the exceptional Dirichlet boundary condition procedure mimics the evaluation of quasimap vertex functions by Jackson  $q$ -integrals. We then extrapolated the result to suggest a general geometric expression for the hemisphere block. It would be interesting to test this proposal from the perspective of Higgs branch localisation. We expect that the results of [36, 37] could be adapted to the hemisphere  $S^1 \times H^2$  with a boundary and in this way directly make contact with the quasimap localisation computation.

**Quantum Hikita conjecture.** In chapter 5 we found a close relationship between the ( $K$ -theoretic) Coulomb branch algebra of  $\text{Hilb}^N(\mathbb{C}^2)$  and the difference operators of Kedem and Di Francesco [140]. We showed that the difference operators add line operator charge to the half index of  $\text{Hilb}^N(\mathbb{C}^2)$ . We expect this is an instance of a  $K$ -theoretic lift of the quantum Hikita conjecture of [229]; we leave a careful study of the relationships between these algebras to future work.

**Black hole entropy.** As discussed in more detail towards the end of chapter 5, it is possible, at least in principle, to extract the entropy functional of certain  $\text{AdS}_4$  black holes from the vertex functions of  $\text{Hilb}^N(\mathbb{C}^2)$ . It would be interesting to interpret this calculation in a more intrinsically geometric setting and elucidate the intriguing connections between black hole entropy and the representation theory of quantum toroidal algebras.





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# APPENDIX A

## SPECIAL FUNCTIONS AND COMBINATORICS

---

Our conventions for  $q$ -functions and enumerative combinatorics follow closely [257, 144, 258, 259], we summarise them here for ease of reference.

### A.1 $q$ -functions

**Definitions.** The  $q$ -Pochhammer function is defined for  $a \in \mathbb{C}$  and  $|q| < 1$  by the infinite product

$$(a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k), \quad (\text{A.1})$$

The analytic continuation to  $|q| > 1$  is given by

$$(a; q^{-1})_{\infty} = \prod_{k=0}^{\infty} \frac{1}{1 - aq^{k+1}}, \quad (\text{A.2})$$

which converges for  $|a| < q$ . For  $z \in \mathbb{C}$  we define the finite  $q$ -Pochhammer symbol by

$$(a; q)_z := \frac{(a; q)_{\infty}}{(aq^z; q)_{\infty}}, \quad (\text{A.3})$$

which for  $z = n > 0$  becomes

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (\text{A.4})$$

and for  $z = n < 0$  we have

$$(a; q)_n = \prod_{k=0}^{-n-1} \frac{1}{1 - aq^{-k-1}}. \quad (\text{A.5})$$

The  $q$ -Gamma function is defined for  $|q| < 1$  by

$$\Gamma_q(z) := \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z}. \quad (\text{A.6})$$

The  $q$ -binomial coefficient is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}. \quad (\text{A.7})$$

The  $q$ -binomial coefficient generalises the binomial expansion

$$(a; q)_n = \sum_{k=0}^n q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k. \quad (\text{A.8})$$

**Identities.** The  $q$ -Pochhammer functions satisfy the  $q$ -binomial theorem for  $|z| < 1$

$$\frac{(az; q)_\infty}{(z; q)_\infty} = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n. \quad (\text{A.9})$$

We note the *fusion identity*

$$(aq^{\frac{n}{2}}; q)_\infty (aq^{-\frac{n}{2}}; q^{-1})_\infty = (aq^{\frac{n}{2}}; q)_{-n+1}, \quad (\text{A.10})$$

and the following basic identities which follow straightforwardly from the definition

$$\begin{aligned} (a; q)_n &= (a^{-1}q^{1-n}; q)_n (-a)^n q^{\binom{n}{2}}, \\ (a; q)_{-n} &= \frac{(-qa^{-1})^n}{(qa^{-1}; q)_n} q^{\binom{n}{2}}. \end{aligned} \quad (\text{A.11})$$

**Plethystic exponential.** Let  $V$  be an  $N$ -dimensional torus  $T$  module with weights  $\omega_1, \dots, \omega_N$ . The *plethystic exponential* of a weight is defined by

$$\text{PE}[\omega] = \frac{1}{1 - \omega}, \quad (\text{A.12})$$

and extended to the module  $V$  by

$$\text{PE}[\omega + \omega'] = \text{PE}[\omega] \text{PE}[\omega']. \quad (\text{A.13})$$

We define

$$\text{PE}[-\omega] = 1 - \omega. \quad (\text{A.14})$$

The plethystic exponential can also be represented by

$$\text{PE}[V] = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \text{ch}_T(t_1^n, \dots, t_N^n) \right). \quad (\text{A.15})$$

## A.2 Partitions and Young diagrams

**Partitions.** A partition is a sequence of non-negative integers

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots), \quad (\text{A.16})$$

such that  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$  and only finitely many  $\lambda_i$  are non-zero. The  $\lambda_i$  are called parts of the partition. The *length* of a partition  $l(\lambda)$  is the number of non-zero entries in  $\lambda$  and the *weight* is defined by  $|\lambda| = \lambda_1 + \dots + \lambda_{l(\lambda)}$ . Setting  $n = |\lambda|$  we say that  $\lambda$  is a partition of  $n$  and write  $\lambda \vdash n$ . We occasionally make use of the norm

$$\|\lambda\|^2 = \sum_{i \geq 1} \lambda_i^2. \quad (\text{A.17})$$

We write  $m_i(\lambda)$  for the *multiplicity* of  $i$  in  $\lambda$  *i.e.* the number of times the part  $i$  appears in  $\lambda$ . This allows us to use the alternative notation for a partition

$$\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \dots). \quad (\text{A.18})$$

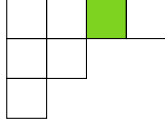
**Young diagrams.** A partition can be represented by a *Young diagram* defined as the set of points  $(i, j) \in \mathbb{Z}^2$  such that  $1 \leq j \leq \lambda_i$ .  $(i, j)$  are coordinates describing the rows and columns respectively of a partition, we denote by  $s \in \lambda$  a box with coordinates  $(i_s, j_s)$ —this is the ‘English’ convention of [144] and is summarised in figure A.1a. The *transpose* of a partition, denoted  $\lambda^\vee$ , is the partition obtained from transposing the diagram of  $\lambda$ . We denote the parts of the transpose partition by  $\lambda^\vee = (\lambda_1^\vee, \lambda_2^\vee, \dots)$ .

The *arm length* and *leg length* of a box  $s \in \lambda$  are defined by

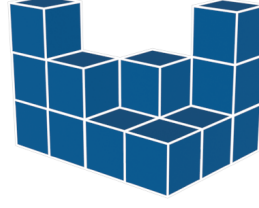
$$a_\lambda(s) := \lambda_i - j, \quad l_\lambda(s) := \lambda_j^\vee - i. \quad (\text{A.19})$$

Staring at the diagram A.1a explains why these functions are called arm and leg lengths. The *arm colength* and *leg colength* are defined by

$$a'_\lambda(s) = j - 1, \quad l'_\lambda(s) = i - 1. \quad (\text{A.20})$$



(a) The partition  $\lambda = (4, 2, 1)$  with the box  $s = (1, 3)$  highlighted.



(b) An example reverse plane partition with base  $\lambda$ . Here  $\pi_{(1,3)} = 2$ .

Figure A.1: Conventions for partitions and reverse plane partitions.

The *hook* and *content* of a box  $s \in \lambda$  are defined by

$$\begin{aligned} h_\lambda(s) &:= a_\lambda(s) + l_\lambda(s) + 1, \\ c_\lambda(s) &:= a'_\lambda(s) - l'_\lambda(s) = j_s - i_s. \end{aligned} \tag{A.21}$$

Summations over hook and content can be expressed in terms of the norm

$$\begin{aligned} \sum_{s \in \lambda} c_\lambda(s) &= \frac{1}{2}(\|\lambda\|^2 - \|\lambda^\vee\|^2), \\ \sum_{s \in \lambda} h_\lambda(s) &= \frac{1}{2}(\|\lambda\|^2 + \|\lambda^\vee\|^2). \end{aligned} \tag{A.22}$$

Or, alternatively, the sums can be expressed over the coordinates of boxes as

$$\begin{aligned} \sum_{s \in \lambda} c_\lambda(s) &= \sum_{(i,j) \in \lambda} j - i, \\ \sum_{s \in \lambda} h_\lambda(s) &= \sum_{(i,j) \in \lambda} i + j + 1. \end{aligned} \tag{A.23}$$

Finally, we define the statistic

$$n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i, \tag{A.24}$$

which satisfies  $n(\lambda) = \frac{1}{2}(\|\lambda\|^2 - |\lambda|)$ .

**Orderings.** The *lexicographic* ordering on partitions is defined by  $\lambda < \mu$  if for some  $i$  we have  $\lambda_i = \mu_j$  for all  $j < i$  and  $\lambda_i < \mu_i$ , *i.e.* the ordering one would put on partitions if one were to put them in the dictionary.

**Skew diagrams.** If  $\lambda$  and  $\mu$  are two partitions then  $\mu \subset \lambda$  means that the diagram for  $\mu$  is a subset of the diagram for  $\lambda$ . The set  $\lambda/\mu = \{\theta_i = \lambda_i - \mu_i \mid i = 1, 2, \dots\}$  is called a *skew diagram*. A skew diagram  $\theta$  is called *connected* if all of the boxes in  $\theta$  share at least

one common side. We say  $\theta$  is a *border strip* of a partition  $\lambda$  if  $\theta$  is contained in  $\lambda$  and  $\theta$  is connected with no  $2 \times 2$  blocks of boxes. The height  $\text{ht}$  of a border strip is defined to be one less than the number of rows it occupies. We further say  $\theta$  is a *maximal border strip* of  $\lambda$  if the box  $s = (i, j) \in \theta$  with maximal content is such that  $s = (i + 1, j)$  is not in  $\lambda$  and the box  $s' = (i', j')$  with minimal content is such that  $(i', j' + 1)$  is not in  $\lambda$ .

Every skew diagram  $\lambda/\mu$  can be uniquely decomposed into maximal border strips. The height of a skew diagram  $\text{ht}(\lambda/\mu)$  is the sum of the heights of the maximal border strips in the decomposition of  $\lambda/\mu$ .

**Reverse plane partitions.** A reverse plane partition (RPP)  $\pi$  with base  $\lambda$  is a 3d partition with non-negative integer heights  $\pi_s$  above each box  $s \in \lambda$  such that  $\pi_s$  weakly decrease along the rows and columns of  $\lambda$ , an example is shown in figure A.1b. We write  $|\pi| = \sum_{s \in \lambda} \pi_s$  for the total number of boxes in the reverse plane partition.

**Height.** A reverse plane partition can be thought of in terms of layers of skew shapes  $\lambda/\mu_i$ , with  $i = 1, 2, \dots$  stacked on top of each other. We define the *height of a reverse plane partition* in terms of the skew counterpart defined in the previous paragraph by:

$$\text{ht}(\pi) := \sum_{i \geq 1} \text{ht}(\lambda/\mu_i). \quad (\text{A.25})$$





# APPENDIX B

## SYMMETRIC FUNCTIONS AND MACDONALD POLYNOMIALS

---

In this appendix we review Macdonald polynomials following [144] and [258]. We leave detailed proofs to these references but spend some time reviewing the main results since this material is quite disjoint from conventionally assumed background in the 3d  $\mathcal{N} = 4$  gauge theory literature.

### B.1 Symmetric functions

We denote by  $\mathbb{Q}[x_1, \dots, x_N]$  the ring of rational polynomials in variables  $x_1, \dots, x_N$ . The symmetric group  $S_N$  acts naturally on this ring and we define the ring of symmetric polynomials in  $N$  variables by

$$\Lambda_N := \mathbb{Q}[x_1, \dots, x_N]^{S_N}. \quad (\text{B.1})$$

The ring is graded by homogeneous symmetric polynomials of degree  $k$  so that

$$\Lambda_N = \bigoplus_{k \geq 0} \Lambda_N^k. \quad (\text{B.2})$$

We write  $\Lambda$  for the ring of symmetric polynomials in infinitely many variables—this can be defined rigorously as an inverse limit of the graded rings with respect to the restriction maps that take  $\Lambda_M$  to  $\Lambda_N$  with  $M > N$  [144].

We use *plethystic notation* and write  $X = x_1 + x_2 + \dots$  for the set of variables  $\{x_1, x_2, \dots\}$ . The Cartesian product is  $XY = (x_1 + x_2 + \dots)(y_1 + y_2 + \dots)$  which represents the set  $XY = \{x_1y_1 + x_2y_1 + \dots + x_1y_2 + x_2y_2 + \dots\}$ . The union of variables is denoted  $X + Y = x_1 + x_2 + \dots + y_1 + y_2 + \dots$  which represents the set  $X + Y = \{x_1, y_1, x_2, y_2, \dots\}$ . In the following we also write  $\frac{1}{1-q}$  for the set of variables  $\{1, q, q^2, \dots\}$ . We write  $\bar{X}$  for the set of inverse variables  $\{x_1^{-1}, x_2^{-1}, \dots\}$ .

**Monomial symmetric functions.** For  $\lambda$  a partition the monomial symmetric functions are defined by

$$m_\lambda(X) := \sum_{\alpha} x^\alpha, \quad (\text{B.3})$$

where the sum is over *distinct* permutations  $\alpha$  of  $\lambda$ . The monomial symmetric functions  $m_\lambda$  with  $|\lambda| = k$  are a basis for  $\Lambda^k$ .

**Homogenous symmetric functions.** We now define the *homogenous symmetric functions*  $h_\lambda$ . Firstly, let

$$h_n = \sum_{\lambda \vdash n} m_\lambda, \quad (\text{B.4})$$

we then define

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \dots \quad (\text{B.5})$$

**Power sum symmetric functions.** The power sum symmetric functions are defined for  $n > 0$  by

$$p_n(X) := \sum_{i \geq 1} x_i^n. \quad (\text{B.6})$$

Similarly for a partition  $\lambda$  we define

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \dots \quad (\text{B.7})$$

The power sum symmetric functions  $p_\lambda$  form a basis for  $\Lambda$ .

**Hall inner product.** The *Hall inner product* is an inner product on  $\Lambda$  defined such that the homogenous and monomial symmetric functions are orthonormal:

$$\langle h_\mu, m_\lambda \rangle = \delta_{\mu\lambda}. \quad (\text{B.8})$$

**Schur polynomials.** The Schur polynomials  $s_\lambda(X)$  are elements of  $\Lambda_N$  defined by

$$s_\lambda(X) = \frac{\det_{1 \leq i, j \leq N} (x_i^{\lambda_j + N - j})}{\det_{1 \leq i, j \leq N} (x_i^{N - j})}, \quad (\text{B.9})$$

with  $l(\lambda) \leq N$ . The Schur polynomials are symmetric, homogeneous of degree  $|\lambda|$ , and form a basis for  $\Lambda_N$ . The elements of the transition matrix between monomial symmetric functions and Schur polynomials are known as *Kostka numbers*

$$s_\lambda = \sum_{\mu} K_{\lambda\mu} m_\mu. \quad (\text{B.10})$$

The *Schur Cauchy kernel* is the plethystic exponential

$$\text{PE}[XY] = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} . \quad (\text{B.11})$$

The Schur polynomials satisfy the Cauchy identity

$$\sum_{\lambda} s_{\lambda}(X) s_{\lambda}(Y) = \text{PE}[XY] . \quad (\text{B.12})$$

Schur polynomials are also orthonormal with respect to the Hall inner product

$$\langle s_{\lambda}, s_{\mu} \rangle = \delta_{\lambda, \mu} . \quad (\text{B.13})$$

The inner product can be written as an integral arising from the Haar measure of  $\mathfrak{gl}_N$

$$\langle f(X), g(X) \rangle = \oint d\mu_S^{(N)}[X] f(X) g(\bar{X}) , \quad (\text{B.14})$$

where the Schur measure in  $N$  variables is

$$d\mu_S^{(N)}[X] = \frac{1}{N!} \prod_{i=1}^N \frac{dx_i}{2\pi x_i} \prod_{i \neq j}^N \left( 1 - \frac{x_i}{x_j} \right) . \quad (\text{B.15})$$

**Hall-Littlewood polynomials.** Hall-Littlewood polynomials are an intermediate step between Schur polynomials and Macdonald polynomials. They are symmetric polynomials in  $\Lambda \otimes \mathbb{Q}(t)$  and, unlike Macdonald polynomials, admit an explicit formula as a sum over the Weyl group:

$$P_{\lambda}^{\text{HL}}(X; t) := \frac{1}{v_{\lambda}(t)} \sum_{\omega \in S_N} x_1^{\lambda_1} \dots x_N^{\lambda_N} \prod_{i < j} \frac{1 - t x_j / x_i}{1 - x_j / x_i} , \quad (\text{B.16})$$

where the Weyl group acts on the variables  $x_1, \dots, x_N$  and the normalisation constant is

$$v_{\lambda}(t) = \prod_{i \geq 0} \frac{(t; t)_{m_i(\lambda)}}{(1 - t)^{m_i(\lambda)}} . \quad (\text{B.17})$$

These polynomials interpolate between Schur polynomials and monomial symmetric functions in the following sense

$$P_{\lambda}^{\text{HL}}(X; 0) = s_{\lambda}(X) , \quad P_{\lambda}^{\text{HL}}(X; 1) = m_{\lambda}(X) . \quad (\text{B.18})$$

Hall-Littlewood polynomials satisfy the Cauchy identity

$$\sum_{\lambda} P_{\lambda}^{\text{HL}}(X; t) Q_{\lambda}^{\text{HL}}(X; t) = \text{PE}[(1 - t)XY] , \quad (\text{B.19})$$

where  $Q_\lambda^{\text{HL}}(X; t)$  is a renormalised Hall-Littlewood polynomial defined by  $Q_\lambda^{\text{HL}}(X; t) = b_\lambda(t) P_\lambda^{\text{HL}}(X; t)$  and the normalisation constant is

$$b_\lambda(t) = \prod_{i \geq 1} (t; t)_{m_i(\lambda)} . \quad (\text{B.20})$$

## B.2 Macdonald polynomials

Macdonald polynomials are symmetric functions in two parameters  $q, t$ . They are elements of<sup>1</sup>  $\Lambda \otimes \mathbb{Q}(q, t)$ .

**Plethystic substitution.** *Plethystic substitutions* of a symmetric functions  $f$  in  $\Lambda \otimes \mathbb{Q}(q, t)$  can be understood in terms of their power sum expansions, for example  $f\left(\frac{1-t}{1-q}X\right)$  denotes the replacement

$$p_n \rightarrow \frac{1 - t^n}{1 - q^n} p_n , \quad (\text{B.21})$$

in the power sum expansion of  $f$ . This agrees with the plethystic notation discussed at the end of section B.1 but we note a subtlety that if  $c \in \mathbb{Q}$  is not an indeterminate, then by definition we have  $p_n(cX) = cp_n(X)$  so in particular  $p_n(-X) = -p_n(X) \neq (-1)^n p_n(X)$ . The variables  $-X = \{-x_1, -x_2, \dots\}$  typically denote the plethystic substitution  $-p_n(X)$  rather than the evaluation  $(-1)^n p_n(X)$ .

**Modified Hall inner product.** The *modified Hall inner product* is defined on  $\Lambda \otimes \mathbb{Q}(q, t)$  by

$$\langle f(X), g(Y) \rangle_{q,t} := \langle f\left(X \frac{1-t}{1-q}\right), g(Y) \rangle . \quad (\text{B.22})$$

**Macdonald polynomials.** Macdonald polynomials are polynomials in  $\Lambda \otimes \mathbb{Q}(q, t)$  that have an upper-triangular expansion in the monomial symmetric functions with respect to the lexicographic ordering on partitions

$$P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu . \quad (\text{B.23})$$

They are orthogonal with respect to the modified Hall inner product

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad \text{with } \lambda \neq \mu . \quad (\text{B.24})$$

Existence and uniqueness of such polynomials was shown by Macdonald in [144]. In that reference Macdonald also shows that  $P_\lambda$  can be realised combinatorially as a sum over

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<sup>1</sup>In fact, as we shall see, suitably normalised Macdonald polynomials turn out to be elements of  $\Lambda \otimes \mathbb{Z}(q, t)$ .

Young tableaux, we do not review this construction here but note that it allows you to compute Macdonald polynomials by hand.<sup>2</sup> The Macdonald polynomials for partitions of three are

$$\begin{aligned} P_{111}(X; q, t) &= m_{111}, \\ P_{21}(X; q, t) &= m_{21} + \frac{(t-1)(2qt + q + t + 2)}{qt^2 - 1} m_{111}, \\ P_3(X; q, t) &= m_3 + \frac{(q^2 + q + 1)(t-1)}{q^2t - 1} m_{21} + \frac{(q+1)(q^2 + q + 1)(t-1)^2}{(qt-1)(q^2t-1)} m_{111}. \end{aligned} \quad (\text{B.25})$$

Macdonald polynomials are a basis for  $\Lambda \otimes \mathbb{Q}(q, t)$  and are homogeneous of degree  $|\lambda|$ . They specialise to Hall-Littlewood polynomials when  $q = 0$

$$P_\lambda(X; 0, t) = P_\lambda^{\text{HL}}(X; t), \quad (\text{B.26})$$

and when  $q = t$  we obtain Schur polynomials

$$P_\lambda(X; t, t) = s_\lambda(X). \quad (\text{B.27})$$

The renormalised Macdonald polynomial  $Q_\lambda$  is defined by

$$Q_\lambda(X; q, t) := b_\lambda(q, t) P_\lambda(X; q, t), \quad (\text{B.28})$$

where

$$b_\lambda(q, t) = \prod_{s \in \lambda} \frac{1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}}{1 - q^{a_\lambda(s)+1} t^{l_\lambda(s)}}. \quad (\text{B.29})$$

The polynomials  $Q_\lambda$  and  $P_\lambda$  are orthonormal with respect to the modified Hall inner product

$$\langle P_\lambda, Q_\mu \rangle_{q,t} = \delta_{\lambda\mu}. \quad (\text{B.30})$$

Macdonald polynomials satisfy the Cauchy identity

$$\sum_\lambda P_\lambda(X; q, t) Q_\lambda(X; q, t) = \text{PE} \left[ \frac{1-t}{1-q} XY \right], \quad (\text{B.31})$$

which follows from orthonormality with respect to the modified Hall inner product. Using the definition of the plethystic exponential (A.12) we can alternatively write the Macdonald Cauchy kernel as

$$\text{PE} \left[ \frac{1-t}{1-q} XY \right] = \prod_{\substack{x \in X \\ y \in Y}} \frac{(txy; q)_\infty}{(txy; q)_\infty}. \quad (\text{B.32})$$

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<sup>2</sup>Or borrow them from the excellent resource <https://www2.math.upenn.edu/~peal/polynomials/> as we have done.

Macdonald polynomials satisfy  $P_\lambda(q, t) = P_\lambda(q^{-1}, t^{-1})$  and we note the following identity

$$P_{\lambda+(s^r)}(x_1, \dots, x_r) = x_1^s x_2^s \dots x_r^s P_\lambda(x_1, \dots, x_r). \quad (\text{B.33})$$

which follows from the uniqueness property.

**Principal specialisation formula.** Macdonald polynomials  $P_\lambda$  satisfy the *principal specialisation* formula

$$P_\lambda\left(\frac{1}{1-t}; q, t\right) = t^{n(\lambda)} \prod_{s \in \lambda} \frac{1}{1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}}. \quad (\text{B.34})$$

**Macdonald  $J$  polynomials.** In section 3.2.4 we make use of *Milne polynomials*. Before discussing these, it is convenient first to study a normalisation of Macdonald polynomials known as Macdonald's  $J$  polynomials

$$J_\lambda(X; q, t) = c_\lambda(q, t) P_\lambda(X; q, t), \quad (\text{B.35})$$

where

$$c_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}). \quad (\text{B.36})$$

When combined with a plethystic substitution, these polynomials have a positive Schur expansion

$$J_\lambda\left(\frac{X}{1-t}; q, t\right) = \sum_{\mu} K_{\mu\lambda}(q, t) s_{\mu}(X), \quad (\text{B.37})$$

where the  $K_{\mu\lambda}(q, t)$  are two parameter generalisations of the Kostka number  $K_{\mu\lambda}$  with  $K_{\lambda\mu}(0, 1) = K_{\lambda\mu}$ . In fact  $K_{\mu\lambda}(q, t)$  have positive integral coefficients [145].

**Milne polynomials.** Milne polynomials are a degeneration limit of the plethystically substituted Macdonald  $J$  polynomial

$$Q'_\lambda(X; t) = \lim_{q \rightarrow 0} J_\lambda\left(\frac{X}{1-t}; t\right) = \sum_{\mu} K_{\mu\lambda}(t) s_{\mu}(X). \quad (\text{B.38})$$

Using the fact that

$$\lim_{q \rightarrow 0} c_\lambda(q, t) \prod_{i \geq 0} (t; t)_{m_i(\lambda)} =: b_\lambda(t), \quad (\text{B.39})$$

then we have that Milne polynomials are a re-normalised Hall-Littlewood polynomial (B.16) with a plethystic substitution, these are traditionally denoted by

$$Q'_\lambda(X; t) = b_\lambda(t) P_\lambda\left(\frac{X}{1-t}; t\right) = Q_\lambda\left(\frac{X}{1-t}; t\right). \quad (\text{B.40})$$

**Skew Macdonald polynomials.** We now turn to *Skew Macdonald polynomials*. First we define the structure constants  $f_{\mu\nu}^\lambda(q, t)$  to be the coefficients in the expansion of a product of two Macdonald polynomials

$$P_\mu P_\nu = \sum_\lambda f_{\mu\nu}^\lambda P_\lambda. \quad (\text{B.41})$$

The coefficients  $f_{\mu\nu}^\lambda$  vanish unless  $|\lambda| = |\mu| + |\nu|$  and  $\nu, \mu \subset \lambda$ . Skew Macdonald polynomials are then defined by

$$Q_{\lambda/\mu}(X; q, t) := \sum_\nu f_{\mu\nu}^\lambda(q, t) Q_\nu(X; q, t), \quad (\text{B.42})$$

and the renormalised  $P_{\lambda/\mu}$  version is defined by

$$Q_{\lambda/\mu}(X; q, t) = \frac{b_\lambda(q, t)}{b_\mu(q, t)} P_{\lambda/\mu}(X; q, t). \quad (\text{B.43})$$

Both of these polynomials are manifestly symmetric and homogeneous of degree  $|\lambda| - |\mu|$ . They vanish unless  $\mu \subset \lambda$ . Skew Macdonald polynomials satisfy a *skew Cauchy identity*

$$\sum_\lambda P_{\rho/\lambda}(X; q, t) Q_{\rho/\mu}(Y; q, t) = \text{PE} \left[ \frac{1-t}{1-q} XY \right] \sum_\rho P_{\mu/\rho}(X; q, t) Q_{\lambda/\rho}(Y; q, t). \quad (\text{B.44})$$

**Integral inner product.** We define an integral inner product  $\langle -, - \rangle'_{N; q, t}$  on  $\Lambda_N \otimes \mathbb{Q}(q, t)$  by

$$\langle f, g \rangle'_{N; q, t} = \oint d\mu_M^{(N)} f(X) g(\bar{X}), \quad (\text{B.45})$$

where the Macdonald measure in  $N$  variables is

$$d\mu_M^{(N)}[X; q, t] = \frac{1}{N!} \prod_{i=1}^N \frac{dx_i}{2\pi x_i} \prod_{i \neq j} \frac{(x_i/x_j; q)_\infty}{(tx_i/x_j; q)_\infty}. \quad (\text{B.46})$$

Macdonald polynomials are orthogonal with respect to this inner product but not orthonormal. The normalisation constant is given by

$$\langle P_\lambda, P_\mu \rangle'_{N; q, t} = \delta_{\lambda\mu} \tilde{c}_N(\lambda; q, t) \frac{1}{b_\lambda(q, t)}, \quad (\text{B.47})$$

where

$$\tilde{c}_N(\nu; q, t) = \prod_{i=1}^N \frac{\Gamma_q(i\beta)}{\Gamma_q(\beta)\Gamma_q((i-1)\beta+1)} \prod_{s \in \lambda} \frac{1 - q^{a'(s)} t^{N-l'(s)}}{1 - q^{a'(s)+1} t^{N-l'(s)-1}}, \quad (\text{B.48})$$

in the above we write  $t = q^\beta$ . The constant  $\tilde{c}_N$  was determined in the work [260].