Gauge/ Gravity Correspondence, Bulk Locality and Quantum Black Holes

by

Debajyoti Sarkar

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2014

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

____________	________________________
Date	Prof. Daniel N. Kabat Chair of Examining Committee
____________	________________________
Date	Prof. Steven Greenbaum Executive Officer

Prof. Dimitra Karabali ________________________

Prof. V. Parameswaran Nair ________________________

Prof. Alexios Polychronakos ________________________

Prof. Martin Schaden ________________________

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

Gauge/ Gravity Correspondence, Bulk Locality and Quantum Black Holes

by

Debajyoti Sarkar

Advisor: Daniel N. Kabat, Professor of Physics

The aim of this dissertation is threefold. We begin by the study of two parallel ideal cosmic strings in the presence of non-minimal scalar fields and spin-1 gauge fields. We show that the contributions of the non-minimal term on the interaction energy between the strings are similar to that of the gauge field for a particular value of non-minimal coupling parameter. In this context we clarify some of the issues that arise when comparing the renormalization of black hole entropy and entanglement entropy using the replica trick.

In the second part of the dissertation we study the process of bound state formation in clusters of Dp- brane collision and Dp shell/ Membrane collapse processes. We consider two mechanisms for bound state formation. The first, operative at weak coupling in the worldvolume gauge theory, is creation of W-bosons. The second, operative at strong coupling, corresponds to formation of a black hole in the dual supergravity. These two processes agree qualitatively at intermediate coupling, in accord with the correspondence principle of Horowitz and Polchinski. We show that the size of the bound state and timescale for formation of a bound state agree at the correspondence point, along with other relevant thermodynamic quantities. The timescale involves matching a parametric resonance in the gauge theory to a quasinormal mode in supergravity.

Finally we study construction of local operators in AdS using the generalized AdS/ CFT correspondence. After briefly sketching previous works on this topic which involve massless and massive scalar fields, we present similar construction for spin- 1 and spin- 2 gauge fields. Working in holographic gauge in the bulk, at leading order in $1/N$ bulk gauge fields are obtained by smearing boundary currents over a sphere on the complexified boundary, while linearized metric fluctuations are obtained by smearing the boundary stress tensor over a ball. This representation respects AdS covariance up to a compensating gauge transformation. We also consider massive vector fields, where the bulk field is obtained by smearing a non-conserved current. We compute bulk two-point functions and show that bulk locality is respected. We show how to include interactions of massive vectors using $1/N$ perturbation theory, and we comment on the issue of general backgrounds. We point out some more recent works on interacting scalar and gauge fields and try to answer the question of what should be the CFT properties to have a dual gravitational descriptions on AdS space. We end with some speculations about finite N and when we have black holes in the bulk.

To My Family

Acknowledgments

On one hand, it is quite impossible for me to properly acknowledge everyone who have contributed to my academic life and stayed beside me leading to my PhD degree and on the other hand it is a pure pleasure. Especially, I think of myself as being exceptionally lucky to be alongside of so many great people and being fortunate enough to have them sometimes as my friend and sometimes as my guide. But being lucky has its own cost: the list is humongous and they come in so many colors and flavors. So without even trying, I will be very precise and will only name those (unsuccessfully) who have been directly involved and have tremendous effect in my academic world leading to my PhD degree.

To begin with, of course I should recall my adviser Prof. Daniel Kabat, a fantastic person, who has shared his knowledge and insights to let me and make me mature in physics and paved my way for the future. Once again there are so many things I can say about him, that I will simply stop. Physics-maturity-wise, he shaped me to who I am, with all the defects being entirely my shortcomings. I would also like to thank him for various financial supports which enabled me to participate in various academic frontiers, which was essential towards my graduation.

Among the other faculties at City University of New York (CUNY), I should of course mention Prof. Dimitra Karabali, Prof. V. Parameswaran Nair, Prof. Alexios Polychronakos for not only creating a vibrant research atmosphere at City College/ Lehman College, but also for teaching me various courses and finally serving as my thesis committee member. I would also like to thank Prof. Martin Schaden and Prof. Alberto Nicolis for serving respectively at my thesis and pre-thesis defense committee. Special thanks also goes to Prof. Peter Orland for couple of his fabulous QFT classes and his various advice.

And finally among the CUNY faculties, I will end with Prof. Justin Vazquez-Poritz. What a wonderful friend, philosopher and guide he has been to me! I will not try to enumerate his contributions because I can't, but will simply acknowledge that I will always regard him as one of the best mentors that I had at CUNY.

I would also like to thank all the CUNY academic and administrative staffs, all the CUNY and outside professors, postdocs and graduate students whom I met at CUNY and elsewhere for helping me out, for sharing their knowledge and for easing my way out. Special thanks goes to Shubho Roy, a postdoc in our group for a couple of years with whom I shared many cups of coffee and learned a lot of things. I would also like to thank Diptarka Das, Norihiro Iizuka, Gilad Lifschytz and Xiao Xiao for past and ongoing collaborations. Special thanks also goes to Mr. Daniel Moy, our Physics APO for many interesting conversations and buckets of help.

Personally (and also pre-graduate study wise), I would like to start off with my parents and stop right there. No thesis is long enough to fully reflect their contributions to my life. What about my wife and her family? No. I will have to stop again. May be I will simply say that I am grateful to her for her love and life (and also so that she smiles a bit more if she reads this acknowledgement). Then if I am to name friends starting with most recent ones, in my graduate life all names will be after Dario Capasso. Thanks for being such a great friend! Also during my undergraduate and Masters studies, I have had the fortune of having great friends and teachers. Among my friends, Nirupam Dutta for motivating me towards high energy physics and introducing me to Rajsekhar Bhattacharyya, a wonderful teacher that I had in my college days. And there are Arindam, Debmalya, two Partha's, Soma, Arko ... simply so many. Among the teachers Prof. Babul Dutta, Prof. Partha Pratim Basu and again so many; and then again in my school life I had some of my greatest class mates and teachers who offered me a very competitive and intelligent environment to grew up with.

The acknowledgment is already nearly two page long now and I am having a feeling of dissatisfaction mixed with satisfaction and nostalgia. But the submission date is near and may be I should simply conclude by saying: "thanks to you all!"

Contents

List of Figures

1

Introduction

No doubt that unifying the four forces of nature has always been one of the ultimate goal of the Physicists. Sometimes the goal has been from pure academic interests, and sometimes it has been driven by experimental evidence, but both these motivations have contributed to its progress. And now with the well built classical General Relativity at one hand and the Standard Model on the other, we have all the reasons to get encouraged towards fulfilling that goal. We even have succeeded appreciably. There are now different programs towards quantizing gravity in a well formulated way. The richest and best-developed program is String Theory [1] which was built with the simple assumption that the fundamental constituents of matter are strings. The modern point of view is to consider them as a well built mathematical formalism which has a much richer structure to incorporate various other constituents like D-branes. String Theory starts as a quantum theory and nicely and naturally incorporates gravity. With the help of an exotic symmetry called Supersymmetry,* we can even build some models within the framework of string theory that incorporate the various particles that we see in nature and explain by the Standard Model. Now as the Standard Model is a gauge theory, the String Theory program is an attempt to account for both gauge and gravity dynamics in terms of the same underlying stringy degrees of freedom. It suggests that the same tools can be used to understand both gauge and gravity dynamics. The duality relation found by Maldacena (for a review see e.g. [2]) was a huge step towards this goal. It provides a precise dictionary between gauge and gravitational dynamics, and

*For a comprehensive discussion on Quantum Field Theory (QFT), Supersymmetry (SUSY), Supergravity (SUGRA) and Conformal Field Theory etc. see e.g. [1].

goes by the name of AdS/CFT duality. It is basically a correspondence that relates string theory in Anti de-Sitter (AdS) space to the conformal field theory (CFT) residing at the timelike boundary of AdS. This duality is most readily visualized in the so called 't Hooft limit (with 't Hooft coupling parameter $\lambda = g_{YM}^2 N \to$ fixed) of the field theory side and can be generalized to non-conformal, finite temperature systems to make closer contact with our everyday world. Here g_{YM} is the field theory coupling constant and N is the number of 'colors' that is taken to be a very large number†. Also, this duality adds to previously known dualities in string theory such as Matrix Models which again is a correspondence between the so called Matrix theories (where the degrees of freedom are matrices and not usual commutative coordinates) and M-theory in a particular frame (for a review see e.g. [4]). Though not proven, till now we don't have any counterexample to these dualities. The main difficulty in proving the conjecture has been the fact that if the gravity side is weakly coupled, the field theory side becomes strongly coupled (and vice-versa) signaling the breakdown of well-known perturbation techniques. But on the positive side, using this conjecture we can give striking predictions of properties of some strongly correlated field theoretic systems (e.g. the Quark Gluon Plasma that has been discovered recently in Brookhaven National Laboratory (BNL)). Also new directions have opened up, and it has enabled us to grow a closer contact to other fields of physics such as Condensed Matter Physics, Nuclear Physics and Atomic Physics where it may finally find an opportunity of experimental verification.

We cover three main topics in this thesis. In chapter 2, we start off by discussing Casimir energy calculation in the presence of two parallel cosmic strings in four dimension. The cosmic strings are taken to be ideal and so they have zero thickness and some length L. We compute the interaction energy between them in the presence of spin-1 gauge fields and non-minimally coupled scalar fields. We find that in such set ups the contribution for the gauge fields towards interaction energy in such spacetime is precisely similar to the contribution from a non-minimally coupled scalar for a specific value of the non-minimal coupling parameter. These type of terms also appear in the renormalization of gravitational coupling constant when calculating the one loop correction of the Bekenstein Hawking entropy of a black hole in the presence of gauge fields. But these terms

†When talking about N as 'colors', we are assuming a color group in the CFT side. This is because the best developed and understood AdS/CFT duality come in terms of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory with $SU(N)$ gauge group on the CFT side. In general we should take a CFT with a matrix model description where the sizes of the matrices are $N \times N$ and for the duality to correspond to a weakly coupled bulk dual, we can think of these matrices to be very large. See e.g. [3].

are absent from the entanglement entropy renormalization. This mismatch of renormalization were subject to some confusion lately and our calculation supports the presence of such 'contact terms' in a gauge invariant quantity like interaction energy. We comment on these issues at the end of this chapter.

In the next couple of chapters, we discuss formation of black holes in terms of gauge/ gravity dualities such as AdS/ CFT and Matrix models. The relevant results are presented in chapter 3 and in chapter 4. First in chapter 3, we study the formation of the black hole in the collapse process of Dp brane clusters and Dp brane shells. The calculation is done in the limit where the dimensionless effective 't Hooft coupling (λ_{eff}) tends to 1. λ_{eff} is a dimension dependent quantity and is given by $\frac{\lambda}{U^{3-p}}$ where U is the relevant energy scale in the study of Dp branes. From the basic ideas of AdS/ CFT we know that $\lambda_{eff} \to 1$ is the correspondence point between the two dual theories and we match our gauge theory calculation with the SUGRA ones at that point. Chapter 4 on the other hand, is devoted to the collapse of a Membrane or Fuzzy spheres. For both types of systems, we show the presence of parametric resonance. It turns out that any matrix like models with a quartic interaction potential with time dependent initial condition have this resonance process and upon calculating various thermodynamic quantities of interests at the correspondence point, they quite naturally match up with the SUGRA side. We end both of the sections with a discussion of various generalization that one can do to such systems to go beyond the constraints of correspondence point.

Chapter 5 and chapter 6 are then devoted to the construction and study of local operators in AdS using the information of the CFT operators at large values of N using generalized AdS/ CFT[‡]. After briefly reviewing some known results, we present new results. These new results contain a study of spin-1 Maxwell fields in chapter 5 and the gravitons in chapter 6. We consider only free fields. Here the statement of locality due to gauge invariance is already subtle, and for gravity the problem is even more non-local due to the diffeomorphism symmetry. We show how can one still have a notion of local bulk operators in such cases. We end these chapters by presenting some new results which captures this construction for gauge fields for the next order in $1/N$ interaction theory in the bulk. We also present some ongoing work which deals with four-point functions and higher.

We conclude in chapter 7 where we collect our findings so far and discuss some of the future

[‡]Here we are basically trying to answer the general question of 'what should be the CFT properties for it to have a local bulk dual?'. Also see the footnote of previous page.

directions and present our outlook. Finally, the appendices then collect some calculation that we left out during the main chapters.

2

Cosmic String Interactions Induced by Gauge and Scalar Fields

As mentioned in the introduction, this chapter stands quite alone from the underlying tool of the rest of the dissertation. Here we calculate the interaction energy between two parallel ideal cosmic strings in the presence of gauge and non-minimally coupled scalars. We point out the relevance of the result with the ideas of black hole entropy calculation using the replica trick and it's difference from the entropy of entanglement. The work described here is based on the paper [5].

For a single cosmic string in four Euclidean dimensions the metric is [6, 7]

$$ds^2 = dr^2 + r^2 d\psi^2 + d\tau^2 + dz^2 \tag{2.1}$$

The string tension produces a deficit angle, $\psi \approx \psi + \beta$ where

$$\beta = 2\pi - 8\pi\lambda \tag{2.2}$$

Here $\lambda = G\mu$ where G is Newton's constant and μ is the mass per unit length of the string.

We will be interested in the interaction between two parallel cosmic strings. At the classical level there is no force between strings,* but (as in the Casimir effect) an interaction potential can be generated at one loop by a quantum field propagating on this background. For simplicity we will

*In classical gravity there is, however, a non-trivial scattering amplitude which results from the conical boundary conditions [8, 9].

take a perturbative approach, and calculate the interaction energy at first order in the product of the two deficit angles. We consider two types of fields – scalar fields with a non-minimal coupling to curvature, and abelian gauge fields – as the main point of this chapter is to highlight a relation between these two cases. Vacuum polarization in the presence of a single cosmic string has been studied before; see for example [10, 11, 12] for scalar fields and [13, 14] for gauge fields. For related calculations in the presence of multiple cosmic strings see [15, 16].

We begin by recalling the argument that, to first order in the background curvature, there should be a relation between gauge fields and scalar fields with specific non-minimal couplings to curvature. To our knowledge this relation was first stated in [17], although the essence of the following argument is taken from [18]. Consider a spacetime which is a product $\mathcal{M}_n \times \mathbb{R}^{d-n}$ of a weakly-curved n-dimensional Einstein manifold $\mathcal{M}_n$ with flat space $\mathbb{R}^{d-n}$. The metric takes the form

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + \delta_{ij} dx^i dx^j \tag{2.3}$$

where x^α are coordinates on $\mathcal{M}_n$ and x^i are coordinates on $\mathbb{R}^{d-n}$. The Einstein manifold has Ricci curvature $R_{\alpha\beta} = \frac{1}{n} g_{\alpha\beta} R$.† Choose a vielbein $g_{\alpha\beta} = e^a_\alpha e^b_\beta \delta_{ab}$ and denote the corresponding spin connection ω_α.

To establish the relation between gauge and scalar fields we compare their equations of motion. For a gauge field, the equations of motion in Feynman gauge are

$$-\nabla_\nu \nabla^\nu A_\mu + R_{\mu\nu} A^\nu = 0 \tag{2.4}$$

where $x^\mu = (x^\alpha, x^i)$. There are ghosts associated with this choice of gauge which behave like a pair of minimally-coupled scalar fields [19]. The components of the gauge field tangent to $\mathbb{R}^{d-n}$ obey

$$-\nabla_\beta \nabla^\beta A_i - \partial_j \partial^j A_i = 0 \tag{2.5}$$

where the covariant derivative ∇_α treats A_i as a singlet of $SO(n)$. That is, the components A_i behave like minimally-coupled scalar fields. The components of the gauge field tangent to $\mathcal{M}_n$, on

†By Einstein manifold we mean a manifold with Ricci curvature locally proportional to the metric, $R_{\alpha\beta}(x) = f(x) g_{\alpha\beta}(x)$. In two dimensions all manifolds are Einstein. In higher dimensions the contracted Bianchi identity $\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = 0$ requires that f be a constant. In either case it follows from the definition that $R_{\alpha\beta} = \frac{1}{n} g_{\alpha\beta} R$.

the other hand, obey

$$-\nabla_\beta \nabla^\beta A_a - \partial_j \partial^j A_a + \frac{1}{n} R A_a = 0 \tag{2.6}$$

Here ∇_α acts on $A_a = e_a{}^\alpha A_\alpha$ in the fundamental representation of $SO(n)$, and we've made use of the fact that $R_{\alpha\beta} = \frac{1}{n} g_{\alpha\beta} R$. So the components A_a are in the fundamental representation of $SO(n)$ and have an explicit non-minimal coupling to curvature.

Physical quantities can be computed perturbatively, as an expansion in powers of the background curvature. As a concrete example imagine computing the effective action for the background which results from integrating out A_μ. The spin connection can appear in the effective action, but only through its field strength $F = d\omega + \omega^2$. In fact the field strength can first appear in the effective action in terms such as $F_{\alpha\beta}F^{\alpha\beta}$ that are quadratic in the curvature. So to first order in the background curvature we can forget about the spin connection and treat A_a as a collection of n scalar fields with a non-minimal coupling to curvature. The equation of motion for a non-minimal scalar is

$$-\nabla_\beta \nabla^\beta \phi - \partial_j \partial^j \phi + \xi R \phi = 0\,, \tag{2.7}$$

and comparing to (2.6) we identify the effective non-minimal coupling parameter $\xi = 1/n$. Thus to first order in the background curvature a gauge field is equivalent to n scalar fields with $\xi = 1/n$, plus $d - n$ minimally-coupled scalars.

This discussion is relevant to parallel cosmic strings because in two dimensions every manifold is an Einstein manifold. The argument suggests that, to first order in the product of the deficit angles, the interaction between two cosmic strings induced by a gauge field should be the same as the interaction induced by an appropriate collection of non-minimal scalars.

In the remainder this chapter we verify this claim, by computing the interaction energy between cosmic strings perturbatively. In section 2.1 we compute the interaction energy for a scalar field, and in section 2.2 we carry out the corresponding computation for a gauge field. We conclude in section 2.3, where we comment on our results and point out the relation to studies of black hole entropy.

2.1 Non-minimal scalar energy

The Euclidean action is

$$S = \int d^4x\,\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\xi R\phi^2\right)$$

For the conical geometry (2.1) the scalar curvature is[‡]

$$R = 16\pi\lambda\delta^2(x)/\sqrt{g} \tag{2.8}$$

The action on a cone can be split into three pieces,

$$S_{\text{cone}} = S_0 + S_{\text{int}}, \qquad S_{\text{int}} = S_{\text{wedge}} + S_{\text{tip}} \tag{2.9}$$

where

$$S_0 = \int d^4x\,\frac{1}{2}\delta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{2.10}$$

is the action in flat space,

$$S_{\text{wedge}} = -\int d\tau dz \int_0^\infty r dr \int_{-4\pi\lambda}^{4\pi\lambda} d\psi\,\frac{1}{2}\delta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{2.11}$$

cancels the flat-space action in the region corresponding to the deficit angle, and

$$S_{\text{tip}} = \int d\tau dz\, 8\pi\lambda\xi\phi^2 \tag{2.12}$$

arises from the non-minimal coupling to curvature. It's straightforward to extend this to a pair of cosmic strings, just by putting the deficit angles in opposite directions as shown in Fig. 2.1.

We will treat S_{int} as a perturbation.[§] To find the interaction energy per unit length along the strings $\mathcal{H}_{\text{int}}$ we use

$$\int d\tau dz\,\mathcal{H}_{\text{int}} = \langle 1 - e^{-S_{\text{int}}}\rangle_{C,0} \tag{2.13}$$

[‡]The easiest way to see this is to note that a truncated cone, i.e. a disc with a conical singularity at the center, has Euler characteristic $\chi = \frac{1}{4\pi}\int d^2x\sqrt{g}R + \frac{1}{2\pi}\beta = 1$.

[§]This is somewhat subtle, since it's not manifest that perturbation theory in S_{int} will enforce the proper conical boundary condition $\phi(r,\psi) = \phi(r,\psi+\beta)$. Fortunately the boundary conditions are controlled by the spin connection on the cone, which as we argued in the introduction can only enter at second order in the deficit angle.

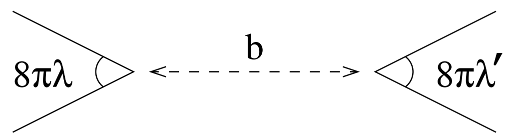

Figure 2.1: Two parallel cosmic strings, separated by a distance b.

where the subscript $C, 0$ denotes a connected correlation function computed in the unperturbed theory (2.10). Expanding in powers of S_{int}, the leading $\mathcal{O}(\lambda\lambda')$ interaction between the strings comes from

$$\int d\tau dz\, \mathcal{H}_{\text{int}} \approx -\langle S^{(1)}_{\text{int}}\, S^{(2)}_{\text{int}} \rangle_{C,0} \tag{2.14}$$

where the superscripts (1), (2) refer to the first and second cosmic string, respectively. Some useful unperturbed correlators are

$$\langle \phi(x)\phi(x') \rangle = \frac{1}{4\pi^2}\frac{1}{(x-x')^2} \tag{2.15}$$

and

$$\begin{aligned}
\left\langle (\partial\phi)^2(x)\,(\partial\phi)^2(x') \right\rangle &= \frac{6}{\pi^4}\frac{1}{(x-x')^8} \\
\left\langle (\partial\phi)^2(x)\,\phi^2(x') \right\rangle &= \frac{1}{2\pi^4}\frac{1}{(x-x')^6} \\
\left\langle \phi^2(x)\,\phi^2(x') \right\rangle &= \frac{1}{8\pi^4}\frac{1}{(x-x')^4}
\end{aligned}$$

There are three types of interactions. For generality we can imagine that the two strings have different non-minimal couplings ξ, ξ'.

<u>wedge – wedge</u>

To first order in λ and λ' the wedges can be treated as very narrow, so that

$$\begin{aligned}
\mathcal{H}_{\text{int}} &= -16\pi^2\lambda\lambda' \int d\tau dz \int_0^\infty x dx \int_0^\infty x' dx'\, \frac{6}{\pi^4}\frac{1}{\left(\tau^2+z^2+(x+x'+b)^2\right)^4} \\
&= -\frac{4\lambda\lambda'}{15\pi b^2}
\end{aligned}$$

wedge – tip

For wedge 1 with tip 2 we have

$$\begin{aligned}\mathcal{H}_{\text{int}} &= 32\pi^2\lambda\lambda'\xi' \int d\tau dz \int_0^\infty x dx\, \frac{1}{2\pi^4}\, \frac{1}{\left(\tau^2+z^2+(x+b)^2\right)^3} \\ &= \frac{4\lambda\lambda'\xi'}{3\pi b^2}\end{aligned}$$

tip – tip

The interaction between the two tips is

$$\begin{aligned}\mathcal{H}_{\text{int}} &= -64\pi^2\lambda\lambda'\xi\xi' \int d\tau dz\, \frac{1}{8\pi^4}\, \frac{1}{\left(\tau^2+z^2+b^2\right)^2} \\ &= -\frac{8\lambda\lambda'\xi\xi'}{\pi b^2}\end{aligned}$$

Assembling these results, to first order in λ and λ' the interaction energy per unit length due to a non-minimally coupled scalar field is

$$\mathcal{H}_{\text{int}} = \frac{\lambda\lambda'}{\pi b^2}\left(-\frac{4}{15} + \frac{4}{3}(\xi+\xi') - 8\xi\xi'\right) \tag{2.16}$$

To check the validity of our perturbative approach consider computing $\langle\phi^2\rangle$ for a minimally-coupled scalar field in the presence of a single cosmic string. At first order in perturbation theory, after subtracting the divergence which is present in flat space, we have

$$\langle\phi^2\rangle = -\langle\phi^2 S_{\text{wedge}}\rangle_{C,0} = \frac{\lambda}{6\pi^2 r^2} \tag{2.17}$$

where r is the distance from the tip of the cone. On the other hand $\langle\phi^2\rangle$ can be computed exactly,

$$\langle\phi^2\rangle = \int_0^\infty ds\, K(s,x,x) \tag{2.18}$$

where the scalar heat kernel on a cone is[¶]

$$K(s,x,x) = -\frac{1}{2\beta}\frac{1}{(4\pi s)^2}\int_{-\infty}^{\infty} dy\, e^{-\frac{r^2}{s}\cosh^2(y/2)}\left(\cot\frac{\pi}{\beta}(\pi+iy)+\cot\frac{\pi}{\beta}(\pi-iy)\right) \tag{2.19}$$

Expanding the heat kernel to first order in the deficit angle and integrating over s reproduces (2.17).

2.2 Gauge field energy

We start from the Euclidean action

$$\begin{aligned} S &= S_{\text{Maxwell}} + S_{\text{gauge fixing}} \\ &= \int d^dx\sqrt{g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}(\nabla_\mu A^\mu)^2\right) \end{aligned}$$

There are ghosts associated with this choice of gauge that behave like a pair of minimally-coupled scalars.

If we smooth out the conical singularities, so that we can freely integrate by parts, the action becomes

$$\begin{aligned} S &= \int d^dx\sqrt{g}\left(\frac{1}{2}\nabla_\mu A_\nu\nabla^\mu A^\nu - \frac{1}{2}A^\mu(\nabla_\mu\nabla_\nu-\nabla_\nu\nabla_\mu)A^\nu\right) \\ &= \int d^dx\sqrt{g}\left(\frac{1}{2}\nabla_\mu A_\nu\nabla^\mu A^\nu + \frac{1}{2}R_{\mu\nu}A^\mu A^\nu\right) \end{aligned}$$

In the second line we used $[\nabla_\mu,\nabla_\nu]A^\nu = -R_{\mu\nu}A^\nu$. We work on a space which is a product of a two-dimensional cone with coordinates x^α and a $(d-2)$-dimensional flat space with coordinates x^i.

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + \delta_{ij}dx^i dx^j$$

In two dimensions the Ricci tensor is proportional to the metric, so from (2.8)

$$R_{\alpha\beta} = 8\pi\lambda g_{\alpha\beta}\delta^2(x)/\sqrt{g} \tag{2.20}$$

[¶]See for example [20]. We dropped the term in the heat kernel $1/(4\pi s)^2$ which is responsible for the divergence in flat space.

where $8\pi\lambda$ is the deficit angle. Thus the action for a gauge field on a cone can be decomposed into

$$S_{\text{cone}} = S_0 + S_{\text{int}}, \qquad S_{\text{int}} = S_{\text{wedge}} + S_{\text{tip}} \tag{2.21}$$

For example in four dimensions

$$S_0 = \int d^4x \, \frac{1}{2}(\partial_\mu A_\nu)^2 \tag{2.22}$$

is the Feynman gauge action in flat space,

$$S_{\text{wedge}} = -\int d\tau dz \int_0^\infty rdr \int_{-4\pi\lambda}^{4\pi\lambda} d\psi \, \frac{1}{2}(\partial_\mu A_\nu)^2 \tag{2.23}$$

cancels the flat-space action in the region corresponding to the deficit angle, and

$$S_{\text{tip}} = 4\pi\lambda \int d\tau dz \, g_{\alpha\beta} A^\alpha A^\beta \tag{2.24}$$

arises from the explicit coupling to curvature. Aside from the sums over photon polarizations, this is identical to the decomposition of the non-minimal scalar action (2.9).

The interaction between two cosmic strings can be calculated perturbatively, just as for a non-minimal scalar field.[‖] In fact the two calculations are identical. There are $d-2$ polarizations transverse to the cone which behave in perturbation theory just like minimally-coupled scalars. Two of these polarizations are canceled by the ghosts, leaving no contribution in four dimensions. The two polarizations tangent to the cone behave like non-minimal scalars with $\xi = 1/2$. So the overall interaction energy coming from a gauge field in four dimensions is simply twice the scalar result (2.16) evaluated at $\xi = 1/2$. That is, for a gauge field in four dimensions

$$\mathcal{H}_{\text{int}} = \frac{2\lambda\lambda'}{\pi b^2}\left(-\frac{14}{15}\right) \tag{2.25}$$

To check the validity of our perturbative approach consider computing $\langle A_\mu A^\mu \rangle$ around a single

[‖] Again it's not manifest that perturbation theory in S_{int} enforces the proper conical boundary conditions on A_α, but this effect is controlled by the spin connection which can only enter at second order in the deficit angle.

cosmic string. In perturbation theory, after subtracting the divergence present in flat space, we have

$$\langle A_\mu A^\mu \rangle = \langle A_\mu A^\mu \left(-S_{\rm wedge} - S_{\rm tip}\right)\rangle_{C,0} = \frac{4\lambda}{6\pi^2 r^2} - \frac{\lambda}{\pi^2 r^2} \tag{2.26}$$

The first term comes from $S_{\rm wedge}$ and is four times the scalar field result (2.17). The second term comes from $S_{\rm tip}$ and reflects the non-minimal coupling to curvature. The same quantity can be computed exactly,

$$\langle A_\mu A^\mu \rangle = \int_0^\infty ds\, g_{\mu\nu} K^{\mu\nu}_{\rm vector}(s,x,x) \tag{2.27}$$

where the vector heat kernel is [20]

$$g_{\mu\nu} K^{\mu\nu}_{\rm vector} = 4K_{\rm scalar}(s,x,x) + \frac{2}{r}\partial_r s K_{\rm scalar}(s,x,x) \tag{2.28}$$

Expanding to first order in the deficit angle and integrating over s reproduces (2.26).[**]

2.3 Conclusions

In this chapter we considered a cosmic string spacetime and argued that to first order in the deficit angle there is an equivalence between a gauge field and a collection of scalar fields with specific non-minimal couplings to curvature. More generally the equivalence holds on the product of any weakly-curved Einstein manifold with flat space. We tested the equivalence by computing the interaction energy between two cosmic strings to first order in perturbation theory, showing that it indeed matched for the appropriate value of the non-minimal coupling parameter.

Throughout this chapter we worked in Feynman gauge, which is adequate for studying gauge-invariant quantities. However it would be interesting to study the relation between gauge and scalar fields in other choices of gauge. Also it would be interesting to study the interaction between strings at higher orders in perturbation theory. Beyond leading order there is no reason to expect an equivalence between gauge and scalar fields, since the spin connection distinguishes between the two types of fields and can appear in the interaction energy at second order in the deficit angle.

Besides their direct application to cosmic strings, our results also have relevance to the ther-

[**] Note that the last term in (2.28), which in the black hole context captures the contact interaction of a gauge field with the horizon, corresponds at first order in perturbation theory to effects associated with $S_{\rm tip}$.

modynamics of black holes. In a Euclidean formalism the entropy of a black hole measures the response of the partition function to an infinitesimal conical deficit angle inserted at the horizon [21, 22]. This has been used to study the renormalization of black hole entropy due to matter fields, with the somewhat surprising conclusion that a gauge field can make a negative contribution to the entropy. In [20] it was shown that this is due to a contact term in the partition function for a gauge field, associated with particle paths that begin and end on the horizon. Here we've shown that, to first order in the deficit angle, a gauge field is equivalent to a collection of non-minimal scalars. So the contact interaction of [20] is visible at the level of the equations of motion, as the explicit non-minimal coupling to curvature seen in (2.6). This makes the negative renormalization of black hole entropy less mysterious, since it maps a gauge field to the well-studied problem of a non-minimally coupled scalar field in a black hole background [23]. Our results also show the physical relevance of these contact interactions: besides contributing to black hole entropy, they make a (finite, observable, gauge invariant) contribution to the force between two cosmic strings. Though there is a cancellation in two dimensions from the ghost zero modes [24], but in general such terms are present and their gauge invariance was shown by Solodukhin [25].

We conclude with some additional evidence in support of the relation between gauge and scalar fields at first order in the background curvature. The partition function for a gauge field on a cone was evaluated in [20]. Including the ghosts, the result is

$$\beta F_{\text{gauge}} = (d-2)\beta F^{\text{minimal}}_{\text{scalar}} + A_\perp(2\pi - \beta)\int_{\epsilon^2}^{\infty} \frac{ds}{(4\pi s)^{d/2}} \tag{2.29}$$

Here d is the total number of spacetime dimensions, $A_\perp$ is the area of the $d-2$ transverse dimensions corresponding to the horizon, s is a Schwinger parameter, and ϵ is a UV cutoff. The partition function for a non-minimal scalar was evaluated to first order in the deficit angle in [23], with the result

$$\beta F^{\xi}_{\text{scalar}} = \beta F^{\text{minimal}}_{\text{scalar}} + \xi A_\perp(2\pi - \beta)\int_{\epsilon^2}^{\infty} \frac{ds}{(4\pi s)^{d/2}} \tag{2.30}$$

Comparing the partition functions again shows that a gauge field corresponds to two non-minimal scalars with $\xi = 1/2$, together with $d-2$ minimal scalars (two of which are canceled by the ghosts).

The same relation can be seen in the one-loop renormalization of Newton's constant,

$$\frac{1}{4G_{N,\mathrm{ren}}} = \frac{1}{4G_N} + \frac{c_1}{(4\pi)^{\frac{d-2}{2}}(d-2)\epsilon^{d-2}} \tag{2.31}$$

where the Seeley – de Witt coefficients are [26]

$$c_1 = \begin{cases} \frac{1}{6} - \xi & \text{non-minimal scalar} \\ \\ \frac{d-2}{6} - 1 & \text{gauge field including ghosts} \end{cases} \tag{2.32}$$

On a d-dimensional Einstein manifold the gauge field result corresponds to d non-minimal scalars with $\xi = 1/d$, plus two minimally-coupled scalar ghosts.

3

Black Hole Formation at the Correspondence Point: Part I

In this chapter and the next we study the formation of a black hole using various gauge/ gravity dualities at a particular limiting value of the energy scale. This chapter is devoted to the formation of black holes made out of collapse processes of various dimensional Dp brane shells and clusters. The next chapter, on the other hand, discusses collapse of fuzzy spheres. We show that at least, at the correspondence point (we clarify the definition of this point in a moment), there is a generic process in the gauge theory side which can describe field theory thermalization and the corresponding black hole formation machanism in gravity. This mechanism gives us a natural time scale which in turn is related to the time scale of the formation of the black holes.

3.1 Introduction and summary

Understanding black holes microstates from a D-brane or fundamental string perspective is a long-standing theme in string theory. The original observation that vibrating strings qualitatively resemble a black hole [21, 27] was followed by a quantitative worldvolume derivation of black hole entropy for certain BPS states [28]. This relationship eventually became a fundamental aspect of the holographic duality between gauge and gravity degrees of freedom [29]. According to this duality, microstates of a black hole are in one-to-one correspondence with microstates of a strongly-coupled gauge theory. This duality also applies to time-dependent processes such as black hole formation and evaporation, leading to the viewpoint that these processes should be unitary, contrary to [30].

To gain insight into black hole formation, and a better understanding of the microstructure of the resulting black hole, in this chapter we study the process of bound state formation from two perspectives: perturbative gauge theory and supergravity. In perturbative gauge theory a D-brane bound state can be formed through a process of open string creation. In supergravity we will see that open string creation is not possible, and one instead forms a bound state through the gravitational or closed-string process of black hole formation.

The perturbative gauge theory and supergravity calculations of bound state formation do not have an overlapping range of validity. But we will show that they agree qualitatively at an intermediate value of the coupling, in accord with the correspondence principle introduced by Horowitz and Polchinski [31]. This suggests that there is a smooth transition between the process of open string creation at weak coupling and black hole formation at strong coupling.

As a first test of these ideas, in section 3.2 we study bound state formation in D0-brane collisions and show that the sizes of the bound states match at the correspondence point. In section 3.3 we extend this analysis to general Dp-branes.

Next we consider the time development of the bound states after they have formed. In section 3.4 we show that the weakly-coupled gauge theory has a parametric resonance which exponentially amplifies the number of open strings present, and we identify the timescale for the production of additional open strings at weak coupling. In the gravitational description, a perturbed black hole approaches equilibrium on a timescale determined by the quasinormal frequencies. In section 3.5 we compare these two timescales and show that they agree at the correspondence point.

In section 3.6 we compare properties of the bound state as initially formed to equilibrium properties of the black hole, and show that at the correspondence point the bound state is created in a state of near-equilibrium. In section 3.7 we study a different initial configuration, in which a bound state is formed by collapse of a spherical shell of D0-branes, and show that the picture of a smooth transition between open string production and black hole formation continues to hold. We conclude in section 3.8.

The present work is related to several studies in the literature. In gauge – gravity duality, a black hole on the gravity side is dual to a thermal state of the gauge theory, where all $\mathcal{O}(N^2)$ degrees of freedom are excited [32, 33]. There have been many studies of 0-brane black hole microstates

Figure 3.1: Colliding stacks of 0-branes.

from matrix quantum mechanics, along with their associated thermalization process. Some previous studies of 0-brane black holes from matrix quantum mechanics include [34, 35, 36, 37, 38, 39, 40]. Also see [41, 42] for studies of black hole formation from the gravity perspective, and [43, 44, 45, 46, 47, 48] for studies from the gauge theory perspective. In particular parametric resonance has been discussed in relation to thermalization in the closely related work [46]. Open string production has been studied as a mechanism for trapping moduli at enhanced symmetry points in [49], while open string production in relativistic D-brane collisions has been studied in [50].

3.2 Bound state formation in 0-brane collisions

Consider colliding two clusters of 0-branes as shown in Fig. 3.1. We'd like to understand whether a bound state is formed during the collision. Two mechanisms for bound state formation have been discussed in the literature.

1. In a perturbative description of D-brane dynamics, open strings can be produced and lead to formation of a bound state. This occurs for impact parameters $b \lesssim \sqrt{v\alpha'}$ [51]. This can be understood as the condition for violating the adiabatic approximation. For a review of the calculation see appendix A.1.

2. At strong coupling the D-brane system has a dual gravitational description [3]. In this description, according to the hoop conjecture of Thorne [52, 53], a black hole should form if the two D-brane clusters are contained within their own Schwarzschild radius.

Our goal is to understand in what regimes these two mechanisms for bound state formation are operative, and whether they are connected in any way.

It will be convenient to work in terms of a radial coordinate U with units of energy, $U = r/\alpha'$. The 't Hooft coupling of the M(atrix) quantum mechanics is $\lambda = g_{\rm YM}^2 N$, which in string and M-theory units can be expressed as

$$\lambda = g_s N/\ell_s^3 = R^3 N/\ell_{11}^6 \,. \tag{3.1}$$

Here g_s is the string coupling, ℓ_s is the string length, R is the radius of the M-theory circle, and ℓ_{11} is the M-theory Planck length. The mass of a single D0-brane is

$$m_0 = \frac{1}{g_s \ell_s} = \frac{1}{R} \,. \tag{3.2}$$

3.2.1 Perturbative string production

We work in the center of mass frame, with momenta

$$p_1 = \frac{N_1}{R} v_1 \qquad p_2 = \frac{N_2}{R} v_2 \qquad p_1 + p_2 = 0 \tag{3.3}$$

We consider a fixed total energy E, which determines the asymptotic relative velocity v.

$$\begin{aligned} & \frac{1}{2}\frac{N_1}{R} v_1^2 + \frac{1}{2}\frac{N_2}{R} v_2^2 = E \,, \\ \Rightarrow \quad & v = v_1 - v_2 \sim \left(\frac{NER}{N_1 N_2}\right)^{1/2} = \left(\frac{\lambda E l_s^4}{N_1 N_2}\right)^{1/2} \,. \end{aligned} \tag{3.4}$$

In terms of the U coordinate, the relative velocity is

$$\dot{U} = \left(\frac{\lambda E}{N_1 N_2}\right)^{1/2} \,. \tag{3.5}$$

As reviewed in appendix A.1, open string production sets in when

$$U \sim \sqrt{\dot{U}} = \left(\frac{\lambda E}{N_1 N_2}\right)^{1/4} \,. \tag{3.6}$$

Note that the radius at which open strings are produced depends on how we split the total D-brane charge. The radius is minimized when $N_1 = N_2 = N/2$, which gives the minimum radius for open string production as

$$U_0 \sim \left(\frac{\lambda E}{N^2}\right)^{1/4} . \tag{3.7}$$

This is the case which is interesting for matching to supergravity.

There are some checks we should perform to make sure this perturbative result is valid. As discussed in [54], the effective action has a double expansion in λ/U^3 and $\dot{U}^2/U^4$. The expansion in powers of λ/U^3 is the Yang-Mills loop expansion, which is valid provided $U_0 > \lambda^{1/3}$. From (3.7) this requires

$$E > N^2\lambda^{1/3} \tag{3.8}$$

At the critical point where the loop expansion breaks down, $U_0 \sim \lambda^{1/3}$, the inequality (3.8) is saturated.

The expansion in powers of $\dot{U}^2/U^4$ is the derivative expansion, which is valid when $\dot{U}^2 < U^4$. Note that the derivative expansion breaks down at the point where open strings are produced. Up to this point, *i.e.* for $U > \sqrt{\dot{U}}$, one can trust the two-derivative terms in the effective action, which means one can ignore corrections to the asymptotic velocity estimate (3.4).* So the only condition for the validity of the perturbative description of open string production is (3.8).

3.2.2 Bound state formation in gravity

The M(atrix) quantum mechanics has a dual gravitational description at strong coupling, meaning for $U < \lambda^{1/3}$. So let's imagine the 0-brane clusters approach to within this distance, and study whether a bound state can form.

At first, one might think a bound state could form via open string production. As noted in [3], the metric factors cancel out of the Nambu-Goto action, and even in the supergravity regime the mass of an open string connecting the two clusters of D-branes is $m_W \sim U$. The adiabatic approximation breaks down, and these open strings should be produced, if $\dot{U}/U^2 > 1$. However this velocity cannot be attained in the regime where supergravity is valid, since it violates the causality

*As we will see, this is not the case in the supergravity regime.

bound [55, 56]. This can be seen in the probe approximation, where the DBI action for a probe is

$$S = \frac{1}{g_{\rm YM}^2} \int dt \, \frac{U^7}{\lambda} \left(1 - \sqrt{1 - \frac{\lambda \dot{U}^2}{U^7}} \right) \tag{3.9}$$

Thus causality bounds the velocity of the probe,

$$\frac{\lambda \dot{U}^2}{U^7} < 1 \,. \tag{3.10}$$

Rather remarkably, the probe has to slow down significantly as $U \to 0$. In any case, in the supergravity regime we have $\frac{\dot{U}^2}{U^4} < \frac{U^3}{\lambda}$, and since $\frac{U^3}{\lambda} < 1$ at strong coupling, open strings can never be produced.

This means black hole formation is the only way to form a bound state in the supergravity regime. Since open string production is ruled out, we reach the sensible conclusion that the formation of a horizon is a purely gravitational closed-string process. The hoop conjecture states that a black hole will form if the energy E is contained within its own Schwarzschild radius. For a 10-dimensional black hole with N units of 0-brane charge, the Schwarzschild radius is

$$U_0 = \left(\frac{\lambda^2 E}{N^2} \right)^{1/7} \tag{3.11}$$

This 10-D supergravity description is only valid if the curvature and string coupling are small at the horizon, which requires

$$\lambda^{1/3} N^{-4/21} < U_0 < \lambda^{1/3} \tag{3.12}$$

For smaller U_0 one must lift to M-theory; for larger U_0 the M(atrix) quantum mechanics is weakly coupled. At the outer radius where the supergravity approximation breaks down, $U_0 \sim \lambda^{1/3}$, eq. (3.11) tells us that $E \sim N^2 \lambda^{1/3}$.

3.2.3 Correspondence point

We've found that open string production is only possible at weak coupling, while black hole formation can only occur within the bubble where supergravity is valid. One could ask if the two phenomena are smoothly connected. Is there a correspondence point where both descriptions are valid?

From the perturbative point of view, the transition happens when the condition (3.8) is saturated, $E = N^2\lambda^{1/3}$. In this case open strings are produced, but at a radius $U_0 \sim \lambda^{1/3}$ where the system is just becoming strongly coupled.

From the supergravity point of view, the transition happens when the energy of the black hole is $E = N^2\lambda^{1/3}$, corresponding to a Schwarzschild radius $U_0 \sim \lambda^{1/3}$. In this case the black hole fills the entire region where supergravity is valid.

This suggests that open string production and black hole formation are indeed continuously connected. Since the transition between the two descriptions happens when the curvature at the horizon is of order string scale,

$$\alpha' R \sim (\lambda/U^3)^{-1/2} \sim 1\,, \tag{3.13}$$

this is an example of the correspondence principle of Horowitz and Polchinski [31]. Note that for a given black hole energy, one can view the condition of being at the correspondence point, $E = N^2\lambda^{1/3}$, as fixing the total 0-brane charge,

$$N = \left(\frac{E^3\ell_s^3}{g_s}\right)^{1/7}\,. \tag{3.14}$$

3.3 Dp-brane collisions

In this section we generalize our 0-brane results and consider Dp-branes wrapped on a p-torus of volume V_p. We first record some general formulas then analyze particular cases.

The Yang-Mills coupling is $g^2_{\rm YM} = g_s/\ell_s^{3-p}$ and the 't Hooft coupling is $\lambda = g^2_{\rm YM}N$. In terms of $U = r/\alpha'$, the effective dimensionless 't Hooft coupling is

$$\lambda_{\rm eff} = \frac{\lambda}{U^{3-p}}\,. \tag{3.15}$$

The Yang-Mills theory is weakly coupled when $\lambda_{\rm eff} < 1$ and has a dual gravitational description when $\lambda_{\rm eff} > 1$ [3].

Imagine colliding two stacks of wrapped Dp-branes at weak coupling, with a fixed energy density ϵ as measured in the Yang-Mills theory. The mass of a wrapped p-brane is $V_p/g_s\ell_s^{p+1}$, so in the center of mass frame the relative velocity is

$$\dot{U} = \left(\frac{\lambda\epsilon}{N_1 N_2}\right)^{1/2} . \tag{3.16}$$

Open string production sets in when

$$U \sim \sqrt{\dot{U}} \sim \left(\frac{\lambda\epsilon}{N_1 N_2}\right)^{1/4} . \tag{3.17}$$

The radius at which open strings are produced depends on how we divide the total D-brane charge. The radius is minimized by setting $N_1 = N_2 = N/2$, which gives the minimum radius for open string production as

$$U_0 \sim \left(\frac{\lambda\epsilon}{N^2}\right)^{1/4} . \tag{3.18}$$

This is the case which is interesting for comparison to supergravity.

Just as for 0-branes, open string production is not possible in the supergravity regime. The DBI action for a probe brane is

$$S = \frac{1}{g_{\rm YM}^2}\int d^{p+1}x\,\frac{U^{7-p}}{\lambda}\left(1-\sqrt{1-\frac{\lambda\dot{U}^2}{U^{7-p}}}\right) \tag{3.19}$$

Thus the causality bound is $\dot{U}^2/U^4 < U^{3-p}/\lambda = 1/\lambda_{\rm eff}$ [55], which rules out open string production (at least in the probe approximation). Instead we have the process of black hole formation, with a horizon radius $U_0 = (g_{\rm YM}^4\epsilon)^{1/(7-p)}$ [3].

Further analysis depends on the dimension of the branes.

$p = 0, 1, 2$

For $p < 3$ the Yang-Mills theory is weakly coupled when $U > \lambda^{1/(3-p)}$ and has a dual gravitational description when $U < \lambda^{1/(3-p)}$. Thus open string production is possible at large distances, while black hole formation is possible at small distances. The correspondence point, where the two descriptions match on to each other, occurs when

$$\epsilon = N^2\lambda^{\frac{1+p}{3-p}}$$

$$U_0 = \lambda^{1/(3-p)}$$

At this energy density open string production occurs just as the Yang-Mills theory is becoming strongly coupled. From the supergravity perspective, the resulting black brane fills the entire region in which supergravity is valid.

$\underline{p = 3}$

In this case the Yang-Mills theory is conformal and dual to $AdS_5 \times S^5$ [29]. The 't Hooft coupling is dimensionless. For $\lambda \lesssim 1$ open string production is possible, while for $\lambda \gtrsim 1$ black holes can form. The two descriptions match on to each other at the correspondence point $\lambda = 1$. Note that, unlike other values of p, the correspondence point is independent of the energy density ϵ.

As a test of this idea, note that the radius at which open strings form is

$$U_0 = (\lambda\epsilon/N^2)^{1/4} \tag{3.20}$$

while for $p = 3$ the horizon radius is

$$U_0 = (g_{\rm YM}^4 \epsilon)^{1/4} \tag{3.21}$$

These two expressions for U_0 agree when $\lambda = 1$. This suggests that the process of open string production for $\lambda \lesssim 1$ smoothly matches on to black hole formation for $\lambda \gtrsim 1$.

$\underline{p = 4,\, 5,\, 6}$

For $p > 3$ the Yang-Mills theory is strongly coupled in the UV and has a dual supergravity description (modulo some subtleties described in [3]). In the IR the Yang-Mills theory is weakly coupled. Black hole production is possible in the supergravity regime, where $U > \lambda^{1/(3-p)}$, while open string production is possible for $U < \lambda^{1/(3-p)}$. The correspondence point where the two descriptions match is at

$$\epsilon = N^2 \lambda^{\frac{1+p}{3-p}} \tag{3.22}$$

$$U_0 = \lambda^{1/(3-p)} \tag{3.23}$$

3.4 Parametric resonance in perturbative SYM

In this section we study the evolution of a bound state formed at weak coupling by open string creation. We show that the number of open strings increases exponentially with time due to a parametric resonance in the gauge theory. For simplicity we consider 0-brane collisions; the generalization to Dp-branes is straightforward and will be mentioned in section 3.5.2.

Suppose a cluster of N_1 incoming 0-branes collides with a stack of N_2 coincident 0-branes at rest. In the collision suppose n open strings are produced. These open strings produce a linear confining potential, so the system will begin to oscillate. The conserved total energy is

$$E = \frac{1}{2}mv^2 + n\tau x \tag{3.24}$$

Here we're adopting a non-relativistic description, appropriate to the form of the D0-brane quantum mechanics, while m is the mass of the incoming 0-branes, v is their velocity, n is the number of open strings created, $\tau = 1/2\pi\alpha'$ is the fundamental string tension, and x is the length of the open strings. The period of oscillation is

$$\Delta t = 4\left(\frac{m}{2}\right)^{1/2} \int_0^{E/n\tau} \frac{dx}{\sqrt{E - n\tau x}} \sim \frac{\sqrt{mE}}{n\tau} \tag{3.25}$$

So up to numerical factors, the frequency of oscillation is

$$\Omega = \frac{n\tau}{\sqrt{mE}} \tag{3.26}$$

while the amplitude of oscillation (the maximum value of x) is

$$L = \frac{E}{n\tau} \tag{3.27}$$

We introduce this as a classical M(atrix) background by setting $X^i = X^i_{\mathrm{cl}} + x^i$ where

$$X^1_{\mathrm{cl}} = \begin{pmatrix} L\sin\Omega t\,\mathbf{1}_{N_1} & 0 \\ 0 & 0 \end{pmatrix} \qquad X^2_{\mathrm{cl}} = \cdots = X^9_{\mathrm{cl}} = 0\,, \tag{3.28}$$

We have decomposed the $N \times N$ matrix into blocks; $\mathbf{1}_{N_1}$ is the $N_1 \times N_1$ unit matrix. Expanding to quadratic order in the fluctuations, the M(atrix) Lagrangian[†]

$$\mathcal{L}_{\rm YM} = \frac{1}{2g_{\rm YM}^2}{\rm Tr}\left(\dot{X}^i\dot{X}^i + \frac{1}{2}[X^i,X^j][X^i,X^j]\right) \tag{3.29}$$

reduces to

$$\mathcal{L}_{\rm YM} = \frac{1}{2g_{\rm YM}^2}{\rm Tr}\left(\dot{x}^1\dot{x}^1\right) + \frac{1}{2g_{\rm YM}^2}\sum_{i=2}^{9}{\rm Tr}\left(\dot{x}^i\dot{x}^i + [x^i,X^1_{\rm cl}][x^i,X^1_{\rm cl}]\right) \tag{3.30}$$

Note that the potential for x^1 vanishes. We also have the Gauss constraint associated with setting $A_0 = 0$, namely

$$\sum_i [X^i,\dot{X}^i] = 0 \tag{3.31}$$

To quadratic order this reduces to $[X^1_{\rm cl},\dot{x}^1] = [\dot{X}^1_{\rm cl},x^1]$ which only constrains x^1. The simplest solution is to set $x^1 = 0$.

To study the remaining degrees of freedom we decompose

$$x^i = \begin{pmatrix} a^i & b^{i\dagger} \\ b^i & c^i \end{pmatrix} \tag{3.32}$$

where a^i is an $N_1 \times N_1$ matrix, b^i is an $N_1 \times N_2$ rectangular matrix and c^i is an $N_2 \times N_2$ matrix. We will often suppress the index $i = 2,\dots,9$. To quadratic order the a and c entries have trivial dynamics, since $[x^i, X^1_{\rm cl}]$ does not involve a and c. On the other hand, the equation of motion for b is

$$\ddot{b} + L^2\sin^2(\Omega t)\,b = 0 \tag{3.33}$$

Defining $s = \Omega t$ this reduces to Mathieu's equation,

$$\frac{d^2b}{ds^2} + (a - 2q\cos 2s)\,b = 0 \tag{3.34}$$

with the particular values $a = 2q = L^2/2\Omega^2$. Mathieu's equation admits Floquet solutions

$$b(t) = e^{i\gamma\Omega t}P(\Omega t) \tag{3.35}$$

[†]We are setting $2\pi\alpha' = 1$ and $A_0 = 0$.

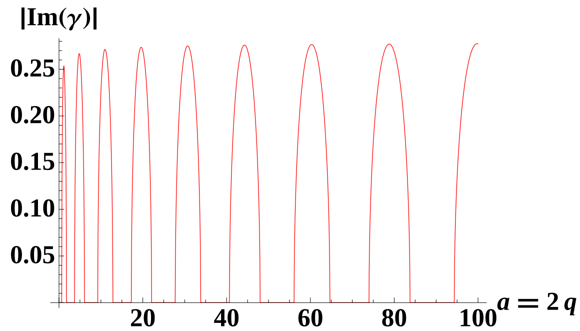

Figure 3.2: The imaginary part of the Mathieu characteristic exponent as a function of $a = 2q$.

where $P(\cdot)$ is a periodic function with period π. As a function of a and q there are intervals where γ has a negative imaginary part and the solution grows exponentially. These intervals correspond to band gaps in the Bloch interpretation of Mathieu's equation. The imaginary part of γ is plotted as a function of $a = 2q$ in Fig. 3.2. There are clearly many intervals where the solution is unstable, with a typical exponent $|\mathrm{Im}\gamma| \sim 0.25$.

This instability corresponds to an exponential growth in the number of open strings present. Note that in our case[‡]

$$a = 2q \sim mE^3/n^4 \tag{3.36}$$

After the initial collision the energy E in the oscillating background will decrease as the system begins to thermalize, while the number n of open strings gets larger. So we expect the value of a to decrease with time. This means the system will scan across the different instability bands available to it.

To summarize, we have found that the oscillating background resulting from a 0-brane collision is unstable. The $16N_1N_2$ real degrees of freedom contained in b^i for $i = 2, \ldots, 9$ behave as parametrically-driven oscillators. Their amplitude grows exponentially, on a timescale

$$t_{\mathrm{YM}} \sim 1/\Omega \sim \sqrt{mE}/n\tau \tag{3.37}$$

Here m is the mass of the N_1 incoming 0-branes, E is the total energy of the system, n is the number of open strings present in the off-diagonal block b and τ is the fundamental string tension.

[‡]Restoring units, we would have $L^2 \to L^2\tau^2$ in (3.33) and $a = 2q \sim mE^3/n^4\tau^2$ in (3.36).

3.5 Comparison of timescales

We compare the timescale associated with parametric resonance to the quasinormal modes of a black hole. We consider parametric resonance for D0-branes in section 3.5.1, generalize to Dp-branes in section 3.5.2, and compare to quasinormal modes in section 3.5.3.

3.5.1 0-brane parametric resonance

As we saw in section 3.4, the timescale for parametric resonance is determined by the period of oscillation. In a 0-brane collision this is given by

$$t_{\rm YM} \sim 1/\Omega \sim \sqrt{mE}/n\tau \tag{3.38}$$

For N_1 incoming D0-branes the mass is $m = N_1/R$, where $R = g_s l_s$ is the radius of the M-theory circle. Also E is the total energy of the system, n is the number of open strings and $\tau \sim 1/l_s^2$ is string tension. We consider the case $N_1 \sim N_2 \sim N$. Then the off-diagonal block b contains $\mathcal{O}(N^2)$ elements, so as shown in appendix A.1 $\mathcal{O}(N^2)$ open strings are created by parametric resonance.

Using $R = g_s\ell_s$, $\tau \sim 1/\ell_s^2$, $n \sim N^2$ and $g_s \sim g_{YM}^2 \ell_s^3$ we obtain

$$t_{\rm YM} \sim \sqrt{\frac{NE}{R}}\frac{1}{n\tau} \sim \frac{\sqrt{E}}{\lambda^{1/2}N}\,. \tag{3.39}$$

At the correspondence point

$$E \sim N^2\lambda^{1/3} \tag{3.40}$$

which means

$$t_{\rm YM} \sim \lambda^{-1/3}\,. \tag{3.41}$$

At the correspondence point the timescale for parametric resonance is independent of N and is set by the 't Hooft scale. As we will see in section 3.5.3, the same holds true for the quasinormal frequencies of a black hole at the correspondence point.

3.5.2 p-brane parametric resonance

It's straightforward to extend this result to Dp-branes. First, the mass of a single D0-brane in the previous section is replaced by the mass of Dp-brane wrapped on a volume V_p. So we should replace

$$1/R \to V_p/g_s l_s^{p+1} \,. \tag{3.42}$$

The energy of the incoming Dp-branes is related to the energy density ϵ by

$$E = \epsilon V_p \,. \tag{3.43}$$

The tension of the strings is the same, $\tau \sim 1/\ell_s^2$. So for Dp-branes, in place of (3.38), the oscillation timescale is

$$t_{\rm YM} \sim \frac{\sqrt{mE}}{n\tau} \to V_p \sqrt{\frac{N\epsilon}{g_s l_s^{p+1}}} \frac{1}{n\tau} \,. \tag{3.44}$$

The number of open strings n is modified. As shown in appendix A.1, for $N_1 \sim N_2$ and $p \neq 3$, the number density of open strings at the correspondence point is set by the 't Hooft scale. Thus

$$n \sim N^2 V_p \lambda^{\frac{p}{3-p}} \,. \tag{3.45}$$

Using this together with $g_s N = g_{YM}^2 N \ell_s^{3-p} = \lambda \ell_s^{3-p}$ we obtain

$$t_{\rm YM} \sim V_p \sqrt{\frac{N\epsilon}{g_s \ell_s^{p+1}}} \frac{1}{n\tau} \sim \frac{\lambda^{-\frac{p}{3-p}} \sqrt{\epsilon}}{\lambda^{1/2} N} \,. \tag{3.46}$$

From (3.22) the energy density at the correspondence point is

$$\epsilon \sim N^2 \lambda^{\frac{1+p}{3-p}} \tag{3.47}$$

so the timescale is

$$t_{\rm YM} \sim \lambda^{-\frac{1}{3-p}} \,. \tag{3.48}$$

Just as for 0-branes, the timescale for parametric resonance is independent of N and set by the 't Hooft scale.

3-branes are a special case since the 't Hooft coupling is dimensionless. The correspondence point is defined by $\lambda \sim 1$. As shown in appendix A.1, for $N_1 \sim N_2$ the number of open strings at the correspondence point is

$$n \sim N^2 V_3 U_0^3 \tag{3.49}$$

where U_0 is the horizon radius of the black brane. The energy density at the correspondence point is $\epsilon \sim N^2 U_0^4$, so the parametric resonance timescale is

$$t_{\mathrm{YM}} \sim V_p \sqrt{\frac{N\epsilon}{g_s \ell_s^{p+1}}} \frac{1}{n\tau} \sim \frac{1}{U_0} \tag{3.50}$$

Thus for D3-branes the parametric resonance timescale is $1/U_0$, which also happens to be the inverse temperature of the black brane.

3.5.3 Comparison to quasinormal modes

Quasinormal modes for non-extremal Dp-branes were studied in [57, 58] following earlier work on AdS-Schwarzschild black holes [59]. The basic idea is to solve the scalar wave equation in the near-horizon geometry of N coincident non-extremal Dp-branes, with a Dirichlet boundary condition at infinity and purely ingoing waves at the future horizon. This gives rise to a discrete set of complex quasinormal frequencies, whose imaginary parts govern the decay of scalar perturbations of the black hole. It was found that the quasinormal frequencies are proportional to the temperature, with a coefficient of proportionality that was found numerically in [57].

Recall that the temperature, energy density and entropy density of these black branes are related to their horizon radius U_0 by [3, 57]

$$\begin{aligned} T &\sim \frac{1}{\sqrt{\lambda}} U_0^{(5-p)/2} \\ \epsilon &\sim \frac{N^2}{\lambda^2} U_0^{7-p} \\ s &\sim \frac{N^2}{\lambda^{3/2}} U_0^{(9-p)/2} \end{aligned}$$

Assuming $p \neq 3$, at the correspondence point we have $U_0 \sim \lambda^{1/(3-p)}$ so that

$$
\begin{aligned}
T &\sim \lambda^{\frac{1}{3-p}} \\
\epsilon &\sim N^2 \lambda^{\frac{p+1}{3-p}} \\
s &\sim N^2 \lambda^{\frac{p}{3-p}}
\end{aligned}
$$

These quantities all obey the expected large-N counting, and since the 't Hooft coupling λ has units of (energy)$^{3-p}$, these results could have been guessed on dimensional grounds. In the special case $p = 3$ the 't Hooft coupling is dimensionless and the correspondence point is defined by $\lambda = 1$. At the correspondence point the horizon radius U_0 remains arbitrary, with

$$
\begin{aligned}
T &= U_0 \\
\epsilon &= N^2 U_0^4 \\
s &= N^2 U_0^3
\end{aligned}
$$

Again these results could have been guessed on dimensional grounds.

As we saw in section 3.5.1 and 3.5.2 the timescale for parametric resonance is

$$
t_{\rm YM} \sim \begin{cases} \lambda^{-1/(3-p)} & \text{for } p \neq 3 \\ 1/U_0 & \text{for } p = 3 \end{cases} \tag{3.51}
$$

For all p this matches the inverse temperature of the black brane, $t_{\rm YM} \sim 1/T$. Thus at the correspondence point the timescale for parametric resonance matches the timescale for the decay of quasinormal excitations of the black brane.

3.6 Comparison to equilibrium properties

It's interesting to compare the properties of the bound state as initially formed to the equilibrium properties of the black hole. This will show us that, at the correspondence point, very little additional evolution is required to reach equilibrium – perhaps just a few e-foldings of parametric resonance will suffice.

First, in a 0-brane collision, note that the total number of open strings produced is $\sim N_1 N_2$. With equal charges $N_1 = N_2 = N/2$ the number of open strings is $\mathcal{O}(N^2)$. At the correspondence point these strings have a mass $\sim \lambda^{1/3}$, so the total energy and entropy in open strings is

$$E \sim N^2 \lambda^{1/3}$$
$$S \sim N^2$$

This matches the equilibrium energy and entropy of the black hole, suggesting that black hole formation at the correspondence point is a simple one-step procedure, in which the open strings that are formed in the initial collision essentially account for the equilibrium properties of the black hole. The analogous result for p-branes is that the number of open strings at the correspondence point is, for $p \neq 3$,

$$n \sim N^2 V_p \lambda^{\frac{p}{3-p}} \tag{3.52}$$

where we have used (A.2) and the fact that $U \sim \lambda^{\frac{1}{3-p}}$. Since the open strings have a mass $\sim U$, this corresponds to a total energy and entropy in open strings

$$E \sim N^2 V_p \lambda^{\frac{p+1}{3-p}}$$
$$S \sim N^2 V_p \lambda^{\frac{p}{3-p}}$$

which again matches the equilibrium energy and entropy of the black brane. This again suggests that the black hole is essentially fully formed in the initial collision, with very little additional evolution required to reach equilibrium.[§]

Another quantity we can compare at the correspondence point is the size of the bound state. At weak coupling, after n open strings have been formed, the amplitude of oscillation of the resulting bound state is, from (3.27),

$$L = \frac{E}{n\tau} \tag{3.53}$$

[§]When $p = 3$ the matching is $n \sim N^2 V_3 U_0^3$, $E \sim N^2 V_3 U_0^4$, $S \sim N^2 V_3 U_0^3$.

At the correspondence point for general p we have

$$E \sim N^2 V_p U_0^{p+1} \tag{3.54}$$

while the initial number of open strings created is

$$n \sim N^2 V_p U_0^p \tag{3.55}$$

Thus the initial amplitude of oscillation as measured in the U coordinate is

$$L/\ell_s^2 = E/n \sim U_0 \tag{3.56}$$

In other words, the initial oscillation amplitude matches the equilibrium horizon radius of the black brane. Again this suggests that after the initial collision, only a small amount of additional evolution is required to reach equilibrium.

3.7 Shell Collapse

So far we have studied bound state formation in a collision between two clusters of D-branes, in the geometry shown in Fig. 3.1. Here we study a different initial configuration, in which N D0-branes are uniformly distributed over a collapsing spherical shell as in Fig. 3.3. We will see that the correspondence principle applies and a similar outcome is obtained in this case.

We consider an initial configuration in which the 0-branes are uniformly spread over an S^8 of radius U in 9 spatial dimensions. The 0-branes are localized but uniformly distributed over the sphere, with velocities directed toward the center. Intuitively we argue as follows. Since the total volume of the sphere scales as U^8, each 0-brane occupies a volume $\sim U^8/N$, and the distance between nearest-neighbor 0-branes scales as $U/N^{1/8}$. This means virtual open strings connecting nearest-neighbor 0-branes are quite light, with a mass $\sim U/N^{1/8}$ that goes to zero at large N. However the typical open string is much heavier, with a mass $\sim U$ that is independent of N. We expect these typical open strings to dominate the bound-state formation process, and therefore expect to have a well-defined correspondence point at large N.

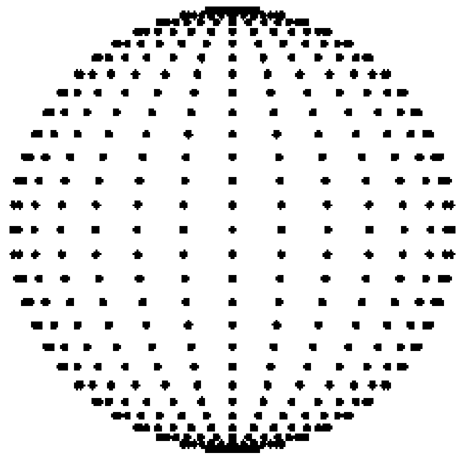

Figure 3.3: A collapsing shell of 0-branes. Initially the 0-branes are spread uniformly over an S^8 with velocities toward the center.

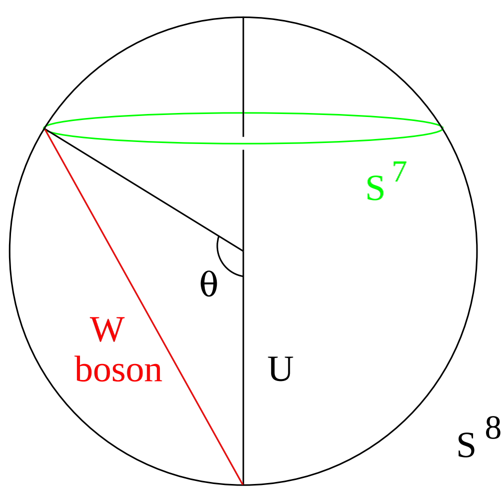

Figure 3.4: The 0-branes are spread over an S^8 of radius U. The S^7 has radius $U \sin\theta$ and the W boson has length $2U \sin\theta/2$.

To argue this in more detail, it is useful to consider a 0-brane located at the south pole and study the number of virtual open strings as a function of the angle θ to the other 0-brane. See Fig. 3.4. The number of distinct open strings dn in the interval $(\theta, \theta + d\theta)$ is

$$dn = \frac{N}{\frac{32\pi^4 U^8}{105}} \times \frac{\pi^4}{3}(U \sin\theta)^7 \times U d\theta \tag{3.57}$$

The first factor $N/\left(\frac{32\pi^4 U^8}{105}\right)$ is the number density of 0-branes on the S^8, the second factor $\frac{\pi^4}{3}(U \sin\theta)^7$ is the volume of an S^7 located at an angle θ from the south pole. Thus the number density of open strings is

$$\frac{dn}{d\theta} = \frac{35}{32} N \sin^7\theta \tag{3.58}$$

We can also find the mass density of open strings $\frac{dm}{d\theta}$. Since an open string subtending an angle θ has a mass $2U \sin\theta/2$, this is given by

$$\frac{dm}{d\theta} = \frac{dn}{d\theta} \cdot 2U \sin\frac{\theta}{2} = \frac{35}{16} NU \sin^7\theta \sin\frac{\theta}{2} \tag{3.59}$$

The W-boson number density $\frac{1}{N}\frac{dn}{d\theta}$ and mass density $\frac{1}{NU}\frac{dm}{d\theta}$ are plotted in Fig. 3.5.

As can be seen in the figure, there are light open strings at large N. However the number of these strings is tiny, since $\frac{dn}{d\theta} \sim \theta^7$ at small angles.¶ Most of the W-bosons are concentrated around $\theta = \pi/2$. Therefore a spherical shell is basically the same as having W-bosons distributed in the interval $\theta_0 < \theta < \pi - \theta_0$, where θ_0 is determined by the fraction of 0-branes pairs we neglect. For example, if we neglect $\frac{dn}{d\theta} \leq 10^{-7}N$, then $\theta_0 \sim 0.1$. Since the masses of the W-bosons near $\theta = \pi/2$ are all $O(U)$, we can simply approximate the entire W-boson spectrum by taking $m_W \sim U$.

We now consider what happens when we give the shell of 0-branes some velocity toward the origin. The analysis is almost identical to the colliding clusters considered in section 3.2. Given N D0-branes with total energy E, the asymptotic relative velocity is

$$\begin{aligned} E &\sim \text{mass} \times v^2 \sim \frac{N}{R} v^2 \\ \Rightarrow \quad v &\sim \left(\frac{ER}{N}\right)^{1/2} = \left(\frac{E\lambda l_s^4}{N^2}\right)^{1/2} \end{aligned} \tag{3.60}$$

¶This is due to the fact that the 0-branes are spread on an S^8. The distribution would be less sharply peaked in lower dimensions, with $\frac{dn}{d\theta} \sim \theta^{d-1}$ on an S^d.

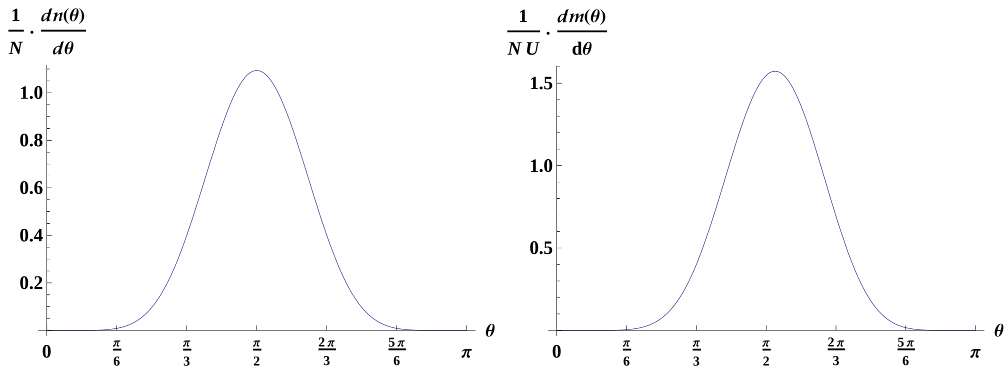

Figure 3.5: On the left, the W-boson number density $\frac{1}{N}\frac{dn}{d\theta}$. On the right, the W-boson mass density $\frac{1}{NU}\frac{dm}{d\theta}$.

In terms of the U coordinate, this becomes

$$\dot{U} = \left(\frac{E\lambda}{N^2}\right)^{1/2} \tag{3.61}$$

This matches the result in section 3.2 for $N_1 = N_2 \sim N$. Since the W-boson masses are concentrated around $m_W \sim U$, open string production again sets in when

$$U \sim \sqrt{\dot{U}} \sim \left(\frac{E\lambda}{N^2}\right)^{1/4} \tag{3.62}$$

At the correspondence point, where the effective gauge coupling becomes order one, we have

$$U \sim \lambda^{1/3} \tag{3.63}$$

and therefore

$$E \sim N^2 \lambda^{1/3}\,. \tag{3.64}$$

Just as in section 3.2, this matches the radius and energy energy of a black hole at the correspondence point.

3.8 Conclusions

In this chapter we studied D-brane collisions. We argued that the process of open string creation, which leads to formation of a D-brane bound state at weak coupling, smoothly matches on to a process at strong coupling, namely black hole formation in the dual supergravity. The transition happens at an intermediate value of the coupling, given by the correspondence principle of Horowitz and Polchinski. The size of the bound state, the timescale for approaching equilibrium, and the thermodynamic properties of the bound state all agree between the two descriptions. The latter agreement happens quickly, which suggests that the bound state is formed by the initial collision in a near-equilibrium configuration.

We considered two types of initial configurations, namely colliding clusters of wrapped Dp-branes and a collapsing shell of D0-branes. The main difference between the two configurations was that the shell had a tail of light open strings which we argued could be neglected. In fact, this distinction between the two configurations is somewhat artificial, since with somewhat more generic initial conditions the 0-branes which make up the clusters could have some small random relative velocities. One would then expect some open string production within the clusters, which would put the two examples on much the same footing.

In the examples we studied the powers of N were fixed by large-N counting, so at the correspondence point there was essentially only a single length scale in the problem, namely the 't Hooft scale (for $p \neq 3$) or the horizon radius (when $p = 3$). In a sense this guaranteed the matching between perturbative gauge theory and gravity results, just on dimensional grounds. To explore this further it would be interesting to study multi-charged black holes, or to deform the background in a way which introduces another length scale, and ask whether there is still a simple transition between perturbative worldvolume dynamics and black hole formation.

A step in this direction would be to consider 0-brane collisions but with $N_1 \neq N_2$. In this case, as we saw in section 3.6, the matching between perturbative gauge and gravity results must be more complicated, because the energy and entropy in open strings that are created in the initial collision do not match the equilibrium energy and entropy of the black hole. This means further dynamical evolution is required before the bound state reaches equilibrium. It would be interesting to study this, perhaps by going beyond the linearized approximation made when studying parametric resonance in section 3.4. There are several related interesting examples to consider, for example a situation in which several concentric layers of shells are collapsing.

Another direction would be to use the present results to better understand the microstructure of black holes. The picture that emerges, is that a black hole is a thermal bound state of D-branes and open strings, is reminiscent of the fuzzball proposal [60]. However the real question, relevant also for firewalls [61], is whether this thermal state could be a dual description of the interior geometry of the black hole.

4

Black Hole Formation at the Correspondence Point: Part II

In the previous chapter [62], we studied bound state formation in D-brane collisions, including the possible formation of a black hole. We considered collisions between clusters of D-branes, as well as a configuration in which D-branes were arranged in a spherical shell with velocities directed toward the center. At weak coupling a bound state forms via a process of open string production. At strong coupling, where the system has a dual supergravity description [3], the collision results in formation of a black hole. We found that the crossover between these two mechanisms for bound state formation is smooth. We did a matrix model calculation on the gauge theory side at an intermediate value of the coupling and matched our results with various thermodynamic quantities associated with the black hole, that form in the gravity side description.

4.1 Introduction

The purpose of the present work [63] is to study a more interesting initial configuration, namely a fuzzy sphere or spherical membrane built out of 0-branes. Starting from rest, a fuzzy sphere will shrink under its own tension. Classically the sphere shrinks to zero size and re-expands. But taking quantum effects into account, as the sphere shrinks open string production can occur at weak coupling, while black hole formation can occur at strong coupling. Our objective is to study these processes in more detail and show that they are smoothly connected at the correspondence point.

An outline of this chapter is as follows. In section 4.2 we review the description of fuzzy spheres and study the spectrum of fluctuations about a fuzzy sphere. In section 4.3 we study the collapse of a fuzzy sphere at weak coupling as open strings are produced. In section 4.4 we argue that there is a smooth match to the process of black hole formation at strong coupling. In section 4.5 we study the perturbative evolution of the sphere in more detail, including back-reaction from open string production. In section 4.6 we provide further evidence for a smooth crossover at the correspondence point.

There is a large literature on fuzzy geometry in various matrix models, for a review see [64]. In particular Berenstein and Trancanelli studied the tachyons which result from fuzzy spheres intersecting at angles in the BMN model, and the role they play in thermalizing the system [46].

4.2 Fuzzy spheres

To describe an ordinary sphere embedded in $\mathbb{R}^d$, we begin by introducing three Cartesian coordinates $x_A = (x, y, z)$ on a unit S^2, subject to the constraint

$$\sum_A x_A^2 = 1$$

The embedding coordinates in $\mathbb{R}^d$, which we denote X^i for $i = 1, \ldots, d$, can then be expanded in powers of the x_A's.

$$X^i = \sum_{\ell=0}^{\infty} c^i_{A_1 \cdots A_\ell} \, x_{A_1} \cdots x_{A_\ell} \tag{4.1}$$

The coefficients $c^i_{A_1 \cdots A_\ell}$ are symmetric and traceless on their lower indices. They transform in the spin-ℓ representation of $SU(2)$. After the traces are removed, the product $x_{A_1} \cdots x_{A_\ell}$ provides a Cartesian basis for the spin-ℓ spherical harmonics [65].

To make the sphere fuzzy or non-commutative we use the dictionary [66, 67, 68]

$$x_A \leftrightarrow \frac{2}{N} J_A \tag{4.2}$$

where the matrices J_A are generators of $SU(2)$ in the N-dimensional representation (i.e. with spin

$j = \frac{N-1}{2}$). They obey

$$[J_A, J_B] = i\epsilon_{ABC} J_C \qquad \sum_A J_A^2 = \frac{N^2-1}{4}\mathbf{1} \tag{4.3}$$

The embedding coordinates become Hermitian matrices, with the expansion

$$X^i = \sum_{\ell=0}^{N-1} c^i_{A_1\cdots A_\ell} \left(\frac{2}{N}\right)^\ell J_{A_1}\cdots J_{A_\ell} \tag{4.4}$$

Note that the expansion terminates at $\ell = N-1$, since beyond this point one no longer gets independent matrices. To make this plausible, note that summing the dimensions of the appropriate $SU(2)$ representations accounts for the N^2 parameters in a Hermitian matrix.

$$\sum_{\ell=0}^{N-1}(2\ell+1) = N^2 \tag{4.5}$$

In fact there is a stronger result: the matrices vanish identically for $\ell \geq N$. To see this it's convenient to work in a basis of raising and lowering operators $J_\pm = J_x \pm iJ_y$ with metric $ds^2 = dx^+dx^- + dz^2$. Note that $(J_+)^\ell$ is traceless and symmetric on its lower indices – it's the highest weight state in the spin-ℓ representation – and with N-dimensional generators it vanishes identically for $\ell \geq N$, $(J_+)^\ell = 0$ for $\ell \geq N$. Then by applying lowering operators a general symmetrized traceless product must vanish for $\ell \geq N$.

This construction of a fuzzy sphere finds a natural home in the BFSS model [69], or the quantum mechanics of N D0-branes, where the bosonic part of the action is*

$$S = \frac{1}{g^2_{\mathrm{YM}}}\int dt\,\mathrm{Tr}\left(\frac{1}{2}(\partial_0 X^i)^2 + \frac{1}{4}[X^i, X^j]^2\right) \tag{4.6}$$

We've fixed the gauge $A_0 = 0$, so the equation of motion

$$\ddot{X}^i + [[X^i, X^j], X^j] = 0 \tag{4.7}$$

*Conventions: the fields X^i have units of energy. They are related to 0-brane positions by $X = (\mathrm{position})/2\pi\alpha'$. The Yang-Mills coupling is $g^2_{\mathrm{YM}} = \frac{g_s}{(2\pi)^2\ell_s^3}$.

must be supplemented with the Gauss constraint

$$[\partial_0 X^i, X^i] = 0 \tag{4.8}$$

At the classical level a simple configuration is a spherical membrane of initial radius U_0, described by setting [70, 71]

$$\begin{aligned} X^A(t) &= U(t)\frac{2}{N}J^A \qquad & A = 1,2,3 \\ X^I &= 0 & I = 4,\cdots,9 \end{aligned} \tag{4.9}$$

The Gauss constraint is trivially satisfied since $[J^A, J^A] = 0$, while the equation of motion reduces to

$$\ddot{U} = -\frac{8}{N^2}U^3 \tag{4.10}$$

Solving this with the initial conditions $U(0) = U_0$, $\dot{U}(0) = 0$ one finds that the sphere collapses after a time

$$\tau = \frac{N\Gamma(1/4)^2}{\sqrt{128\pi}\,U_0} \tag{4.11}$$

This construction of a spherical membrane is based on the pioneering work of de Wit et al. [72]. The collapsing sphere solution was first described by Collins and Tucker [73].

In the quantum theory we'll be interested in fluctuations about this solution, so we set[†]

$$\begin{aligned} X^A(t) &= U(t)\frac{2}{N}J_A + x^A(t) \\ X^I(t) &= x^I(t) \end{aligned} \tag{4.12}$$

At linearized order the Gauss constraint (4.8) reduces to

$$\dot{U}[J^A, x^A] = U[J^A, \dot{x}^A] \tag{4.13}$$

This constraint removes roughly N^2 degrees of freedom from the $3N^2$ degrees of freedom contained

[†]The results for the spectrum in the remainder of this section have also been obtained by Harold Steinacker and Jochen Zahn [74].

in x^A.‡ However to linearized order it puts no constraint on x^I.

The linearized equation of motion for x^I is

$$\ddot{x}^I + \frac{4}{N^2}U^2[J^A, [J^A, x^I]] = 0 \tag{4.14}$$

To solve this we expand the field in fuzzy spherical harmonics,

$$x^I = \sum_{\ell=0}^{N-1} c^I_{A_1 \cdots A_\ell} \left(\frac{2}{N}\right)^\ell J_{A_1} \cdots J_{A_\ell} \tag{4.15}$$

With the $SU(2)$ algebra $[J_A, J_B] = i\epsilon_{ABC} J_C$ and the identity $\epsilon_{ABC}\epsilon_{ADE} = \delta_{BD}\delta_{CE} - \delta_{BE}\delta_{CD}$ one can show that (assuming the indices $A_1 \cdots A_\ell$ are contracted with a symmetric traceless tensor)

$$[J_A, [J_A, J_{A_1} \cdots J_{A_\ell}]] = \ell(\ell+1) J_{A_1} \cdots J_{A_\ell} \tag{4.16}$$

In other words, fuzzy spherical harmonics are angular momentum eigenstates, with the expected eigenvalue of the total angular momentum. The linearized equation of motion (4.14) then reduces to

$$\ddot{c}^I_{A_1 \cdots A_\ell} + \frac{4\ell(\ell+1)}{N^2} U^2 c^I_{A_1 \cdots A_\ell} = 0 \tag{4.17}$$

This determines the spectrum of fluctuations in the transverse dimensions $I = 4, \ldots, 9$. In each of these dimensions there are fluctuations with $\ell = 0, \ldots, N-1$. A fluctuation with angular momentum ℓ is $(2\ell+1)$-fold degenerate and has frequency

$$\omega_\ell = \frac{2}{N}\sqrt{\ell(\ell+1)}\, U \tag{4.18}$$

The spectrum of fluctuations in the dimensions $A = 1, 2, 3$ is studied in appendix A.2. Here we just summarize the results. Decomposing x^A into $SU(2)$ representations we find that there are s-type fluctuations with spin $\ell + 1$ for $\ell = 0, \ldots, N-1$. These fluctuations are $(2\ell+3)$-fold degenerate and have frequency

$$\omega_\ell = \frac{2}{N}\sqrt{\ell(\ell-1)}\, U \tag{4.19}$$

‡More precisely it removes $N^2 - 1$ degrees of freedom: the trace of a commutator vanishes, so the trace of the Gauss constraint is trivially satisfied.

name	labels	spin	degeneracy	frequency
transverse	$I = 4, \ldots, 9$ $\ell = 0, \ldots, N-1$	ℓ	$2\ell + 1$	$\frac{2}{N}\sqrt{\ell(\ell+1)}\,U$
s-type	$\ell = 0, \ldots, N-1$	$\ell + 1$	$2\ell + 3$	$\frac{2}{N}\sqrt{\ell(\ell-1)}\,U$
u-type	$\ell = 1, \ldots, N-1$	$\ell - 1$	$2\ell - 1$	$\frac{2}{N}\sqrt{(\ell+1)(\ell+2)}\,U$

Table 4.1: Spectrum of fluctuations about a fuzzy sphere of radius U.

There are also u-type fluctuations with spin $\ell - 1$ for $\ell = 1, \ldots, N-1$. These fluctuations are $(2\ell - 1)$-fold degenerate and have frequencies

$$\omega_\ell = \frac{2}{N}\sqrt{(\ell+1)(\ell+2)}\,U \tag{4.20}$$

In the rest of this chapter the distinction between these various types of frequencies will not matter, and from now on we will ignore the differences between the formulas (4.18), (4.19), (4.20). When we write explicit formulas we will make use of the transverse frequencies (4.18).

4.3 Perturbative sphere collapse

Assuming the 0-brane quantum mechanics is weakly coupled, let's study the collapse of a fuzzy sphere in a little more detail. The conserved total energy of the quantum mechanics is

$$E_{\rm YM} = \frac{1}{g_{\rm YM}^2}\mathrm{Tr}\left(\frac{1}{2}(\partial_0 X^i)^2 - \frac{1}{4}[X^i, X^j]^2\right) \tag{4.21}$$

which at large N for the classical solution (4.9) reduces to

$$E_{\rm YM} \approx \frac{1}{g_{\rm YM}^2}\left(\frac{N}{2}\dot{U}^2 + \frac{2}{N}U^4\right) \tag{4.22}$$

So the radial velocity $\dot{U}$ is related to the initial radius of the sphere U_0 by

$$\dot{U}^2 \approx \frac{4}{N^2}\left(U_0^4 - U^4\right) \tag{4.23}$$

Classically a fuzzy sphere remains spherical as it collapses, but quantum mechanically other modes will get excited. This happens when the adiabatic approximation breaks down. For a mode with frequency ω_ℓ, the adiabatic approximation fails when

$$\frac{\dot{\omega}_\ell}{\omega_\ell^2} \gtrsim 1 \tag{4.24}$$

Given the frequencies (4.18), adiabaticity breaks down when

$$\frac{N\dot{U}}{U^2\sqrt{\ell(\ell+1)}} \gtrsim 1 \tag{4.25}$$

which using (4.23) can be rewritten as

$$U \lesssim \frac{U_0}{\big(\ell(\ell+1)+1\big)^{1/4}} \tag{4.26}$$

So a large fuzzy sphere evolves adiabatically. As the sphere shrinks modes with more and more angular momentum become excited. The mode with the largest angular momentum, $\ell_{\text{max}} \sim N$, gets excited when the fuzzy sphere reaches the inner radius for open string production

$$U_{\text{inner}} \sim U_0/\sqrt{N} \tag{4.27}$$

At this point the adiabatic approximation has completely broken down, and all N^2 degrees of freedom in the matrices have become excited, or equivalently all possible open strings have been produced. The subsequent evolution of the sphere will be studied in section 4.5.

4.4 Black hole formation and the correspondence point

At large N and strong coupling the 0-brane quantum mechanics has a dual description in terms of IIA supergravity [3]. Introducing the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$ and a radial coordinate with units of energy $U = r/\alpha'$, the 0-brane quantum mechanics is weakly coupled when $U > \lambda^{1/3}$ and has a dual supergravity description when $U < \lambda^{1/3}$.[§]

In the supergravity regime one would expect a fuzzy sphere to collapse and form a black hole

[§]The radial coordinate U introduced here differs by a factor 2π from the radius of the sphere introduced in (4.9): see footnote $*$. We will ignore this difference from now on.

with N units of 0-brane charge. The Schwarzschild radius of such a black hole is [3]

$$U_S \sim \left(g_{\rm YM}^4 E\right)^{1/7} \tag{4.28}$$

Here E is the energy above extremality, identified with the Hamiltonian of the quantum mechanics. Given (4.22), the Schwarzschild radius is related to the initial radius of the sphere by

$$U_S \sim \left(\frac{g_{\rm YM}^2 U_0^4}{N}\right)^{1/7} \tag{4.29}$$

Of course this discussion only makes sense if the black hole fits in the region where supergravity is valid. This requires $U_S < \lambda^{1/3}$ or equivalently $E < N^2 \lambda^{1/3}$, which means

$$U_0 < N^{1/2} \lambda^{1/3} \tag{4.30}$$

The perturbative description of fuzzy sphere collapse worked out in section 4.3, on the other hand, is only valid if the quantum mechanics is weakly coupled. We followed the evolution of the sphere perturbatively down to the radius $U_{\rm inner}$ given in (4.27), at which point all N^2 degrees of freedom have gotten excited. This perturbative description is only valid if $U_{\rm inner} > \lambda^{1/3}$, or equivalently

$$U_0 > N^{1/2} \lambda^{1/3} \tag{4.31}$$

We now see that there is a smooth crossover between the perturbative description of fuzzy sphere collapse and the non-perturbative process of black hole formation. The crossover occurs when the initial radius and total energy are

$$\begin{aligned} U_0 &\sim N^{1/2} \lambda^{1/3} \\ E &\sim N^2 \lambda^{1/3} \end{aligned} \tag{4.32}$$

At the crossover point the Schwarzschild radius and inner radius for open string production agree,

$$U_S \sim U_{\rm inner} \sim \lambda^{1/3} \tag{4.33}$$

For a black hole of this size the curvature at the horizon is of order string scale.

$$\alpha' R \sim (U^3/\lambda)^{1/2} \sim 1 \tag{4.34}$$

So this crossover, once again, is an example of the correspondence principle of Horowitz and Polchinski at work [31].

4.5 Back-reaction and parametric resonance

In section 4.3 we followed the evolution of a weakly-coupled fuzzy sphere down to the radius $U_{\rm inner} \sim U_0/\sqrt{N}$. At this radius adiabaticity has broken down for all of the fluctuation modes, so $\mathcal{O}(N^2)$ open strings have been produced. In this section we study the subsequent evolution of the sphere, still assuming weak coupling, but taking into account back reaction from open string production. We'll show that a parametric resonance is present in the weakly-coupled field theory which exponentially amplifies the number of open strings present.

To study the back-reaction from open string production, we begin by estimating the total energy in open strings. Suppose that as the sphere collapses roughly one open string is produced in each of the fluctuation modes (4.18). This is justified in appendix A.3. Then once the sphere has crossed the radius $U_{\rm inner}$, the total energy in open strings is

$$\begin{aligned} E_{\rm open} &\sim \sum_{\ell=0}^{N-1} (2\ell+1)\,\omega_\ell \\ &= \sum_{\ell=0}^{N-1} (2\ell+1)\frac{2}{N}\sqrt{\ell(\ell+1)}\,U \\ &\sim N^2 U \end{aligned} \tag{4.35}$$

For this description of the collapse process to make sense, we should check that back-reaction from open string production can be neglected down to the radius $U_{\rm inner}$. To do this we compare the energy in open strings (4.35) to the total energy of the sphere (4.22). At the radius $U_{\rm inner}$ we have $E_{\rm open} \sim N^2 U_{\rm inner}$, while the total energy $E_{\rm YM} \sim U_0^4/\lambda \sim N^2 U_{\rm inner}^4/\lambda$, so

$$\frac{E_{\rm open}}{E_{\rm YM}} \sim \frac{\lambda}{U_{\rm inner}^3} \tag{4.36}$$

Indeed, provided the field theory remains weakly coupled down to the radius U_{inner}, we have $U_{\text{inner}} > \lambda^{1/3}$ (or equivalently $U_0 > N^{1/2}\lambda^{1/3}$) and back reaction can be neglected during the initial collapse of the sphere.

Even though back-reaction can be neglected during the initial collapse of the sphere, it is not necessarily negligible when the sphere subsequently re-expands. To decide this issue we compare the potential energy in open strings (4.35), $E_{\text{open}} \sim N^2 U$, to the classical potential energy of a fuzzy sphere, which from (4.22) is given by $E_{\text{classical}} = \frac{2}{\lambda}U^4$. Thus

$$\frac{E_{\text{open}}}{E_{\text{classical}}} \sim \frac{N^2\lambda}{U^3} \tag{4.37}$$

The linear potential from open strings dominates at small radius, while the classical U^4 potential dominates at large radius. The two energies are comparable when $U \sim N^{2/3}\lambda^{1/3}$.

We can now identify three qualitatively different behaviors, depending on the initial radius of the sphere.

large initial radius, $U_0 > N^{2/3}\lambda^{1/3}$

In this case the classical potential energy of the sphere is dominant near the turning point, which is located at $U \approx U_0$. The classical evolution of the sphere described in section 4.2 is a good approximation to the true behavior. In particular the sphere collapses to zero size on the timescale $\tau \sim N/U_0$ given in (4.11).

intermediate initial radius, $N^{1/2}\lambda^{1/3} < U_0 < N^{2/3}\lambda^{1/3}$

In this case the field theory remains weakly coupled down to the radius U_{inner}, but when the sphere subsequently re-expands it's the linear potential arising from open string production which is dominant near the turning point. The classical U^4 potential can be neglected, and overall energy conservation reads (in place of (4.22))

$$\frac{N}{2g_{\text{YM}}^2}\dot{U}^2 + cN^2U = \frac{2U_0^4}{\lambda} \tag{4.38}$$

Here c is an $\mathcal{O}(1)$ constant reflecting the number of open strings present in each mode. In this linear potential the turning point is located at $U = 2U_0^4/cN^2\lambda$, which fortunately is in the weakly coupled regime of the field theory. After reaching the turning point, the sphere re-collapses to zero size in a time

$$\tau = \frac{2U_0^2}{cN\lambda} \tag{4.39}$$

small initial radius, $U_0 < N^{1/2}\lambda^{1/3}$

In this case the sphere enters the regime where supergravity is valid and falls within its own Schwarzschild radius to form a black hole.

We can now describe the subsequent evolution of the sphere in a little more detail. At weak coupling the sphere pulsates with a frequency

$$\Omega \sim 1/\tau \sim \begin{cases} U_0/N & \text{large initial radius} \\ N\lambda/U_0^2 & \text{intermediate initial radius} \end{cases} \tag{4.40}$$

One can approximate this as an oscillating classical background $U(t) = \widetilde{U}_0 \sin \Omega t$, where the back-reacted amplitude of oscillation

$$\widetilde{U}_0 \sim \begin{cases} U_0 & \text{large initial radius} \\ U_0^4/N^2\lambda & \text{intermediate initial radius} \end{cases} \tag{4.41}$$

Plugging this oscillating background into the fluctuation equation (4.17) for the transverse fluctuations, one finds that small fluctuations are governed by the Mathieu equation. As in [62], this means there is a parametric resonance which makes the number of open strings grow exponentially with time, on a timescale set by the period of oscillation τ.[¶]

4.6 More on the correspondence point

The collapse of a fuzzy sphere appears qualitatively different depending on whether the initial radius is large, intermediate or small. In this section we study the transitions between these different regimes, and argue that they are in fact smoothly connected.

One can smoothly continue from large to intermediate initial radius in the formulas (4.40),

[¶]Similarly, for s-type and u-type fluctuations, we obtain Mathieu equations with ω_l given by (4.19) and (4.20). However the derivation of (4.19) and (4.20) in appendix A.2 is under the adiabatic approximation, $\dot{U} \to 0$. Therefore we expect the Mathieu equations for s-type and u-type fluctuations are modified once the adiabatic approximation breaks down and parametric resonance occurs.

(4.41) for the frequency and amplitude of oscillation, since the expressions agree at the large-to-intermediate crossover point $U_0 \sim N^{2/3}\lambda^{1/3}$. In a way this is not surprising. At large and intermediate initial radius open string production takes place while the field theory is still weakly coupled. As the initial radius is decreased open string production becomes more important. The resulting linear potential smoothly takes over from the classical U^4 potential, and this is responsible for modifying the frequency and amplitude of oscillation.

Now let's see if we can continue from intermediate to small initial radius. This intermediate-to-small crossover occurs when $U_0 \sim N^{1/2}\lambda^{1/3}$, which corresponds to a total energy $E \sim U_0^4/\lambda \sim N^2\lambda^{1/3}$. This amounts to working at the correspondence point of Horowitz and Polchinski [31], since the Schwarzschild radius of the resulting black hole

$$U_S \sim (g_{\rm YM}^4 E)^{1/7} \sim \lambda^{1/3} \tag{4.42}$$

which means the curvature at the horizon is of order string scale.

$$\alpha' R \sim (U_S^3/\lambda)^{1/2} \sim 1 \tag{4.43}$$

In other words, the resulting black hole just fits in the region where supergravity is valid [3].

There are various quantities we can compare at the Horowitz-Polchinski correspondence point which suggest that the crossover is smooth.

classical size

In the weakly-coupled field theory the classical background is a pulsating sphere with a maximum size given in (4.41). Evaluating this at $U_0 = N^{1/2}\lambda^{1/3}$ we find that the back-reacted amplitude of oscillation is set by the 't Hooft scale, $\widetilde{U}_0 \sim \lambda^{1/3}$. This matches the Schwarzschild radius (4.42) of a black hole at the correspondence point, $U_S \sim \lambda^{1/3}$.

size of quantum fluctuations

For the classical background (4.9), the size of the sphere can be measured by

$$\frac{1}{N}{\rm Tr}\left(X^A X^A\right) = U^2\left(1 + \mathcal{O}(1/N^2)\right) \tag{4.44}$$

Let's compare this to the spread in the 0-brane positions due to quantum fluctuations, measured by

$$(\Delta X)^2 \equiv \frac{1}{N}\langle \mathrm{Tr}\left(X^I X^I\right)\rangle \qquad I = 4,\ldots,9 \tag{4.45}$$

To evaluate this, recall that for a harmonic oscillator

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right) \tag{4.46}$$

We can adapt this to the problem at hand by identifying $\hbar/m$ with g^2_{YM}. Then assuming small fluctuations and using the frequencies (4.18) we have

$$\begin{aligned} (\Delta X)^2 &= \sum_I \frac{1}{N} \sum_{\ell=1}^{N-1} \sum_{m=-\ell}^{\ell} \frac{g^2_{\mathrm{YM}}}{\omega_\ell}\left(n^I_{\ell m} + \frac{1}{2}\right) \\ &\sim \frac{1}{N}\sum_{\ell=1}^{N-1}(2\ell+1)\frac{g^2_{\mathrm{YM}}N}{\sqrt{\ell(\ell+1)}\,U} \\ &\sim \frac{\lambda}{U} \end{aligned} \tag{4.47}$$

In the first line we suppressed the $\ell = 0$ modes which describe center of mass position. In the second line we dropped the sum on I and took the quantum numbers $n^I_{\ell m} \sim \mathcal{O}(1)$, appropriate to having one open string per mode. To compare the size of these quantum fluctuations to the size of the classical background, we set $U = \widetilde{U}_0$ and consider the ratio

$$\frac{(\Delta X)^2}{(\widetilde{U}_0)^2} \sim \frac{\lambda}{(\widetilde{U}_0)^3} \tag{4.48}$$

Provided the maximum size of the sphere is larger than the 't Hooft scale, $\widetilde{U}_0 > \lambda^{1/3}$ or equivalently $U_0 > N^{1/2}\lambda^{1/3}$, then the quantum fluctuations in the 0-brane positions are small compared to the radius of the sphere. This shows that at large and intermediate initial radius a classical fuzzy sphere provides a good description of the quantum state.[‖] It also shows that as we go to the Horowitz-Polchinski correspondence point, $\widetilde{U}_0 = \lambda^{1/3}$, the classical background merges into the quantum fluctuations. This fits with a general expectation in gravity-gauge duality, that at strong coupling

[‖]Although as we saw in section 4.5, for intermediate initial radius one must take back-reaction into account to find the correct frequency and amplitude for the classical background.

the D-brane positions have quantum fluctuations which are comparable in size to the region in which supergravity is valid [75, 76].

thermalization time

On the weakly-coupled side we identified a parametric resonance which leads to open string production on a timescale set by the frequency (4.40). Evaluating this at $U_0 = N^{1/2}\lambda^{1/3}$ we find that the frequency of oscillation is set by the 't Hooft scale, $\Omega \sim \lambda^{1/3}$.

What does this correspond to on the supergravity side? The black hole has a spectrum of quasinormal frequencies which govern the approach to equilibrium. The quasinormal frequencies are set by the Hawking temperature [59, 57, 58], namely $T \sim \frac{1}{\sqrt{\lambda}} U_S^{5/2}$, which at the correspondence point is of order the 't Hooft scale, $T \sim \lambda^{1/3}$. Thus at the correspondence point the timescale for parametric resonance agrees with the relaxation time of the black hole. This suggests that the weak-coupling process of open string production via parametric resonance smoothly matches on to the strong-coupling process of black hole formation.

entropy production

At weak coupling, during the initial collapse of a fuzzy sphere, we saw that $\mathcal{O}(N^2)$ open strings are produced. These strings have an entropy $S_{\text{string}} \sim N^2$. On the other hand, on the supergravity side, the equilibrium entropy of the black hole is [3]

$$S_{\text{bh}} \sim N^2 U_S^{9/2} / \lambda^{3/2} \tag{4.49}$$

Evaluating this at the correspondence point $U_S \sim \lambda^{1/3}$ we see that $S_{\text{bh}} \sim N^2$. So at the correspondence point the entropy produced during the initial collapse of a fuzzy sphere is close to the equilibrium entropy of the black hole. This suggests that very little additional evolution – perhaps just a few e-foldings of parametric resonance – is required for the system to reach equilibrium.

5

Holographic Representation of Bulk Fields: Case for Spin-1

In this chapter and the next, we steer our interests towards the construction of local operators in the bulk of AdS spaces in terms of gauge theory information, as mentioned in the introduction [77]. We especially look at the situation where there is a gauge symmetry on the bulk side. We start off with the spin -1 Maxwell fields in this chapter and construct the corresponding smearing function which later helps us to write down a gauge invariant local bulk operator. We show its locality property by computing a commutator of that operator with a spacelike separated boundary operator. We do the same for spin-2 graviton fields in the next chapter.

5.1 Introduction

The question of locality and causality in quantum gravity is an old and unresolved issue. AdS/CFT implies that at best locality and causality are approximate notions. However it is vital to understand in what situations and in what way the notion of bulk locality arises. One approach to this issue, pursued since the early days of AdS/CFT, is to construct operators in the CFT which can mimic the local field operators of bulk supergravity.

In [78, 79, 80, 81] free scalar fields in the bulk were expressed as CFT operators, and it was shown that bulk locality was obeyed in the leading large-N limit. This approach was refined to obtain CFT expressions that are covariant and convenient in [82, 83, 84]. In particular it was shown that one can represent bulk scalar fields as smeared operators in the CFT, where the smearing has support on a ball on the complexified boundary. See e.g. appendix A.4 for an overview of calculation for the case of scalars. In [85] (see also [86]) it was shown that for scalar fields this construction can be extended to include interactions using $1/N$ perturbation theory. The construction of bulk operators in asymptotically AdS spacetimes has been further extended and clarified in [87].

In this chapter we build upon two approaches that have been successfully used to construct scalar fields in the bulk.

1. Given a bulk Lagrangian one can solve the bulk equations of motion perturbatively, to express the Heisenberg picture field operators in terms of boundary data. This leads to an expression for the bulk field as a sum of smeared CFT operators. The bulk operator constructed in this way of course respects locality, assuming one starts from a local Lagrangian in the bulk, but the construction seems tied to knowing the bulk equations of motion.

2. Alternatively one can start in the CFT with a candidate bulk operator, constructed by solving free equations of motion, then demand that bulk micro-causality holds at the level of three point functions. This can be achieved order-by-order in the $1/N$ expansion, by modifying the definition of the bulk field in the CFT to include a sum of appropriately-smeared higher dimension operators. In this construction the guiding principle is bulk micro-causality.

The later construction can be carried out fully inside the CFT, without knowing the bulk Lagrangian. Hence it may enable one to see the limitations of bulk perturbation theory, and understand the way in which micro-causality breaks down at the non-perturbative level*. A difficulty of extending the second approach to gauge fields is that the correct statement of bulk micro-causality is necessarily somewhat subtle [87].

An outline of this chapter is as follows. In the first part of this chapter we extend the program of [82, 83, 84] to free fields with spin one. A closely related construction has been carried out by Heemskerk [89]. In section 5.2 we derive the smearing function for a bulk gauge field and show that it is covariant under conformal transformations. We compute the bulk-to-boundary two point function and show that, although the gauge field does not obey micro-causality, the corresponding field strength does. In section 5.3 we derive the smearing function for a massive vector field, and

*By micro-causality, we basically mean that the operators will commute at space like separation [88].

show that a massive vector directly obeys micro-causality. The helps clarify the relation between gauge symmetry and locality.

5.2 Gauge smearing functions

In this section we develop the representation of an abelian bulk gauge field as a non-local observable in the dual CFT. Our basic result is given in (5.4) below: the bulk gauge field at a point (x, z) in the bulk is obtained by integrating the boundary current over a sphere of radius z on the complexified boundary.

Our conventions are as follows. We work in Poincaré coordinates in AdS_{d+1} with metric

$$ds^2 = G_{MN} dX^M dX^N = \frac{R^2}{z^2}\left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2\right) \qquad \mu, \nu = 0, \ldots, d-1$$

The boundary at $z = 0$ carries a flat Minkowski metric, $\eta_{\mu\nu} = \mathrm{diag}(- + \cdots +)$. Boundary indices μ, ν are raised and lowered with $\eta_{\mu\nu}$.

Our goal is to solve the source-free Maxwell equations in the bulk, $\nabla_M F^{MN} = 0$, with the boundary conditions

$$F_{z\mu}(x, z) \sim (d-2) z^{d-3} j_\mu(x) \qquad \text{as } z \to 0 \tag{5.1}$$

The factor $d - 2$ is inserted for later convenience.† From the bulk perspective this defines $j_\mu(x)$ as the coefficient of the leading small-z behavior of the bulk field. But in the dual CFT $j_\mu(x)$ is interpreted as a conserved current. So if we can solve for the bulk field in terms of its near-boundary behavior, via a kernel of the form

$$A_M(x, z) = \int d^d x' \, K_M{}^\mu(x, z|x') j_\mu(x') \,, \tag{5.2}$$

then we will have succeeded in representing the bulk gauge field as a non-local observable in the dual CFT. We'll refer to $K_M{}^\mu$ as a smearing function, although as we'll see below, smearing distribution might be more appropriate.

A few comments are in order.

†The special case $d = 2$ will be discussed in section 5.2.2.1.

- The smearing function we are after should not be confused with Witten's bulk-to-boundary propagator, which relates a non-normalizeable field in the bulk to a source in the dual CFT [32]. Rather we wish to express a *normalizeable* field in the bulk in terms of an *operator* in the CFT.

- The AdS boundary is timelike, so this is not a standard Cauchy problem. Nonetheless, in all cases of interest, it seems an explicit solution is possible. There is some discussion of this fact in [87]. Also note that we will construct smearing functions with compact support on the complexified boundary, along the lines of [84]. For a construction with support on a real section of the boundary see [89].

Of course the CFT doesn't know about bulk gauge symmetries – it only sees global conservation laws – so in order to reconstruct a bulk gauge field we will need to make some choice of gauge in the bulk. It's convenient to work in "holographic gauge" and set

$$A_z(x, z) = 0\,.$$

This allows a residual gauge freedom

$$A_\mu(x, z) \rightarrow A_\mu(x, z) + \partial_\mu \lambda(x)$$

where the gauge parameter λ is independent of z. The equation of motion from varying A_z is

$$\partial_z \left(\eta^{\mu\nu} \partial_\mu A_\nu\right) = 0\,.$$

Thus $\partial_\mu A^\mu$ is independent of z, and we can use a residual gauge transformation to set $\partial_\mu A^\mu = 0$ everywhere.[‡] The remaining Maxwell equations then simplify to

$$\partial_\mu \partial^\mu A_\nu + z^{d-3} \partial_z \frac{1}{z^{d-3}} \partial_z A_\nu = 0\,.$$

[‡]From the CFT point of view this is guaranteed by the boundary conditions at $z = 0$, where the bulk gauge field approaches a conserved current in the CFT.

Defining $\phi_\mu(x,z) = zA_\mu(x,z)$ one finds that[§]

$$\partial_\mu \partial^\mu \phi_\nu + z^{d-1}\partial_z \frac{1}{z^{d-1}} \partial_z \phi_\nu + \frac{d-1}{z^2}\phi_\nu = 0\,. \tag{5.3}$$

This shows that each component of ϕ obeys the usual scalar wave equation,[¶] and from the mass term we can read off $m^2R^2 = 1-d$.

Although tachyonic, the scalar satisfies the Breitenlohner-Freedman (BF) bound [90]. It is dual to an operator of conformal dimension

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2R^2} = d-1$$

The normalizeable near-boundary behavior for such a scalar field is

$$\phi_\mu(x,z) \sim z^{d-1} j_\mu(x) \qquad \text{as } z \to 0$$

In appendix A.4 we show how to construct a smearing function for such a scalar field. The result, given in (A.43), can be used to represent a bulk gauge field in terms of the boundary current.

$$zA_\mu(t,\mathbf{x},z) = \frac{1}{\mathrm{vol}(S^{d-1})} \int\limits_{t'^2+|\mathbf{y}'|^2=z^2} dt' d^{d-1}y'\, j_\mu(t+t', \mathbf{x}+i\mathbf{y}') \tag{5.4}$$

$$\mathrm{vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

Here we're splitting the boundary coordinates $x^\mu = (t;\mathbf{x})$ into a time coordinate t and $d-1$ spatial coordinates $\mathbf{x}$. Note that the boundary current is evaluated at complex values of the spatial coordinates. The integral is over a sphere of radius z on the complexified boundary, with the center of the sphere located at $(t;\mathbf{x})$.

The basic claim is that (5.4) gives a gauge field that satisfies Maxwell's equations and has the boundary behavior

$$A_\mu(x,z) \sim z^{d-2} j_\mu(x) \qquad \text{as } z \to 0 \tag{5.5}$$

[§]This amounts to expressing the gauge field in a vielbein basis, setting $A_a = e_a{}^\mu A_\mu$ where $e_a{}^\mu = \frac{z}{R}\delta_a{}^\mu$.

[¶]The mass term actually represents a non-minimal coupling to curvature, $(\Box + \xi R)\,\phi = 0$ where $\xi = -\frac{d-1}{d(d+1)}$.

The fact that A_μ satisfies Maxwell's equations follows from appendix A.4, while the boundary conditions are easy to check. As $z \to 0$ the integration region shrinks to a point, so we can bring the current outside the integral; the factors of $\text{vol}(S^{d-1})$ cancel and we're left with (5.5). The corresponding field strength then satisfies (5.1). This is one nice feature of working on the complexified boundary: it's manifest that local fields in the bulk go over to local operators in the CFT, in the limit that the bulk point approaches the boundary.

Finally note that (5.4) can be written in a covariant form. The invariant distance between two points in AdS is

$$\sigma(x, z|x', z') = \frac{z^2 + z'^2 + (x - x')_\mu (x - x')^\mu}{2zz'} .$$

The invariant distance diverges as $z' \to 0$. However we can define a regulated bulk - boundary distance

$$(\sigma z')_{z' \to 0} = \frac{z^2 + (x - x')_\mu (x - x')^\mu}{2z} \tag{5.6}$$

In terms of $\sigma z'$, the smearing integral (5.4) can be written as

$$z A_\mu(t, \mathbf{x}, z) = \frac{1}{\text{vol}(S^{d-1})} \int dt' d^{d-1}y' \, \delta(\sigma z') \, j_\mu(t + t', \mathbf{x} + i\mathbf{y}') \tag{5.7}$$

5.2.1 AdS covariance for gauge fields

It's instructive to check that the smearing function (5.7) behaves covariantly under conformal transformations. First note that it's manifestly covariant under Poincaré transformations of the x^μ coordinates. Under a dilation, which corresponds to the bulk isometry

$$x^\mu \to x'^\mu = \lambda x^\mu \qquad z \to z' = \lambda z$$

we have

$$A_\mu \to A'_\mu = \frac{1}{\lambda} A_\mu \qquad A_z \to A'_z = \frac{1}{\lambda} A_z$$

Thus holographic gauge is preserved, $A'_z = 0$, and the quantity $z A_\mu$ appearing on the left hand side of (5.7) transforms like a scalar. This is consistent with the right hand side of (5.7), since under a dilation $d^d x$ has dimension $-d$, $\delta(\sigma z')$ has dimension 1, and j_μ has dimension $d - 1$.

Special conformal transformations are a little more subtle. These correspond to the bulk isometry

$$x^\mu \to x'^\mu = \frac{x^\mu - b^\mu(x^2+z^2)}{1 - 2b\cdot x + b^2(x^2+z^2)} \tag{5.8}$$

$$z \to z' = \frac{z}{1 - 2b\cdot x + b^2(x^2+z^2)} \tag{5.9}$$

Starting from holographic gauge $A_z = 0$ and working to first order in b^μ we find

$$A'_z = 2zb\cdot A \tag{5.10}$$

$$A'_\mu = A_\mu + 2x_\mu b\cdot A - 2b_\mu x\cdot A - 2b\cdot x A_\mu \tag{5.11}$$

So holographic gauge isn't preserved. To restore it we make a compensating gauge transformation $A \to A + d\lambda$ where

$$\lambda = -\frac{1}{\mathrm{vol}(S^{d-1})}\int d^dx'\,\theta(\sigma z')\,2b\cdot j$$

The gauge parameter λ has been chosen so that

$$\partial_z\lambda = -\frac{1}{\mathrm{vol}(S^{d-1})}\int d^dx'\,\delta(\sigma z')\,2b\cdot j = -2zb\cdot A \tag{5.12}$$

and

$$\partial_\mu\lambda = -\frac{1}{\mathrm{vol}(S^{d-1})}\int d^dx'\,\delta(\sigma z')\,\frac{1}{z}(x-x')_\mu\,2b\cdot j \tag{5.13}$$

$$= -2x_\mu b\cdot A + \frac{1}{\mathrm{vol}(S^{d-1})}\int d^dx'\,\delta(\sigma z')\,\frac{1}{z}x'_\mu\,2b\cdot j \tag{5.14}$$

The gauge transformation restores holographic gauge, $A'_z = 0$, while combining (5.11) and (5.13) we find

$$\big(zA_\mu\big)' = zA_\mu - 2zb_\mu x\cdot A + \frac{1}{\mathrm{vol}(S^{d-1})}\int d^dx'\,\delta(\sigma z')\,x'_\mu\,2b\cdot j \tag{5.15}$$

$$= zA_\mu + \frac{1}{\mathrm{vol}(S^{d-1})}\int d^dx'\,\delta(\sigma z')\,2(x'_\mu\,b\cdot j - b_\mu x\cdot j) \tag{5.16}$$

Current conservation implies $\int d^dx'\,\theta(\sigma z')\,\partial_\mu j^\mu = 0$, which after integrating by parts means

$$\int d^dx'\,\delta(\sigma z')\,(x - x')_\mu j^\mu = 0\,. \tag{5.17}$$

So we can replace x with x' in the last term of (5.16) to obtain

$$\big(zA_\mu\big)' = zA_\mu + \frac{1}{\mathrm{vol}(S^{d-1})}\int d^dx'\,\delta(\sigma z')\,2(x'_\mu\, b\cdot j - b_\mu x'\cdot j) \tag{5.18}$$

This establishes how the left hand side of (5.7) behaves under a special conformal transformation. Now let's look at the right hand side. Under a special conformal transformation

$$x^\mu \to x'^\mu = x^\mu + 2b\cdot x x^\mu - b^\mu x^2 \tag{5.19}$$

a vector of dimension Δ transforms according to

$$j_\mu \to j'_\mu = j_\mu + 2x_\mu b\cdot j - 2b_\mu x\cdot j - 2\Delta b\cdot x j_\mu \tag{5.20}$$

The measure $d^dx'\,\delta(\sigma z')$ has dimension $1-d$ and transforms according to

$$d^dx'\,\delta(\sigma z') \to d^dx'\,\delta(\sigma z')\big[1 - 2(1-d)b\cdot x\big] \tag{5.21}$$

Combining (5.20) and (5.21) for $\Delta = d-1$ reproduces the transformation seen in (5.18).

This shows explicitly that the smearing function we have defined behaves covariantly under conformal transformations.‖ Indeed it seems that, aside from the freedom to choose a different gauge in the bulk, the smearing function is uniquely fixed by the requirement of AdS covariance, at least if one works on the complexified boundary. This means that, even though we derived the smearing function by solving Maxwell's equations, it actually has a more general scope of validity. It can be used whenever one seeks a linear map from a conserved current on the boundary to a gauge field in the bulk.

‖This was probably guaranteed to be the case by the construction in section 5.2.

5.2.2 Two-point functions and bulk causality for gauge fields

In this section we use the smearing functions we have constructed to study bulk locality and causality for gauge fields. Since we are working at leading order in the $1/N$ expansion of the CFT, we are restricted to studying bulk physics at the level of two-point functions. We consider two basic cases: in section 5.2.2.1 we consider Chern-Simons theory in AdS_3, and in section 5.2.2.2 we consider Maxwell theory in AdS_4 and higher.

5.2.2.1 Chern-Simons fields in AdS_3

AdS_3 is something of a special case, since a conserved current in the CFT is dual to a Chern-Simons gauge field in the bulk [91]. Fortunately we can still use our smearing functions in this context, since they're essentially fixed by AdS covariance.

From the smearing function (5.4) we have

$$zA_\mu(t,x,z) = \frac{1}{2\pi}\int_0^{2\pi} z d\theta\, j_\mu(t + z\sin\theta, x + iz\cos\theta) \tag{5.22}$$

It's convenient to introduce light-front coordinates $x^\pm = t \pm x$ and write the AdS_3 metric as

$$ds^2 = \frac{R^2}{z^2}\left(-dx^+dx^- + dz^2\right)$$

For concreteness consider a CFT with a right-moving abelian current $j_- = j_-(x^-)$. We assume the left-moving current vanishes, $j_+ = 0$. Then the only non-trivial smearing integral is

$$A_-(x^+, x^-, z) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, j_-(x^- - ize^{i\theta})$$

Defining $\xi = e^{i\theta}$ the contour integral picks up the pole at $\xi = 0$ and gives $A_-(x^+, x^-, z) = j_-(x^-)$. So a right-moving current in the CFT is dual to a bulk gauge field

$$\begin{aligned} &A_+ = 0 \\ &A_-(x^+, x^-, z) = j_-(x^-) \\ &A_z = 0 \end{aligned} \tag{5.23}$$

This is the world's simplest example of holography: the boundary current is lifted to be z-independent, and declared to be a gauge field in the bulk.

Although "reading the hologram" in this case is almost trivial, there are a few things to check. First of all, (5.23) defines a flat gauge field in AdS, which satisfies the Chern-Simons equations of motion.** Working backwards, the boundary conditions on the gauge field are a bit different from (5.1), since we have

$$A_\mu(x,z) \sim j_\mu(x) \qquad \text{as } z \to 0$$

We can use this framework to compute 2-point functions in the bulk. The boundary correlator is fixed by conformal invariance. With a Wightman $i\epsilon$ prescription

$$\langle j_-(x^-) j_-(x^{-\prime}) \rangle = -\frac{k}{8\pi^2} \frac{1}{(x^- - x^{-\prime} - i\epsilon)^2} \tag{5.24}$$

where k is the level of the current algebra. This lifts to a bulk correlator

$$\langle A_-(x^+, x^-, z) A_-(x^{+\prime}, x^{-\prime}, z') \rangle = -\frac{k}{8\pi^2} \frac{1}{(x^- - x^{-\prime} - i\epsilon)^2}$$

Note that the bulk 2-point function is independent of x^+ and z, which is perhaps not so surprising in a topological theory.

We can also study bulk locality and causality in this framework. The correlator (5.24) implies that the CFT currents obey the standard current algebra

$$i[j_-(x^-), j_-(x^{-\prime})] = -\frac{k}{4\pi} \delta'(x^- - x^{-\prime})\,.$$

This lifts to a bulk commutator

$$i[A_-(x^+, x^-, z), A_-(x^{+\prime}, x^{-\prime}, z')] = -\frac{k}{4\pi} \delta'(x^- - x^{-\prime}) \tag{5.25}$$

This bulk commutator is clearly non-local, being independent of both x^+ and z. But causality is respected: the field strength vanishes, so all local gauge-invariant quantities obey causal (in fact

**The smearing functions were constructed by solving Maxwell's equations, but they are essentially fixed by AdS covariance and therefore hold more generally. In AdS_3 the smearing functions seem to know that a current in the CFT is dual to a Chern-Simons gauge field in the bulk.

trivial) commutation relations.

We obtained these results by applying our smearing functions to the current algebra on the boundary. In appendix A.5 we show that they can also be obtained from the bulk point of view, by quantizing Chern-Simons theory in holographic gauge.

5.2.2.2 Maxwell fields in AdS_4 and higher

We now consider Maxwell fields in AdS_4 and higher, where a bulk gauge field obeying Maxwell's equations is dual to a conserved current on the boundary.[††]

Our starting point is the current – current correlator in a d-dimensional CFT,

$$\langle\, j_\mu(x)\, j_\nu(0)\,\rangle = \left(\frac{1}{x^2}\right)^{d-1}\left(\eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}\right) . \tag{5.26}$$

Up to an overall normalization, this correlator is fixed by current conservation and conformal invariance. We will be interested in Wightman correlators, defined by the $i\epsilon$ prescription

$$x^2 \equiv -(t - i\epsilon)^2 + |\mathbf{x}|^2 .$$

Our goal is to apply the smearing function (5.4) to the first operator in (5.26), to obtain a bulk - boundary correlator

$$\langle\, A_\mu(t, \mathbf{x}, z)\, j_\nu(0)\,\rangle .$$

To deal with the vector indices it's useful to note that for any Lorentz-invariant function $f(x) = f\left(\sqrt{-t^2 + |\mathbf{x}|^2}\right)$ we have

$$\partial_\mu \partial_\nu f = \eta_{\mu\nu}\, \frac{1}{x} f'(x) + \frac{x_\mu x_\nu}{x^2}\Big(f''(x) - \frac{1}{x} f'(x)\Big) \tag{5.27}$$

This lets us write the current – current correlator in the form

$$\langle\, j_\mu(x)\, j_\nu(0)\,\rangle = \frac{d-2}{d-1}\, \eta_{\mu\nu} \left(\frac{1}{x^2}\right)^{d-1} - \frac{1}{2(d-1)(d-2)}\, \partial_\mu \partial_\nu \left(\frac{1}{x^2}\right)^{d-2}$$

[††]Low dimensions are special, for example in AdS_3 a bulk Maxwell field is dual to a gauge field in the CFT [92, 91]. Strictly speaking AdS_4 Maxwell is also special since the boundary currents only capture the "electric" sector of the bulk theory [93].

Applying the smearing function (5.4) gives the bulk – boundary correlator in terms of two scalar integrals,

$$\langle zA_\mu(t,\mathbf{x},z)j_\nu(0)\rangle = \frac{\Gamma(d/2)}{2\pi^{d/2}}\left(\frac{d-2}{d-1}\,\eta_{\mu\nu}I_1 - \frac{1}{2(d-1)(d-2)}\,\partial_\mu\partial_\nu I_2\right) \tag{5.28}$$

where

$$I_n = \int\limits_{t'^2+|\mathbf{y}'|^2=z^2} dt' d^{d-1}y' \,\frac{1}{\big(-(t+t')^2+|\mathbf{x}+i\mathbf{y}'|^2\big)^{d-n}} \tag{5.29}$$

The integral is over a $(d-1)$-sphere of radius z on the boundary. We write the metric on this sphere as

$$ds^2 = \frac{z^2}{z^2-y^2}\,dy^2 + (z^2-y^2)\,d\Omega_{d-2}^2$$

Here $-z < y < z$ and $d\Omega_{d-2}^2$ is the metric on a unit S^{d-2}. To take advantage of spherical symmetry on S^{d-2} we work at spacelike separation in the x_1 direction, setting

$$x_1 = x \qquad t = x_2 = \cdots = x_{d-1} = 0$$

Then I_n reduces to a one-dimensional integral.

$$I_n = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}\int_{-z}^{z} dy\,\frac{z(z^2-y^2)^{(d-3)/2}}{(x^2-z^2+2ixy)^{d-n}}$$

The prescription for defining this integral is to begin at large spacelike separation, $x \gg 0$, where the operators are well-separated on the boundary and the integral is well-defined. It can be extended to smaller values of x by analytic continuation, as described in Fig. 5.1. This prescription gives I_n in terms of a hypergeometric function.

$$I_n = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,\frac{z^{d-1}}{(x^2)^{d-n}}F\Big(d-n, \frac{d}{2}-n+1, \frac{d}{2}, -\frac{z^2}{x^2}\Big) \tag{5.30}$$

When $n = 1$ this reduces to

$$I_1 = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,\frac{z^{d-1}}{(x^2+z^2)^{d-1}}\,. \tag{5.31}$$

Note that I_1 is only singular on the bulk lightcone, at $x^2+z^2 = 0$. It has an AdS-covariant form, with $I_1 \sim 1/(\sigma z')^{d-1}$. These properties could have been anticipated since, up to an overall coefficient, I_1

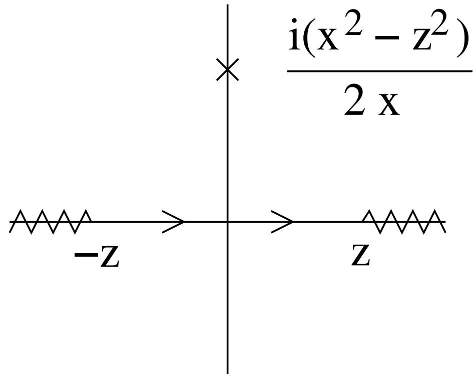

Figure 5.1: Integration contour for I_n. At large spacelike separation the pole is far up the imaginary axis. The pole moves down and crosses the integration contour when $x = z$; one can continue to smaller values of x by deforming the contour. The integral may be singular when $x \to 0^+$ and the pole moves to $-i\infty$. There are singularities when $x \to \pm iz$ and the pole hits an endpoint of the integration contour.

d	I_2
3	$-\frac{2\pi iz}{x}\log\frac{x+iz}{x-iz}$
4	$\frac{2\pi^2 z^3}{x^2(x^2+z^2)}$
5	$-\frac{i\pi^2 z}{2x^3}\log\frac{x+iz}{x-iz} - \frac{\pi^2 z^2(x^2-z^2)}{x^2(x^2+z^2)^2}$
6	$\frac{\pi^3 z^5(z^2+3x^2)}{3x^4(x^2+z^2)^3}$

Table 5.1: I_2 in various dimensions.

is the bulk - boundary correlator for a scalar field with dimension $\Delta = d - 1$.

We are also interested in $n = 2$. In any given dimension I_2 can be reduced to elementary functions, see for example Table 5.1, however the expressions become unwieldy as d increases. For our purposes a key observation is that I_2 is singular on the boundary lightcone, with

$$I_2 \sim \frac{\pi^{(d+1)/2}}{2^{d-4}\Gamma((d-1)/2)}\frac{z}{x^{d-2}} \qquad \text{as } x \to 0$$

I_2 is also singular on the bulk lightcone, at $x^2 + z^2 = 0$.

Bulk – boundary correlators follow from (5.28) and (5.30); the identity (5.27) is useful for taking derivatives of I_2. For example in AdS_4 we find

$$\begin{aligned}\langle\, A_\mu(t,\mathbf{x},z)\, j_\nu(0)\,\rangle &= \eta_{\mu\nu}\left[\frac{z(3x^2+z^2)}{4x^2(x^2+z^2)^2}-\frac{i}{8x^3}\log\frac{x+iz}{x-iz}\right]\\ &\quad -x_\mu x_\nu\left[\frac{z(5x^2+3z^2)}{4x^4(x^2+z^2)^2}-\frac{3i}{8x^5}\log\frac{x+iz}{x-iz}\right]\end{aligned}$$

while in AdS_5 we have

$$\langle\, A_\mu(t,\mathbf{x},z)\, j_\nu(0)\,\rangle = \eta_{\mu\nu}\,\frac{z^2(6x^4+3x^2z^2+z^4)}{6x^4(x^2+z^2)^3} - x_\mu x_\nu\,\frac{2z^2(3x^4+3x^2z^2+z^4)}{3x^6(x^2+z^2)^3}$$

Explicit expressions in higher dimensions become rather unwieldy. In general the $A-j$ correlators inherit the singularity structure of I_2: they are singular on the boundary lightcone $x^2=0$, as well as on the bulk lightcone $x^2+z^2=0$. Correlators involving field strengths are both simpler and better behaved. In any dimension we find

$$\begin{aligned}\langle\, F_{\lambda\mu}(t,\mathbf{x},z)\, j_\nu(0)\,\rangle &= -\frac{2(d-2)z^{d-2}}{(x^2+z^2)^d}\,\left(x_\lambda\eta_{\mu\nu}-x_\mu\eta_{\lambda\nu}\right)\\ \langle\, F_{z\mu}(t,\mathbf{x},z)\, j_\nu(0)\,\rangle &= \frac{(d-2)z^{d-3}}{(x^2+z^2)^d}\left(\eta_{\mu\nu}(x^2-z^2)-2x_\mu x_\nu\right)\end{aligned} \tag{5.32}$$

Note that $F-j$ correlators are only singular on the bulk lightcone.

Finally we can use these results to discuss bulk locality and causality. The expectation value of a commutator $\langle\,[A_\mu(t,\mathbf{x},z),\, j_\nu(0)]\,\rangle$ is given by the difference in the prescriptions $t\to t-i\epsilon$ and $t\to t+i\epsilon$. It follows that the commutator of a bulk gauge field with a boundary current is non-zero at lightlike separation on the boundary. Lightlike separation on the boundary implies spacelike separation in the bulk, so we appear to have non-local or acausal correlators. Of course there is no real violation of causality here, since $A-j$ correlators are gauge dependent. For Maxwell fields we can test causality by looking at gauge-invariant quantities, and indeed field strengths have causal correlators: they commute with the boundary currents at bulk spacelike separation.

5.3 Massive vector fields

We start with the Lagrangian for a massive vector field in Lorentzian AdS_{d+1}.

$$S = \int dz d^d x \sqrt{-G} \big[-\frac{1}{4} F^{MN} F_{MN} - \frac{1}{2} m^2 A_M A^M \big] \tag{5.33}$$

The equations of motion $\nabla_M F^{MN} - m^2 A^N = 0$ imply

$$\nabla_M A^M = 0\,. \tag{5.34}$$

Decomposing $A_M = (A_z, A_\mu)$, the equations of motion for A_z are

$$[\partial_z^2 + \partial_\mu \partial^\mu - \frac{1}{z}(d-1)\partial_z - \frac{m^2 - d + 1}{z^2}] A_z = 0 \tag{5.35}$$

This is identical to the equation of motion for a scalar field with $(\text{mass})^2 = m^2 - d + 1$. For the other components one has (defining $\phi_\mu = z A_\mu$)

$$[\partial_z^2 + \partial_\nu \partial^\nu - \frac{1}{z}(d-1)\partial_z - \frac{m^2 - d + 1}{z^2}] \phi_\mu = 2\partial_\mu A_z \tag{5.36}$$

Let

$$\Delta = \frac{d}{2} + \sqrt{\frac{(d-2)^2}{4} + m^2} \tag{5.37}$$

and define the boundary value of A_z by

$$A_z \sim z^\Delta A_z^0 \qquad \text{as } z \to 0$$

The equation of motion for A_z can be solved in the same way as for a scalar field (see appendix A.4)

$$A_z(t, \mathbf{x}, z) = \int_{t'^2 + \mathbf{y}'^2 < z^2} dt' d\mathbf{y}' \left(\frac{z^2 - t'^2 - \mathbf{y}'^2}{z} \right)^{\Delta - d} A_z^0(t + t', \mathbf{x} + i\mathbf{y}') \tag{5.38}$$

What is the boundary value of A_z^0 in terms of CFT data? Since $\phi_\mu(z \to 0) \sim z^\Delta$ then $A_\mu \sim z^{\Delta-1} j_\mu$, and inserting this in (5.34) gives

$$A_z^0 = \frac{1}{d - \Delta - 1} \, \partial_\mu j^\mu \tag{5.39}$$

So A_z^0 is sourced by the divergence of the boundary current.

Now let's solve (5.36). First note that a solution to the homogeneous equation (5.35) can be expanded in modes as

$$A_z = \int_{|\omega|>|k|} d\omega d^{d-1}k \, a^z_{\omega k} e^{-i\omega t + i\mathbf{k}\mathbf{x}} z^{d/2} J_\nu(z\sqrt{\omega^2 - \mathbf{k}^2}) \tag{5.40}$$

where $\nu = \Delta - d/2$ and $J_\nu(y)$ is a Bessel function. A similar solution would hold for (5.36) if the right hand side was zero. The complete solution to (5.36) can then be written in the form [94]

$$\begin{aligned} \phi_\mu(t, \mathbf{x}, z) \;&=\; \int_{|\omega|>|k|} d\omega d^{d-1}k \, z^{d/2} e^{-i\omega t + i\mathbf{k}\mathbf{x}} \\ &\times \;\; [a^\mu_{\omega k} J_\nu(z\sqrt{\omega^2 - \mathbf{k}^2}) + a^z_{\omega k} \frac{izk_\mu}{\sqrt{\omega^2 - \mathbf{k}^2}} J_{\nu+1}(z\sqrt{\omega^2 - \mathbf{k}^2})] \end{aligned} \tag{5.41}$$

Now from the boundary behavior of A_z one has

$$a^z_{\omega k} = \frac{2^\nu \Gamma(\nu+1)}{(2\pi)^d (\omega^2 - \mathbf{k}^2)^{\nu/2}} \int dt' d^{d-1}x' e^{i\omega t' - i\mathbf{k}\mathbf{x}'} A_z^0(t', \mathbf{x}') \tag{5.42}$$

and since the term proportional to $a^z_{\omega k}$ in (5.41) is subleading as $z \to 0$ one also has

$$a^\mu_{\omega k} = \frac{2^\nu \Gamma(\nu+1)}{(2\pi)^d (\omega^2 - \mathbf{k}^2)^{\nu/2}} \int dt' d^{d-1}x' e^{i\omega t' - i\mathbf{k}\mathbf{x}'} z j_\mu(t', \mathbf{x}') \tag{5.43}$$

By inserting the expressions for $a^\mu_{\omega k}$ and $a^z_{\omega k}$ into (5.41) one gets an expression for the bulk field in terms of boundary data. The first term looks just like the smearing function for a scalar field of dimension Δ, while the second term (aside from a factor $\frac{izk_\mu}{2(\nu+1)}$) is just the smearing function for a scalar field of dimension $\Delta + 1$ [84]. As a result we get the following expression

$$\phi_\mu(t, \mathbf{x}, z) = \int K_\Delta(x, x') j_\mu(x') + \frac{z}{2(\nu+1)} \int K_{\Delta+1}(x, x') \, \partial_\mu A_z^0(x') \tag{5.44}$$

More explicitly

$$zA_\mu(t,\mathbf{x},z) = \frac{\Gamma(\Delta - d/2+1)}{\pi^{d/2}\Gamma(\Delta-d+1)}\int_{t'^2+\mathbf{y}'^2<z^2} dt' d^{d-1}y' (\frac{z^2-t'^2-\mathbf{y}'^2}{z})^{\Delta-d} A^0_\mu(t+t', x+i\mathbf{y}')$$
$$+\frac{z\Gamma(\Delta - d/2+1)}{2\pi^{d/2}\Gamma(\Delta-d+2)}\int_{t'^2+\mathbf{y}'^2<z^2} dt' d^{d-1}y' (\frac{z^2-t'^2-\mathbf{y}'^2}{z})^{\Delta-d+1} \partial_\mu A^0_z(t+t', x+i\mathbf{y}')$$

5.3.1 Two-point functions and bulk causality

In this section we compute the two point function of a massive vector. The CFT two point function for a spin-1 field is

$$< j_\mu(x) j_\nu(0) >= (\eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2})\frac{1}{(x^2)^\Delta} \tag{5.45}$$

It can also be written in the form

$$< j_\mu(x) j_\nu(0) >= \frac{\Delta-1}{\Delta}\eta_{\mu\nu}\frac{1}{(x^2)^\Delta} - \frac{1}{2\Delta(\Delta-1)}\partial_\mu\partial_\nu\frac{1}{(x^2)^{\Delta-1}} \tag{5.46}$$

Since our expression for the bulk operator involves the divergence of the current we will also need

$$< \partial_\mu j^\mu(x) j_\nu(0) >= \frac{d-\Delta-1}{\Delta}\partial_\nu\frac{1}{(x^2)^\Delta} \tag{5.47}$$

The correlator of a bulk field A_z with a boundary current j_ν is easy to read off from the smearing function for A_z, which as we showed is just the smearing function of a scalar field of dimension Δ. Since $A_z(x) = \frac{1}{d-\Delta-1}\partial_\mu j^\mu(x)$ we have

$$< A_z(z,x) j_\nu(0) >= \frac{1}{\Delta}\partial_\nu(\frac{z}{x^2+z^2})^\Delta \tag{5.48}$$

This two-point function respects bulk causality.

For the other components of the bulk field we have

$$< zA_\mu(t,\mathbf{x},z) j_\nu(0) >= \int_{t'^2+|\mathbf{y}'|^2<z^2} dt' d^{d-1}y' \Big[\tag{5.49}$$

$$\frac{\Gamma(\Delta-d/2+1)}{\pi^{d/2}\Gamma(\Delta-d+1)} \left(\frac{z^2-t'^2-|\mathbf{y}'|^2}{z}\right)^{\Delta-d} < j_\mu(t+t',\mathbf{x}+i\mathbf{y}') j_\nu(0) >$$

$$+\frac{z\Gamma(\Delta-d/2+1)}{2\pi^{d/2}\Gamma(\Delta-d+2)} \left(\frac{z^2-t'^2-|\mathbf{y}'|^2}{z}\right)^{\Delta-d+1} \partial_\mu < A_z(t+t',\mathbf{x}+i\mathbf{y}') j_\nu(0) > \Big]$$

Using (5.46) and (5.48) we write this as

$$< zA_\mu(x,z) j_\nu(0) >= \frac{\Delta-1}{\Delta}\eta_{\mu\nu}\left(\frac{z}{z^2+x^2}\right)^{\Delta}$$

$$- \quad \frac{\Gamma(\Delta-d/2+1)}{2\Delta\pi^{d/2}\Gamma(\Delta-d+1)}\partial_\mu\partial_\nu\left(\frac{1}{\Delta-1}f_\Delta(z,x) - \frac{z}{\Delta-d+1}f_{\Delta+1}(z,x)\right)$$

where

$$f_\Delta(z,x) = \int_{t'^2+|y'|^2<z^2} dt' d^{d-1}y' \left(\frac{z^2-t'^2-|y'|^2}{z}\right)^{\Delta-d} \times$$

$$\left(\frac{1}{-(t+t')^2+(x_1+iy_1)^2+\cdots+(x_{d-1}+iy_{d-1})^2}\right)^{\Delta-1}$$

We set $t=0$, $x_1=x$, $x_2=\cdots=x_{d-1}=0$. We will compute f_Δ for this case then restore the dependence on the other coordinates using Lorentz invariance. Switching from (t',y') to spherical coordinates we get

$$f_\Delta = \mathrm{vol}(S^{d-2})\int_0^z dr r^{d-1}\left(\frac{z^2-r^2}{z^2}\right)^{\Delta-d}\int_0^\pi \frac{\sin^{d-2}\theta}{(x^2+2ixr\cos\theta-r^2)^{\Delta-1}} \tag{5.50}$$

We use the integrals

$$\int_0^\pi \frac{\sin^{2\mu-1}\theta}{(1+2a\cos\theta+a^2)^\nu} = \frac{\Gamma(\mu)\Gamma(\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})}F(\nu,\nu-\mu+\frac{1}{2},\mu+\frac{1}{2},a^2)$$

$$\int_0^1 (1-x)^{\mu-1}x^{\gamma-1}F(\alpha,\beta,\gamma,ax) = \frac{\Gamma(\mu)\Gamma(\gamma)}{\Gamma(\mu+\gamma)}F(\alpha,\beta,\gamma+\mu,a) \tag{5.51}$$

to find

$$f_\Delta = \frac{\pi^{d/2}\Gamma(\Delta-d+1)}{\Gamma(\Delta-\frac{d}{2}+1)}\frac{z^\Delta}{x^{2\Delta-2}}F(\Delta-1,\Delta-\frac{d}{2},\Delta-\frac{d}{2}+1,-\frac{z^2}{x^2}) \tag{5.52}$$

Then we use the identity

$$\gamma F(\alpha,\beta,\gamma,x) - \gamma F(\alpha,\beta+1,\gamma,x) + x\alpha F(\alpha+1,\beta+1,\gamma+1,x) \tag{5.53}$$

and restore Lorentz invariance to find

$$< zA_\mu(x,z)j_\nu(0) >= \frac{\Delta-1}{\Delta}\eta_{\mu\nu}\left(\frac{z}{x^2+z^2}\right)^{\Delta} - \frac{z^\Delta}{2\Delta(\Delta-1)}\partial_\mu\partial_\nu\left(\frac{1}{x^2+z^2}\right)^{\Delta-1} \tag{5.54}$$

Note that the final answer is only non-analytic on the bulk lightcone. This however was achieved by a cancellation of terms that are non-analytic on the boundary lightcone between f_Δ and $f_{\Delta+1}$. So the locality of a massive vector field in the bulk is made possible by the fact that the dual boundary current isn't conserved, which allowed us to cancel non-analytic terms in the correlator. This mechanism is not available for a gauge field since it is dual to a conserved current.

5.4 Conclusions

In this chapter we worked out the smearing functions which describe linearized spin-1 excitations in AdS. We showed that bulk locality is respected: although gauge fields have non-local commutators when one works in holographic gauge, the corresponding curvature – the field strength for A_μ – is causal. We also studied massive vector fields, where the vector field itself is causal due to the non-conserved nature of the dual boundary current.

These results could be extended in several directions. For example we computed the smearing function for a Chern-Simons gauge field in AdS_3. It would be interesting to work out the smearing function for a Maxwell field in AdS_3, dual to a CFT with a dynamical gauge field [92, 91] (see however [95]). This would allow one to study the Maxwell-Chern-Simons theory recently analyzed in [96]. Since the smearing functions are basically fixed by AdS covariance, in principle our results could be applied to a duality between AdS_2 and CFT_1, although the physical interpretation in this context is not clear.

Perhaps a more automatic direction is to extend our results to include interactions. For massive vector fields we showed how this works in section 6.3: in a $1/N$ expansion one adds appropriately smeared higher-dimension vector operators, with coefficients that are fixed by the requirement of bulk causality. It has already been extended to gauge fields [86] and very recently for gravitons [97]. We discuss their results briefly in the next chapter.

6

Holographic Representation of Bulk Fields: Case for Spin-2

6.1 Introduction

In this chapter we extend our program of last chapter for spin- 2 graviton fields. In section 6.2 we obtain results for gravity analogous to the Maxwell case. We first work out the smearing function for a graviton and show that the graviton has non-local correlators with a spacelike separated operator. We find out the 'gauge invariant' role in our context is played by Weyl tensor and as expected, it obeys bulk micro-causality.

In the second part of this chapter we discuss interactions and general backgrounds. In section 6.3 we take a step back to spin- 1 fields and show how to extend our construction for massive vector fields to include interactions, using perturbation theory in $1/N$, and we discuss the difficulty with gauge and gravity fields resulting from the existence of conserved charges. The detailed calculation appear in [86] and [97]. In section 6.4 we provide a framework for extending the construction to general backgrounds and for going beyond the approximation of having a fixed background. We also explain the necessary conditions for the existence of approximately local operators in the bulk. This last part is quite speculative and we comment on it briefly in the conclusions section 7.

6.2 Graviton smearing functions

We now turn our attention to constructing a smearing function that describes a fluctuation of the bulk metric. To this end we consider a linearized perturbation of the AdS metric,

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + g_{\mu\nu}dx^\mu dx^\nu\right) \qquad (6.1)$$
$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{z^2}{R^2}h_{\mu\nu}$$

Here we are working in "holographic gauge" (or Fefferman - Graham coordinates [98]) in which

$$g_{zz} = g_{z\mu} = 0$$

The source-free Einstein equations in this coordinate system can be found in [99].* Working to linear order in $h_{\mu\nu}$ the zz, the $z\nu$, and the trace of the $\mu\nu$ components of the Einstein equations read

$$zz: \quad \left(\partial_z^2 + \frac{3}{z}\partial_z\right)h = 0 \qquad (6.2)$$
$$z\nu: \quad \left(\partial_z + \frac{2}{z}\right)\left(\partial_\mu h^{\mu\nu} - \partial^\nu h\right) = 0 \qquad (6.3)$$
$$\text{trace}: \quad \left(\partial_z^2 - \frac{2d-5}{z}\partial_z - \frac{4(d-1)}{z^2}\right)h + 2\left(\partial_\mu\partial^\mu h - \partial_\mu\partial_\nu h^{\mu\nu}\right) = 0 \qquad (6.4)$$

Here $h \equiv h^\mu{}_\mu$. The only solution to this system of equations compatible with normalizeable behavior as $z \to 0$ is to set†

$$h = 0 \qquad \partial_\mu h^{\mu\nu} = 0 \qquad (6.5)$$

Thus $h_{\mu\nu}$ is traceless and conserved, which enables us to consistently identify its boundary behavior with the stress tensor of the CFT.

It only remains to solve the $\mu\nu$ components of the Einstein equations, which given (6.5) can be

*Ref. [99] uses $\rho = z^2/R^2$ as a radial coordinate.

†To see this note that (6.3) implies $\partial_\mu h^{\mu\nu} - \partial^\nu h \sim 1/z^2$. To avoid this non-normalizeable behavior we must set $\partial_\mu h^{\mu\nu} - \partial^\nu h = 0$. The divergence of this equation means the last term in (6.4) drops out. Then the difference of (6.2) and (6.4) gives $\left(\partial_z + \frac{2}{z}\right)h = 0$ which requires that we set $h = 0$.

simplified to

$$\left(\partial_\alpha \partial^\alpha + \partial_z^2 + \frac{5-d}{z}\partial_z - \frac{2(d-2)}{z^2}\right) h_{\mu\nu} = 0$$

Following the procedure that worked for Maxwell fields, we define $\phi_{\mu\nu} = z^2 h_{\mu\nu}$ and find that‡

$$\left(\partial_\alpha \partial^\alpha + z^{d-1}\partial_z \frac{1}{z^{d-1}}\partial_z\right) \phi_{\mu\nu} = 0$$

That is, each component of $\phi_{\mu\nu}$ obeys the massless scalar wave equation. A massless scalar is dual to an operator of dimension $\Delta = d$ in the CFT, and has the asymptotic fall-off

$$\phi_{\mu\nu}(x,z) \sim z^d T_{\mu\nu}(x) \qquad \text{as } z \to 0$$

We identify $T_{\mu\nu}$ with the stress tensor of the CFT. To reconstruct the bulk metric perturbation from the stress tensor we use the scalar smearing function (A.36) given in appendix A.4. Setting $\Delta = d$, this gives

$$z^2 h_{\mu\nu}(t,\mathbf{x},z) = \frac{1}{\mathrm{vol}(B^d)} \int_{t'^2+|\mathbf{y}'|^2<z^2} dt' d^{d-1}y'\, T_{\mu\nu}(t+t',\mathbf{x}+i\mathbf{y}') \qquad (6.6)$$

$$\text{volume of a unit } d\text{-ball} = \mathrm{vol}(B^d) = \frac{2\pi^{d/2}}{d\Gamma(d/2)}$$

Thus the bulk metric perturbation is obtained by smearing the stress tensor over a ball of radius z on the complexified boundary.

6.2.1 AdS covariance

It's instructive to check that the smearing function (6.6) respects AdS covariance. We will be somewhat brief, since the steps are very similar to those in section 5.2.1. Covariance under Poincaré transformations of x^μ is manifest. A dilation corresponds to the bulk isometry

$$x^\mu \to x'^\mu = \lambda x^\mu \qquad z \to z' = \lambda z\,.$$

‡This amounts to working in a vielbein basis, $h_{ab} = e_a{}^\mu e_b{}^\nu h_{\mu\nu}$ where $e_a{}^\mu = \frac{z}{R}\delta_a{}^\mu$.

Holographic gauge is preserved since $h'_{zz} = h'_{z\mu} = 0$, while the combination $z^2 h_{\mu\nu}$ which appears on the left hand side of (6.6) transforms like a scalar. This matches the behavior of the right hand side: the stress tensor has dimension d, while the measure $d^d x'$ has dimension $-d$.

Special conformal transformations are a little more involved. A special conformal transformation corresponds to an infinitesimal bulk isometry

$$\begin{aligned} x^\mu &\to x'^\mu = x^\mu + 2b\cdot x x^\mu - b^\mu(x^2+z^2) \\ z &\to z' = z + 2b\cdot x z \end{aligned}$$

Under this isometry

$$\begin{aligned} h'_{zz} &= 0 \\ h'_{z\mu} &= 2zb^\alpha h_{\alpha\mu} \\ h'_{\mu\nu} &= h_{\mu\nu} + 2b^\alpha(x_\mu h_{\alpha\nu} + x_\nu h_{\alpha\mu}) - 2x^\alpha(b_\mu h_{\alpha\nu} + b_\nu h_{\alpha\mu}) - 4b\cdot x h_{\mu\nu} \end{aligned} \tag{6.7}$$

Holographic gauge isn't preserved, so to restore it we make a compensating diffeomorphism $x^\mu \to x^\mu + \epsilon^\mu(x,z)$, under which

$$\begin{aligned} \delta h_{\mu\nu} &= \frac{R^2}{z^2}\left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) \\ \delta h_{z\mu} &= \frac{R^2}{z^2}\partial_z \epsilon_\mu \\ \delta h_{zz} &= 0 \end{aligned}$$

The appropriate diffeomorphism is

$$\epsilon^\mu = -\frac{1}{R^2 \mathrm{vol}(B^d)} \int d^d x'\, \theta(\sigma z')\sigma z z'\, 2b_\alpha T^{\alpha\mu} \tag{6.8}$$

for which

$$\begin{aligned} \delta h_{z\mu} &= -2zb^\alpha h_{\alpha\mu} \\ \delta h_{\mu\nu} &= -\frac{1}{z^2 \mathrm{vol}(B^d)} \int d^d x'\, \theta(\sigma z')\, 2b^\alpha (x-x')_\mu T_{\alpha\nu} + (\mu \leftrightarrow \nu) \end{aligned} \tag{6.9}$$

This restores holographic gauge. Combining (6.7) and (6.9) we find

$$\left(z^2 h_{\mu\nu}\right)' = z^2 h_{\mu\nu} + \frac{1}{\mathrm{vol}(B^d)} \int d^d x' \, \theta(\sigma z') \, \left[2b^\alpha x'_\mu T_{\alpha\nu} - 2x^\alpha b_\mu T_{\alpha\nu} + (\mu \leftrightarrow \nu)\right] \tag{6.10}$$

Current conservation in the form $\int d^d x' \, \theta(\sigma z') \sigma z z' \, \partial_\mu T^{\mu\nu} = 0$ implies

$$\int d^d x' \, \theta(\sigma z') \, (x - x')^\mu T_{\mu\nu} = 0$$

This means we can replace x^α with x'^α in (6.10), to obtain the transformation of the left hand side of (6.6). The result exactly matches the transformation of the right hand side, since under a special conformal transformation

$$T_{\mu\nu} \to T'_{\mu\nu} = T_{\mu\nu} + 2b^\alpha (x_\mu T_{\alpha\nu} + x_\nu T_{\alpha\mu}) - 2x^\alpha (b_\mu T_{\alpha\nu} + b_\nu T_{\alpha\mu}) - 2d b \cdot x T_{\mu\nu}$$

The last term cancels the transformation of the measure $d^d x' \, \theta(\sigma z')$.

6.2.2 Two-point functions and bulk causality for gravity

We now use the smearing functions we have constructed to compute 2-point functions for the graviton. We consider gravity in AdS_3 in section 6.2.2.1, and gravity in AdS_4 and higher in section 6.2.2.2.

6.2.2.1 Gravity in AdS_3

AdS_3 is special because there is no propagating graviton [100]. Rather the bulk curvature is completely determined by the vacuum Einstein equations

$$R_{MN} = \frac{\Lambda}{d-1} G_{MN} \tag{6.11}$$

where the cosmological constant $\Lambda = -d(d-1)/R^2$. This uniquely fixes the geometry. So in AdS_3 we expect the smearing function to generate a metric perturbation which corresponds to an infinitesimal (but non-normalizeable) diffeomorphism of the background AdS metric.

We work in light-front coordinates $x^{\pm} = t \pm x$ and write the perturbed AdS metric as

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 - dx^+ dx^-\right) + h_{\mu\nu} dx^\mu dx^\nu \tag{6.12}$$

From the smearing function (6.6) we have for instance

$$z^2 h_{--} = \frac{1}{\pi} \int\limits_{t'^2 + y'^2 < z^2} dt' dy' \, T_{--}(t + t', x + iy') \tag{6.13}$$

Since T_{--} only depends on x^- this becomes ($t' = r \sin\theta$, $y' = r \cos\theta$)

$$z^2 h_{--} = \frac{1}{\pi} \int_0^z r dr \int_0^{2\pi} d\theta \, T_{--}\left(x^- - ire^{i\theta}\right) \tag{6.14}$$

Defining $\xi = e^{i\theta}$ the contour integral picks up the pole at $\xi = 0$ and ends up giving $h_{--} = T_{--}$. So at the linearized level a stress tensor in the CFT corresponds to a bulk metric perturbation

$$\begin{aligned} h_{--} &= T_{--}(x^-) \\ h_{++} &= T_{++}(x^+) \\ h_{+-} &= 0 \end{aligned} \tag{6.15}$$

This provides a remarkably simple example of holography: the boundary stress tensor is lifted to be z-independent and re-interpreted as a metric perturbation in the bulk. Not surprisingly, this is very reminiscent of the Chern-Simons correspondence (5.23).

We can use this to compute the bulk 2-point function for the graviton. For instance the CFT 2-point function

$$\langle T_{--}(x^-) \, T_{--}(x'^-) \rangle = \frac{c}{8\pi^2} \frac{1}{(x^- - x'^- - i\epsilon)^4} \tag{6.16}$$

lifts to a bulk correlator

$$\langle h_{--}(x^+, x^-, z) \, h_{--}(x'^+, x'^-, z') \rangle = \frac{c}{8\pi^2} \frac{1}{(x^- - x'^- - i\epsilon)^4}$$

Here we have used a Wightman $i\epsilon$ prescription and c is the central charge of the CFT.

To study bulk locality and causality in this framework, note that the CFT correlator (6.16) corresponds to a Virasoro algebra

$$i[T_{--}(x^-), T_{--}(x'^-)] = \frac{c}{24\pi}\delta'''(x^- - x'^-)$$

This lifts to the bulk commutator

$$i[h_{--}(x^+, x^-, z), h_{--}(x'^+, x'^-, z')] = \frac{c}{24\pi}\delta'''(x^- - x'^-)$$

Metric perturbations in the bulk have non-local commutators; this behavior is acceptable since metric perturbations are coordinate dependent. One might ask if there is a quantity – analogous to the field strength for a gauge field – which obeys causal commutation relations. In the next section we will claim that, for gravity, such a quantity is provided by the Weyl tensor. This claim becomes vacuous in three dimensions since the Weyl tensor vanishes identically.

We began this section by recalling that the source-free Einstein equations fix the bulk geometry to be pure AdS. So to complete the story, one might ask for a coordinate transformation which brings the perturbed metric (6.12), (6.15) back to the canonical form $ds^2 = \frac{R^2}{z^2}\big(dz^2 - dx^+dx^-\big)$. The required transformation is

$$\begin{aligned}
\delta x^+ &= -\frac{2}{R^2}\frac{1}{\partial_+^3}T_{++} - \frac{z^2}{R^2}\frac{1}{\partial_-}T_{--} \\
\delta x^- &= -\frac{2}{R^2}\frac{1}{\partial_-^3}T_{--} - \frac{z^2}{R^2}\frac{1}{\partial_+}T_{++} \\
\delta z &= -\frac{z}{R^2}\left(\frac{1}{\partial_+^2}T_{++} + \frac{1}{\partial_-^2}T_{--}\right)
\end{aligned} \tag{6.17}$$

Note that the transformation does not vanish at the boundary, so it does not correspond to a (normalizeable) gauge symmetry of the bulk theory.

6.2.2.2 Gravity in AdS_4 and higher

Our starting point for gravity in AdS_4 and higher is the 2-point function of the stress tensor in a general CFT. Up to an overall coefficient proportional to the central charge, this has the form[§]

$$\langle T_{\mu\nu}(x)T_{\alpha\beta}(0)\rangle = X_{\mu\nu\alpha\beta}\frac{1}{(x^2)^d} + Y_{\mu\nu\alpha\beta}\frac{1}{(x^2)^{d-1}} + Z_{\mu\nu\alpha\beta}\frac{1}{(x^2)^{d-2}} \tag{6.18}$$

where we've introduced

$$\begin{aligned} X_{\mu\nu\alpha\beta} &= -2d\,\eta_{\mu\nu}\eta_{\alpha\beta} + d(d-1)\big(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}\big) \\ Y_{\mu\nu\alpha\beta} &= \frac{1}{d-1}\big(\eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu\big) - \frac{1}{2}\big(\eta_{\mu\alpha}\partial_\nu\partial_\beta + \eta_{\mu\beta}\partial_\nu\partial_\alpha + \eta_{\nu\alpha}\partial_\mu\partial_\beta + \eta_{\nu\beta}\partial_\mu\partial_\alpha\big) \\ Z_{\mu\nu\alpha\beta} &= \frac{1}{2(d-1)(d-2)}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta \end{aligned} \tag{6.19}$$

Up to an overall normalization this correlator is uniquely determined by requiring that the stress tensor be traceless and conserved with the correct scaling dimension. Applying the smearing function (6.6) gives the bulk – boundary correlator

$$z^2\langle h_{\mu\nu}(t,\mathbf{x},z)T_{\alpha\beta}(0)\rangle = X_{\mu\nu\alpha\beta}J_0 + Y_{\mu\nu\alpha\beta}J_1 + Z_{\mu\nu\alpha\beta}J_2 \tag{6.20}$$

where

$$J_n = \frac{1}{\mathrm{vol}(B^d)}\int_{t'^2+|\mathbf{y}'|^2<z^2} dt' d^{d-1}y' \,\frac{1}{\big(-(t+t')^2 + |\mathbf{x}+i\mathbf{y}'|^2\big)^{d-n}} \tag{6.21}$$

Note that J_n is related to the integral (5.29) we encountered for gauge fields.

$$\frac{d}{dz}J_n = \frac{1}{\mathrm{vol}(B^d)}I_n$$

This can be integrated using (5.30) to give

$$J_n = \frac{z^d}{(x^2)^{d-n}}F\Big(d-n, \frac{d}{2}-n+1, \frac{d}{2}+1, -\frac{z^2}{x^2}\Big) \tag{6.22}$$

[§]See for example (2.37) and (A5) in Ref. [101].

d	J_1	J_2
3	$-\frac{3i}{4x}\log\frac{x+iz}{x-iz}-\frac{3z}{2(x^2+z^2)}$	$-\frac{3i(x^2+z^2)}{4x}\log\frac{x+iz}{x-iz}-\frac{3z}{2}$
4	$\frac{z^4}{x^2(x^2+z^2)^2}$	$-2\log\frac{x^2+z^2}{x^2}+\frac{2z^2}{x^2}$
5	$-\frac{5i}{32x^3}\log\frac{x+iz}{x-iz}-\frac{5z(3x^4+8x^2z^2-3z^4)}{48x^2(x^2+z^2)^3}$	$\frac{15i}{32x^3}(3x^2-z^2)\log\frac{x+iz}{x-iz}+\frac{15z(3x^2+z^2)}{16x^2(x^2+z^2)}$
6	$\frac{z^6(4x^2+z^2)}{4x^4(x^2+z^2)^4}$	$\frac{z^6}{x^4(x^2+z^2)^2}$

Table 6.1: J_1 and J_2 in low dimensions.

In general J_n has singularities on both the boundary lightcone (where $x^2 = 0$) and the bulk lightcone (where $x^2 + z^2 = 0$). The case $n = 0$ is an exception to this general rule, since

$$J_0 = \frac{z^d}{(x^2+z^2)^d}$$

J_0 is only singular on the bulk lightcone, and in fact has an AdS-covariant form $J_0 \sim 1/(\sigma z')^d$. This was to be expected since, up to an overall normalization, J_0 is the bulk – boundary correlator for a massless scalar field. Some other cases of interest can be found in table 6.1.

At this stage we have an expression for the $h - T$ correlator in terms of differential operators acting on J_n's. We will stop here, since explicitly evaluating the derivatives in (6.20) leads to lengthy expressions. But one important observation we can make is that the $h - T$ correlator inherits the singularity structure of J_1 and J_2: it has singularities on both the bulk and boundary lightcones. This means the commutator $[h_{\mu\nu}(t, \mathbf{x}, z), T_{\alpha\beta}(0)]$ will be non-zero at lightlike separation on the boundary (where $x^2 = 0$), even though this corresponds to spacelike separation in the bulk (since $x^2+z^2 > 0$). This shows that in holographic gauge metric perturbations have acausal commutators. This is acceptable because the commutator is gauge dependent.

This raises an interesting question, whether there is a quantity one can define in linearized gravity which obeys causal commutation relations. That is, whether there is something analogous to the Maxwell field strength $F_{\mu\nu}$, which as we saw in (5.32) has correlators that are only singular on the bulk lightcone. At first one might think the gravitational analog is provided by the Riemann tensor. However this can't be right: perturbing the source-free Einstein equations (6.11) shows that

$\delta R_{\mu\nu} = -\frac{d}{R^2} h_{\mu\nu}$. Since we've already shown that the metric perturbation has acausal commutators, the same must be true for the Ricci tensor.

This suggests that we split off the Ricci part of the curvature and work with the Weyl tensor. In fact the Weyl tensor commutes with the boundary stress tensor at bulk spacelike separation. We will show this in two ways: first by an intuitive argument, then by an explicit calculation in holographic gauge.

The intuitive argument runs as follows. Imagine quantizing the bulk theory perturbatively using a covariant gauge condition. Then locality would be manifest, and all fields (including the metric perturbation) would obey canonical local commutation relations. It follows that in covariant gauge the Weyl tensor commutes with the boundary stress tensor at spacelike separation. But since the Weyl tensor transforms homogeneously under changes of coordinates, if the commutator vanishes in covariant gauge it should also vanish in holographic gauge.[¶] This fits with the fact that, at the linearized level, the Weyl tensor is gauge invariant around an AdS background.

The explicit calculation proceeds as follows. Linearizing around an AdS background the non-trivial components of the Weyl tensor are

$$\begin{aligned} z^2 C_{\alpha\beta\gamma\delta} &= \frac{1}{2}\big(\partial_\alpha\partial_\gamma\phi_{\beta\delta} - \partial_\alpha\partial_\delta\phi_{\beta\gamma} - \partial_\beta\partial_\gamma\phi_{\alpha\delta} + \partial_\beta\partial_\delta\phi_{\alpha\gamma}\big) \\ &\quad -\frac{1}{2z}\partial_z\big(\eta_{\alpha\gamma}\phi_{\beta\delta} - \eta_{\alpha\delta}\phi_{\beta\gamma} - \eta_{\beta\gamma}\phi_{\alpha\delta} + \eta_{\beta\delta}\phi_{\alpha\gamma}\big) \\ z^2 C_{z\beta\gamma\delta} &= \frac{1}{2}\partial_z\big(\partial_\gamma\phi_{\beta\delta} - \partial_\delta\phi_{\beta\gamma}\big) \end{aligned} \tag{6.23}$$

Here $\phi_{\alpha\beta} = z^2 h_{\alpha\beta}$, and we have used the fact that $\phi_{\alpha\beta}$ obeys the massless scalar wave equation $\big(\partial_\alpha\partial^\alpha + \partial_z^2\big)\phi_{\mu\nu} = \frac{d-1}{z}\partial_z\phi_{\mu\nu}$. The remaining components of the Weyl tensor $C_{z\beta z\delta}$ are not independent by the trace-free condition.

In principle it is straightforward to compute $C-T$ correlators. Consider for example $z^2\langle C_{z\beta\gamma\delta}(x)T_{\rho\sigma}(0)\rangle$. Using the $\phi - T$ correlator (6.20) and the operators (6.19) one obtains a rather long expression. However many terms drop out when you antisymmetrize on γ and δ. What survives has the form

[¶] This argument breaks down for the Riemann tensor. In an AdS background the Riemann tensor acquires a vev, and a perturbation $\delta R_{\alpha\beta\gamma\delta}$ transforms inhomogeneously under changes of coordinates.

('stuff' meaning metrics and derivatives tangent to the boundary)

$$
\begin{aligned}
z^2\langle C_{z\beta\gamma\delta}(x)T_{\rho\sigma}(0)\rangle &= \partial_z \int_{x'^2<z^2} d^dx' \left\{(\text{stuff})\cdot\frac{1}{(x^2)^d} + (\text{stuff})\cdot\frac{1}{(x^2)^{d-1}}\right\} \\
&= \int_{x'^2=z^2} d^dx' \left\{(\text{stuff})\cdot\frac{1}{(x^2)^d} + (\text{stuff})\cdot\frac{1}{(x^2)^{d-1}}\right\} \\
&= (\text{stuff})\cdot I_0 + (\text{stuff})\cdot I_1
\end{aligned}
\tag{6.24}
$$

As we saw in (5.31) I_1 is analytic on the boundary lightcone. It turns out that I_0 is also analytic at $x^2 = 0$:

$$
I_0 = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,\frac{z^{d-1}(x^2 - z^2)}{(x^2 + z^2)^{d+1}}
\tag{6.25}
$$

So the correlator (6.24) is analytic at $x^2 = 0$, and $C_{z\beta\gamma\delta}$ obeys causal commutation relations with the boundary stress tensor.

Now consider $z^2\langle C_{\alpha\beta\gamma\delta}(x)T_{\rho\sigma}(0)\rangle$. Again one obtains a rather long expression. However many terms drop out when you antisymmetrize on α and β, or on γ and δ. Also many terms involve either J_0, I_0 or I_1 which we know are analytic at $x^2 = 0$. Dropping all such contributions, up to an overall coefficient we find that only two terms survive:

$$
\begin{aligned}
& z^2\langle C_{\alpha\beta\gamma\delta}(x)T_{\rho\sigma}(0)\rangle \\
\sim\; & \partial_{[\alpha}\eta_{\beta][\gamma}\partial_{\delta]}\partial_\rho\partial_\sigma\Big(\int_{x'^2<z^2} d^dx'\,\frac{1}{(x^2)^{d-1}} - \frac{1}{2(d-2)z}\int_{x'^2=z^2} d^dx'\,\frac{1}{(x^2)^{d-2}}\Big) \\
=\; & \partial_{[\alpha}\eta_{\beta][\gamma}\partial_{\delta]}\partial_\rho\partial_\sigma\Big(\mathrm{vol}(B^d)J_1 - \frac{1}{2(d-2)z}I_2\Big)
\end{aligned}
\tag{6.26}
$$

With the help of one of Gauss' recursion relations for hypergeometric functions one can show that the quantity in parenthesis is

$$
-\frac{\pi^{d/2}}{(d-2)\Gamma(d/2)}\,\frac{z^{d-2}}{(x^2+z^2)^{d-2}}
$$

This is analytic on the boundary lightcone, so $C_{\alpha\beta\gamma\delta}$ obeys causal commutation relations with the boundary stress tensor.

6.3 Interactions

In this section we make some remarks on constructing bulk operators at higher orders in $1/N$. For scalar fields it was shown in [85] that one can construct interacting local bulk fields without any knowledge of the bulk Lagrangian. Rather, by adopting bulk micro-causality as a guiding principle, one can construct the appropriate bulk operators just from knowing CFT correlators. Here we show that something similar can be done for a massive vector field in AdS_3: a local bulk operator can be constructed, even in the presence of interactions. However for a gauge field in AdS_3 we show that the analogous procedure breaks down. In this section, to avoid notational complexity, we denote

$$w = x^+ = t + x$$
$$\bar{w} = x^- = t - x$$

Up to an overall coefficient, the three point function of three primary operators in a two dimensional CFT is

$$\begin{aligned} &< \mathcal{O}_{1,h_1,\bar{h}_1}(w_1,\bar{w}_1)\mathcal{O}_{2,h_2,\bar{h}_2}(w_2,\bar{w}_2)\mathcal{O}_{3,h_3,\bar{h}_3}(w_3,\bar{w}_3) > \\ =\ &\frac{1}{w_{12}^{h_1+h_2-h_3} w_{23}^{h_2+h_3-h_1} w_{13}^{h_3+h_1-h_2}} \frac{1}{\bar{w}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{w}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{w}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}} \end{aligned} \tag{6.27}$$

Here $w_{ij} = w_i - w_j$. Let us for simplicity assume that $\mathcal{O}_2$ and $\mathcal{O}_3$ are scalar operators so $h_2 = \bar{h}_2$ and $h_3 = \bar{h}_3$, but $\mathcal{O}_1$ has spin 1 with $h_1 = \bar{h}_1 + 1$. To explore bulk locality we smear $\mathcal{O}_2$ into a bulk operator using the free field smearing function

$$\mathcal{O}_2(z, w_2, \bar{w}_2) = \int_0^z r dr (\frac{z^2 - r^2}{z})^{2h-2} \int_{|\alpha|=1} \frac{d\alpha}{i\alpha} \mathcal{O}(w_2 + r\alpha, \bar{w}_2 - r\alpha^{-1}). \tag{6.28}$$

We can get the CFT three point function with $h_1 \to h_1 + 1$ (as long as $h_1 \neq 0$) by acting on a three point correlator with the operator

$$\frac{1}{h_3 - h_2 - h_1} \frac{\partial}{\partial w_{12}} - \frac{1}{h_2 - h_3 - h_1} \frac{\partial}{\partial w_{13}} \tag{6.29}$$

So the result for $h_1 = \bar{h}_1 + 1$ can be gotten from the result for $h_1 = \bar{h}_1$ by acting with the operator

(6.29). The situation with $h_1 = \bar{h}_1$ was analyzed in [85]. It was found that for scalar operators one can add a series of appropriately smeared higher dimension scalar operators that will cancel the causality-violating terms in the three point function. Here we see that this is still true if one of the boundary operator has spin. Note however that for the special case of conserved current (meaning $h = 0$, $\bar{h} = 1$ or $h = 1$, $\bar{h} = 0$) this argument does not apply. This is not only because acting with the operator (6.29) is not possible, but also because if $\mathcal{O}_1$ is a conserved current then Ward identities restrict its three point function. For instance for a conserved current the three point function will vanish unless the two point function $< \mathcal{O}_2\mathcal{O}_3 >$ is non-zero. So for a conserved current adding smeared higher dimension primaries is not in general possible.

We now consider the case where $\mathcal{O}_1$ is smeared into the bulk. We'll work in terms of the OPE, similarly to what was done in [85]. For simplicity we denote $h_1 = n$, $\bar{h}_1 = n - 1$ and assume that $h_2 = \bar{h}_2 = 1$. We look at terms in the OPE proportional to the scalar operator

$$\begin{aligned} j^{n,n-1}(w,\bar{w})\mathcal{O}^{1,1}(0) &= \frac{\mathcal{O}^{1,1}(0)}{w^n \bar{w}^{n-1}} + \cdots \\ j^{n-1,n}(w,\bar{w})\mathcal{O}^{1,1}(0) &= \frac{\mathcal{O}^{1,1}(0)}{w^{n-1} \bar{w}^{n}} + \cdots \end{aligned} \tag{6.30}$$

When $n = 1$ the smearing function (5.23) for a massless gauge field in AdS_3 gives

$$A^{1,0}(z,w,\bar{w})\mathcal{O}^{1,1}(0) = \frac{1}{w}\mathcal{O}^{1,1}(0) + \cdots \tag{6.31}$$

On the other hand for a massive vector the smearing function (5.44) leads to

$$A^{n,n-1}(z,w,\bar{w})\mathcal{O}^{1,1}(0) = \left(-\frac{2}{\pi}\frac{d}{dw} I_1^{(n-1)} + \frac{z}{\pi}\frac{d}{dw} I_2^{(n)} \right) \mathcal{O}^{1,1}(0) + \cdots \tag{6.32}$$

where

$$\begin{aligned} I_1^{(n-1)} &= \int_0^z r dr \left(\frac{z^2 - r^2}{z} \right)^{2n-3} \int_{|\alpha|=1} \frac{d\alpha}{\alpha (w + r\alpha)^{n-1} (\bar{w} - r/\alpha)^{n-1}} \\ I_2^{(n)} &= \int_0^z r dr \left(\frac{z^2 - r^2}{z} \right)^{2n-2} \int_{|\alpha|=1} \frac{d\alpha}{\alpha (w + r\alpha)^{n} (\bar{w} - r/\alpha)^{n}} \end{aligned} \tag{6.33}$$

Using (5.51) one gets

$$\begin{aligned} I_1^{(n-1)} &= \frac{\pi z^{2n-1}}{(2n-2)(w\bar{w})^{n-1}} F(n-1, n-1, 2n-1, -\frac{z^2}{w\bar{w}}) \\ I_1^{(n-1)} &= \frac{\pi z^{2n}}{(2n-1)(w\bar{w})^{n}} F(n, n, 2n, -\frac{z^2}{w\bar{w}}) \end{aligned} \tag{6.34}$$

and finally using (5.53) one gets

$$A^{n,n-1}(z, w, \bar{w}) \mathcal{O}^{1,1}(0) = -\mathcal{O}^{1,1}(0) \frac{d}{dw} \left(\frac{z^{2n-1}}{(n-1)(w\bar{w})^{n-1}} F(n-1, n, 2n-1, -\frac{z^2}{w\bar{w}}) \right) \tag{6.35}$$

A similar result holds for $A^{n-1,n}$ by replacing $w \to \bar{w}$. The quantity in parenthesis in (6.35) is non-analytic due to terms of the form

$$\left(\frac{w\bar{w}}{z^2}\right)^m \ln \frac{z^2 + w\bar{w}}{w\bar{w}} \tag{6.36}$$

with $n \geq m \geq 1$.

Suppose we have a massless gauge field in the bulk. The singular term in (6.31) leads to a non-vanishing commutator at bulk spacelike separation, and must be canceled if the gauge field is to commute at spacelike separation. But given the structure (6.36) there is no massive vector we can add to our definition of a bulk gauge field that will cancel the divergent term in (6.31). This means that it is not possible to promote a boundary conserved current to a local bulk field.‖

On the other hand, starting from a non-conserved current in the CFT, there is no obstacle to restoring bulk locality. One can cancel non-analytic terms of the form (6.36) by adding a tower of higher-dimension spin-1 fields with appropriately chosen masses and coefficients to our definition of a bulk vector field. This will leave a non-analytic term of the form

$$(\frac{w\bar{w}}{z^2})^{n_{\max}} \ln(w\bar{w}) \tag{6.37}$$

where $n_{\max}$ the largest n used in the sum over higher dimension primaries. So, just as in the scalar

‖The lesson here is not that causality is violated. For example in AdS_3 the field strength associated with (5.23) vanishes identically, and in this sense micro-causality is trivially satisfied even in the presence of interactions. Rather the lesson is that there is an obstacle to constructing bulk gauge fields which have local commutators. This is a feature, not a bug, since as we discuss in section 6.3.1 gauge fields are expected to have non-local commutators.

case [85], we can make a massive vector field in the bulk as local as we wish. For spin- 1 Maxwell fields, one finds the same conclusion which has been sketched out in [86] .

6.3.1 A comment on gauge fields

If there is a gauge symmetry in the bulk, i.e. a conserved current on the boundary, the issue of constructing bulk operators become a bit more involved. Of course one could start from the bulk equations of motion and solve them perturbatively, to express bulk fields in terms of boundary data. If one starts from a local bulk Lagrangian, this procedure is guaranteed to describe a local theory in the bulk (at least perturbatively). But if one wants to construct bulk operators purely in terms of the CFT, without making reference to bulk equations of motion, then having bulk gauge symmetries complicates matters. If there is a gauge symmetry in the bulk then the corresponding charge can be expressed as a surface term and identified with a conserved quantity in the CFT. The charge generates global gauge transformations, so as discussed in [87, 89], charged fields in the bulk must have non-local commutators in order to properly implement the Gauss constraint. In the context of gravity this discussion applies to time evolution, since the CFT Hamilton should generate time translation everywhere in the bulk. While these non-local commutators do not actually violate causality, they do complicate the CFT construction, in the sense that the guiding principle of bulk causality must be stated more carefully. It's tempting to speculate that the good causal properties we found for the field strength and Weyl tensor at the linearized level can provide a basis for constructing the interacting theory, at least in perturbation theory. Indeed it does turn out that if e.g. one considers charged scalar fields in the bulk (charged under a $U(1)$ gauge symmetry or diffeomorphism), even though the scalars commute among themselves at spacelike separation, they fail to satisfy micro-causality with A_μ and $h_{\mu\nu}$ perturbation**. On the other hand, upon adding an infinite tower of smeared higher dimensional operators (albeit non-primary this time), the above-mentioned 'gauge-invariant' tensors do satisfy micro-causality principle. The fact that the operators are non-primary precisely gives the transformation rule for the bulk operator under AdS isometries, that one should expect if it is to be charged under the gauge field. The relevant calculations have been done in [86] and [97] in greater details.

**Although the amount of non-commutativity is the same as required by the Gauss law.

6.4 General backgrounds

Using the Green's function approach one can write an expression for the Heisenberg picture fields in the bulk in terms of the boundary values of those same fields, now interpreted as operators in the dual CFT. Correlation function of these CFT operators then reproduce bulk correlation functions. The computations are done from the bulk point of view in a particular gauge $G_{z\mu} = 0$, $G_{zz} = R^2/z^2$. With gauge fields one also sets $A_z = 0$. These conditions completely fix the gauge. The resulting computations are thus physical since all redundant degrees of freedom have been eliminated. In a fixed gauge one can reproduce bulk calculations using boundary data, and since the boundary data comes from a unitary field theory this constitutes holography. From the CFT point of view one corrects the naive smeared operator (constructed to represent a free field in the bulk) by adding higher dimension smeared operators to get a local bulk operator. However these calculations as presented are done in a fixed background metric with a fixed causal structure. This causal structure cannot be circumvented or changed in perturbation theory since it is built in to the hardware of the approach. The approach based on micro-causality and CFT correlators has the same difficulty. One must define a smearing function which is determined by the background metric, and this smearing function cannot be changed in perturbation theory, aside from corrections to incorporate anomalous dimensions.

Besides the question: how local can bulk operators be in this formalism?, one can ask how this formalism could work without an a priori notion of a background. Here we make a few comments on these issues.

In a fixed background the equations of motion for the bulk fields come from a radial Hamiltonian H_r. (By radial Hamiltonian we mean the operator which generates radial evolution of fluctuations about this particular background.) Schematically (ϕ stands for any perturbative field including gravitons on this background)

$$\frac{\partial \phi}{\partial z} = -[H_r, \phi] \tag{6.38}$$

We also need to impose an initial condition, given by normalizable falloff as $z \to 0$ for each field. The radial Hamiltonian can be explicitly written down in the supergravity approximation. If we had a different background metric then the radial Hamiltonian would be some different operator, but for each background we can think of the radial Hamiltonian as some operator in the CFT, generating the transformation from boundary operators to bulk operators via the map

$$\mathcal{O}(x,t) \to e^{-\int_0^z H_r} \mathcal{O}(x,t) e^{\int_0^z H_r} \tag{6.39}$$

However the idea that we will just get a different smearing function for each background is still problematic. The construction of smearing functions relies on having a classical spacetime (perhaps with a few perturbative quantum fluctuations). This clearly does not have to be the case for a generic state in the CFT.

The approximation of getting a fixed background with a few supergravity excitations on it involves two steps. First one needs to integrate out all the bulk stringy modes, which in the CFT means integrating out all high dimension operators. Second one must do a semiclassical approximation to get a well-defined background metric. We won't have much to say about the first step, other than that one has to be careful later on when discussing high dimension operators. For instance, in the promotion of a boundary operator to a field in the bulk, one one needs to include from the CFT perspective a tower of high dimension operators. If one includes high dimension operators only up to some $\Delta_{\max}$ then, according to [85], a good estimate of the commutator of a bulk operator with a boundary operator (taken to be scalars in AdS_3), which are spacelike separated in the bulk but not on the boundary, is

$$[\phi(t,\mathbf{x},z), \mathcal{O}(0)] \sim \frac{1}{\Delta_{\max}} \Big(\frac{t^2 - |\mathbf{x}|^2}{z^2}\Big)^{\Delta_{\max}} \tag{6.40}$$

Although non-zero, the commutator is exponentially suppressed away from the bulk lightcone provided $\Delta_{\max}$ is large. A nice way to characterize the bulk non-locality associated with a finite value of $\Delta_{\max}$ is to ask how far from the bulk lightcone one can go before the commutator becomes exponentially small. This is given by

$$\delta S \sim R/\Delta_{\max} \tag{6.41}$$

where R is the AdS radius and S is proper length in the bulk. For $\Delta_{\max} \sim (g^2_{\rm YM}N)^{1/4}$ – appropriate for stringy modes – one gets $\delta S \sim l_s$.

Even if the approximation of integrating out the stringy modes is good it does not mean the CFT state describes a semiclassical space time. In the supergravity approximation we can write down the equations of motion for the metric and matter fields in holographic gauge without choosing a particular background. This is done by replacing the radial Hamiltonian in (6.38) with the appropriate Hamiltonian for the supergravity system, namely $H_g = \int d^d x \frac{1}{z^2} H_{WD}$ where H_{WD} is the Wheeler – de Witt operator. The radial evolution equations are

$$\frac{\partial \mathcal{O}}{\partial z} = -[H_g, \mathcal{O}] \quad \frac{\partial g_{\mu\nu}}{\partial z} = -[H_g, g_{\mu\nu}] \tag{6.42}$$

and similarly for the conjugate momenta. Once the constraints are satisfied on the initial slice ($z = 0$) the equations of motion guarantee that they are obeyed at any z. We assume here that

$$g_{\mu\nu}(z \to 0) = \eta_{\mu\nu} \tag{6.43}$$

So corrections to the bulk metric come from normalizable modes, with the leading correction for small z being proportional to $T_{\mu\nu}$. This together with $\partial^\mu T_{\mu\nu} = 0$ and $T^\mu_\mu = 0$ gives enough initial data to solve the equations.[††]

The equations of motion can formally be solved to give the bulk fields as functionals of the boundary data.

$$\begin{aligned} \phi(x,z) &= \phi(x,z)\big[T_{\mu\nu}(x'), \mathcal{O}(x'')\big] \\ g_{\mu\nu}(x,z) &= g_{\mu\nu}(x,z)\big[T_{\mu\nu}(x'), \mathcal{O}(x'')\big] \end{aligned} \tag{6.44}$$

So far this is independent of the state of the CFT. But now, given some state of the CFT, we would like to obtain a set of bulk operators which look like fields propagating on some semiclassical space time. To do this, to a good approximation one needs to be able to substitute

$$T_{\mu\nu} = \langle T_{\mu\nu} \rangle + \delta T_{\mu\nu}. \tag{6.45}$$

If this approximation is valid then we are guarantied that correlators of our bulk operators, calculated in the CFT, will look like correlation function of supergravity fields on a background which

[††]We are ignoring the question of whether holographic gauge can be extended all the way to $z = \infty$. Also since we are working in a Poincaré patch we are ignoring any anomalous trace of the stress tensor.

solves the Einstein equations with asymptotics set by $\langle T_{\mu\nu} \rangle$.

Clearly such an approximation is valid in a CFT state if connected correlation functions of CFT operators obey large N factorization. Thus CFT states with large N factorization will be dual to semiclassical spacetimes, while those which do not obey large N factorization will not have a local spacetime interpretation.

Finally we want to speculate about a natural construction for bulk operators purely inside the CFT. It seems possible from the above considerations that one can define "bulk operators" in the CFT regardless of the state of the CFT or any low energy approximation. These operators will not generically have a bulk interpretation, except for a restricted set of states where large N factorization holds. What are these master bulk operators? It seems natural to extrapolate from the supergravity situation (6.42). A natural guess is that one should replace the radial Hamiltonian in (6.39) with a more fundamental gauge theory operator, such as the exact RG Hamiltonian or Fokker-Planck Hamiltonian (see for instance [102, 103]).

6.5 Conclusions and Ongoing Work

In this chapter we worked out the smearing functions for spin-2 excitations in AdS, analogous to the case for spin- 1 in the last chapter. We showed again that the bulk locality is respected: although metric perturbations have non-local commutators when one works in holographic gauge, part of the corresponding curvature, namely the Weyl tensor is causal. We also discussed how the case for interactions goes and what are the subtleties that one can expect in these cases. Ultimately one might hope to make contact between the 'bottom-up' approach of constructing bulk observables in $1/N$ perturbation theory, and the 'top-down' approach of section 6.4 where bulk operators are constructed using the Fokker-Planck Hamiltonian of the boundary theory.

Finally let's end the chapter with a brief overview of the ongoing work that is going on along these lines. After studying the three point function, the only non-trivial order in which one will want to check the locality is $\mathcal{O}(N^{-2})$, which will mean e.g. a ϕ^4 interaction term in the bulk or equivalently the study of four-point functions[‡‡]. At this point the previous calculations of constructing local operators for gauge fields come in handy, as we now have to necessarily deal with them.

[‡‡]Note that we don't really need to worry about the higher point functions, as we can always use the operator product expansion (OPE) between two operators and ultimately deal with the case at the level of four-point functions.

But, now that one has a construction of local operators for fields charged under gauge or gravity fields at the level of 3-point function, it's quite understandable that locality should hold even at $\mathcal{O}(N^{-2})$. Because whenever we have a four-point function (here for simplicity, let's assume four scalar operators) and we smear one of the scalar operator, to get

$$\langle [\phi(x_1, z), \mathcal{O}(x_2)] \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle \tag{6.46}$$

we can always use

$$\mathcal{O}(x_3)\mathcal{O}(x_4) = \sum_k c_k(x_3, x_4) T^k(x_3)$$

type OPE where $T^k(x_3)$ is a complete set of operators in the given CFT with arbitrary spins. But [86] and [97] already shows that (6.46) is zero for $T^k(x_3)$ being a scalar or a conserved (or non-conserved) current or a conserved (or non-conserved) stress tensor and these are basically the relevant operators one deal with in a low energy bulk theory. But of course, how this works for a gauge or gravity field and how in particular the cancellation in the commutator works out remain to be seen. This is currently in progress. As our starting point is micro-causality, a good place to start will be a simple representation of boundary four-point function and an educated guess could be to work with the Mellin representation of operators [104]. This could also serve as a probe to verify the conjecture made by Heemskerk et al. [105], [106] which states that to achieve bulk locality, a CFT need to have a planar or large N expansion and moreover, all single trace operators with spins greater than two have to have parametrically large conformal dimensions. Similar suggestions have been made by using Mellin representations of correlators in [107] where an added condition has been the boundedness of Mellin amplitudes for CFT correlation functions for large values of Mellin space variables. It will be nice to see how these different constraints on CFT to have a local bulk dual fit together with our approach from micro-causality.

7

Comments and Outlooks

As is clear from the title, the most used tool in this dissertation was gauge/ gravity correspondence. The second string revolution starting with open-closed string dualities and discovery of D-branes led to this current generalized understanding of quantum gravity theories in AdS spaces and undoubtedly since then it has become a much used tool to theoretical physicists with application ranging to even condensed matter systems and strongly coupled field theory systems. This type of duality is so useful because it can shed light on both the perspectives of field theory and of quantum gravity. Hence it's no surprise that we have tried to make use of both sides of this duality; on one hand, we tried to understand the fast thermalization in field theory from a black hole creation process in the bulk, and on the other hand we studied bulk locality from a gauge theory point of view. Of course the understanding of even this duality is far from complete and some of the most pressing questions pertain to cultivate various other limits of duality, such as $\lambda \to 0$ limit (which corresponds to higher spin gravity in the bulk) or the study of dS/CFT dualities. It will be e.g. exciting to understand the study of black hole formation and local operator construction in the framework of these dualities.

But some of the immediate technical questions are the following, which we also discussed, to some extent, in their respective chapters: along the line of black hole formation process, one of the important limitations we had was that we were precisely at the correspondence point with equal number of D-brane clusters colliding with each other. Being at the correspondence point is strictly necessary to be able to match our results with the gauge theory side (and thereby providing another instance where the duality works). However, it will be interesting to study how can we go beyond the equal momentum splitting case and take $N_1 \neq N_2$ (or study similar cases like multicharged black hole etc.). For $N_1 \neq N_2$, the total number of excited degrees of freedoms are $\sim \mathcal{O}(N_1 N_2)$ after parametric resonance and we seem to need a secondary step to excite the remaining $\sim \mathcal{O}(N^2 - N_1 N_2)$ degrees of freedoms (dof) of the black hole. But this is technically difficult to do perturbatively to first order in quantum fluctuations as the approximation breaks down as soon as we have parametric resonance in the first place. Also, even though all the black holes we consider have horizon curvature of the order of string scale, it is interesting to see that they are made out of the open string degrees of freedoms. Similar stringy description of black holes is available only for certain black hole charges and in the presence of supersymmetry, namely in the fuzzballs [60].

Similarly there are various directions one can go from the current study of bulk locality. We already talked about some of the previous and ongoing work on CFT four-point functions that could be used to study the locality issues in more details. But all these descriptions are valid only for perturbation in $1/N$. Note that this construction is quite custom made for complementarity like arguments. Black hole complementarity basically tells us that the degrees of freedom inside and outside the horizon are in fact not independent, even though we can define states and operators on a nice slice passing through the horizon to have a local field theoretic description. A good idea will then be to construct such local operators for Schwarzschild black holes in AdS, but it turns out to be highly non-trivial. This would also help us in answering the black hole firewall paradox recently addressed in Almheiri et al. [61], [108] which argues that the horizon should not be structureless, but does not give up the independence of degrees of freedom inside and outside the horizon. This firewall argument is already an intensely studied problem and we will not try to list all the relevant references here. But the closest work that addresses the local operator construction in such framework is by Raju and Papadodimas ([109], [110], [111]) who are able to construct local operators beyond horizon at large N and argue that it eliminates the need for firewalls. But one of their other outcomes is that one can't seem to have a background independent construction any more as in their construction the smearing operator itself is CFT state dependent. One other related study that supports non-locality between d.o.fs inside and outside black hole horizon, but give up firewalls is by Maldacena and Susskind [112]. There the dependence of operators inside and outside the horizon comes in the form of a wormhole. It will thus be interesting to extend our construction to finite N, study the possibility of building a background independent statement of local operator construction and study the link between such construction and wormholes to understand these issues in finer details. Other (rather different) future direction would be to extend our construction with dS space and investigate the possibility of a gauge theory construction of dS quantum gravity [113], [114] despite it's various criticisms [115], [116].

Appendix A

Appendix

A.1 String production in a D-brane collision

We review the process of open string production in a D-brane collision, following [51, 117].

Consider colliding two 0-branes with relative velocity v and impact parameter b. Setting $2\pi\alpha' = 1$, the virtual open strings connecting the two 0-branes have an energy or frequency $\omega = \sqrt{v^2t^2 + b^2}$. As long as this frequency is changing adiabatically open strings will not be produced. The adiabatic approximation breaks down when $\dot{\omega}/\omega^2 \gtrsim 1$. The peak value of this quantity is $\dot{\omega}/\omega^2 \sim v/b^2$ when $vt \sim b$, so (restoring units) open strings are produced for $b \lesssim \sqrt{v\alpha'}$.

Now consider colliding two p-branes wrapped on a torus of volume V_p, with relative velocity v and impact parameter b in the transverse dimensions. Consider a virtual open string that connects the two p-branes and has momentum k along the p-brane worldvolumes. Setting $2\pi\alpha' = 1$, this virtual open string has an energy or frequency

$$\omega = \sqrt{k^2 + v^2t^2 + b^2}$$

If $k = 0$ then the condition for open string production is just what it was for 0-branes, $b \lesssim \sqrt{v}$. Having non-zero k increases ω and suppresses open string production. Effectively there is a cutoff, that open strings are produced up to a maximum momentum $k \sim b \sim \sqrt{v}$. Restoring units, the maximum momentum is $k \sim \sqrt{v/\alpha'} = \dot{U}^{1/2}$. This cutoff corresponds to a number density of open strings on the p-brane worldvolume

$$\frac{\#\text{ open strings}}{\text{volume}} \sim \dot{U}^{p/2}$$

If we collide two stacks of Dp-branes with charges N_1 and N_2 respectively, it's easy to estimate the total number of open strings that are produced. At weak coupling the individual brane collisions are independent events. So for 0-branes the total number of open strings produced is

$$n \sim N_1 N_2$$

while for p-branes the total number of open strings produced is

$$n \sim N_1 N_2 V_p \dot{U}^{p/2} \tag{A.1}$$

or equivalently

$$n \sim N_1 N_2 V_p U^p \tag{A.2}$$

There is, however, an important consistency check on this result: we need to make sure the incoming D-branes have enough kinetic energy to produce this number of open strings. Equivalently, we need to make sure that the back-reaction of open string production on the velocities of the D-branes is under control. Given the number of open strings (A.2), the energy in open strings is

$$E_{\text{string}} = nU = N_1 N_2 V_p \left(\frac{\lambda \epsilon}{N_1 N_2}\right)^{\frac{p+1}{4}}$$

where we have used (3.17). On the other hand the kinetic energy of the incoming branes is

$$E = \epsilon V_p$$

Thus the ratio

$$\frac{E_{\text{string}}}{E} = \lambda \left(\frac{\lambda \epsilon}{N_1 N_2}\right)^{\frac{p-3}{4}} \tag{A.3}$$

and the consistency condition $E_{\text{string}}/E < 1$ is equivalent to

$$\lambda U^{p-3} < 1$$

This is nothing but the condition $\lambda_{\text{eff}} < 1$. Thus at weak coupling energy conservation does not limit the number of open strings that are produced and the simple estimate (A.2) can be trusted.

A.2 Fluctuations in the X^A dimensions

In this appendix we study the spectrum of fluctuations in the directions $A = 1, 2, 3$. We need to solve the linearized Gauss constraint

$$\dot{U}[J^A, x^A] = U[J^A, \dot{x}^A] \tag{A.4}$$

along with the linearized equation of motion

$$\ddot{x}^A + \frac{4}{N^2}U^2[[x^A, J^B], J^B] + \frac{4}{N^2}U^2[[J^A, x^B], J^B] + \frac{4}{N^2}U^2[[J^A, J^B], x^B] = 0 \tag{A.5}$$

These expressions can be simplified somewhat. In the adiabatic approximation we study the spectrum of fluctuations treating U as constant. Then the fluctuation modes can be taken to have definite frequency, $x^A \sim e^{-i\omega t}$, so the Gauss constraint amounts to the requirement that

$$[J^A, x^A] = 0 \tag{A.6}$$

Also we can simplify the equation of motion using

$$\begin{aligned} [[J^A, x^B], J^B] &= -[[x^B, J^B], J^A] - [[J^B, J^A], x^B] \qquad \text{(Jacobi identity)} \\ &= [[J^A, J^B], x^B] \qquad \text{(Gauss constraint)} \end{aligned}$$

This reduces the equation of motion to

$$\ddot{x}^A + \frac{4}{N^2}U^2[[x^A, J^B], J^B] + \frac{8}{N^2}U^2[[J^A, J^B], x^B] = 0 \tag{A.7}$$

To go further we expand the fluctuations in fuzzy vector spherical harmonics. These are constructed as follows. Expanding in a complete set of matrices we can set*

$$x^A = \sum_{\ell=0}^{N-1} x_{AA_1\cdots A_\ell} J_{A_1} \cdots J_{A_\ell} \tag{A.8}$$

The tensor $x_{AA_1\cdots A_\ell}$ is symmetric and traceless on the indices $A_1 \cdots A_\ell$, so taking all indices into account it transforms as a (spin 1) $\otimes$ (spin ℓ) product representation of $SU(2)$. Decomposing this product, the irreducible pieces correspond to tensors s, t, u that have spin $\ell+1$, ℓ, $\ell-1$ respectively. These tensors can be constructed explicitly.†

$$\begin{aligned} s_{A_0A_1\cdots A_\ell} &\sim (x_{A_0A_1\cdots A_\ell} + \text{cyclic permutations of } A_0 \cdots A_\ell) \\ &\quad -\frac{2}{2\ell+1} \sum_{\substack{i,j=0 \\ i<j}}^{\ell} \delta_{A_iA_j} x_{BBA_0\cdots\widehat{A}_i\cdots\widehat{A}_j\cdots A_\ell} \end{aligned} \tag{A.9}$$

$$t_{A_1\cdots A_\ell} \sim \sum_{i=1}^{\ell} \epsilon_{A_iAB} x_{ABA_1\cdots\widehat{A}_i\cdots A_\ell} \tag{A.10}$$

$$u_{A_2\cdots A_\ell} \sim \delta_{AB} x_{ABA_2\cdots A_\ell} \tag{A.11}$$

The tensors s, t, u are constructed to be symmetric and traceless on all indices, so that they correspond to the appropriate irreducible $SU(2)$ representations.

This decomposition helps in understanding the Gauss constraint (A.6), since

$$\begin{aligned} [J^A, x^A] &= x_{AA_1\cdots A_\ell}[J^A, J_{A_1} \cdots J_{A_\ell}] \\ &= i\left(\epsilon_{A_1AB} x_{ABA_2\cdots A_\ell} + \epsilon_{A_2AB} x_{AA_1BA_3\cdots A_\ell} + \cdots + \epsilon_{A_\ell AB} x_{AA_1\cdots A_{\ell-1}B}\right) J_{A_1} \cdots J_{A_\ell} \\ &\sim it_{A_1\cdots A_\ell} J_{A_1} \cdots J_{A_\ell} \end{aligned}$$

Thus the Gauss constraint requires that we set the spin-ℓ irreducible piece to zero, $t_{A_1\cdots A_\ell} = 0$.

Now let's study the equation of motion (A.7). Using (4.16) in the middle term, and evaluating

*To save writing we're adopting a different normalization convention in expanding x^A, without the factor $\left(\frac{2}{N}\right)^\ell$ present in (4.15).

†A hat denotes a missing index. There's an overall normalization in these formulas which we leave unspecified.

the commutators in the last term, the equation of motion becomes

$$\begin{aligned}
&\ddot{x}_{AA_1\cdots A_\ell} J_{A_1}\cdots J_{A_\ell} + \frac{4}{N^2}U^2\ell(\ell+1)x_{AA_1\cdots A_\ell}J_{A_1}\cdots J_{A_\ell} \\
&+\frac{8}{N^2}U^2 x_{BBA_2\cdots A_\ell}\left(J_A J_{A_2}\cdots J_{A_\ell} + J_{A_2}J_A J_{A_3}\cdots J_{A_\ell} + \cdots\right) \\
&-\frac{8}{N^2}U^2 x_{BAA_2\cdots A_\ell}\left(J_B J_{A_2}\cdots J_{A_\ell} + J_{A_2}J_B J_{A_3}\cdots J_{A_\ell} + \cdots\right) \\
&= 0
\end{aligned} \tag{A.12}$$

(there are ℓ terms in the second and third lines, where the generators J_A and J_B are inserted at different positions). We consider the different irreducible pieces in turn.

s-type fluctuations

To study the irreducible piece with spin $\ell+1$ we take x to be symmetric and traceless on all indices,

$$x_{AA_1\cdots A_\ell} = s_{AA_1\cdots A_\ell} \tag{A.13}$$

For such a tensor the Gauss law is automatically satisfied, while the equation of motion (A.12) reduces to

$$\ddot{s}_{AA_1\cdots A_\ell} + \frac{4}{N^2}U^2\ell(\ell-1)s_{AA_1\cdots A_\ell} = 0 \tag{A.14}$$

We read off the frequencies

$$\omega_\ell = \frac{2}{N}U\sqrt{\ell(\ell-1)} \tag{A.15}$$

These modes are $(2\ell+3)$-fold degenerate. There are two zero-frequency modes: $\ell = 0$ is a translation zero mode in the X^A directions, while $\ell = 1$ is an energy-preserving quadrupole deformation of the sphere.

t-type fluctuations

These exist for $\ell \geq 1$. We can reconstruct the tensor x from its spin-ℓ irreducible piece t by setting

$$x_{AA_1\cdots A_\ell} = \epsilon_{AA_1B}t_{BA_2\cdots A_\ell} + \epsilon_{AA_2B}t_{A_1BA_3\cdots A_\ell} + \cdots + \epsilon_{AA_\ell B}t_{A_1\cdots A_{\ell-1}B} \tag{A.16}$$

This map has been constructed so that x is symmetric and traceless on the indices $A_1\cdots A_\ell$. In other words, it defines the embedding of (spin ℓ) $\hookrightarrow$ (spin 1) $\otimes$ (spin ℓ). Given (A.16), the corresponding

Hermitian matrix x^A can be written as a commutator,

$$\begin{aligned} x^A &\equiv x_{AA_1\cdots A_\ell} J_{A_1} \cdots J_{A_\ell} \\ &= it_{A_1\cdots A_\ell}[J_A, J_{A_1} \cdots J_{A_\ell}] \end{aligned} \tag{A.17}$$

As we saw earlier, these fluctuations fail to satisfy the Gauss constraint, since from (4.16)

$$[J^A, x^A] = i\ell(\ell+1) t_{A_1\cdots A_\ell} J_{A_1} \cdots J_{A_\ell} \tag{A.18}$$

Again the only solution to the Gauss constraint is to set $t = 0$.

u-type fluctuations

These exist for $\ell \geq 1$. We can reconstruct x from its spin-$(\ell - 1)$ irreducible piece using

$$x_{AA_1\cdots A_\ell} = \sum_{i=1}^{\ell} \delta_{AA_i} u_{A_1\cdots \widehat{A}_i \cdots A_\ell} - \frac{2}{2\ell - 1} \sum_{\substack{i,j=1 \\ i<j}}^{\ell} \delta_{A_i A_j} u_{AA_1\cdots \widehat{A}_i \cdots \widehat{A}_j \cdots A_\ell} \tag{A.19}$$

This map is constructed so that x is symmetric and traceless on $A_1 \cdots A_\ell$. For such a tensor the Gauss law is automatically satisfied. Substituting the expression for x into the equation of motion (A.12), we find after some algebra that

$$\ddot{u}_{A_2\cdots A_\ell} + \frac{4}{N^2} U^2 (\ell+1)(\ell+2) u_{A_2\cdots A_\ell} = 0 \tag{A.20}$$

From this we read off the frequencies

$$\omega_\ell = \frac{2}{N} U \sqrt{(\ell+1)(\ell+2)} \tag{A.21}$$

These modes are $(2\ell - 1)$-fold degenerate. The $\ell = 1$ mode is a monopole deformation of the sphere, $U \to U + \delta U$. The frequency ω_1 agrees with what one obtains by perturbing the background equation of motion (4.10).

A.3 Open string production

In this appendix we study the process of open string production in more detail. Our goal is to show that, during the initial collapse of a fuzzy sphere, roughly one open string is produced in each of the fluctuation modes. We assume the fluctuations are weakly coupled, which as discussed in section 4.5 means $U_0 > N^{1/2}\lambda^{1/3}$.

We focus on a particular fluctuation mode. For concreteness we consider a transverse mode (4.18) with frequency

$$\omega_\ell = \frac{2}{N}\sqrt{\ell(\ell+1)}\,U \tag{A.22}$$

For this mode, the adiabatic approximation breaks down when $\dot{\omega}_\ell/\omega_\ell^2 \sim 1$ or

$$\frac{N\dot{U}}{2\sqrt{\ell(\ell+1)}\,U^2} \sim 1 \tag{A.23}$$

Energy conservation (4.23) fixes $\dot{U}^2 \approx \frac{4}{N^2}\big(U_0^4 - U^4\big)$. By the time the adiabatic approximation has broken down we can neglect the U^4 term, so the velocity is

$$\dot{U} \approx \frac{2}{N}U_0^2 \tag{A.24}$$

and the adiabatic approximation fails at

$$U \approx \frac{U_0}{\big(\ell(\ell+1)\big)^{1/4}} \tag{A.25}$$

At the point where the adiabatic approximation fails the mode can be thought of as a harmonic oscillator in its ground state, with a frequency

$$\omega \approx \frac{2}{N}\big(\ell(\ell+1)\big)^{1/4}U_0 \tag{A.26}$$

and a ground state wavefunction (identifying $\hbar/m$ with $g_{\rm YM}^2$)

$$\psi_0(x) = \left(\frac{\omega}{\pi g_{\rm YM}^2}\right)^{1/4} e^{-\frac{1}{2}\omega x^2/g_{\rm YM}^2} \tag{A.27}$$

After the adiabatic approximation breaks down the sphere continues to shrink. We must follow the evolution of the mode through the non-adiabatic regime, until the sphere re-expands to the radius (A.25) at which adiabaticity is restored. In the non-adiabatic regime the frequency is so low that it seems reasonable to neglect the potential energy for the mode, in other words, to treat it as a free particle. In this approximation the Gaussian wavefunction (A.27) undergoes free diffusion, spreading to a width

$$\Delta x^2 = \Delta x_0^2 + \frac{g_{\rm YM}^4 \Delta t^2}{4 \Delta x_0^2} \tag{A.28}$$

Here the initial position uncertainty is $\Delta x_0^2 = g_{\rm YM}^2/2\omega$, while the time spent in the non-adiabatic regime is

$$\Delta t = \frac{\Delta U}{\dot{U}} \approx \frac{N}{\big(\ell(\ell+1)\big)^{1/4} U_0} \tag{A.29}$$

This gives $\Delta x^2 \approx 5 \Delta x_0^2$: the wavefunction spreads by a factor of roughly $\sqrt{5}$ as the sphere transits the non-adiabatic regime. This factor is independent of the parameters N, ℓ, U_0, which suggests that of order one open string is produced in each of the fluctuation modes.

To argue this more precisely we recall some properties of squeezed states [118]. For a harmonic oscillator these are defined by

$$|\xi\rangle = \exp\left[\frac{\xi}{2}\left(\hat{a}^\dagger \hat{a}^\dagger - \hat{a}\hat{a}\right)\right]|0\rangle \tag{A.30}$$

where the squeezing parameter $0 < \xi < \infty$. An equivalent expression is

$$|\xi\rangle = \left(1-\gamma^2\right)^{1/4} \exp\left[\frac{\gamma}{2}\hat{a}^\dagger \hat{a}^\dagger\right]|0\rangle \tag{A.31}$$

where $\gamma = \tanh \xi$. A squeezed state has a Gaussian wavefunction with a width

$$\Delta x = e^{\xi} \Delta x_0 \tag{A.32}$$

so we identify $e^{\xi} \approx \sqrt{5}$. Expanding the exponential in (A.31), the probability of finding $2n$ strings present is

$$P(2n \text{ strings}) = \left(1-\gamma^2\right)^{1/2} \frac{(2n)!}{(n!)^2} \left(\frac{\gamma}{2}\right)^{2n} \tag{A.33}$$

The probability decreases monotonically with n. The average number of strings present is

$$\sum_{n=0}^{\infty} 2nP(2n \text{ strings}) = \frac{\gamma^2}{1-\gamma^2} \approx \frac{4}{5} \tag{A.34}$$

So the simple approximation of free diffusion in the non-adiabatic regime supports the claim that roughly one open string is produced in each fluctuation mode.

A.4 Scalar smearing functions

Consider a scalar field of mass m in AdS_{d+1}. It's dual to an operator of dimension Δ in the CFT, where $m^2R^2 = \Delta(\Delta - d)$. The mode expansion is

$$\phi(t, \mathbf{x}, z) = \int_{|\omega|>|\mathbf{k}|} d\omega d^{d-1}k \, a_{\omega\mathbf{k}} e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}} z^{d/2} J_\nu(z\sqrt{\omega^2 - |\mathbf{k}|^2}) \tag{A.35}$$

where $\nu = \Delta - d/2$. As $z \to 0$ we have $\phi(t, \mathbf{x}, z) \sim z^\Delta \phi_0(t, \mathbf{x})$ where the boundary field

$$\phi_0(t, \mathbf{x}) = \frac{1}{2^\nu \Gamma(\nu+1)} \int_{|\omega|>|\mathbf{k}|} d\omega d^{d-1}k \, a_{\omega\mathbf{k}} e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}} (\omega^2 - |\mathbf{k}|^2)^{\nu/2}$$

Our basic goal is to express the bulk field in terms of the boundary field. A straightforward way to do this is to express the coefficients $a_{\omega\mathbf{k}}$ as a Fourier transform of ϕ_0,

$$a_{\omega\mathbf{k}} = \frac{2^\nu \Gamma(\nu+1)}{(2\pi)^d (\omega^2 - |\mathbf{k}|^2)^{\nu/2}} \int dt d^{d-1}x \, e^{i\omega t} e^{-i\mathbf{k}\cdot\mathbf{x}} \phi_0(t, \mathbf{x}) \, .$$

Substituting this back in (A.35) leads to an integral representation of the smearing function. Generically one obtains a smearing function with support on the entire boundary of the Poincaré patch, however by complexifying the boundary spatial coordinates one can obtain a smearing function with compact support. As shown in [84] this leads to

$$\phi(t, \mathbf{x}, z) = \frac{\Gamma(\Delta - \frac{d}{2} + 1)}{\pi^{d/2} \Gamma(\Delta - d + 1)} \int\limits_{t'^2 + |\mathbf{y}'|^2 < z^2} dt' d^{d-1}y' \left(\frac{z^2 - t'^2 - |\mathbf{y}'|^2}{z} \right)^{\Delta - d} \phi_0(t + t', \mathbf{x} + i\mathbf{y}') \tag{A.36}$$

This expression is fine for $\Delta > d - 1$. However when $\Delta = d - 1$ it's ill-defined: the integral diverges, and the coefficient in front goes to zero.

To construct a smearing function for $\Delta = d-1$ we return to the mode expansion (A.35). As a warm-up example take a massless field in AdS_2 with $\Delta = 0$. The mode expansion is $\phi(t,z) = \int d\omega\, a_\omega e^{-i\omega t}\cos(\omega z)$. Then $a_\omega = \frac{1}{2\pi}\int dt\, e^{i\omega t}\phi_0(t)$ and

$$\begin{aligned}\phi(t,z) &= \int dt' \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\cos(\omega z)\phi_0(t') \\ &= \frac{1}{2}\left(\phi_0(t+z) + \phi_0(t-z)\right)\end{aligned} \tag{A.37}$$

This clearly satisfies the wave equation $(\partial_t^2 - \partial_x^2)\phi = 0$ and obeys the boundary condition $\phi(t,z) \to \phi_0(t)$ as $z \to 0$. It can be written in the covariant form

$$\phi(t,z) = \frac{1}{2}\int dt'\, \delta(\sigma z')\phi_0(t')$$

where $\sigma z' = \frac{z^2-(t-t')^2}{2z}$.

We now consider the general case of a field with $\Delta = d-1$. In any dimension solving for $a_{\omega\mathbf{k}}$ in terms of ϕ_0 and plugging back into the mode expansion gives

$$\phi(t,\mathbf{x},z) = \int_{|\omega|>|\mathbf{k}|} d\omega d^{d-1}k\, \frac{2^\nu \Gamma(d/2) z^{d/2}}{(2\pi)^d(\omega^2-|\mathbf{k}|^2)^{\nu/2}}\, J_\nu\big(z\sqrt{\omega^2-|\mathbf{k}|^2}\big)e^{-i\omega t}e^{i\mathbf{k}\cdot\mathbf{x}}\phi_0(\omega,\mathbf{k}) \tag{A.38}$$

Here $\nu = \frac{d}{2}-1$ and $\phi_0(\omega,\mathbf{k})$ is the Fourier transform of the boundary field. The Bessel function has an integral representation

$$J_\nu(a) = \frac{1}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{a}{2}\right)^\nu \int_0^\pi d\theta\, e^{-ia\cos\theta}\sin^{2\nu}\theta \tag{A.39}$$

or equivalently

$$J_\nu(a) = \frac{1}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{a}{2}\right)^\nu \frac{1}{\mathrm{vol}(S^{d-2})}\int_{|\mathbf{n}|=1} d\mathbf{n}\, e^{-i\mathbf{a}\cdot\mathbf{n}} \tag{A.40}$$

Here $\mathbf{a}$ is a d-component vector with Euclidean norm a and $\mathbf{n} \in S^{d-1}$ is a unit vector. Setting $\mathbf{a} = z(\omega, -ik_1, \ldots, -ik_{d-1})$ and using

$$\mathrm{vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}\right)\mathrm{vol}(S^{d-2})}{\Gamma(d/2)} \tag{A.41}$$

this becomes

$$\frac{2^\nu \Gamma(d/2) z^{d/2}}{(\omega^2 - |\mathbf{k}|^2)^{\nu/2}} J_\nu(z\sqrt{\omega^2 - |\mathbf{k}|^2}) = \frac{1}{\mathrm{vol}(S^{d-1})} \int\limits_{t'^2 + |\mathbf{y}'|^2 = z^2} dt' d^{d-1}y' \, e^{-i\omega t'} e^{-\mathbf{k}\cdot\mathbf{y}'}$$

Using this representation in (A.38) leads to[‡]

$$\phi(t, \mathbf{x}, z) = \frac{1}{\mathrm{vol}(S^{d-1})} \int\limits_{t'^2 + |\mathbf{y}'|^2 = z^2} dt' d^{d-1}y' \int \frac{d\omega d^{d-1}k}{(2\pi)^d} e^{-i\omega(t+t')} e^{i\mathbf{k}\cdot(\mathbf{x}+i\mathbf{y}')} \phi_0(\omega, \mathbf{k}) \tag{A.42}$$

We interpret the Fourier transforms in (A.42) as defining the analytic continuation of $\phi_0(t, \mathbf{x})$ to complex $\mathbf{x}$. Thus the smearing function for a scalar field with $\Delta = d - 1$ is

$$\phi(t, \mathbf{x}, z) = \frac{1}{\mathrm{vol}(S^{d-1})} \int\limits_{t'^2 + |\mathbf{y}'|^2 = z^2} dt' d^{d-1}y' \, \phi_0(t + t', \mathbf{x} + i\mathbf{y}') \tag{A.43}$$

This can be written in a covariant form

$$\phi(t, \mathbf{x}, z) = \frac{1}{\mathrm{vol}(S^{d-1})} \int dt' d^{d-1}y' \, \delta(\sigma z') \, \phi_0(t + t', \mathbf{x} + i\mathbf{y}') \tag{A.44}$$

in terms of the bulk - boundary distance (5.6).

It's clear that (A.43), (A.44) satisfy the correct boundary conditions. As $z \to 0$ the integration region on the boundary shrinks to a point, so we can bring the boundary field outside the integral and recover

$$\phi(t, \mathbf{x}, z) \sim z^{d-1} \phi_0(t, \mathbf{x}) \qquad \text{as } z \to 0$$

One can also check that (A.44) satisfies the wave equation. Acting on a function of the AdS-invariant distance σ, the wave equation $(\Box - m^2)\phi = 0$ reduces to

$$(\sigma^2 - 1)\phi'' + (d+1)\sigma\phi' - \Delta(\Delta - d)\phi = 0$$

With a small fixed cutoff z', the smearing kernel appearing in (A.44) is $\frac{1}{z'}\delta(\sigma)$. We want to check that this is annihilated by the wave operator in the limit $z' \to 0$. To do this we act with the wave

[‡]The boundary field ϕ_0 only has Fourier components with $|\omega| > |\mathbf{k}|$, so we can integrate over ω and $\mathbf{k}$ without restriction.

operator and integrate against a test function $f(\sigma z')$ (the test function can be thought of as the boundary field). For $\Delta = d-1$ this gives

$$\begin{aligned} & \int d(\sigma z')\, f(\sigma z') \left[(\sigma^2-1)\frac{d^2}{d\sigma^2} + (d+1)\sigma\frac{d}{d\sigma} + (d-1) \right] \frac{1}{z'}\delta(\sigma) \\ = & \int d(\sigma z') \frac{1}{z'}\delta(\sigma) \left[\frac{d^2}{d\sigma^2}(\sigma^2-1) - (d+1)\frac{d}{d\sigma}\sigma + (d-1) \right] f(\sigma z') \\ = & -z'^2 f''(0) \end{aligned}$$

This vanishes as $z' \to 0$, which shows that the wave equation is satisfied when the regulator is removed.

A.5 Chern-Simons in holographic gauge

Our goal in this appendix is to quantize Chern-Simons theory in holographic gauge. We want to show that we recover the bulk commutator (5.25) obtained in section 5.2.2.1 by applying our smearing functions to the current algebra on the boundary.

We begin from the abelian Chern-Simons action[§]

$$S_{\text{bulk}} = \int d^3x\, \frac{1}{2}\kappa\, \epsilon^{ABC} A_A \partial_B A_C$$

To obtain a right-moving current algebra on the boundary we supplement this with a surface term [119]

$$S_{\text{bdy}} = \int d^2x\, \kappa A_+ A_-$$

The surface term leads to a well-defined variational principle provided we impose the boundary condition that A_- is fixed (that is, $\delta A_- = 0$) on the boundary.

In light-front coordinates one can integrate by parts to find (the surface terms cancel against S_{bdy})

$$S_{\text{bulk+bdy}} = \int dx^+ dx^- dz\, \kappa A_z \partial_+ A_- + \kappa A_+ \left(\partial_- A_z - \partial_z A_-\right) .$$

[§]Conventions: light-front coordinates are $x^\pm = t \pm x$. We take $\epsilon_{012} = +1$ and relate the bulk and boundary orientations by $\int d^3x\, \partial_z f = -\int d^2x f|_{z=0}$.

We adopt x^+ as light-front time [120] and read off the Poisson bracket [121]

$$\{A_z(x^-, z), A_-(x^{-\prime}, z')\} = \frac{1}{\kappa}\delta(x^- - x^{-\prime})\delta(z - z')$$

A_+ is a Lagrange multiplier that enforces the Chern-Simons Gauss law. Thus we have a (first-class) constraint

$$\chi_1 = \partial_z A_- - \partial_- A_z \approx 0\,.$$

The constraint generates the expected gauge transformation

$$\begin{aligned}
\delta A_z &= \left\{\int dx^{-\prime}dz'\,\lambda_1\chi_1,\; A_z(x^-, z)\right\} = \frac{1}{\kappa}\partial_z\lambda_1 \\
\delta A_- &= \left\{\int dx^{-\prime}dz'\,\lambda_1\chi_1,\; A_-(x^-, z)\right\} = \frac{1}{\kappa}\partial_-\lambda_1
\end{aligned}$$

To preserve the boundary condition $\delta A_-|_{z=0} = 0$, we require that the gauge parameter satisfy $\lambda_1|_{z=0} = 0$. We wish to work in holographic gauge, so we impose an additional constraint (a gauge-fixing condition)

$$\chi_2 = A_z \approx 0\,.$$

The constraints obey

$$\Delta_{ij} \equiv \{\chi_i,\, \chi_j\} = \begin{pmatrix} 0 & -\frac{1}{\kappa}\delta(x^- - x^{-\prime})\delta'(z - z') \\ -\frac{1}{\kappa}\delta(x^- - x^{-\prime})\delta'(z - z') & 0 \end{pmatrix}$$

Acting on functions $\binom{\lambda_1}{\lambda_2}$ this operator has zero modes, but as we will see the zero modes can be eliminated by requiring

$$\lambda_1(x^-, z = 0) = 0 \qquad \lambda_2(x^-, z = \infty) = 0$$

Then Δ has a well-defined inverse,

$$\Delta^{-1} = \begin{pmatrix} 0 & -\kappa\delta(x^- - x^{-\prime})\theta(z - z') \\ \kappa\delta(x^- - x^{-\prime})\theta(z' - z) & 0 \end{pmatrix}$$

Note that Δ^{-1} is antisymmetric. One can easily check the basic property

$$\Delta^{-1}\Delta \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1(x^-, z) - \lambda_1(x^-, 0) \\ \\ \lambda_2(x^-, z) - \lambda_2(x^-, \infty) \end{pmatrix}$$

which shows that Δ is invertible given our boundary conditions. The constraints can be eliminated by defining Dirac brackets. The Dirac bracket of A_z with anything will vanish, while the Dirac bracket of A_- with itself is

$$\begin{aligned} \left\{A_-(x^-, z),\, A_-(x^{-\prime}, z')\right\} &= 0 - \{A_-,\, \chi_i\}\, \Delta^{-1}_{ij}\, \{\chi_j,\, A_-\} \\ &= -\frac{1}{\kappa}\delta'(x^- - x^{-\prime}) \end{aligned}$$

Quantizing via $\{\cdot,\, \cdot\} \to i[\cdot,\, \cdot]$ reproduces the bulk commutator (5.25) and fixes the normalization $\kappa = 4\pi/k$.

Bibliography

[1] M. Dine. Supersymmetry and string theory: Beyond the standard model. 2007.

[2] Ofer Aharony, Steven S. Gubser, Juan Martin Maldacena, Hirosi Ooguri, and Yaron Oz. Large N field theories, string theory and gravity. *Phys.Rept.*, 323:183–386, 2000.

[3] Nissan Itzhaki, Juan Martin Maldacena, Jacob Sonnenschein, and Shimon Yankielowicz. Supergravity and the large N limit of theories with sixteen supercharges. *Phys.Rev.*, D58:046004, 1998.

[4] D. Bigatti and Leonard Susskind. Review of matrix theory. 1997.

[5] Daniel Kabat and Debajyoti Sarkar. Cosmic string interactions induced by gauge and scalar fields. *Phys.Rev.*, D86:084021, 2012.

[6] A. Vilenkin. Gravitational Field of Vacuum Domain Walls and Strings. *Phys.Rev.*, D23:852–857, 1981.

[7] III Gott, J. Richard. Gravitational lensing effects of vacuum strings: Exact solutions. *Astrophys.J.*, 288:422–427, 1985.

[8] Gerard 't Hooft. Nonperturbative Two Particle Scattering Amplitudes in (2+1)-Dimensional Quantum Gravity. *Commun.Math.Phys.*, 117:685, 1988.

[9] Stanley Deser and R. Jackiw. Classical and Quantum Scattering on a Cone. *Commun.Math.Phys.*, 118:495, 1988.

[10] T.M. Helliwell and D.A. Konkowski. VACUUM FLUCTUATIONS OUTSIDE COSMIC STRINGS. *Phys.Rev.*, D34:1918–1920, 1986.

[11] B. Linet. Quantum Field Theory in the Space-time of a Cosmic String. *Phys.Rev.*, D35:536–539, 1987.

[12] Bruce Allen and Adrian C. Ottewill. Effects of curvature couplings for quantum fields on cosmic string space-times. *Phys.Rev.*, D42:2669–2677, 1990.

[13] Valeri P. Frolov and E.M. Serebryanyi. Vacuum Polarization in the Gravitational Field of a Cosmic String. *Phys.Rev.*, D35:3779–3782, 1987.

[14] J.S. Dowker. Vacuum Averages for Arbitrary Spin Around a Cosmic String. *Phys.Rev.*, D36:3742, 1987.

[15] D.V. Galtsov, Yu.V. Grats, and A.B. Lavrentev. Vacuum polarization and topological selfinteraction of a charge in multiconic space. *Phys.Atom.Nucl.*, 58:516–521, 1995.

[16] A.N. Aliev. Casimir effect in the space-time of multiple cosmic strings. *Phys.Rev.*, D55:3903–3904, 1997.

[17] Finn Larsen and Frank Wilczek. Renormalization of black hole entropy and of the gravitational coupling constant. *Nucl.Phys.*, B458:249–266, 1996.

[18] Leonard Susskind and John Uglum. Black hole entropy in canonical quantum gravity and superstring theory. *Phys.Rev.*, D50:2700–2711, 1994.

[19] Dmitri V. Vassilevich. QED on curved background and on manifolds with boundaries: Unitarity versus covariance. *Phys.Rev.*, D52:999–1010, 1995.

[20] Daniel N. Kabat. Black hole entropy and entropy of entanglement. *Nucl.Phys.*, B453:281–302, 1995.

[21] Leonard Susskind. Some speculations about black hole entropy in string theory. 1993.

[22] Steven Carlip and Claudio Teitelboim. The Off-shell black hole. *Class.Quant.Grav.*, 12:1699–1704, 1995.

[23] Sergei N. Solodukhin. One loop renormalization of black hole entropy due to nonminimally coupled matter. *Phys.Rev.*, D52:7046–7052, 1995.

[24] William Donnelly and Aron C. Wall. Do gauge fields really contribute negatively to black hole entropy? *Phys.Rev.*, D86:064042, 2012.

[25] Sergey N. Solodukhin. Remarks on effective action and entanglement entropy of Maxwell field in generic gauge. *JHEP*, 1212:036, 2012.

[26] N.D. Birrell and P.C.W. Davies. Quantum Fields in Curved Space. *Cambridge Monogr.Math.Phys.*, 1982.

[27] Ashoke Sen. Extremal black holes and elementary string states. *Mod.Phys.Lett.*, A10:2081–2094, 1995.

[28] Andrew Strominger and Cumrun Vafa. Microscopic origin of the Bekenstein-Hawking entropy. *Phys.Lett.*, B379:99–104, 1996.

[29] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Adv.Theor.Math.Phys.*, 2:231–252, 1998.

[30] S. W. Hawking. Breakdown of Predictability in Gravitational Collapse. *Phys. Rev.*, D14:2460–2473, 1976.

[31] Gary T. Horowitz and Joseph Polchinski. A Correspondence principle for black holes and strings. *Phys.Rev.*, D55:6189–6197, 1997.

[32] Edward Witten. Anti-de Sitter space and holography. *Adv.Theor.Math.Phys.*, 2:253–291, 1998.

[33] Edward Witten. Anti-de Sitter space, thermal phase transition, and confinement in gauge theories. *Adv.Theor.Math.Phys.*, 2:505–532, 1998.

[34] Norihiro Iizuka, Daniel N. Kabat, Gilad Lifschytz, and David A. Lowe. Probing black holes in nonperturbative gauge theory. *Phys.Rev.*, D65:024012, 2002.

[35] Naoyuki Kawahara, Jun Nishimura, and Shingo Takeuchi. Phase structure of matrix quantum mechanics at finite temperature. *JHEP*, 0710:097, 2007.

[36] Simon Catterall and Toby Wiseman. Towards lattice simulation of the gauge theory duals to black holes and hot strings. *JHEP*, 0712:104, 2007.

[37] Konstantinos N. Anagnostopoulos, Masanori Hanada, Jun Nishimura, and Shingo Takeuchi. Monte Carlo studies of supersymmetric matrix quantum mechanics with sixteen supercharges at finite temperature. *Phys.Rev.Lett.*, 100:021601, 2008.

[38] Masanori Hanada, Akitsugu Miwa, Jun Nishimura, and Shingo Takeuchi. Schwarzschild radius from Monte Carlo calculation of the Wilson loop in supersymmetric matrix quantum mechanics. *Phys.Rev.Lett.*, 102:181602, 2009.

[39] Masanori Hanada, Yoshifumi Hyakutake, Jun Nishimura, and Shingo Takeuchi. Higher derivative corrections to black hole thermodynamics from supersymmetric matrix quantum mechanics. *Phys.Rev.Lett.*, 102:191602, 2009.

[40] Simon Catterall and Toby Wiseman. Extracting black hole physics from the lattice. *JHEP*, 1004:077, 2010.

[41] Sayantani Bhattacharyya and Shiraz Minwalla. Weak Field Black Hole Formation in Asymptotically AdS Spacetimes. *JHEP*, 0909:034, 2009.

[42] David Garfinkle and Leopoldo A. Pando Zayas. Rapid Thermalization in Field Theory from Gravitational Collapse. *Phys.Rev.*, D84:066006, 2011.

[43] Guido Festuccia and Hong Liu. The Arrow of time, black holes, and quantum mixing of large N Yang-Mills theories. *JHEP*, 0712:027, 2007.

[44] Norihiro Iizuka and Joseph Polchinski. A Matrix Model for Black Hole Thermalization. *JHEP*, 0810:028, 2008.

[45] Norihiro Iizuka, Takuya Okuda, and Joseph Polchinski. Matrix Models for the Black Hole Information Paradox. *JHEP*, 1002:073, 2010.

[46] David Berenstein and Diego Trancanelli. Dynamical tachyons on fuzzy spheres. *Phys.Rev.*, D83:106001, 2011.

[47] Curtis Asplund, David Berenstein, and Diego Trancanelli. Evidence for fast thermalization in the plane-wave matrix model. *Phys.Rev.Lett.*, 107:171602, 2011.

[48] Curtis T. Asplund, David Berenstein, and Eric Dzienkowski. Large N classical dynamics of holographic matrix models. *Phys.Rev.*, D87:084044, 2013.

[49] Lev Kofman, Andrei D. Linde, Xiao Liu, Alexander Maloney, Liam McAllister, et al. Beauty is attractive: Moduli trapping at enhanced symmetry points. *JHEP*, 0405:030, 2004.

[50] Liam McAllister and Indrajit Mitra. Relativistic D-brane scattering is extremely inelastic. *JHEP*, 0502:019, 2005.

[51] C. Bachas. D-brane dynamics. *Phys.Lett.*, B374:37–42, 1996.

[52] G.W. Gibbons. Birkhoff's invariant and Thorne's Hoop Conjecture. 2009.

[53] M. Cvetic, G.W. Gibbons, C.N. Pope, G.W. Gibbons, and C.N. Pope. More about Birkhoff's Invariant and Thorne's Hoop Conjecture for Horizons. *Class.Quant.Grav.*, 28:195001, 2011.

[54] Katrin Becker, Melanie Becker, Joseph Polchinski, and Arkady A. Tseytlin. Higher order graviton scattering in M(atrix) theory. *Phys.Rev.*, D56:3174–3178, 1997.

[55] Daniel N. Kabat and Gilad Lifschytz. Gauge theory origins of supergravity causal structure. *JHEP*, 9905:005, 1999.

[56] Eva Silverstein and David Tong. Scalar speed limits and cosmology: Acceleration from D-cceleration. *Phys.Rev.*, D70:103505, 2004.

[57] Norihiro Iizuka, Daniel N. Kabat, Gilad Lifschytz, and David A. Lowe. Stretched horizons, quasiparticles and quasinormal modes. *Phys.Rev.*, D68:084021, 2003.

[58] Kengo Maeda, Makoto Natsuume, and Takashi Okamura. Quasinormal modes for nonextreme Dp-branes and thermalizations of super-Yang-Mills theories. *Phys.Rev.*, D72:086012, 2005.

[59] Gary T. Horowitz and Veronika E. Hubeny. Quasinormal modes of AdS black holes and the approach to thermal equilibrium. *Phys.Rev.*, D62:024027, 2000.

[60] Samir D. Mathur. The Fuzzball proposal for black holes: An Elementary review. *Fortsch.Phys.*, 53:793–827, 2005.

[61] Ahmed Almheiri, Donald Marolf, Joseph Polchinski, and James Sully. Black Holes: Complementarity or Firewalls? *JHEP*, 1302:062, 2013.

[62] Norihiro Iizuka, Daniel Kabat, Shubho Roy, and Debajyoti Sarkar. Black Hole Formation at the Correspondence Point. *Phys.Rev.*, D87:126010, 2013.

[63] Norihiro Iizuka, Daniel Kabat, Shubho Roy, and Debajyoti Sarkar. Black Hole Formation in Fuzzy Sphere Collapse. *Phys.Rev.*, D88:044019, 2013.

[64] Harold Steinacker. Non-commutative geometry and matrix models. *PoS*, QGQGS2011:004, 2011.

[65] Jun John Sakurai and Jim Napolitano. *Modern quantum mechanics*. Addison-Wesley, 2011. See section 3.11.

[66] J. Madore. The fuzzy sphere. *Class.Quant.Grav.*, 9:69–88, 1992.

[67] A.P. Balachandran, S. Kurkcuoglu, and S. Vaidya. Lectures on fuzzy and fuzzy SUSY physics. 2005.

[68] Paolo Aschieri, Theodoros Grammatikopoulos, Harold Steinacker, and George Zoupanos. Dynamical generation of fuzzy extra dimensions, dimensional reduction and symmetry breaking. *JHEP*, 0609:026, 2006. See appendix 5.1.

[69] Tom Banks, W. Fischler, S.H. Shenker, and Leonard Susskind. M theory as a matrix model: A conjecture. *Phys.Rev.*, D55:5112–5128, 1997.

[70] Daniel N. Kabat and Washington Taylor. Spherical membranes in matrix theory. *Adv.Theor.Math.Phys.*, 2:181–206, 1998.

[71] Soo-Jong Rey. Gravitating M(atrix) Q balls. 1997.

[72] B. de Wit, J. Hoppe, and H. Nicolai. On the quantum mechanics of supermembranes. *Nucl.Phys.*, B305:545, 1988.

[73] P.A. Collins and R.W. Tucker. Classical and quantum mechanics of free relativistic membranes. *Nucl.Phys.*, B112:150, 1976.

[74] Harold Steinacker, private communication.

[75] Leonard Susskind. Holography in the flat space limit. pages 98–112, 1998.

[76] Joseph Polchinski. M theory and the light cone. *Prog.Theor.Phys.Suppl.*, 134:158–170, 1999.

[77] Daniel Kabat, Gilad Lifschytz, Shubho Roy, and Debajyoti Sarkar. Holographic representation of bulk fields with spin in AdS/CFT. *Phys.Rev.*, D86:026004, 2012.

[78] Vijay Balasubramanian, Per Kraus, and Albion E. Lawrence. Bulk vs. boundary dynamics in anti-de Sitter spacetime. *Phys. Rev.*, D59:046003, 1999.

[79] Tom Banks, Michael R. Douglas, Gary T. Horowitz, and Emil J. Martinec. AdS dynamics from conformal field theory. 1998.

[80] V. K. Dobrev. Intertwining operator realization of the AdS/CFT correspondence. *Nucl. Phys.*, B553:559–582, 1999.

[81] Iosif Bena. On the construction of local fields in the bulk of AdS(5) and other spaces. *Phys. Rev.*, D62:066007, 2000.

[82] Alex Hamilton, Daniel Kabat, Gilad Lifschytz, and David A. Lowe. Local bulk operators in ads/cft: A boundary view of horizons and locality. *Phys. Rev.*, D73:086003, 2006.

[83] Alex Hamilton, Daniel Kabat, Gilad Lifschytz, and David A. Lowe. Holographic representation of local bulk operators. *Phys. Rev.*, D74:066009, 2006.

[84] Alex Hamilton, Daniel Kabat, Gilad Lifschytz, and David A. Lowe. Local bulk operators in ads/cft: A holographic description of the black hole interior. *Phys. Rev.*, D75:106001, 2007.

[85] Daniel Kabat, Gilad Lifschytz, and David A. Lowe. Constructing local bulk observables in interacting AdS/CFT. *Phys.Rev.*, D83:106009, 2011.

[86] Daniel Kabat and Gilad Lifschytz. CFT representation of interacting bulk gauge fields in AdS. *Phys.Rev.*, D87:086004, 2013.

[87] Idse Heemskerk, Donald Marolf, and Joseph Polchinski. Bulk and Transhorizon Measurements in AdS/CFT. 2012.

[88] Sergei Dubovsky, Alberto Nicolis, Enrico Trincherini, and Giovanni Villadoro. Microcausality in curved space-time. *Phys.Rev.*, D77:084016, 2008.

[89] Idse Heemskerk. Construction of Bulk Fields with Gauge Redundancy. 2012.

[90] Peter Breitenlohner and Daniel Z. Freedman. Stability in Gauged Extended Supergravity. *Annals Phys.*, 144:249, 1982.

[91] Kristan Jensen. Chiral anomalies and AdS/CMT in two dimensions. *JHEP*, 1101:109, 2011.

[92] Donald Marolf and Simon F. Ross. Boundary Conditions and New Dualities: Vector Fields in AdS/CFT. *JHEP*, 0611:085, 2006.

[93] Edward Witten. SL(2,Z) action on three-dimensional conformal field theories with Abelian symmetry. 2003.

[94] Wolfgang Mueck and K.S. Viswanathan. Conformal field theory correlators from classical field theory on anti-de Sitter space. 2. Vector and spinor fields. *Phys.Rev.*, D58:106006, 1998.

[95] Tomas Andrade and Donald Marolf. AdS/CFT beyond the unitarity bound. *JHEP*, 1201:049, 2012.

[96] Tomas Andrade, Juan I. Jottar, and Robert G. Leigh. Boundary Conditions and Unitarity: the Maxwell-Chern-Simons System in AdS3/CFT2. 2011.

[97] Daniel Kabat and Gilad Lifschytz. Decoding the hologram: Scalar fields interacting with gravity. 2013.

[98] C. Fefferman and C. R. Graham. *Conformal Invariants.* Asterisque, 1985. In *Elie Cartan et les Math ?matiques d?Aujourdhui.*

[99] M. Henningson and K. Skenderis. The Holographic Weyl anomaly. *JHEP*, 9807:023, 1998.

[100] Stanley Deser and R. Jackiw. Three-Dimensional Cosmological Gravity: Dynamics of Constant Curvature. *Annals Phys.*, 153:405–416, 1984.

[101] J. Erdmenger and H. Osborn. Conserved currents and the energy momentum tensor in conformally invariant theories for general dimensions. *Nucl.Phys.*, B483:431–474, 1997.

[102] Gilad Lifschytz and Vipul Periwal. Schwinger-Dyson = Wheeler-DeWitt: Gauge theory observables as bulk operators. *JHEP*, 04:026, 2000.

[103] Idse Heemskerk and Joseph Polchinski. Holographic and Wilsonian Renormalization Groups. *JHEP*, 06:031, 2011.

[104] Joao Penedones. Writing CFT correlation functions as AdS scattering amplitudes. *JHEP*, 1103:025, 2011.

[105] Idse Heemskerk, Joao Penedones, Joseph Polchinski, and James Sully. Holography from Conformal Field Theory. *JHEP*, 0910:079, 2009.

[106] Idse Heemskerk and James Sully. More Holography from Conformal Field Theory. *JHEP*, 1009:099, 2010.

[107] A. Liam Fitzpatrick and Jared Kaplan. AdS Field Theory from Conformal Field Theory. *JHEP*, 1302:054, 2013.

[108] Ahmed Almheiri, Donald Marolf, Joseph Polchinski, Douglas Stanford, and James Sully. An Apologia for Firewalls. *JHEP*, 1309:018, 2013.

[109] Kyriakos Papadodimas and Suvrat Raju. An Infalling Observer in AdS/CFT. 2012.

[110] Kyriakos Papadodimas and Suvrat Raju. The Black Hole Interior in AdS/CFT and the Information Paradox. 2013.

[111] Kyriakos Papadodimas and Suvrat Raju. State-Dependent Bulk-Boundary Maps and Black Hole Complementarity. 2013.

[112] Juan Maldacena and Leonard Susskind. Cool horizons for entangled black holes. 2013.

[113] Andrew Strominger. The dS / CFT correspondence. *JHEP*, 0110:034, 2001.

[114] Dionysios Anninos, Thomas Hartman, and Andrew Strominger. Higher Spin Realization of the dS/CFT Correspondence. 2011.

[115] Lisa Dyson, James Lindesay, and Leonard Susskind. Is there really a de Sitter/CFT duality? *JHEP*, 0208:045, 2002.

[116] Naureen Goheer, Matthew Kleban, and Leonard Susskind. The Trouble with de Sitter space. *JHEP*, 0307:056, 2003.

[117] Michael R. Douglas, Daniel N. Kabat, Philippe Pouliot, and Stephen H. Shenker. D-branes and short distances in string theory. *Nucl.Phys.*, B485:85–127, 1997.

[118] S. M. Barnett and P. M. Radmore. *Methods in theoretical quantum optics.* Oxford Univ. Press, 1997.

[119] Per Kraus. Lectures on black holes and the AdS(3) / CFT(2) correspondence. *Lect.Notes Phys.*, 755:193–247, 2008.

[120] Thomas Heinzl. Light cone quantization: Foundations and applications. *Lect.Notes Phys.*, 572:55–142, 2001.

[121] L.D. Faddeev and R. Jackiw. Hamiltonian Reduction of Unconstrained and Constrained Systems. *Phys.Rev.Lett.*, 60:1692, 1988.

Publications

[1] Daniel Kabat, Gilad Lifschytz, Shubho Roy, and Debajyoti Sarkar. Holographic representation of bulk fields with spin in AdS/CFT. *Phys.Rev.*, D86:026004, 2012.

[2] Daniel Kabat and Debajyoti Sarkar. Cosmic string interactions induced by gauge and scalar fields. *Phys.Rev.*, D86:084021, 2012.

[3] Dario Capasso and Debajyoti Sarkar. Spinning Fluids: A Group Theoretical Approach. 2012.

[4] Norihiro Iizuka, Daniel Kabat, Shubho Roy, and Debajyoti Sarkar. Black Hole Formation in Fuzzy Sphere Collapse. *Phys.Rev.*, D88:044019, 2013.

[5] Norihiro Iizuka, Daniel Kabat, Shubho Roy, and Debajyoti Sarkar. Black Hole Formation at the Correspondence Point. *Phys.Rev.*, D87:126010, 2013.