ON THE RELATION BETWEEN GEOMETRICAL QUANTUM MECHANICS AND INFORMATION GEOMETRY

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Abstract. Let (M, g) be a compact, connected and oriented Riemannian manifold with volume form $d\mathrm{vol}_g$. We denote by $\mathcal{D}$ the space of smooth probability density functions on M, i.e. $\mathcal{D} := \{\rho \in C^\infty(M, \mathbb{R}) \,|\, \rho > 0 \text{ and } \int_M \rho \cdot d\mathrm{vol}_g = 1\}$. We regard $\mathcal{D}$ as an infinite dimensional manifold.

In this paper, we consider the almost Hermitian structure on $T\mathcal{D}$ associated, via Dombrowski's construction, to the Wasserstein metric $g^{\mathcal{D}}$ and a natural connection $\nabla^{\mathcal{D}}$ on $\mathcal{D}$. Using geometric mechanical methods, we show that the corresponding fundamental 2-form on $T\mathcal{D}$ leads to the Schrödinger equation for a quantum particle living in M. Geometrically, we exhibit a map which pulls back the Fubini-Study symplectic form to the 2-form on $T\mathcal{D}$. The integrability of the almost complex structure on $T\mathcal{D}$ is also discussed.

These results echo other papers of the author where it is stressed that the Fisher metric and exponential connection are related (via Dombrowski's construction) to Kähler geometry and the quantum formalism in finite dimension.

1. **Introduction – reconstruction of quantum mechanics.** Over the past two decades, the rapid development of quantum information theory has shed new light on the foundations of quantum mechanics, and has initiated a paradigm shift : instead of trying to *interpret* the "strangeness" of the quantum formalism (Bohmian interpretation [22], Consistent Histories [28], Many-Worlds [77], Transactional [17], etc.) one now focuses on *reconstructing*[1] the theory from a set of simple and physically motivated principles [30]. In Rovelli's words [70],

"...quantum mechanics will cease to look puzzling only when we will be able to derive the formalism of the theory from a set of simple physical assertions ("postulates", "principles") about the world. Therefore, we should not try to append a

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[1]Following Grinbaum [30], a reconstruction of a physical theory is a three step process : (1) give a set of physical principles, (2) formulate their mathematical representations, (3) derive from them the formalism of the theory. In his classic paper *Quantum Theory From Five Reasonable Axioms* [32], Hardly proposed such a reconstruction (of quantum mechanics) which greatly influenced further works on quantum foundations, and largely contributed to initiate the present wave of quantum reconstructions (see, for example, [19, 51, 72] and references therein).

reasonable interpretation to the quantum mechanics formalism, but rather to derive the formalism from a set of experimentally motivated postulates."

The first axiomatic derivation of quantum mechanics goes back to the late 20's with the work of von Neumann [35, 62, 63, 64, 65], quickly followed by many others, among which [36, 42, 46, 69, 73, 78, 84] (see also [30] for a nice discussion). It was however at the cost of highly abstract mathematical hypotheses of obscure physical origins. The novelty, with the advent of information theory, is the possibility –or at least the belief– that quantum mechanics could be derived from a set of "crisp" and "compelling" fundamental postulates originating in the notion of information [23, 70, 80]. By now, several reconstructions, based on information-theoretical principles, have successfully reproduced the full-blown structure of quantum mechanics *in finite dimension* [18, 15, 29, 12, 26, 27, 51, 70]. They all have their own merits and respective successes, but to our knowledge, no consensus has emerged yet.

In our paper [58] (see also [57, 59]), we touched upon the problem of reconstructing quantum mechanics, but from a slightly different perspective. We do not provide a list of postulates from which we derive the quantum formalism, nor do we propose a "methatheory" having quantum mechanics as a particular case. No, our approach is more basic and consists in a geometric construction, that we call *Kählerification*, which associates to a given statistical model (of exponential type) a Kähler manifold (see our discussion below and Section 2). For example, if $\Omega := \{x_1, ..., x_n\}$ is a finite set and if $\mathcal{P}_n^\times$ is the space of probabilities $p : \Omega \to \mathbb{R}$, $p > 0$, $\sum_{k=1}^n p(x_k) = 1$, defined over Ω, then the corresponding Kählerification is an open dense subset of the complex projective space $\mathbb{P}(\mathbb{C}^n)$, the latter being recovered as a natural completion (see Example 2 and [58]).

Many authors have stressed the importance of Kähler geometry in relation to the quantum formalism [13, 14, 33, 34, 40]. As it has been shown, a quantum system, with Hilbert space $\mathbb{C}^n$, can be entirely described by means of the Kähler structure of $\mathbb{P}(\mathbb{C}^n)$; this is the so-called *geometrical formulation of quantum mechanics* [5]. Hence, by realizing $\mathbb{P}(\mathbb{C}^n)$ as a natural completion of the Kählerification of $\mathcal{P}_n^\times$, one obtains a kind of geometric reconstruction of the quantum state space in finite dimension, whose starting point is a purely statistical object, $\mathcal{P}_n^\times$. Schematically :

$$
\begin{array}{ccc}
\mathcal{P}_n^\times & \xrightarrow[\text{+ completion}]{\text{Kählerification}} & \mathbb{P}(\mathbb{C}^n) \\
\updownarrow & & \updownarrow \\
\text{Statistics} & \xrightarrow{\text{"reconstruction"}} & \text{Quantum state space}
\end{array}
\qquad (1)
$$

In [58], we carefully analysed the quantum formalism in the light of the Kählerification construction and observed that, in finite dimension, all the ingredients of the geometrical formulation of quantum mechanics (quantum state space, observables, probabilistic interpretation, etc.) can be expressed in terms of the statistical structure of $\mathcal{P}_n^\times$ (+ completion arguments). This is a crucial observation, for it allows to somewhat "enlarge" the geometrical formulation of quantum mechanics and gives new geometrical insight. For example, by Kählerifying the space $\mathcal{B}(n)$ of binomial distributions $p(k) = \binom{n}{k}q^k(1-q)^{n-k}$, $q \in]0,1[$, defined over $\{0, ..., n\}$ (see Example 3), we were naturally led to a characterization of the so-called *spin coherent*

states[2] in terms of the *Veronese embedding* $S^2 \hookrightarrow \mathbb{P}(\mathbb{C}^{n+1})$, the latter map being well-known in the context of algebraic geometry (see [9, 58]).

Regarding the reconstruction problem, what is important for us is that, altogether, the Kählerification construction and its consequences strongly suggest that *the quantum formalism is grounded on the Kähler geometry which naturally emerges from statistics* (at least in finite dimension). As such, it is clear that this won't lead us directly to the fundamental principles of quantum mechanics (if they even exist!), but at least it provides clues on what to focus on, namely the relation between Kähler geometry and statistics.

Although it is not our purpose, let us briefly discuss one of these possibilities. As we will explain shortly, the Kählerification construction takes as a starting point the natural geometry that every statistical manifold possesses (incarnated by the Fisher metric and α-connections), and one surprising fact about this geometry is that it is unique due to statistical constraints ("invariance with respect to sufficient statistics"). This is the so-called Chentsov Theorem [3, 6, 10]. Hence, the Kähler geometry that naturally emerges in statistics –and which is intimately related to the quantum formalism– finds its origin in an invariance principle. What is the physical meaning of this? It is likely that a reconstruction should answer such a question.

To our knowledge, a reconstruction of the quantum formalism following the above line of thoughts has still to be undertaken (see however [83] for earlier partial results in this direction).

In the present paper, our objective is not to pursue further the reconstruction problem, but rather to extend some aspects of the Kählerification construction to the *infinite dimensional* case, through a particular but important example. Our quantum system is now a non-relativistic quantum particle, mathematically represented by a wave function $\psi : M \to \mathbb{C}$, living on a compact and connected Riemannian manifold (M, g), and whose dynamics is governed by the Schrödinger equation

$$i\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\,\Delta\,\psi + V\psi\,. \tag{2}$$

Here $\hbar$ is the Planck constant, Δ is the Laplacian operator and $V : M \to \mathbb{R}$ is a given potential.

To this system, we attach, as a statistical model, the space $\mathcal{D}$ of smooth density probability functions on M :

$$\mathcal{D} := \left\{\rho \in C^\infty(M, \mathbb{R}) \,\middle|\, \rho > 0\,,\ \int_M \rho\,d\mathrm{vol}_g = 1\right\}, \tag{3}$$

where $d\mathrm{vol}_g$ denotes the Riemannian volume form associated to g (M is assumed oriented). We regard the space $\mathcal{D}$ as an infinite dimensional analog of $\mathcal{P}_n^\times$.

Following the "Kählerification point of view", we would like to exploit the statistical nature of $\mathcal{D}$ and to associate to $\mathcal{D}$ an infinite dimensional Kähler manifold, or more generally an almost Hermitian manifold, whose geometry would entail the

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[2] Spin coherent states are a particular case of what physicists call coherent states, historically discovered in 1926 by Schrödinger in relation to the quantum harmonic oscillator [71], and later on rediscovered by Glauber [24] who used them to explain coherence phenomena in quantum optics (for example laser light can be thought of as an appropriate coherent state). Nowadays, the concept of coherent states has been generalized in various directions, leading to many non-equivalent definitions (see for example [2, 16, 41, 68]).

quantum aspects of our particle (for example the dynamics), like in the finite dimensional case. Since it is the core of this paper, it is necessary, at this point, to discuss in more detail the Kählerification construction and its link with information theory (see Section 2).

The Kählerification construction is a slight refinement of a geometric construction which already goes back to the 60's with the work of Dombrowski [21]. In his paper, Dombrowski shows that if a manifold M is endowed with a Riemannian metric h and an affine connection ∇ (not necessarily the Levi-Civita connection), then its tangent bundle TM becomes naturally an almost Hermitian manifold. Also, if the couple (h, ∇) meets some analytical conditions, then this Hermitian structure is actually Kähler (see Proposition 2).

In the context of statistics, and more specifically in *information geometry*, Dombrowski's construction has important consequences. Let us recall, in this respect, that information geometry is a branch of statistics characterized by its use of differential geometrical techniques [3, 61]. Its basic objects of study are *statistical manifolds*, i.e. manifolds whose points can be identified with probability density functions over some fixed measured space. For example, Gaussian distributions over $\mathbb{R}$ form a 2-dimensional statistical manifold parameterized by the mean μ and deviation σ. In general –and this is what information geometry is about– a statistical manifold S possesses a rich geometry that encodes many of its statistical properties. It has a Riemannian metric h_F, called *Fisher metric*, and a couple of dual connections $\nabla^{(e)}, \nabla^{(m)}$, respectively called *exponential connection* and *mixture connection*, which can be used, for example, to give lower bounds in estimation problems (compare e.g. the Cramér-Rao inequality). Together, $(h_F, \nabla^{(e)}, \nabla^{(m)})$ forms what is called a *dualistic structure*, and it is the most important geometric structure in information geometry. As a consequence of Dombrowski's construction (applied to h_F and $\nabla^{(e)}$), the tangent bundle of a statistical manifold S is naturally an almost Hermitian manifold. If S is an *exponential family* (see Definition 2.1), then TS is even Kähler, in which case we define the Kählerification of S as the quotient of TS by the action of a discrete group which acts via holomorphic isometries (see Definition 2.2).

In the case of an infinite dimensional statistical manifold like $\mathcal{D}$, there are no intrinsic definitions for the Fisher metric h_F and the exponential connexion $\nabla^{(e)}$ (see however [39, 54]). Consequently, one cannot directly invoke Dombrowski's construction and obtain an almost Hermitian structure on $T\mathcal{D}$ in a canonical way. First, one needs to find infinite dimensional analogs of the Fisher metric and exponential connection that would be, somehow, quantum mechanically relevant. Now, the main observation of this paper is that such analogs exist (in a sense to be explained below).

In more detail, we show the following: first, that it is possible to rewrite the Schrödinger equation (2) into a genuine system of Lagrangian equations on $T\mathcal{D}$ for an appropriate Lagrangian $\mathcal{L} : T\mathcal{D} \to \mathbb{R}$. Second, that this Lagrangian system can be reformulated in a symplectic way on $T\mathcal{D}$ using geometric mechanical methods. Finally, that the corresponding symplectic form $\Omega_{\mathcal{L}}$ on $T\mathcal{D}$ is nothing but the fundamental form of the almost Hermitian structure coming from Dombrowski's construction with respect to a natural metric $g^{\mathcal{D}}$ and a connection $\nabla^{\mathcal{D}}$ living on $\mathcal{D}$.

The couple $(g^{\mathcal{D}}, \nabla^{\mathcal{D}})$ on $\mathcal{D}$ is thus –and this is the main observation of this paper– an infinite dimensional analog of $(h_F, \nabla^{(e)})$ on $\mathcal{P}_n^{\times}$ which encodes the dynamics[3], exactly as in the finite dimensional case (see [58]). Regarding the geometrical formulation of quantum mechanics[4], we show the existence of a smooth map $T : T\mathcal{D} \to \mathbb{P}(\mathcal{H})$, $\mathcal{H} := L^2(M, \mathbb{C})$, which pulls backs the Fubini-Study symplectic form on $\mathbb{P}(\mathcal{H})$ to the fundamental form of $T\mathcal{D}$ coming from Dombrowski's construction (see Proposition 10). Additionally, we observe that the almost complex structure of $T\mathcal{D}$ is not integrable and that, contrary to $\nabla^{(e)}$, the connection $\nabla^{\mathcal{D}}$ on $\mathcal{D}$ has a non-trivial torsion (this proves in particular that $\nabla^{\mathcal{D}}$ is *not* the Levi-Civita connection associated to $g^{\mathcal{D}}$).

Surprisingly[5], the Riemannian metric $g^{\mathcal{D}}$ is well-known in the context of *optimal transport theory*[6], and is called *Wasserstein metric* (see Remark 10). In contrast, we don't know if the connection $\nabla^{\mathcal{D}}$ has already been considered in the literature.

To conclude, the results of the present paper comfort some of the ideas formulated in [58] in relation to the quantum formalism, but also point toward differences. Among the similarities, we note the importance of Dombrowski's construction in relation to geometric objects having a statistical or probabilistic origin (Fisher and Wasserstein metrics, exponential connection). Among the differences, we note that in the infinite dimensional case, the Wasserstein metric replaces the Fisher metric. On one hand this is a good thing, for both metrics (Fisher and Wasserstein) are of great statistical/probabilistic significance, which should make easier their derivations from first principals. But, on the other hand, it is known that they are not equivalent (see for example [76]), which addresses the following question: what is the link between them, especially on physical grounds? Another issue is the definition of $\mathcal{D}$ which imposes the condition $\rho = |\psi|^2 > 0$, hence excluding the possibility for a wave function to have nodes. This is a serious limitation of our approach, for excited states are precisely characterized by the existence of zeros in the wave function. Finally, we don't consider the question of the observables, measurements or spectral properties (in particular in view of Remark 15).

In our opinion, the quantum formalism may very well have a statistical or informational origin, but it seems unavoidable, before envisaging seriously a reconstruction, to solve the above problems and to get a better understanding of the "natural geometries" that a statistical model, or a space of probabilities, possesses. Then, in a second step, one should clarify their links with Hermitian geometry.

This paper is organized as follows. In §2, we recall the main concepts of information geometry discussed throughout this paper, and review some of the results obtained in [58] in relation to Kähler geometry. In §3 we describe the geometry of $\mathcal{D}$ and its tangent bundle; that will allow us, in §4 and §5, to recast the Schrödinger

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[3]We stress the fact that this is about the dynamics. In this paper we don't discuss other aspects of the quantum formalism like the probabilistic interpretation or the observables.

[4]Recall that in the geometrical formulation of quantum mechanics, the dynamical aspects are determined by the Fubini-Study symplectic form on the complex projective space.

[5]We were not aware of this fact when we wrote the ArXiv version [56] of the present paper.

[6]Very roughly, optimal transport theory is an area of mathematics motivated by the following problem: *how to realize the transportation of some material (sand, soil, water,...) from one prescribed place to another, at minimal cost?* The subject originates in the work of Gaspard Monge who published in 1781 his *Mémoire sur la théorie des déblais et des remblais*, and since then, it has been rediscovered several times in many guises and different contexts (economics, computer science, mechanics,...). Nowadays, optimal transport has become a very active branch of mathematics with various and unexpected applications (see [79]).

equation directly on $T\mathcal{D}$, in a Lagrangian form (§4) and in a Hamiltonian form (§5). Finally, in §6 we observe that the symplectic form $\Omega_{\mathcal{L}}$ on $T\mathcal{D}$ describing the dynamics of the quantum particle is nothing but the fundamental form of the almost Hermitian structure associated to $(g^{\mathcal{D}}, \nabla^{\mathcal{D}})$ on $\mathcal{D}$, and discuss its link with the Fubini-Study symplectic form.

Some of our results are expressed in the category of *tame Fréchet manifolds* introduced by Hamilton in [31]. The relevant definitions are recalled in an appendix (see §7).

2. **Information geometry and Kähler structures.** In this section, we recall the basic concepts of information geometry discussed throughout this paper, and review some of the results obtained in [58] in relation to Kähler geometry.

Our brief presentation (regarding information geometry) follows the currently reference book [3] (see also [61]).

2.1. **Information geometry.** A *statistical manifold* (or *statistical model*), is a couple (S, j) where S is a manifold and where j is an injective map from S to the space of all probability density functions p defined on a fixed measured space (Ω, dx)[7] :

$$j : S \hookrightarrow \left\{ p : \Omega \to \mathbb{R} \,\middle|\, p \text{ is measurable, } p \geq 0 \text{ and } \int_{\Omega} p(x)\, dx = 1 \right\}. \qquad (4)$$

In the case of a discrete space Ω, it will be implicitly assumed that dx is the counting measure, i.e. $dx(A) = \text{card}(A)$, where $\text{card}(A)$ denotes the cardinality of a given subset $A \subset \Omega$.

As a matter of notation, if $(\xi : U \subseteq S \to \mathbb{R}^n)$ is a chart of a statistical manifold S with local coordinates $\xi = (\xi_1, ..., \xi_n)$, then we shall indistinctly write $p(x; \xi)$ or $p_{\xi}(x)$ for the probability density function determined by ξ and in the variable $x \in \Omega$.

Now, given a "reasonable" statistical manifold S, it is possible to define a metric h_F and a family of connections $\nabla^{(\alpha)}$ on S ($\alpha \in \mathbb{R}$) in the following way: for a chart $\xi = (\xi_1, ..., \xi_n)$ of S, define

- $(h_F)_{\xi}(\partial_i, \partial_j) := E_{p_{\xi}}(\partial_i \ln(p_{\xi}) \cdot \partial_j \ln(p_{\xi}))$,
- $\Gamma_{ij,k}^{(\alpha)}(\xi) := E_{p_{\xi}}\left[\left(\partial_i \partial_j \ln(p_{\xi}) + \dfrac{1-\alpha}{2} \partial_i \ln(p_{\xi}) \cdot \partial_j \ln(p_{\xi}) \right) \partial_k \ln(p_{\xi}) \right]$,

where $E_{p_{\xi}}$ denotes the mean, or expectation, with respect to the probability $p_{\xi}\, dx$, and where ∂_i is a shorthand for $\partial/\partial_{\xi_i}$.

It can be shown that if the above expressions are defined and smooth for every chart of S (this is not always the case), then h_F is a well defined metric on S called the *Fisher metric*, and that the $\Gamma_{ij,k}^{(\alpha)}$'s define a connection $\nabla^{(\alpha)}$ via the formula $\Gamma_{ij,k}^{(\alpha)}(\xi) = (h_F)_{\xi}\left(\nabla_{\partial_i}^{(\alpha)} \partial_j, \partial_k \right)$ which is called the α-*connection*. Among the α-connections, the (± 1)-connections are particularly important; the 1-connection is usually referred to as the *exponential connection*, also denoted $\nabla^{(e)}$, while the (-1)-connection is referred to as the *mixture connection*, denoted $\nabla^{(m)}$.

One particularity of the $(\pm\alpha)$-connections is that they are *dual* of each other with respect to the Fisher metric h_F, or equivalently, that they form a *dualistic structure* on S. The general definition of a dualistic structure on an arbitrary manifold M is

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[7]Depending on the symbol we use for the variable living in Ω, for example "x", "k", etc., we shall use the notation "dx", "dk", etc., for the measure on Ω.

as follows: a dualistic structure on M is a triple (h, ∇, ∇^*) where h is a Riemannian metric on M and where ∇ and ∇^* are connections satisfying

$$X\big(h(Y, Z)\big) = h\big(\nabla_X Y, Z\big) + h\big(Y, \nabla_X^* Z\big), \tag{5}$$

for all vector fields X, Y, Z on M. The connection ∇^* is called the *dual connection*, or *conjugate connection*, of the connection ∇ (and vice versa)[8].

An important class of dualistic structures is that of *dually flat structures*. A dually flat structure on a manifold M is a dualistic structure (h, ∇, ∇^*) for which both connections are flat, meaning that their torsions and curvature tensors vanish. As conventions are not uniform in the literature, let us agree that the torsion T and the curvature tensor R of a connection ∇ on M are defined as

$$\begin{aligned} T(X, Y) &:= \nabla_X Y - \nabla_Y X - [X, Y], \\ R(X, Y)Z &:= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \end{aligned} \tag{6}$$

where X, Y, Z are vector fields on M.

Let us introduce an important class of dually flat statistical manifolds.

Definition 2.1. An exponential family $\mathcal{E}$ on a measured space (Ω, dx) is a set of probability density functions $p(x; \theta)$ of the form

$$p(x; \theta) = \exp\left\{ C(x) + \sum_{i=1}^{n} \theta_i F_i(x) - \psi(\theta) \right\}, \tag{7}$$

where $C, F_1, ..., F_n$ are measurable functions on Ω, $\theta = (\theta_1, ..., \theta_n)$ is a vector varying in an open subset Θ of $\mathbb{R}^n$ and where ψ is a function defined on Θ.

In the above definition, it is assumed that the family $\{1, F_1, ..., F_n\}$ is linearly independent, so that the map $p(x, \theta) \mapsto \theta \in \Theta$ becomes a bijection, hence defining a global chart of $\mathcal{E}$.

Exponential families are found among the most common probability distributions: *Bernoulli, beta, binomial, chi-square, Dirichlet, exponential, gamma, geometric, multinomial, normal, Poisson*, to name but just a few.

Here is a more detailed example.

Example 1. For a finite set $\Omega = \{x_1, ..., x_n\}$, define

$$\mathcal{P}_n^\times := \left\{ p : \Omega \to \mathbb{R} \,\big|\, p(x) > 0 \text{ for all } x \in \Omega \text{ and } \sum_{k=1}^{n} p(x_k) = 1 \right\}. \tag{8}$$

The space $\mathcal{P}_n^\times$ is clearly a statistical manifold of dimension $n - 1$, and it can be turned into an exponential family by means of the following parameterization:

$$p(x; \theta) = \exp\left\{ \sum_{i=1}^{n-1} \theta_i F_i(x) - \psi(\theta) \right\}, \tag{9}$$

where $x \in \Omega$, $\theta = (\theta_1, ..., \theta_{n-1}) \in \mathbb{R}^{n-1}$, $F_i(x_j) = \delta_{ij}$ and where $\psi(\theta) = -\ln\big(1 + \sum_{i=1}^{n-1} \exp(\theta_i)\big)$.

Proposition 1 ([3]). *Let $\mathcal{E}$ be an exponential family such as in Definition 2.1. Then $(\mathcal{E}, h_F, \nabla^{(e)}, \nabla^{(m)})$ is dually flat.*

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[8]Given a connection ∇ on a Riemannian manifold (M, h), there exists a unique connection ∇^* on M such that (5) holds; it is thus justified to call ∇^* the dual connection of ∇.

2.2. **Dombrowski's construction.** Recall that if M is a manifold endowed with an affine connection ∇, then Dombrowski splitting Theorem holds (see [21, 44]) :

$$T(TM) \cong TM \oplus TM \oplus TM, \tag{10}$$

this splitting being viewed as an isomorphism of vector bundles over M, and the isomorphism, say Φ, being

$$T_{u_x}TM \ni A_{u_x} \overset{\Phi}{\longmapsto} \left(u_x, \pi_{*_{u_x}} A_{u_x}, K A_{u_x}\right), \tag{11}$$

where $\pi : TM \to M$ is the canonical projection and where $K : T(TM) \to TM$ is the canonical connector associated to the connection ∇ (see [44]). Observe that if $TM \cong M \times E$ is a trivial vector bundle, then the map

$$TM \oplus TM \oplus TM \to M \times E \times E \times E, \quad \left((x,u),(x,v),(x,w)\right) \mapsto (x,u,v,w) \tag{12}$$

is a vector bundle isomorphism over M.

Having $A_{u_x} = \Phi^{-1}\left((u_x, v_x, w_x)\right) \in T_{u_x}TM$, we shall write, for simplicity, $A_{u_x} = (u_x, v_x, w_x)$ instead of $\Phi^{-1}\left((u_x, v_x, w_x)\right)$, i.e., we will drop Φ. The second component v_x is usually referred to as the horizontal component of A_{u_x} (with respect to the connection ∇) and w_x the vertical component.

With the above notation, and provided that M is endowed with a Riemannian metric h, it is a simple matter to define on TM an almost Hermitian structure. Indeed, we define a metric g, a 2-form ω and an almost complex structure J by setting

$$
\begin{aligned}
g_{u_x}\left((u_x, v_x, w_x), (u_x, \overline{v}_x, \overline{w}_x)\right) &:= h_x\left(v_x, \overline{v}_x\right) + h_x\left(w_x, \overline{w}_x\right), \\
\omega_{u_x}\left((u_x, v_x, w_x), (u_x, \overline{v}_x, \overline{w}_x)\right) &:= h_x\left(v_x, \overline{w}_x\right) - h_x\left(w_x, \overline{v}_x\right), \\
J_{u_x}\left((u_x, v_x, w_x)\right) &:= \left(u_x, -w_x, v_x\right),
\end{aligned}
\tag{13}
$$

where $u_x, v_x, w_x, \overline{v}_x, \overline{w}_x \in T_xM$.

Clearly, $J^2 = -\text{Id}$ and $g(J\,.\,, J\,.\,) = g(\,.\,,\,.\,)$, which means that (TM, g, J) is an almost Hermitian manifold, and one readily sees that g, J and ω are compatible, i.e., that $\omega = g(J\,.\,,\,.\,)$; the 2-form ω is thus the fundamental 2-form of the almost Hermitian manifold (TM, g, J). This is Dombrowski's construction.

Observe that the map $\pi : (TM, g) \to (M, h)$ is a Riemannian submersion.

Proposition 2 ([21, 58]). *Let (h, ∇, ∇^*) be a dualistic structure on a manifold M and (g, J, ω) the almost Hermitian structure on TM associated to (h, ∇) via Dombrowski's construction. Then,*

$$J \text{ is integrable} \quad \Leftrightarrow \quad \nabla \text{ is flat}. \tag{14}$$

In this case (i.e. ∇ is flat or J is integrable), then we also have

$$d\omega = 0 \quad \Leftrightarrow \quad T^* = 0, \tag{15}$$

where T^* denotes the torsion of ∇^*.

Recall that an almost Hermitian structure (g, J, ω) on a given manifold is Kähler when the following two analytical conditions are met: (1) J is integrable; (2) $d\omega = 0$. Having this in mind, Proposition 2 readily implies the following corollaries.

Corollary 1. *Let (h, ∇, ∇^*) be a dualistic structure on a manifold M and (g, J, ω) the almost Hermitian structure on TM associated to (h, ∇) via Dombrowski's construction. Then,*

$$(TM, g, J, \omega) \text{ is Kähler} \quad \Leftrightarrow \quad (M, h, \nabla, \nabla^*) \text{ is dually flat}. \tag{16}$$

Corollary 2. *The tangent bundle $T\mathcal{E}$ of an exponential family $\mathcal{E}$ is a Kähler manifold for the Kähler structure (g, J, ω) associated to $(h_F, \nabla^{(e)})$ via Dombrowski's construction.*

Let us now discuss the Kählerification construction. Let $\mathcal{E}$ be an exponential family and let (g, J, ω) be the Kähler structure of $T\mathcal{E}$. We define a subgroup $\Gamma(\mathcal{E})$ of the group of all diffeomorphisms $\mathrm{Diff}(T\mathcal{E})$ of $T\mathcal{E}$ by letting

$$\Gamma(\mathcal{E}) \quad := \quad \big\{ \phi \in \mathrm{Diff}(T\mathcal{E}) \,\big|\, \phi^* g = g \,,\ \phi_* J = J \phi_* \text{ and}$$
$$f \circ \phi = f \text{ for all } f \in \mathscr{K}(T\mathcal{E}) \big\}, \qquad (17)$$

where $\mathscr{K}(T\mathcal{E})$ is the space of smooth functions $f : T\mathcal{E} \to \mathbb{R}$ whose associated Hamiltonian vector field X_f is Killing with respect to g.

Definition 2.2 ([58])**.** Let $\mathcal{E}$ be an exponential family having a discrete $\Gamma(\mathcal{E})$ and whose natural action[9] on $T\mathcal{E}$ is free and proper. The quotient space $T\mathcal{E}/\Gamma(\mathcal{E})$ is thus naturally a Kähler manifold for which the quotient map $T\mathcal{E} \to T\mathcal{E}/\Gamma(\mathcal{E})$ becomes a holomorphic Riemannian submersion. We shall call this quotient the Kählerification of $\mathcal{E}$, and use the following notation:

$$\mathcal{E}^{\mathbb{C}} := T\mathcal{E}/\Gamma(\mathcal{E}) \,. \qquad (18)$$

Under reasonable hypotheses, one can show that a Kählerification comes with a natural Riemannian submersion $\pi_{\mathcal{E}} : \mathcal{E}^{\mathbb{C}} \to \mathcal{E}$.

Example 2. Let $\mathcal{P}_n^{\times}$ be the statistical manifold defined in Example 1. For an appropriate normalization of the Fubini-Study metric and symplectic form, we have a natural identification of Kähler manifolds:

$$(\mathcal{P}_n^{\times})^{\mathbb{C}} \cong \mathbb{P}(\mathbb{C}^n)^{\times} \,, \qquad (19)$$

where $\mathbb{P}(\mathbb{C}^n)^{\times} := \big\{ [z_1, ..., z_n] \in \mathbb{P}(\mathbb{C}^n) \,|\, z_k \neq 0 \text{ for all } k = 1, ..., n \big\}$ (we use homogeneous coordinates). Moreover, in terms of the above identification, the canonical projection $\pi_{\mathcal{P}_n^{\times}} : (\mathcal{P}_n^{\times})^{\mathbb{C}} \to \mathcal{P}_n^{\times}$ becomes

$$\pi_{\mathcal{P}_n^{\times}} : \mathbb{P}(\mathbb{C}^n)^{\times} \to \mathcal{P}_n^{\times} \,, \quad \pi_{\mathcal{P}_n^{\times}}([z])(x_k) := \frac{z_k \overline{z_k}}{\langle z, z \rangle} \,, \qquad (20)$$

where $\langle z, w \rangle = \sum_{k=1}^{n} \overline{z}_k w_k$ is the Hermitian product on $\mathbb{C}^n$.

Example 3. Let $\mathcal{B}(n)$ be the space of binomial distributions defined over $\Omega := \{0, ..., n\}$, i.e.,

$$p \in \mathcal{B}(n) \ \Leftrightarrow\ \exists q \in \,]0, 1[\,:\ p(k) = \frac{n!}{(n-k)!k!}\, q^k (1-q)^{n-k}, \ k = 0, ..., n. \qquad (21)$$

The set of binomial distributions forms a 1-dimensional statistical manifold (parameterized by q) and is easily seen to be an exponential family. Let $S^2 := \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^3 = 1\}$ be the unit sphere in $\mathbb{R}^3$. If the natural Kähler structure of S^2 is multiplied by n, then

$$\mathcal{B}(n)^{\mathbb{C}} \cong (S^2)^{\times} \,, \qquad (22)$$

where $(S^2)^{\times} := S^2 - \{(1, 0, 0), (-1, 0, 0)\}$. Moreover, in terms of the above identification, the map $\pi_{\mathcal{B}(n)} : \mathcal{B}(n)^{\mathbb{C}} \to \mathcal{B}(n)$ becomes

$$\pi_{\mathcal{B}(n)} : (S^2)^{\times} \to \mathcal{B}(n) \,, \quad \pi_{\mathcal{B}(n)}(x, y, z)(k) = \frac{1}{2^n} \binom{n}{k} (1+x)^k (1-x)^{n-k} \,. \qquad (23)$$

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[9]The natural action of $\Gamma(\mathcal{E})$ on $T\mathcal{E}$ is simply given by $\gamma \cdot u_x := \gamma(u_x)$, where $\gamma \in \Gamma(\mathcal{E})$ and $u_x \in T\mathcal{E}$.

Example 4. Let $\mathcal{N}(\mu, 1)$ be the set of all probability density functions defined over $\Omega := \mathbb{R}$ by

$$p(\xi; \mu) := \frac{1}{\sqrt{2\pi}} \exp\left\{ - \frac{(\mu - \xi)^2}{2} \right\}, \tag{24}$$

where $\xi \in \Omega$ and $\mu \in \mathbb{R}$ ($\mathbb{R}$ is endowed with the Lebesgue measure). One can show that $\mathcal{N}(\mu, 1)$ is an exponential family, and that,

$$\mathcal{N}(\mu, 1)^{\mathbb{C}} \cong \mathbb{C}. \tag{25}$$

Moreover, the canonical projection $\pi_{\mathcal{N}(\mu,1)} : \mathbb{C} \to \mathcal{N}(\mu, 1)$ is easily seen to be

$$\pi_{\mathcal{N}(\mu,1)} : \mathbb{C} \to \mathcal{N}(\mu, 1), \quad \pi_{\mathcal{N}(\mu,1)}(z)(\xi) = \frac{1}{\sqrt{2\pi}} \exp\left\{ - \frac{(x - \xi)^2}{2} \right\}, \tag{26}$$

where $z = x + iy \in \mathbb{C}$.

3. **The manifold structure of $\mathcal{D}$ and its tangent bundle.** Let (M, g) be a compact, connected and oriented Riemannian manifold with Riemannian volume form $d\mathrm{vol}_g$, and let $\mathcal{D}$ be the space of smooth density probability functions on M:

$$\mathcal{D} := \left\{ \rho \in C^\infty(M, \mathbb{R}) \,\middle|\, \rho > 0, \; \int_M \rho \, d\mathrm{vol}_g = 1 \right\}. \tag{27}$$

Throughout this section, we shall use the category of *tame Fréchet manifolds* introduced by Hamilton in [31]. The relevant definitions are recalled in §7.

Regarding the notation, we shall write $C^\infty(M)$ instead of $C^\infty(M, \mathbb{R})$ (and similar for subspaces of $C^\infty(M, \mathbb{R})$), and use the symbol $\mathbb{R}^*_+ := \{r \in \mathbb{R} \,|\, r > 0\}$.

Let us start with the differentiable structure of $\mathcal{D}$.

Proposition 3. *The space $\mathcal{D}$ is a tame Fréchet submanifold of the tame Fréchet space $C^\infty(M)$, and for $\rho \in \mathcal{D}$,*

$$T_\rho \mathcal{D} \cong C_0^\infty(M), \tag{28}$$

where

$$C_0^\infty(M) := \left\{ f \in C^\infty(M) \,\middle|\, \int_M f \, d\mathrm{vol}_g = 0 \right\}. \tag{29}$$

Observe that we have the following L^2-orthogonal decomposition,

$$C^\infty(M) = C_0^\infty(M) \oplus \mathbb{R}, \tag{30}$$

the decomposition being given, for $f \in C^\infty(M)$, by

$$f = f - \frac{1}{\mathrm{Vol}(M)} \int_M f \cdot d\mathrm{vol}_g + \frac{1}{\mathrm{Vol}(M)} \int_M f \cdot d\mathrm{vol}_g, \tag{31}$$

where $\mathrm{Vol}(M) := \int_M d\mathrm{vol}_g$ denotes the Riemannian volume of M. In particular, the space $C_0^\infty(M)$ is a tame Fréchet space (it is a Fréchet space because $C_0^\infty(M)$ is closed in $C^\infty(M)$ and it is also a tame space because $C^\infty(M)$ is tame, see [31], Definition 1.3.1 and Corollary 1.3.9).

Proof of Proposition 3. The proof relies on the following tame diffeomorphim of tame Féchet manifolds:

$$\Phi : \left\{ \begin{array}{l} C^\infty(M) \to C_0^\infty(M) \times \mathbb{R}, \\ f \mapsto \left(f - \mathrm{Vol}(M)^{-1} \int_M f \, d\mathrm{vol}_g, \mathrm{Vol}(M)^{-1} \int_M f \, d\mathrm{vol}_g - \mathrm{Vol}(M)^{-1} \right). \end{array} \right. \tag{32}$$

Using Φ, it is possible to define splitting charts for $C^\infty(M)$; indeed, the space $C^\infty(M, \mathbb{R}^*_+)$ being clearly an open subset of $C^\infty(M)$ for its natural Fréchet space

topology, every $\rho \in \mathcal{D}$ possesses an open neighborhood in $C^\infty(M)$, say U_ρ, such that $\rho \in U_\rho \subseteq C^\infty(M, \mathbb{R}_+^*)$, and, restricting U_ρ if necessary, we may assume that $\Phi(U_\rho) = V_\rho \times W_\rho$ where V_ρ and W_ρ are open subsets of $C_0^\infty(M)$ and $\mathbb{R}$ respectively. But now, $(U_\rho, \Phi|_{U_\rho})$ is a chart of $C^\infty(M)$ and it is easy to see that

$$(\Phi|_{U_\rho})(U_\rho \cap \mathcal{D}) = V_\rho \times \{0\}. \tag{33}$$

The proposition follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now want to give a geometrical description of the tangent space of $\mathcal{D}$. Recall that if $X \in \mathfrak{X}(M)$ is a vector field on M, then its divergence with respect to the volume form $d\mathrm{vol}_g$ is the unique function $\mathrm{div}(X) : M \to \mathbb{R}$ satisfying $\mathscr{L}_X(d\mathrm{vol}_g) = \mathrm{div}(X) \cdot d\mathrm{vol}_g$, $\mathscr{L}_X$ being the Lie derivative in direction X.

Using the divergence operator, we define, for $f : M \to \mathbb{R}_+^*$, an elliptic differential operator $\mathrm{P}_f : C^\infty(M) \to C^\infty(M)$ via the formula

$$\mathrm{P}_f(u) := \mathrm{div}(f \cdot \nabla u), \tag{34}$$

where $u : M \to \mathbb{R}$ is a smooth function. Observe that

- $\mathrm{P}_1 = \Delta$ is the Laplacian operator,
- P_f takes values in $C_0^\infty(M)$ since the integral with respect to the Riemannian volume form of a divergence is always zero by application of Stokes' Theorem.
- The kernel of P_f reduces to the constant functions. This is due to the fact that P_f is a second order elliptic differential operator whose constant term $\mathrm{P}_f(1)$ is zero, and it is well-known that for such differential operators on compact manifolds, the kernel reduces to the constant functions (see [37]).

Lemma 3.1. *For $f \in C^\infty(M)$, $f > 0$, the restriction $\overline{\mathrm{P}}_f$ of the operator P_f to $C_0^\infty(M)$,*

$$\overline{\mathrm{P}}_f : C_0^\infty(M) \to C_0^\infty(M), \tag{35}$$

is an isomorphism of Fréchet spaces. Moreover, its family of inverses

$$C^\infty(M, \mathbb{R}_+^*) \times C_0^\infty(M) \to C_0^\infty(M), \ (f, h) \mapsto (\overline{\mathrm{P}}_f)^{-1}(h) \tag{36}$$

forms a smooth tame map.

Proof. The operator $\overline{\mathrm{P}}_f$ is injective since its kernel is the intersection of the kernel of P_f with the space $C_0^\infty(M)$, which is zero.

For the surjectivity, take $\tilde{f} : [0, 1] \to C^\infty(M, \mathbb{R}_+^*)$ a continuous path such that $\tilde{f}_0 \equiv 1$ and $\tilde{f}_1 = f$. As one may see, $\mathrm{P}_{\tilde{f}_t}$ defines a continuous path of elliptic operators (acting on a suitable Sobolev space), and by the topological invariance of the analytic index Ind of an elliptic operator together with the fact that the analytic index of $\Delta : C^\infty(M) \to C^\infty(M)$ is zero, we have :

$$\mathrm{Ind}(\mathrm{P}_f) = \mathrm{Ind}(\mathrm{P}_{\tilde{f}_1}) = \mathrm{Ind}(\mathrm{P}_{\tilde{f}_0}) = \mathrm{Ind}(\Delta) = 0. \tag{37}$$

Hence, the codimension of the image $\mathrm{Im}(\mathrm{P}_f)$ of P_f is 1, and since $\mathrm{Im}(\mathrm{P}_f) \subseteq C_0^\infty(M)$, this later space being of codimension 1, $\mathrm{Im}(\mathrm{P}_f) = C_0^\infty(M)$. It follows that $\overline{\mathrm{P}}_f : C_0^\infty(M) \to C_0^\infty(M)$ is a bijection.

Finally, $\overline{\mathrm{P}}_f$ is continuous since it is a differential operator, and its inverse is also continuous by application of the open mapping Theorem.

The fact that the family of inverses defined in (36) forms a smooth tame map is a consequence of a result due to Hamilton (see [31], Theorem 3.3.3) about the family

of inverses of a family of invertible (up to something of finite dimension) elliptic differential operators, applied to the following map :

$$\begin{cases} C^\infty(M, \mathbb{R}_+^*) \times C^\infty(M) \times \mathbb{R} \to C^\infty(M) \times \mathbb{R}\,, \\ \quad (f, h, x) \mapsto \left(\mathrm{P}_f(h) + x, \int_M h\, d\mathrm{vol}_g \right). \end{cases} \tag{38}$$

The result of Hamilton implies the existence of a smooth Green operator G : $C^\infty(M, \mathbb{R}_+^*) \times C^\infty(M) \to C_0^\infty(M)$ whose restriction to $C^\infty(M, \mathbb{R}_+^*) \times C_0^\infty(M)$ coincides with the family considered in (36). The lemma follows. $\qquad\square$

Remark 1. Another way to prove Lemma 3.1 without invoking the tame category of Hamilton would be to extend the operator field P_f as an operator between Sobolev spaces $P_f : H^s(M) \to H^{s-2}(M)$, to show the desired properties of this extension for any s, and then to use the fact that $\bigcap_s H^s(M) = C^\infty(M)$. For more details on this type of reasoning, see for example [66] and Remark 17.

Proposition 4. *Let $X \in \mathfrak{X}(M)$ be a vector field and let $\rho \in \mathcal{D}$ be a smooth density. For $h \in T_\rho \mathcal{D} \cong C_0^\infty(M)$, there exists a unique function $\phi : M \to \mathbb{R}$ (defined up to an additive constant), such that*

$$h = \mathrm{div}\left(\rho \left(\nabla\phi + X \right) \right). \tag{39}$$

Moreover, the map

$$T\mathcal{D} \to \mathcal{D} \times \nabla C^\infty(M),\ h = \mathrm{div}\left(\rho \left(\nabla\phi + X \right) \right) \mapsto (\rho, \nabla\phi)\,, \tag{40}$$

is a non-linear tame isomorphism of tame Fréchet vector bundles, $\mathcal{D} \times \nabla C^\infty(M)$ being the trivial vector bundle over $\mathcal{D}$.

Proof. For $\rho \in \mathcal{D}$ and $h \in T_\rho \mathcal{D} \cong C_0^\infty(M)$, define $\phi \in C_0^\infty(M)$ by letting

$$\phi := (\overline{P}_\rho)^{-1}\left[h - \mathrm{div}(\rho\, X) \right] \tag{41}$$

(note that $\mathrm{div}(\rho\, X) \in C_0^\infty(M)$, and thus $h - \mathrm{div}(\rho\, X) \in C_0^\infty(M)$).

By applying the operator $\overline{P}_\rho$ to (41), we see that

$$\begin{aligned} \overline{P}_\rho(\phi) = h - \mathrm{div}(\rho\, X) \quad &\Rightarrow \quad \mathrm{div}(\rho\, \nabla\phi) = h - \mathrm{div}(\rho\, X) \\ &\Rightarrow \quad h = \mathrm{div}\left(\rho \left(\nabla\phi + X \right) \right). \end{aligned} \tag{42}$$

Moreover, if $\phi' : M \to \mathbb{R}$ satisfies $h = \mathrm{div}\left(\rho \left(\nabla\phi' + X \right)\right.$, then $P_\rho(\phi - \phi') = 0$, and thus, $\phi - \phi'$ is a constant function. The first assertion of the proposition follows.

For the second assertion, it is clear that the map defined in (40) is a fiber preserving bijection; its smoothness is a consequence of the smoothness of the family of inverses (36) (that one may apply in charts such as defined in the proof of Proposition 3, or directly using the convenient calculus developed in [43]); this map is also tame for the same reason and its inverse is clearly a smooth tame map. The proposition follows. $\qquad\square$

Remark 2. The space of all gradients $\nabla C^\infty(M)$ is a tame Fréchet space. This comes from the fact that the Helmholtz-Hodge decomposition

$$\mathfrak{X}(M) = \mathfrak{X}_{d\mathrm{vol}_g}(M) \oplus \nabla C^\infty(M)\,, \tag{43}$$

where $\mathfrak{X}_{d\mathrm{vol}_g}(M) := \{ X \in \mathfrak{X}(M) \,|\, \mathrm{div}(X) = 0 \}$, is a topological direct sum (see [31]). As a consequence, the space $\mathcal{D} \times \nabla C^\infty(M)$ is a tame Fréchet space, and in particular, it is a trivial tame Fréchet vector bundle over $\mathcal{D}$.

Remark 3. In connection with electromagnetism, if we allow the vector field $X \in \mathfrak{X}(M)$ of Proposition 4 to be time-dependent, then an obvious modification of the proof of Proposition 4 shows that the map

$$\begin{cases} T\mathcal{D} \times \mathbb{R} \to \mathcal{D} \times \nabla C^\infty(M) \times \mathbb{R}, \\ \left(\rho, h = \operatorname{div}\left(\rho\left(\nabla\phi_t + X_t\right)\right), t\right) \mapsto \left(\rho, \nabla\phi_t, t\right), \end{cases}$$

is a smooth tame diffeomorphism.

Remark 4. If $X \equiv 0$, then the map $T\mathcal{D} \to \mathcal{D} \times \nabla C^\infty(M)$ defined in (40) becomes a *linear* tame isomorphism of tame Fréchet vector bundles.

Remark 5. In this section we were working in the category of tame Fréchet spaces, but in the sequel we will relax this hypothesis and simply work with the usual Fréchet category.

4. **Euler-Lagrange equations on $\mathcal{D}$ and the Schrödinger equation.** Having a precise and geometric description of the tangent bundle of $\mathcal{D}$, it is easy to write interesting Lagrangians on $\mathcal{D}$. Indeed, for a time-dependent vector field $X_t \in \mathfrak{X}(M)$, a time-dependent potential $V_t : M \to \mathbb{R}$, and using the diffeomorphism $T\mathcal{D} \times \mathbb{R} \to \mathcal{D} \times \nabla C^\infty(M) \times \mathbb{R}$ of Remark 3, we can consider, with an abuse of notation, the following time-dependent Lagrangian :

$$\mathcal{L}\left(\rho, h = \operatorname{div}\left(\rho\left(\nabla\phi_t + X_t\right)\right), t\right) = \mathcal{L}(\rho, \nabla\phi_t, t) :=$$

$$\int_M \left(\frac{1}{2}\|\nabla\phi_t\|^2 - \|X_t\|^2 - V_t\right)\rho \cdot d\mathrm{vol}_g - \frac{\hbar^2}{2}\int_M \left\|\nabla(\sqrt{\rho})\right\|^2 \cdot d\mathrm{vol}_g. \quad (44)$$

Note that $\mathcal{L}$ is smooth by application of the convenient calculus together with Remark 3.

By using the formula

$$\frac{1}{4}\frac{\|\nabla u\|^2}{u^2} - \frac{1}{2}\frac{\Delta u}{u} = -\frac{\Delta(\sqrt{u})}{\sqrt{u}}, \quad (45)$$

which is valid for every smooth positive function $u : M \to \mathbb{R}_+^*$, and by doing a usual fixed end-point variation of the Lagrangian $\mathcal{L}$, one easily finds the following Euler-Lagrange equations :

Proposition 5. *The Euler-Lagrange equations associated to the Lagrangian $\mathcal{L}$ defined in (44), are given by*

$$\begin{cases} \dfrac{\partial\phi}{\partial t} &= \dfrac{1}{2}\|\nabla\phi + X\|^2 + V - \dfrac{\hbar^2}{2}\dfrac{\Delta\left(\sqrt{\rho}\right)}{\sqrt{\rho}} + c_t, \\ \dfrac{\partial\rho}{\partial t} &= \operatorname{div}\left(\rho\left(\nabla\phi + X\right)\right), \end{cases} \quad (46)$$

where $\rho : I \subseteq \mathbb{R} \to \mathcal{D}$ is a smooth curve in $\mathcal{D}$ and where c_t is a time-dependent constant.

Remark 6. The second equation in (46) has actually nothing to do with variational principles; it is just the geometric way to express tangent vectors in $\mathcal{D}$, such as described in Proposition 4.

Remark 7. The appearance of the time-dependent constant c_t in (46) is due to the L^2-orthogonal decomposition (30).

Remark 8. By doing the change of variable $\phi' := \phi - \int c_t dt$, one may assume $c_t \equiv 0$.

Using vector calculus identities, it is possible to rewrite the system (46) in a more "suggestive way", as follows.

Recall that in dimension $n = 3$, one can use the Hodge operator $*$ on differential forms to express the cross product and the curl operator as (see [50]):

$$u \times v = [*(u^\flat \wedge v^\flat)]^\sharp, \qquad \mathrm{curl}(X) = [*(dX^\flat)]^\sharp, \tag{47}$$

where u, v are tangent vectors and where X is a vector field on M (here "$\flat = \sharp^{-1}$" stands for the usual musical isomorphism between vectors and forms on M associated to the Riemannian metric g).

Lemma 4.1. *Let X, Y be two vector fields on M. Then,*

(i) $\mathscr{L}_X(X^\flat) = (\nabla_X X)^\flat + \frac{1}{2}d(\|X\|^2)$, *where $\mathscr{L}$ is the Lie derivative and where $\nabla_X X$ is the Riemannian covariant derivative of X in the direction X,*

(ii) *if $n = 3$, then $X \times \mathrm{curl}(Y) = -(i_X dY^\flat)^\sharp$, where "$i_X$" is the contraction operator on forms.*

Proof. (i) See [4]. (ii) Given a vector field X and a p-form α, one has $X^\flat \wedge *\alpha = (-1)^{p-1} * i_X \alpha$ and $** \alpha = (-1)^{p(n+1)} \alpha$, where n is the dimension of M (the first identity follows easily from [53] while the second is well-known). Taking into account (47), we thus have, for $n = 3$,

$$X \times \mathrm{curl}(Y) \;\; = \;\; \left[*\left(X^\flat \wedge *(dY^\flat)\right)\right]^\sharp = -\left[** i_X dY^\flat\right]^\sharp = -[i_X dY^\flat]^\sharp. \tag{48}$$

The lemma follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Taking the gradient of the first equation in (46), and setting $u := -(\nabla\phi + X)$, it follows from the first item above together with "Cartan's Magic Formula" that

$$\frac{\partial u}{\partial t} + \nabla_u u = -\nabla V - \frac{\partial X}{\partial t} - (i_u dX^\flat)^\sharp - \nabla V_{qu}, \tag{49}$$

where the quantity

$$V_{qu} := -\frac{\hbar^2}{2} \frac{\Delta\left(\sqrt{\rho}\right)}{\sqrt{\rho}} \tag{50}$$

is known as the *Bohm quantum potential* [7, 8].

In the particular case $n = 3$, a direct application of the second item yields:

Corollary 3. *Assume $n = 3$ and set $u := -(\nabla\phi + X)$. Then the system (46) is equivalent to*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \nabla_u u = E + u \times B - \nabla V_{qu}, \\[2mm] \qquad \dfrac{\partial \rho}{\partial t} + \mathrm{div}\left(\rho u\right) = 0 \,, \end{cases} \tag{51}$$

where $E := -\nabla V - \frac{\partial X}{\partial t}$ and $B := \mathrm{curl}(X)$.

When $\hbar = 0$ (i.e. when $V_{qu} = 0$), the system (51) describes the dynamics of a fluid of density ρ and velocity field u subjected to an electromagnetic field (E, B) characterized by the scalar potential V and vector potential X. In the general case ($\hbar \neq 0$), these equations are known to be equivalent (up to a physically irrelevant

phase factor) to the Schrödinger equation for a quantum charged particle in an electromagnetic field (see for example [52]):

$$i\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\,\Delta\,\psi - \frac{\hbar}{i}\,g(X,\nabla\psi) + \frac{1}{2}\left(-\frac{\hbar}{i}\,\mathrm{div}\,(X) + \|X\|^2\right)\psi + V\psi\,, \qquad (52)$$

where

$$\psi := \sqrt{\rho}\,e^{-\frac{i}{\hbar}\phi}\,. \qquad (53)$$

Physicists regard the fluid equations (51) (with $\hbar = 0$) as a classical approximation of the Schrödinger equation (52). It was Madelung who, in the 20's, was the first to put forth the above similarities between quantum mechanics and fluid mechanics [47, 48], leading to a reformulation of quantum mechanics which is nowadays referred to as the *hydrodynamical formulation of quantum mechanics* (see [38, 74, 75, 81, 82]).

In general, a straightforward computation shows the following result.

Corollary 4. *Let ρ be a solution in $\mathcal{D}$ of the Euler-Lagrange equations associated to the Lagrangian $\mathcal{L} : T\mathcal{D} \to \mathbb{R}$ (see (44)), with $\partial\rho/\partial t = \mathrm{div}\left(\rho\,(\nabla\phi + X)\right)$. Then the wave function associated to ρ,*

$$\psi := \sqrt{\rho}\,e^{-\frac{i}{\hbar}\left(\phi - \int c_t\,dt\right)}\,, \qquad (54)$$

(see (46) for the definition of c_t), satisfies the Schrödinger equation (52).

Note that we are not assuming $n = 3$ in the above corollary.

Remark 9. For a smooth function $\psi : M \to \mathbb{C}$, let us denote by $[\psi]$ the complex line generated by ψ in the complex Hilbert space $\mathcal{H} := L^2(M, \mathbb{C})$ (the latter being endowed with its natural L^2-scalar product). Let us also consider the following map

$$T : T\mathcal{D} \to \mathbb{P}(\mathcal{H})\,, \quad (\rho, \nabla\phi) \mapsto \left[\sqrt{\rho}\,e^{-\frac{i}{\hbar}\phi}\right]\,, \qquad (55)$$

where $\mathbb{P}(\mathcal{H})$ denotes the complex projective space of complex lines in $\mathcal{H}$. As one may easily see, this map is well defined, it is injective, and since

$$\left[\sqrt{\rho}\,e^{-\frac{i}{\hbar}\left(\phi - \int c_t\,dt\right)}\right] = \left[\sqrt{\rho}\,e^{-\frac{i}{\hbar}\phi}\right]\,, \qquad (56)$$

Corollary 4 implies that T maps solutions of the Euler-Lagrange equations (46) to solutions of the Schrödinger equation (52), projected on $\mathbb{P}(\mathcal{H})$. To understand this from the point of view of the geometrical formulation of quantum mechanics, see Proposition 10.

5. **Hamiltonian formulation.** In this section, we continue our study of the dynamics of a quantum particle initiated in §4, but we will now focus on the Hamiltonian formulation. We will still assume that (M, g) is a compact, connected and oriented Riemannian manifold, but for simplicity, we will assume that the particle is only under the influence of a time-independent potential $V : M \to \mathbb{R}$. The Lagrangian thus reads

$$\mathcal{L}(\rho, \nabla\phi)$$
$$= \frac{1}{2}\int_M \|\nabla\phi\|^2\,\rho\cdot d\mathrm{vol}_g - \int_M V\,\rho\cdot d\mathrm{vol}_g - \frac{\hbar^2}{2}\int_M \left\|\nabla(\sqrt{\rho})\right\|^2\cdot d\mathrm{vol}_g, \quad (57)$$

where $(\rho, \nabla\phi) \in \mathcal{D} \times \nabla C^\infty(M)$.

This Lagrangian is of the form kinetic energy minus two potential terms, the corresponding Riemannian metric being

$$(g^{\mathcal{D}})_{\rho}\big((\rho, \nabla\phi), (\rho, \nabla\phi')\big) := \int_{M} g(\nabla\phi, \nabla\phi')\, \rho \cdot d\mathrm{vol}_g\,. \tag{58}$$

Remark 10. In the context of optimal transport theory, the metric $g^{\mathcal{D}}$ is known as the Wasserstein metric. As it has been stressed by Otto in his seminal paper [67], its importance stems from the fact that many diffusion equations (e.g. the heat equation) can be interpreted geometrically as gradient flows[10] on appropriate spaces of probability measures endowed with the Wasserstein metric, leading naturally to a rich interplay of geometry, functional analysis and partial differential equations (see [79], Chap.3). For the geometrical properties of $g^{\mathcal{D}}$, we refer the reader to [45].

In order to find a Hamiltonian description of the above Lagrangian system, let us digress a little and recall some basic facts about dynamical systems in finite dimension (see for example [1, 25, 60, 49] for details).

Let Q be a finite dimensional manifold and let $L : TQ \to \mathbb{R}$ be a Lagrangian of the form $L(u_x) = \frac{1}{2}\, h(u_x, u_x) - U(x)$, where h is a Riemannian metric and where $U : Q \to \mathbb{R}$ is a potential. Let also θ_L be the following 1-form on TQ :

$$(\theta_L)_{u_x}(A_{u_x}) = h_x\big(u_x, (\pi^{TQ})_{*u_x} A_{u_x}\big)\,, \tag{59}$$

where $x \in Q$, $u_x \in T_x Q$, $A_{u_x} \in T_{u_x} TQ$ and where $\pi^{TQ} : TQ \to Q$ is the canonical projection.

In this context, it this well-known that the 2-form $\omega_L := -d\theta_L$ is a symplectic form, and that the integral curves of the Hamiltonian vector field[11] X_H associated to the Hamiltonian $H(u_x) := \frac{1}{2}\, h(u_x, u_x) + U(x)$ project, via π^{TQ}, to the solutions of the Euler-Lagrange equations associated to L on Q. Hence, the symplectic form ω_L yields, together with H, a Hamiltonian description on TQ of the initial Lagrangian system.

In our situation, it is thus natural to define the following 1-form on $T\mathcal{D}$:

$$(\Theta_{\mathcal{L}})_{(\rho, \nabla\phi)}(A_{(\rho, \nabla\phi)}) := (g^{\mathcal{D}})_{\rho}\Big(\nabla\phi, (\pi^{T\mathcal{D}})_{*(\rho, \nabla\phi)} A_{(\rho, \nabla\phi)}\Big)\,, \tag{60}$$

where $A_{(\rho, \nabla\phi)} \in T_{(\rho, \nabla\phi)} T\mathcal{D}$ and where $\pi^{T\mathcal{D}} : T\mathcal{D} \to \mathcal{D}$ denotes the canonical projection, and then to define

$$\Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}}\,. \tag{61}$$

Our goal is now to compute explicitly $\Omega_{\mathcal{L}}$ and to show that it is indeed a symplectic form on $T\mathcal{D}$, like in the finite dimensional case. To this end, we will use the following identification

$$T(T\mathcal{D}) \cong \mathcal{D} \times \nabla C^{\infty}(M) \times \nabla C^{\infty}(M) \times \nabla C^{\infty}(M)\,, \tag{62}$$

where the diffeomorphism is given by

$$\frac{d}{dt}\bigg|_0\, \big(\rho_t, \nabla\phi + t\nabla\psi_2\big) \mapsto (\rho_0, \nabla\phi, \nabla\psi_1, \nabla\psi_2)\,, \tag{63}$$

---

[10]A gradient flow on a Riemannian manifold (M, g) is simply the flow generated by the Riemannian gradient ∇f of a function $f : M \to \mathbb{R}$.

[11]Recall that if ω is a symplectic form on a manifold N, then the Hamiltonian vector field of a function $F : N \to \mathbb{R}$ with respect to ω is by definition the unique vector field X_F satisfying $\omega(X_F, .) = dF(.)$. It is also called the symplectic gradient of F.

and where ρ_t is a smooth curve in $\mathcal{D}$ satisfying

$$\frac{d}{dt}\bigg|_0 \rho_t = \text{div}\left(\rho_0 \cdot \nabla\psi_1\right). \tag{64}$$

Using (58) and (62), it is clear that (60) may be rewritten

$$(\Theta_{\mathcal{L}})_{(\rho,\nabla\phi)}(\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2) = \int_M g(\nabla\phi, \nabla\psi_1)\,\rho \cdot d\text{vol}_g. \tag{65}$$

Our strategy to compute the differential of $\Theta_{\mathcal{L}}$ at a point $(\rho, \nabla\phi)$, will be to use the formula

$$(d\Theta_{\mathcal{L}})_{(\rho,\nabla\phi)}(X, Y)$$
$$= X_{(\rho,\nabla\phi)}\big(\Theta_{\mathcal{L}}(Y)\big) - Y_{(\rho,\nabla\phi)}\big(\Theta_{\mathcal{L}}(X)\big) - (\Theta_{\mathcal{L}})_{(\rho,\nabla\phi)}([X, Y]), \tag{66}$$

where X, Y are vector fields on $T\mathcal{D}$.

As the above formula is tensorial in X and Y, we are free to choose X and Y arbitrary at a given point $(\rho, \nabla\phi)$, and to extend these vector fields as simply as possible elsewhere. A natural choice is to set, for any $(\rho, \nabla\phi) \in T\mathcal{D}$,

$$X_{(\rho,\nabla\phi)} := (\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2) \quad \text{and} \quad Y_{(\rho,\nabla\phi)} := (\rho, \nabla\phi, \nabla\alpha_1, \nabla\alpha_2), \tag{67}$$

where $\nabla\psi_1, \nabla\psi_2, \nabla\alpha_1, \nabla\alpha_2$ are held fixed.

In view of (66), we now have to compute $X_{(\rho,\nabla\phi)}\big(\Theta_{\mathcal{L}}(Y)\big)$ and $(\Theta_{\mathcal{L}})_{(\rho,\nabla\phi)}([X, Y])$, with X and Y as defined in (67).

Lemma 5.1. *We have :*

$$X_{(\rho,\nabla\phi)}\big(\Theta_{\mathcal{L}}(Y)\big)$$
$$= \int_M g(\nabla\phi, \nabla\alpha_1)\,\text{div}(\rho \cdot \nabla\psi_1) \cdot d\text{vol}_g + \int_M g(\nabla\psi_2, \nabla\alpha_1)\,\rho \cdot d\text{vol}_g. \tag{68}$$

Proof. Let ρ_t be a curve in $\mathcal{D}$ satisfying

$$\rho_0 = \rho \quad \text{and} \quad \frac{\partial\rho_t}{\partial t} = \text{div}\left(\rho_t \cdot \nabla\psi_1\right). \tag{69}$$

If $c(t) := (\rho_t, \nabla\phi + t\nabla\psi_2)$, then

$$\dot{c}(0) = (\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2) = X_{(\rho,\nabla\phi)}, \tag{70}$$

and thus,

$$X_{(\rho,\nabla\phi)}\big(\Theta_{\mathcal{L}}(Y)\big)$$
$$= \frac{d}{dt}\bigg|_0 (\Theta_{\mathcal{L}})_{c(t)}(Y_{c(t)}) = \frac{d}{dt}\bigg|_0 \Theta_{\mathcal{L}}\big(\rho_t, \nabla\phi + t\nabla\psi_2, \nabla\alpha_1, \nabla\alpha_2\big)$$
$$= \frac{d}{dt}\bigg|_0 \left[\int_M g\big(\nabla\phi, \nabla\alpha_1\big)\rho_t \cdot d\text{vol}_g + t\int_M g\big(\nabla\psi_2, \nabla\alpha_1\big)\rho_t \cdot d\text{vol}_g\right]$$
$$= \int_M g\big(\nabla\phi, \nabla\alpha_1\big)\,\text{div}\left(\rho \cdot \nabla\psi_1\right) \cdot d\text{vol}_g + \int_M g\big(\nabla\psi_2, \nabla\alpha_1\big)\rho \cdot d\text{vol}_g. \tag{71}$$

The lemma follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For the term $(\Theta_{\mathcal{L}})_{(\rho,\nabla\phi)}([X, Y])$, we need to compute the Lie bracket $[X, Y]$ of X and Y, and this can be done with a good description of the flow φ_t^X of X. This description may be obtained with the following map

$$D : \text{Diff}(M) \to C^\infty(M), \tag{72}$$

which is defined, for a φ belonging to the group of all diffeomorphims $\mathrm{Diff}(M)$, via the formula

$$\varphi^* d\mathrm{vol}_g = D(\varphi) \cdot d\mathrm{vol}_g. \tag{73}$$

As a matter of notation, we shall write $D(\varphi) = \varphi^* d\mathrm{vol}_g / d\mathrm{vol}_g$.

It may be shown that the map D is smooth (see [31]), and that

$$D(\varphi \circ \psi) = D(\varphi) \circ \psi \cdot D(\psi), \tag{74}$$

where $\varphi, \psi \in \mathrm{Diff}(M)$.

Observe also that if a diffeomorphism φ preserves the orientation of $(M, d\mathrm{vol}_g)$, then $1/\mathrm{Vol}(M) \cdot D(\varphi) \in \mathcal{D}$.

Lemma 5.2. *The flow φ_t^X of X is given, for $\rho \in \mathcal{D}$ and $\nabla\phi \in \nabla C^\infty(M)$, by*

$$\varphi_t^X(\rho, \nabla\phi) := \left(\frac{1}{\mathrm{Vol}(M)} \cdot D(\varphi \circ \varphi_t^{\nabla\psi_1}), \nabla\phi + t\nabla\psi_2 \right), \tag{75}$$

where $\varphi \in \mathrm{Diff}(M)$ is chosen such that $D(\varphi) = \mathrm{Vol}(M)\rho$ (such φ necessarily exits according to Moser's Theorem).

Proof. According to (73), we have :

$$D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot d\mathrm{vol}_g = (\varphi_t^{\nabla\psi_1})^* \varphi^* d\mathrm{vol}_g = (\varphi_t^{\nabla\psi_1})^* D(\varphi) \cdot d\mathrm{vol}_g$$

$$\Rightarrow \quad D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot d\mathrm{vol}_g = (\varphi_t^{\nabla\psi_1})^* D(\varphi) \cdot d\mathrm{vol}_g$$

$$\Rightarrow \quad \frac{d}{dt} D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot d\mathrm{vol}_g = \mathscr{L}_{\nabla\psi_1}\left((\varphi_t^{\nabla\psi_1})^* D(\varphi) \cdot d\mathrm{vol}_g \right), \tag{76}$$

and, in view of (74),

$$\mathscr{L}_{\nabla\psi_1}\left((\varphi_t^{\nabla\psi_1})^* D(\varphi) \cdot d\mathrm{vol}_g \right)$$

$$= \quad \mathscr{L}_{\nabla\psi_1}\left((D(\varphi) \circ \varphi_t^{\nabla\psi_1}) \cdot D(\varphi_t^{\nabla\psi_1}) \cdot d\mathrm{vol}_g \right)$$

$$= \quad \mathscr{L}_{\nabla\psi_1}\left(D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot d\mathrm{vol}_g \right)$$

$$= \quad \left(g(\nabla\psi_1, \nabla D(\varphi \circ \varphi_t^{\nabla\psi_1})) + D(\varphi \circ \varphi_t^{\nabla\psi_1}) \mathrm{div}(\nabla\psi_1) \right) \cdot d\mathrm{vol}_g$$

$$= \quad \mathrm{div}\left(D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot \nabla\psi_1 \right) \cdot d\mathrm{vol}_g. \tag{77}$$

Collecting (76) and (77), we thus get

$$\frac{d}{dt} D(\varphi \circ \varphi_t^{\nabla\psi_1}) = \mathrm{div}\left(D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot \nabla\psi_1 \right), \tag{78}$$

from which we see, having in mind the identification (62), that

$$\frac{d}{dt}\left(\frac{1}{\mathrm{Vol}(M)} \cdot D(\varphi \circ \varphi_t^{\nabla\psi_1}), \nabla\phi + t\nabla\psi_2 \right)$$

$$= \quad \left(\frac{1}{\mathrm{Vol}(M)} \cdot D(\varphi \circ \varphi_t^{\nabla\psi_1}), \nabla\phi + t\nabla\psi_2, \nabla\psi_1, \nabla\psi_2 \right). \tag{79}$$

Equation (79) exactly means that φ_t^X, such as defined in (75), is the flow of X. The lemma follows. $\qquad\square$

We are now almost able to compute the Lie bracket $[X, Y]$. But for this, we still need, for $\rho \in \mathcal{D}$, the following continuous map of Fréchet spaces

$$\mathbb{P}_\rho : \left\{ \begin{array}{c} \mathfrak{X}(M) = \mathfrak{X}_{d\mathrm{vol}_g}(M) \oplus \rho \nabla C^\infty(M) \to \nabla C^\infty(M) \,, \\ X = \overline{X} + \rho \nabla \phi \mapsto \nabla \phi \,, \end{array} \right. \tag{80}$$

where $\overline{X} \in \mathfrak{X}_{d\mathrm{vol}_g}(M) = \{Z \in \mathfrak{X}(M) \,|\, \mathrm{div}(Z) = 0\}$, and where the topological direct sum $\mathfrak{X}(M) = \mathfrak{X}_{d\mathrm{vol}_g}(M) \oplus \rho \nabla C^\infty(M)$ is simply a slight generalisation of the Helmholtz-Hodge decomposition (see (43) and [55] for a proof of this generalization).

Remark 11. Using Stokes' Theorem, it is easy to show the following convenient formula :

$$\int_M g\big(\nabla \phi, \mathbb{P}_\rho(\rho X)\big) \rho \cdot d\mathrm{vol}_g = \int_M g(\nabla \phi, X) \, \rho \cdot d\mathrm{vol}_g \,, \tag{81}$$

where $\phi \in C^\infty(M)$ and where $X \in \mathfrak{X}(M)$.

Lemma 5.3. *For $\rho \in \mathcal{D}$ and $\nabla \phi \in \nabla C^\infty(M)$, we have :*

$$[X, Y]_{(\rho, \nabla \phi)} = \Big(\rho, \nabla \phi, \mathbb{P}_\rho\big(\rho \left[\nabla \alpha_1, \nabla \psi_1\right]\big), 0\Big) \,. \tag{82}$$

Proof. Let us choose $\varphi, \psi_t, \beta_{t,s} \in \mathrm{Diff}(M)$ such that

$$D(\varphi) = \mathrm{Vol}(M) \cdot \rho \,, \quad D(\psi_t) = D(\varphi \circ \varphi_t^{\nabla \psi_1}) \,, \quad D(\beta_{t,s}) = D(\psi_t \circ \varphi_s^{\nabla \alpha_1}) \,. \tag{83}$$

According to Lemma 5.2, we have

$$
\begin{aligned}
[X, Y]_{(\rho, \nabla \phi)} &= \frac{d}{dt}\bigg|_0 \left((\varphi_{-t}^X)_{*\varphi_t^X(\rho, \nabla \phi)} Y_{\varphi_t^X(\rho, \nabla \phi)} \right) \\
&= \frac{d}{dt}\bigg|_0 \frac{d}{ds}\bigg|_0 \left(\varphi_{-t}^X \circ \varphi_s^Y \circ \varphi_t^X \right)(\rho, \nabla \phi) \\
&= \frac{d}{dt}\bigg|_0 \frac{d}{ds}\bigg|_0 \left(\varphi_{-t}^X \circ \varphi_s^Y \right) \left(\frac{1}{\mathrm{Vol}(M)} \cdot D(\varphi \circ \varphi_t^{\nabla \psi_1}), \nabla \phi + t \nabla \psi_2 \right) \\
&= \frac{d}{dt}\bigg|_0 \frac{d}{ds}\bigg|_0 \varphi_{-t}^X \left(\frac{1}{\mathrm{Vol}(M)} \cdot D(\psi_t \circ \varphi_s^{\nabla \alpha_1}), \nabla \phi + t \nabla \psi_2 + s \nabla \alpha_2 \right) \\
&= \frac{d}{dt}\bigg|_0 \frac{d}{ds}\bigg|_0 \left(\frac{1}{\mathrm{Vol}(M)} \cdot D(\beta_{t,s} \circ \varphi_{-t}^{\nabla \psi_1}), \nabla \phi + t \nabla \psi_2 + s \nabla \alpha_2 - t \nabla \psi_2 \right) \\
&= \frac{d}{dt}\bigg|_0 \frac{d}{ds}\bigg|_0 \left(\frac{1}{\mathrm{Vol}(M)} \cdot D(\beta_{t,s} \circ \varphi_{-t}^{\nabla \psi_1}), \nabla \phi + s \nabla \alpha_2 \right) . \tag{84}
\end{aligned}
$$

From (84), we already see that the bracket $[X, Y]_{(\rho, \nabla \phi)}$ is of the form $(\rho, \nabla \phi, *, 0)$, where "$*$" has to be determined by computing the derivatives of $D(\beta_{t,s} \circ \varphi_{-t}^{\nabla \psi_1})$ with respect to s and t, and by putting it in a divergence form.

Using (74) and (83), we see that

$$
\begin{aligned}
D(\beta_{t,s} \circ \varphi_{-t}^{\nabla \psi_1}) &= D(\beta_{t,s}) \circ \varphi_{-t}^{\nabla \psi_1} \cdot D(\varphi_{-t}^{\nabla \psi_1}) = D(\psi_t \circ \varphi_s^{\nabla \alpha_1}) \circ \varphi_{-t}^{\nabla \psi_1} \cdot D(\varphi_{-t}^{\nabla \psi_1}) \\
&= D(\psi_t) \circ \varphi_s^{\nabla \alpha_1} \circ \varphi_{-t}^{\nabla \psi_1} \cdot D(\varphi_s^{\nabla \alpha_1}) \circ \varphi_{-t}^{\nabla \psi_1} \cdot D(\varphi_{-t}^{\nabla \psi_1}) \\
&= D(\varphi \circ \varphi_t^{\nabla \psi_1}) \circ \varphi_s^{\nabla \alpha_1} \circ \varphi_{-t}^{\nabla \psi_1} \cdot D(\varphi_s^{\nabla \alpha_1}) \circ \varphi_{-t}^{\nabla \psi_1} \cdot D(\varphi_{-t}^{\nabla \psi_1}) \\
&= \Big(D(\varphi \circ \varphi_t^{\nabla \psi_1}) \circ \varphi_s^{\nabla \alpha_1} \cdot D(\varphi_s^{\nabla \alpha_1}) \Big) \circ \varphi_{-t}^{\nabla \psi_1} \cdot D(\varphi_{-t}^{\nabla \psi_1}) \\
&= D(\varphi \circ \varphi_t^{\nabla \psi_1} \circ \varphi_s^{\nabla \alpha_1}) \circ \varphi_{-t}^{\nabla \psi_1} \cdot D(\varphi_{-t}^{\nabla \psi_1}) \\
&= D(\varphi \circ \varphi_t^{\nabla \psi_1} \circ \varphi_s^{\nabla \alpha_1} \circ \varphi_{-t}^{\nabla \psi_1}) \,, \tag{85}
\end{aligned}
$$

and thus,

$$
\begin{aligned}
& \left.\frac{d}{dt}\right|_0 \left.\frac{d}{ds}\right|_0 \frac{1}{\mathrm{Vol}(M)} \cdot D(\beta_{t,s} \circ \varphi_{-t}^{\nabla\psi_1}) \\
=\ & \left.\frac{d}{dt}\right|_0 \left.\frac{d}{ds}\right|_0 \frac{1}{\mathrm{Vol}(M)} \cdot D(\varphi \circ \varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1}) \\
=\ & \left.\frac{d}{dt}\right|_0 \left.\frac{d}{ds}\right|_0 \rho \circ \varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1} \cdot D(\varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1}) \\
=\ & \left.\frac{d}{dt}\right|_0 \left.\frac{d}{ds}\right|_0 \rho \circ \varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1} + \rho \cdot \left.\frac{d}{dt}\right|_0 \left.\frac{d}{ds}\right|_0 D(\varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1}) \\
=\ & g\big(\nabla\rho, [\nabla\alpha_1, \nabla\psi_1]\big) + \rho \cdot \left.\frac{d}{dt}\right|_0 \left.\frac{d}{ds}\right|_0 \big(\varphi_t^{\nabla\psi_1} \circ \varphi_s^{\nabla\alpha_1} \circ \varphi_{-t}^{\nabla\psi_1}\big)^* d\mathrm{vol}_g / d\mathrm{vol}_g \\
=\ & g\big(\nabla\rho, [\nabla\alpha_1, \nabla\psi_1]\big) + \rho \cdot \left.\frac{d}{dt}\right|_0 \mathscr{L}_{(\varphi_t^{\nabla\psi_1})_{*_{\varphi_{-t}^{\nabla\psi_1}}}(\nabla\alpha_1)_{\varphi_{-t}^{\nabla\psi_1}}} \big(d\mathrm{vol}_g\big) / d\mathrm{vol}_g \\
=\ & g\big(\nabla\rho, [\nabla\alpha_1, \nabla\psi_1]\big) + \rho \cdot \left.\frac{d}{dt}\right|_0 \mathrm{div}\left((\varphi_t^{\nabla\psi_1})_{*_{\varphi_{-t}^{\nabla\psi_1}}}(\nabla\alpha_1)_{\varphi_{-t}^{\nabla\psi_1}}\right) \\
=\ & g\big(\nabla\rho, [\nabla\alpha_1, \nabla\psi_1]\big) + \rho \cdot \mathrm{div}\big([\nabla\alpha_1, \nabla\psi_1]\big) \\
=\ & \mathrm{div}\big(\rho \cdot [\nabla\alpha_1, \nabla\psi_1]\big) = \mathrm{div}\big(\rho \cdot \mathbb{P}_\rho\big(\rho \cdot [\nabla\alpha_1, \nabla\psi_1]\big)\big).
\end{aligned}
\tag{86}
$$

The lemma follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 12. Let $Z_\rho := (\rho, \nabla\beta)$ be the "constant" vector field on $\mathcal{D}$ associated to a fixed $\nabla\beta \in \nabla C^\infty(M)$. By inspection of the proof of Lemma 5.2, one sees that $\varphi_t^Z(\rho) = (1/\mathrm{Vol}(M))D(\varphi \circ \varphi_t^{\nabla\beta})$, where $\varphi \in \mathrm{Diff}(M)$ satisfies $D(\varphi) = \mathrm{Vol}(M)\,\rho$. Moreover, by inspection of the proof of Lemma 5.3, one also sees that if $W_\rho := (\rho, \nabla\gamma)$ is another constant vector field on $\mathcal{D}$, then $[Z, W]_\rho = \big(\rho, \mathbb{P}_\rho\big(\rho\,[\nabla\gamma, \nabla\beta]\big)\big)$.

Proposition 6. *The form* $\Omega_\mathcal{L} := -d\Theta_\mathcal{L}$ *(see (65) for the definition of $\Theta_\mathcal{L}$), is a symplectic form on $T\mathcal{D}$, and for $\rho \in \mathcal{D}$ and $\nabla\phi \in \nabla C^\infty(M)$,*

$$
\begin{aligned}
& (\Omega_\mathcal{L})_{(\rho, \nabla\phi)}\big((\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2), (\rho, \nabla\phi, \nabla\alpha_1, \nabla\alpha_2)\big) \\
=\ & \int_M g(\nabla\psi_1, \nabla\alpha_2)\,\rho \cdot d\mathrm{vol}_g - \int_M g(\nabla\alpha_1, \nabla\psi_2)\,\rho \cdot d\mathrm{vol}_g,
\end{aligned}
\tag{87}
$$

where $\nabla\psi_1, \nabla\psi_2, \nabla\alpha_1, \nabla\alpha_2 \in \nabla C^\infty(M)$.

Proof. The fact that $\Omega_\mathcal{L}$ is a symplectic form, i.e., that $\Omega_\mathcal{L}$ is non-degenerate (the closedness being clear), is a simple consequence of formula (87) that we are now going to show.

Equation (66), together with Lemma 5.1 and Lemma 5.3, yield

$$
\begin{aligned}
& (\Omega_\mathcal{L})_{(\rho, \nabla\phi)}\big((\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2), (\rho, \nabla\phi, \nabla\alpha_1, \nabla\alpha_2)\big) \\
=\ & -(d\Theta_\mathcal{L})_{(\rho, \nabla\phi)}\big(X_{(\rho, \nabla\phi)}, Y_{(\rho, \nabla\phi)}\big) \\
=\ & -X_{(\rho, \nabla\phi)}\big(\Theta_\mathcal{L}(Y)\big) + Y_{(\rho, \nabla\phi)}\big(\Theta_\mathcal{L}(X)\big) + (\Theta_\mathcal{L})_{(\rho, \nabla\phi)}([X, Y]), \\
=\ & \int_M g(\nabla\psi_1, \nabla\alpha_2)\,\rho \cdot d\mathrm{vol}_g - \int_M g(\nabla\alpha_1, \nabla\psi_2)\,\rho \cdot d\mathrm{vol}_g
\end{aligned}
$$

$$+ \int_M g(\nabla \psi_1, \nabla \phi) \operatorname{div} \left(\rho \cdot \nabla \alpha_1 \right) \cdot d\mathrm{vol}_g$$

$$- \int_M g(\nabla \alpha_1, \nabla \phi) \operatorname{div} \left(\rho \cdot \nabla \psi_1 \right) \cdot d\mathrm{vol}_g$$

$$- \int_M g\big(\nabla \phi, \mathbb{P}_\rho\big(\rho \cdot [\nabla \psi_1, \nabla \alpha_1]\big)\big) \rho \cdot d\mathrm{vol}_g \,. \tag{88}$$

Clearly, we have to show that the last three lines in (88) vanish.

Using Remark 11, one may rewrite the last term in (88) as

$$\int_M g\big(\nabla \phi, \mathbb{P}_\rho\big(\rho \cdot [\nabla \psi_1, \nabla \alpha_1]\big)\big) \rho \cdot d\mathrm{vol}_g = \int_M g\big(\nabla \phi, [\nabla \psi_1, \nabla \alpha_1]\big) \rho \cdot d\mathrm{vol}_g \,. \tag{89}$$

Using this last equation, one observes that the last three terms in (88) may be rewritten :

$$\int_M g(\nabla \psi_1, \nabla \phi) \operatorname{div} \left(\rho \cdot \nabla \alpha_1 \right) \cdot d\mathrm{vol}_g - \int_M g(\nabla \alpha_1, \nabla \phi) \operatorname{div} \left(\rho \cdot \nabla \psi_1 \right) \cdot d\mathrm{vol}_g$$

$$- \int_M g\big(\nabla \phi, [\nabla \psi_1, \nabla \alpha_1]\big) \rho \cdot d\mathrm{vol}_g \,.$$

$$= \int_M \bigg(- g\Big(\nabla \alpha_1, \nabla g(\nabla \psi_1, \nabla \phi)\Big) + g\Big(\nabla \psi_1, \nabla g(\nabla \alpha_1, \nabla \phi)\Big)$$

$$- g(\nabla \phi, [\nabla \psi_1, \nabla \alpha_1]) \bigg) \rho \cdot d\mathrm{vol}_g$$

$$= \int_M \bigg(- (\nabla \alpha_1) \, d\phi(\nabla \psi_1) + (\nabla \psi_1) \, d\phi(\nabla \alpha_1) - d\phi([\nabla \psi_1, \nabla \alpha_1]) \bigg) \rho \cdot d\mathrm{vol}_g$$

$$= \int_M d(d\phi)(\nabla \psi_1, \nabla \alpha_1) \, \rho \cdot d\mathrm{vol}_g = 0 \,. \tag{90}$$

The proposition follows. $\qquad\square$

With such simple expression for the symplectic form $\Omega_{\mathcal{L}}$ (see (87)), it is possible the compute explicitly the symplectic gradient of interesting functions, as well as their Poisson brackets. Indeed, we define, for $F : TM \to \mathbb{R}$, the following function on $T\mathcal{D}$:

$$\widehat{F}(\rho, \nabla \phi) := \int_M F(\nabla \phi) \, \rho \cdot d\mathrm{vol}_g \,. \tag{91}$$

We also denote by $\mathcal{H} : T\mathcal{D} \to \mathbb{R}$ the Hamiltonian associated to the Lagrangian $\mathcal{L}$:

$$\mathcal{H}(\rho, \nabla \phi) := \int_M \left(\frac{1}{2} \|\nabla \phi\|^2 + V \right) \rho \cdot d\mathrm{vol}_g + \frac{\hbar^2}{2} \int_M \|\nabla(\sqrt{\rho})\|^2 \cdot d\mathrm{vol}_g \,. \tag{92}$$

We shall denote by $X_{\widehat{F}}$ and $X_{\mathcal{H}}$ the symplectic gradients associated to $\widehat{F}$ and $\mathcal{H}$ via the symplectic form $\Omega_{\mathcal{L}}$ (recall that these two vector fields are defined on $T\mathcal{D}$ via the relations $\Omega_{\mathcal{L}}(X_{\widehat{F}}, \,.\,) = d\widehat{F}$ and $\Omega_{\mathcal{L}}(X_{\mathcal{H}}, \,.\,) = d\mathcal{H}$).

On $T\mathcal{D}$, we shall use the Poisson bracket $\{\,.\,,\,.\,\}_{\mathcal{L}}$ associated to the symplectic form $\Omega_{\mathcal{L}}$ (of course, this Poisson bracket is only defined for functions having a symplectic gradient), and on TM we shall use the Poisson bracket, denoted $\{\,.\,,\,.\,\}_L$, canonically associated to the Lagrangian $L(u_x) := 1/2 \cdot g(u_x, u_x) - V(x)$.

Proposition 7. *For $F, G : TM \to \mathbb{R}$, $\rho \in \mathcal{D}$ and $\nabla \phi \in \nabla C^\infty(M)$, we have :*

1. $(X_{\mathcal{H}})_{(\rho, \nabla \phi)} = \left(\rho, \nabla \phi, \nabla \phi, \nabla \left[\dfrac{1}{2} \|\nabla \phi\|^2 + V - \dfrac{\hbar^2}{2} \dfrac{\triangle(\sqrt{\rho})}{\sqrt{\rho}} \right] \right),$

2. $(X_{\widehat{F}})_{(\rho,\nabla\phi)} = \left(\rho, \nabla\phi, \mathbb{P}_\rho\big(\rho\left(\pi_*^{TM} \circ X_F \circ \nabla\phi\right), \nabla\big(F(\nabla\phi)\big)\big)\right),$

3. $\{\widehat{F},\widehat{G}\}_{\mathcal{L}} = -\widehat{\{F,G\}}_L \,.$

We will show Proposition 7 with a series of Lemmas.

Lemma 5.4. *For $\rho \in \mathcal{D}$ and $\nabla\phi \in \nabla C^\infty(M)$, we have :*

$$(X_{\mathcal{H}})_{(\rho,\nabla\phi)} = \left(\rho, \nabla\phi, \nabla\phi, \nabla\left[\frac{1}{2}\,\|\nabla\phi\|^2 + V - \frac{\hbar^2}{2}\,\frac{\triangle\left(\sqrt{\rho}\right)}{\sqrt{\rho}} \right] \right). \tag{93}$$

Proof. We will use the vector field $X \in \mathfrak{X}(T\mathcal{D})$ introduced in (67), and especially its flow φ_t^X which is given in Lemma 5.2.

We have :

$$(d\mathcal{H})_{(\rho,\nabla\phi)} X_{(\rho,\nabla\phi)} = \frac{d}{dt}\Big|_0 (\mathcal{H} \circ \varphi_t^X)(\rho,\nabla\phi)$$

$$= \frac{d}{dt}\Big|_0 \mathcal{H}\left(\frac{1}{\mathrm{Vol}(M)} \cdot D(\varphi \circ \varphi_t^{\nabla\psi_1}), \nabla\phi + t\nabla\psi_2 \right)$$

$$= \frac{d}{dt}\Big|_0 \left[\int_M \left(\frac{1}{2}\,\|\nabla\phi + t\nabla\psi_2\|^2 + V \right) \frac{1}{\mathrm{Vol}(M)}\, D(\varphi \circ \varphi_t^{\nabla\psi_1}) \cdot d\mathrm{vol}_g \right.$$

$$\left. + \frac{\hbar^2}{2} \int_M \left\| \nabla\left(\sqrt{\frac{1}{\mathrm{Vol}(M)} D(\varphi \circ \varphi_t^{\nabla\psi_1})} \right) \right\|^2 \cdot d\mathrm{vol}_g \right]$$

$$= \frac{d}{dt}\Big|_0 \int_M \left(\frac{1}{2}\,\|\nabla\phi\|^2 + t\, g(\nabla\phi,\nabla\psi_2) + \frac{t^2}{2}\,\|\nabla\psi_2\|^2 + V \right) \rho_t \cdot d\mathrm{vol}_g$$

$$+ \frac{d}{dt}\Big|_0 \frac{\hbar^2}{2} \int_M \|\nabla(\sqrt{\rho_t})\|^2 \cdot d\mathrm{vol}_g \,, \tag{94}$$

where $\rho_t := 1/\mathrm{Vol}(M) \cdot D(\varphi \circ \varphi_t^{\nabla\psi_1})$.

But, according to (78),

$$\frac{\partial \rho_t}{\partial t} = \mathrm{div}\left(\rho_t \cdot \nabla\psi_1 \right), \tag{95}$$

and thus,

$$(d\mathcal{H})_{(\rho,\nabla\phi)} X_{(\rho,\nabla\phi)}$$

$$= \frac{1}{2} \int_M \|\nabla\phi\|^2 \,\mathrm{div}\left(\rho \cdot \nabla\psi_1 \right) \cdot d\mathrm{vol}_g + \int_M g(\nabla\phi,\nabla\psi_2)\, \rho \cdot d\mathrm{vol}_g$$

$$+ \int_M V\, \mathrm{div}\left(\rho \cdot \nabla\psi_1 \right) \cdot d\mathrm{vol}_g + \frac{d}{dt}\Big|_0 \frac{\hbar^2}{2} \int_M \|\nabla(\sqrt{\rho_t})\|^2 \cdot d\mathrm{vol}_g \,. \tag{96}$$

Let us compute the last term in (96) :

$$\frac{d}{dt}\Big|_0 \int_M \|\nabla(\sqrt{\rho_t})\|^2 d\mathrm{vol}_g = 2\int_M g\big(\nabla\tfrac{\partial}{\partial t}\big|_0 \sqrt{\rho_t}, \nabla(\sqrt{\rho})\big) d\mathrm{vol}_g$$

$$= \int_M g\big(\nabla\big[\tfrac{1}{\sqrt{\rho}}\mathrm{div}(\rho\nabla\psi_1)\big], \nabla(\sqrt{\rho})\big) d\mathrm{vol}_g = \int_M g\big(\mathrm{div}(\rho\nabla\psi_1)\big(-\tfrac{1}{\rho}\tfrac{1}{2\sqrt{\rho}}\nabla\rho\big)$$

$$+ \tfrac{1}{\sqrt{\rho}}\nabla\mathrm{div}(\rho\nabla\psi_1), \nabla(\sqrt{\rho})\big) d\mathrm{vol}_g$$

$$= -\int_M g\big(\nabla\rho, \nabla(\sqrt{\rho})\big)\mathrm{div}(\rho\nabla\psi_1)\tfrac{1}{\rho}\tfrac{1}{2\sqrt{\rho}} d\mathrm{vol}_g + \int_M g\big(\nabla\mathrm{div}(\rho\nabla\psi_1),$$

$$\nabla(\sqrt{\rho}))\,\tfrac{1}{\sqrt{\rho}}d\mathrm{vol}_g$$

$$= -\int_M \|\nabla\rho\|^2 \mathrm{div}(\rho\nabla\psi_1)\tfrac{1}{4}\tfrac{1}{\rho^2}d\mathrm{vol}_g + \int_M g\big(\nabla\mathrm{div}(\rho\nabla\psi_1),\nabla\rho\big)\tfrac{1}{2\rho}d\mathrm{vol}_g$$

$$= -\int_M \|\nabla\rho\|^2 \mathrm{div}(\rho\nabla\psi_1)\tfrac{1}{4}\tfrac{1}{\rho^2}d\mathrm{vol}_g - \int_M \mathrm{div}(\rho\nabla\psi_1)\mathscr{L}_{\nabla\rho}\big(\tfrac{1}{2\rho}d\mathrm{vol}_g\big)$$

$$= -\int_M \|\nabla\rho\|^2 \mathrm{div}(\rho\nabla\psi_1)\tfrac{1}{4}\tfrac{1}{\rho^2}d\mathrm{vol}_g - \int_M \mathrm{div}(\rho\nabla\psi)g\big(\nabla\rho,\nabla\big(\tfrac{1}{2\rho}\big)\big)d\mathrm{vol}_g$$

$$\quad - \int_M \mathrm{div}(\rho\nabla\psi_1)\tfrac{1}{2\rho}\mathrm{div}(\nabla\rho)d\mathrm{vol}_g$$

$$= \int_M \Big[\tfrac{1}{4}\tfrac{\|\nabla\rho\|^2}{\rho^2} - \tfrac{1}{2}\tfrac{\Delta\rho}{\rho}\Big]\mathrm{div}(\rho\nabla\psi_1)d\mathrm{vol}_g = -\int_M \tfrac{\Delta(\sqrt{\rho})}{\sqrt{\rho}}\mathrm{div}(\rho\nabla\psi_1)d\mathrm{vol}_g. \quad (97)$$

In the above computation, we have used the formula given in (45).

Now, (96), (97) and Proposition 6 yield

$$(d\mathcal{H})_{(\rho,\nabla\phi)}X_{(\rho,\nabla\phi)}$$

$$= \int_M \left[\frac{1}{2}\|\nabla\phi\|^2 + V - \frac{\hbar^2}{2}\frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}}\right]\mathrm{div}\,(\rho\cdot\nabla\psi_1)\cdot d\mathrm{vol}_g$$

$$\quad + \int_M g(\nabla\phi,\nabla\psi_2)\,\rho\cdot d\mathrm{vol}_g$$

$$= -\int_M g\left(\nabla\psi_1,\nabla\left[\frac{1}{2}\|\nabla\phi\|^2 + V - \frac{\hbar^2}{2}\frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}}\right]\right)\rho\cdot d\mathrm{vol}_g$$

$$\quad + \int_M g(\nabla\phi,\nabla\psi_2)\,\rho\cdot d\mathrm{vol}_g$$

$$= (\Omega_{\mathcal{L}})_{(\rho,\nabla\phi)}(X_{\mathcal{H}},X). \quad (98)$$

The lemma follows. $\qquad\square$

Remark 13. We observe (as it was intended to), that the flow generated by the symplectic gradient $X_{\mathcal{H}} \in \mathfrak{X}(T\mathcal{D})$ corresponds exactly to the solutions of the Euler-Lagrange equations on $\mathcal{D}$ associated to the Lagrangian $\mathcal{L} : T\mathcal{D} \to \mathbb{R}$ introduced in (57), i.e., it satisfies the system of equations (46) (with $X \cong 0$).

We thus have a rigorous symplectic formulation of the Schrödinger equation via its hydrodynamical formulation which agrees with the corresponding Lagrangian formulation given in Corollary 4.

Lemma 5.5. *For $\rho \in \mathcal{D}$, $\nabla\phi \in \nabla C^\infty(M)$ and $F : TM \to \mathbb{R}$, we have :*

$$(X_{\widehat{F}})_{(\rho,\nabla\phi)} = \left(\rho,\nabla\phi,\mathbb{P}_\rho\big(\rho\left(\pi_*^{TM}\circ X_F\circ\nabla\phi\right),\nabla\big(F(\nabla\phi)\big)\right). \quad (99)$$

Proof. As for the proof of Lemma 5.4, we will use the vector field $X \in \mathfrak{X}(T\mathcal{D})$ introduced in (67), its flow φ_t^X which is given in Lemma 5.2, and the curve ρ_t defined in the proof of Lemma 5.4 (see (95)). We have :

$$(d\widehat{F})_{(\rho,\nabla\phi)}X_{(\rho,\nabla\phi)} = \left.\frac{d}{dt}\right|_0 \widehat{F}(\rho_t,\nabla\phi + t\nabla\psi_2)$$

$$= \left.\frac{d}{dt}\right|_0 \int_M F(\nabla\phi + t\nabla\psi_2)\,\rho_t\cdot d\mathrm{vol}_g$$

$$= \int_M \left[\mathbb{F}F(\nabla\phi)(\nabla\psi_2)\,\rho + F(\nabla\phi)\,\mathrm{div}\,(\rho\cdot\nabla\phi)\right]\cdot d\mathrm{vol}_g$$

$$= \int_M \left[\mathbb{F}F(\nabla\phi)(\nabla\psi_2) - g\big(\nabla\psi_1, \nabla\big(F(\nabla\phi)\big)\big) \right] \rho \cdot d\mathrm{vol}_g \,, \qquad (100)$$

where $\mathbb{F}F : TM \to T^*M$ is the Legendre transform of F, i.e., $\mathbb{F}F(u_x)(v_x) = \frac{d}{dt}|_0 F(u_x + tv_x)$.

We need to transform the term $\mathbb{F}F(\nabla\phi)(\nabla\psi_2)$ into a scalar product; to this end, we will use the following formula

$$\mathbb{F}F(u_x)(v_x) = g_x\big(\pi^{TM}_{*u_x}(X_F)_{u_x}, v_x\big), \qquad (101)$$

which holds whenever $u_x, v_x \in T_xM$, and where X_F is the symplectic gradient of F with respect to the symplectic form $\omega := \omega_L$ on TM associated to the Lagrangian $L(u_x) := \frac{1}{2}g(u_x, u_x)$ as in (59). This formula may be seen as follows. Recall that the symplectic form ω may be written (see for example [44]) :

$$\omega_{u_x}(A_{u_x}, B_{u_x}) = g_x(\pi^{TM}_{*u_x} A_{u_x}, KB_{u_x}) - g_x(\pi^{TM}_{*u_x} B_{u_x}, KA_{u_x}), \qquad (102)$$

where $u_x \in T_xM$, $A_{u_x}, B_{u_x} \in T_{u_x}TM$ and where $K : T(TM) \to TM$ is the connector associated to the Riemannian metric g. With (102), it is a simple matter to derive (101) :

$$
\begin{aligned}
\mathbb{F}F(u_x)(v_x) &= \frac{d}{dt}\bigg|_0 F(u_x + tv_x) = (dF)_{u_x}\frac{d}{dt}\bigg|_0 (u_x + tv_x) \\
&= \omega_{u_x}\left((X_F)_{u_x}, \frac{d}{dt}\bigg|_0 (u_x + tv_x)\right) \\
&= g_x\left(\pi^{TM}_{*u_x}(X_F)_{u_x}, K\frac{d}{dt}\bigg|_0 (u_x + tv_x)\right) \\
&\quad -g_x\left(\pi^{TM}_{*u_x}\frac{d}{dt}\bigg|_0 (u_x + tv_x), K(X_F)_{u_x}\right) \\
&= g_x\big(\pi^{TM}_{*u_x}(X_F)_{u_x}, v_x\big). \qquad (103)
\end{aligned}
$$

Of course, in the above computation we have used the following simple formulas:

$$K\frac{d}{dt}\bigg|_0 (u_x + tv_x) = v_x \quad \text{and} \quad \pi^{TM}_{*u_x}\frac{d}{dt}\bigg|_0 (u_x + tv_x) = 0. \qquad (104)$$

Taking into account (101), we may rewrite (100) as

$$
\begin{aligned}
&\int_M \left[\mathbb{F}F(\nabla\phi)(\nabla\psi_2) - g\big(\nabla\psi_1, \nabla\big(F(\nabla\phi)\big)\big) \right] \rho \cdot d\mathrm{vol}_g \\
=\ &\int_M \left[g\big(\pi^{TM}_* \circ X_F \circ \nabla\phi, \nabla\psi_2\big) - g\big(\nabla\psi_1, \nabla\big(F(\nabla\phi)\big)\big) \right] \rho \cdot d\mathrm{vol}_g \\
=\ &\int_M \left[g\Big(\mathbb{P}_\rho\big(\rho\,(\pi^{TM}_* \circ X_F \circ \nabla\phi)\big), \nabla\psi_2\Big) - g\big(\nabla\psi_1, \nabla\big(F(\nabla\phi)\big)\big) \right] \rho \cdot d\mathrm{vol}_g, \quad (105)
\end{aligned}
$$

from which we see that $(d\widehat{F})X = \Omega_{\mathcal{L}}(X_{\widehat{F}}, X)$, with $X_{\widehat{F}}$ such as defined in the right hand side of (99). The vector field $X_{\widehat{F}}$ is thus the symplectic gradient of F with respect to the symplectic form $\Omega_{\mathcal{L}}$. The lemma follows. $\square$

Lemma 5.6. *For $F, G : TM \to \mathbb{R}$, we have :*

$$\{\widehat{F}, \widehat{G}\}_{\mathcal{L}} = -\{\widehat{F, G}\}_L. \qquad (106)$$

Proof. For $\rho \in \mathcal{D}$, $\nabla\phi \in \nabla C^\infty(M)$, and, in view of Lemma 5.5, we have :

$$\{\widehat{F}, \widehat{G}\}_{\mathcal{L}}(\rho, \nabla\phi) = (\Omega_{\mathcal{L}})_{(\rho, \nabla\phi)}(X_{\widehat{F}}, X_{\widehat{G}})$$

$$= \int_M g\Big(\mathbb{P}_\rho\big(\rho\,(\pi_*^{TM} \circ X_F \circ \nabla\phi)\big), \nabla\big(G(\nabla\phi)\big)\Big)\, \rho \cdot d\mathrm{vol}_g$$

$$- \int_M g\Big(\mathbb{P}_\rho\big(\rho\,(\pi_*^{TM} \circ X_G \circ \nabla\phi)\big), \nabla\big(F(\nabla\phi)\big)\Big)\, \rho \cdot d\mathrm{vol}_g$$

$$= \int_M g\Big(\pi_*^{TM} \circ X_F \circ \nabla\phi, \nabla\big(G(\nabla\phi)\big)\Big)\, \rho \cdot d\mathrm{vol}_g$$

$$- \int_M g\Big(\pi_*^{TM} \circ X_G \circ \nabla\phi, \nabla\big(F(\nabla\phi)\big)\Big)\, \rho \cdot d\mathrm{vol}_g\,. \tag{107}$$

Moreover, we observe that if X is a vector field on M, then

$$g\big(\nabla\big(G(\nabla\phi)\big), X\big) = G_*(\nabla\phi)_*X = \omega\big(X_G \circ \nabla\phi, (\nabla\phi)_*X\big)$$

$$= g\big(\pi_*^{TM} \circ X_G \circ \nabla\phi, K(\nabla\phi)_*X\big) - g\big(\pi_*^{TM} \circ (\nabla\phi)_*X, KX_G \circ \nabla\phi\big)$$

$$= g\big(\pi_*^{TM} \circ X_G \circ \nabla\phi, \nabla_X\nabla\phi\big) - g\big(X, KX_G \circ \nabla\phi\big)\,, \tag{108}$$

and thus, denoting $\widetilde{X}_F := \pi_*^{TM} \circ X_G \circ \nabla\phi$ and $\widetilde{X}_G := \pi_*^{TM} \circ X_F \circ \nabla\phi$ for simplicity, we may rewrite (107) as :

$$\{\widehat{F}, \widehat{G}\}_{\mathcal{L}}(\rho, \nabla\phi) =$$

$$\int_M g\big(\widetilde{X}_G, \nabla_{\widetilde{X}_F}\nabla\phi\big)\, \rho \cdot d\mathrm{vol}_g - \int_M g\big(\widetilde{X}_F, KX_G \circ \nabla\phi\big)\, \rho \cdot d\mathrm{vol}_g$$

$$- \int_M g\big(\widetilde{X}_F, \nabla_{\widetilde{X}_G}\nabla\phi\big)\, \rho \cdot d\mathrm{vol}_g - \int_M g\big(\widetilde{X}_G, KX_F \circ \nabla\phi\big)\, \rho \cdot d\mathrm{vol}_g$$

$$= -\left[\int_M g(\widetilde{X}_F, KX_G \circ \nabla\phi)\, \rho \cdot d\mathrm{vol}_g - \int_M g(\widetilde{X}_G, KX_G \circ \nabla\phi)\, \rho \cdot d\mathrm{vol}_g\right]$$

$$+ \int_M g\big(\widetilde{X}_G, \nabla_{\widetilde{X}_F}\nabla\phi\big)\, \rho \cdot d\mathrm{vol}_g - \int_M g\big(\widetilde{X}_F, \nabla_{\widetilde{X}_G}\nabla\phi\big)\, \rho \cdot d\mathrm{vol}_g$$

$$= -\{\widehat{F,G}\}_L(\rho, \nabla\phi) + \int_M g\big(\widetilde{X}_G, \nabla_{\widetilde{X}_F}\nabla\phi\big)\, \rho \cdot d\mathrm{vol}_g$$

$$- \int_M g\big(\widetilde{X}_F, \nabla_{\widetilde{X}_G}\nabla\phi\big)\, \rho \cdot d\mathrm{vol}_g\,. \tag{109}$$

Clearly, we have to show that the last line in (109) vanishes. But this can be done easily with the help of the following formula

$$g\big(X, \nabla_Y Z\big) - g\big(Y, \nabla_X Z\big) = -d(Z^\flat)(X, Y)\,, \tag{110}$$

which holds for every vector fields $X, Y, Z \in \mathfrak{X}(M)$, and where $Z^\flat$ is the 1-form on M obtained via the musical isomorphism.

Using (110) and the fact that $d(d\phi) = 0$, one easily sees that the last line in (109) vanishes. The lemma follows. □

6. **The almost Hermitian structure of** $T\mathcal{D}$**.** In §4 and §5, we used the usual techniques of geometric mechanics to find a Lagrangian and Hamiltonian description of the Schrödinger equation, and we eventually arrived at the symplectic form $\Omega_{\mathcal{L}}$ on $T\mathcal{D}$ which encodes the dynamics of a quantum particle and whose explicit description is given in Proposition 6.

In this section, we show that $\Omega_{\mathcal{L}}$ is the fundamental 2-form of the almost Hermitian structure on $T\mathcal{D}$ coming from Dombrowski's construction [21] applied to the Wasserstein metric $g^{\mathcal{D}}$ and a (non-metric) connection $\nabla^{\mathcal{D}}$ on $\mathcal{D}$. We then discuss the integrability of this almost Hermitian structure as well as its relation to the Fubini-Study symplectic form on $\mathbb{P}(\mathcal{H})$, where $\mathcal{H} = L^2(M, \mathbb{C})$.

Recall that $T\mathcal{D}$, regarded as a vector bundle, is trivial and that $T\mathcal{D} \cong \mathcal{D} \times \nabla C^{\infty}(M)$ (isomorphism of Fréchet vector bundles, see Proposition 4 and Remark 4). Taking the tangent bundle on both side of this identification thus yields another isomorphism

$$T(T\mathcal{D}) \cong \mathcal{D} \times \nabla C^{\infty}(M) \times \nabla C^{\infty}(M) \times \nabla C^{\infty}(M) \qquad (111)$$

which is actually the map we considered in (62). Obviously, this defines an affine connection $\nabla^{\mathcal{D}}$ on $\mathcal{D}$ whose associated connector is

$$K^{\mathcal{D}} : T(T\mathcal{D}) \to T\mathcal{D}, \;\; (\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2) \mapsto (\rho, \nabla\psi_2). \qquad (112)$$

Together with the Wasserstein metric $g^{\mathcal{D}}$, we thus have a couple $(g^{\mathcal{D}}, \nabla^{\mathcal{D}})$ which yields, via Dombrowski's construction, an almost Hermitian structure $(g^{T\mathcal{D}}, J^{T\mathcal{D}}, \omega^{T\mathcal{D}})$ on $T\mathcal{D}$. For example,

$$(g^{T\mathcal{D}})_{(\rho, \nabla\phi)}\big((\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2), (\rho, \nabla\phi, \nabla\alpha_1, \nabla\alpha_2)\big)$$
$$= \int_M g(\nabla\psi_1, \nabla\alpha_1)\, \rho \cdot d\mathrm{vol}_g + \int_M g(\nabla\psi_2, \nabla\alpha_2)\, \rho \cdot d\mathrm{vol}_g. \qquad (113)$$

Comparing with Proposition 6 immediately yields the following result.

Proposition 8. *The fundamental 2-form $\omega^{T\mathcal{D}}$ of the almost Hermitian structure of $T\mathcal{D}$ associated to $(g^{\mathcal{D}}, \nabla^{\mathcal{D}})$ via Dombrowski's construction is $\Omega_{\mathcal{L}}$, i.e.*

$$\omega^{T\mathcal{D}} = \Omega_{\mathcal{L}}, \qquad (114)$$

where $\Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}}$ has been defined in (60).

Remark 14. As we saw in §5, the flow generated by the Hamiltonian vector field $X_{\mathcal{H}} \in \mathfrak{X}(T\mathcal{D})$ with respect to the symplectic form $\Omega_{\mathcal{L}}$ gives the dynamics of a quantum particle under the influence of a potential V (see (92) for the definition of $\mathcal{H} : T\mathcal{D} \to \mathbb{R}$). Hence, and since $\Omega_{\mathcal{L}} = \Omega^{T\mathcal{D}}$, we deduce that the dynamics of a quantum particle is encoded in $(\mathcal{D}, g^{\mathcal{D}}, \nabla^{\mathcal{D}})$. This is analogous to the fact that the dynamics of a finite dimensional quantum system is encoded in the triple $(\mathcal{P}_n^{\times}, h_F, \nabla^{(e)})$, where h_F and $\nabla^{(e)}$ are respectively the Fisher metric and the exponential connection on $\mathcal{P}_n^{\times}$ (see [58] and Section 2). In this sense, $g^{\mathcal{D}}$ and $\nabla^{\mathcal{D}}$ are infinite dimensional analogs of h_F and $\nabla^{(e)}$.

Let $T^{\mathcal{D}}$ and $R^{\mathcal{D}}$ be the torsion and the curvature tensor associated to the connection $\nabla^{\mathcal{D}}$ (as defined in (6)). Taking into account Remark 12, one easily finds that

Lemma 6.1. *We have:*

1. $T^{\mathcal{D}}\big((\rho, \nabla\phi), (\rho, \nabla\psi)\big) = \Big(\rho, \mathbb{P}_{\rho}\big(\rho\,[\nabla\phi, \nabla\psi]\big)\Big),$

2. $R^{\mathcal{D}} \equiv 0,$

where $\rho \in \mathcal{D}$ and $\nabla\phi, \nabla\psi \in \nabla C^{\infty}(M)$, and where the operator $\mathbb{P}_{\rho}$ has been defined in (80). In particular, $\nabla^{\mathcal{D}}$ is not the Levi-Civita connection associated to $g^{\mathcal{D}}$ (its torsion is not trivial).

Let $N^{T\mathcal{D}}$ be the Nijenhuis tensor of $J^{T\mathcal{D}}$, i.e.,

$$N^{T\mathcal{D}}(X,Y) :=$$
$$[X,Y] - [J^{TM}X, J^{TM}Y] + J^{T\mathcal{D}}[J^{T\mathcal{D}}X,Y] + J^{T\mathcal{D}}[X, J^{T\mathcal{D}}Y], \quad (115)$$

where $X,Y \in \mathfrak{X}(T\mathcal{D})$.

Using Lemma 5.3, a straightforward calculation shows that

Proposition 9. *Let $J^{T\mathcal{D}}$ be the almost complex structure on $T\mathcal{D}$ associated to $(g^{\mathcal{D}}, \nabla^{\mathcal{D}})$ via Dombrowski's construction, and let $N^{T\mathcal{D}}$ be its Nijenhuis tensor. Then,*

$$N^{T\mathcal{D}}\big((\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2), (\rho, \nabla\phi, \nabla\alpha_1, \nabla\alpha_2)\big)$$
$$= \Big(\rho, \nabla\phi, \mathbb{P}_\rho\big\{\rho[\nabla\alpha_1, \nabla\psi_1] - \rho[\nabla\alpha_2, \nabla\psi_2]\big\}, 0\Big)$$
$$+ \Big(\rho, \nabla\phi, 0, \mathbb{P}_\rho\big\{\rho[\nabla\psi_2, \nabla\alpha_1] + \rho[\nabla\psi_1, \nabla\alpha_2]\big\}\Big), \quad (116)$$

where $\rho \in \mathcal{D}$ and where $\nabla\phi, \nabla\psi_1, \nabla\psi_2, \nabla\alpha_1, \nabla\alpha_2 \in \nabla C^\infty(M)$.

Corollary 5. *The almost Hermitian structure $J^{T\mathcal{D}}$ of $T\mathcal{D}$ is not integrable, i.e., $N^{T\mathcal{D}} \not\equiv 0$.*

We now want to relate $\omega^{T\mathcal{D}}$ to the Fubini-Study symplectic form ω_{FS} of the complex projective space $\mathbb{P}(\mathcal{H})$, where $\mathcal{H} := L^2(M, \mathbb{C})$ is the space of square-integrable complex functions on M endowed with the L^2-inner product $\langle f, g\rangle := \int_M \bar{f} g \cdot d\mathrm{vol}_g$.

Define

$$F : T\mathcal{D} \to \mathcal{H}, \quad (\rho, \nabla\phi) \mapsto \sqrt{\rho}\, e^{-\frac{i}{\hbar}\phi_0}, \quad (117)$$

where $\phi_0 \in C^\infty(M)$ is uniquely determined by the conditions $\nabla\phi = \nabla\phi_0$ and $\int_M \phi_0 \cdot d\mathrm{vol}_g = 0$.

Regarding $T\mathcal{D}$ as a Fréchet manifold and $\mathcal{H}$ as a Hilbert manifold, we have the following result.

Lemma 6.2. *The map F is smooth and its derivative at a point $(\rho, \nabla\phi)$ is given by*

$$F_{*(\rho,\nabla\phi)}(\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2) = \left[\frac{\mathrm{div}(\rho\nabla\psi_1)}{2\rho} - \frac{i}{\hbar}(\psi_2)_0\right] \cdot F(\rho, \nabla\phi), \quad (118)$$

where $\nabla\psi_1, \nabla\psi_2 \in \nabla C^\infty(M)$.

Proof. To see the smoothness, write $F = i \circ \widetilde{F}$, where $\widetilde{F} : T\mathcal{D} \to C^\infty(M, \mathbb{C})$, $(\rho, \nabla\phi)$ $\mapsto \sqrt{\rho}\, e^{-\frac{i}{\hbar}\phi_0}$ and where $i : C^\infty(M, \mathbb{C}) \hookrightarrow L^2(M, \mathbb{C})$ is the inclusion map. The C^∞-topology being finer than the L^2-topology, i is a continuous map between Fréchet spaces, and consequently it is smooth. The smoothness of $\widetilde{F}$ is a consequence of the convenient calculus together with the fact that $C_0^\infty(M, \mathbb{R}) \to \nabla C^\infty(M, \mathbb{R})$, $\phi \mapsto \nabla\phi$ is an isomorphism of Fréchet spaces (recall that $C_0^\infty(M, \mathbb{R})$ is the space of smooth functions whose integrals are zero). We now compute the derivative of F. Let ρ_t be a smooth curve in $\mathcal{D}$ satisfying $\rho_0 = \rho$ and $\frac{\partial \rho_t}{\partial t} = \mathrm{div}(\rho_t \nabla\psi_1)$. The curve $c(t) = (\rho_t, \nabla\phi + t\nabla\psi_2)$ is thus a smooth curve in $T\mathcal{D}$ which satisfies $c(0) = (\rho, \nabla\phi)$ and $\frac{d}{dt}\big|_0 c(t) = (\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2)$, and we have:

$$F_{*_{(\rho,\nabla\phi)}}(\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2) = \frac{d}{dt}\Big|_0 F\big(c(t)\big)$$

$$= \frac{d}{dt}\Big|_0 F(\rho_t, \nabla\phi + t\nabla\psi_2) = \frac{d}{dt}\Big|_0 \sqrt{\rho_t}\, e^{-\frac{i}{\hbar}\left(\phi_0 + t(\psi_2)_0\right)}$$

$$= \frac{1}{2\sqrt{\rho}}\Big(\frac{\partial}{\partial t}\Big|_0 \rho_t\Big) e^{-\frac{i}{\hbar}\phi_0} - \frac{i}{\hbar}(\psi_2)_0 \sqrt{\rho}\, e^{-\frac{i}{\hbar}\phi_0}$$

$$= \frac{\operatorname{div}(\rho\nabla\psi_1)}{2\rho} F(\rho, \nabla\phi) - \frac{i}{\hbar}(\psi_2)_0 F(\rho, \nabla\phi). \tag{119}$$

The lemma follows. $\qquad\square$

Let us now recall a few facts about the complex projective space $\mathbb{P}(\mathcal{H})$. Let $B := \{f \in \mathcal{H} \,|\, \langle f, f \rangle = 1\}$ be the unit sphere and $j : B \hookrightarrow \mathcal{H}$ the associated inclusion map. We denote by $\pi : B \to \mathbb{P}(\mathcal{H})$ the projection induced by the action of the circle $S^1 := \{e^{i\theta} \,|\, \theta \in \mathbb{R}\}$ on B (the action being $e^{i\theta} \cdot f := f\, e^{i\theta}$).

Regarded as a real vector space, it is known that $\mathcal{H}$ is a symplectic manifold whose symplectic form is the imaginary part of the Hermitian inner product $\langle\,,\,\rangle$ (see [11]), and that

$$\pi^*\omega_{FS} = j^* \operatorname{Im}(\langle\,,\,\rangle). \tag{120}$$

Since π is a submersion, this formula characterizes the Fubini-Study symplectic form[12].

Having this in mind, let us now return to the injective map $T : T\mathcal{D} \to \mathbb{P}(\mathcal{H})$ that we defined in Remark 9.

Proposition 10. *The map T is smooth and satisfies*

$$T^*\omega_{FS} = \frac{1}{2\hbar}\, \omega^{T\mathcal{D}}, \tag{121}$$

where $\omega^{T\mathcal{D}}$ is the symplectic form coming from Dombrowski's construction applied to $(g^{\mathcal{D}}, \nabla^{\mathcal{D}})$.

Proof. Smoothness follows from the fact that $T = \pi \circ F$. To see (121), observe that

$$T^*\omega_{FS} = (\pi \circ F)^*\omega_{FS} = F^*\pi^*\omega_{FS} = F^*j^* \operatorname{Im}(\langle\,,\,\rangle) = (j \circ F)^* \operatorname{Im}(\langle\,,\,\rangle). \tag{122}$$

We thus have to show that $(j \circ F)^* \operatorname{Im}(\langle\,,\,\rangle) = \frac{1}{2\hbar} \omega^{T\mathcal{D}}$. Taking into account Lemma 6.2, we have

$$\Big((j \circ F)^* \operatorname{Im}(\langle\,,\,\rangle)\Big)_{(\rho,\nabla\phi)} \Big((\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2), (\rho, \nabla\phi, \nabla\alpha_1, \nabla\alpha_2)\Big)$$

$$= \operatorname{Im}\Big\langle j_* F_*(\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2),\ j_* F_*(\rho, \nabla\phi, \nabla\alpha_1, \nabla\alpha_2) \Big\rangle$$

$$= \operatorname{Im}\Big\langle \Big[\frac{\operatorname{div}(\rho\nabla\psi_1)}{2\rho} - \frac{i}{\hbar}(\psi_2)_0\Big] \cdot F(\rho, \nabla\phi),\ \Big[\frac{\operatorname{div}(\rho\nabla\alpha_1)}{2\rho} - \frac{i}{\hbar}(\alpha_2)_0\Big] \cdot F(\rho, \nabla\phi) \Big\rangle$$

---

[12]Similarly, one has $\pi^* g_{FS} = j^* \operatorname{Real}(\langle\,,\,\rangle)$, where g_{FS} is the Fubini-Study metric on $\mathbb{P}(\mathcal{H})$. Depending on the convention, the Fubini-Study metric and symplectic form may appear in the literature multiplied by a positive constant.

$$
\begin{aligned}
&= \quad \frac{1}{2\hbar} \left\{ \int_M (\psi_2)_0 \operatorname{div}(\rho \nabla \alpha_1) \, d\mathrm{vol}_g - \int_M (\alpha_2)_0 \operatorname{div}(\rho \nabla \psi_1) \, d\mathrm{vol}_g \right\} \\
&= \quad \frac{1}{2\hbar} \left\{ - \int_M g(\nabla \alpha_1, \nabla \psi_2) \, \rho \, d\mathrm{vol}_g + \int_M g(\nabla \alpha_2, \nabla \psi_1) \, \rho \, d\mathrm{vol}_g \right\} \\
&= \quad \frac{1}{2\hbar} \, \omega^{T\mathcal{D}} \Big((\rho, \nabla\phi, \nabla\psi_1, \nabla\psi_2), (\rho, \nabla\phi, \nabla\alpha_1, \nabla\alpha_2) \Big).
\end{aligned}
\tag{123}
$$

The proposition follows. $\qquad\square$

Remark 15. One may ask if a similar result holds for the Fubini-Study metric g_{FS} on $\mathbb{P}(\mathcal{H})$, i.e., if $T^* g_{FS} = \lambda \, g^{T\mathcal{D}}$ for some $\lambda \in \mathbb{R}$. By keeping track of the real part in (123), one sees that it is not the case (which is not surprising in view of Corollary 5).

7. **Appendix: Tame Fréchet manifolds.** In this section, we review very briefly the category of tame Fréchet manifolds introduced by Hamilton in [31].

Definition 7.1.   1. A graded Fréchet space $(F, \{\| \, . \, \|_n\}_{n \in \mathbb{N}})$, is a Fréchet space F whose topology is defined by a collection of seminorms $\{\| \, . \, \|_n\}_{n \in \mathbb{N}}$ which are increasing in strength:

$$
\|x\|_0 \le \|x\|_1 \le \|x\|_2 \le \cdots
\tag{124}
$$

for all $x \in F$.

2. A linear map $L : F \to G$ between two graded Fréchet spaces F and G is tame (of degree r and base b) if for all $n \ge b$, there exists a constant $C_n > 0$ such that for all $x \in F$,

$$
\|L(x)\|_n \le C_n \|x\|_{n+r}.
\tag{125}
$$

3. If $(B, \| \, . \, \|_B)$ is a Banach space, then $\Sigma(B)$ denotes the graded Fréchet space of all sequences $\{x_k\}_{k \in \mathbb{N}}$ of B such that for all $n \ge 0$,

$$
\|\{x_k\}_{k \in \mathbb{N}}\|_n := \Sigma_{k=0}^\infty e^{nk} \|x_k\|_B < \infty.
\tag{126}
$$

4. A graded Fréchet space F is tame if there exist a Banach space B and two tame linear maps $i : F \to \Sigma(B)$ and $p : \Sigma(B) \to F$ such that $p \circ i$ is the identity on F.

5. Let F, G be two tame Fréchet spaces, U an open subset of F and $f : U \to G$ a map. We say that f is a smooth tame map if f is smooth[13] and if for every $k \in \mathbb{N}$ and for every $(x, u_1, ..., u_k) \in U \times F \times \cdots F$, there exist a neighborhood V of $(x, u_1, ..., u_k)$ in $U \times F \times \cdots F$ and $b_k, r_0, ..., r_k \in \mathbb{N}$ such that for every $n \ge b_k$, there exists $C_{k,n}^V > 0$ such that

$$
\begin{aligned}
\|d^k f(y)\{v_1, ..., v_k\}\|_n \; &\le \\
C_{k,n}^V \big(1 &+ \|y\|_{n+r_0} + \|v_1\|_{n+r_1} + \cdots + \|v_k\|_{n+r_k} \big),
\end{aligned}
\tag{127}
$$

for every $(y, v_1, ..., v_k) \in V$, where $d^k f : U \times F \times \cdots \times F \to G$ denotes the kth derivative of f.

Remark 16. In this paper, we use interchangeably the notation $(df)(x)\{v\}$ or $f_{*_x} v$ for the first derivative of f at a point x in direction v.

---

[13]By smooth we mean that $f : U \subseteq F \to G$ is continuous and that for all $k \in \mathbb{N}$, the kth derivative $d^k f : U \times F \times \cdots \times F \to G$ exists and is jointly continuous on the product space, such as described in [31].

Example 5. Let M be a compact manifold of dimension n. Take an atlas $\mathcal{A} = \{(U_i, \varphi_i) \mid i = 1, ..., r\}$ for M and a family of compact subsets $\{K_i \mid i = 1, ..., r\}$ such that $K_i \subseteq U_i$ and $\cup_{i=1}^{r} K_i = M$. For $q \in \mathbb{N}$, we define a seminorm on the space $C^\infty(M)$ of smooth real valued functions on M as follows:

$$\|f\|_q := \sum_{i=1}^{r} \sum_{|k| \leq q} \sup_{x \in \varphi_i(K_i)} \left| \frac{\partial^{|k|}(f \circ \varphi_i^{-1})}{\partial x_1^{k_1} ... \partial x_n^{k_n}}(x) \right|, \tag{128}$$

where $k = (k_1, ..., k_n) \in \mathbb{N}^n$ is a multi-index with length $|k| := k_1 + ... + k_n$. It is well-known that the family $\{\|\ \|_q\}_{q \in \mathbb{N}}$ defines a Fréchet topology on $C^\infty(M)$ which is independent of the choices of $\mathcal{A}$ and $\{K_i\}_{i=1,...,n}$ (see for example [20], page 237). Obviously, this family of seminorms is also increasing in strength. Therefore, $C^\infty(M)$ endowed with $\{\|\ \|_q\}_{q \in \mathbb{N}}$ is a graded Fréchet space, and one can show that it is actually tame (see [31], Theorem 1.3.6). More generally, the space of smooth sections $\Gamma(E)$ of a vector bundle $\pi : E \to M$ over M can also be given the structure of a tame space (see [31], Corollary 1.3.9).

As one may notice, tame Fréchet spaces and smooth tame maps form a category, and it is thus natural to define a tame Fréchet manifold as a Hausdorff topological space with an atlas of coordinates charts taking their value in tame Fréchet spaces, such that the coordinate transition functions are all smooth tame maps (see [31]). The definition of a tame smooth map between tame Fréchet manifolds is then straightforward, and we thus obtain a subcategory of the category of Fréchet manifolds.

In order to avoid confusion, let us also make precise our notion of submanifold. A subset $\mathcal{M}$ of a tame Fréchet manifold $\mathcal{N}$, endowed with the trace topology, is a submanifold, if for every point $x \in \mathcal{M}$, there exists a chart $(\mathcal{U}, \varphi)$ of $\mathcal{N}$ such that $x \in \mathcal{U}$ and such that $\varphi(\mathcal{U} \cap \mathcal{M}) = U \times \{0\}$, where $\varphi(\mathcal{U}) = U \times V$ is a product of two open subsets of tame Fréchet spaces. Note that a submanifold of a tame Fréchet manifold is also a tame Fréchet manifold.

For the sake of completeness, let us state here the raison d'être of tame Fréchet spaces and tame Fréchet manifolds (see [31]) :

Theorem 7.2 (Nash-Moser inverse function Theorem). *Let F, G be two tame Fréchet spaces, U an open subset of F and $f : U \to G$ a smooth tame map. If there exists an open subset $V \subseteq U$ such that*

  1. $df(x) : F \to G$ is an linear isomorphism for all $x \in V$,
  2. the map $V \times G \to F$, $(x, v) \mapsto \left(df(x)\right)^{-1}\{v\}$ is a smooth tame map,

then f is locally invertible on V and each local inverse is a smooth tame map.

Remark 17. The Nash-Moser inverse function Theorem is important in geometric hydrodynamics, since one of its most important geometric objects, namely the group of all smooth volume preserving diffeomorphims $\mathrm{SDiff}_\mu(M) := \{\varphi \in \mathrm{Diff}(M) \mid \varphi^*\mu = \mu\}$ of an oriented manifold (M, μ), can only be given a rigorous Fréchet Lie group structure by using an inverse function theorem (at least up to now). To our knowledge, only two authors succeeded in doing this. The first was Omori who showed and used an inverse function theorem in terms of ILB-spaces ("inverse limit of Banach spaces", see [66]), and later on, Hamilton with his category of tame Fréchet spaces together with the Nash-Moser inverse function Theorem (see [31]). Nowadays, it is nevertheless not uncommon to find mistakes or big gaps in the literature

when it comes to the differentiable structure of $\mathrm{SDiff}_\mu(M)$, even in some specialized textbooks in infinite dimensional geometry.

Finally, and quite apart from the category of Hamilton, let us remind one of the most useful result of the convenient calculus (see [43]) :

Lemma 7.3. *Let F, G be two Fréchet spaces, U an open subset of F and $f : U \to G$ a map. Then f is smooth is the sense of Hamilton (see footnote 13), if and only if $f \circ c : I \to \mathbb{R}$ is a smooth curve in G whenever $c : I \to U$ is a smooth curve in U.*

If $F = \Gamma(E)$ is the space of smooth sections of a vector bundle $\pi : E \to M$ over a compact manifold M, then the set of smooth curves in $\Gamma(E)$ is naturally identified with the set of smooth maps $c : I \times M \to E$ satisfying $\pi(c(t, x)) = x$ for all $t \in I$ and all $x \in M$. As a consequence, if M, N are manifolds, M being compact, then a smooth curve in the Fréchet manifold $C^\infty(M, N)$ may by identified with a smooth map $f : I \times M \to N$, its time derivative being identified with the partial derivative of f with respect to t. From this together with Lemma 7.3, it is usually easy to show that a map defined between submanifolds of spaces of maps is smooth: it suffices to compose this map with a smooth curve, and then to check that the result is smooth in the "finite dimensional sense" with respect to all the "finite dimensional" variables (see [43]).

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