

A NOTE ON THE SOLUTION OF THE EQUATION

$$g_{\mu\nu;\lambda} - \frac{2}{3} (\Gamma_\nu g_{\mu\lambda} + \Gamma_\lambda g_{\mu\nu}) = 0$$

BY L. J. GREGORY AND A. H. KLOTZ

Department of Applied Mathematics, University of Sydney*

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The affine connection of Einstein's weak and strong unified field theories can be expressed in terms of the fundamental tensor by solving a system of sixty-four algebraic equations. In the Einstein-Kaufman theory, there is again a system of sixty-four algebraic equations but it is shown that the rank of this system is only sixty.

1. Introduction

Mme Tonnelat obtained (Ref. [1]) a general solution of the equation

$$g_{\mu\nu;\lambda} \equiv g_{\mu\nu,\lambda} - \Gamma_{\mu\lambda}^\sigma g_{\sigma\nu} - \Gamma_{\lambda\nu}^\sigma g_{\mu\sigma} = 0, \quad (1)$$

of Einstein's non-symmetric unified field theory, expressing the components $\Gamma_{\mu\nu}^\lambda$ of the affine connection in terms of the components $g_{\mu\nu}$ of the fundamental tensor and their first derivatives. Further work (Ref. [2]) on the above theory revealed that it may be more appropriate to consider first the equation

$$g_{\mu\nu;\lambda} - \frac{2}{3} (\Gamma_\nu g_{\mu\lambda} + \Gamma_\lambda g_{\mu\nu}) = 0. \quad (2)$$

Since these may be written in the form

$$g_{\mu\nu;\lambda}(\Delta_{\beta\gamma}^\alpha) = 0, \quad (3)$$

where

$$\Delta_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^\alpha + \frac{2}{3} (\delta_\beta^\alpha \Gamma_\gamma), \quad (4)$$

for which

$$\Delta_\beta \equiv \Delta_{\beta\alpha}^\alpha = 0, \quad (5)$$

* Address: Department of Applied Mathematics, University of Sydney, N. S. W. 2006 Sydney Australia.

(in Einstein's and Tonnelat's notation adopted throughout, a hook denoting the skew symmetric part), it appears that equation (2) cannot determine Γ_λ . However, an explicit proof of this, namely that the equation (2) considered as an algebraic equation for $\Gamma_{\mu\nu}^\lambda$ is of rank 60, can be obtained only by following through the Tonnelat solution for the new equations. The purpose of this note is to sketch such a proof while emphasising the generality of the Tonnelat method.

2. The Tonnelat method

The general method depends on splitting the equation (2) into forty equations for the symmetric part $\Gamma_{\mu\nu}^\lambda$ of the affine connection and twenty four equations for its skew symmetric part (a tensor) $\Gamma_{\mu\nu}^\lambda$, and solving the latter by purely tensorial techniques. If we write

$$g_{\mu\nu} = h_{\mu\nu} = h_{\nu\mu}, \quad g_{\mu\nu} = k_{\mu\nu} = -k_{\nu\mu}, \tag{6}$$

$h^{\mu\nu}$, $k^{\mu\nu}$ respectively, for their tensorial inverses and h , k for their determinants (so that $h < 0$ and $k \geq 0$) we readily find that

$$\Gamma_{\mu\nu}^\lambda = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} + u_{\mu\nu}^\lambda, \tag{7}$$

where

$$u_{\mu\nu,\rho} = h_{\rho\lambda} u_{\mu\nu}^\lambda = \Gamma_{\mu\rho}^\sigma k_{\sigma\nu} + \Gamma_{\nu\rho}^\sigma k_{\sigma\mu} - \frac{1}{3} \Gamma_\mu g_{\rho\nu} - \frac{1}{3} \Gamma_\nu g_{\rho\mu}, \tag{8}$$

and $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$ are the Christoffel brackets formed from $h_{\mu\nu}$. Also

$$\Gamma_{\mu\nu,\rho}^\lambda \equiv h_{\rho\lambda} \Gamma_{\mu\nu}^\lambda = -\frac{1}{2} k_{\mu\nu\rho} + \nabla_\rho k_{\mu\nu} - (u_{\mu\rho}^\sigma k_{\sigma\nu} - u_{\nu\rho}^\sigma k_{\sigma\mu}) + \frac{1}{3} \Gamma_\mu g_{\rho\nu} - \frac{1}{3} \Gamma_\nu g_{\rho\mu} - \frac{2}{3} \Gamma_\rho k_{\mu\nu}, \tag{9}$$

where $k_{\mu\nu\rho} = k_{\mu\nu,\rho}^{\cdot}$ (the cycle sum) (we shall use a similar notation for other three index quantities, $\Gamma_{\mu\nu\rho}$ and $u_{\mu\nu\rho}$), and Δ_ρ is the covariant differentiation operation with respect to $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$. Then

$$u_{\mu\nu\rho} = -\frac{2}{3} \Gamma_\mu h_{\nu\rho} \quad \text{and} \quad \Gamma_{\mu\nu\rho} = -\frac{1}{2} k_{\mu\nu\rho}. \tag{10}$$

Algebraic equations for $\Gamma_{\mu\nu,\rho}^\lambda$ are obtained from the equations (8) and (9) in a self evident way but their solutions relies crucially on the introduction of two linearly independent operations. The first is a repeated raising of a covariant index with $h^{\mu\nu}$ and lowering of it with $k_{\mu\nu}$, which is called the "barring" of that index. Thus

$$\begin{aligned} A^{\dots\bar{\lambda}\dots} &= k_{\lambda\alpha} h^{\alpha\beta} A^{\dots\beta\dots}; \\ A^{\dots\bar{\lambda}\dots} &= k_{\lambda\alpha} h^{\alpha\beta} k_{\beta\gamma} h^{\gamma\epsilon} A^{\dots\epsilon\dots}, \end{aligned} \tag{11}$$

etc. This, in a way, avoids raising indices with $k^{\mu\nu}$ (which may not exist, although we can always write

$$k^{\mu\nu} = \frac{1}{2\sqrt{k}} \varepsilon^{\mu\nu\alpha\beta} k_{\alpha\beta}, \quad k_{\mu\nu} = \frac{1}{2}\sqrt{k} \varepsilon_{\mu\nu\alpha\beta} k^{\alpha\beta}. \quad (12)$$

Since

$$\sqrt{k} = \frac{1}{8} \varepsilon^{\mu\nu\varrho\sigma} k_{\mu\nu} k_{\varrho\sigma},$$

and

$$g = h + k + \frac{1}{2} h^{\mu\varrho} h^{\nu\sigma} k_{\mu\nu} k_{\varrho\sigma},$$

it is not difficult to show that

$$A_{\varrho}^{\bar{\bar{}}} = -\frac{k}{h} A_{\varrho} - \frac{1}{h} (g - h - k) A_{\varrho}^{\bar{\bar{}}}, \quad (13)$$

$$\equiv \omega^2 A_{\varrho}^{\bar{\bar{}}} + (1 - j^2) A_{\varrho}^{\bar{\bar{}}}, \quad \text{say, where} \quad \omega = \sqrt{-\frac{k}{h}} \quad \text{and} \quad j^2 = \frac{g - k}{h}.$$

The second operation is the well known duality one:

$$A_{\dots\mu\nu\dots}^* = \frac{\sqrt{-h}}{2} \varepsilon_{\mu\nu\alpha\beta} h^{\alpha\varrho} h^{\beta\sigma} A_{\dots\varrho\sigma\dots} \quad (14)$$

We have a useful relation between these, valid for any vector A_{λ} :

$$2h^{\sigma\lambda} k_{\lambda\varrho} k_{\sigma[\mu}^* A_{\nu]} = -\sqrt{-h} h^{\sigma\lambda} \varepsilon_{\mu\nu\varrho\sigma} A_{\lambda}, \quad (15)$$

the square bracket this time denoting the skew symmetric part and the star referring (as throughout the sequel) to the pair $(\mu\nu)$ only. The barring and starring operations can be performed on both sides of a tensor equation (the latter, of course, only with respect to a skew symmetric pair of indices), leading in general to additional equations. There are some other identities which it is worthwhile to recall in the reduction of the equations of interest to us. These are

$$h \varepsilon_{\mu\nu\alpha\beta} h^{\alpha\lambda} h^{\beta\sigma} = \varepsilon^{\lambda\sigma\alpha\beta} h_{\alpha\mu} h_{\beta\nu}, \quad (16)$$

and

$$\begin{aligned} \varepsilon_{\alpha\lambda\mu\nu} \varepsilon^{\alpha\beta\gamma\eta} &= (\delta_{\alpha}^{\beta} \delta_{\mu}^{\gamma} \delta_{\nu}^{\eta} - \delta_{\mu}^{\beta} \delta_{\alpha}^{\gamma} \delta_{\nu}^{\eta}), \\ \varepsilon_{\alpha\beta\mu\nu} \varepsilon^{\alpha\beta\gamma\eta} &= 2!(\delta_{\mu}^{\gamma} \delta_{\nu}^{\eta} - \delta_{\nu}^{\gamma} \delta_{\mu}^{\eta}), \quad \varepsilon_{\alpha\beta\gamma\nu} \varepsilon^{\alpha\beta\gamma\eta} = 3!\delta_{\nu}^{\eta}. \end{aligned} \quad (17)$$

It follows now from the equations (8) and (9) that

$$\begin{aligned} \Gamma_{\check{\mu}\check{\nu},\varrho} - h^{\sigma\lambda} [\bar{\Gamma}_{\mu\lambda\varrho} k_{\sigma\nu} - \bar{\Gamma}_{\nu\lambda\varrho} k_{\sigma\mu}] - 2h^{\sigma\lambda} [\Gamma_{\varrho\lambda,\check{\mu}} k_{\sigma\nu} - \Gamma_{\varrho\lambda,\check{\nu}} k_{\sigma\mu}] - \bar{\Gamma}_{\mu\nu\varrho} + \Gamma_{\mu\nu,\check{\varrho}} \\ = -\frac{1}{2} k_{\mu\nu} + \nabla_{\varrho} k_{\mu\nu} + \frac{1}{3} [\Gamma_{\mu}(h_{\nu\varrho} + h^{\sigma A} k_{\lambda\varrho} k_{\sigma\nu}) - \Gamma_{\nu}(h_{\mu\varrho} + h^{\sigma\lambda} k_{\lambda\varrho} k_{\sigma\mu})], \end{aligned} \quad (18)$$

where $\bar{\Gamma}_{\mu\nu\varrho} = \Gamma_{\check{\mu},\check{\nu},\check{\varrho}}$, $\bar{\Gamma}_{\mu\lambda\varrho} = \Gamma_{\check{\mu},\check{\lambda},\check{\varrho}}$, the bars not shifting, of course, in the cyclic summation.

The quantities in the square brackets on the left hand side of the equation (18) and $\bar{\Gamma}$ have been calculated in Tonnelat's paper directly from definitions to give the equation

$$\begin{aligned} \Gamma_{\mu\nu, \varrho} - 2\omega\Gamma_{\mu\nu, \varrho}^* + \Gamma_{\mu\nu, \bar{\varrho}} = & -\frac{1}{2}k_{\mu\nu\varrho} + \nabla_{\varrho}k_{\mu\nu} - \frac{1}{2}(1-j^2)k_{\mu\nu\varrho} \\ & + \frac{1}{2}h^{\sigma\lambda}h^{\tau\pi}(\sqrt{k}\varepsilon_{\mu\nu\sigma\tau} - k_{\sigma\tau}k_{\mu\nu})k_{\varrho\lambda\pi} - \frac{\sqrt{k}}{4}\varepsilon_{\mu\nu\varrho\lambda}h^{\alpha\sigma}h^{\lambda\tau}k_{\sigma\tau}k^{\beta\gamma}k_{\alpha\beta\gamma} \\ & + \frac{1}{3}[\Gamma_{\mu}(h_{\nu\varrho} + h^{\sigma\lambda}k_{\lambda\varrho}k_{\sigma\nu}) - \Gamma_{\nu}(h_{\mu\varrho} + h^{\sigma\lambda}k_{\lambda\varrho}k_{\sigma\mu})] \\ & - \sqrt{k}\varepsilon_{\mu\nu\lambda\pi}h^{\lambda\sigma}h^{\pi\tau}k_{\sigma\varrho}B_{\tau} + \sqrt{k}\varepsilon_{\mu\nu\varrho\sigma}k^{\lambda\sigma}A_{\lambda} - 2k_{\mu\nu}A_{\varrho}, \end{aligned} \quad (19)$$

where

$$A_{\varrho} = \frac{1}{2}h^{\mu\lambda}h^{\nu\tau}k_{\mu\nu}\Gamma_{\lambda\tau, \varrho} \quad \text{and} \quad B_{\varrho} = \frac{1}{2}k^{\mu\nu}\Gamma_{\mu\nu, \varrho}. \quad (20)$$

3. Calculation of A_{ϱ} & B_{ϱ} and the final equation

Contracting equations (2) with $g^{\mu\nu}$ gives

$$\Gamma_{\varrho\sigma}^{\sigma} = \partial_{\varrho} \ln \sqrt{-g} - \frac{5}{3}\Gamma_{\varrho}. \quad (21)$$

Using equations (21) and (9) we obtain directly from the definition of B_{ϱ}

$$B_{\varrho} = -\frac{1}{4}k^{\mu\nu}k_{\mu\nu\varrho} + \partial_{\varrho} \ln \sqrt{-\frac{k}{g}} + \frac{1}{6}k^{\mu\nu}(\Gamma_{\mu}g_{\nu\varrho} - \Gamma_{\nu}g_{\mu\varrho}) + \frac{1}{3}\Gamma_{\varrho}. \quad (22)$$

The terms proportional to Γ_{λ} are absent from the corresponding formula of Mme Tonnelat.

Similarly, contracting equation (18) with $k^{\mu\nu}$ and substituting from (22), gives

$$A_{\varrho} = -\frac{1}{4}h^{\alpha\mu}h^{\beta\nu}k_{\alpha\beta}k_{\mu\nu\varrho} + \partial_{\varrho} \ln \sqrt{\frac{g}{h}} + \frac{1}{3}h^{\sigma\lambda}\Gamma_{\sigma}k_{\lambda\varrho}. \quad (23)$$

We may note that since B_{ϱ} appears in the equation (19) multiplied by $\sqrt{k}$, the result which we are about to write down is still valid when the determinant of $k_{\mu\nu}$ vanishes.

We now substitute for A_{ϱ} and B_{ϱ} into the equation (19) and obtain

$$\Gamma_{\mu\nu, \varrho} - 2\omega\Gamma_{\mu\nu, \varrho}^* + \Gamma_{\mu\nu, \bar{\varrho}} = J_{\mu\nu, \varrho} + W_{\mu\nu, \varrho}, \quad (24)$$

where

$$\begin{aligned} J_{\mu\nu, \varrho} = & -\frac{1}{2}k_{\mu\nu\varrho} + \nabla_{\varrho}k_{\mu\nu} + \frac{1}{2}\omega k_{\mu\nu\varrho}^* + \frac{1}{4}\omega k_{\mu\nu}^*k^{\sigma\tau}k_{\sigma\tau\varrho} \\ & - (k_{\mu\nu}\delta_{\varrho}^{\lambda} - \frac{1}{2}\sqrt{k}\varepsilon_{\mu\nu\varrho\sigma}k^{\lambda\sigma})\partial_{\lambda} \ln(j^2 - \omega^2) - \frac{1}{2}\omega(\varepsilon_{\mu\nu\varrho\sigma}^*k^{\lambda\sigma} - k_{\mu\nu}^*\delta_{\varrho}^{\lambda})\partial_{\lambda} \ln\left(\frac{j^2 - \omega^2}{\omega^2}\right), \end{aligned} \quad (25)$$

and is the same as the corresponding expression of Mme Tonnelat, and

$$W_{\mu\nu, \varrho} = \frac{2}{3}h^{\sigma\lambda}[k\varepsilon_{\mu\nu\varrho\sigma}\Gamma_{\lambda} + k_{\lambda\varrho}k_{\sigma[\mu}\Gamma_{\nu]}] + \frac{2}{3}\Gamma_{[\mu}h_{\nu]\varrho}. \quad (26)$$

Equation (24) is now in the form in which it can be readily solved by Tonnelat's method. The main difference is that previously Γ_λ was known and now it can only be determined, if at all, from equation (24) itself.

4. Solution of the equation (24)

Let us apply to the equation (24) in succession the star operation, the double bar operation and the two operations combined. Remembering that

$$A^{**}{}_{\mu\nu\dots} = -A_{\mu\nu\dots}, \quad (27)$$

and recalling the identity (13) we obtain besides (24) the following additional equations for the four unknowns $\Gamma_{\mu\nu,q}$, $\Gamma_{\mu\nu,q}^*$, $\Gamma_{\mu\nu,\bar{q}}$, and $\Gamma_{\mu\nu,\bar{q}}^*$

$$\begin{aligned} 2\omega\Gamma_{\mu\nu,q} + \Gamma_{\mu\nu,q}^* + \Gamma_{\mu\nu,\bar{q}}^* &= J_{\mu\nu,q}^* + W_{\mu\nu,q}^*, \\ \omega^2\Gamma_{\mu\nu,q} + (2-j^2)\Gamma_{\mu\nu,\bar{q}} - 2\omega\Gamma_{\mu\nu,\bar{q}}^* &= J_{\mu\nu,\bar{q}} + W_{\mu\nu,\bar{q}}, \\ \omega^2\Gamma_{\mu\nu,q}^* + 2\omega\Gamma_{\mu\nu,\bar{q}} + (2-j^2)\Gamma_{\mu\nu,\bar{q}}^* &= J_{\mu\nu,\bar{q}}^* + W_{\mu\nu,\bar{q}}^*. \end{aligned} \quad (28)$$

From (24) and (28), we get

$$(a^2 + b^2)\Gamma_{\mu\nu,q} = U_{\mu\nu,q} + (2\omega b - ac)W_{\mu\nu,q} + (2\omega a + bc)W_{\mu\nu,q}^* + aW_{\mu\nu,\bar{q}} - bW_{\mu\nu,\bar{q}}^*, \quad (29)$$

where

$$a = 5\omega^2 + j^2 - 2, \quad b = 2\omega(3 - j^2), \quad c = 2 - j^2,$$

and

$$U_{\mu\nu,q} = (2\omega b - ac)J_{\mu\nu,q} + (2\omega a + bc)J_{\mu\nu,q}^* + aJ_{\mu\nu,\bar{q}} - bJ_{\mu\nu,\bar{q}}^*,$$

is the Tonnelat solution for $\Gamma_{\mu\nu,q}$ in the case when $\Gamma_\lambda = 0$.

Since then $\Gamma_{\mu\nu}^\lambda$ is determined by the equations (3) (with $A_{\mu\nu}^\lambda \equiv \Gamma_{\mu\nu}^\lambda$) for which

$$\sqrt{-g} \Gamma_\mu = h_{\mu\nu} g^{\nu\sigma}, \quad (30)$$

it follows that if

$$g \neq 0 \quad \text{and} \quad g^{\mu\nu}_{;\nu} = 0,$$

then

$$U_\mu = h^{\nu q} U_{\mu\nu,q} = 0, \quad (31)$$

Unless $a = 0$ and $b = 0$, the Tonnelat solution is unique.

If we now let

$$A_{\mu\nu,\lambda} = -\frac{2}{3} \Gamma_\nu h_{\mu\lambda}, \quad (32)$$

so that

$$A_{\mu\nu,\lambda} = \frac{2}{3} \Gamma_{[\mu} h_{\nu]\lambda}, \quad (33)$$

then

$$\begin{aligned} A_{\mu\nu,e}^* &= \frac{\sqrt{-h}}{2} \varepsilon_{\mu\nu\alpha\beta} h^{\alpha\gamma} h^{\beta\delta} A_{\gamma\delta,e} \\ &= \frac{\sqrt{-h}}{6} \varepsilon_{\mu\nu\alpha\beta} h^{\alpha\gamma} h^{\beta\delta} (\Gamma_\gamma h_{\delta e} - \Gamma_\delta h_{\gamma e}) \\ &= -\frac{\sqrt{-h}}{3} \varepsilon_{\mu\nu e\sigma} h^{\sigma\lambda} \Gamma_\lambda, \end{aligned}$$

or

$$-2\omega A_{\mu\nu,e}^* = \frac{2}{3} \sqrt{k} h^{\sigma\lambda} \varepsilon_{\mu\nu e\sigma} \Gamma_\lambda. \quad (34)$$

Also

$$\begin{aligned} A_{\mu\nu,\bar{e}} &= k_{e\sigma} h^{\delta\gamma} k_{\gamma\beta} h^{\beta\alpha} A_{\mu\nu,\alpha} \\ &= \frac{1}{3} k_{e\delta} h^{\delta\gamma} k_{\gamma\beta} h^{\beta\alpha} (\Gamma_\mu h_{\gamma\alpha} - \Gamma_\nu h_{\mu\alpha}) \\ &= \frac{1}{3} k_{e\delta} h^{\delta\gamma} k_{\gamma\nu} \Gamma_\mu - \frac{1}{3} k_{e\delta} h^{\delta\gamma} k_{\gamma\mu} \Gamma_\nu \\ &= \frac{2}{3} h^{\sigma\lambda} k_{\lambda e} k_{\sigma[\mu} \Gamma_{\nu]}. \end{aligned} \quad (35)$$

Hence

$$A_{\mu\nu,e} - 2\omega A_{\mu\nu,e}^* + A_{\mu\nu,\bar{e}} = W_{\mu\nu,e}.$$

Consequently, equation (29) is equivalent to

$$(a^2 + b^2) \Gamma_{\mu\nu,e} = U_{\mu\nu,e} + \frac{2}{3} (a^2 + b^2) \Gamma_{[\mu} h_{\nu]e}$$

from which Γ_μ cannot be determined.

In conclusion, we note that the above method of solution applies to any equation with scalar coefficients α , β , γ , δ , of the form

$$\alpha A_{\mu\nu e} + \beta A_{\mu\nu e}^* + \gamma A_{\mu\nu \bar{e}} + \delta A_{\mu\nu \bar{e}}^* = W_{\mu\nu e}.$$

REFERENCES

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