

# Application of trapped ions in quantum-computing: Fidelity, scalability and integrated technology

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**Abstract.** Quantum computing, which is based on the rules of quantum mechanics, has great potential in many fields for its theoretical ability to simulate any natural process based on quantum mechanics. Many types of physical implementations of quantum computers have been developed, and the method based on trapped ions is a promising one. However, there is still a lot of work that need to be done, like how to improve the fidelity and scalability of those quantum computers. Besides laser light, microwave and waveguides have also been used in some experiments.

**Keywords:** Quantum computing, apped ions, microwave, integrated optics.

## 1. Introduction

Built on the basis of quantum mechanical elements, quantum computers simulate quantum systems whose properties such as superposition and entanglement enable them to gain advantages over classical computers, such as significant increase in computing speed and potentials in cybersecurity[1]. There are many physical implementations of quantum computing process, such as superconducting circuits and nuclear magnetic resonance[2]. Among them, trapped-ion method has a lot advantages. For example, coherence times of trapped-ion qubits can be much longer than that of other types of qubits, and fidelity of gates realized on trapped-ion qubits is better than any other technique. Additionally, straightforward state preparation and read out process, limited number of calibration steps, and the ability to work at room temperature are also advantages of trapped-ion method.

Ion trap is a typical physical implementation of the quantum computing process. Paul trap is the most common one, in which we use electric fields provided by electrodes to trap the particles, create vacuum environment to prevent the ion from getting knocked out, make the laser through the window to cool the particles, and then single atoms can be seen through imaging lens[3]. Potential for scalability and long coherence times which can reach the range of minutes enable trapped ions to have great potential in the implementation of quantum computing process.

Precise control of the ions is required for realization of quantum computing based on ion trap, in order to perform initialization, gate operations and read out final states. Ion-trap computers need to perform gate operations in order to perform calculations, and different schemes have been developed, based on laser or microwave. Ultrafast pulse laser is the most frequently used method to manipulate the ions. Different types of lasers have their corresponding uses to manipulate the ions and produce different



results. Optimized sequences of coupling pulses can be used for rapid cooling, with the scheme having potential application for various systems[4]. A pair of counter-propagating, resonant ultrafast pulses can be used for achieving fast entangling gates, which solved the limit of gate speed caused by the secular motion[5]. Continuous wave can be used for cooling the atom's motion and localize the ions[6]. Optical frequency combs(OFC) is a type of ultrafast pulse laser generated from mode-locked lasers. OFC has properties such as homogeneous spacing and having coherent frequency components with stable phase that enables it to be used as a ruler for distant frequencies, which allow us to build connections between different energy levels in a controlled manner[7]. Mode-locked lasers can be used in many ways in coherent control and entanglement of qubits according to their properties.

According to the pair of ion states that define the qubit levels, qubits can be divided into four categories: fine-structure qubit, optical qubit, Zeeman qubit and hyperfine qubit. Hyperfine qubits' greater splitting allow them to have advantages in more straightforward state detection besides long coherence times[8]. With certain conditions satisfied, including the pulse bandwidth being much larger than the qubit splitting and much smaller than the detuning, laser pulses can be used to perform hyperfine qubit operations in an effective way. Slow decoherence rates can be realized by ultrafast operations in an atomic system[9]. In the fast regime, one-qubit gate operation can be achieved by controlling the intensity and phase-of-the-phase between two fast pulses, with a duration less than tens of picoseconds, and by driving simulated Raman transitions from a picosecond mode-locked laser, single-qubit operation can be achieved with a gate speed much faster than the frequency of the motion of a trapped atom[10]; In the slow regime, approximations can be made according to the condition that the ratio of the trap cycle time to the pulse train approaches zero in order to achieve atom-atom entanglement[9]. Multi-ion quantum logic gates are commonly mediated by shared motional modes. Coherent single-qubit rotations result from pulse resonances with the qubit transition following state preparation and cooling.

In this review, we will focus our discussion on the application of trapped ions in quantum computing. We will discuss sources of error and methods to improve fidelity, compare different schemes to scale up the quantum computing system, then introduce integration technologies and new laser-free methods. At last, we will look into the future research direction of trapped-ion quantum computers.

## 2. Fidelity

Experimental progress in single-qubit and double-qubit atoms has reached a relatively high level, while how to maintain the high-fidelity property property in multi-qubit atoms still remains a problem. Fault-tolerant quantum computers require total errors to be below 0.01 to function normally[11]. For the Bell states, motional mode heating, frequency drifts and laser frequency noise are three main sources of estimated error; two-ion readout error, Kerr cross-coupling and spectator mode occupancies are also three sources that contribute to gate errors. Near-ground-state cooling are required for high-fidelity implementations. To meet the requirements for fault-tolerant quantum computation, techniques for qubit manipulation need to be improved to reduce the errors caused by field noise or fluctuations in general, as in most cases motional heating and AMH are the main sources of decoherence[8].

High-fidelity single-qubit and two-qubit logic gates have been performed on trapped-ion hyperfine qubits of calcium-43 ions. For two-qubit gate on calcium-43 ions, a method has been developed to measure the gate error which include generating the ion to the bell state by using it together with single-qubit rotations and then excluding the qubit state-preparation and measurement(SPAM) errors as well as the dominant single qubit errors. There are many sources of error for two-qubit quantum gates, and the trade-off between gate fidelity and gate time has been studied. Photon scattering which include both Raman and Rayleigh contributions is the dominant source of error for gate times between 3 microseconds and 100 microseconds. For photon scattering, if the intensity of the Raman beam is held constant, as the laser detuning decreases, both the gate speed and the photon scattering error increase with it. The Raman beams should be able to make balance between a proper gate speed and a gate fidelity as high as possible. Spin dephasing from magnetic field fluctuations is the dominant source of error for gate times above 200 microseconds. This type of error can be reduced by improving

experimental apparatus in order to reduce noise uncorrelated between the two halves of the spin-echo as much as possible. In addition, motional heating and dephasing as well as other errors relevant to the ions' motional states also contribute a lot to two-qubit gate error. Intensity drift of the laser beam and phase shifts added by a.c. Stark shifts also have contributions to the gate errors, and the latter can be suppressed greatly by pulse shaping[11]. There are many other sources of error for two-qubit gates, and most of them are negligible compared with those dominant ones.

Besides manipulation techniques, different methods for improving the fidelity of trapped-ion quantum systems have been explored, include different gate methods, ion types and the states chosen. Progress in the control of Rydberg excitations with a high fidelity has been made. Successfully reduced laser phase noise bring a significant improvement in coherence properties. In this case, the fidelity for a two-atom entangled state can exceed 0.97[12]. This fidelity is above the threshold, but shows the prospect of high-fidelity multi-qubit quantum gates reached by improving experimental techniques. Besides, experiments of all single-qubit operations with errors more than one order of magnitude below the threshold for surface-code quantum error correction performed on  $^{43}\text{Ca}^+$  in hyperfine atomic clock states with high fidelities have been implemented successfully, by choosing the proper states for qubit and designing a scalable microfabricated surface-electrode ion trap[13].

**Table 1.** Architectures for scaling quantum computing systems.

Architecture	Advantages	Limitation	Causes	Corresponding schemes
1 linear arrays	Simplicity	Gate speed and fidelity decrease with increasing number of ions	Resolving every motional mode and entangling distant ions will be more difficult when there are more ions	Use pulse modulation to entangle arbitrary pairs of ions
		Techniques for fast and low-error swap operations need to be developed	More modules require more swap operations	Swapping the ions' positions in a physical way can reach a high fidelity, but it is still not fast enough
2D arrays	1 the potential for transporting quantum information with higher speed and fidelity	The cooling process for higher fidelity costs too much	Higher temperature means additional heating motion that will reduce the gate fidelity	"Diabatic" and "adiabatic" transport schemes, and the latter performs better in terms of speed and fidelity
	2 QCDD can realize various functions in different regions	Difficult implementation of the arrangement	The basic arrangement and junctions required for 2D arrays increase the need for electrodes a lot	A promising technique__microfabricated surface-electrode ion traps
photonic interconnects	1 Besides different modules as the 2D array, connection between different chambers is also allowed	The rate for generation still need to be increased	The highest rate achieved so far is around 5Hz	Adopt a larger number of ions in the modules in parallel
	2 Independent connection speed	High cost that comes with the rate-increasing strategy	Larger number of ions and more complexity in manipulating the photons lead to heavier overheads	Improve the photon collection efficiency by coupling between the ion and a specially designed cavity
	3 Extensibility to more modules			

### 3. Scalability

Scalability of the whole quantum system is also a significant element, and the main challenge lies in how to improve operation fidelities while scaling quantum computing systems. There are many types of architectures and techniques developed for scaling to larger numbers of ions, like linear arrays, 2D arrays and photonic interconnects. Their relevant information is listed as follows.

These are some of the methods developed for increasing the number of ions and discussions about the problems arise with them. There are still a lot of work that need to be done to improve them.

### 4. Integration

Integrated technologies for ion control is an essential element for the construction large-scale quantum systems. Integrated optics can be used to drive multi-ion entangling quantum logic, like an eight-channel fibre array attached to the ion trap, and optical fibres are widely used for delivery. This type of assembly is usually placed in a cryogenic vacuum apparatus and cooled. Waveguide or free space fiber for light deliver are also widely used for integration. In many methods, waveguides and gratings are formed in the lower platinum where the substrate reflect the light to increase the grating efficiency. Gratings for certain wavelength of light can also be designed to illuminate certain ions uniformly.

A theoretically scalable microfabricated trap based on diffractive optics which use integrated diffractive mirrors to interface a single  $^{174}\text{Yb}^+$  ion with single-mode optical fiber as well as free space was designed and realised, and the total coupling efficiency was measured to be 4.1(6) [14]. A microfabricated planar ion trap on a glass wafer with an integrated waveguide was fabricated, and the detection of particle secular motion by using the integrated waveguide as well as evaluation of the particle's characteristics can be realised [15]. Designation and fabrication of a surface trap with integrated wave guides as well as the producing and cooling process of the  $^{171}\text{Yb}^+$  ion was demonstrated. Besides, the heating rates and its relations with position, the axial motion and the ion-surface distance, the waveguide beam profile and electrostatic charging were also measured [16]. An scalable ion trap with great potential in miniaturizing was made from a monolithic microchip by using MEMS technology [17].  $^{24}\text{Mg}^+$  ions were trapped in a surface-electrode Paul trap where fluorescence light from the ion were efficiently collected by using integrated fiber optics, and the trap has the ability to position the ion at a certain range of distances [18]. A scalable system that combines ion traps and specially-designed lenses was built, their collection performance was measured, and the manageable impact of proximity was shown [19].

Besides, integrated optics can be used to provide laser light source for a surface-electrode ion trap, and based on this technique, integrated multi-wavelength control can be realized later. Both of them demonstrate the integration of a  $^{88}\text{Sr}^+$  qubit ion trap. They delivered the fiber required through fibers, made use of various voltages applied to the electrodes to change the ion position, but the wavelength of light change from merely 674nm that drives the quadrupole transition of the ion to 674nm, 461nm, 422/405nm and 1092/1033nm which were used to perform all basic ion-qubit operations, and the diffraction caused by vertical grating couplers was demonstrated to couple different beams of light to the integrated photonic trap chip. Besides, there are many differences between them. To achieve fiber-trap integration, the fiber of the former was embedded in the substrate, while the latter made use of a fiber-array block where the facet of the chip and the fibers were bonded and inverse-taper waveguide couplers couple the light with different wavelengths from fibers to the ion-trap chip. Their characterization of the profiles of the beams were also different. The former realized the imaging process through an in situ micromotion-free optimization technique, and the latter measured the relationship between position of the ion and the interacting strength of laser and light to achieve this characterization [20][21].

Integrated optics and detectors can also be used for light collection and detection of trapped ions, and many methods of integration have been developed by making use of optical techniques like diffractive mirrors and lens. In one technique, as integrating photon collection and detection elements with the trap has promise in scalability and high collection efficiency, and single-photon avalanche photodiodes can be operated at room temperature, the use of room temperature surface ion trap that is monolithically

integrated with SPADs for measuring ion fluorescence can avoid the size and power consumption while keeping the ability for scaling up. Different sources of counts were measured, the ion/non-ion fidelity was characterized by timestamped digital pulses using the adaptive Bayesian technique[22]. In another technique, a scalable photon-ion interface on a multi-zone micro-fabricated surface trap with integrated diffractive mirrors was realised. Monolithic diffractive mirrors are integrated, and diffractive optics has great potential for scalability. The collection efficiency of the integrated optics was measured by using a single photon generation protocol, and the protocol was triggered enough times for counting. The mirrors have potential for the implementation of quantum networks and remote entanglement sharing with trapped ions as well as other quantum light sources[23].

### 5. Laser-free control methods

Laser light has been commonly used in trapped-ion quantum systems, but primary sources of limitations like laser intensity and phase noise always limit the fidelity. Laser-free operations using static M-field gradients or microwave frequency can avoid these sources of infidelity based on laser systems. However, the fidelity and speed have been proved to perform worse in experiments than laser-based interactions. The fidelity of the newly-invented laser-free method making use of oscillating near-field rf frequency M-field gradient has reached almost the same level as fidelity of the highest fidelity of laser-based operations, and the speed of this method also performed better. The experimental apparatus included rf and microwave control currents and trapping voltages, and qubit states are  $|F=3, mF=3\rangle$  and  $|F=2, mF=2\rangle$  states within the two  $^{25}\text{Mg}^+$  ions'  $^2\text{S}_{1/2}$  ground-state hyperfine manifolds. Currents were applied to generate M-field gradient as well as entangling interaction, and the transformation from the initial state to the symmetric Bell state took 740 micrometers. Antisymmetric Bell states were created from the symmetric state by using the universal qubit control which was achieved from individual qubit addressing enabled by tolerance to qubit frequency offsets, and the fidelity  $0.9977^{+0.0010}_{-0.0013}$  is fairly high. It has been estimated that decoherence of the ion motion is the main source of Bell-state infidelity, which was around  $7 \times 10^{-4}$  totally, and increasing the interaction strength can help to reduce this source of errors. This scheme shows great potential in scalability, as simultaneous entangling operations can be carried out in different zones of a multizone trap, and its insensibility to offsets or drifts in the driving parameters[24].

Increasing the fidelity of quantum logic gate operations and scaling up the techniques are two main challenges for realizing useful quantum computation. Besides widely-used laser light, long-wavelength radiation in the microwave or radiofrequency regime are also potential candidates for constructing large-scale quantum computers. Both single-qubit operations[25] and a two-qubit logic gate[26] have been applied to trapped-ion qubits stored in hyperfine states in the ground level of  $^{43}\text{Ca}^+$ . They have been implemented with the same apparatus, and the two-qubit scheme was carried out on the basis of those single qubit accomplishments. They were both implemented in room-temperature microfabricated surface ion traps with exceptionally high fidelities which were above the  $\approx 99\%$  threshold required by fault-tolerant quantum computation. They were driven by near-field microwaves, and the latter was especially driven by microwave near-field gradient. Microwave methods show their great prospect in performing coherent operations for they can be purely electronic, and the simplicity as well as stability of microwave. Atomic clock transitions are first-order-independent to magnetic field at 146G, and this feature allows the realization of sharing the same noise-immune frequency among all qubits. With regard to the former, Ramsey experiments on the “atomic clock” qubit transition are performed to measure the qubit coherence time, and its limiting factors are estimated to be residual magnetic field drift, fluctuations of the rf voltage amplitude and instability of the local oscillator; About the latter, in order to measure a.c. Zeeman shift fluctuations, single-ion Ramsey experiment is also carried out, which shows the advantage of the DDMS gate over previously developed MS gate. For the single-qubit operations, the combined SPAM(state preparation and measurement) error is measured by repeating the preparing and reading out process for 150000 times, and the result is  $6.8(5) \times 10^{-4}$ , which is better than the results of other measurements performed on single physical qubits. In this process, both microwave and optical techniques are included. Circularly polarized light, microwave pulses, optical pumping

pulses and lasers used for Doppler Cooling are used. Nearly all sources of error of these two microwave-based methods could be reduced by improving pure techniques, and technical noise can also be reduced for less error by employing composite pulse techniques. Single-qubit gate errors were measured by randomized benchmarking, off-resonant effects and detuning offset as well as pulse area error of microwave were estimated to be main sources of gate error, while standard tomography is used to measure the fidelity of the generated Bell state  $|\Phi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + i|\downarrow\downarrow\rangle)$  produced by the four-loop gate, and the fidelity is calculated to be 99.7(1)%. Beside  $^{43}\text{Ca}^+$ , there are also microwave-based techniques realized on other ions. For  $^{25}\text{Mg}^+$ , coherent manipulation of its internal states as well as entanglement between internal degrees of freedom of the two atoms have been achieved with a gate fidelity of 0.76(3) [4]. Like the near-field microwaved-based techniques described above, apparatus of this method includes a surface-electrode ion trap working at the room temperature. Similar to the qubit on  $\text{Ca}^+$  mentioned above, first-order magnetic-field-independent transition was realized in this technique which is between states  $|3, 1\rangle$  and  $|2, 1\rangle$ . What's more, Rabi flopping on the  $|3, 1\rangle \rightarrow |2, 1\rangle$  transition can demonstrate single-qubit operation speed, which is much higher than the typical value obtained with laser beams. In addition, global radiation fields were also used in implementation of quantum states and generation of entangled states. The overhead as well as difficulty of previous schemes which were based on long-wavelength radiation or laser increase exponentially when the number of ions goes up. This correlation between the number of ions and the number of radiation fields was once a challenge for realization of large-scale quantum computers. To solve this problem, quantum-engineered noise-resilient clock qubits were used and a new method was developed. This new type of state which came from this newly-created entangling gate was achieved by using a static magnetic field gradient as well as long-wavelength radiation. Voltages were controlled individually and applied to each location of logic gate, and this enabled parallel gate operations to be facilitated in different zones. This technique merely required a small number of global radiation fields, and this property was not affected by the number of ions. The gate operation was applied to two  $^{171}\text{Yb}^+$  ions, and their different positions led to different magnetic field exerted on them because of the field gradient. Addressing the ions individually was enabled by the tunable frequency of qubit transition, which shows the advantage over standard clock transitions. Scheme of  $^{25}\text{Mg}^+$  and scheme of  $^{171}\text{Yb}^+$  realized individual ion addressing, especially the former, where ions in different zones were individually addressed by using electric field and voltages and ions in the same entangling zones were individually addressed by applying magnetic field gradient. For all the schemes mentioned above, fidelity can be increased by improving technical apparatus, and laser light can not only be used further for cooling the ions, but also combined with long-wavelength radiation to realize more functions in improving scalability as well as fidelity[25-28].

## 6. Outlook

Quantum computers' computational power allows it to have great potential in a wide variety of fields. It could be used to simulate any natural process such as quantum systems, dynamics within complex particles and black holes, which far surpasses the capability of classical computers[29]. Besides, quantum algorithms also have significant potential in cryptography, information theory, mathematics and language theory[30]. Efficient randomized algorithms were given for prime factorization and discrete logarithms by Peter Shor in 1997, which demonstrated quantum computers' potential in increasing computing speed significantly[29]. The research on building general purpose quantum computers still has a long way to go, and there are challenges in terms of fidelity, scalability, size and speed. How to increase all of them at the same time still remains a problem. To build a quantum computer reliable enough, both reducing the effects caused by the surroundings and some sort of error correction is needed. According to the threshold theorem, as long as the error rate or time step is limited to a certain degree, creating a quantum computer to perform arbitrary quantum calculations is possible[31]. Fault-tolerant quantum computing is expected in the future, and more application scenarios for quantum computers will be explored.

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